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ADVANCED ORBIT PROPAGATION METHODS APPLIED TO
ASTEROIDS AND SPACE DEBRIS

PH.D. DISSERTATION

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RESUMEN

Los asteroides y la *basura espacial* plantean amenazas relevantes para la civilización, tanto en el espacio como en la tierra. Al mismo tiempo, presentan varios desafíos científicos y ingenieriles en común, que tienen que ser enfrentados en el contexto de la *Space Situational Awareness (SSA)*. Para mejorar las tecnologías de SSA actuales y futuras, se necesitan métodos de propagación de órbita robustos y eficientes. El objetivo principal de esta tesis es demostrar que métodos basados en formulaciones regularizadas de la dinámica aseguran importantes ventajas en los problemas de propagación de asteroides y basura espacial más difíciles.

Las formulaciones regularizadas se obtienen eliminando la singularidad $1/r^2$ en las ecuaciones de movimiento Newtonianas a través de un procedimiento analítico. Las ecuaciones regularizadas resultantes exhiben prestaciones numéricas excelentes. En esta tesis, consideramos la formulación Kustaanheimo-Stiefel y varios métodos de la familia *Dromo*, que representan la trayectoria con un conjunto de elementos orbitales no clásicos.

En la primera parte, nos concentramos en la propagación orbital de encuentros cercanos planetarios, y consideramos distintos casos de prueba. Como escenarios de alta relevancia aplicativa, propagamos encuentros resonantes de varios asteroides ficticios, midiendo el error en las coordenadas en el *b-plane*. Para generalizar los resultados, llevamos a cabo simulaciones a gran escala en el problema de los tres cuerpos restringido circular usando una parametrización bidimensional. Analizamos el caso del asteroide (99942) Apophis, dedicando atención particular a la amplificación del error numérico consecuente a su encuentro cercano profundo en el 2029.

La segunda parte está dedicada a la predicción de largo plazo de órbitas de satélites terrestres. Comparamos formulaciones regularizadas a un método semianalítico en elementos equinocciales para distintos regímenes orbitales y perturbaciones. Los parámetros que afectan la eficiencia de la propagación semianalítica se calibran con un análisis de las diferentes contribuciones al error de integración, que también expone los límites de aplicación de estos métodos. Las formulaciones regularizadas tienen evidentes ventajas para órbitas altamente elípticas y super-síncronas, y tienen prestaciones prometedoras para análisis de tiempo de vida y exploraciones numéricas del espacio cislunar.

En la tercera y última parte, se presentan unas aplicaciones a la prevención de impactos asteroidales. Exponemos los resultados de una deflexión geográfica del asteroide ficticio 2015PDC obtenida con un sistema Ion Beam Shepherd. Finalmente, desarrollamos un estudio sistemático de los potenciales retornos resonantes del asteroide ficticio 2017PDC después de su deflexión con un artefacto nuclear.

ABSTRACT

Asteroids and space debris pose relevant menaces to civilization, both on ground and in space. Simultaneously, they present a number of common engineering and scientific challenges that must be tackled in the realm of Space Situational Awareness (SSA). As to improve current and future SSA technologies, robust and efficient orbit propagation methods are required. The main goal of the present thesis is to demonstrate that regularized formulations of dynamics entail significant advantages in the most demanding orbit propagation problems for asteroids and space debris.

Regularized formulations are obtained by eliminating the $1/r^2$ singularity in Newtonian equations of motion through an analytical procedure. The resulting regularized equations exhibit an excellent numerical performance. In this thesis, we consider the Kustaanheimo-Stiefel formulation and several methods of the *Dromo* family, which represent the trajectory through a set of non-classical orbital elements.

In the first part, we focus on the orbit propagation of planetary close encounters, taking into account several test cases. As scenarios of relevant practical importance, we propagate resonant returns of several fictitious asteroids and measure the error in the b-plane coordinates. To generalize the results, we carry out large-scale simulations in the Circular, Restricted Three-Body Problem by means of a bi-dimensional parametrization. We analyse the case of the asteroid (99942) Apophis, devoting particular attention to the amplification of the numerical error consequent to its deep close encounter in 2029.

The second part is dedicated to the long-term prediction of Earth satellite orbits. We compare regularized formulations to a semi-analytical method based on equinoctial elements for several orbital regimes and perturbations. The parameters affecting the semi-analytical propagation efficiency are fine-tuned by analysing the different contributions to the integration error, which also gives insight on the limits of applicability of semi-analytical methods. Regularized formulations compare very favourably for highly elliptical and super-synchronous orbits, which encourages their application to lifetime analyses and numerical explorations of the cislunar space.

Applications to asteroid impact avoidance are presented in the third part. We show the results of a geographical deflection of the fictitious asteroid 2015PDC obtained with an Ion Beam Shepherd spacecraft. Finally, we perform a systematic study of the potential resonant returns of the fictitious asteroid 2017PDC after its deflection by a nuclear device.

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INTRODUCTION

Every generation experiences the feeling of being at the end of history, and perceive certain shared risks to society as *existential*, that is to say that they threaten the very survival of our species. There are today several menaces that our generation is posed to face. Centuries of technological development, resulting in an unprecedented increase of atmospheric greenhouses gases, have caused a steady rise in global temperatures that will put millions of lives at risk in the next decades. Networks of increasingly sophisticated Artificial Intelligences (AIs) make everyday life easier by enabling services that were unimaginable only a few years ago. But the same exponential pace of development that originated AIs spurs unsettling questions: Will the exponential growth in computing technology slow down into an S-curve, or will it reach a quasi-vertical *singularity*? In the latter event, what will be the consequences?

The human brain, which is only biologically wired to deal with immediate physical threats, is not accustomed to ponder on these sobering prospects. Thus, it often resorts to instinctual reactions that obfuscate rational judgement. These might have been effective in the environment in which the first humans emerged hundreds of thousands of years ago, but they are counter-productive in the highly complex world of today.

Science gives us the tools necessary to rationalize these fears and to concentrate on how to solve the challenges of tomorrow, or even on how to turn them into opportunities. Rather than surrendering to the instinct, a scientist welcomes trialling circumstances by recognizing their potential for creating further knowledge. In fact, there is no better example of this process than exploiting the threats posed by impacts from *potentially hazardous asteroids* (PHAs) and from *space debris* for the further advancement of celestial mechanics and of the space sciences, which is the theme underlying the present thesis.

1.1 NEAR-EARTH ASTEROIDS

Asteroids are small, rocky celestial bodies whose sizes range from several kilometres to few metres. They are minor planets remnant from the initial stages of formation of the Solar System, and are widely believed to be planetesimals that failed to accrete into a larger body. The majority of asteroids is concentrated in the Main Belt, roughly between the orbits of Mars and Jupiter; this region encompasses low-eccentricity orbits with semi-major axes ranging from 2.0 au to 3.5 au, shown in [Figure 1](#). The Main Belt contains an estimated population of a few million asteroids bigger than 1 km [94], of which more than half a million are catalogued in public databases such as AstDys¹.

1.1.1 Origin and classification of NEAs

[Figure 1](#) shows that the asteroid population in the main belt has sharp minima for values of the semi-major axis corresponding to orbital periods that

¹ URL: <http://hamilton.dm.unipi.it/astdys/index.php?pc=0>, last visited May 23rd, 2017.

Figure 1: Histogram showing the asteroid distribution in the main belt as a function of semi-major axis. The mean-motion resonances corresponding to the Kirkwood gaps are shown on the plot. Credit: A. Chamberlin, JPL/Caltech (2007).

are integer multiples of that of Jupiter, i.e. to *mean-motion resonances*. These locations, the *Kirkwood gaps*, are the sources of Near-Earth Asteroids (NEAs), which are defined as having perihelion distances of less than 1.3 au. The dynamical genesis of NEAs can be understood by considering that Main Belt asteroids are subject to non-gravitational perturbations. The most relevant one is the Yarkovsky effect, which is due to the anisotropic emission of thermal photons from the surface of the asteroid that ultimately results in a secular change of the semi-major axis. Main Belt asteroids subject to the Yarkovsky effect migrate along the semi-major axis, and eventually might cross one of the Kirkwood gaps. At this point, powerful mean-motion resonances with Jupiter cause the eccentricity to grow and the perihelion distance to decrease to values under the 1.3 au threshold. The lifetime of NEAs is relatively short due to planetary close encounters causing either a collision with a planet or the Sun, or an ejection from the Solar System [46]. NEAs are classified in different groups according to their values of semi-major axis, perihelion, and apohelion; these are shown in Table 1. Potentially Hazardous Asteroids (PHAs) are NEAs whose orbits bring them particularly close to the Earth, and whose absolute magnitude H indicates that they are large enough to cause regional devastation in case of impact. A NEA has to satisfy two criteria to be in the PHA category. Firstly, its Minimum Orbit Intersection Distance with the Earth, i.e. the minimum distance between any point of its orbit and of the Earth's, has to be less than 0.05 au. Secondly, its absolute magnitude H must be less than 22.

