TESIS DOCTORAL

COMPUTATIONAL ALGORITHMS FOR CYCLIC PLASTICITY BASED ON PRAGER'S TRANSLATION RULE

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Meijuan Zhang

DIRECTOR DE LA TESIS DOCTORAL:

Francisco J. Montáns Leal

CODIRECTOR DE LA TESIS DOCTORAL:

José María Benítez Baena

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Based on Prager's Translation Rule

Meijuan Zhang
Máster en Ingeniería Industrial

Director: Francisco J. Montáns Leal
Doctor Ingeniero Industrial

Codirector: José María Benítez Baena
Doctor Ingeniero de Caminos, Canales y Puertos

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Based on Prager's Translation Rule.

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Make days worth remembering.

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Resumen

El objetivo de esta tesis es introducir la teoría y un algoritmo de integración totalmente implícito para la plasticidad de superficies multiples utilizando la regla de Prager con endurecimiento cinemático e isotrópo.

El modelo utiliza la idea de Mróz de discretizar la curva de tension-deformación uniaxial en varios segmentos, resultando un campo de superficies de endurecimiento. Este enfoque es muy sencillo para la obtención de parámetros de material del modelo. Sin embargo, en contraste con la propuesta de Mróz, el modelo se basa en la regla tradicional de Prager. La superficie de fluencia es siempre la superficie más interna. Las superficies externas son superficies de endurecimiento, y constituyen una herramienta para describir un módulo de endurecimiento anisotrópo dependiente de la historia. La formulación y el algoritmo numérico recuperan naturalmente la plasticidad clásica de von Mises con endurecimiento mixto en el caso de las curvas tensión-deformación bilineal uniaxial, independientemente del número de superficies que se utilicen. Por lo tanto, las predicciones son autoconsistentes.

El algoritmo de integración se basa en el algoritmo de retorno radial. Se incluyen linealizaciones consistentes locales y globales con el fin de conservar la convergencia asintótica de segundo orden de los esquemas de Newton. Algunas simulaciones de puntos de tensión y elementos finitos muestran tanto la aplicabilidad como la robustez del algoritmo de integración de tensiones. El modelo se valida con predicciones de los resultados experimentales de ensayos uniaxiales y multiaxiales tomados de la literatura, que incluyen caminos de carga muy diferentes. La eficiencia numérica de la formulación algorítmica Backward-Euler totalmente implícita se demuestra mediante la implementación en un programa de elementos finitos y la simulación de un ejemplo de carga cíclica en una estructura formato de la literatura.

El uso de este modelo ha dado lugar a dos observaciones importantes.

Mediante la predicción de los desplazamientos exactos de la tensión de plastificación, los mismos caminos de carga de sondaje de experimentos reales sobre la evolución de superficies de rendimiento subsecuentes, se obtienen resultados similares sobre superficies de rendimiento aparentes en comparación con los de experimentos, lo que resalta el propósito

más importante de esta tesis : Mostrar que una parte relevante de las observaciones en los experimentos puede atribuirse a (y por lo tanto modelado por) endurecimiento cinemático anisotrópico desarrollado durante la precarga.

Otra, que no se suele comentar en la literatura, es la influencia del tamaño de la superficie de plastificación real en las predicciones bajo algunas caminos de carga, incluso si se emplea la misma curva tensión-deformación. Puesto que una determinación precisa del tamaño real de la superficie de plastificación es difícil en algunos materiales, la trayectoria de tensiones durante la carga multiaxial puede facilitar esta determinación.

Abstract

The focus of this thesis is to introduce the theory and a fully implicit integration algorithm for multisurface plasticity using Prager's rule with mixed isotropic and nonlinear kinematic hardening.

The model uses the idea from Mróz of discretizing the uniaxial stress-strain curve in several segments, resulting in a field of hardening surfaces. This approach is very simple in obtaining material parameters of the model. However, in contrast to Mróz's proposal, the model is based on the traditional Prager's rule and the yield surface is always the innermost surface. The outer surfaces are just hardening surfaces, a tool to describe an anisotropic history-dependent hardening modulus. The formulation and numerical algorithm naturally recover classical von Mises plasticity with mixed hardening in the case of uniaxial bilinear stress-strain curves, regardless of the number of surfaces being employed. Hence, the predictions are self-consistent.

The integration algorithm is based in the closest point projection algorithm. Both local and global consistent linearizations are included in order to preserve the asymptotic second order convergence of Newton schemes. Some stress-point and finite element simulations show both the applicability and the robustness of the stress integration algorithm. And the model is validated with predictions of the experimental results from both uniaxial and multiaxial tests taken from the literature, which include very different loading paths. The numerical efficiency of the fully implicit, Backward-Euler algorithmic formulation is demonstrated by the implementation in a finite element program and the simulation of a cyclic loading example from the literature.

Using this model has resulted in two important observations.

By predicting the exact offsets of probing plastic strain, the same probing loading paths of actual experiments about the evolution of subsequent yield surfaces, similar results about apparent yield surfaces are got compared to those from experiments, which brings out the most important purpose of this thesis: to show that a relevant part of the observations in the experiments may be attributed to (and hence modelled by anisotropic

kinematic hardening developed during preloading.

Another one, not usually commented in the literature, is the influence of the size of the actual yield surface in the predictions under some loading paths, even if the same stress-strain curve is employed. Since an accurate determination of the actual size of the yield surface is difficult in some materials, the stress path during multiaxial loading may facilitate this determination.

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Chapter 1

Introduction

1.1 Objective of this thesis

The objective of this work is to show that instead of being only applicable to linear kinematic hardening model, the Prager's translation rule can also be used to modeling non-linear cyclic plasticity through a multilinear kinematic hardening model. To achieve this objective, the consistency of the model in describing multi-axial behaviors is proved, the robustness of finite element implementation is shown, and more importantly, the effectiveness in predicting multiaxial responses under non-proportional loadings is demonstrated with multiple simulations of experiments from literature.

A fully implicit algorithm is developed to enable the description of cyclic hardening/softening, which means that theory and structure of the model allow the extension to include such history-dependent effects. Furthermore, the model has the potential of predicting the ratchetting effect by adding a dynamic recovery item.

More importantly, this work brings out new perspectives about some well accepted notions: the distortion of the shape of yield surface and the nonproportional hardening. Does the yield surface really change shape after being subjected to loadings? Or it is only a observation due to assumptions caused by the limitation of experimental conditions? Does nonproportional hardening exist? Or it is just a result of an illy assumed size of yield surface shown through multiaxial behaviors?

1.2 Structure of the thesis

This thesis contains three main parts.

The first part is this introduction. Below an introduction to some physical phenomena in cyclic plasticity is addressed, to which computational formulations are addressed for engineering purposes. Second, detailed background about cyclic plasticity theories that are related to each aspect of modelling is explained. Third, an introduction about existing rules and special means of modelling that are related to the model in this work is performed. Fourth, an explanation about some typical and well-known models as well as a brief review about the evolution of the shape of yield surfaces is carried out.

In the second part, the theories that are related to the model are explained, the development of the numerical algorithm to include the combined hardening, the finite element implementation as well as the procedure for the determination of the material parameters are illustrated. It also focuses on the effectiveness of the model on the prediction of cyclic plastic behaviors under nonproportional loadings. In addition, the influence of the size of the actual yield surface on the prediction of material's behaviors under multiaxial loadings is proved.

In the third part, a new way to understand the observed evolution of yield surfaces in experiments is shown. Then its validity is proved through a reasonably accurate prediction of an experimental observation that supports our understanding about the evolution of the shape of yield surface.

1.3 Phenomenological aspects and mathematical theories about cyclic plasticity

1.3.1 General introduction about plasticity

Plastic behavior is the behavior of irreversible deformation of materials and the physical nature can be different depending on the material. For materials like soils, plastic behavior is the dominant behavior.

The theory of plasticity is about the mathematical description of mechanical behaviors of materials under irreversible strains. It can be used to predict the stability or security of structures, to calculate the deformation of structures under loading, or on the contrary, to anticipate the loading needed in order to achieve a certain deformation, for example, in metal forming.

Theoretically, the deformation of the material is in an elastic regime before a certain value of stress is reached, after this stress is reached, permanent deformations occur. In monotonic loading, the stress that defines the limit is called the plasticity threshold or the yield stress σ_Y .

Figure 1.1: The three possible phenomena after the yield stress is reached

In practice, specially with very ductile materials, it is very difficult to detect the value of σ_Y . One conventional and approximate way is defining a certain amount of permanent strain as an offset strain value, and carrying out experiments to measure the stress value under this offset strain value.

A commonly used value for this permanent strain value is 0.2%, but if it is too high compared to the corresponding elastic strain, a more strict offset strain value should be used, a conventional more strict one is 0.02%. Another procedure to define this offset value is using a relative value of the permanent strain value compared to the elastic strain range.

Under plastic regime, materials can exhibit different mechanical behaviors. Depending on the material type, different plastic behaviors may be observed. For

instance, in the stress-strain curve of a material by an uniaxial test, after the elastic regime, there may be a flat region, or a raising or falling curve as is shown in the Fig 1.1, which are the three simplest linear cases. In reality, the stress-strain curves are usually more complicated and does not only consist of only two simple segments, mostly are non-linear, and that is why even though linear models are still widely used, multilinear or non-linear models can have more precise description of the behaviors of materials.

1.3.2 Cyclic plastic phenomena

The term "cyclic" can be a little bit misleading, as long as the loading is not holding forever, there is the possibility of showing cyclic behavior because of unloading. Depending on the materials and types of loading, the cyclic phenomena can be very different and complex. Depending on the focus of cyclic plastic models, cyclic plastic phenomena should be modelled in a practical and controllable way. Thereafter, some basic and well-known cyclic plastic phenomena are described.

Bauschinger effect

It is well-known that when most polycrystalline metal materials are subjected to cyclic loading, the Bauschinger effect can be observed. After the material is plastified due to the loading in one direction, if the loading is changed to the reverse direction, it can be observed that the yield stress during the later loading is smaller than that during the former loading, as shown in Figure 1.2. Being capable of predicting the Bauschinger effect is one of the fundamental requirements for cyclic plastic models.

Masing behavior

A material shows Masing behavior when the upper part of hysteresis stress-strain loops with different strain amplitudes after alignment in lower peaks overlap, as shown in Fig 1.3 (a). Masing behavior usually suggest a stable microstructure in the fatigue process, on the other hand, materials that does not obey Masing behavior may show some dependence on the amplitude of plastic strain or the accumulated plastic strain, for example, materials that have cyclic hardening/softening effect do not obey the Masing behavior.

Cyclic hardening/softening

Due to the increase or decrease of the material's resistance to deformation under cyclic loading, the material can show the cyclic hardening/softening effect. This can be seen in Fig 1.4, during cyclic loading, when the strain amplitude is prescribed

Figure 1.2: The Bauschinger effect

Figure 1.3: Hysteresis loops after alignment of a material that obeys the Masing rule (a) and a example of a material that doesn't obey the Masing rule (b)

Figure 1.4: A demonstration of the cyclic stress-strain curve with cyclic hardening(a) or cyclic softening(b) effect

as constant, the amplitude of the stress response may be increasing or decreasing continuously depending on the material, and the hysteresis strain-stress loops are closed. This phenomenon is especially notable during the first loading loops, and it usually becomes less obvious after a certain number of loops, as if the stress amplitude is stabilizing. In most cases the stabilized material keeps relatively stable until the formation of fatigue crack. Cyclic hardening/softening effect can also be illustrated by the hysteresis loops under stress controlled loading.

Under strain controlled loading, it is interesting to relate the number of loops to the increment/decrement of the peaks of stresses, to get the relation between them usually by curve fitting. Since the accumulated plastic strain or the plastic work can be approximately represented by the number of loading loops, the change of the peak-value of stress is usually related to the accumulated plastic strain or plastic work. Furthermore, this relations allows to model general multi-cycle loading cases.

For modelling this phenomenon, data from cyclic experiments are used to determine parameters for cyclic plastic models with isotropic hardening rule.

Ratchetting

For some materials, during cyclic loading, if the prescribed stress limits are stable and unsymmetrical, it will produce a "creep" effect in the direction of the mean stress, as shown in Figure 1.5. This effect can be generally considered as an accumulation of any component of the strain tensor due to the increase of loading cycles.

It is worth noticing that other effects like cyclic hardening/softening can also produce a ratchetting behavior.

To model ratchetting, data from cyclic stress-strain experiments are needed to determine parameters for kinematic hardening parameters, for example, the dynamic recovery item of Chaboche non-linear kinematic hardening model. The ratchetting behavior can usually be described with parameters like the mean stress, the number of loading cycles and the mean strain(the mean value between two peaks of stress), etc.

1.3.3 Yield function and flow rule

Since the stress is a tensor, it is not straightforward to use the stress tensor to check the current state or the change of the state of the material. So a scalar function based on the stress is used, which is usually called the "yield function", usually denoted by f . The purpose of function f is to give a scalar value that represents the stress state, so as long as the produced value is positive for non-elastic states, negative for elastic states and zero for the yield state. The yield function f can have

Figure 1.5: An example of the ratchetting effect

different expressions informed from expressions observed in different materials.

The flow rule determines the plastic strain evolution and contains the magnitude as well as the direction of the rate of plastic strain.

1.3.4 Loading/unloading and consistency condition

We denote $\dot{\gamma}$ as a scalar related to the rate of plastic strain, and $f(\boldsymbol{\sigma}, \sigma_Y)$ as the yield function defined by current stress state $\boldsymbol{\sigma}$ and the yield stress σ_Y . Thus, to characterize the behavior of a elastoplastic material, the Karush-Kuhn-Tucker conditions are used:

$$\dot{\gamma} \geq 0, f(\boldsymbol{\sigma}, \sigma_Y) \leq 0 \quad (1.1)$$

Two possibilities hold: First, if the process is elastic:

$$\dot{\gamma} = 0, f(\boldsymbol{\sigma}, \sigma_Y) < 0 \quad (1.2)$$

Second, if the process is plastic:

$$\dot{\gamma} > 0, f(\boldsymbol{\sigma}, \sigma_Y) = 0 \quad (1.3)$$

In both ways, the relation holds:

$$\dot{\gamma} f(\boldsymbol{\sigma}, \sigma_Y) = 0 \quad (1.4)$$

In the plastic case, the last time-derivative of the last equation leads to:

$$\dot{\gamma} \dot{f}(\boldsymbol{\sigma}, \sigma_Y) = 0 \quad (1.5)$$

and this equation is called the consistency condition, and the parameter $\dot{\gamma}$ is calculated based on this condition.

Since the stress point is always characterized by a point on the yield surface when it is not in the elastic regime, we always have $f(\boldsymbol{\sigma}, \sigma_Y) = 0$, when $\dot{f}(\boldsymbol{\sigma}, \sigma_Y) < 0$ the material is under elastic unloading, when $\dot{f}(\boldsymbol{\sigma}, \sigma_Y) = 0$, the material is under plastic loading [35].

It is interesting to notice that the definition for loading and unloading may also vary depending on the model. For example, for models with "vanishing" elastic regime, $\dot{f}(\boldsymbol{\sigma}, \sigma_Y)$ doesn't have a valid form, so different definitions for loading and unloading may be applied.

1.3.5 J2 plasticity

For materials like ductile metals, experiments show that the yielding is decided by the second invariant (J_2) of the deviatoric stress tensor. This was brought up by von Mises at the year of 1913, thus it is called the von Mises criterion. Theories and implementation of this criterion is simply called "J₂ plasticity".

For a symmetric Cauchy stress tensor $\boldsymbol{\sigma}$, it can be written into its volumetric part and deviatoric part as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^v + \boldsymbol{\sigma}^d = p\mathbf{I} + \boldsymbol{\sigma}^d \quad (1.6)$$

where $\boldsymbol{\sigma}^v$ is its volumetric part. $p = \text{tr}(\boldsymbol{\sigma})/3$ is the mean normal stress, and $\mathbf{I}$ is the second rank identity tensor. tr is the trace operator. To write $\boldsymbol{\sigma}^d$ in a tensor form:

$$\boldsymbol{\sigma}^d = \text{dev}(\boldsymbol{\sigma}) = \mathbb{P}_1 : \boldsymbol{\sigma}^f = [\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}] : \boldsymbol{\sigma}^f \quad (1.7)$$

And for a strain tensor, it can be written into a similar way:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^v + \boldsymbol{\epsilon}^d = \epsilon^v \mathbf{I} + \boldsymbol{\epsilon}^d \quad (1.8)$$

$\boldsymbol{\epsilon}^v$ and $\boldsymbol{\epsilon}^d$ are the volumetric and deviatoric strain tensors, and ϵ^v is the volumetric strain:

$$\epsilon^v = \text{tr}(\boldsymbol{\epsilon}^f) \quad (1.9)$$

Elastic constitutive equations are:

$$p = K\epsilon^v$$

and

$$\boldsymbol{\sigma}^d = 2\mu\boldsymbol{\epsilon}^d \quad (1.10)$$

where K is the elastic bulk and μ is the shear moduli. so:

$$\boldsymbol{\sigma} = p\mathbf{I} + \boldsymbol{\sigma}^d = K\epsilon^v \mathbf{I} + 2\mu\boldsymbol{\epsilon}^d = \mathbb{C}^e : \boldsymbol{\epsilon} \quad (1.11)$$

so the elastic moduli $\mathbb{C}^e$ is:

$$\mathbb{C}^e = K\mathbf{I} \otimes \mathbf{I} + 2\mu(\mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}) \quad (1.12)$$

It is a fourth rank tensor which relates the total strain to the total stress.

If we use J_2 to represent the second invariant of the deviatoric stress tensor:

$$J_2 = \frac{1}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{2}\|\boldsymbol{\sigma}^d\|^2 \quad (1.13)$$

when it is expressed in the terms of the principle stresses, it can be written as:

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2] \quad (1.14)$$

The J_2 yield criterion, which is the von Mises yield criterion, determines whether a stress state is at yield or not by comparing the value of J_2 with a material constant k :

$$\begin{cases} J_2 = k^2 & \text{on the yield surface} \\ J_2 < k^2 & \text{inside the yield surface} \end{cases} \quad (1.15)$$

thus, in the space of principle stresses, the yield surface is:

$$\frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2] = k^2 \quad (1.16)$$

which indicates that in the space of principle stresses, the yield surface is a cylinder with radius $\sqrt{2}k$, and the hydrostatic axial $\sigma_1 = \sigma_2 = \sigma_3$, in another word, the cylinder has axial direction $[1, 1, 1]$.

The constant k for a certain material can be detected by uniaxial tests, and it should be equal to the scalar value of yield stresses got by those tests. The yield stress is usually determined by a offset value from the linearity of the stress-strain curve. Since the uniaxial tests can be torsion or tension-compression, attention should be paid here that the equivalent strain and stress values should be used here.

With f as the yield function, the von Mises criterion can be expressed as:

$$\begin{cases} f = J_2 - k^2 = 0 & \text{on the yield surface} \\ f = J_2 - k^2 < 0 & \text{inside the yield surface} \end{cases} \quad (1.17)$$

The yield function f can have different forms, in J_2 plasticity, A commonly used form is:

$$f = \sigma_{eq} - Y \quad (1.18)$$

Y is the yield stress and σ_{eq} is the well-known von Mises equivalent stress:

$$\sigma_{eq} = \sqrt{\frac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} = \sqrt{3J_2} \quad (1.19)$$

The equivalent plastic strain can be expressed in a similar form, a quick way to decide the expression of equivalent plastic strain is by considering the computation for plastic work, so the scalar constant for equivalent plastic strain is $\sqrt{\frac{2}{3}}$ instead of $\sqrt{\frac{3}{2}}$. And this definition is quite important for detecting the subsequent yield surface, and it will be mentioned again at that section.

For the model of this work, the yield surface is a hardening surface with radius $r = \sqrt{\frac{2}{3}}Y$, before any strain hardening happens, the form of the yield function used is:

$$f = \|\boldsymbol{\sigma}^d\| - r \quad (1.20)$$

1.3.6 Isotropic, kinematic and combined hardening

The essential difference between isotropic hardening and kinematic hardening is the treatment of the elastic strain during loading and unloading. In isotropic hardening model, the elastic regime increases with plastic straining; in kinematic model, the elastic regime is always constant.

In the elastic-perfectly plastic case, there is a plateau in the stress-strain curve right after the elastic regime. But actually, materials usually show a strain-hardening effect after the initial yielding, so strain-hardening models are needed to predict this strain-stress response.

Isotropic hardening

Isotropic hardening rule is featured with the increasing/decreasing of the yield stress according to the hardening/softening behavior of the material, thus resulting in the increment/decrement of the elastic region, as is shown in Figure 1.6. However, because it cannot describe many of the plastic phenomena, for example, it couldn't describe the Bauschinger effect, it is rarely used alone.

Although the application of isotropic hardening began from about 1950, the theories were developed much earlier around 1870. Theories based on isotropic hardening have a significant application on proportional loading. However, it cannot describe behaviors of materials under nonproportional loading especially cyclic loading.

Kinematic hardening

Around 1950, Prager gave the first formulation of kinematic hardening, and most of the modern theories and models are based on it. Kinematic hardening rules can be generally classified as linear and non-linear. The multilayer rules, which is the focus of this thesis, can be categorized into non-linear rules that are realized in a multilinear way.

The main feature of linear kinematic hardening is that the plastic stress-strain slope is constant. Despite the fact that linear kinematic hardening models are still widely used, they cannot predict many of the plastic phenomena.

Figure 1.6: The description of the stress-strain curve with the change of size of the yield surface according to the isotropic hardening rule

Figure 1.7: The description of the stress-strain curve with the movement of the yield surface according to the kinematic hardening rule

The non-linear kinematic hardening rules are represented by well-known models such as Mroz's model, Frederick-Armstrong model, Chaboche model and Ohno-Wang model.

For kinematic hardening, the sizes of the hardening surfaces/yield surfaces are assumed constant. It is featured with the moving of yield surface in the stress space according to the loading condition, as is shown in Figure 1.7. For tracking the movement of the yield surface as well as other hardening surfaces in models with multiple surfaces, the concept of "backstress" is used, which represent the location of the center of the yield/hardening surfaces in the stress space. The rule for predicting the backstress of the yield surface is the translation rule.

Combined hardening

Combined hardening models can realize the prediction of more complex and complete plastic behaviors, usually for cyclic plasticity. Although kinematic hardening is a more realistic way of modelling, there are cases in which the change of the size

Figure 1.8: The description of stress-strain curve with combined isotropic-kinematic hardening rule

or even the shape of yield surface is observed, as shown in 1.8. Experiments have also shown a dependence of strain-hardening on the loading history, and different variables, like the strain amplitude or the mean strain, can be used to represent the strain history. And these variable are usually concluded as parameters of isotropic hardening, which is used in conjunction with kinematic hardening.

1.3.7 Multilayer kinematic hardening rule

In reality, the stress-stain curve of a elastoplastic material is usually non-linear, but linear kinematic hardening has its advantages and potentials. The multilayer kinematic hardening rule is based on the idea of simulating the non-linear behavior in a multilinear way, and extending the linear rule to a multilinear one.

The multilayer plasticity is based on the idea of the use of several hardening surfaces. Hardening surfaces are a section of nested circles, whose the radii depend on the stress-strain points that are selected from the stress-strain curve of uniaxial

tests of materials. In J_2 plasticity, the yield surface and other hardening surfaces are circles in the first-octant octahedral plane of principle stresses. They are used to compute of effective hardening modulus of materials during loading or unloading in a multilinear way. The stress-strain curve is simplified as a multilinear curve with segments divided by those stress points that are chosen to relate to the radii of the hardening surfaces.

The innermost hardening surface is usually assumed as the yield surface, at least the intimal yield surface. The hardening surfaces need to follow a translation rule to translate without overlapping during loading, and the amount of translation of a certain hardening surface is decided by its associated plastic modulus which may change in combined kinematic-isotropic hardening models due to cyclic softening/hardening. In this case, when plastic deformation happens, because of the change of elastic zone and stress values, the sizes of hardening surfaces decrease/increase depending on the value of cyclic softening/hardening.

1.3.8 Radial return algorithm

Despite the hardening rule used, the yield surface for J_2 plasticity is a circular cylinder in the principle stress space, and the radial return algorithm was proposed based on this shape [1] .

In numerical algorithms, one way to predict whether a process is elastic or plastic is by calculating the trial state.

During plastic loading, the stress point is always on the yield surface, it can never be out of the yield surface, so if the rate of the deviatoric trial stress points to the outside of the yield surface, it means that the process is plastic and the trail stress point should "return" to the location on the yield point along the radial direction, otherwise, it's elastic and the trial state is the actual state.