1.1.2 Surveys and impact monitoring activities

Impacts from NEAs and Near-Earth Objects (NEOs, which include comets) have shaped the history of life on Earth. One of the pre-eminent hypotheses on its origin postulates that the delivery of water and complex molecules by cometary impacts facilitated the genesis of the first living beings. Later on

Table 1: Classification of the NEA groups according to their values of semi-major axis a , perihelion q , and apohelion Q . Credit: Center for Near-Earth Object Studies (CNEOS)/JPL.

Group	Definition	Description
Atras	$a < 1.0 \text{ au}$ $Q < 0.983 \text{ au}$	NEAs whose orbits are contained entirely within the orbit of the Earth.
Atens	$a < 1.0 \text{ au}$ $Q > 0.983 \text{ au}$	Earth-crossing NEAs with semi-major axes smaller than the Earth's.
Apollos	$a > 1.0 \text{ au}$ $q < 1.017 \text{ au}$	Earth-crossing NEAs with semi-major axes larger than the Earth's.
Amors	$a > 1.0 \text{ au}$ $1.017 \text{ au} < q < 1.3 \text{ au}$	Earth-approaching NEAs with orbits exterior to Earth's but interior to Mars'.
PHAs	$\text{MOID} \leq 0.05 \text{ au}$ $H \leq 22.0$	Potentially Hazardous Asteroids.

in Earth's history, an impact close to the present-day peninsula of Yucatan by a celestial body some 10 km across caused the Cretaceous-Tertiary extinction event. While wiping out about three quarters of the species of life, it paved the way for the evolution of mammals that eventually spawned *Homo Sapiens Sapiens*. In recent history, the destructive power of asteroids has been reminded by the 1908 Tunguska event and the 2013 Chelyabinsk event. These were caused by impactors several tens of metres wide, and the latter event also resulted in harm to the population due to the shockwave generated by the disintegration of the asteroid in the atmosphere.

Impacts capable of disrupting life on a regional or global scale are exceedingly rare. On average, impacts from asteroids in the 100 m size range (such as Tunguska's) take place every few centuries, while those from km-sized asteroids only happen every few million years [21]. However, the consequent damage to human civilization is potentially devastating, as even an air-burst from a Tunguska-like asteroid over a populated region could result in a high number of fatalities. Therefore, the impact risk is comparable to that of other natural disasters, and requires a cautious assessment.

In this context, NASA launched at the end of the 1990s a comprehensive survey of the NEA population, named *Project Spaceguard*. The stated aim of the project was to find 90% of the population of asteroids of 1 km and larger, which has been recently met. As of now, the goal has been updated to detect the 90% of asteroids larger than 140 m by 2020. These activities have incited the creation of several global astronomical surveys conducted using facilities across the world. As a consequence, the rate of detection of NEAs is accelerating, as shown in Figure 2.

The observational data generated by the surveys is relayed to the Minor Planet Center of the Smithsonian Astronomical Observatory, which takes care of their public dissemination on behalf of the International Astronomi-

Figure 2: NEA discoveries as a function of time from 1980 to present. Credit: CNEOS/JPL.

cal Union. Impact monitoring systems propagate the observational data into the future to scan for potential close encounters and impacts, and assigning probabilities based on observational uncertainties. Two of these systems are currently operational: *SENTRY*, operated by NASA's Jet Propulsion Laboratory (JPL), and *CLOMON2*, operated by the University of Pisa (Italy) under the auspices of the European Space Agency.

1.1.3 Orbit propagation requirements for NEAs

As of May 23rd, 2017, 16133 NEOs are catalogued in the NEODyS database². Each of these objects is propagated for 100 years by the *CLOMON2* system, which automatically detects and assigns probabilities to close encounters and potential impacts. These propagations are repeated periodically, as new observations increase the number of catalogued asteroids and improve the orbits of those already in the catalogue. Moreover, due to the unavoidable observational uncertainties, thousands of Virtual Asteroids (VAs) are generated and propagated for each of the asteroids undergoing a close encounter. Therefore, the computational cost of impact monitoring activities is quite demanding. Next-generation sky surveys, which will further increase the rate of discoveries, imply that this situation will be aggravated in the coming years.

Only resorting to Moore's law, thus relying on increasingly advanced hardware, might be insufficient to cope with these stringent requirements. New orbit propagation methods must have excellent characteristics of speed, due to the large number of initial conditions to be propagated, and of accuracy, as to reliably predict close encounters.

1.2 SPACE DEBRIS

The achievements of the space age have a great impact on many facets of our lives and in some cases have become even indispensable. However, decades of space activity have littered Earth's orbits with debris, i.e. man-made non-

² URL: <http://newton.dm.unipi.it/neodys/index.php?pc=1.0>

functioning objects³. These pose a significant economic and scientific threat to all spacefaring nations, as their presence could lead to hypervelocity collisions with operational satellites.

A dramatic example of the origins of space debris took place in 2009, when Iridium 33, an operational US communications satellite, collided with Cosmos 2251, a retired Russian satellite, at an altitude of approximately 790 km over the Great Siberian Plain. Just two years prior to this accidental event, the deliberate destruction of the Fengyun-1C weather satellite by a Chinese anti-satellite test on 11 January 2007 created the most severe debris cloud in history, which is predicted to remain in orbit until 2090 [78]. These two events underscored the consequences of the space debris risks not only to operational spacecraft, but also to the near-Earth space environment as a whole.

As we continue to increase activities in space the chance for additional collisions increases correspondingly, which may lead to an exponential growth of the entire debris population; this is the nature of the so-called *Kessler syndrome*. The onset of the Kessler syndrome might involve economic losses in the order of billions of euros, and significant societal impacts due to the demise of essential satellites such as those dedicated to Earth observation and navigation. Moreover, the drastic increase in collision risk could make access to space significantly harder for future generations. A swift action is required to confront this taxing problem, as any further collision or breakup event makes space activities more demanding.

1.2.1 Mitigation measures

Many spacefaring nations and organizations outline clear objectives for mitigating orbital debris and sustaining the space environment. As of now, the prevention of in-orbit collisions is the primary measure of debris mitigation.

Preventing collisions, however, can only happen on about 1200 of the 18 000+ space objects greater in size than 10 centimetres, since only operational (active) satellites have manoeuvring capabilities. These considerations do not account for the hundreds of thousands of smaller, untrackable detritus (particles between 1 and 10 cm in diameter) predicted from breakup modelling and simulation [59]. Post-mission disposal, such as the Inter-Agency Space Debris Coordination Committee (IADC) recommended 25-year decay rule for satellites in low-Earth orbit (LEO), will help limit growth [54], but will be insufficient in preventing the self-generating collisional cascading phenomenon from happening. Indeed, a recent study on the evolution of the orbital debris populations in certain preferential LEO regions indicates that the “critical density” has been reached [64], and thus the LEO environment is unstable and population growth is inevitable (see, e.g., Figure 3).

Active debris removal (ADR) is currently seen as the most viable remediation measure to limit, and possibly even stop, the growth of the debris population. Several concepts aimed to the active de-orbiting and repositioning of derelict spacecraft are under study or development. Among these, the e.Deorbit mission by ESA, based on capturing a target by means of either a harpoon or a net, is in development with a planned launch for 2023⁴

³ The IADC defines space or orbital debris as “all man-made objects, including fragments and elements thereof, that are orbiting the Earth or re-entering the Earth’s atmosphere, that are non-functional.”

⁴ URL: http://www.esa.int/Our_Activities/Space_Engineering_Technology/Clean_Space/e.Deorbit, last visited May 22nd, 2017.

Figure 3: Simulated evolution of the population of LEO objects up to 2140. The purple curve represents a “Business As Usual” case in which spacecraft comply to the 25-year de-orbiting rule, no in-orbit explosions take place, but some mission-related debris are left in orbit. The red curve is a “Mitigated” case in which no mission-related debris are left in orbit. The thin blue curves are the number of objects plus or minus one standard deviation. Reproduced from Rossi, A., and Anselmo, L. and Pardini, C. and Jehn, R. and Valsecchi, G. B. [88].