The rate of the trial stress is computed assuming only with plastic behavior:

$${}^{tr}\dot{\boldsymbol{\sigma}} = \mathbb{C}^e : \dot{\boldsymbol{\epsilon}} \quad (1.21)$$

The deviatoric part is:

$${}^{tr}\dot{\boldsymbol{\sigma}}^d = \mathbb{P}_I : {}^{tr}\dot{\boldsymbol{\sigma}} \quad (1.22)$$

with the fourth-order tensor $\mathbb{P}_I = \mathbb{I} - \hat{\mathbf{I}} \otimes \hat{\mathbf{I}}$ as the projector over the deviatoric stress space. $\mathbb{C}^e$ is the elastic tangent of the material, and $\dot{\boldsymbol{\epsilon}}$ is the rate of strain.

The criteria for yielding leads to this computational algorithm of radial return. The radial return algorithm is the foundation for the stress-point integration scheme, and it is used for integrating the rate-constitutive equations.

When the rate of the deviatoric trial stress is computed with the assumption of elasticity, we can write the trial of deviatoric stress as:

$${}^{tr}\dot{\boldsymbol{\sigma}}^d = {}^{tr}\dot{\boldsymbol{\sigma}}^d + {}^t\boldsymbol{\sigma} \quad (1.23)$$

${}^t\boldsymbol{\sigma}$ is the known deviatoric stress before the beginning of this loading step.

$$J_2^{tr} = \frac{1}{2} {}^{tr}\boldsymbol{\sigma}^d : {}^{tr}\boldsymbol{\sigma}^d \quad (1.24)$$

If $J_2^{tr} - k^2 \leq 0$, the assumption of elasticity is correct, so that the trial stress state is the actual stress state. Thus, the elastic strain rate equals the total strain rate, and the plastic strain rate is zero.

If $J_2^{tr} - k^2 > 0$, the trial stress point is placed outside the yield surface, which shouldn't be the one for the current stress state. In this case, the assumption of elasticity is not correct, and the plastic strain rate is not zero. To get the actual stress, the point of the trial stress should return to the yield surface along the radial direction from the trial stress point to the center of the yield surface. Which means the rate of the stress should be scaled down with a factor to put the updated stress point on the yield surface.

Using general Hooke's law, which is valid no matter there is the plastic deformation or not:

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}^e : \dot{\boldsymbol{\epsilon}}^e \quad (1.25)$$

So the elastic strain can be computed, since for small strain problem:

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p \quad (1.26)$$

in this way we can get the plastic strain rate.

The above is not the unique procedure for radial return algorithm, the plastic strain rate can be computed first as a corrector to the actual strain rate, and the actual stress can be calculated when the actual plastic strain rate is defined. The model in this work corresponds to this later case.

1.3.9 Stress integration algorithm

Depending on the model, the stress integration algorithm can be based on different methods, for example, the forward or backward Euler method. In the model of this work, the stress integration algorithm is based on the backward Euler return method.

Figure 1.9: Stress integration algorithm

If we take a combined kinematic-isotropic hardening model as an example, as shown in the Figure 1.9, with the known information from step n , the stress integration algorithm for step $n + 1$ can be carried out in the following way:

First, calculate the trial state according to the assumption of elasticity:

$$\boldsymbol{\sigma}_{n+1}^{tr} = \boldsymbol{\sigma}_n + \mathbb{C}^e : \Delta \boldsymbol{\varepsilon} \quad (1.27)$$

Second, use the trial state to the yield function to decide the actual state.

If $f^{tr} \leq 0$, the assumption of elasticity is correct, $\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{tr}$. Otherwise, calculate the plastic flow and update the state variables for kinematic and/or isotropic hardening.

1.3.10 Prager's translation rule

In the present model, the classical Prager's translation rule is used. This rule is suggested by Prager [2] and Ziegler [3]. It is valid for bilinear models, and has been implemented into almost all the finite element programs, however, when a non-linear hardening function is introduced directly to the original model, the predicted convexity of the loops is nonphysical [5]. The validity of this model and its advantages over the Mroz's rule in both explicit and implicit forms are proved in [6].

The model that we have been working on obeys the principle of maximum dissipation, which is the principle that determines the actual stress points from all the points that satisfy the yield criterion, to preserve the principle of maximum plastic dissipation, associative plastic flow and hardening rules (with the same direction) are used.

As shown in Figure 1.10, Prager's translation rule defines that the direction of the rate of backstress of the yield surface is the same as the plastic flow direction $\hat{\mathbf{n}}$:

$$\frac{\dot{\boldsymbol{\alpha}}^1}{\|\dot{\boldsymbol{\alpha}}^1\|} = \hat{\mathbf{n}} \quad (1.28)$$

Here the backstress $\boldsymbol{\alpha}^1$ represents the location of the yield surface in the stress space; the subscripts n and $n + 1$ indicate the previous step and the current step. This rule forces $\hat{\mathbf{n}}(\boldsymbol{\sigma}_{n+1}^{tr}, \boldsymbol{\alpha}_{n+1}^1) = \hat{\mathbf{n}}(\boldsymbol{\alpha}_n^1, \boldsymbol{\alpha}_{n+1}^1)$, and because of this, the direction $\hat{\mathbf{n}}$ can be obtained "explicitly" with the trial stress state and the backstress at the previous step,

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\sigma}_{n+1}^{tr} - \boldsymbol{\alpha}_n^1}{\|\boldsymbol{\sigma}_{n+1}^{tr} - \boldsymbol{\alpha}_n^1\|} \quad (1.29)$$

Figure 1.10: The implicit Prager rule for multilayer plasticity

1.4 Finite element program DULCINEA

Dulcinea is a finite element program coded in the language of Fortran.

It can realize static or dynamic analysis for finite element problems with different element types and different material behaviors. It can realize the modelling of element groups, the boundary condition, the loading condition by generation or individual indication. It allows different forms of storage for vectors, and it is equipped with different solvers, as well as different procedures for checking and optimization.

Dulcinea allows the user to have a maximum controllability over tolerances, procedures for approximation, maximum number of iterations, etc. Besides, by using a big blanket array in memory to keep track of all the interested information, it allows the recording of any information of any nodal point or integration point at any specific time step. It is used in conjunction with matlab program to realize the storage of the history of interested information and the plotting of outputs in matlab figures for a straight forward demonstration of the result as well as for checking the finite element modelling.

In conclusion, because of the total controllability over almost everything for running a finite element problem, Dulcinea is a program utterly suitable for investigation. Also, because of the convenience of the implementation of new subroutines,

the program is full of potential. The present cyclic plastic model is implemented as a material type to the program.

The simulations of experiments applied to tubular specimens in this thesis are modelled with single-element problems in Dulcinea.

1.5 Shape of subsequent yield surface

The shape and the size of yield surface play an essential rule in constitutive models.

Theoretically, according to the Von Mises criterion, the yield surface should be a cylinder in the space of principle stresses, with the hydrostatic axial parallel to the direction of vector $[1,1,1]$, and should be a circle in the $\sigma - \sqrt{3}\tau$ plane and in the deviatoric stress “ π ” plane. While in experiments, as is shown in Fig 5.11 from 1.11, which is redrawn in Figure 1.11, when there is no pre-loading, the detected yield surface is in the shape of a circle and centered at the zero point, yet it is frequently observed that if the material is loaded in one uniaxial direction prior to being loaded in another direction perpendicular to the previous one, a distorted yield surface could be detected. The detected yield surface shows a “nose” in the former direction and an almost flat line in the opposite direction, resulting in an often called “egg” or “corner” effect. This anisotropy is usually taken as a specific cyclic plastic effect.

In [5], this effect is considered complex to predict, and can only be described by anisotropic hardening rules. It is taken as related to the material property as well as the loading path, which may be probably true, but this complex anisotropy can actually be predicted solely with a kinematic hardening rule model. As will be further illustrated in the third chapter.

With the model of this work, to understand the reason for the distortion of the observed subsequent yield surface, the most straightforward way may be by observing the geographic locations of the hardening surfaces during the probing loading, 1.12 shows how the hardening surfaces are located in the stress space at the very end of the pre-loading. As is shown, the pre-loading has changed the distribution of the hardening surfaces significantly, and the influence of the initial loading lasts all the time during the following loading sequences, which are typically perpendicular loading sequences in experiments that are aimed at capturing the subsequent yield surfaces. The change of the distribution of hardening surfaces leads to the difference of hardening at the direction of each probing path, which causes the difference in the norm of the change of equivalent stress needed to achieve the same offset plastic strain value.

Figure 1.11: Redrawn from [5], Fig 5.11. Elastic regime in the plane of normal-shear stresses. Tension-torsion tests on tubular specimen: 2024 Aluminium alloy

Figure 1.12: Geometric locations of the hardening surfaces at the end of the preloading in simulating the subsequent yield surface and the predicted subsequent yield surface

1.6 State of the art: constitutive models for cyclic plasticity

Since the first formulation is given by Prager around 1950, kinematic hardening has been the foundation of most models for cyclic plasticity. Depending on the behaviors of the material as well as the phenomena to capture, different models have been introduced. Because of the availability of computers and experimental data, more and more complex and combined models are used. In the following part, some of the symbolic models are introduced, and these models are also the bases for a major part of the present models.

1.6.1 Mroz's model

After the first formulation of kinematic hardening given by Prager, it is natural to improve this linear kinematic hardening formulation by adding more surfaces to produce a multilayer model. One of the most famous multilayer kinematic models is the Mroz model [4]. It can describe Bauschinger effect precisely, but was proved problematic in [6]. In the following its features and problems will be discussed briefly.

The basic idea of multilayer kinematic models is the discretization of stress-strain curve from an uniaxial test. A set of nested surfaces are used to model the change of the hardening modulus, and each of these surfaces has an associated kinematic hardening modulus in order to properly describe Masing rules. The surfaces translate according to a specific hardening rule based on geometric requirements, i.e. to avoid overlapping of surfaces. Mroz's initial proposal presents inconsistencies in avoiding overlapping and has been modified by Garud [9]. Moreover, the original model is explicit and it shows the difficulty of fully implicit finite element implementation. However, within some restrictions, it is possible [10].

In J_2 plasticity, the surfaces could be written as:

$$f^i = \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}^i\| - r^i \quad (1.30)$$

Where $\boldsymbol{\sigma}^d$ is the deviatoric stress, $\boldsymbol{\alpha}^i$ is the backstress of surface i , and r^i is the radius of surface i .

This rule requires that the active surfaces contact at the stress point, so for surface $j(j < a)$, a is the active surface, $f^j = 0$, so:

$$f^1 = f^2 = \dots = f^a = 0 \quad (1.31)$$

The rate of plastic strain can be calculated for associative plasticity.

$$d\boldsymbol{\varepsilon}^p = d\lambda^1(\partial f^1/\partial \boldsymbol{\sigma}) = \dots = d\lambda^a(\partial f^a/\partial \boldsymbol{\sigma}) \quad (1.32)$$

where $\lambda^i (i = 1, a)$ is the plastic multiplier, and the rule requires that:

$$d\lambda^1 = d\lambda^2 = \dots = d\lambda^a = d\lambda \quad (1.33)$$

$d\lambda$ can be an arbitrary function of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}^a$:

$$d\lambda = \frac{H(f^a)}{C^a(\boldsymbol{\sigma})} \frac{\langle (\partial f^a / \partial \boldsymbol{\sigma}) : d\boldsymbol{\sigma} \rangle}{(\partial f^a / \partial \boldsymbol{\sigma}) : (\partial f^a / \partial \boldsymbol{\sigma})} \quad (1.34)$$

where $\langle \cdot \rangle$ is the Macaulay bracket function: $\langle x \rangle = \frac{1}{2}(x + \|x\|)$. The consistency conditions:

$$df^1 = df^2 = \dots = df^a = 0 \quad (1.35)$$

And the movements of the location of the centers of surfaces:

$$d\boldsymbol{\alpha}^1 = d\boldsymbol{\alpha}^2 = \dots = d\boldsymbol{\alpha}^a = (\boldsymbol{\sigma}^{a+1} - \boldsymbol{\sigma})d\mu \quad (1.36)$$

Here the multiplier $d\mu$ is determined by $df^a = 0$:

$$d\mu = H(f^a) \frac{\langle (\partial f^a / \partial \boldsymbol{\sigma}) : d\boldsymbol{\sigma} \rangle}{\partial f^a / \partial \boldsymbol{\sigma} : (\boldsymbol{\sigma}^{a+1} - \boldsymbol{\sigma})} \quad (1.37)$$

In Fig 1.13, ${}^t\boldsymbol{\alpha}^a$ is the current backstress of the active surface a , ${}^t\boldsymbol{\alpha}^{a+1}$ is the current backstress of surface $a + 1$, ${}^t\boldsymbol{\sigma}$ is the current stress point, ${}^t\boldsymbol{\sigma}^c$ is the updated stress point as well as the contact point of the surfaces. The Mroz rule defines that the outward normal direction at stress current stress point ${}^t\boldsymbol{\sigma}$ to surface a is identical to the normal direction of stress point ${}^t\boldsymbol{\sigma}^c$ to surface $a + 1$. The kinematic hardening occurs in the direction of ${}^t\boldsymbol{m}({}^t\boldsymbol{\sigma} \text{ to } {}^t\boldsymbol{\sigma}^c)$. In this way, the rule forces the surfaces to meet only at the targeted stress point, with identical normals.

The flow direction for Mroz's rule is:

$${}^{t+\Delta t} \hat{\boldsymbol{n}} = \frac{{}^{t+\Delta t} \boldsymbol{\sigma}^{tr} - {}^t \boldsymbol{\alpha}^a}{\| {}^{t+\Delta t} \boldsymbol{\sigma}^{tr} - {}^t \boldsymbol{\alpha}^a \|} \quad (1.38)$$

where ${}^{t+\Delta t} \boldsymbol{\sigma}^{tr}$ is the elastic predictor for time step $t + \Delta t$, which is the trial deviatoric stress. ${}^t \boldsymbol{\alpha}^a$ is the backstress of the activated surface at the previous time step.

The introduction of multilayer concept to the kinematic hardening model is very original. The main advantages of Mroz models are the simplest characterization of the cyclic stress-strain behavior, the correct description of the Masing effect and the prediction of stabilized cycles.

However, the Mroz translation rule is based on purely geometric restrictions and has been questioned for its inconsistency and insufficient multiaxial, nonproportional

Figure 1.13: The geographic illustration of the explicit Mroz's model

predictive capabilities. The limitations of Mroz's model and its comparison with Prager's model are discussed in [6].

First, the model doesn't hold multiaxial consistency. The stress response to a specific strain path varies unpredictably depending on the number of surfaces used, here it's not referring to the degree of discretization of the stress-strain curve. Second, though it doesn't predict the ratchetting effect under uniaxial cyclic loading, under multiaxial cyclic loading, the implicit Mroz model produces an unpredictable and uncontrolled ratchetting effect, while the explicit formulation doesn't describe the ratchetting effect. Third, the original implementation of the model is explicit, and also because of this, it is difficult to include the cyclic isotropic hardening/softening to the explicit Mroz's model. Fourth, in order to fulfill the graphic requirement that the stress point is also the contact point of hardening surfaces, the Mroz's model has a restriction on the sizes of the surfaces, so that each surface should be no bigger than twice the size of its previous one, which makes it not applicable for models with a "vanishing" elastic region, because the number of surfaces needed couldn't be determined. Moreover, in order to make the surface contact at the stress point, Mroz's rule doesn't follow the principle of maximum dissipation, so it's not an associative rule. Although non-associative plastic flow is not proved to be less realistic, only it's not based on experimental evidence, but solely the graphic restriction.

Essentially, the inadequacy lies in the way of dealing with the surfaces. It is assumed that once one surface becomes activated, it turns into the yield surface(the last activated surface), and consistency is forced at this active surface. This assumption about the change of yield surface is not only illogical, but also causes inconsistency in the algorithm.

1.6.2 Borja's model

This model was brought out in [11], based on the bounding surface(a surface that stress points never approach) model by [12]. It is intended for describing the non-linear stress-strain behavior of soils at low strain levels, whose deformations are predominantly plastic, so it allows vanishing elastic regime. This model uses Prager's rule as the translation rule for backstress. Because of the assumption of vanishing elastic area, the stress and strain paths are closely related.

In this model, two functions are considered, F as the yield function, and B represents the bounding surface. They are both functions of deviatoric stress tensor. The surface F is always inside of the surface B .

This models assumes that the latest elastic unloading point F_0 has a hardening

modulus which is infinitely big, and the bounding surface B has a hardening modulus which is zero. In another word, the material performs instantaneously elastic F_0 , and instantaneously perfectly plastic on the bounding surface B . The surface with the instantaneous hardening modulus is centered around this point F_0 with infinity as its hardening modulus. The instantaneous hardening modulus is decided by an interpolation rule, which is generated by an one-dimensional non-linear model for soils.

Hardening rules are essential to bounding surface formulation. The point σ^B on the bounding surface B is defined as:

$$\sigma^B = \sigma' + k(\sigma' - \sigma'_0) \quad (1.39)$$

As shown in the Figure 1.14, σ'_0 is the stress tensor at F_0 . k is a scalar parameter that varies with the hardening modulus H' .

The specific forms for H' can be given using, for example, the exponential function, the Hyperbolic function or the Davidenkov model. The key is to express H' as a function of stress.

The yield function is:

$$F = \xi' : \xi' - r^2 = 0 \quad (1.40)$$

Where r is the radius of the yield surface, and $\xi' = \sigma' - \alpha$ is the translated deviatoric stress tensor. And the deviatoric backstress tensor α represents the location of the center of the yield surface. The bounding surface is defined in a similar way as the yield function:

$$B = \sigma' : \sigma' - R^2 = 0 \quad (1.41)$$

The radius R depends on the material type. The rate of constitutive equation:

$$\dot{\sigma} = C^e : (\dot{\epsilon} - \dot{\epsilon}^p) = \dot{\sigma}^{tr} - 2\mu\lambda\hat{n} \quad (1.42)$$

where $\dot{\epsilon}^p = \lambda\hat{n}$, λ is the consistency parameter, μ is the shear modulus, and $\hat{n}$ is the outward normal to F .

The Prager's translation rule is applied here:

$$\dot{\alpha} = \|\dot{\alpha}\| \hat{n} \quad (1.43)$$

When the elastic regime is "vanishing", $r \rightarrow 0$, the direction of the stress rate corresponds with the direction of the strain rate.

$$\hat{n} \rightarrow \frac{\dot{\sigma}'}{\|\dot{\sigma}'\|} \quad (1.44)$$

Figure 1.14: The surfaces of Borja's model. F_0 is the latest elastic unloading point, F is the yield surface and B is the bounding surface.

so the deviatoric part of the rate of constitutive equation becomes:

$$\dot{\boldsymbol{\sigma}}' = 2\mu\dot{\boldsymbol{\varepsilon}}' - 2\mu\lambda\frac{\dot{\boldsymbol{\sigma}}'}{\|\dot{\boldsymbol{\sigma}}'\|} \quad (1.45)$$

The norm of the rate of backstress is:

$$\|\dot{\boldsymbol{\alpha}}\| = \frac{2}{3}H'\lambda \quad (1.46)$$

and it can also be written as:

$$\|\dot{\boldsymbol{\alpha}}\| = 2\mu(\|\dot{\boldsymbol{\varepsilon}}'\| - \lambda) \quad (1.47)$$

so the consistency parameter is calculated as:

$$\lambda = \frac{3\mu}{3\mu + H'} \|\dot{\boldsymbol{\varepsilon}}'\| \quad (1.48)$$

The rate of consistency equation can finally be written as:

$$\dot{\boldsymbol{\sigma}} = Ktr(\dot{\boldsymbol{\varepsilon}})\mathbf{I} + 2\mu\left(1 + \frac{3\mu}{H'}\right)^{-1}\dot{\boldsymbol{\varepsilon}}' \quad (1.49)$$

where K is the elastic bulk modulus, $\mathbf{I}$ is the second-order tensor.

In multiaxial loading cases, the unloading can be difficult to define. Because when the elastic area vanishes from one surface to one point, when the loading is multiaxial, the direction of $\hat{\mathbf{n}}$ which should be normal to the yield surface cannot be determined, so the loading and unloading of this model is determined by the change of hardening modulus. Because the bounding surface has a hardening modulus of zero and the point F_0 has a hardening modulus of infinite, so it is assumed that the increment of hardening modulus implies unloading, on the other hand, the decrement of hardening modulus implies loading.

The assumption of the "vanishing" elastic regime describes the some dependence of the direction of the stress rate on the direction of the strain rate, but this assumption may not be very physically reliable.

An essential issue of Borja's model is caused by the definition of loading and unloading, since it is assumed that when H' reduces, loading occurs, and when H' increases, unloading occurs. This will cause an problem in consistency. For example, for closed loading path, since the at the initial state and the final state the H' are the same so it means that the whole precess can be seen in a way that no loading and unloading have happened, which is not physically logical. Because in the stress

space, around the line where the hardening modulus is consistent, a small of change of direction of stress can vary the result from loading to unloading, or vise versa, which is not physically logical and can make algorithms based on this assumption very sensitive to numerical errors.

An important point is that when an unloading occurs, the center of contour of constant H' must be changed from point F^0 to F .

Since the bounding surface can never be reached by stress point, the size of this surface becomes very important, depending on the material, it needs to be defined specifically. Also, for multiaxial loading, the existence of the bounding surface may cause some problems. In multiaxial loading, the size of the bounding surface may have a significant influence in the description of the multiaxial behavior of material. Also, the assumption that the bounding surface is fixed may also make a big difference on the prediction of the response to multiaxial loading.

Another problem is that as an model for predicting behaviors of soils, it cannot describe strain softening.

Also, the this model is not specifically for small-strain behaviors, but it is based on the strain rate equation that can only be true under small stain conditions, which is $\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p$.

1.6.3 Armstrong-Frederick's model

Linear kinematic hardening model is the simplest among all the kinematic hardening models, and it is based on the assumption that the yield surface moves linearly with the plastic strain, as is the idea of Melan-Prager rule:

$$\dot{\alpha} = \frac{2}{3} C \dot{\epsilon}_p \quad (1.50)$$

where $\dot{\alpha}$ is the backstress rate that represents the change of location of the yield surface, and C is a material parameter.

When the "fading memory term" is added to 1.50, the non-linear kinematic hardening model called Armstrong-Frederick model is developed[40], which may be the most well-known non-linear kinematic hardening model. The model uses two surfaces(the loading surface and the limit surface) and can be considered bounding model as shown in Fig 1.15, but the hardening modulus continuously evolves, and the hardening rule is defined by a differential equation which includes a dynamic recovery term in order to improve the cyclic behavior allowing for non-linear kinematic hardening using only two surfaces, the second one reached only in the limit.

The von Mises criterion is used, so the yield function is:

$$f = J_2(\sigma - \alpha) - k - R \quad (1.51)$$

α stands for the backstress tensor. k is the initial size of the surface, R is the isotropic variable, if $R = 0$, the model is simplified as pure kinematic hardening model.

$$\dot{\alpha} = \frac{2}{3} C \dot{\epsilon}_p - \gamma \alpha \dot{p} \quad (1.52)$$

where $\dot{p}$ is the rate of the accumulative plastic strain:

$$\dot{p} = \sqrt{\frac{2}{3} \dot{\epsilon}_p : \dot{\epsilon}_p} = d\lambda \quad (1.53)$$

The outward normal is:

$$\mathbf{n} = \frac{\partial f / \partial \boldsymbol{\sigma}}{(\partial f / \partial \boldsymbol{\sigma} : \partial f / \partial \boldsymbol{\sigma})^{1/2}} = \sqrt{\frac{3}{2}} \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}'}{R + k} \quad (1.54)$$

$$d\boldsymbol{\epsilon}^p = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2} \frac{1}{h} \left\langle \frac{3}{2} \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}'}{R + k} : d\boldsymbol{\sigma} \right\rangle \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}'}{R + k} = \frac{3}{2} \frac{1}{h} \langle \mathbf{n} : d\boldsymbol{\sigma} \rangle \mathbf{n} \quad (1.55)$$

The hardening modulus h depends on the kinematic stress:

$$h = \frac{2}{3} C \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \gamma \boldsymbol{\alpha} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \left(\frac{2}{3} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)^{1/2} \quad (1.56)$$

The parameters k, C and γ are determined by tension-compression test from stabilized hysteresis loops under different strain amplitudes. k is determined by the elastic regime, C/γ is the value of $\Delta\sigma/2 - k$, where C and γ are determined by a curve fitting relation got from two half cycles of the cyclic curves of tension and compression: (the curve of the relation between $\Delta\epsilon_p/2$ and $\Delta\sigma/2 - k$.)

$$\frac{\Delta\sigma}{2} = \alpha_{\max} + k + R = \frac{C}{\gamma} \tan h\left(\gamma \frac{\Delta\epsilon_p}{2}\right) + k + R \quad (1.57)$$

The advantage of the Armstrong-Frederick model is that it represents the Bauschinger effect well with its smooth tangent modulus, also, when the mean stress is nonzero, it predicts a ratchetting effect which is sometimes observed in experiments.

One of the main inadequacies of the model is that when there are small cycles of loading-unloading/unloading-loading in big cycle, the small stress-strain loops are not closed correctly, which means that the model couldn't describe the Masing behavior.

The model is highly valued because of its ability for the prediction of the ratchetting behavior, under uniaxial or multiaxial loadings, which cannot be achieved by

Figure 1.15: The yield surface and bounding for Armstrong-Frederick model, redrawn from [5] Fig 5.36

linear models. But the Armstrong-Frederick model over-predicts ratchetting effect, and the ratchetting rate is always constant, so the uncontrollability of the ratchetting effect is also a disadvantage of the model. This is because the model realizes the prediction of ratchetting behavior by the dynamic recovery term, yet at the same time this term is coupled with the description of cyclic behavior. Also, it is noteworthy the fact that the translation rule for the model is equivalent to that of the Mroz's model with two surfaces, see [13], Sec. III.3.2, and the Mroz's model has been proved problematic in [6].

Moreover, fitting is used for generating the material parameters and this model cannot reproduce the uniaxial curve from which the parameters are generated precisely.

1.6.4 Chaboche's model and Ohno-Wang's model

Chaboche's model and Ohno-Wang's mode are widely used. A lot of newer models use the idea that a non-linear kinematic hardening rule is used in conjunction with a isotropic hardening rule, or strain history factor or nonproportional loading parameter, etc.

They are both extensions for Armstrong-Frederick model, especially for correcting the dynamic recovery part to avoid over-predicting of the ratchetting effect. Since the description of cyclic plasticity actually is not independent from the description

of ratchetting effect for all those non-linear kinematic hardening models, different means have been taken to control the dynamic recovery in Chaboche's model as well as Ohno-Wang's model.

One of the most important improvement for Armstrong-Frederick model is from Chaboche by generating the backstress as the sum of multiple individual backstresses components [7] and [8]. So the real shape of the stress-strain curve can be better predicted. The generalized model can be written as:

$$\dot{\alpha} = \sum \dot{\alpha}_i \quad (1.58)$$

Each of the component is identical to the basic model:

$$\dot{\alpha}_i = \frac{2}{3}C_i\dot{\epsilon}_p - \gamma_i\alpha_i\dot{p} \quad (1.59)$$

The form of the individual backstress offers a fading memory effect of the strain path. The generation of the model makes the backstress a sum of individual backstresses, and each component doesn't necessary need to follow the non-linear kinematic hardening rule. For example, one of the component could follow a linear rule, to make the model applicable to large strain problem.

One improvement for Chaboche's model is the superposition of isotropic hardening for predicting cyclic hardening/softening, the main variables considered are the accumulated plastic strain and the maximum plastic strain range(strain history) [13]. Given that R is the isotropic hardening variable, it is assumed that R is a function of p (the accumulated plastic strain), $R(0) = 0$, and:

$$dR = b(Q - R)dp \quad (1.60)$$

b is a parameter for the speed of stabilization, and Q is the asymptotic value for a regime of stabilized cycles. R is usually decided by the peak stress values of the first cycle and the stabilized cycle, and the relation between them is usually an exponential equation got by curve fitting. For example (with cyclic hardening):

$$\frac{\sigma_{\max} - \sigma_{\max 0}}{\sigma_{\max s} - \sigma_{\max 0}} = 1 - \exp(-2b\Delta\epsilon_p N) \quad (1.61)$$

Where N is the number of cycle, and $\Delta\epsilon_p$ is the approximate amplitude of the change of plastic strain (the actually value will change slightly because of the cyclic hardening/softening).

About the consideration of the strain history, a parameter(usually q) is introduced to remember the maximum plastic strain by using a surface in the space of plastic strain is introduced to track the change of the maximum plastic strain.

More important modifications about Chaboche's model are about the dynamic recovery item for correcting excessive prediction of ratchetting effect [58]: (1) introducing a power function to the dynamic recovery term (MILL); (2) introducing a threshold in the dynamic recovery term (NLK-T).

Ohno and Wang suggested two ways for the constraint of the dynamic recovery term. The first way [15] is by setting a critical value for the magnitude of each backstress item α_i , and the dynamic recovery item of α_i is activated only when the critical value is attained. The second way [16] is by assuming that the α_i becomes significant nonlinearly as α_i approaches the surface $f_i = 0$.

By studying the Ohno-Wang's model, it is found that with the implicit implementation of the model, non-unique solution can be produced by iteration.

It should be noted that the Chaboche's model and the two forms of Ohno-Wang's model have different control over the prediction for the ratchetting behaviors, but referring to the nonproportional deformation without ratchetting, they produce very similar results [17].

The fact that modifications have been made to Chaboche's model and Ohno-Wang's model to predict multiaxial responses has shown the deficiency of these models in describing multiaxial behaviors. They can give a reasonable prediction of multiaxial ratchetting, but may not be able to capture behaviors of materials under multiaxial especially nonproportional loadings.

Chapter 2

Cyclic plasticity using Prager's translation rule and both non-linear kinematic and isotropic hardening: theory, validation and algorithmic implementation

A paper based on this chapter has been submitted to the journal *Methods in Applied Mechanics and Engineering*.

Finite element analysis of structures under elasto-plastic nonproportional cyclic loadings is useful in seismic engineering, fatigue analysis and ductile fracture. Usual models with non-linear stress-strain curves in cyclic behavior are based on Mroz multisurface plasticity, bounding surface models or models derived from the Armstrong-Frederick rule. These models depart from the associative Prager's rule with the main purpose of modeling aspects of cyclic non-linear hardening.

A model for cyclic plasticity within the framework of the associative classical plasticity theory using Prager's rule accounting for anisotropic non-linear kinematic hardening coupled with non-linear isotropic hardening will be introduced in this chapter, as well as the validation of the theory against several uniaxial and multi-axial cyclic experiments and an efficient fully implicit radial return algorithm. The parameters of the model are obtained directly by a discretization of the uniaxial stress-strain behavior. Remarkably, both the presented theory and the computational algorithm automatically recover classical bi-linear plasticity and the Krieg

and Key algorithm if the user-prescribed stress-strain curve is bilinear.

2.1 Introduction

The finite element analysis of structures under nonproportional and cyclic non-linear behavior is very important in many applications, among them seismic engineering [11], [18], [19], fatigue and fracture analyses [20], [21], [22], [23] and plastic conformation [24], [25], [26], [27]. The correct description of multiaxial hardening effects have proved to be critical for an accurate prediction of the effective displacements, accelerations and safety of the structures [28]. In fatigue analysis of notched specimens, the plastic loading at the notch edge induces nonproportional multiaxial loading [29], frequently studied through cyclic plasticity models which, thereafter, are related to nominal strains and stresses through incremental Neuber rules or related strain or energy methods [21], [30], [31], [32], [33], [34].

In cyclic behavior, the accurate representation of the Bauschinger effect is crucial, a behavior which is approximately represented by Masing's rules. The classical linear kinematic hardening reproduces accurately such rules, has a very efficient integration algorithm and is available in most finite element programs. This model uses Prager's translation rule for the backstress, which is consistent with the principle of maximum dissipation [35]. However, a priori, the direct inclusion of a non-linear anisotropic kinematic hardening function produces unphysical loops. Therefore, finite element programs use Prager's rule exclusively in the case of *linear* kinematic hardening, and for non-linear kinematic hardening, which is needed to better describe the cyclic loops, they resort to other types of formulations (if available to the user), typically based on the Armstrong and Frederick rule [36], [37].

One of the procedures to account for anisotropic, history-dependent non-linear kinematic hardening is based on the history of observable quantities by hereditary integrals, as in the endochronic theory, similar to those used in viscoelasticity [38]. However, since this approach is more cumbersome for computational application and for finite element analysis, the approach based on internal variables [5], where only information of the previous step is needed, is usually preferred.

Remarkably, the theories to model non-linear kinematic hardening based on internal variables have departed from the classical linearly hardened framework, introducing different non-associative translation rules for the backstress, as Mróz's rule [39], Garud's rule [9] or the commented Armstrong and Frederick rule [40]. These rules have resulted in different models as the Mróz model [39], Chaboche's model [5], [13], bounding surface models [41], [12], [11], and non-linear kinematic hardening models with the addition of multiple backstresses [5], [15]. The algorithmic imple-

mentation of some of these models is usually more elaborate than that of classical plasticity [42], [43], [44], [45], [46], and even though some recent efficient algorithms are available for some cases [36], [37], they do not constitute a natural extension of classical Prager’s plasticity. To improve the cyclic multiaxial plastic behavior, sometimes non-proportionality parameters obtained for certain materials under certain loading paths by fitting experiments are employed, like in [47], [48], [49], [50], [51], [52], [53], [46], [54]. However, despite the common believe, it is possible to develop non-linear kinematic hardening models using Prager’s translation rule and preserving Masing’s rules [10], [55]. These models are similar in conception to Mróz’s model, but with some crucial differences, as the preservation of Prager’s rule, the use of a single yield surface (outer surfaces are hardening surfaces) and the simplicity of the integration algorithms. Furthermore, they lack the inconsistencies present in Mróz’s model under multiaxial loading [56], [57]. As will be shown in this work, the predictions for multiaxial plastic behavior using Prager’s rule are similar to the behavior observed in the experiments, at least for the different materials addressed below.

Despite of the employed kinematic hardening rule, mixed kinematic-isotropic hardening models have been increasingly used in cyclic loading analysis [58], [59], [60], [61], [62], [63], [64], [65], [26]. For a better description of cyclic hardening/softening in certain materials, the effect of a memory for the strain amplitude can be included in the isotropic hardening rule [13], [66], [67], [53], and the inclusion of isotropic cyclic softening may be used to model the effects of damage by fatigue. However, as seen below, the presence of isotropic hardening also modifies the anisotropic kinematic hardening moduli, an observation that must be considered in the formulation and computational algorithm.

Therefore, it is valuable, and the purpose of this work, to extend the classical von Mises associative theory of plasticity using Prager’s translation rule to account for history-dependent non-linear anisotropic kinematic hardening combined with cyclic isotropic hardening/softening. In the next sections the theory will be introduced firstly, showing that it is proposed a simple extension of the classical framework by allowing a dependence of the effective hardening moduli on some internal variables. Then the method to compute such dependence will be introduced, i.e. the effective hardening modulus at each instant. Thereafter the efficient radial return algorithm will be explained, which for the bilinear case naturally recovers the solution from Krieg and Key [68]. With some examples it will be shown that the theory predicts rather well the experimentally observed multiaxial behavior in several materials and that the finite element implementation preserves the asymptotic quadratic convergence of Newton schemes.

2.2 Classical J_2 –plasticity using Prager’s rule with non-linear anisotropic mixed hardening

2.2.1 Classical associative plasticity with anisotropic, cyclically modified hardening

The stored “observable” (elastic) energy depends on the elastic strains $\boldsymbol{\varepsilon}_e$, which are a function of the total strains and some internal strains $\boldsymbol{\varepsilon}_p$ (the plastic strains), i.e. $\boldsymbol{\varepsilon}_e(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p)$. The chain rule derived from the explicit dependencies $\boldsymbol{\varepsilon}_e(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p)$, gives immediately the rate as

$$\dot{\boldsymbol{\varepsilon}}_e = \left. \frac{\partial \boldsymbol{\varepsilon}_e}{\partial \boldsymbol{\varepsilon}} \right|_{\dot{\boldsymbol{\varepsilon}}_p=0} : \dot{\boldsymbol{\varepsilon}} + \left. \frac{\partial \boldsymbol{\varepsilon}_e}{\partial \boldsymbol{\varepsilon}_p} \right|_{\dot{\boldsymbol{\varepsilon}}=0} : \dot{\boldsymbol{\varepsilon}}_p = \dot{\boldsymbol{\varepsilon}}_e|_{\dot{\boldsymbol{\varepsilon}}_p=0} + \dot{\boldsymbol{\varepsilon}}_e|_{\dot{\boldsymbol{\varepsilon}}=0} \quad (2.1)$$

$$\equiv {}^{tr}\dot{\boldsymbol{\varepsilon}}_e + {}^{ct}\dot{\boldsymbol{\varepsilon}}_e \equiv \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}_p \quad (2.2)$$

where the last identity accomodates the rate form of the usual additive decomposition ansatz $\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p$. We defined the trial and corrector partial derivative contributions ${}^{tr}\dot{\boldsymbol{\varepsilon}}_e \equiv \dot{\boldsymbol{\varepsilon}}_e|_{\dot{\boldsymbol{\varepsilon}}_p=0}$ and ${}^{ct}\dot{\boldsymbol{\varepsilon}}_e \equiv \dot{\boldsymbol{\varepsilon}}_e|_{\dot{\boldsymbol{\varepsilon}}=0}$. In the new formulation of large strain elasto-plasticity presented in [69] and [70], this framework based on the chain rule remains additive and unaltered at large strains even when using the multiplicative decomposition. The stored energy is —see Figure 2.1

$$\Psi(\boldsymbol{\varepsilon}_e, \boldsymbol{\xi}, \gamma) = \mathcal{W}(\boldsymbol{\varepsilon}_e) + \mathcal{H}(\boldsymbol{\xi}, \gamma) = \frac{1}{2} \boldsymbol{\varepsilon}_e : \mathbb{C} : \boldsymbol{\varepsilon}_e + \mathcal{H}(\boldsymbol{\xi}, \gamma) \quad (2.3)$$

where $\mathcal{W}(\boldsymbol{\varepsilon}_e)$ is the elastic energy, immediately recovered upon unloading, and $\mathcal{H}(\boldsymbol{\xi}, \gamma)$ is the internal, microstructurally-blocked “elastic” energy of which part can be recovered following certain plastic deformation paths. This energy may be function of some macroscopic internal variable $\boldsymbol{\xi}$; and in turn $\boldsymbol{\xi}$ itself may be a function of some internal variables $\boldsymbol{\xi}_i$ of microscopic origin, mapped to the continuum level. For elastically isotropic materials, and using the usual observation that plastic flow is isochoric, we write

$$\Psi(\boldsymbol{\varepsilon}_e, \boldsymbol{\xi}, \gamma) = \frac{1}{2} \kappa (\varepsilon^v)^2 + \frac{1}{2} 2\mu \boldsymbol{\varepsilon}_e^d : \boldsymbol{\varepsilon}_e^d + \mathcal{H}(\boldsymbol{\xi}, \gamma) \quad (2.4)$$

where ε^v are the volumetric strains (elastic), $\boldsymbol{\varepsilon}_e^d$ are the elastic deviatoric strains, μ is the shear modulus, κ is the bulk modulus. We define the backstress $\boldsymbol{\alpha}$ and the overstress $\hat{r}$ as

$$\boldsymbol{\alpha}(\boldsymbol{\xi}, \gamma) := \frac{\partial \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \boldsymbol{\xi}}, \text{ and } \hat{r}(\boldsymbol{\xi}, \gamma) := \frac{\partial \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \gamma} \quad (2.5)$$

Figure 2.1: Decomposition of the input energy

With these definitions, the rate of the stored energy is

$$\dot{\Psi} = \kappa \varepsilon^v \dot{\varepsilon}^v + 2\mu \varepsilon_e^d : \dot{\varepsilon}_e^d + \boldsymbol{\alpha} : \dot{\boldsymbol{\xi}} + \hat{r} \dot{\gamma} \quad (2.6)$$

The rate $\dot{\boldsymbol{\xi}} = -{}^{ct}\dot{\boldsymbol{\varepsilon}}_e = \dot{\boldsymbol{\varepsilon}}_p$, accounting for the evolution of internal variables, only takes place during plastic flow. The definition $\dot{\boldsymbol{\xi}} = -{}^{ct}\dot{\boldsymbol{\varepsilon}}_e$ is just a traditional, convenient definition so $\boldsymbol{\alpha}$ results in the usual macroscopic, continuum backstress, that can be directly subtracted from the stress in the yield function as seen below. In a similar fashion, to accommodate the usual definitions, we take $\dot{\gamma}$ as the norm of $\dot{\boldsymbol{\xi}}$, $\dot{\gamma} = \|\dot{\boldsymbol{\xi}}\|$, so $\dot{\boldsymbol{\xi}} = \dot{\gamma} \hat{\boldsymbol{n}}$ and we define $\hat{\boldsymbol{n}} = \dot{\boldsymbol{\xi}} / \|\dot{\boldsymbol{\xi}}\|$, and the last term in Eq. (2.6) adopts the familiar form for isotropic hardening. Note also that sometimes the sign of both $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}$ are opposite than in our definition here to understand $\boldsymbol{\xi}$ as internal elastic strains mapped to the continuum (because of the “elastic” nature of $\mathcal{H}(\boldsymbol{\xi}, \gamma)$), but in practice the numerical result remains to be the same. Denoting the stress tensor by $\boldsymbol{\sigma}$, the stress power is $\mathcal{P} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$. Using $\dot{\boldsymbol{\varepsilon}} \equiv {}^{tr}\dot{\boldsymbol{\varepsilon}}_e$, the dissipation is

$$\mathcal{D} = \mathcal{P} - \dot{\Psi} = \boldsymbol{\sigma} : {}^{tr}\dot{\boldsymbol{\varepsilon}}_e - K(\varepsilon^v) \dot{\varepsilon}^v - 2\mu \varepsilon_e^d : \dot{\varepsilon}_e^d - \boldsymbol{\alpha} : \dot{\boldsymbol{\xi}} - \hat{r} \dot{\gamma} \geq 0 \quad (2.7)$$

which can be interpreted as $\mathcal{D} \equiv -\partial \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p) / \partial \boldsymbol{\varepsilon}_p|_{\dot{\boldsymbol{\varepsilon}}=0} \equiv -{}^{ct}\dot{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p) = {}^{tr}\dot{\Psi} - \dot{\Psi}$. Following the usual Coleman procedure, for purely elastic incremental deformations, when $\mathcal{D} = 0$, we also have $\dot{\boldsymbol{\varepsilon}}_p = \mathbf{0}$ and by definition ${}^{ct}\dot{\boldsymbol{\varepsilon}}_e = -\dot{\boldsymbol{\xi}} = \mathbf{0}$. Thus by

chain differentiation we immediately identify the necessary condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^v + \boldsymbol{\sigma}^d = K\varepsilon^v \mathbf{I} + 2\mu\boldsymbol{\varepsilon}_e^d$, where $\boldsymbol{\sigma}^v$ and $\boldsymbol{\sigma}^d$ are the volumetric and deviatoric parts, respectively. Then, since $\dot{\gamma} = (d\gamma/d\xi) : \dot{\boldsymbol{\xi}} = \hat{\mathbf{n}} : \dot{\boldsymbol{\xi}} = -\hat{\mathbf{n}} : {}^{ct}\dot{\boldsymbol{\varepsilon}}_e$ and ${}^{tr}\dot{\boldsymbol{\varepsilon}}_e = \dot{\boldsymbol{\varepsilon}}_e - {}^{ct}\dot{\boldsymbol{\varepsilon}}_e$, for the plastic case, the inequality $\mathcal{D} > 0$ can be written as

$$\mathcal{D} = - [\boldsymbol{\sigma}^d - \boldsymbol{\alpha} - \hat{r}\hat{\mathbf{n}}] : {}^{ct}\dot{\boldsymbol{\varepsilon}}_e > 0 \quad (2.8)$$

Hence, the dissipation is maximum, and the inequality is always guaranteed when ${}^{ct}\dot{\boldsymbol{\varepsilon}}_e$ and $[\boldsymbol{\sigma}^d - \boldsymbol{\alpha} - \hat{r}\hat{\mathbf{n}}]$ have the opposite direction, i.e. in this case $\hat{\mathbf{n}}$ may also be defined as

$$\hat{\mathbf{n}} = \frac{\boldsymbol{\sigma}^d - \boldsymbol{\alpha}}{\|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}\|} \quad (2.9)$$

so the flow rule $- {}^{ct}\dot{\boldsymbol{\varepsilon}}_e \equiv \dot{\boldsymbol{\varepsilon}}_p = \dot{\gamma}\hat{\mathbf{n}}$ and the hardening rule $\dot{\boldsymbol{\xi}} = - {}^{ct}\dot{\boldsymbol{\varepsilon}}_e = \dot{\gamma}\hat{\mathbf{n}}$ are defined in terms of this tensor $\hat{\mathbf{n}}$. The multiplier $\dot{\gamma} \geq 0$ is now interpreted as the norm of the plastic strain rate (the equivalent plastic strain rate, up to a constant). This variable is non-negative because $\mathcal{D} \geq 0$. Then we can write

$$\mathcal{D} = \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha} - \hat{r}\hat{\mathbf{n}}\| \dot{\gamma} \equiv {}^0r\dot{\gamma} \equiv \sigma_Y \dot{p} \geq 0 \quad (2.10)$$

where we have defined ${}^0r := \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}\| - \hat{r} \geq 0$, and $\|\cdot\|$ denotes the 2-norm. The term $\sigma_Y \dot{p}$ is the uniaxial equivalence, where $\sigma_Y = \sqrt{3/2} {}^0r$ is the uniaxial yield stress and $\dot{p} = \sqrt{2/3} \dot{\gamma}$ is the uniaxial plastic strain rate, obtained from

$$p = \sqrt{\frac{2}{3}} \int_0^t \|\dot{\boldsymbol{\varepsilon}}^p\| dt = \sqrt{\frac{2}{3}} \gamma \quad (2.11)$$

Equation (2.10) may be written in the following convenient format for identification of the yield surface

$$\mathcal{D} = (\|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}\| - \hat{r} - {}^0r) \dot{\gamma} + {}^0r\dot{\gamma} \geq 0 \quad (2.12)$$

which implies that (case of plastic loading)

$$f := \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}\| - r = 0 \quad \text{if } \dot{\gamma} > 0 \quad (2.13)$$

where we defined the current radius $r := {}^0r + \hat{r}$, but we can have (case of elastic loading or neutral loading)

$$f \leq 0 \quad \text{if } \dot{\gamma} = 0 \quad (2.14)$$

During plastic deformations, the consistency parameter $\dot{\gamma} > 0$ can be obtained from the consistency condition which must be $\dot{f} = 0$ during plastic flow ($\dot{\gamma} > 0$), so $\dot{f} = 0$ (or equivalently $\dot{\gamma}\dot{f} = 0$ in any case, elastic or plastic)

$$\dot{f}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p) = \underbrace{\hat{\mathbf{n}} : {}^{tr}\dot{\boldsymbol{\sigma}}^d}_{{}^{tr}\dot{f} \equiv \dot{f}|_{\dot{\boldsymbol{\varepsilon}}_p=0}} + \underbrace{\hat{\mathbf{n}} : ({}^{ct}\dot{\boldsymbol{\sigma}}^d - \dot{\boldsymbol{\alpha}}) - \dot{r}}_{{}^{ct}\dot{f} \equiv \dot{f}|_{\dot{\boldsymbol{\varepsilon}}=0}} = 0 \quad (2.15)$$

where we define the trial deviatoric stress rate as

$${}^{tr}\dot{\boldsymbol{\sigma}}^d := 2\mu {}^{tr}\dot{\boldsymbol{\varepsilon}}^d \equiv 2\mu \dot{\boldsymbol{\varepsilon}}^d \quad (2.16)$$

and the corrector one as

$${}^{ct}\dot{\boldsymbol{\sigma}}^d = 2\mu {}^{ct}\dot{\boldsymbol{\varepsilon}}_e \equiv -2\mu \dot{\boldsymbol{\varepsilon}}_p = -2\mu \dot{\gamma} \hat{\mathbf{n}} \quad (2.17)$$

The latter is the rate of the stress tensor during plastic flow which produce the maximum dissipation. Note that this novel continuum framework presented in [69], [70] mimics the usual algorithmic implementation, even at large strains, but it is just an immediate application of the chain rule when considering the dependencies $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p)$, i.e.