Also, contactless technologies such as the Ion Beam Shepherd are particularly promising due to their reduced sensitivity to the unknown attitude motion of the targets [11].

Even the most promising ADR concepts are still at a relatively low technology readiness level, due to the daunting engineering obstacles posed by these types of missions. Therefore, there has been considerable activity also on developing improved mitigation measures for current or future spacecraft. In LEO, the presence of air drag acts as a natural sink for space debris, and this effect can be enhanced by launching specific drag-enhancing devices leading to shorter lifetimes. In medium Earth orbits and high Earth orbits (MEO and HEO respectively), somewhat more subtle approaches are required. By accurately choosing the launch epoch and the location of a spacecraft in the phase space, it is possible to exploit gravitational perturbations by the Moon and the Sun to achieve post-mission disposal with minimum fuel expense [22]. This concept has also motivated cartographic studies which have improved our understanding of the near-Earth phase space, and have evidenced a plethora of dynamical phenomena [45, 87].

1.2.2 Evolutionary models

As to plan reliable mitigation measures and to understand the criticality posed by space debris in different orbital regimes, it is necessary to use population models predicting the spatial and mass distribution of debris for a long period into the future (usually, 100 to 200 years). Such models, as ESA’s MASTER and NASA’s LEGEND, are based on estimates of the debris population which are propagated on the basis of probabilistic assumptions, given the large number of uncertain parameters involved⁵. A Monte Carlo approach is therefore mandatory, and estimates for the rate of growth of

⁵ For instance, the number of spacecraft collisions and breakups per year, the yearly traffic, the ballistic coefficient, and the atmospheric density, are all parameters which are inherently aleatory.

space debris are usually given as an average over hundreds of simulation runs.

1.2.3 *Future trends*

Perhaps the single most important factor in determining the future evolution of space debris will be the introduction of satellite “mega-constellations” of hundreds or thousands of spacecraft. These would satisfy the growing demand for fast, broadband communications, especially in developing countries and rural areas. Plans for mega-constellations have been put forward by a number of companies such as OneWeb and SpaceX. The impact of the injection of such an impressive number of spacecraft in LEO on collision probabilities is yet not completely understood, but preliminary analyses show that the collision risk might indeed increase without a corresponding improvement of mitigation measures [83].

At the same time, a number of improvements in Space Situational Awareness (SSA) are predicted in the coming years. Notably, Lockheed’s Space Fence will increase the number of catalogued objects by one order of magnitude, due to the decrease in the minimum detectable debris size and a field of view wider than current systems⁶. Optical observations and space-based observatories will also improve the debris detection capability in the critical GEO region.

1.2.4 *Orbit propagation requirements for space debris*

The current scenario for SSA activities in the context of space debris imposes demanding requirements on orbit propagation techniques.

On one hand, an orbit propagator has to be sufficiently fast as to handle propagations of several thousands of objects for time spans in the order of a century. In fact, the need for Monte Carlo approaches and the future introduction of more sensitive surveys which will drastically increase the number of catalogued objects will only increase the importance of this requirement. Long-term orbit propagation of large numbers of objects poses a formidable challenge for current methods, and has been identified as one of the key issues in the last IADC report [60].

On the other hand, orbit propagation software has to be sufficiently reliable as not to incur in numerical instabilities, and it has to work for all types of orbits regardless of zero-eccentricity and zero-inclination singularities. The accumulation of numerical error has to be effectively contained because of the long propagation time spans and of the sensitive and possibly chaotic dynamics stemming from orbits in the MEO and GEO regions. In addition, the predicted outstanding increase in LEO satellites will make improved estimations of lifetime in the presence of atmospheric drag mandatory.

1.3 GOAL OF THE THESIS

The present thesis tackles two important challenges in modern numerical orbit propagation, namely asteroid close encounters with a planet and the long-term evolution of Earth satellite orbits. We aim to advance both the un-

⁶ See *Space Fence*, <http://www.lockheedmartin.com/us/products/space-fence.html>, last visited June 1, 2017.

derstanding of the issues inherent to these problems and the performance of the state-of-the-art numerical methods by employing regularized formulations of the perturbed two-body problem. The work finds significant applications in the prevention of asteroidal impacts by deflection technologies.

The fundamentals of the theory of regularization are expounded in [chapter 2](#), along with an historical perspective and descriptions of the regularized formulations employed in the thesis. These are first applied to the close encounter problem in [chapter 3](#), where they are employed to improve the accuracy in the propagation of hyperbolic trajectories and in the orbit prediction of resonant returns. This analysis is expanded in [chapter 4](#) through large-scale simulations, which aid in redefining the concept of sphere of influence for numerical propagation, and by considering the long-term propagation of asteroid (99942) Apophis with several types of methods. In [chapter 5](#), we describe in detail the method of averaging employed in semi-analytical propagators, and we perform a systematic study of their integration error to optimize their performance. We use a semi-analytical propagator as a benchmark for the propagation of Earth satellite orbits in [chapter 6](#), against which we compare non-averaged, regularized formulations. An innovative approach to mitigate damage due to a possible asteroid impact through an impact point retargeting is shown in [chapter 7](#). The possible resonant returns consequent to an asteroid deflection manoeuvre are characterized in [chapter 8](#). In the last chapter, we summarize the conclusions of the thesis.

This unsophisticated approach throws away all of our knowledge of the two-body problem and its integrals.

Victor R. Bond and Mark C. Allman on Cowell's method, *Modern Astrodynamics* (1996)

In special perturbations methods, the most straightforward way of solving the perturbed two-body problem is by numerically integrating the equations of motion in Cartesian coordinates,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{F}, \quad (1)$$

where $\mathbf{r}$ is the position with respect to the primary body, μ is its gravitational parameter and $\mathbf{F}$ is the perturbing acceleration. This approach, known as *Cowell's method* or *Cowell's formulation* [1, p. 447], is simple and robust, especially in situations where the magnitude of the perturbation $\mathbf{F}$ is comparable to that of the main gravitational acceleration. However, the direct integration of Equation 1 can be disadvantageous from the computational point of view. In fact, its solutions are unstable even in the unperturbed case, i.e. for Keplerian motion. This implies that the propagation of numerical error is quite fast. It also exhibits a singularity for $r = 0$ that poses limits on the step-size when close to the primary body, and makes the integration of collisional orbits impossible.

For the above reasons, sophisticated and adaptive numerical solvers are required to reach satisfactory levels of accuracy, in particular when highly elliptical orbits or planetary close encounters are to be integrated. All of these issues can be ameliorated by *regularizing* the equations of motion, that is by analytically removing the singularity in Equation 1. In this chapter, we provide a brief outline of regularized formulations of the perturbed two-body problem. We will give some historical remarks and describe the analytical developments underlying each of the formulations used in this thesis, with particular attention to their implementation aspects.

2.1 HISTORICAL REMARKS

The special perturbations techniques presented in this work revolve around two key developments in celestial mechanics, namely the *regularization* of the equations of motion and the *Variation Of Parameters* technique (VOP).

2.1.1 Regularization

The origins of regularization can be traced back to Karl Sundman's seminal paper on the solution of the three-body problem [92]. He accomplished the objective of writing a formal power series solution that was valid for all sets of initial conditions with non-zero angular momentum, although the convergence of this series is so slow to hinder any practical computation [29]. A relevant aspect of his solution is the elimination of the singularity due to a two-body collision, which is achieved by using a differential transformation

of the independent variable from the physical to a fictitious time, which is nowadays known as the *Sundman transformation*.

Notwithstanding the formal importance of Sundman's solution, the concept of regularization remained somewhat withdrawn from practical applications until the dawn of the space age. The new numerical problems presented by space mission design and the paucity of computational resources at the time stimulated a renaissance in celestial mechanics. A key figure in reintroducing regularization into the spotlight was the Swiss mathematician Eduard Stiefel. By applying the spinor formalism, as originally introduced by Paul Kustaanheimo, to find an extension of Levi-Civita's transformation for three-dimensional space, he succeeded in the full regularization and linearization of the two-body problem. Although his and Gerhard Scheifele's monograph on the topic [91] was criticized for not giving enough credit to previous contributions on the topic [27, 105], one cannot underestimate its importance in disseminating the idea of regularization to the scientific community. Indeed, work on the *Kustaanheimo-Stiefel (K-S)* formulation continues to the present day, and its ideas have been used fruitfully to improve both special and general perturbations techniques. Following Stiefel and Scheifele [91], a cornucopia of works exploring the properties and numerical applications of regularizations through the Sundman transformation appeared in the literature. We limit ourselves to those which are particularly relevant to the approaches followed in this thesis.

Bettis and Szebehely [8] gave one of the first applications of regularizations in the numerical integration of close encounters. Baumgarte [7] used the Sundman transformation to stabilize Keplerian motion using Cartesian coordinates. Janin [56] compared this method and those presented by Stiefel and Scheifele [91], recommending them in the integration of highly eccentric orbits. Velez [101] analysed in more detail the numerical implications of regularizations, and evidenced its positive impact on the truncation error propagation properties. The impact of Sundman transformations of different orders¹ on the magnitude of the local truncation error was studied by Nacozy [75], who also presented a Sundman transformation with variable order along the orbit.

Many more works involving regularizations are present in the literature, but their full historical review is outside of the scope of this thesis.

2.1.2 Variation Of Parameters

The Variation Of Parameters technique, originally formulated by Leonhard Euler and Joseph-Louis Lagrange, is one of the foundations of the edifice of celestial mechanics. The technique is based on deriving a set of integrals of motion that are rigorously constant in the unperturbed two-body problem. These integrals, or *orbital elements*, are assumed to be time-varying in the perturbed problem, their rates of change being described by a set of first-order Ordinary Differential Equations (ODEs). The advantage of this representation stems from the fact that the orbital elements, unlike Cartesian coordinates, are slowly-varying functions of time if the perturbations are weak. This is a common situation in celestial mechanics, as this hypothesis is satisfied by the majority of Earth satellite orbits and asteroid orbits.

The power of the VOP technique allowed to reach many of the most impressive achievements of celestial mechanics. Thus, astrodynamists and

¹ As it will be shown in the following, the *order* of the Sundman transformation is the exponent of the orbital radius in its expression.

space engineers naturally turned to this fundamental device to build the analytical and numerical tools necessary to cope with the challenges of space mission design. Burdet was perhaps the first to realize that these needs could be satisfied by a VOP technique together with a Sundman transformation of first order. He introduced a universal (i.e., valid for any type of orbit) and regularized set of *focal* orbital elements [16], which was later improved on by Sperling [13, ch. 9]. The importance of this type of formulation in the computation of ephemerides for artificial satellites was soon realized, e.g. in the integration of geostationary orbits by Flury and Janin [40]. Stiefel and Scheifele [91] also obtained a set of non-singular orbital elements derived from the K-S formulation. Moreover, they introduced a *time element*, akin to the time of pericentre passage for the classical orbital elements, which is also valid for the K-S coordinates. Basing himself on Hansen's concept of the *ideal frame*, Deprit [26] proposed a set of non-singular orbital elements. In this formulation, the size and shape of the orbit is described through a set of three parameters, while the remaining four are a set of Euler angles describing the orientation of the orbital plane with respect to the ideal frame.

As noted by [6], use of VOP methods diminished in the last two decades. Perhaps this has been due to a common belief that the advancement of computing hardware, together with sophisticated numerical integration schemes, would render the mathematical complexities of VOP techniques futile. In the following chapters, we will show that this is not the case.

The reader is invited to refer to the Introduction of Baù et al. [6] for a thorough, comprehensive overview of other important methods based on VOP techniques, which we have omitted here for the sake of conciseness.

2.1.3 *Dromo formulations*

In the year 2000, the need for an in-house orbit propagator for electrodynamic tethers in the *Grupo de Dinámica de Tethers* (now the *Space Dynamics Group*) of the Technical University of Madrid spurred the development of a new special perturbations method based on non-singular orbital elements, which was originally called *Dromo* [79]. The method consists of seven spatial elements similar to those presented by Deprit. While in Deprit's ideal elements the independent variable is the physical time, in *Dromo* it is a fictitious time obtained through a Sundman transformation of second order. In unperturbed motion, the fictitious time collapses to the true anomaly measured from the departure point of the ideal frame. The method was then improved by the introduction of a perturbing potential [5] and time elements [4].

Following similar analytical developments but using a first-order Sundman transformation instead, Baù et al. [6] developed a related formulation in non-singular elements which is also endowed with a perturbing potential and time elements. The latter formulation is only valid for negative values of the total energy and shows a superior performance for highly elliptical orbits, in which the perturbations at apoapsis may be particularly relevant. By considering positive values of the orbital energy, Baù et al. [3] also obtained another version of the formulation that works only for hyperbolic orbits. In the rest of the work, we will recall the latter two formulations, for closed and open orbits, by *EDromo* and *HDromo* respectively. Roa and Peláez [85] approached the problem in the framework of Minkowskian geometry,

ultimately arriving to equations that are similar to those obtained by Baù et al.

We will refer to all of the above methods as regularized formulations, since they are well-defined for zero eccentricity and inclination of the osculating orbit. However, it is important to stress that only K-S and Stiefel-Scheifele's elements can deal with collision orbits (i.e., with vanishing angular momentum), and so avoid both topological and physical singularities. The singularity for vanishing angular momentum was characterized in detail in [86] for the *Dromo* formulation, who proposed possible solutions to mitigate this issue.

2.2 KEY ASPECTS OF REGULARIZATION

First of all, we consider a dimensionless system of units in all of the equations. The reference mass, length and time are chosen so that the gravitational parameter of the primary attracting body is unitary.

Regularization is achieved in two steps. The first one is to change the independent variable from the physical time t to a fictitious time s by means of the generalized Sundman transformation

$$\frac{dt}{ds} = f(\mathbf{y}, s)r^\alpha, \quad f > 0, \quad (2)$$

where f , in general, is a function of the state vector $\mathbf{y}$ and is constant if the motion is unperturbed, and α is the *order*, which is a positive constant.

The fictitious time s is an angle-like variable. In unperturbed motion, s reduces to the eccentric anomaly for $f = 1/\sqrt{-2\tilde{e}}, \alpha = 1$ and to the true anomaly for $f = 1/h, \alpha = 2$, where h is the angular momentum. The case $f = 1, \alpha = 3/2$ corresponds to the *intermediate anomaly*, which is related to the true anomaly through an incomplete elliptic integral of the first kind [77].

The second step in regularization is to represent the two-body problem with linear differential equations that do not contain the singularity $r = 0$. This result is usually achieved by embedding some Keplerian integrals in the equations of motion [13, ch. 9] and introducing new state variables, rather than position and velocity.

2.2.1 Linearization and VOP methods

Linearity is a desirable feature for improving the numerical performance and can be obtained also without eliminating the singularity for $r = 0$. In Burdet's focal method [16] the Keplerian motion is decomposed into the radial displacement with respect to the primary and the free rotation of the radial direction in space. Then, if the true anomaly is taken as independent variable ($\alpha = 2$ in Equation 1), the inverse of the orbital radius and the radial unit vector satisfy linear differential equations with constant coefficients.

By applying the VOP technique to the solution of the linearized equations, we can define new orbital elements, and use them as state variables. These quantities, being integrals of the motion, exhibit a smooth evolution for weakly-perturbed problems. Due to this characteristic, element formulations are highly efficient when the magnitude of the perturbing acceleration F is small. Moreover, the same beneficial properties of the parent variables are inherited by the derived elements: they may be well-defined for circular and equatorial orbits, and at collision.

We will refer to all the schemes transforming the two-body problem into a set of linear differential equations for elements or coordinates as regularized methods.

2.2.2 Numerical implications of regularizations

Regularized formulations have shown excellent numerical performances in numerous tests, due to several factors that we consider here.

First, the solution is well-behaved also close to the primary body. The right-hand-side of the Newtonian equations of motion presents periodic maxima at each periapsis passage because of the presence of the $1/r^2$ factor. The elimination of this factor by regularization results in a solution that is smoother. The more eccentric the orbit that is being integrated, the higher the resulting advantage with respect to the Cowell formulation.

The change of the independent variable to an angle-like quantity results in an *analytical step-size regulation* in physical time. A uniform step-size distribution in s results in a step-size distribution in t that is denser around the periapsis². The order α controls the shape of this distribution; the higher the order the smaller the steps in t close to the periapsis. An important consequence of the analytical step-size regulation is that it allows one to use fixed step-size integrators for moderately and highly eccentric orbits with a satisfactory accuracy.

Another advantage of regularized formulations is that they may induce a stabilization of the Keplerian motion. In fact, Keplerian motion is unstable in the Lyapunov sense, a fact that can be understood by virtue of a simple example. Consider at a certain epoch a particle following a Keplerian ellipse and another particle having a slightly different position and velocity. The two orbital energies will differ by some amount, and since the mean motion depends on the energy, for arbitrary small variations of the position and velocity after some time the distance between the two particles will become bigger than a given bound. In the space of regularized variables this distance assumes an oscillatory behaviour, rather than diverging [91, sec. 16].