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p) = \left. \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \right|_{\dot{\boldsymbol{\varepsilon}}_p=0} : \dot{\boldsymbol{\varepsilon}} + \left. \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}_p} \right|_{\dot{\boldsymbol{\varepsilon}}=0} : \dot{\boldsymbol{\varepsilon}}_p = {}^{tr}\dot{\boldsymbol{\sigma}} + {}^{ct}\dot{\boldsymbol{\sigma}} \quad (2.18)$$

This type of decomposition is also recovered at large strains and its backward-Euler discretization results immediately in the typical Closest Point Projection algorithms.

The component of the rate of $\boldsymbol{\alpha}$ perpendicular to $\hat{\mathbf{n}}$ does not affect the yield condition and does not change the dissipation, so as it is well known the direction of minimum accumulation of internal elastic energy, and maximum dissipation for $\dot{\boldsymbol{\alpha}}$, is the one given by Prager's rule; see e.g. [35]. Accordingly, we assume Prager's translation rule, so $\dot{\boldsymbol{\alpha}}$ is proportional to $\hat{\mathbf{n}}$

$$\dot{\boldsymbol{\alpha}} = \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}} : \dot{\boldsymbol{\xi}} + \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \boldsymbol{\xi} \partial \gamma} \dot{\gamma} \quad (2.19)$$

$$= \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{\xi}} : \dot{\boldsymbol{\xi}} + \frac{\partial \boldsymbol{\alpha}}{\partial \gamma} \dot{\gamma} =: \dot{\lambda} \hat{\mathbf{n}} =: \frac{2}{3} \bar{H}(\boldsymbol{\xi}, \gamma) \dot{\gamma} \hat{\mathbf{n}} \quad (2.20)$$

where $\partial \boldsymbol{\alpha} / \partial \gamma = \partial r / \partial \boldsymbol{\xi}$. The backstress rate modulus is given as $\dot{\lambda} = \|\dot{\boldsymbol{\alpha}}\| =: \frac{2}{3} \bar{H} \dot{\gamma}$, or equivalently

$$\dot{\gamma} = \frac{\langle \dot{\boldsymbol{\alpha}} : \hat{\mathbf{n}} \rangle}{\frac{2}{3} \bar{H}(\boldsymbol{\xi}, \gamma)} \quad (2.21)$$

and we defined

$$\frac{2}{3}\bar{H}(\boldsymbol{\xi}, \gamma) := \hat{\mathbf{n}} : \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \boldsymbol{\xi} \partial \boldsymbol{\xi}} : \hat{\mathbf{n}} + \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \gamma \partial \boldsymbol{\xi}} : \hat{\mathbf{n}} \quad (2.22)$$

as the effective kinematic hardening. The Macaulay bracket in $\langle \dot{\boldsymbol{\alpha}} : \hat{\mathbf{n}} \rangle$ implies that $\dot{\gamma} \geq 0$. Then, from the consistency condition, Eq. (2.15), we restrict the formulation to a dependence $r(\gamma)$ —a dependence of r on $\boldsymbol{\xi}$ would include a path-dependent isotropic hardening, found in some materials [47], but which for simplicity won't be considered in this work— so

$$\dot{r} := r'(\gamma) \dot{\gamma} := \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \gamma \partial \boldsymbol{\xi}} : \dot{\gamma} \hat{\mathbf{n}} + \frac{\partial^2 \mathcal{H}(\boldsymbol{\xi}, \gamma)}{\partial \gamma \partial \gamma} \dot{\gamma} \quad (2.23)$$

where, note, both types of hardening are implicitly coupled through the term $\partial^2 \mathcal{H} / \partial \gamma \partial \boldsymbol{\xi}$ as it will be seen below. Then, we can factor out the consistency parameter as usual

$$\dot{\gamma} = \frac{2\mu}{2\mu + \frac{2}{3}\bar{H}(\boldsymbol{\xi}, \gamma) + r'(\gamma)} \langle \hat{\mathbf{n}} : \dot{\boldsymbol{\epsilon}}^d \rangle \quad (2.24)$$

where $\bar{H}(\boldsymbol{\xi}, \gamma)$ and $r'(\gamma)$ are the functions to be obtained from experiments. Then, it is immediate to obtain the continuum tangent

$$\mathbb{C}_{ep} = \kappa \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{P} - \frac{(2\mu)^2}{2\mu + \frac{2}{3}\bar{H}(\boldsymbol{\xi}, \gamma) + r'(\gamma)} \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} \quad (2.25)$$

Remarkably, these are the classical J_2 plasticity expresions given, for example, in Sec. 2.3.2 of [35] or in Sec. 4.3 of [73], where the previous equations are obtained from an equivalent procedure *but assuming a constant $\bar{H}$* . Then, the single important difference in Eqs. (2.24) and (2.25) respect to the classical ones is that we do not assume that $\bar{H}(\boldsymbol{\xi}, \gamma)$ is constant, but instead that it changes according to the flow direction and the history of deformation by way of the implicit dependences on $\boldsymbol{\xi}$ and γ , as shown below. However, in the particular case that $\bar{H}$ results constant, both the theory and computational algorithms in [35] (see also [74]) are naturally recovered regardless of the number of surfaces employed, and without user intervention; i.e. the solution is reached in just one iteration if r' is also constant. Therefore, our model constitutes a natural extension of the classical computational algorithms in finite element software to account for history-dependent non-linear kinematic hardening and anisotropic hardening effects.

2.2.2 Description of anisotropic hardening

The anisotropic hardening is described in our model with the aid of different, initially concentric, *hardening surfaces* (i.e. *not* macroscopic yield surfaces). The idea of Mróz and Iwan in using concentric surfaces for describing the hardening is followed, but the theory is different from Mróz's theory because our hardening surfaces are not yield surfaces at the macroscopic level, but just a tool to compute an effective macroscopic anisotropic hardening modulus $\bar{H}$ which take into account microstructural changes due to the deformation history in the material. Furthermore, Prager's hardening rule is kept instead of being changed to Mróz's or other alternative rule. These hardening surfaces may be written as, see Figure 2.2a

$$f_i = \|\hat{\boldsymbol{\sigma}}_i - \boldsymbol{\alpha}_i\| - r_i = 0, \quad i = 1, \dots, N \quad (2.26)$$

where N is the number of surfaces and $\hat{\boldsymbol{\sigma}}_i$ is the contact point in the deviatoric stress space of surface i with surface $(i - 1)$, and the first, innermost surface is defined as the (macroscopic) yield surface, i.e.

$$f_1 \equiv f, \quad \hat{\boldsymbol{\sigma}}_1 \equiv \boldsymbol{\sigma}^d, \quad \boldsymbol{\alpha}_1 \equiv \boldsymbol{\alpha}, \quad r_1 \equiv r \quad (2.27)$$

As another phenomenological difference with Mróz's proposal, the contact point between surfaces is not, in general, the same point, i.e. $\hat{\boldsymbol{\sigma}}_i \neq \hat{\boldsymbol{\sigma}}_j$ for $i \neq j$. Equations (2.26) are convenient for visualization. However, they are fully equivalent to the following equations, more convenient for mathematical derivations, see Figure 2.2b

$$\begin{cases} \bar{f}_1 \equiv f_1 = \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}_1\| - r_1 = 0 \\ \bar{f}_i = \|\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}\| - \rho_i = 0, \quad i = 2, \dots, N \end{cases} \quad (2.28)$$

where it has been defined that $\rho_i = r_i - r_{i-1} > 0$ (and $\rho_1 = r_1$). Hence, the hardening surfaces can be alternatively interpreted as yield surfaces for internal microstructural stresses $\boldsymbol{\alpha}_i$. These internal stresses are a result of internally stored (blocked) energy, which is returned upon stress reversal. Interestingly, the view in Figure 2.2b shows an interpretation of the theory close to that of hereditary models, as for example the endochronic theory [38], [77].

2.2.3 Consistency condition in hardening surfaces

For those surfaces such that $f_i < 0$ (or equivalently $\bar{f}_i < 0$), the hardening surfaces remain fixed because the inner surfaces are fully inside, not touching surface i . However, if $f_i = 0$ for any $i > 1$, this implies that surface $(i - 1)$ is touching surface

i. In such a case, it must be guaranteed that $\dot{f}_i \leq 0$ (or equivalently $d\bar{f}_i/dt \leq 0$) in order to avoid overlapping surfaces which would yield an inconsistent definition of the hardening function in the stress domain. From Eq. (2.28₂)

$$\frac{d\bar{f}_i}{dt} = \frac{\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}}{\|\boldsymbol{\alpha}_i - \boldsymbol{\alpha}_{i-1}\|} : (\dot{\boldsymbol{\alpha}}_i - \dot{\boldsymbol{\alpha}}_{i-1}) - \dot{\rho}_i = 0 \quad (2.29)$$

Define —this tensor can be interpreted as the flow direction for the internal stresses $\boldsymbol{\alpha}_i$ of microstructural origin, see Fig. 2.2b

$$\hat{\mathbf{m}}_i := \frac{\mathbf{m}_i}{\|\mathbf{m}_i\|} := \frac{\boldsymbol{\alpha}_{i-1} - \boldsymbol{\alpha}_i}{\|\boldsymbol{\alpha}_{i-1} - \boldsymbol{\alpha}_i\|} \quad \text{and} \quad \hat{\mathbf{m}}_1 \equiv \hat{\mathbf{n}} \quad (2.30)$$

so

$$-\frac{d\bar{f}_i}{dt} = \hat{\mathbf{m}}_i : (\dot{\boldsymbol{\alpha}}_i - \dot{\boldsymbol{\alpha}}_{i-1}) + \dot{\rho}_i = 0 \quad (2.31)$$

The first surface is the actual yield surface, so as seen above, it translates according to Prager's rule, $\dot{\boldsymbol{\alpha}}_1 = \|\dot{\boldsymbol{\alpha}}_1\| \hat{\mathbf{n}}$. Then the consistency on the second surface is

$$\hat{\mathbf{m}}_2 : \dot{\boldsymbol{\alpha}}_2 - \|\dot{\boldsymbol{\alpha}}_1\| (\hat{\mathbf{n}} : \hat{\mathbf{m}}_2) + \dot{\rho}_2 = 0 \quad (2.32)$$

Note that as in the case of the yield surface, the component of $\dot{\boldsymbol{\alpha}}_2$ perpendicular to $\hat{\mathbf{m}}_2$ is irrelevant here, but the minimum translation for a given $\|\dot{\boldsymbol{\alpha}}_1\|$ takes place for translation in direction $\hat{\mathbf{m}}_2$, i.e. $\dot{\boldsymbol{\alpha}}_2 = \|\dot{\boldsymbol{\alpha}}_2\| \hat{\mathbf{m}}_2$ —this can be seen as Prager's rule applied to $\bar{f}_i$, interpreted as yield conditions at different microstructural levels

$$\|\dot{\boldsymbol{\alpha}}_2\| = \|\dot{\boldsymbol{\alpha}}_1\| (\hat{\mathbf{n}} : \hat{\mathbf{m}}_2) - \dot{\rho}_2 \quad (2.33)$$

In general,

$$\|\dot{\boldsymbol{\alpha}}_i\| = \|\dot{\boldsymbol{\alpha}}_{i-1}\| (\hat{\mathbf{m}}_{i-1} : \hat{\mathbf{m}}_i) - \dot{\rho}_i \geq 0 \quad (2.34)$$

It is important to note that if the position of the inner surface is known and contact is taking place, from $\bar{f}_i = 0$, Eq. (2.28₂), and using $\dot{\boldsymbol{\alpha}}_i = \|\dot{\boldsymbol{\alpha}}_i\| \hat{\mathbf{m}}_i$, the position of the outer surface can be obtained in an explicit manner, see Figure 2.2

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{i-1} - \rho_i \hat{\mathbf{m}}_i \quad (2.35)$$

which is a condition easier to use in the algorithmic (discrete) implementation. Note that defining $\boldsymbol{\rho}_i = \rho_i \hat{\mathbf{m}}_i$, the backstress can also be interpreted as the addition of several over stresses (which could in turn be interpreted as an internal equilibrium equation)

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_{a+1} + \sum_{i=2}^a \boldsymbol{\rho}_i \quad \text{and} \quad \boldsymbol{\sigma} = \boldsymbol{\alpha}_{a+1} + \sum_{i=1}^a \boldsymbol{\rho}_i \quad (2.36)$$

where a is the outermost active surface.

2.2.4 Consistency parameter from multiple hardening surfaces

Equation (2.21) is directly applicable for one surface. In the presence of N hardening surfaces, we assume the existence of some strain-like internal variables ξ_i , $i = 1, \dots, N$ (of microstructural origin) describing an anisotropic hardening field, such that we have the dependency $\xi(\xi_i)$, and in particular we assume that they follow the compatibility equation

$$\dot{\xi} = \sum_{i=1}^N \dot{\xi}_i \quad (2.37)$$

In this work ratchetting effect is not included. However, these effects may be included in the theory by the addition of ratchetting internal strains in the previous equation. From the compatibility equation of internal strains, we have at the continuum observable level that the consistency parameter rate is

$$\dot{\gamma} = \dot{\xi} : \hat{n} = \sum_{i=1}^a \dot{\xi}_i : \hat{n} = \sum_{i=1}^a \dot{\gamma}_i \quad (2.38)$$

where it has been defined that $\dot{\gamma}_i := \dot{\xi}_i : \hat{n}$ and $a \leq N$ is the outermost active surface, i.e.

$$a = \arg \left(\max_i (f_i = 0) \right) \quad (2.39)$$

If we associate a hardening modulus $H_i(\gamma)$ to each one of the hardening surfaces such that

$$\dot{\alpha}_i = \frac{2}{3} H_i(\gamma) \dot{\xi}_i \quad \text{so} \quad \dot{\xi}_i = \frac{\dot{\alpha}_i}{\frac{2}{3} H_i(\gamma)} \quad (2.40)$$

then by Eq. (2.38)

$$\dot{\gamma} = \frac{\langle \dot{\alpha} : \hat{n} \rangle}{\frac{2}{3} \bar{H}(\xi, \gamma)} = \sum_{i=1}^a \dot{\gamma}_i = \sum_{i=1}^a \frac{\langle \dot{\alpha}_i : \hat{n} \rangle}{\frac{2}{3} H_i(\gamma)} \quad (2.41)$$

In this work the dependence of the moduli $H_i(\gamma)$ is restricted to depend only on the cumulative parameter γ . The other dependency in $\bar{H}(\xi, \gamma)$ is implicitly defined by the history of the backstresses α_i and their flow $\dot{\xi}_i$. Using Eq. (2.34) recursively

$$\begin{aligned} \dot{\gamma}_i &= \frac{\langle \dot{\alpha}_i : \hat{n} \rangle}{\frac{2}{3} H_i} = \frac{\|\dot{\alpha}_i\|}{\frac{2}{3} H_i} \langle \hat{m}_i : \hat{n} \rangle \\ &= \frac{\langle \hat{m}_i : \hat{n} \rangle}{\frac{2}{3} H_i} \|\dot{\alpha}\| \prod_{j=2}^i \langle \hat{m}_j : \hat{m}_{j-1} \rangle - \frac{\langle \hat{m}_i : \hat{n} \rangle}{\frac{2}{3} H_i} \sum_{j=2}^i \dot{\rho}_j \prod_{k=j}^{i-1} (\hat{m}_k : \hat{m}_{k+1}) \end{aligned} \quad (2.42)$$

The first addend is the most important one, whereas the other addends represent the coupling between isotropic and kinematic hardening, and vanish even with isotropic hardening in the case that $\dot{r}_j = \dot{r}_{j-1}$. Since

$$\sum_{j=2}^i \dot{\rho}_j = \dot{r}_i - \dot{r}_1 \quad (2.43)$$

for the case of *uniaxial loading*, we have $\hat{\mathbf{m}}_i = \hat{\mathbf{n}}$ for all i ,

$$\dot{\gamma}_i = \frac{\|\dot{\boldsymbol{\alpha}}\|}{\frac{2}{3}H_i} - \frac{\dot{r}_i - \dot{r}_1}{\frac{2}{3}H_i} \quad (2.44)$$

$$\dot{\gamma}_i = \frac{\|\dot{\boldsymbol{\alpha}}\|}{\frac{2}{3}H_i} - \|\dot{\boldsymbol{\alpha}}\| \frac{(\dot{r}_i - \dot{r}_1) / \|\dot{\boldsymbol{\alpha}}\|}{\frac{2}{3}H_i} = \frac{\|\dot{\boldsymbol{\alpha}}\|}{\frac{2}{3}H_i} [1 - \zeta_i^a(p)] \quad (2.45)$$

where we have defined

$$\zeta_i^a(p) := \frac{\dot{r}_i - \dot{r}_1}{\|\dot{\boldsymbol{\alpha}}\|} = \frac{\hat{\sigma}'_i(p) - \hat{\sigma}'_1(p)}{\bar{H}^a} \quad (2.46)$$

in which $\hat{\sigma}'_i(p)$ is the derivative of the uniaxial-equivalent size of surface i as a function of the uniaxial plastic strain p , and $\bar{H}^a$ (with *superindex* a) is the uniaxial equivalent hardening modulus when the outermost active surface is surface a . Then

$$\|\dot{\boldsymbol{\alpha}}\| = \underbrace{\frac{\|\dot{\boldsymbol{\alpha}}_i\|}{2}}_{\frac{2}{3}H_i\dot{\gamma}_i} + (\dot{r}_i - \dot{r}_1) \quad (2.47)$$

and, from Eq. (2.41), in the particular case of proportional loading

$$\dot{\gamma} = \frac{3}{2} \|\dot{\boldsymbol{\alpha}}\| \frac{1}{\bar{H}^a} = \sum_{i=1}^a \dot{\gamma}_i = \frac{3}{2} \|\dot{\boldsymbol{\alpha}}\| \sum_{i=1}^a \frac{1}{H_i} (1 - \zeta_i^a) \quad (2.48)$$

This equation results in

$$\frac{1}{\bar{H}^a} = \sum_{i=1}^a \frac{1 - \zeta_i^a}{H_i} \quad (2.49)$$

which is the expression that may be used to determine the hardening moduli associated to the surfaces from a uniaxial stress-strain curve. It is noted that during a monotonic loading it is impossible to distinguish which part of the hardening corresponds to isotropic and which part to kinematic hardening. Only upon stress

reversal, identifying the position of the backstress (at half the elastic region), both components can be identified. For a given surface

$$\frac{1}{H_a} = \frac{1}{(1 - \zeta_a^a) \bar{H}^a} - \sum_{i=1}^{a-1} \frac{1 - \zeta_i^a}{1 - \zeta_a^a} \frac{1}{H_i} \quad \text{for } a = 1, \dots, N \quad (2.50)$$

This expression simplifies in the usual case where isotropic hardening is small respect to kinematic hardening or all the surfaces harden equally, i.e. $\zeta_i^a \ll 1$, to

$$\frac{1}{H_a} = \frac{1}{\bar{H}^a} - \sum_{i=1}^{a-1} \frac{1}{H_i} \quad \text{for } a = 1, \dots, N \quad (2.51)$$

In this case the modulus is given by the uniaxial equivalence

$$\frac{E \bar{H}^a}{E + \bar{H}^a} \simeq E_{ep}^{(a)} = \frac{\hat{\sigma}_{a+1} - \hat{\sigma}_a}{\hat{\varepsilon}_{a+1} - \hat{\varepsilon}_a} \quad (2.52)$$

in which E is the elastic Young modulus, $E_{ep}^{(a)}$ is the uniaxial elastoplastic modulus while surface a is the outermost active surface, and $\hat{\sigma}_a$ and $\hat{\varepsilon}_a$ are the stress-strain data points in which the uniaxial stress-strain curve has been discretized, see Figures 2.3 and 2.4. Then

$$\bar{H}^a = \frac{E - E_{ep}^{(a)}}{E E_{ep}^{(a)}} \quad (2.53)$$

For the case $a = N$, the value of $E_{ep}^{(N)}$, and hence H_N , is given by the desired limiting hardening modulus.

Remarkably, if we insert a hardening surface in a straight section of the stress-strain curve, we have $E_{ep}^{(i+1)} = E_{ep}^{(i)}$, so $\bar{H}^{i+1} = \bar{H}^i$ and $1/H_{i+1} = 0$. Then note that $\dot{\gamma}_{i+1} = 0$ and the surface becomes irrelevant in terms of plastic work. Furthermore, if the uniaxial stress-strain curve happens to be bi-linear, naturally all $1/H_i = 0$ except $1/H_1$, and $\dot{\gamma}_i = 0$ except for $\dot{\gamma}_1 = \dot{\gamma}$. Therefore, the classical linearly-hardened J_2 -plasticity is recovered naturally without explicitly accounting for it, just as a consequence of the prescribed stress-strain curve. This is extremely important in order to obtain consistent predictions for different, arbitrary discretizations, a property not enjoyed, for example, by Mróz's model [78].

2.2.5 Cyclic isotropic hardening/softening

There are many possible implementations for isotropic hardening/softening in the multisurface model. Since we are dealing with multiple surfaces to describe

Figure 2.3: Example of a discretization of a stress-strain curve and the generation of the hardening surfaces; depicted in the $\sqrt{3}\tau - \sigma$ plane.

hardening, these surfaces may be used even to change the shape of the stress-strain curve during cyclic loading, for example from the monotonic backbone curve to a very different skeleton one. However, the purpose of the present work is to show the capabilities of the model to describe multiaxial loading with cyclic hardening/softening. Therefore, since in the examples a relevant change of shape in the skeleton curve won't be accounted for, to keep the presentation simple, an isotropic hardening function such that all surfaces remain proportional in size is considered

$$r_i(p) = R(p) {}^0r_i \quad (2.54)$$

where ${}^0r_i = r_i(0)$, $R(p)$ is the isotropic hardening function and we made, as usual, an abuse of notation in $r_i(p) = r_i(\gamma)$ to avoid proliferation of symbols. Then

$$\dot{r}_i(p) = \frac{dr_i}{dp} \dot{p} = {}^0r_i R'(p) \dot{p} = \sqrt{\frac{2}{3}} R'(p) {}^0r_i \dot{\gamma} \quad (2.55)$$

and obviously

$$\dot{\rho}_i(p) = \dot{r}_i(p) - \dot{r}_{i-1}(p) = {}^0\rho_i R'(p) \dot{p} \quad (2.56)$$

An scheme of the surface growth may be seen in Figure 2.4.

Figure 2.4: Procedure for modification of the skeleton curve due to cyclic hardening.

The procedure that is used to determine the function $R(p)$ is the follow-up of the peak stresses during subsequent cycles, which is a procedure useful in fatigue analysis. With the data from the change of peak stresses and the accumulated plastic strain during a cycle, we obtain the relation between R and p . With the knowledge of $R(p)$, and hence of the change in the stresses $\hat{\sigma}_a$, the derivative of the radius of any hardening surface dr_i/dp can be obtained. Hence, the relevant key function is $R(p)$.

A popular way of expressing the function $R(p)$ is in relative terms, i.e. R/R_s ([13], [67]), where R_s is the saturated value. During cyclic loading, we can approximate $R(p)$ measuring the peak stresses in successive strain-controlled cycles. Assume that when the strain range of a loop is known, the accumulative plastic strain of that first loop is approximately p_0 . Then, during cyclic loading, the accumulated plastic strain is approximately $p_n = 0.25p_0 + np_0$, where n is the number of loops. The change of $R(p)$ at different loop counting n , gives pairs $\{p_n, R_n\}$, which can be immediately used to establish the relation $R(p)$ in any desired format (piecewise linear, spline or adjusting a model function).

Once the relation $R(p)$, or more generally, the relation $r_i(p)$ is known, the expression for the derivative of the kinematic hardening moduli H_i can be obtained, see Figure 2.4. The inverse of the uniaxial elastoplastic tangent when surface a is

active is, for a given p

$$\frac{1}{E_{ep}^{(a)}(p)} = \frac{1}{E} + \frac{1}{\bar{H}^a(p)} = \frac{\varepsilon_{a+1}(p) - \varepsilon_a(p)}{\hat{\sigma}_{a+1}(p) - \hat{\sigma}_a(p)} \quad (2.57)$$

Since the elastic behavior is constant, the variations during a cycle are related by—see Figure 2.4

$$\frac{\Delta[\varepsilon_{a+1}(p) - \varepsilon_a(p)]}{\Delta[\hat{\sigma}_{a+1}(p) - \hat{\sigma}_a(p)]} = \frac{1}{E} \quad (2.58)$$

Then, using the chain rule, taking the derivative of Eq. (2.57) respect to p —note that as it should be expected, this derivative vanishes if all surfaces harden equally in an isotropic manner, so $d\hat{\sigma}_{a+1}/dp = d\hat{\sigma}_a/dp$ —

$$\frac{d(1/\bar{H}^a(p))}{dp} = \left(\frac{1}{E} - \frac{\varepsilon_{a+1}(p) - \varepsilon_a(p)}{[\hat{\sigma}_{a+1}(p) - \hat{\sigma}_a(p)]^2} \right) \left(\frac{d\hat{\sigma}_{a+1}(p)}{dp} - \frac{d\hat{\sigma}_a(p)}{dp} \right) \quad (2.59)$$

where

$$\frac{d\hat{\sigma}_a}{dp} = \sqrt{\frac{2}{3}} \frac{dr_a}{dp} \quad (2.60)$$

Thus, from Eq. (2.49) we can compute in a recursive manner

$$\frac{d(1/H_a(p))}{dp} = \frac{d(1/\bar{H}^a(p))}{dp} - \sum_{i=1}^{a-1} \frac{d(1/H_i(p))}{dp} \quad (2.61)$$

and, obviously

$$\frac{dH_i}{dp} = -H_i^2(p) \frac{d(1/H_i(p))}{dp} \quad (2.62)$$

In summary, from $R(p)$ we can immediately compute $r_i(p)$, dr_i/dp and dH_i/dp for all surfaces (the hardening of the last surface may be set constant). If a change of the shape of the stress-strain curve from monotonic to cyclic behavior is desired, different functions for $r_i(p)$ can be selected to model such effect.

2.3 Fully implicit algorithmic implementation

The superposition of isotropic hardening brings some changes to the algorithm, the total increment of consistency parameter is still the sum of the increment of consistency parameters of each surface, as in 2.38, but since 3.2, the increment of plastic strain depends on the total increment of consistency parameter, thus the increment

of accumulated plastic strain depends on the total increment of the consistency parameter. On the other hand, it is assumed that the sizes of hardening surfaces change due to the scalar value of the accumulated plastic strain, which forces the radii and hardening moduli related to each hardening surface to relate to the total increment of the consistency parameter.

For a more clear illustration, if we rewrite the discrete yield function at the step $n + 1$ with the accumulated plastic strain p :

$$f_{n+1} \equiv \|\boldsymbol{\sigma}_{n+1}^{tr} - \boldsymbol{\alpha}_n^1\| - 2\mu\Delta\gamma - H_1(p)\Delta\gamma_1 - r_1(p) = 0 \quad (2.63)$$

It is evident that when the information of step n is known, the increment of the consistency parameter $\Delta\gamma_1$ of the yield surface depends on p , which increment is decided by $\Delta\gamma$, so now with the superposition of isotropic hardening, besides the fact that $\Delta\gamma$ is a function of $\Delta\gamma_1$ ($\Delta\gamma$ is the sum of each $\Delta\gamma_i$ that could be taken as a function of $\Delta\gamma_1$) :

$$\Delta\gamma = \sum_{i=1}^a \Delta\gamma_i(\Delta\gamma_1)$$

which is defined by the pure kinematic hardening rule, now $\Delta\gamma_1$ is also a function of $\Delta\gamma$. To conclude, $\Delta\gamma_1$ and $\Delta\gamma$ depend on each other in a mutual way, so the former iterative procedure on the sole variable $\Delta\gamma_1$ is no longer valid.

An upgraded Newton-Raphson iterative procedure is used on the dual variables $\Delta\gamma_1$ and $\Delta\gamma$ to minimize the residual of the yield function as well as the numerical difference between the assumed and calculated values of $\Delta\gamma$.

The fully implicit algorithm is based on the Closest Point Projection (CPP) Algorithm [79].

2.3.1 Incremental formulation

The stress integration algorithm computes the stress ${}^{t+\Delta t}\boldsymbol{\sigma}$ at “time” (or step) $t + \Delta t$ with the knowledge of: the elastic strains ${}^t\boldsymbol{\varepsilon}_e$ at time t , the increment of total deformations $\Delta\boldsymbol{\varepsilon}$ during the step, the history variables ${}^t\boldsymbol{\alpha}_i$, and either tp or ${}^t\gamma$. The time-integration scheme, based on the classical radial return algorithm is depicted in Figure 2.2c. Using the trial stress rate Eq. (2.16), and defining the *algorithmic* trial stress ${}^{tr}\boldsymbol{\sigma} = {}^t\boldsymbol{\sigma} + \Delta{}^{tr}\boldsymbol{\sigma}$,

$${}^{tr}\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}({}^t\boldsymbol{\varepsilon}_e + \Delta\boldsymbol{\varepsilon}) = {}^t\boldsymbol{\sigma} + \kappa\Delta\varepsilon_v\mathbf{I} + 2\mu\Delta\boldsymbol{\varepsilon}^d \quad (2.64)$$

Since during the absence of plastic flow the backstresses do not change (i.e. ${}^{tr}\dot{\boldsymbol{\alpha}}_i = \mathbf{0}$), we can define the *algorithmic* backstresses

$${}^{tr}\boldsymbol{\alpha}_i := {}^t\boldsymbol{\alpha}_i, \quad i = 1, \dots, N \quad (2.65)$$

The corrector stress increment, using a Backward-Euler (radial-return) integration algorithm, is

$$\Delta {}^{ct}\boldsymbol{\sigma} = 2\mu\Delta {}^{ct}\boldsymbol{\varepsilon}_e = -2\mu\Delta\gamma {}^{t+\Delta t}\hat{\mathbf{n}} \quad (2.66)$$

where ${}^{t+\Delta t}\hat{\mathbf{n}}$ is the normal to the yield surface at the end of the step. Then

$${}^{t+\Delta t}\boldsymbol{\sigma} = {}^{tr}\boldsymbol{\sigma} - 2\mu\Delta\gamma {}^{t+\Delta t}\hat{\mathbf{n}} \quad (2.67)$$

Using the same integration rule, from Eq. (2.20) for surface 1

$${}^{t+\Delta t}\boldsymbol{\alpha} = {}^t\boldsymbol{\alpha} + {}^{t+\Delta t}\tilde{H}_1\Delta\gamma_1 {}^{t+\Delta t}\hat{\mathbf{n}} \quad (2.68)$$

where for convenience, we have absorbed the uniaxial equivalence factor $2/3$ in

$${}^{t+\Delta t}\tilde{H}_i := \frac{2}{3} {}^{t+\Delta t}H_i \quad (2.69)$$

Then, note that as in classical plasticity, we can obtain the flow direction directly from the deviatoric trial state—recall that the volumetric part is elastic

$${}^{t+\Delta t}\hat{\mathbf{n}} = \frac{{}^{tr}\boldsymbol{\sigma}^d - {}^t\boldsymbol{\alpha}}{\|{}^{tr}\boldsymbol{\sigma}^d - {}^t\boldsymbol{\alpha}\|} =: \frac{{}^{t+\Delta t}\mathbf{n}}{\|{}^{t+\Delta t}\mathbf{n}\|} \quad (2.70)$$

However, to obtain ${}^{t+\Delta t}\boldsymbol{\alpha}$ in Eq. (2.68), which accounts for the correction to the dissipation (the energy not dissipated but microstructurally stored), we need to compute the consistency parameters which guarantee material compatibility. In order to do so, we need to develop an integration algorithm. We will be able to establish below a very efficient iterative algorithm based just on two scalars, namely $\Delta\gamma_1$ and $\Delta\gamma$ (i.e. a two-parameter CPP iterative algorithm), regardless of the number of surfaces being employed. The two functions to be minimized are the yield function f and the compatibility requirement

$$g(\Delta\gamma, \Delta\gamma_1) = \Delta\gamma - \Delta\gamma^a \rightarrow 0 \quad (2.71)$$

where we defined —note the different meaning of sub- and superindices

$$\Delta\gamma^a(\Delta\gamma_1) := \sum_{i=1}^a \Delta\gamma_i(\Delta\gamma_1) = \Delta\gamma_a(\Delta\gamma_1) + \Delta\gamma^{a-1}(\Delta\gamma_1) \quad (2.72)$$

The last recursive identity is given for future reference. In order to do so, we need the derivatives respect to these two basic iterative variables, namely $\Delta\gamma$ and $\Delta\gamma_1$.

From Equation (2.35), using also Eq. (2.30), after some straightforward algebra, we can obtain

$$\frac{d^{t+\Delta t}\boldsymbol{\alpha}_i}{d\Delta\gamma_1} = {}^{t+\Delta t}\mathbb{D}_{1i} : \frac{d^{t+\Delta t}\boldsymbol{\alpha}_{i-1}}{d\Delta\gamma_1} - {}^{t+\Delta t}\mathbf{D}_{2i} \frac{d\Delta\gamma^a}{d\Delta\gamma_1} \quad (2.73)$$

where the algorithmic projectors and algorithmic radii vector changes are

$${}^{t+\Delta t}\mathbb{D}_{1i} = \left(1 - \frac{{}^{t+\Delta t}\rho_i}{\|{}^{t+\Delta t}\mathbf{m}_i\|}\right) \mathbb{P} + \frac{{}^{t+\Delta t}\rho_i}{\|{}^{t+\Delta t}\mathbf{m}_i\|} {}^{t+\Delta t}\hat{\mathbf{m}}_i \otimes {}^{t+\Delta t}\hat{\mathbf{m}}_i \quad (2.74)$$

$${}^{t+\Delta t}\mathbf{D}_{2i} = \sqrt{\frac{2}{3}} \frac{d^{t+\Delta t}\rho_i}{d^{t+\Delta t}p} {}^{t+\Delta t}\hat{\mathbf{m}}_i \quad \text{with} \quad \frac{d^{t+\Delta t}\rho_i}{d^{t+\Delta t}p} = {}^0\rho_i \frac{d^{t+\Delta t}R}{d^{t+\Delta t}p} \quad (2.75)$$

in which it is defined—note that ${}^{t+\Delta t}\hat{\mathbf{m}}_i$ is also recursively defined

$${}^{t+\Delta t}\mathbf{m}_i := {}^{t+\Delta t}\boldsymbol{\alpha}_{i-1} - {}^t\boldsymbol{\alpha}_i \quad \text{and} \quad {}^{t+\Delta t}\hat{\mathbf{m}}_i := \frac{{}^{t+\Delta t}\mathbf{m}_i}{\|{}^{t+\Delta t}\mathbf{m}_i\|} \quad (2.76)$$

and $\mathbb{P}$ is the fourth order deviatoric projector tensor; i.e. $\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}$, where $\mathbb{I}$ is the fully symmetric 4th-order identity tensor. Then, taking the derivative of Eq. (2.41) respect to $\Delta\gamma_1$,

$$\begin{aligned} \frac{d\Delta\gamma_i}{d\Delta\gamma_1} &= \frac{{}^{t+\Delta t}Z_i}{{}^{t+\Delta t}\tilde{H}_i} {}^{t+\Delta t}\hat{\mathbf{n}} : \left({}^{t+\Delta t}\mathbb{D}_{1i} : \frac{d^{t+\Delta t}\boldsymbol{\alpha}_{i-1}}{d\Delta\gamma_1} - {}^{t+\Delta t}\mathbf{D}_{2i} \frac{d\Delta\gamma^a}{d\Delta\gamma_1} \right) \\ &\quad - {}^{t+\Delta t}Z_i \sqrt{\frac{2}{3}} \frac{{}^{t+\Delta t}\hat{\mathbf{n}} : {}^{t+\Delta t}\hat{\mathbf{m}}_i}{{}^{t+\Delta t}\tilde{H}_i^2} \frac{d^{t+\Delta t}\tilde{H}_i}{d^{t+\Delta t}p} \frac{d\Delta\gamma^a}{d\Delta\gamma_1} \end{aligned} \quad (2.77)$$

we can compute $d\Delta\gamma^a/d\Delta\gamma_1$ in a recursive manner using Eq. (2.72)₂.

$$\frac{d\Delta\gamma^a}{d\Delta\gamma_1} = \sum_{i=1}^a \frac{d\Delta\gamma_i}{d\Delta\gamma_1} \quad (2.78)$$

Consider

$$\frac{d\Delta\gamma^b}{d\Delta\gamma_1} = \left(\sum_{i=1}^{b-1} \frac{d\Delta\gamma_i}{d\Delta\gamma_1} \right) + \frac{d\Delta\gamma_b}{d\Delta\gamma_1} = \frac{d\Delta\gamma^{b-1}}{d\Delta\gamma_1} + \frac{d\Delta\gamma_b}{d\Delta\gamma_1} \quad (2.79)$$

After some straightforward algebra we obtain

$$\frac{d\Delta\gamma^b}{d\Delta\gamma_1} = \frac{\frac{{}^{t+\Delta t}Z_b}{{}^{t+\Delta t}\tilde{H}_b} {}^{t+\Delta t}\hat{\mathbf{n}} : {}^{t+\Delta t}\mathbb{D}_{1b} : \frac{d{}^{t+\Delta t}\boldsymbol{\alpha}_{b-1}}{d\Delta\gamma_1} + \frac{\partial\Delta\gamma^{b-1}}{\partial\Delta\gamma_1}}{1 + \frac{{}^{t+\Delta t}Z_b}{{}^{t+\Delta t}\tilde{H}_b} {}^{t+\Delta t}\hat{\mathbf{n}} : {}^{t+\Delta t}\mathbf{D}_{2b} + {}^{t+\Delta t}Z_b\Delta\gamma_b\sqrt{\frac{2}{3}}\frac{{}^{t+\Delta t}\hat{\mathbf{n}} : {}^{t+\Delta t}\hat{\mathbf{m}}_b}{{}^{t+\Delta t}\tilde{H}_b}\frac{d{}^{t+\Delta t}\tilde{H}_b}{d{}^{t+\Delta t}p}} \quad (2.80)$$

whih is applied repeatedly for $b = 1, \dots, a$. In this equation, to account in Eq. (2.41) for $\dot{\gamma}_i \geq 0$, we defined for convenience the Heaviside function

$${}^{t+\Delta t}Z_b = \frac{1 + \text{sign}({}^{t+\Delta t}\hat{\mathbf{m}}_b : {}^{t+\Delta t}\hat{\mathbf{n}})}{2} \in \{0, 1\} \quad (2.81)$$

Remarkably, if $E_{ep}^{(b)} = E_{ep}^{(b-1)}$ we have $\bar{H}^b = \bar{H}^{b-1}$ and $\tilde{H}_b \rightarrow \infty$; then $\Delta\gamma^b = \Delta\gamma^{b-1}$, $d\Delta\gamma^b/d\Delta\gamma_1 = d\Delta\gamma^{b-1}/d\Delta\gamma_1$ and surface b has no effect in the formulation or the iterations.

2.3.2 Local iterative algorithm

As mentioned, the implicit algorithm is based on the search for the values of $\Delta\gamma_1$ and $\Delta\gamma$ such that the following conditions are met

$$\begin{cases} {}^{t+\Delta t}f(\Delta\gamma, \Delta\gamma_1) := \|{}^{tr}\boldsymbol{\sigma} - {}^t\boldsymbol{\alpha}\| - 2\mu\Delta\gamma - {}^{t+\Delta t}\tilde{H}_1\Delta\gamma_1 - {}^{t+\Delta t}r_1 \rightarrow 0 \\ {}^{t+\Delta t}g(\Delta\gamma, \Delta\gamma_1) := \Delta\gamma^a(\Delta\gamma_1) - \Delta\gamma \rightarrow 0 \end{cases} \quad (2.82)$$

The solution for $[\Delta\gamma, \Delta\gamma_1]^T$ is obtained by a Newton-Raphson algorithm until convergence is achieved in both f and g ,

$$\begin{bmatrix} \Delta\gamma \\ \Delta\gamma_1 \end{bmatrix}^{(j+1)} = \begin{bmatrix} \Delta\gamma \\ \Delta\gamma_1 \end{bmatrix}^{(j)} - \begin{bmatrix} \partial\Delta\gamma/\partial f & \partial\Delta\gamma/\partial g \\ \partial\Delta\gamma_1/\partial f & \partial\Delta\gamma_1/\partial g \end{bmatrix}^{(j)} \begin{bmatrix} f \\ g \end{bmatrix}^{(j)} \quad (2.83)$$

where $[\cdot]^{(j)}$ implies values at iteration j . The derivatives in the tangent matrix in Eq. (2.83) are obtained as usual from its inverse matrix, whose components are

$$\begin{cases} \frac{\partial f}{\partial\Delta\gamma_1} = - {}^{t+\Delta t}\tilde{H}_1 \\ \frac{\partial f}{\partial\Delta\gamma} = -2\mu - \sqrt{\frac{2}{3}}\Delta\gamma_1\frac{d{}^{t+\Delta t}\tilde{H}_1}{d{}^{t+\Delta t}p} - \sqrt{\frac{2}{3}}\frac{d{}^{t+\Delta t}r_1}{d{}^{t+\Delta t}p} \\ \frac{\partial g}{\partial\Delta\gamma_1} = \frac{d\Delta\gamma^a}{d\Delta\gamma_1} \\ \frac{\partial g}{\partial\Delta\gamma} = -1 \end{cases} \quad (2.84)$$

which right-hand-side terms have already been computed above, see Eqs. (2.80) and (2.55). Remarkably, if linear hardening is considered, in which case only one surface is relevant, $d\tilde{H}_i/dp = 0$, dr_i/dp is constant and $\Delta\gamma^a = \Delta\gamma_1$. Then, the function g is identically satisfied and the algorithm gives the exact solution in just one iteration because all coefficients in the tangent are constants. As expected, this result is in line with the well-known classical radial return algorithm of Krieg and Key [68], so both the continuum and algorithmic classical plasticity are recovered. In the case when only isotropic hardening is non-linear, g is also identically satisfied and the classical scalar-based algorithm is also recovered, see [35].

For the case when multiple surfaces are present, a guess $(\cdot)^{(0)}$ is needed to start the iterations. We always take $\Delta\gamma_1^{(0)} = 0$. A simplest guess for the active surface is to assume $a^{(0)} = 1$. If after the computation of the solution we obtain $\bar{f}_2 > 0$, then the index is increased $a \leftarrow a + 1$, and the solution is recomputed; and so on. However, if a large number of surfaces is employed, a better guess to save computational time is to set a for the first iterations as the outermost surface such that

$$\bar{f}_a = 0 \text{ and } {}^{t+\Delta t}\hat{\mathbf{n}} : {}^t\hat{\mathbf{m}}_a > 0 \quad (2.85)$$

If after reaching a solution $\bar{f}_{a+1} > 0$, then the active surface index is increased. On the contrary, if a solution with $\Delta\gamma_a > 0$ is not obtained, the active surface index is decreased.

A layout of the computational algorithm is shown in Table 2.1

2.3.3 Consistent tangent for the global iterative algorithm

To preserve the efficient asymptotic quadratic convergence rate of implicit Newton algorithms in global equilibrium iterations in finite element simulations, the algorithmic (consistent) tangent moduli must be used. These moduli relate the increment of stresses to the increment of strains according to the algorithmic implementation, i.e.

$${}^{t+\Delta t}\mathbb{C} = \frac{d^{t+\Delta t}\boldsymbol{\sigma}}{d^{t+\Delta t}\boldsymbol{\varepsilon}} = \kappa \mathbf{I} \otimes \mathbf{I} + \frac{d^{t+\Delta t}\boldsymbol{\sigma}^d}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} : \frac{d\boldsymbol{\varepsilon}^d}{d\boldsymbol{\varepsilon}} \quad (2.86)$$

where $\kappa \mathbf{I} \otimes \mathbf{I}$ is the volumetric part (purely elastic and constant), $d\boldsymbol{\varepsilon}^d/d\boldsymbol{\varepsilon} \equiv \mathbb{P}$ is the also constant deviatoric projector tensor and $d^{t+\Delta t}\boldsymbol{\sigma}^d/d^{t+\Delta t}\boldsymbol{\varepsilon}^d$ is the relevant non-trivial algorithmic part. This tensor may be obtained taking the derivative of—see Figure 2.2c

$${}^{t+\Delta t}\boldsymbol{\sigma}^d = {}^t\boldsymbol{\alpha} + ({}^{t+\Delta t}\tilde{H}_1\Delta\gamma_1 + {}^{t+\Delta t}r_1) {}^{t+\Delta t}\hat{\mathbf{n}} \quad (2.87)$$

as

Local iterative algorithm

1. Compute trial stress ${}^{tr}\boldsymbol{\sigma}$, Eq. (2.64) and both ${}^{t+\Delta t}\mathbf{n}$ and ${}^{t+\Delta t}\hat{\mathbf{n}}$, Eq. (2.70)
2. If ${}^{tr}f := \|{}^{t+\Delta t}\mathbf{n}\| - {}^tr_1 \leq 0$, step is elastic; accept trial solution and exit.
Else, step is plastic: continue
3. Set $\Delta\gamma = \Delta\gamma_i = 0$ and guess a ; either $a = 1$ or by Eq. (2.85)
4. For each iteration (j)

Compute ${}^{t+\Delta t}\tilde{H}_i({}^{t+\Delta t}p)$ and ${}^{t+\Delta t}r_i({}^{t+\Delta t}p)$ with ${}^{t+\Delta t}p = \sqrt{2/3}({}^t\gamma + \Delta\gamma)$
 Set ${}^{t+\Delta t}\boldsymbol{\alpha}_1 = {}^t\boldsymbol{\alpha}_1 + {}^{t+\Delta t}\tilde{H}_1\Delta\gamma_1 {}^{t+\Delta t}\hat{\mathbf{n}}$ and initialize $\Delta\gamma^a = \Delta\gamma_1$
 For each hardening surface $i = 2, \dots, a$

Compute ${}^{t+\Delta t}\mathbf{m}_i$ and ${}^{t+\Delta t}\hat{\mathbf{m}}_i$ via Eq. (2.76)
 By Eq. (2.35) ${}^{t+\Delta t}\boldsymbol{\alpha}_i = {}^{t+\Delta t}\boldsymbol{\alpha}_{i-1} - {}^{t+\Delta t}\rho_i {}^{t+\Delta t}\hat{\mathbf{m}}_i$
 $\Delta\gamma_i = \left(1/{}^{t+\Delta t}\tilde{H}_i\right) ({}^{t+\Delta t}\boldsymbol{\alpha}_i - {}^t\boldsymbol{\alpha}_i) : {}^{t+\Delta t}\hat{\mathbf{n}}$
 $\Delta\gamma^a \leftarrow \Delta\gamma^a + \Delta\gamma_i$

Compute ${}^{t+\Delta t}f = \|{}^{t+\Delta t}\mathbf{n}\| - 2\mu\Delta\gamma - {}^{t+\Delta t}\tilde{H}_1\Delta\gamma_1 - {}^{t+\Delta t}r_1$
 Compute ${}^{t+\Delta t}g = \Delta\gamma^a - \Delta\gamma$
 If $|{}^{t+\Delta t}f| < tol_f$ and $|{}^{t+\Delta t}g| < tol_g$ accept the solution, go to step 5.
 Perform loop $i = 2, \dots, a$ to compute $d\Delta\gamma^a/d\Delta\gamma_1$ via Eq. (2.80)
 Compute the local tangent matrix, Eqs. (2.84) and (2.80)
 Update solution solving 2×2 linear system of Equations, e.g. Eq. (2.83)

Table 2.1: Local stress integration algorithm

$$\begin{aligned} \frac{d^{t+\Delta t} \boldsymbol{\sigma}^d}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} &= ({}^{t+\Delta t} \tilde{H}_1 \Delta \gamma_1 + {}^{t+\Delta t} r_1) \frac{d^{t+\Delta t} \hat{\mathbf{n}}}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} + {}^{t+\Delta t} \tilde{H}_1 {}^{t+\Delta t} \hat{\mathbf{n}} \otimes \frac{d\Delta \gamma_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \\ &+ \Delta \gamma_1 {}^{t+\Delta t} \hat{\mathbf{n}} \otimes \frac{d {}^{t+\Delta t} \tilde{H}_1}{d {}^{t+\Delta t} \boldsymbol{\varepsilon}^d} + {}^{t+\Delta t} \hat{\mathbf{n}} \otimes \frac{d^{t+\Delta t} r_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \end{aligned} \quad (2.88)$$

Taking into account that $\Delta \boldsymbol{\varepsilon}_p = \Delta \gamma {}^{t+\Delta t} \hat{\mathbf{n}} = \sqrt{3/2} \Delta p {}^{t+\Delta t} \hat{\mathbf{n}}$, we obtain

$$\frac{d^{t+\Delta t} \boldsymbol{\varepsilon}_p}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \Delta \gamma \frac{d^{t+\Delta t} \hat{\mathbf{n}}}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} + \frac{d\Delta \gamma}{d\Delta \gamma_1} {}^{t+\Delta t} \hat{\mathbf{n}} \otimes \frac{d\Delta \gamma_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \quad (2.89)$$

and

$$\frac{d^{t+\Delta t} p}{d^{t+\Delta t} \boldsymbol{\varepsilon}_p} = \sqrt{\frac{2}{3}} {}^{t+\Delta t} \hat{\mathbf{n}}, \quad \text{so} \quad \frac{d^{t+\Delta t} p}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \frac{d^{t+\Delta t} p}{d^{t+\Delta t} \boldsymbol{\varepsilon}_p} : \frac{d^{t+\Delta t} \boldsymbol{\varepsilon}_p}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \quad (2.90)$$

Using the chain rule

$$\frac{d^{t+\Delta t} \tilde{H}_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \frac{d^{t+\Delta t} \tilde{H}_1}{d^{t+\Delta t} p} \frac{d^{t+\Delta t} p}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \quad \text{and} \quad \frac{d^{t+\Delta t} r_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \frac{d^{t+\Delta t} r_1}{d^{t+\Delta t} p} \frac{d^{t+\Delta t} p}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \quad (2.91)$$

Taking into account that by Eq. (2.70)

$$\frac{d \| {}^{t+\Delta t} \mathbf{n} \|}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = 2\mu {}^{t+\Delta t} \hat{\mathbf{n}} \quad (2.92)$$

$$\frac{d^{t+\Delta t} \hat{\mathbf{n}}}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \frac{2\mu}{\| {}^{tr} \boldsymbol{\sigma} - {}^t \boldsymbol{\alpha} \|} (\mathbb{I} - {}^{t+\Delta t} \hat{\mathbf{n}} \otimes {}^{t+\Delta t} \hat{\mathbf{n}}) \quad (2.93)$$

and that by Eq. (2.82)

$$\begin{aligned} 0 \equiv \frac{d^{t+\Delta t} f}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} &= 2\mu {}^{t+\Delta t} \hat{\mathbf{n}} - 2\mu \frac{d\Delta \gamma}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} - {}^{t+\Delta t} \tilde{H}_1 \frac{d\Delta \gamma_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \\ &- \frac{d^{t+\Delta t} \tilde{H}_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \Delta \gamma_1 - \frac{d^{t+\Delta t} r_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} \end{aligned} \quad (2.94)$$

after some lengthy but straightforward algebra, we can factor-out the key derivative

$$\frac{d\Delta \gamma_1}{d^{t+\Delta t} \boldsymbol{\varepsilon}^d} = \frac{2\mu}{(2\mu + {}^{t+\Delta t} \theta) \frac{d\Delta \gamma}{d\Delta \gamma_1} + {}^{t+\Delta t} \tilde{H}_1} {}^{t+\Delta t} \hat{\mathbf{n}} \quad (2.95)$$

where

$${}^{t+\Delta t}\theta = \sqrt{\frac{2}{3}} \left(\Delta\gamma_1 \frac{d^{t+\Delta t}\tilde{H}_1}{d^{t+\Delta t}p} + \frac{d^{t+\Delta t}r_1}{d^{t+\Delta t}p} \right) \quad (2.96)$$

Because at local convergence $\Delta\gamma^a \equiv \Delta\gamma$, then $d\Delta\gamma/d\Delta\gamma_1$ is given by Eq. (2.80). Remarkably if only one surface is used we have $d\Delta\gamma/d\Delta\gamma_1 = 1$, $\tilde{H}_1 = 2/3\bar{H}$ and we naturally recover the usual expression for classical von Mises plasticity with mixed hardening if kinematic hardening is linear ($d\bar{H}/dp = 0$).