The physical time t can be computed at each step of the propagation by integration of Equation 2. Its right-hand-side behaves non-linearly in a general case, which complicates the integration. However, rather than considering t as one of the dependent variables, it is possible to use the VOP technique to introduce a *time element* in some formulations. This is another dependent variable, from which the physical time can be recovered through algebraic relations. Two types of time elements are usually introduced, corresponding to either a constant or a linear behaviour for Keplerian motion. Since a time element behaves more regularly than the physical time t , its integration is more efficient. The integration of a constant time element entails a smaller magnitude of the local truncation error, which accumulates quadratically. With a linear time element, the rate of growth of the global truncation error behaves linearly instead, but the local truncation error is bigger. Thus, it is advised to use the latter in long-term integrations involving several millions orbits [4, 76].

Ultimately, all the numerical advantages brought about by regularized formulations can be traced back to one physical cause and a numerical cause. The physical cause is the aforementioned stabilization of Keplerian motion, which involves a slower accumulation of the global truncation error. The

² This is analogous to the concept of “slowing down physical time” when getting closer to a collision, as presented by Waldvogel [102]

numerical cause is that, due to the phenomena mentioned above, the rate of change of the regularized state variables is usually smaller than that of the Cartesian coordinates. This implies that higher derivatives will also be smaller, often by a significant amount. As the local truncation error at each step of numerical integration is proportional to a sufficiently high derivative³ of the state variables, it will also be smaller as a consequence. The latter cause is of particular importance when considering VOP techniques, in which the rate of change of the orbital elements is usually of the same order of magnitude of the perturbations. Therefore, VOP techniques will be particularly efficient for weak perturbations. This is a fact of the utmost importance, as weakly-perturbed trajectories arise in many problems of celestial mechanics.

In some applications, and especially for extremely long propagations, the accumulation of round-off error might be of particular importance. This can be often mitigated by taking appropriate measures in the implementation of the numerical integration scheme [49, 84]. The impact of different formulations on the accumulation of round-off error has been not yet explored completely.

2.3 OVERVIEW OF REGULARIZED FORMULATIONS

A complete review of regularized formulations of the two-body problem would require a separate work, therefore we only provide an overview of the regularized formulations that are used in this thesis. We will outline the analytical developments underlying each of the formulations, with particular attention to their implementation aspects. The reader is invited to consult Baù, Bombardelli, and Peláez [2], Baù et al. [3], Baù and Bombardelli [4], Baù et al. [6], Peláez, Hedó, and Rodríguez de Andrés [79], Roa and Peláez [85], Stiefel and Scheifele [91], and Urrutxua, Sanjurjo-Rivo, and Peláez [98] for more details on the formulations that we expound on in the following.

2.3.1 Kustaanheimo-Stiefel

The K-S regularization is based on the classical Sundman transformation, i.e. $f = \alpha = 1$ in Equation 2, and on a mapping from $\mathbf{u} \in \mathbb{R}^4$ to $\mathbf{r} \in \mathbb{R}^3$ given by

$$\mathbf{x} = \mathbf{L}(\mathbf{u})\mathbf{u}, \quad \mathbf{x} = (\mathbf{r}; 0), \quad (3)$$

where the matrix $\mathbf{L}(\mathbf{u})$ is made by the four parameters $\{u_1, u_2, u_3, u_4\}^\top = \mathbf{u}$ as follows:

$$\mathbf{L}(\mathbf{u}) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}. \quad (4)$$

Let the perturbation $\mathbf{F}$ stem from a potential V and a non-conservative acceleration $\mathbf{P}$:

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{r}} + \mathbf{P}. \quad (5)$$

The total energy $\tilde{\varepsilon}$ is the sum of the Keplerian energy ε and the perturbing potential V ,

$$\tilde{\varepsilon} = \varepsilon + V. \quad (6)$$

³ If an integration scheme is of order p , the local truncation error is proportional to the derivative of order $p + 1$.

Denoting with a prime differentiation with respect to s , the differential equation for the K-S parameters is written as

$$\mathbf{u}'' = \frac{\tilde{\varepsilon}}{2} \mathbf{u} - \frac{1}{4} \frac{\partial}{\partial \mathbf{u}} \left(\|\mathbf{u}\|^2 V \right) + \frac{\|\mathbf{u}\|^2}{2} (\mathbf{L}^\top(\mathbf{u}) \cdot \mathbf{P}). \quad (7)$$

Note that Equation 7 becomes linear for Keplerian motion, furthermore if $\tilde{\varepsilon} < 0$ it represents four scalar harmonic oscillators of the same frequency. Time must be obtained by the Sundman transformation

$$t' = \|\mathbf{u}\|^2, \quad (8)$$

or by introducing a time element [91]. A key operation for improving the efficiency of K-S regularization is to regard the total energy $\tilde{\varepsilon}$ as a state variable instead of computing it from $\mathbf{u}$, $\mathbf{u}'$, t . The reason becomes clear if we consider the case in which $\mathbf{P} = \mathbf{0}$ and V does not depend on time. Then, from the differential equation

$$\tilde{\varepsilon}' = \|\mathbf{u}\|^2 \frac{\partial V}{\partial t} + 2 [\mathbf{u}' \cdot (\mathbf{L}^\top(\mathbf{u}) \cdot \mathbf{P})], \quad (9)$$

we have $\tilde{\varepsilon}(s) = \tilde{\varepsilon}(0)$, while on the other hand the function $\tilde{\varepsilon}(\mathbf{u}, \mathbf{u}', t)$ will deviate from $\tilde{\varepsilon}(0)$ due to errors affecting time and the K-S parameters.

The state vector is $\mathbf{y} = \{\mathbf{u}^\top, (\mathbf{u}')^\top, t, \tilde{\varepsilon}\}$, and its dimension is 10. The set of differential equations to be integrated is given by Equations (7) (rewritten as 8 first-order equations), (8), (9). The transformations between the Cartesian and K-S state vectors are possible through explicit algebraic relations which are given in Stiefel and Scheifele [91, p. 33].

2.3.2 Dromo

The method Dromo [2, 79] consists of seven orbital elements: three of them allow us to recover the motion along the radial direction and the remaining four fix the orientation of the orbital plane and a departure point on it. The angular displacement between the radial unit vector and this point is provided by the independent variable, which is the true anomaly when the motion is unperturbed. Dromo elements can be derived from the focal method developed by [16] through the VOP technique [for more details, see the Introduction 6]. Dromo employs a Sundman transformation of second order,

$$\frac{dt}{ds} = \frac{r^2}{h}, \quad (10)$$

where s is the fictitious time, and h is the angular momentum.

The quaternion $(\zeta_4, \zeta_5, \zeta_6, \zeta_7)$ defines the orientation of an *ideal* frame (x, y, z) with respect to the inertial frame. The orbital plane (x, y) is attached to the ideal frame, defined as the one whose angular velocity in perturbed motion has no vertical component along z .

The size and shape of the orbit are defined through the orbital elements $(\zeta_1, \zeta_2, \zeta_3)$. The first two elements are the projections of the eccentricity vector (or its generalized form, see [5]) on the (x, y) axes of the ideal frame. The third element is either the inverse of the pseudo angular momentum $c = \sqrt{h^2 + 2r^2 V}$, or the total energy $\tilde{\varepsilon}$.

The perturbations are split into conservative and non-conservative terms according to Equation 5, and projected on the radial, local-horizontal and

out-of-plane axes as $\mathbf{F} = \{R, T, N\}^\top$ and $\mathbf{P} = \{R_p, T_p, N_p\}^\top$. The differential equations for the quaternion are:

$$\zeta_4' = \frac{1}{2\sigma} \left[\frac{N}{\zeta_3 \sigma \lambda} (\zeta_7 \cos \Delta s - \zeta_6 \sin \Delta s) + \zeta_5 (\lambda - \sigma) \right] \quad (11)$$

$$\zeta_5' = \frac{1}{2\sigma} \left[\frac{N}{\zeta_3 \sigma \lambda} (\zeta_6 \cos \Delta s + \zeta_7 \sin \Delta s) - \zeta_4 (\lambda - \sigma) \right] \quad (12)$$

$$\zeta_6' = -\frac{1}{2\sigma} \left[\frac{N}{\zeta_3 \sigma \lambda} (\zeta_5 \cos \Delta s - \zeta_4 \sin \Delta s) - \zeta_7 (\lambda - \sigma) \right] \quad (13)$$

$$\zeta_7' = -\frac{1}{2\sigma} \left[\frac{N}{\zeta_3 \sigma \lambda} (\zeta_4 \cos \Delta s + \zeta_5 \sin \Delta s) + \zeta_6 (\lambda - \sigma) \right], \quad (14)$$

where

$$\sigma = \frac{1}{c} + \zeta_1 \cos s + \zeta_2 \sin s,$$

$$\lambda = \sqrt{\sigma^2 - 2V},$$

$$\Delta s = s - s_0,$$

and s_0 is the initial value of the fictitious time.