With the key Equation (2.95), all the quantities in the tangent Eq. (2.88) are known. The flow of the computational procedure for the tangent is

$$\left. \begin{array}{c} \xrightarrow{\text{Eq. (2.80)}} \frac{d\Delta\gamma}{d\Delta\gamma^1} \xrightarrow{\text{Eq. (2.95)}} \frac{d\Delta\gamma_1}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \\ \xrightarrow{\text{Eq. (2.93)}} \frac{d^{t+\Delta t}\hat{\mathbf{n}}}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \end{array} \right\} \xrightarrow{\text{Eq. (2.89)}} \frac{d^{t+\Delta t}\boldsymbol{\varepsilon}_p}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \xrightarrow{\text{Eq. (2.90)}} \frac{d^{t+\Delta t}p}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \xrightarrow{\text{Eq. (2.91)}} \frac{d^{t+\Delta t}\tilde{H}_1}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d}, \frac{d^{t+\Delta t}r_1}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \xrightarrow{\text{Eq. (2.88)}} \frac{d^{t+\Delta t}\boldsymbol{\sigma}^d}{d^{t+\Delta t}\boldsymbol{\varepsilon}^d} \xrightarrow{\text{Eq. (2.86)}} {}^{t+\Delta t}\mathbb{C} \quad (2.97)$$

Obviously this diagram can also be employed to obtain the tangent in closed form. As it should be expected, it is straightforward to verify that if only one surface is present, the classical tangent in [35] is recovered.

2.4 Large strains formulation

As mentioned, the current formulation is a special case of classical von Mises plasticity with mixed hardening in which the effective kinematic hardening modulus is computed with the aid of internal variables. Therefore, as a remarkable advantage, the large strains implementation is immediate. To this end, we can use the algorithm of Eterovic and Bathe [75] based on logarithmic strains. This algorithm formulated in the full tensorial space is valid for kinematic hardening and uses a purely geometric pre- and post-processor, keeping the infinitesimal plastic stress integration unmodified. Using the multiplicative decomposition of the deformation gradient $\mathbf{X} = \mathbf{X}_e\mathbf{X}_p$ into elastic $\mathbf{X}_e$ plastic $\mathbf{X}_p$ parts, the elastic logarithmic strain in the intermediate configuration is obtained from the stretch tensor $\mathbf{U}_e$ of the right polar decomposition $\mathbf{X}_e = \mathbf{R}_e\mathbf{U}_e$ as

$$\mathbf{E}_e = \ln(\mathbf{U}_e) \quad (2.98)$$

The resulting *incrementally* additive framework using logarithmic strains mimics the infinitesimal theory, in which small elastic strains $\boldsymbol{\varepsilon}_e$ are interpreted as logarithmic

elastic strains $\mathbf{E}_e$ in the intermediate configuration, and Cauchy stresses $\boldsymbol{\sigma}$ as generalized Kirchhoff stresses $\mathbf{T}$ (or rotated Kirchhoff stresses in the elastic-isotropic case at hand). The hyperelastic stored energy is written in terms of these stress-strain measures as

$$\mathbf{T} = J\mathcal{U}'(J)\mathbf{I} + 2\mu\mathbf{E}_e^d \quad (2.99)$$

where $J\mathcal{U}'(J)$ and $\mathbf{E}_e^d$ are respectively the volumetric and distortional parts, the former obtained from the volumetric stored energy $\mathcal{U}'(J)$ as a function of the Jacobian determinant J . Algorithmically, the trial state is given by

$${}^{tr}\mathbf{X}_e = {}^{t+\Delta t}_t\mathbf{X} {}^t_0\mathbf{X}_e \quad (2.100)$$

and ${}^{tr}\mathbf{T}^d = 2\mu {}^{tr}\mathbf{E}_e^d$ with ${}^{tr}\mathbf{E}_e^d$ being the distortional component of the trial elastic stretch tensor. Then, interpreting the backstress as a Kirchhoff-alike backstresses $\mathbf{B}$, the small strains algorithm may be used without modification. Upon convergence, the final geometric update is performed from the converged elastic strains ${}^{t+\Delta t}_0\mathbf{E}_e \leftrightarrow {}^{t+\Delta t}\boldsymbol{\epsilon}_e$, as

$${}^{t+\Delta t}_0\mathbf{X}_e = {}^{tr}\mathbf{R}_e \exp({}^{t+\Delta t}_0\mathbf{E}_e) \quad (2.101)$$

where we used the usual assumption that ${}^{tr}\mathbf{R}_e \approx {}^{t+\Delta t}_0\mathbf{R}_e$ [75], [73], [76], [69]. By systematic use of the chain rule, the tangent of the small strains algorithm may be converted to the typical tangent employed in finite element codes which relates second Piola-Kirchhoff stresses with Green-Lagrange strains in the reference configuration. Since the large strains pre- and post-processors are unchanged from the well-established classical plasticity and the experimental data used below are in the range of small strains, we refer for example to [75], [73] (Table 7.3.1), [76], [70], [69] for further details on the large strain numerical implementation using geometrical mappings.

2.5 Numerical examples

In this section several numerical examples are included to demonstrate the predictive capabilities of the model, to show the influence of the actual size of the yield surface (sometimes difficult to determine accurately) and to demonstrate the performance of the fully implicit computational algorithm in actual finite element simulations.

2.5.1 Uniaxial examples

The purpose of this example is to show the capabilities to model cyclic hardening/softening during uniaxial cycles. The material for the simulation is steel BLY160. The material data from [67] is used to predict the uniaxial cyclic experiments performed in Ref. [67]. In [67], the isotropic hardening is described with the relation between the normalized isotropic hardening parameter R/R_s and the accumulated plastic strain p , where R_s is the stabilized value. This function R/R_s is obtained in [67] by curve fitting from cyclic experiments shown in Fig. 18a of [67]. Since the saturated value R_s is also needed for our simulations, we take that data from Fig. 4 of [67], which contains experimental data of saturated stress values. Hence, from both figures in [67] we can obtain the needed relation $R(p)$. However, it is noted that the shape of the skeleton curve is a saturated one with strain range 3% so the stress values have been adjusted by a ratio that is obtained by comparing the estimated yield stress of the saturated curve with the estimated yield stress of the first half loop curve. The resultant curve is the basic backbone which is used to obtain the initial kinematic hardening moduli field.

With this data from [67], simulations are performed to predict the experimental observations shown in Fig. 3 of the same paper, which we reproduce in Figure 2.5a for the reader's convenience. These experiments on three different specimens show the cyclic hardening behavior of the material at different deformation levels. In Figure 2.5b it is shown the predictions from our model. It can be seen that the predictions correlate quite well with the experimental observations.

2.5.2 Examples under multiaxial, nonproportional loading

Self-consistency of the formulation and recovery of classical bi-linear plasticity

In order to show that classical von Mises plasticity with linear kinematic hardening is recovered in multiaxial cases regardless of the number of surfaces employed, multiaxial simulations are performed using as uniaxial stress-strain curve the bilinear one shown in Fig. 2.6a. In this figure it is also shown the locations of the different surfaces employed in the discretization and the slight modification to better appreciate the three cases in the simulations. The deformation path shown in Figure 2.6b is prescribed. The result of the simulations using the three cases are shown in Figure 2.6c. It is seen that there is no relevant difference in the results. In all cases, the solution is obtained in just one iteration for if $\tilde{H}_i/\tilde{H}_{i-1}$ is high enough to overcome the required tolerances in the local system of equations, except when a change of

Figure 2.5: (a) Experimental stress-strain cycles for prescribed strains of $\Delta\epsilon = 1\%$, 2% and 3% for steel BLY160; adapted from [67], with permission from Elsevier. (b) Predictions from our model for the same experiments.

active surface is encountered. In such case an additional iteration is needed just to update the position of the surface.

Experiments of Lamba and Sidebottom [71]

Now it will be shown the predictions for the well-known nonproportional experiments of Lamba and Sidebottom [71] on oxygen-free high-conductivity (OFHC) copper. The nonproportional loading sequence ($0 \rightarrow 1 \rightarrow \dots \rightarrow 8$) is shown in Figure 2.7. In this path, the positive and negative strain peaks are equal, and the angles between the different segments take very different values in order to test the response under change of direction, including a variety of angles [71]. To model these experiments an isotropic hardening contribution is not considered. According to the loading sequence of Lamba and Sidebottom, the first segment $0 - 1$ of the path is uniaxial shear loading, so the experimental stress-strain curve of this segment was used for generating the kinematic-hardening parameters of our model, see Eqs. (2.51) and (2.52), and the Poisson ratio to $\nu = 0.3$.

Figure 2.8 shows the comparison between our predictions (a) and the experimental results (b), the latter redrawn from Lamba and Sidebottom [71]. From the comparison of predicted and experimental results we can conclude that the model is capable of capturing most of the details of the loading history. Recall that the only data employed in the model is the proportional stress-strain curve for segment $0 - 1$, which is shown in Figure 2.8 (1.a).

Experiments of Hamala et al [46]

For further demonstration of the ability of the model for the description of multiaxial loading, it is also predicted the behavior of specimens under several typical strain-controlled tension-torsion loading paths from Hamala et al [46]. Then, the results of the simulations are compared with the experimental results in [46]. The uniaxial cyclic curve of the material (2124-T851 aluminum) from Figure 8 of the same paper is used to determine the kinematic hardening parameters of our model. Four multiaxial strain-loading paths (the rhombus, the circular, square and two-block) are simulated with a strain-driven procedure. The maximum amplitude in the prescribed strain cycles shown in the small details in Fig. 2.9 is 1%. The comparison of the results from simulations and experiments as well as the prescribed strain paths are shown in Fig. 2.9. Since isotropic hardening is not considered in this example, the first full cycle corresponds to a stabilized cycle in our model, and it is compared with the stabilized experimental results given in [46].

Figure 2.6: (a) Uniaxial stress-strain curve with different discretizations. Additional data: Poisson ratio $\nu = 0.3$, $\tilde{H}_i = 10^3 \tilde{H}_{i-1}$. (b) Prescribed strain path. (c) Predicted stress paths

Figure 2.7: Prescribed strain path in the tests from Lamba and Sidebottom [71]

Figure 2.8: Left column (a): Simulations of the Lamba-Sidebottom experiments using our model. The prescribed strain path is given in Figure 2.7. Right column (b): Experimental data from Lamba and Sidebottom, redrawn from [71].

As shown in Fig. 2.9, the model predicts well the observed experimental stress-paths, in both the amplitudes and different experimental shapes of the axial-shear stress paths of the experiments.

In Figure 2.9a, and specially in Figure 2.9c corresponding to a rhombus path, it can be seen that the predictions deviate from experimental results at some locations. This accuracy is affected by the actual size of the yield surface being employed in the model, which for this example was probably larger than the actual size of the material during loading. Remarkably, even though the monotonic stress-strain curve remains the same, the size of the yield surface has proven to be relevant in the predictions of the stress and strain multiaxial paths because the plastic flow is normal to the yield surface. Therefore, in the following examples, the influence of the size of the yield surface in the predictions of multiaxial paths will be addressed. Specifically, for this discussion the rhombus path of the experiments of Tanaka is choosed, which according to Figure 2.9, it is probably the most critical path for this discussion.

Experiments of Tanaka [50]

A series of experiments with nonproportional loadings were carried out also by Tanaka et al [47], [50]. Material parameters obtained from the experiments of Tanaka et al with loading paths of certain shapes have been used in different constitutive models for cyclic plasticity to derive a better description of the multiaxial behaviors of materials under those loading paths, see for example [50], [51], [52].

The prescribed loading path is given in Fig. 2.10a and the experimental result of the initial loading loop from [47] is redrawn in Fig 2.10b for comparison purposes. We note that the experimentally prescribed path given in [47], idealized in Fig. 2.10a, is given in terms of plastic strains, from machine-prescribed total strains with an iterative estimation therein done for the elastic part. We have performed our simulations using total strains, so an estimation of the maximum elastic contribution has been performed from stress values and the reported mean Young and shear modulus of the specimens, given in [47] as $E = 203$ MPa and $G = 78$ MPa. From Fig. 2.10b, it can be seen that the experimentally measured stresses in the initial strain-proportional path are not proportional, an effect also observed in other experiments in [47]. This can be attributed to experimental errors or to lack of compliance of the material state to the initial isotropic assumption.

It is shown in Figure 2.10c the stress-strain curve employed in the simulations taken from Fig. 2 of [47], and in Fig. 2.10d the result of the simulations for the first cycle. Whereas there are several commented uncertainties that may affect the accuracy of the simulations, it is remarkable that the model captures well the shape

Figure 2.9: Comparison of the predictions from our model to experimental cyclic data published in [46] for several prescribed multiaxial tension-torsion strain paths. (a) Square path. (b) Two-block path. (c) Rhombic path. (d) Circular path.

Figure 2.10: Multi-axial cyclic experiments from [47]. (a) Prescribed strain path, $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. (b) Measured axial-shear stress path for the first cycle, redrawn from [47]. (c) Uniaxial monotonic stress-strain curve, obtained from Fig. 2 of [47] using a Young modulus of 203GPa and a shear modulus of 78GPa , which are average values given in Eq. (4) of [47]. (d) Predicted stress path for the first cycle. (e) Cyclic hardening curve $R(p) - 1$, obtained fitting approximate values of peak stress for subsequent cycles in Fig. 7 of [47]. (f) Predictions for several cycles including isotropic hardening (compare to Fig. 3d of [47])

of the stress path, including the special corner effects in the path. To include cyclic isotropic hardening in the simulations, from the peak stresses in the cycles in Fig 2 of [47], a function $R(p)$ has been adjusted, shown in Fig 2.10e. This curve, along with curve 2.10c was used for the simulations in Fig. 2.10d, which can be compared to those in Fig.3d of [47]. It is noted that path-dependent (directional) isotropic hardening has not been included. However, as commented, such effect could be included in the model allowing for a dependency $r(\gamma, \xi)$.

It is shown in Figure 2.11d the locations of the three stress-strain points selected as the ones from the experimental data which determine the yield points in the monotonic curve. To make the comparison more clearly, the cyclic hardening effect is neglected here, and only the stress response of the first loop is shown.

These points are in the almost linear regime of the discrete stress-strain curve; note that the largest plastic strain of all selected points is about 0.025%. After that yield point, the material curve and the hardening field are the same for the three simulations, being the size of the yield surface the relevant change. The predictions for the three cases studied a, b, c are shown respectively in Figures 2.11a, 2.11b, 2.11c. It is apparent from the comparison of the three predictions that the shape and amplitude of the simulated responses are influenced significantly by the size of the yield surface, which means that the assumed size of the elastic regime is relevant for an accurate prediction of the shape of the multiaxial paths. Furthermore, it can be seen that the smaller the yield surface, the closer the relation between strain and stress paths, as it should be expected from physical grounds because incremental stresses and strains correlate more. Finally, it can also be seen that the predictions shown in Figure 2.11c are the ones that better reproduce the shape of the experimental observation given in Fig. 2.10.

Ramp-up loading experiments of Madrigal et al [81].

In [81], a 90° tension-torsion out-of-phase experiment with continuously increasing strain amplitude is performed in AISI 4140 quenched and tempered steel, which is a material frequently used in the aerospace and automotive industries in elements that are subject to fatigue [81]. These experiments, along those from Lamba and Sidebottom, were used to test a previously proposed model [81], [82], [83], [84], similar to bounding surface models, but including a metric tensor which allows for the inclusion of different yield criteria, e.g. Tresca's criterion [81]. This metric tensor also allows them to loosen the correlation between proportional axial and shear behaviors through additional material parameters in order to obtain better predictions for multiaxial loading. The 90° out-of-phase loading experiments with continuously in-

Figure 2.11: Predicted axial-shear stress paths for the experiments of Tanaka [47] (Figure 2.10) using our model for different sizes of the yield stress, but using the same skeleton uniaxial stress-strain curve. (a), (b), (c): Results of the simulation when the selected yield surface is, respectively, at points a, b, c in Figure 2.11d. (d): Skeleton stress-strain curve and location of the points determining the size of the yield surface for the simulations given in Figures 2.11a to 2.11c.

creasing amplitude are challenging because they produce a large amount of additional multiaxial hardening and predictions are very sensitive to the uniaxial stress-strain curve and the size of the yield surface [81].

The experimental prescribed strain path of [81] is shown in Figure 2.12a. The experimental measured stress path obtained by Madrigal et al [81] is shown in Figure 2.12b.

In order to perform the simulations, the monotonic uniaxial stress-strain curve has been discretized, given in Figure 5 of [81] and reproduced in Fig. 2.12c. It is noted that this material has a slightly different behavior in axial than in shear, which was the reason because of which Madrigal et al [81] developed a model including tensorial metrics. However, the simulations were performed using a von Mises criterion and taking a the monotonic stress-strain curve for the model from axial loading; e.g. from Fig. 5 of [81]. In Figure 2.12d it is shown the results from the simulations for the ramp-up experiments with increasing amplitude. It is seen that the model reproduces the overall behavior, and quite accurately the axial stress values, having a difference precisely in the shear stress values, specially in the first loop. This difference in shear values may be attributed to the commented slightly different behavior of the material in shear, which was the reason for the constitutive model proposal in [81].

In this example, the effect of the size of the yield surface is shown. In Figure 2.13 it is shown the results from our simulations for the ramp-up experiments with increasing amplitude. Figure 2.13d shows the uniaxial, monotonic stress-strain curve employed along with the stresses determining the sizes of the yield surfaces in the three cases *a*, *b*, *c* studied, which results are shown respectively in Figures 2.13a, 2.13b and 2.13c. Note that the three points are located in the almost linear zone of the curve, so uniaxial stress-strain predictions will be almost the same for all three cases. It is seen that in this case, the size given in case *a* gives a better correlation with the experimental results shown in Figure 2.12b. Hence, again, the size of the yield surface becomes important in the prediction of multiaxial stress-strain paths. The predictions in Figure 2.13a can be compared with the predictions given in Figs 20 and 21 of [81] for the model of [81]. Overall, the cyclic behavior, i.e. the change in amplitude, is better predicted by the present model. However, it is seen that given one loop, the shape of the path is better predicted by the model of [81]. As discussed in their paper, the material behaves slightly different in shear than in the axial direction, so they introduced this modelling possibility through a metric tensor, obtaining excellent results in capturing the experimentally observed effect. Of course, the present model could be also enhanced using metrics in line with the proposal of [81].

Figure 2.12: Experimental and simulated ramp-up 90° out-of-phase strain path with continuously increasing amplitude. (a) Prescribed experimental axial-shear strain path [81]. (b) Experimental axial-shear stress path [81]. (c) Axial monotonic stress-strain curve employed in the simulation, obtained from Fig. 5 of [81]. (d) Predicted stress path

Figure 2.13: Evolution of the axial-shear stress response under the prescribed loading path given in Figure 2.12a with different yield surface sizes. (a), (b), (c): Results of the simulation when the selected yield surface is, respectively, at points a, b, c in Figure 2.13d. (d): Skeleton stress-strain curve and location of the points determining the size of the yield surface for the simulations given in Figures 2.13a to 2.13c.

2.5.3 Finite element examples

One of the main purposes of multiaxial cyclic models is to compute stress redistribution during loading in a component subjected to multiaxial cyclic loading. Then, an efficient finite element implementation of the model, that allows for relatively large steps, still giving accurate predictions, is needed [21]. Hence, the described implicit computational algorithm has been implemented into our in-house finite element code Dulcinea, used in simulations of other constitutive models [85], [86].

The results of two finite element simulations are shown the the following part. The first one is about a simply-supported beam subjected to cyclic bending, which material shows the cyclic softening effect. The second example is about a single-hole plate fixed at one end and subjected to a constant tension and a cyclic bending at the free end, and the material of the plate shows the cyclic hardening effect. For both simulations, the global convergence is fast and stable and expected results are obtained.

Beam under cyclic bending

To check the implementation of the algorithm in the finite element program Dulcinea, a relatively simple structure of half of the simply-supported beam subjected to cyclic bending is simulated. To make sure that the model can work correctly for both cyclic hardening and softening conditions, the material of the beam is assumed to have the cyclic softening property, as is shown in 2.14.

The loading of bending is simulated with displacement distributing progressively along the direction perpendicular to the neutral axial. And the element type used is the 27-node brick element with mixed formulation (27/4). The loading steps set for the whole loading loop is 20, which means the step size is big.

As shown in 2.15, from the plot of the von Mises stress condition of step 5 and step 65 which strain conditions are identical, the softening phenomenon is very obvious. Considering the global convergence condition from table 2.2, we can see that the quadratic convergence is achieved for the norm of the residual nodal force as well as for the energy, even with a relatively big stepping size.

Single-hole plate under cyclic bending and continuous tension

One of the main purposes of multiaxial cyclic models is to compute stress redistribution during loading in a component subjected to multiaxial plastic cyclic loading. Then, an efficient finite element implementation of any proposed model, that allows for relatively large steps, still giving accurate predictions, is needed [21].

Figure 2.14: (a) the finite element model of half of the simply-supported beam under cyclic bending; (b) the material of the beam which shows the cyclic softening effect

Iteration	Residual norm		
	Step 10	Step 30	Step 50
(1)	$3.111E + 02$	$3.105E + 02$	$3.099E + 02$
(2)	$2.456E + 00$	$3.166E + 00$	$3.954E + 00$
(3)	$8.145E - 04$	$1.150E - 02$	$8.119E - 03$
(4)	$3.699E - 07$	$6.205E - 06$	$5.409E - 06$
Iteration	Energy norm		
	Step 10	Step 30	Step 50
(1)	$2.898E + 02$	$2.892E + 02$	$2.885E + 02$
(2)	$1.672E - 02$	$2.712E - 02$	$4.622E - 02$
(3)	$1.454E - 09$	$7.391E - 07$	$2.530E - 07$
(4)	$4.336E - 16$	$2.652E - 13$	$1.145E - 13$

Table 2.2: Global equilibrium convergence values at steps 10, 30, 50 of the finite element simulation

Figure 2.15: The von Mises stress condition of half of the beam at two loading steps with the identical strain condition (a) at the 5th step; (b) at the 65th step

Accordingly, the described implicit computational algorithm has been implemented into our in-house finite element code Dulcinea, already used in simulations with other constitutive models [85], [86].

In order to verify the performance of the algorithm and, at the same time, to show a comparison with finite element simulations using the model of Ohno, we simulate a finite element example taken from the paper of Ohno et al [45], who used data from annealed OFHC copper. Ohno et al used this example to show the performance of the stress integration algorithm presented therein for the well-known model of Ohno. The simulation is performed on the finite element model of half of a single-hole plate fixed at one end and subjected to cyclic bending and a constant tension at the opposite end. The cyclic bending value u_y^{end} , which is varied from 0 mm to 5 mm, is added after the axial tension σ_{xx}^{end} is increased from 0 MPa to 5 MPa. All the displacements are imposed in our model by means of a penalty formulation. The axial stresses are imposed by equivalent uniformly distributed loads at the nodes. Details of the example are given in Figure 2.16.

In the simulations the hardening surfaces are obtained from the stress-strain curve shown in Figure 2 of [45] following the same procedure as in the previous examples. As in the simulations performed by Ohno, in this example cyclic isotropic hardening obtained from Table 4 of [45] is included, so all the presented capabilities are active and the efficiency of the consistent tangents may be verified in the general case. In [45] the authors made the implementation of their model in the commercial code Abaqus via user subroutine. The elements they used are the typical 20-node serendipity elements C3D20R with reduced integration to avoid volumetric locking. However, in our case, in order to include more integration points per element (important in stress concentration cases as the one at hand) and to use a mixed u/p formulation, fully integrated 27/4 tri-quadratic elements with linear pressure interpolation are selected. These are reliable elements which are known to pass the inf-sup condition and give optimal performance [87]. The finite element mesh is shown in Figure 2.17. A finer layer of elements is included to compare the simulated integration-point stress-strain results with the nodal ones in [45]. Since both the specimen and the loading pattern are symmetric, we used symmetry conditions in the finite element model, so only one half of the specimen has been modelled.

In the simulations the same step increment for cyclic bending is used as the one used in [45]. Even though this time step is too large for an accurate description of the behavior of the structure, this step size shows the robustness of the numerical implementation and facilitates the comparison with [45]. However, for the initial tension loading, to better describe the incursion from the elastic into the plastic regime and to show the effect in the convergence rate, 5 steps are used instead of the

Figure 2.16: Plate with a hole, clamped at one end and subjected to an axial constant tension and a cyclic bending displacement at the other end; example taken from [45]. (a) Details of the geometry with dimensions in mm. (b) Prescribed cyclic bending displacement applied to the free end.

single step used by Ohno et al [45]. In Figure 2.17 the von Mises stress distribution in the upper face of the specimen (in MPa) at the end of the loading cycle is shown. Note that the band plot results shown in the figure are unaveraged, nonsmoothed stresses extrapolated at nodes in order to show the quality of the mesh. The stress jumps at both ends of the specimen are due to the approximate approach used in prescribing the boundary conditions, and for our purposes those jumps are irrelevant in the simulation according to the Saint Venant principle.

The stress-strain cyclic results shown in Figure 2.17c are obtained at the integration point closest to the middle point of the half circle in the direction of width and closest to the upper surface in thickness.

As shown in Figure 2.17c, our predictions of the stress-strain hysteresis loops show the feature of cyclic hardening. These predictions are very similar to those given by Ohno et al in Figure 8b of [45] using their model. Apart from being different models (although both use a multilinear approximation obtained from stress-strain curves), the small discrepancies may also be attributed to the ratchetting effect included in [45], the different element type used, the finer mesh used and the actual location of the corresponding integration points, which is specially important in that zone due to the stress concentration promoted by the hole.

The number of global equilibrium iterations employed for each step is shown in Figure 2.18. As usual, the first iteration corresponds to the imposition of the incremental loads of the step, giving the reference non-equilibrium values, whereas the remaining iterations are the actual iterative search for the equilibrium configuration. Since the first two steps are elastic, just two iterations (one for imposing load and one to regain equilibrium) give the solution to machine precision. The rest of the steps involve plastic strains, so more iterations are needed to reach the prescribed relative tolerances in forces of 10^{-7} and in energy of 10^{-14} . However note that usually only 4 iterations (3 equilibrium iterations) are needed to reach such tolerances, which is a performance similar to that of classical von Mises plasticity. Furthermore, in Table 2.3 force and energy convergence values for three typical steps are shown. It can be observed that asymptotic quadratic convergence is obtained.

2.6 Conclusions

In this chapter, a completely implicit algorithm is introduced for general combined kinematic-isotropic hardening plasticity cases.

From both the uniaxial and multiaxial simulations, as well as the finite element example, we can see that the model has good performance referring to the simulated responses as well as the convergence.

Figure 2.17: Finite element mesh used for our simulations of the example given in [45] (a); elements are 27/4 u/p mixed elements (tri-quadratic interpolation in displacements and linear in pressure) with full $3 \times 3 \times 3$ integration. Band plots correspond to nodal-unaveraged von-Mises stresses at the top layer (band-plot jumps are an estimate of the local error in the solution) (b). It is also indicated the location of node nearest to the integration point which corresponds to the stress-strain history plot (c)

Residual norm			
Iteration	Step 14	Step 34	Step 54
(1)	$4.366E + 06$	$4.366E + 06$	$4.366E + 06$
(2)	$4.706E + 02$	$4.270E + 02$	$4.839E + 02$
(3)	$1.711E + 01$	$1.539E + 01$	$1.663E + 01$
(4)	$1.973E - 01$	$1.021E - 01$	$2.542E - 01$
Energy norm			
Iteration	Step 14	Step 34	Step 54
(1)	$3.063E + 07$	$3.063E + 07$	$3.063E + 07$
(2)	$6.833E - 01$	$6.133E - 01$	$6.889E - 01$
(3)	$1.713E - 03$	$1.302E - 03$	$1.125E - 03$
(4)	$2.282E - 07$	$8.144E - 08$	$1.213E - 07$

Table 2.3: Global equilibrium convergence values at steps 14, 34, 54 of the finite element simulation

Figure 2.18: Number of global iterations used for each loading step

Despite the fact that the model has a descent performance in the description of cyclic plasticity and its logical prediction on the shape of the detected yield surface [72] as well as the influence of the size of the yield surface on the prediction of multiaxial response, it has the perfect potential for the description of ratchetting behavior by adding a dynamic recovery item. Unlike in the other models, the dynamic recovery item can be used solely for the description of ratchetting, instead of for ratchetting as well as cyclic plastic behaviors, thus the model can be more versatile while maintaining its controllability and simplicity.

Chapter 3

Multilinear anisotropic kinematic hardening model for capturing yield surface evolution

A paper based on this chapter has been published on the International Journal of Solids and Structures.

Many authors have observed experimentally that the macroscopic yield surface changes substantially its shape during plastic flow, specially in metals which suffer significant work hardening. The evolution is frequently characterized by a corner effect in the stress direction of loading, and a flatter shape in the opposite direction. In order to incorporate this effect many constitutive models for yield surface evolution have been proposed in the literature. But is the evolution of observed yield surface really due to the change of the actual yield surface?

This chapter is focus on some numerical predictions for experiments similar to the ones performed in the literature using a multilayer kinematic hardening model which employs the associative Prager's translation rule. The simulations of experiments for the study of the evolution of yield surfaces as well as the change of hardening modulus after preloading will be introduced.

To simulate the evolution of the yield surface, offsets of probing plastic strain are prescribed, so apparent yield surfaces can be determined in a similar way as it is performed in the actual experiments. It will be shown that similar shapes to those reported in experiments are obtained. From the simulations it can be concluded that a relevant part of the apparent yield surface evolution may be related to the anisotropic kinematic hardening field.

3.1 Introduction

Classical phenomenological theories of plasticity for metals are based on the existence of an elastic domain characterized by a yield surface. For polycrystal isotropic metals, the Maxwell-von Mises yield criterion has been verified by a number of authors, starting with the tension-torsion experiments of Taylor and Quinney [88]. This yield surface is a circle in the $(\sigma - \sqrt{3}\tau)$ tension stress σ -torsion stress τ plane and in the deviatoric stress “ π ” plane. However, for at least some hardening materials, upon plastic straining in one direction the measured yield surface not only translates due to kinematic hardening, but also changes its shape. This change of shape has been observed by many authors in different metals, see [89], [90], [92], [91], [93], [94], [95], [96], [97], [99], [100], [98], [101], [102], [103], [104], among others. As observed in these experiments, the actual shape of the measured yield surface depends on several factors as the material itself, the amount of prestress, and the permanent plastic strain (probing strain) after which the onset of plasticity (i.e. the limit of the elastic domain) is established. The relevance of this change of shape is unquestionable because it largely affects nonproportional loading and the springback behavior.

Many experiments show similar conclusions on the evolution of the measured shape of the yield surface. Upon prestressing in one direction in the $\sigma - \sqrt{3}\tau$ (axial-torsion) plane, the yield surface shows a “nose” in that direction and an almost flat line in the opposite direction [93], [94], [95], [96], [97], [99], [102], [103], [104], resulting in an often named “egg” effect [106]. This nose (and the opposite flat part) changes according to new substantial prestressing. Some experiments have also observed that in the direction perpendicular to the prestressing one (in the axial-torsion plane) the measured elastic domain becomes wider than in the direction of pre-loading [92], [91], [93], [94], [95], [96], [97], [99], [102], [103], [104]. Furthermore, although it is rarely accentuated (usually neglected) some experiments also show symmetric slightly concave parts in the surface behind the nose, an effect clearly seen in the experimental data of Wu and Yeh [99] and also present in some of the tests of Kahn et al [93], [94], [95].

Because of the major importance of all these effects, many material models have been proposed or extended in order to account for the shape evolution of the yield surface. Some of the models are phenomenological [107], [108], [109], [110], [111], [112], [113], [114], [115], [116] and some of them micromechanically-based or motivated [117], [118], [119], [97]. These models are substantially more complex than traditional models [106] and despite of their complexity, they are usually not able to capture some details. Probably a recent crystal plasticity model is the first one to capture small concavities sometimes present in experiments [97].

The purpose of this work is to perform some predictions of experiments to detect *apparent* yield surfaces employing a special multilinear (or multilayer) nested yield surfaces model. The model is based on the original ideas of Iwan [120] and Mroz [4], [121] of employing several nested yield surfaces that discretize the uniaxial stress-strain curve. The procedure does not require any parameter-fitting procedure; the prescribed stress-strain data are exactly captured in the uniaxial case in a similar way as in our hyperelastic [122], [123] and damage models [124]. From a theoretical standpoint, there is a clear and remarkable difference of our model with the Mroz proposal. In our case the outer surfaces are *not* yield surfaces, but only hardening surfaces; i.e. they are simply a tool to compute the effective anisotropic hardening modulus. The actual yield surface is always the innermost one. The plastic strains are always normal to that yield surface and the hardening direction of the yield surface follows Prager’s associative hardening rule. From a computational standpoint, whereas for the Mroz model there are some relevant restrictions when formulating a fully implicit closest point projection algorithm [121], [125], [126], in the case of Prager’s rule a closest point projection algorithm is possible without restriction, and this algorithm reduces to the solution of a non-linear scalar function [10], [127]. Furthermore, it is remarkable that in the case of linear kinematic hardening, the model exactly reduces to classical J_2 -plasticity regardless of the number of surfaces employed [126] not only from a theoretical but also from a computational point of view (i.e. the global iterations are the same up to round-off errors).

Therefore, during the predictions given below, it is noted that the actual (analytical) yield surface is always the same, i.e. the innermost von Mises surface. Both the plastic flow and the hardening (i.e. translation of the yield surface) follows associative rules. The stress-driven simulations have been performed using a fully implicit algorithm with very small steps and a restrictive tolerance. However, it is shown that employing the typical probing plastic strains and directions we observe *apparent* (measured) yield surfaces with similar characteristics as those observed in experiments; i.e. a “nose” in the preloading direction, a more flat surface in the opposite direction, a relatively wider “elastic” domain in a direction perpendicular to the preloading direction and even small concave zones behind the nose.

It is obvious that yield surface evolution may be due to many different aspects as isotropic hardening, texture evolution [128], [129], ratchetting [130], etc. Viscous effects and yield stress relaxation may also have an important impact on the observed yield surfaces. However, these effects won’t be included, but only anisotropic kinematic hardening. The purpose of this work is to show that a relevant part of the observations in the experiments may be attributed to (and hence modelled by) anisotropic kinematic hardening developed during preloading.

3.2 Summary of the model

The main objective of the model is to account for multiaxial non-linear anisotropic kinematic hardening during nonproportional loading. In order to meet this goal, several nested (initially concentric) surfaces are employed. The innermost one is the yield surface, the boundary of the elastic domain, taken as the von Mises one

$$f_1 := \|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}_1\| - r_1 \leq 0 \quad (3.1)$$

where $\boldsymbol{\sigma}^d$ is the deviatoric stress tensor, $\boldsymbol{\alpha}_1$ is the backstress tensor, $\|\cdot\|$ is the Euclidean norm and $r_1 = \sqrt{2/3}\sigma_Y$ is the radius of the yield surface for the corresponding yield stress σ_Y . We apply the principle of maximum dissipation and assume associativity of both the plastic flow and of the hardening, i.e.

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\partial f_1}{\partial \boldsymbol{\sigma}} = \dot{\gamma} \hat{\mathbf{n}} \quad \text{and} \quad \dot{\boldsymbol{\alpha}}_1 = -\dot{\lambda} \frac{\partial f_1}{\partial \boldsymbol{\alpha}_1} = \dot{\lambda} \hat{\mathbf{n}} \quad \text{with} \quad \hat{\mathbf{n}} := \frac{\boldsymbol{\sigma}^d - \boldsymbol{\alpha}_1}{\|\boldsymbol{\sigma}^d - \boldsymbol{\alpha}_1\|} \quad (3.2)$$

where $\dot{\boldsymbol{\varepsilon}}^p$ is the plastic strain rate and $\dot{\boldsymbol{\alpha}}_1$ is the rate of the backstress, which translates according to the associative Prager's rule. The multipliers $\dot{\gamma}$ and

$$\dot{\lambda} \equiv \langle \dot{\boldsymbol{\alpha}}_1 : \hat{\mathbf{n}} \rangle = \|\dot{\boldsymbol{\alpha}}_1\| \quad (3.3)$$

are computed from the hardening pattern and the consistency condition. Let $\bar{H}$ be the effective hardening modulus. Then we have the usual relation

$$\dot{\lambda} = \frac{2}{3} \bar{H} \dot{\gamma} \quad \text{so} \quad \dot{\gamma} = \frac{\dot{\lambda}}{\frac{2}{3} \bar{H}} = \frac{\|\dot{\boldsymbol{\alpha}}_1\|}{\frac{2}{3} \bar{H}} \quad (3.4)$$

and Prager's rule results in

$$\dot{\boldsymbol{\alpha}}_1 = \frac{2}{3} \bar{H} \dot{\gamma} \hat{\mathbf{n}} \quad (3.5)$$

From the constitutive equation for the deviatoric stress rate, using Eq. (3.2)

$$\dot{\boldsymbol{\sigma}}^d = 2\mu (\dot{\boldsymbol{\varepsilon}}^d - \dot{\boldsymbol{\varepsilon}}^p) = 2\mu \dot{\boldsymbol{\varepsilon}}^d - 2\mu \dot{\gamma} \hat{\mathbf{n}} \quad (3.6)$$

where $\dot{\boldsymbol{\varepsilon}}^d$ are the deviatoric strain rates and μ is the shear modulus. The consistency conditions are

$$\begin{cases} f_1 = 0, \dot{f}_1 = 0 & \text{if } \dot{\gamma} > 0 \\ f_1 \leq 0 & \text{if } \dot{\gamma} = 0 \end{cases} \quad (3.7)$$

Using $\partial f_1 / \partial \boldsymbol{\sigma} = -\partial f_1 / \partial \boldsymbol{\alpha}_1 = \hat{\mathbf{n}}$, we readily obtain the consistency parameter

$$\dot{f}_1 = 0 \Rightarrow \hat{\mathbf{n}} : (\dot{\boldsymbol{\sigma}}^d - \dot{\boldsymbol{\alpha}}_1) = 0 \Rightarrow \dot{\gamma} = \frac{2\mu \langle \hat{\mathbf{n}} : \dot{\boldsymbol{\varepsilon}} \rangle}{2\mu + \frac{2}{3} \bar{H}} \quad (3.8)$$

The elastoplastic tangent moduli $\mathbb{C}^{ep}$ relate stress rates $\dot{\boldsymbol{\sigma}}$ with total strain rates $\dot{\boldsymbol{\varepsilon}}$ by $\dot{\boldsymbol{\sigma}} = \mathbb{C}^{ep} : \dot{\boldsymbol{\varepsilon}}$. These moduli are obtained from the same classical expression employing Eq. (3.8) in the constitutive equation of the deviatoric stress rates $\dot{\boldsymbol{\sigma}}^d$

$$\dot{\boldsymbol{\sigma}}^d = 2\mu\dot{\boldsymbol{\varepsilon}}^d - 2\mu\dot{\gamma}\hat{\mathbf{n}} = 2\mu\dot{\boldsymbol{\varepsilon}}^d - \frac{(2\mu)^2 \langle \hat{\mathbf{n}} : \dot{\boldsymbol{\varepsilon}} \rangle}{2\mu + \frac{2}{3}\bar{H}} \hat{\mathbf{n}} \quad (3.9)$$

so —c.f. Eq. (2.3.9) of Ref. [131]

$$\mathbb{C}^{ep} = K\mathbf{I} \otimes \mathbf{I} + 2\mu \left(\mathbb{I}^s - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} \right) - \frac{(2\mu)^2}{2\mu + \frac{2}{3}\bar{H}} \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} \quad (3.10)$$

where $\mathbb{I}^s$ is the fourth order fully symmetric identity tensor, $\mathbf{I}$ is the second order one and K is the bulk modulus.

From the previous equations it is apparent that the present model is the widely known classical kinematically hardened J_2 –plasticity [132]. In fact, the only difference with the usually employed model is how we compute the effective hardening $\bar{H}$, a procedure that is going to be explained now. In the case $\bar{H}$ is constant, there is not a single theoretical or computational difference with the classical formulation. However, we want $\bar{H}$ to change according to the load level and load path, preserving Masing’s rules and also describing the hardening field for the case of nonproportional loading. We further want $\bar{H}$ to be explicitly given and determined by a uniaxial test. To this end, the idea of Mroz of using several concentric surfaces is employed to describe the hardening field. However, in our case these surfaces are just a tool to compute the effective hardening $\bar{H}$, they are not successive yield surfaces and they do not change the hardening translation rule of the yield surface; the translation rule is still Prager’s rule.

Then, in order to compute the effective $\bar{H}$ which accounts for kinematic hardening, several outer surfaces are employed which can be written as:

$$f_i := \|\bar{\boldsymbol{\sigma}}^d - \boldsymbol{\alpha}_i\| - r_i \quad \text{with } i > 1 \quad (3.11)$$

where $\bar{\boldsymbol{\sigma}}^d$ is the stress at the contact point with the inner surface $i - 1$ and the actual stresses for $i = 1$. An alternative equivalent expression (which differs from Mroz’s setting but is arguably better for developing the formulation and which could allow for the interpretation of the surfaces as yield surfaces for internal variables) is

$$\hat{f}_i := \|\boldsymbol{\alpha}_{i-1} - \boldsymbol{\alpha}_i\| - (r_i - r_{i-1}) \leq 0 \quad \text{with } i > 1 \quad (3.12)$$

As mentioned, these outer surfaces are merely a tool to compute the effective non-linear kinematic hardening preserving Masing’s cyclic behavior and allowing for consistent nonproportional loading [126]. The translation of the hardening surfaces

Figure 3.1: Geometric relations of the model in the deviatoric plane. Left: stress tensor σ^d , flow direction $\hat{n}$, hardening surfaces f_i , contact points and translation directions $\hat{m}_i$. Right: equivalent surfaces $\hat{f}_i$.

follow their specific rule which may be derived from the condition $df_i/dt = 0$ (i.e. they do not overlap) when they are active. Taking the derivative of Eq. (3.12)

$$\frac{\partial \hat{f}_i}{\partial \alpha_i} = -\hat{\mathbf{m}}_i \quad \text{with} \quad \hat{\mathbf{m}}_i = \frac{\alpha_{i-1} - \alpha_i}{\|\alpha_{i-1} - \alpha_i\|} \quad \text{and} \quad \hat{\mathbf{m}}_1 = \hat{\mathbf{n}} \quad (3.13)$$

so for any active surface we obtain the following geometric expression from the non-overlapping condition

$$\frac{d\hat{f}_i}{dt} = \hat{\mathbf{m}}_i : (\dot{\alpha}_{i-1} - \dot{\alpha}_i) = 0 \Leftrightarrow \|\dot{\alpha}_i\| = \|\dot{\alpha}_{i-1}\| \langle \hat{\mathbf{m}}_i : \hat{\mathbf{m}}_{i-1} \rangle \quad (3.14)$$

where $\langle \cdot \rangle$ is the Macaulay bracket function. The Lagrange multiplier $\dot{\gamma}$ is computed from the hardening multipliers $\dot{\lambda}_i$ of the active surfaces $i = 1, \dots, a$. These values are computed from the projection of the translation of the surfaces on the hardening direction $\hat{\mathbf{n}}$ given by Prager's rule

$$\dot{\lambda}_i = \langle \dot{\alpha}_i : \hat{\mathbf{n}} \rangle \quad (3.15)$$

For each $\dot{\lambda}_i$, the contribution $\dot{\gamma}_i$ to $\dot{\gamma}$ is then —see Eq. (3.4)

$$\dot{\gamma}_i = \frac{\langle \dot{\alpha}_i : \hat{\mathbf{n}} \rangle}{\frac{2}{3}\bar{H}_i} = \frac{\|\dot{\alpha}_i\|}{\frac{2}{3}\bar{H}_i} \langle \hat{\mathbf{m}}_i : \hat{\mathbf{n}} \rangle = \frac{\langle \hat{\mathbf{m}}_i : \hat{\mathbf{n}} \rangle}{\frac{2}{3}\bar{H}_i} \|\dot{\alpha}_1\| \prod_{j=2}^i \langle \hat{\mathbf{m}}_j : \hat{\mathbf{m}}_{j-1} \rangle \quad (3.16)$$

where $\bar{H}_i$ is the hardening modulus associated to surface i . In the previous expression we have used repeatedly Eq. (3.14) and defined for notational comfort $\hat{\mathbf{m}}_1 \equiv \hat{\mathbf{n}}$. Then

$$\dot{\gamma} = \sum_{i=1}^a \dot{\gamma}_i = \|\dot{\alpha}_1\| \sum_{i=1}^a \frac{\langle \hat{\mathbf{m}}_i : \hat{\mathbf{n}} \rangle}{\frac{2}{3}\bar{H}_i} \prod_{j=2}^i \langle \hat{\mathbf{m}}_j : \hat{\mathbf{m}}_{j-1} \rangle \quad (3.17)$$

so by comparison with Eq. (3.4) we arrive at the expression of the effective hardening moduli to be employed in Eqs. (3.8) and (3.10), i.e. in the formulation of classical J_2 -plasticity model

$$\frac{1}{\bar{H}} = \sum_{i=1}^a \frac{\langle \hat{\mathbf{m}}_i : \hat{\mathbf{n}} \rangle}{\bar{H}_i} \prod_{j=2}^i \langle \hat{\mathbf{m}}_j : \hat{\mathbf{m}}_{j-1} \rangle \quad (3.18)$$

where a is the outermost active surface index, that for which $\dot{\gamma}_i > 0$ and $\hat{f}_i = 0$ for $i \leq a$, but either $\hat{f}_{a+1} < 0$ or $a = N$, the total number of surfaces.