If the element ζ_3 is defined as the inverse of the pseudo angular momentum, its equation is:

$$\zeta_3' = -\frac{1}{\sigma^4} \left[\frac{u\sigma}{c} \left(2V - \frac{1}{c} \frac{\sigma}{\sigma + 1/c} \frac{\partial U}{\partial (1/c)} \right) + \lambda T + \frac{\partial V}{\partial t} \right], \quad (15)$$

while if ζ_3 is the total energy:

$$\zeta_3' = \frac{c}{\sigma^2} \left(uR + T\sqrt{\sigma^2 - 2V} + \frac{\partial V}{\partial t} \right), \quad (16)$$

where $u = \zeta_1 \sin s - \zeta_2 \cos s$. The equations for the elements (ζ_1, ζ_2) are:

$$\zeta_1' = \frac{\sin s}{\sigma} \left(c \frac{R}{s} - 2V \right) - (c\sigma + 1) \cos s \frac{dc}{ds} \quad (17)$$

$$\zeta_2' = \frac{\cos s}{\sigma} \left(2V - c \frac{R}{s} \right) - (c\sigma + 1) \sin s \frac{dc}{ds} \quad (18)$$

and dc/ds is given by [Equation 15](#).

As to recover the physical time, Dromo does not require the solution of the Kepler equation. Instead, the time is computed either by direct integration of [Equation 10](#), rewritten as

$$\frac{dt}{ds} = \frac{c}{\sigma^2},$$

or by integration of an equation for a time element. The equation for the constant time element is:

$$\zeta_{0,c}' = a^{3/2} \left\{ \frac{d\epsilon}{ds} \left[6a \arctan \left(\frac{u}{f+w} \right) - 3as + k_1 \right] + \left[\frac{Rc}{\sigma} - 2V \right] k_2 \right\}, \quad (19)$$

where $d\epsilon/ds$ is given by [Equation 16](#), $a = -1/2\epsilon$, and:

$$k_1 = \frac{\sqrt{au}}{\sigma^2} \left(\frac{1/c + \sigma}{f} + 2wc + 1 \right),$$

$$k_2 = \frac{1}{\sigma^2} \left(fc + \frac{w}{f} + \frac{u^2}{f\sigma} \right),$$

$$w = \zeta_1 \cos s + \zeta_2 \sin s,$$

$$f = \frac{1}{c} + \sqrt{-2\epsilon}.$$

Finally, the equation of the linear time element is given by:

$$\zeta'_{0,l} = a^{3/2} \left\{ 1 + \frac{d\varepsilon}{ds} \left[6a \arctan \left(\frac{u}{f+w} \right) - 3as + k_1 \right] + \left[\frac{Rc}{\sigma} - 2V \right] k_2 \right\}. \quad (20)$$

The only singularity present is for vanishing generalized angular momentum $c = 0$, which takes place when both the angular momentum and the perturbing potential vanish.

As a rule of thumb, the total energy should be chosen as the element ζ_3 when conservative perturbations are dominating. In fact, the total energy is conserved in this formulation if only conservative perturbations are present, which is advantageous for the numerical stability [91].

The formulation obtained by integrating Equation 10, Equations (11)-(14), and Equation (16) or (15) is universal, in the sense that any orbit can be integrated regardless of the sign of the orbital energy. The Equations (19) and (20) for the time elements are only valid for $\varepsilon < 0$, and we refer to [4] for the corresponding expressions for $\varepsilon \geq 0$.

2.3.3 EDromo

EDromo was born from the idea of applying the same decomposition of the dynamics as in Dromo but with the time transformation

$$t' = \frac{r}{\sqrt{-2\tilde{\varepsilon}}}, \quad \tilde{\varepsilon} < 0, \quad (21)$$

where $\tilde{\varepsilon}$ is the total energy (Equation 6). This choice was made to improve the performance of Dromo for highly eccentric motion under third-body perturbations. Analogously to Dromo, the main role is played by an intermediate reference frame (x, y, z) with the axis z oriented as the angular momentum vector. The evolution of this frame is given by the four components $(\lambda_4, \lambda_5, \lambda_6, \lambda_7)$ of a unit quaternion. From the differential equation of the orbital radius, which becomes linear in the two-body problem, two orbital elements (λ_1, λ_2) can be defined by applying the VOP method. These turn out to be the projections of the eccentricity vector [or its generalized version, see 6] on the axes x, y of the intermediate frame. The quantities λ_1, λ_2 , and $\lambda_3 = -1/(2\tilde{\varepsilon})$ allow us to compute the radial solution and the angle v between the position vector $\mathbf{r}$ and the departure axis x , as follows:

$$r = \lambda_3 \rho, \quad r' = \lambda_3 \zeta, \quad (22)$$

$$v = s + 2 \arctan \left(\frac{\zeta}{m + \rho} \right), \quad (23)$$

where

$$\rho = 1 - \lambda_1 \cos s - \lambda_2 \sin s, \quad \zeta = \lambda_1 \sin s - \lambda_2 \cos s, \quad (24)$$

$$m = \sqrt{1 - \lambda_1^2 - \lambda_2^2}, \quad (25)$$

and s is the fictitious time.

The differential equations for the spatial elements are written as:

$$\lambda'_1 = (Rr - 2V) r \sin s + \Lambda_3 [(1 + \rho) \cos s - \lambda_1], \quad (26)$$

$$\lambda'_2 = (2V - Rr) r \cos s + \Lambda_3 [(1 + \rho) \sin s - \lambda_2], \quad (27)$$

$$\lambda'_3 = 2\lambda_3^3 \left(R_p \zeta + T_p n + \frac{\partial V}{\partial t} \sqrt{\lambda_3 \rho} \right), \quad (28)$$

$$\lambda'_4 = N \frac{r^2}{2n} (\lambda_7 \cos v - \lambda_6 \sin v) + \frac{\omega_z}{2} \lambda_5, \quad (29)$$

$$\lambda'_5 = N \frac{r^2}{2n} (\lambda_6 \cos v + \lambda_7 \sin v) - \frac{\omega_z}{2} \lambda_4, \quad (30)$$

$$\lambda'_6 = N \frac{r^2}{2n} (-\lambda_5 \cos v + \lambda_4 \sin v) + \frac{\omega_z}{2} \lambda_7, \quad (31)$$

$$\lambda'_7 = N \frac{r^2}{2n} (-\lambda_4 \cos v - \lambda_5 \sin v) - \frac{\omega_z}{2} \lambda_6, \quad (32)$$

where

$$\Lambda_3 = \frac{1}{2\lambda_3} \lambda'_3, \quad n = \sqrt{m^2 - 2\lambda_3 \rho^2 V}, \quad (33)$$

and

$$\omega_z = \frac{n - m}{\rho} + \frac{1}{m(1 + m)} [(2V - Rr)(2 - \rho + m)r + \Lambda_3 \zeta(\rho - m)]. \quad (34)$$

Physical time is computed by either a constant or a linear time element. We have, respectively

$$\lambda'_{0,c} = \lambda_3^{3/2} [(Rr - 2V)r + \Lambda_3(2\zeta - 3s)], \quad (35)$$

$$\lambda'_{0,l} = \lambda'_{0,c} + \lambda_3^{3/2}(1 + 3\Lambda_3 s). \quad (36)$$

In unperturbed motion the elements $\lambda_{0,c}, \lambda_1, \dots, \lambda_7$ are constants, while $\lambda_{0,l}$ is a linear function of the independent variable. As for the K-S formulation, the conversion between the EDromo elements and the Cartesian state vector is managed by explicit algebraic relations [6].

Note that the EDromo formulation is only defined for $\tilde{\varepsilon} < 0$. Analogous formulations for $\tilde{\varepsilon} > 0$ have been developed by [85] and [3]. Also, equations (29) to (32) are singular for $n = 0$. This condition, when $V = 0$, is satisfied if the angular momentum vanishes.

2.4 ADDITIONAL CONSIDERATIONS AND CAVEATS

As to implement regularized formulations in an operational context, it is necessary to get acquainted to some of their particularities.