The material parameters are identified from the monotonic uniaxial curve. Obviously, under proportional loading we have $\hat{\mathbf{m}}_i = \hat{\mathbf{n}}$ for all i , so Eq. (3.18) simplifies in this case to

$$\frac{1}{\bar{H}} = \sum_{i=1}^a \frac{1}{\bar{H}_i} \quad (3.19)$$

In a uniaxial test, the tangent modulus E_a^{ep} is obtained as always by

$$\frac{1}{E_a^{ep}} = \frac{1}{E} + \frac{1}{\bar{H}} = \frac{1}{E} + \sum_{i=1}^a \frac{1}{\bar{H}_i} = \frac{1}{E_{a-1}^{ep}} + \frac{1}{\bar{H}_a} \quad (3.20)$$

Given a uniaxial stress-strain curve $\sigma(\varepsilon)$, some points $\{\hat{\sigma}_i, \hat{\varepsilon}_i\}$ may be taken as the discretization of such curve, where $\hat{\sigma}_0 = \hat{\varepsilon}_0 = 0$ and $\hat{\sigma}_1 = \sigma_Y$. Then the parameters $\bar{H}_i$ are uniquely and explicitly computed in a recursive form

$$E_i^{ep} = \frac{\hat{\sigma}_{i+1} - \hat{\sigma}_i}{\hat{\varepsilon}_{i+1} - \hat{\varepsilon}_i} \Rightarrow \bar{H}_i = \frac{E_i^{ep} E_{i-1}^{ep}}{E_{i-1}^{ep} - E_i^{ep}} \quad \text{with } E_i^{ep} < E_{i-1}^{ep} \quad (3.21)$$

where $E_0^{ep} = E$ and E_N^{ep} (where N is the total number of surfaces) is given directly by the user as the remanent hardening for $\varepsilon \rightarrow \infty$.

Note that if $\bar{H}_i \rightarrow \infty$ (or $E_i^{ep} = E_{i-1}^{ep}$) then $\gamma_i \rightarrow 0$ and surface i has no influence on $\bar{H}$ (it is like it does not exist), so the predictions employing different arbitrary discretizations are consistent [126] and the case of classical kinematically hardened J_2 -plasticity is recovered for bilinear stress-strain curves as a particular case. Because of the simple structure of the model, it is possible to develop a fully implicit closest point projection algorithm in which the local algorithm reduces to solving a non-linear scalar equation as in the case of classical J_2 plasticity with mixed hardening. This equation is

$$R(\Delta\gamma) := \Delta\gamma - \sum_{i=1}^a \Delta\gamma_i (\Delta\gamma) \rightarrow 0 \quad (3.22)$$

where $\Delta(\cdot)$ is the increment during the finite step. Note also that once $\dot{\gamma}$ is determined, the update of the surfaces is readily given by Eq. (3.5) and by Eq. (3.14) along with Eq.(3.13). Of course when developing a fully implicit, radial return computational algorithm the derivatives of $\bar{H}$ must also be taken into account. This is the major algorithmic difference with the usual algorithm of J_2 plasticity.

Further details of the model can be found in the previous chapter, including its effectiveness in prediction of nonproportional experiments and its consistency of the multiaxial behavior. A plane stress projected algorithm can be found in [127]. A bounding surface model following Prager's translation rule with simulations in

nonproportional multiaxial loading of soils during an earthquake can also be found in [133].

The purpose of the next sections is to show that the model is capable of predicting some aspects of the yield surface evolution observed in many experiments if those experiments are simulated numerically employing the model.

3.3 The experiments of Wu and Yeh

As mentioned in the introduction many experiments have measured the yield surface evolution in different materials, some at small strains and some at large strains. We have considered only small strains so there is no relevant difference among strain measures and questionable shear effects [134]. From those we have selected the experiments of Wu and Yeh [99] performed on 304 stainless steel, one of the most versatile and widely used stainless steels.

Wu and Yeh [99] performed tests with pre-straining in different directions, but the reported measured yield surface evolution is similar for all cases. Figure 3.2 shows some of their experimental results, redrawn from Figure 6 of their paper. Measured surfaces have a nose in the pre-loading direction and a more flat part in the opposite direction. The width of the yield surface seems to be always larger in the direction perpendicular to the preloading direction than in that direction. Many of the surfaces seem to have a small, slightly concave zone behind the nose if experimental dots are to be trusted, a zone that Wu and Yeh (as most authors) have neglected when tracing a continuous yield function probably because yield surfaces should be convex due to Drucker stability [132]. However this experimental observation is repeated in most of their figures, and also found in some of the experiments in [93], [94], [95].

The experimental observations of Wu and Yeh have been obtained using probing paths perpendicular to the pre-loading direction, as shown in Figure 3.2. The probing paths are also relevant because the reported experimental observations change slightly for different probing paths [93], [94], [95], [96], [97]. Usually when the probing path is radial, the flat part of the surface seems to become slightly curved [96], [97].

3.4 Numerical predictions

Wu and Yeh do not report the actual stress-strain curve of their experiments, although some points can be obtained from the published results. Hence the uniaxial data from Ishikawa and Sasaki [91] have been adapted so the stress-strain data have

Figure 3.2: Subsequent yield surfaces of 304 stainless steel after pre-straining. Experimental data redrawn from Wu and Yeh [99].

similar values as those that can be inferred from the Wu and Yeh experiments. Since our target is not to study this specific material but to simulate the evolution of the yield surface in general and to relate it to non-linear kinematic hardening, the accuracy of the material data is not very important because we are mostly interested in the overall effects. The stress-strain curve employed in the experiments, obtained via a digitalization software, is shown in Figure 3.3. In Figure 3.3 we also show the specific hardening surfaces employed in the simulations which corresponds to the different marks in the curve. The size of the actual yield surface has been estimated from the minimum yield stresses measured by Wu and Yeh for the minimum probing strain of $5 \mu\epsilon$ and from the different reported yield surfaces. However, Wu and Yeh note that the actual plastic strain incurred before a yield point can be confirmed is approximately $10 \mu\epsilon$. Then, we note that the experimental determination of the actual yield surface (for zero probing strain) is impossible and can only be estimated.

In a numerical model, we of course know the yield surface and then we need no probing plastic strain. In fact in our model it is always the innermost surface. However, the point is that in experiments the yield surface is detected after some plastic strain has already occurred, and that the *apparent* yield surface does not coincide necessarily with the analytical one. In this case the definition of the equivalent

Figure 3.3: Uniaxial equivalent stress-strain curve of 304 stainless steel and associated hardening surfaces used in the numerical simulations.

plastic strain used is

$$\bar{\varepsilon}^p = \int \sqrt{\frac{2}{3}} \|\dot{\boldsymbol{\varepsilon}}^p\| dt \rightarrow \sqrt{(\varepsilon^p)^2 + (\gamma^p)^2 / 3} \text{ under proportional loading} \quad (3.23)$$

where $\dot{\boldsymbol{\varepsilon}}^p$ is the plastic deformation rate tensor, ε^p is the axial plastic strain and γ^p is the engineering plastic torsional strain. For the reverse yield point at the direction of pre-loading, a smaller offset plastic strain value is selected in order to guarantee that the detection of other yield points does not exceed the area of the actual yield surface at the direction of pre-loading. For 304 stainless steel, $10\mu\varepsilon$ is closed to be the analytical value for $\bar{\varepsilon}^p$ (experimental measurements are usually between $5\mu\varepsilon$ and $10\mu\varepsilon$) in addition to $3.5\mu\varepsilon$, which is the value for $\bar{\varepsilon}^p$ used to find the reverse yield point [99].

The results of the simulations are shown in Figure 3.4. From the figure, we can observe that the main characteristics of the evolution of the apparent yield surface are captured. For instance, following the same probing path than Wu and Yeh, a nose is obtained in the direction of preloading and, in the opposite direction, a flat zone is also predicted. Interestingly, the model predicts symmetric small concave zones behind the nose connecting the nose and the flatter part. This concave zone may vary depending on the pre-loading amount, the size of the actual yield surface, and on the non-linear hardening. Note also that the predicted yield surface is also wider in the perpendicular direction respect to the preloading direction. The results employing different preloading paths are similar, and within the same level of agreement to the experiments of Wu and Yeh as it can be easily inferred. It is also noticeable a sharper point in the loading direction than that predicted by the model, specially in the middle figure. This disagreement may be possibly attributable to several factors, as a smaller 0–offset yield surface or to viscous effects and yield stress relaxation at the prestress point. The latter effects are often experimentally found in this material and they are not included in the numerical simulations.

Figure 3.5a shows the actual position of the hardening surfaces for one of the cases. From this figure the reason for the predictions obtained can be easily deduced. First it must emphasized again that the actual yield surface is the innermost one. The effective hardening modulus in the direction of the preloading is much lower than the hardening modulus in the opposite or normal directions (see below). Hence, in the preloading direction the probing plastic strain is obtained with a very small increment in the stress, i.e. the offset from the yield surface is very small. However, in the direction perpendicular to the preloading one, a larger stress offset value from the yield surface is needed for the same plastic strain.

The size and shape of these detected yield surfaces are largely decided by the

Figure 3.4: Subsequent yield surfaces obtained by the model under the same pre-loading and probing paths as the experiments of Wu and Yeh [99]

definition of yield, the amount of prestressing and partially by the probing direction. When the offset equivalent plastic strain $\bar{\varepsilon}^p$ is larger, the yield surfaces will be extended as one could obviously expect. This is also observed in the experiments of Hu et al [96]. With this model, if the $\bar{\varepsilon}^p$ value is increased not only the size of the yield surface along the probing direction will be larger, the shape will also be sharper, as shown in Fig. 3.5b. If the $\bar{\varepsilon}^p$ value is very small, the detected yield surface is closer to the innermost hardening surface. Furthermore, the actual procedure to define the probing plastic strain may also have a relevant influence in the form and shape of the measured yield surface.

The shape and size of the yield surface also depend on the probing path, and even on the probing sequence and number of probings performed. If the probing path is starting from the center of the yield surface and going at radial directions, and the other conditions are the same, as shown in Fig. 3.5c, the yield surface will be less distorted and slightly more expanded in the pre-loading direction. Even though probing at the radial directions is also used in experiments, and these experiments seem to show a more rounded surface [96], [97], the perpendicular probing path used by Wu and Yeh (among others) seems to be preferred because this perpendicular probing path can guarantee that the yield surface detected doesn't exceed the actual yield surface at the direction of pre-loading, see discussion in [99]. Note that in Figure 3.5c accumulation of plastic strains also swift the original pre-loading point and the reverse one. Furthermore, the influence of an increased probing value for detecting the reverse yield has also an effect on the roundness of that part and on

Figure 3.5: a) Relative position of the hardening surfaces. b) Influence of the offset microstrain used to detect the yield surfaces: $5\mu\epsilon$, $10\mu\epsilon$ and $20\mu\epsilon$ (from inside to outside) and the reverse yield point with $10\mu\epsilon$ (outermost apparent yield surface, in red). c) Influence of the loading path and of the microstrain used to detect the reverse yield point: perpendicular (in blue) and radial (in red). d) Influence of the elimination of one surface.

the size of the yield surface.

One of the questions that may be raised is the consistency of the predictions given by the model when the number and size of surfaces change. Precisely, this is one of the attributes of the model. In Figure 3.5*d* it is shown the differences in results when the fifth innermost surface is eliminated. Because the number of surfaces to discretize the non-linear stress-strain curve was adequate and because surfaces are just a tool to compute the effective multiaxial hardening, the results obtained are very similar. The elimination of one surface simply brings a less accurate equivalent hardening, but does not change the overall anisotropy, the flow or the translation rule of the yield surface. This is a property that has been proven to be critical in the consistency of the predictions and which is not found in other multisurface models [126].

Therefore, since the effects shown by the model are due to anisotropic hardening, it is relevant to test if the hardening curves are well approximated when reloading in different directions. In Figure 3.6*a* it is shown some experimental results from Ishikawa and Sasaki, redrawn from their paper, Ref. [91]. In these experiments, after some proportional cyclic loading, they performed an unloading to a point which they estimated to be the center of the yield surface, see details in [91]. Then they loaded in a different direction in the $(\sigma, \sqrt{3}\tau)$ plane at an angle away from the preloading direction. Different tests were conducted at angles from 0° to 180° in steps of 30° , see Figure 3.6*b*. The same experiments have been numerically conducted and the results are shown in Figure 3.6*b*. The monotonic curve used is a hardened stress-strain curve because cyclic isotropic hardening hasn't been modeled. However, note that it is not clear how cyclic isotropic hardening would evolve in the nonproportional reloading. Furthermore, the center of the theoretical yield surface is used as the initial reloading point. From a comparison of both sides of the figure, one can deduce that, in general, the model yields a good description of the anisotropic hardening developed in these materials after prestressing. In fact this behavior is also behind the previously given arguments. It is noted that the crossing of the numerical curves in the figure is due to the use of the same definition of equivalent strain as that given by Ishikawa and Sasaki [91] in their results in order to make them comparable. There are of course still some discrepancies which may be attributed to the effects not taken into account in the model as viscous effects, cyclic hardening or further multiaxial effects. Some discrepancies may also be due to the accuracy in the determination of the starting point for reloading.

Figure 3.6: Simulation of the experiments of Fig. 6 of Ishikawa and Sasaki [91]. (a) Experimental results redrawn from [91], which consist of stress-strain curves obtained loading at different angles with the preloading direction from the assumed center of the yield surface. (b) Results of the numerical simulations following the same loading paths from the center of the theoretical yield surface.

3.5 Conclusions

Many experiments performed in metals measure a characteristic evolution of the yield surface when kinematic hardening is important. Some of the distinctive aspects of these yield surfaces in some materials are: the presence of a nose or corner effect in the loading direction when a substantial preloading is applied, a usually flatter zone in the opposite direction, wider measured yield surfaces in the direction perpendicular to the preloading one, and in some experiments small concave zones behind the nose.

Many constitutive models have been proposed or enhanced in order to take into account the observed distortion. These models are usually very complex and have different success in capturing some of the experimentally observed details. In this work we show some numerical predictions using a simple multilayer model for non-linear kinematic hardening. To obtain these predictions the usual experimental procedures have been followed. The actual yield surface of the model is always a von Mises circle and the model can be considered traditional J_2 -plasticity with kinematic hardening in which the effective multiaxial kinematic hardening modulus is computed employing several surfaces as a tool.

However, the results presented herein show that with this model similar shapes to those measured in experiments may be obtained if the numerical experiments are performed in a similar way as those experiments. These *apparent* yield surfaces obtained in the numerical simulation can only be attributed to the developed anisotropic hardening. Therefore we can conclude that anisotropic kinematic hardening itself may be one of the major players (among others) in the observed phenomena.

Chapter 4

Conclusion and future work

4.1 Conclusion

Structures in working condition usually undergo elastic deformations, so elastic analysis is the most common analysis in engineering. However, there are a large number of situations where plastic analyses of structures and materials are crucial. For example, when studying the safety of structures under seismic loading, when manufacturing metal products, when studying the safety of structures under impacts, in plastic dissipation devices for shocks or alternate loading, for propagation of cracks or in fatigue analysis, in crash worthiness, etc. In many of these applications, the loading is nonproportional. Under nonproportional loading, the directions change, and can even be reversed several times. Therefore, computer programs should include constitutive laws and algorithms capable of accurately and efficiently compute the behavior of structures under plastic regime; and those models and algorithms should naturally include the possibility of reversed, alternate and nonproportional loading. In all these cases, the predictions of the team model/algorithm/structure should be not only physically sound and accurate, but also computationally efficient, since cyclic loading usually involve thousands of steps with their local and global iterations at each integration point of the finite element mesh.

The purpose of this thesis has been to develop efficient models and algorithms for the finite element analysis of structures under cyclic nonproportional loading.

Hardening plasticity consists of two basic ingredients. The first one is the uniaxial hardening “law” or model. The simplest model is to set a constant work-hardening, which results in bi-linear stress-strain curves. However, this model is exceedingly simplistic to describe the actual behavior of metals after the yield stress has been reached. Therefore, it is usual to use a more complex model to better describe the

stress-strain uniaxial curve. Some well known models are the Hollomon, Ramberg-Osgood and Voce models. These models consist in an analytical expression which follow the tendency of the stress-strain curve observed in many metals. In order to adapt to the experimental observation for a given material, these analytical expressions have some material parameters. These material parameters are computed as to *best-fit* that experimental observation. These analytical expressions have been long used in engineering because they allow to perform derivations of some manufacturing and deformation processes by hand, and furthermore, represent a complex curve using a reduced number of quantities. However, nowadays, most computations in engineering are performed using computers, typically through the finite element method. Then, the convenience of analytical formulae is not so relevant, and the procedure to determine or *fit* those material parameters become more important. The determination of the material parameters in multilinear discretization of stress-strain curves is straightforward and explicit, and the procedure is valid for the observations in most materials, because it does not impose the shape of the stress-strain curve.

In this work we have develop our models with a simplest determination of the material parameters in mind. Hence, the user simply introduces discrete values of the stress-strain curve which result in a multilinear uniaxial stress-strain law.

The Bauschinger effect states that when loading and unloading incurring in plastic strains in one “direction” (e.g. tension) the yield stress in the opposite direction (e.g. compression) reduces its magnitude. A usual observation is that when performing cyclic loading, the stress-strain cycles close approximately at the previous unloading stress, and then follow the monotonic curve again. Furthermore, the unloading-reloading stress-strain curve follow an homological relation of two with the monotonic (“backbone” or “skeleton”) curve. These rules are known as Masing rules. Although this observation is not exactly followed in metals, it is a rather accurate description of the experimental observations, so they constitute the basic behavior over which further refinements may be added.

In order to represent the Bauschinger effect, Prager developed his model of kinematic hardening. Since an isotropic hardening of the yield stress was far from the experimental observations, he stated that the stress in the opposite direction increased the same amount as the stress in the loading direction. This type of hardening is known as kinematic hardening, and consists basically in a translation of the elastic domain. The amount of translation is given by a scalar. The resulting stress-strain curve is bilinear. Bilinear kinematic hardening fulfills the Masing rules. However, whereas isotropic hardening models may be arbitrarily nonlinear, following for example the mentioned Hollomon or Voce models, kinematic hardening as formulated by Prager must be linear. If a nonlinear expression for the hardening is inserted in the

model of Prager, unphysical cycles, which do not follow Masing's rules and are concave in the unloading branch are obtained. Therefore, Prager's kinematic hardening is linear in all finite element programs.

For describing nonlinear hardening, many models have been proposed. The typical ones are multilinear kinematic hardening model based on Mroz's rule and models based on the nonlinear kinematic hardening Armstrong-Frederick's model, which are the widely used Chaboche's model and Ohno-Wang's model. These models don't follow the principle of maximum dissipation, on which the Prager's rule is based. In order to preserve Prager's rule in nonlinear kinematic hardening, in our model, the multilinear kinematic hardening rule is applied and the equivalent hardening modulus is calculated with internal parameters. For describing cyclic hardening/softening, the nonlinear isotropic hardening is implemented. Since the algorithm is fully implicit, kinematic hardening moduli change when cyclic hardening/softening occurs. So the model can be seen as a model based on the kinematic hardening frame, but with an implicitly upgrading material curve. Multiple experimental paths have been simulated, including some typical integrated multiaxial paths as well as simple multiaxial paths in common shapes, the simulations show quite close results compared to the experimental ones. Moreover, the procedure for generating the isotropic hardening parameter is simple and straightforward, which is by curve-fitting the change of the amplitudes of stress to the accumulated plastic strain, and cyclic hardening can be predicted very well while maintaining the efficiency of the algorithm.

For models using kinematic hardening rules, the kinematic hardening parameters are generated from the uniaxial curve, and the prediction of uniaxial behavior is not sensitive to the discretization of the uniaxial curve used, including the assumed size of yield surface. But experimental observations have shown that only the information from the uniaxial curve is not enough to capture the multiaxial behaviors properly, and because of this, many authors propose nonproportional hardening to capture multiaxial paths. But simply by changing the assumed size of the yield surface, both our model and the Ohno-Wang's model show different results to the same prescribed strain path, when the same uniaxial material curve is used for generating parameters. It means that the assumed size of the elastic regime is important in predicting multiaxial behavior, and on the other way, comparing the simulated and experimental multiaxial results can be a way to get the actual size of the yield surface.

A fully implicit algorithm for this combined hardening model is constructed. When the number of activated hardening surface is known or predicted correctly, the algorithm converges quadratically. When the algorithm is implemented in finite element program, the global convergence is also quadratic. This algorithm is consistent despite the number of hardening surfaces employed. And it can be recovered to

classical bilinear plasticity when using a bilinear stress-strain curve.

The current model is based on $J2$ plasticity, so isotropic yield criteria is assumed. Since anisotropic yield criteria is often observed in experiments with the evolution of yield surface when the loading procedure is nonproportional, many authors have proposed to model the evolution of the observed yield surface. When simulating the experiment for capturing the change of yield surface with our model, if experimental procedures are followed, very similar "egg-shaped" yield surfaces as observed in experiments can be predicted. So the anisotropic kinematic hardening can already predict the evolution of the yield surface while the assumed yield criteria maintaining isotropic.

To conclude, In this work, a new way for general cyclic plasticity modeling is proposed. It can recover classical linear hardening case, and maintain consistency regardless of the number of hardening surfaces used. Despite the fact that its procedure for generating material parameter is simple and straightforward, the model is capable of predicting the exact uniaxial curve from which the material parameters are generated, and is robust in predicting multiaxial behaviors of multiple materials compared to the experimental results. In general, this work has questioned the existing notion that Prager's translation rule can only be used in linear hardening cases, and offers a new perspective about the modeling of cyclic plasticity.

4.2 Future work

There are several aspects of the modelling of metals under cyclic loading which have not been considered so far in this thesis. For example: ratchetting, elastic and plastic damage, anisotropy, large strains and a micromechanical motivation.

One obvious improvement for the model is the capacity for predicting ratchetting. For models that don't follow the Masing rule, in cyclic loading, the hysteresis loops are not closed, so when the mean stress is not zero, a permanent increase of plastic strain will be produced, which is called the ratchetting effect. Some multilinear kinematic hardening model, like Mroz's model, doesn't produce ratchetting under uniaxial loading, but produces an constant uncontrollable ratchetting under multiaxial loading. Nonlinear kinematic hardening models produce ratchetting under both uniaxial and multiaxial loading, for example, the Armstrong-Frederek model produces constant ratcheting under both cases. This is because for nonlinear kinematic hardening models, the description of cyclic plasticity behavior is connected to the description of ratchetting through the use of the dynamic recovery term. And improved versions of nonlinear kinematic hardening models, for example the Chaboche's model and Ohno-Wang's model are reducing the influence of the dynamic recovery term by

setting an limitation to it or only activate it under certain conditions. So a promising improvement of the model of this work is to add a dynamic recovery term that could predict controllable rathcetting.

After combining the isotropic hardening rule to the model, it is capable of describing cyclic hardening/softening as well as the theoretical degradation of the Young's modulus. Since cyclic softening is one of the main causes of the damage of structures due to fatigue, for practical engineering purpose, it would be useful to establish a relation between the softening and the prediction of damage.

The current model is with the anisotropic kinematic hardening rule, but the yield criteria is isotropic. In this thesis, it has been shown that even the assumed yield criteria is isotropic, under certain experimental conditions, the observed yield criteria may be anisotropic. But for materials with valid proof of anisotropic yield criteria, it would be necessary to change the yield function for the modelling of anisotropic elasticity. So modeling the possible anisotropy of yield criteria can also be one part of the future work.

Since the current model is actually a special case of classical J2 plasticity, the implementation of large strain formulation is immediate. Especially in some formulations like that of Eterovic and Bathe which uses a geometric pre- and post-processor, thus, the infinitesimal plastic stress integration part can maintain unchanged. Future work can be carried out to validate the performance of the current model implemented in large strain formulation.

To future study the validity of current model, one good way is to find the corresponding micromechanical motivation for its formulation. According to the assumption, each hardening surface can be taken as related to a class of grains with a certain orientation. The consistency parameter rate of each surface follows the compatibility equation, while the relation between the backstress and over stresses can be interpreted as the internal equilibrium equation. So both the two equations of the current formulation can be interpreted in a micromechanical perspective, and future work about this can be meaningful and persuasive.

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