Regularized sets of equations are redundant, i.e. more than six first-order equations (or three of second-order) are needed to represent the motion of a point test mass in space. This poses no particular complications, since in practical applications the computational time is dominated by the calculation of the perturbations at each step.

Also, position and velocity must be obtained from the state variables through algebraic formulas. Although it is possible to carry out this operation after the integration process has been concluded, it is usually performed stepwise as to allow the computation of perturbations that are not expressed in terms of the state variables.

We also highlight an observation that, even if not specifically linked to regularized formulations themselves, is important to consider in an implementation phase. All of the equations considered in the above sections are

non-dimensionalized. As a consequence, analytical developments are considerably streamlined and the orders of magnitude of the perturbations can be estimated quickly. Non-dimensionalizing the equations is also mandatory to achieve a satisfactory numerical accuracy. In fact, with dimensional quantities a simple change of units could have a significant impact on the numerical results, due to the presence of round-off. Non-dimensionalization avoids this issue, and permits the treatment of physical problems on different scales of time and space using the same formalism.

Perhaps the most important consideration for operational implementations of regularized formulations concerns the calculation of the state variables at prescribed values of the physical time. Since the independent variable is the fictitious time, an iterative procedure must be employed. In a general case, it will require the knowledge of the state variables also *inside* the integration steps, which can be obtained either by direct integration or by interpolation of the numerical solution. In the latter case, one should employ interpolators of an order at least equal to that of the integration as not to incur in an additional numerical error. Multi-step integration schemes are particularly suited for building this kind of interpolators.

We describe an iterative algorithm for the calculation of the state variables at prescribed values of the physical time in the following section.

2.4.1 Output at prescribed values of physical time

Since t increases with s (see Equation 2), the function

$$g(s) = t(s) - t^* \quad (37)$$

has only one root s^* in the integration interval. Let the j -th ($j \geq 1$) integration step be the range of values of $s \in [s_{j-1}, s_j]$. During the propagation we check the condition

$$g(s_j)g(s_{j-1}) \leq 0. \quad (38)$$

If 38 is verified for $j = i$ and $g(s_i), g(s_{i-1}) \neq 0$, we search for a root of $g(s)$ in the open interval $\Sigma = (s_{i-1}, s_i)$. For this purpose, we find successively better approximations τ_n to s^* by Newton's method

$$\tau_{n+1} = \tau_n - \frac{t(\tau_n) - t^*}{t'(\tau_n)}, \quad (39)$$

until the value of $|g(\tau_{n+1})|$ is sufficiently small. Note that t' is the derivative of t with respect to s and its expression is given in Equation 2. The initial guess τ_0 is chosen through the following procedure. Since the integration step is much shorter than the orbital period for highly accurate propagations, we generically have that the second derivative $t''(s)$ changes sign at most once in Σ . If $t''(s)$ is strictly monotonic, we directly set

$$\begin{aligned} \tau_0 &= s_i, & \text{for } t''(s) > 0, \\ \tau_0 &= s_{i-1}, & \text{for } t''(s) < 0. \end{aligned} \quad (40)$$

Assume that the function $t(s)$ has an inflection point at $s_f \in \Sigma$ (i.e. $t''(s_i)t''(s_{i-1}) < 0$). We divide Σ in the two subintervals $\Sigma_\ell = (s_{i-1}, s_1]$, $\Sigma_r = [s_1, s_i)$ where

$$s_1 = \frac{s_i + s_{i-1}}{2}. \quad (41)$$

After computing $g(s_1), t''(s_1)$, we know in which subinterval s^* and s_f are located. If it is the same one for both of them, the bisection is applied again

to such subinterval by introducing s_2 . This operation is carried on until s_k ($k \geq 1$) is between s^* and s_f . Then, we choose τ_0 as follows:

$$\begin{aligned} \tau_0 &= s_i, & \text{if } s^* > s_k \text{ and } t''(s_i) > 0, \\ \tau_0 &= s_{i-1}, & \text{if } s^* < s_k \text{ and } t''(s_{i-1}) < 0, \\ \tau_0 &= s_k, & \text{if } s^* > s_k \text{ and } t''(s_i) < 0, \\ & & s^* < s_k \text{ and } t''(s_{i-1}) > 0. \end{aligned} \tag{42}$$

If $s_f = s^*$, the bisection method itself allows us to find s^* after 52 iterations working in double precision. Finally, the case in which $t''(s_i)t''(s_{i-1}) = 0$ is handled as in [Equation 40](#).

Note that we need to evaluate the state variable $t(s)$, and its derivatives $t'(s)$, $t''(s)$, at several points $\tilde{s} \in (s_{i-1}, s_i)$. This can be accomplished by either taking integration substeps from s_{i-1} or s_i , or by interpolation of the solution inside the step. In the latter choice, the interpolating function should be at least of the same order of integration as not to generate an additional contribution to the numerical error.

Part I

PLANETARY CLOSE ENCOUNTERS

ACCURATE PROPAGATION TO THE ENCOUNTER B-PLANE

The reliable and accurate prediction of planetary close encounters is paramount in Space Situational Awareness (SSA), dynamical astronomy and planetary science.

Interest on this topic was first spurred in the XIX century by the search for the origin of short-periodic comets. According to the capture hypothesis, originally attributed to Laplace and further developed by Le Verrier [33], they hail from long-periodic comets experiencing close encounters with Jupiter that decrease their orbital energy. It was one of these encounters – ended up with the disintegration of Comet Shoemaker-Levy 9 in Jupiter’s atmosphere in July 1994 – that drew attention on the possibility that such a fierce event could happen on Earth. The episode built momentum for modern large-scale asteroid surveys, which spectacularly increased the number of catalogued main belt and Near-Earth Asteroids (NEAs) and posed the basis for impact monitoring activities.

Close encounters have been found to induce exponential growth of the distance between initially close trajectories, leading to the decrease of Lyapunov times [93]. In fact, initially close trajectories diverge linearly after one encounter due to different post-encounter orbital periods. However, the divergence accumulates multiplicatively after each encounter, eventually resulting in an exponential behaviour [100]. The latter is a necessary condition for chaotic trajectories to arise [28, p.50].

Discovering NEAs with highly nonlinear dynamics requires to routinely propagate thousands of initial conditions, following orbital updates coming from observations. The required computational load might become critical when new-generation NEA surveys will settle into place and greatly increase the frequency of observations. For NEAs in which the orbit determination process is nonlinear, some of the currently operating impact monitoring systems propagate the orbital probability density function by using a Monte Carlo approach in a unidimensional subspace, the Line of Variations (LOV). The LOV is sampled, generating several thousands of Virtual Asteroids (VAs) that need to be numerically propagated with a sophisticated dynamical model for subsequent analysis [71]. Using efficient numerical techniques might bring about significant savings in computational cost. Increasing the accuracy of numerical propagation can also improve the estimation of impact probabilities.

It is evident that accurate orbit prediction when close encounters are involved is of great importance to improve the quality of astrometric data, and of asteroid threat assessment and mitigation [31]. For instance, 99942 Apophis, which has an impact probability greater than 10^{-6} in 2068, could enter a 7:6 resonance with the Earth in its 2029 close encounter [37]. These events, along with the influence of the Yarkovsky effect on the dynamics, make its orbit prediction quite challenging. In fact, when considering numerical orbit propagation of interplanetary trajectories with close encounters, it can be intuitively understood that the mechanisms that amplify the position uncertainty have the same effect on the numerical errors. This was noticed, for example, as a limiting factor for the time span in which cometary orbit

predictions can be considered reliable [17]. Besides, the exponential increase of the numerical error generates different orbital evolutions depending on the integration method, like in the case of 4179 Toutatis, which experiences a chaotic evolution driven by close encounters. Therefore, long-term numerical propagation with different methods often give different results, so that only a statistical study of the orbit evolution makes sense [66, 95].

As concerns astrodynamics, the accurate prediction of the motion in presence of close encounters is certainly mandatory in interplanetary missions analysis and design. Direct transfers to the outer planets and to Mercury impose very demanding propellant requirements, so that it is often impossible to achieve such transfers even in the best launch opportunities with current high-thrust propulsion systems [58]. Gravity assist maneuvers, which exploit the mechanics of planetary close encounters, are necessary in these cases to provide the variation in the heliocentric orbital energy of the spacecraft. In the near future, the European Space Agency's Jupiter Icy Moons Explorer mission will make extensive use of flybys of the Jovian Moons both for scientific purposes and to insert the spacecraft into its final orbit around Ganymede. In particular, successive flybys of Callisto are planned in order to increase the orbit inclination for scientific purposes [48]. The orbit propagation of a spacecraft that experiences one or several close encounters with a massive body poses the same kind of difficulties as in the case of an asteroid.

The propagation of trajectories which involve one or more close encounters is still a challenging problem for numerical methods. If high accuracy in position and velocity is necessary, it is advisable to resort to the integration of regularized formulations of the two-body problem, which show superior accuracy in comparison to the direct integration of the Newtonian equations for the reasons described in chapter 2.

In this chapter, we will investigate the possibility of minimizing the numerical error in the propagation of interplanetary trajectories with regularized formulations. The chapter is structured as follows: in the following section the dynamics of close encounters under the approximated analytical theory by Valsecchi et al. [100] will be summarized, and the mechanism leading to the divergence of close trajectories in the b-plane will be described. Next we will assess the efficiency in the propagation of several geocentric hyperbolic orbits, which are characteristic of an encounter, with several formulations of orbital dynamics. Then, we will present the results from the propagation of interplanetary trajectories by using different formulations for the heliocentric and geocentric phases, and we will assess which formulations obtain the best numerical efficiency. Finally, we will summarize the conclusions of our study.

3.1 CLOSE ENCOUNTERS AND AMPLIFICATION OF THE PROPAGATION ERROR

We review here some concepts from the extended Öpik theory of planetary close encounters, which are necessary to compute the initial conditions for the interplanetary trajectory and to measure the numerical error in b-plane. The reader can find an extensive review of the theory in the works by Carusi, Valsecchi, and Greenberg [18] and Valsecchi et al. [100].

Figure 4: Orientation of the pre-encounter geocentric velocity $\mathbf{U}$ of the asteroid and of the reference frame (ξ, η, ζ) with respect to the fixed-axes geocentric frame (X, Y, Z) .

3.1.1 *B-plane coordinates in the close encounter*

Let (X, Y, Z) be a geocentric coordinate system where the axis X is oriented along the Earth's position relative to the Sun, Z is oriented along the Earth's angular momentum and Y completes the right-handed frame. Since we suppose that the Earth's orbit is circular, Y identifies the direction of the velocity. Assume a close encounter occurs at one node and let $\mathbf{U}$ be the pre-encounter (unperturbed) asteroid's velocity with respect to the Earth. If $\mathbf{v}_{\text{Ast}}$ and $\mathbf{v}_{\oplus}$ are the heliocentric velocities of the asteroid and Earth, respectively, then $\mathbf{U}$ is defined as

$$\mathbf{U} = \mathbf{v}_{\text{Ast}} - \mathbf{v}_{\oplus}.$$

The orientation of $\mathbf{U}$ in the geocentric reference frame is determined by the angles (ϕ, θ) so that

$$\begin{Bmatrix} U_X \\ U_Y \\ U_Z \end{Bmatrix} = \begin{Bmatrix} U \sin \theta \sin \phi \\ U \cos \theta \\ U \sin \theta \cos \phi \end{Bmatrix}, \quad (43)$$

as shown in Figure 4. Consequently

$$\begin{Bmatrix} \cos \theta \\ \tan \phi \end{Bmatrix} = \begin{Bmatrix} U_Y/U \\ U_X/U_Z \end{Bmatrix}. \quad (44)$$

The *b-plane* is defined as the plane orthogonal to $\mathbf{U}$. We also define a new reference frame by means of the axes (ξ, η, ζ) , where η is along $\mathbf{U}$, ζ is opposite to the projection of the Earth's velocity on the *b-plane*, and ξ completes the right-handed frame. The transformation from (X, Y, Z) to (ξ, η, ζ) is achieved through a rotation of ϕ around Y , followed by a rotation of θ around the rotated X -axis, that is ξ . Assuming counter-clockwise rotations as positive, we have

$$\begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} = \mathbf{R}_{\xi}(\theta) \mathbf{R}_Y(\phi) \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad (45)$$

Figure 5: Geometry of the rotation between the modified b-plane (orthogonal to η_p) and the b-plane (orthogonal to η). In this sketch, the orbit of the asteroid lies on the ecliptic.

where

$$\mathbf{R}_\xi(\theta)\mathbf{R}_Y(\phi) = \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ \sin(\theta)\sin(\phi) & \cos(\theta) & \cos(\phi)\sin(\theta) \\ \cos(\theta)\sin(\phi) & -\sin(\theta) & \cos(\theta)\cos(\phi) \end{bmatrix}.$$

Some remarks are necessary to understand this choice of coordinates. In Öpik's approximation, the particle follows a rectilinear motion in the direction of $\mathbf{U}$ until it crosses the b-plane. In this point the particle reaches the minimum approach distance and its coordinates (ξ, ζ) have a particular physical meaning. The ξ coordinate is the distance between the orbit of the Earth and that of the asteroid, and so it is the Minimum Orbit Intersection Distance (MOID). It depends only on the mutual geometry of the two orbits and is not influenced by the locations of the two bodies along their respective orbits. The ζ coordinate strongly depends on the timing of the approach, and it increases if the asteroid is late for the encounter. Thus, by means of the coordinate system (ξ, η, ζ) we decouple the two factors governing an encounter, i.e. the relative geometry of the orbits and the timing of the encounter.

3.1.2 From inertial to b-plane coordinates

The output of our numerical propagation is expressed in the heliocentric inertial frame (x, y, z) . For computing the b-plane coordinates at the close encounter, it is first necessary to express the geocentric position and velocity of the asteroid in the system (X, Y, Z) . We shall calculate θ and ϕ from the components of $\mathbf{U}$ by means of Equation 44 and then use Equation 45. However, at no point in the propagation we are explicitly provided with $\mathbf{U}$, which is the incoming asymptotic velocity of the hyperbolic orbit. This difficulty is avoided by introducing the *modified* b-plane, which is orthogonal to the velocity $\mathbf{V}$ at the minimum approach distance (as defined in [96], see also Figure 5). A coordinate system (ξ_p, η_p, ζ_p) can be defined also for the modified b-plane, where η_p is oriented as $\mathbf{V}$. Let $\mathbf{d} = \{X_m, Y_m, Z_m\}^T \equiv \{\xi_p, 0, \zeta_p\}^T$ be the asteroid's position at the closest approach distance. By employing the

angles θ_p, ϕ_p to describe the components of $\mathbf{V}$ with respect to (X, Y, Z) , the modified b-plane coordinates can be expressed as

$$\begin{pmatrix} \xi_p \\ \zeta_p \end{pmatrix} = \begin{pmatrix} X_m \cos \phi_p - Z_m \sin \phi_p \\ X_m \cos \theta_p \sin \phi_p - Y_m \sin \theta_p + Z_m \cos \theta_p \cos \phi_p \end{pmatrix}. \quad (46)$$

We are now ready to obtain the coordinates (ξ, ζ) in the b-plane. Let $\mathbf{b} = \{\xi, 0, \zeta\}^T$, then, the direction of $\mathbf{b}$ can be recovered by applying to $\mathbf{d}$ a clockwise rotation of magnitude $\gamma/2$ around the unit vector $\hat{\mathbf{h}} = (\mathbf{d} \times \mathbf{V})/|\mathbf{d} \times \mathbf{V}|$ as sketched in Figure 5. Note that the angular momentum vector $\mathbf{h}$ of the hyperbolic orbit is not affected by such rotation. Since both $\mathbf{d}$ and $\mathbf{b}$ are perpendicular to $\hat{\mathbf{h}}$, the direction of $\mathbf{d}$ in the modified b-plane is the same as that of $\mathbf{b}$ in the b-plane. By exploiting the conservation of angular momentum, the magnitudes of $\mathbf{b}$ and $\mathbf{d}$ are related by

$$b = \frac{V}{U} d,$$

where $V = |\mathbf{V}|$ is known, and $U = |\mathbf{U}|$ is derived from the conservation of energy

$$U = \sqrt{V^2 - 2\frac{\mu_\oplus}{d}}. \quad (47)$$

The deflection angle $\gamma/2$ and the components of the vector $\mathbf{b}$ on the b-plane are given by

$$\tan \frac{\gamma}{2} = \frac{\mu_\oplus}{bU^2}, \quad (48)$$

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \frac{V}{\sqrt{V^2 - 2\mu_\oplus/d}} \begin{pmatrix} \xi_p \\ \eta_p \end{pmatrix}. \quad (49)$$

By following Carusi, Valsecchi, and Greenberg [18], we can express θ, ϕ through the relations

$$\cos \theta = \cos \theta_p \cos(\gamma/2) - \sin \theta_p \sin(\gamma/2) \cos \psi, \quad (50)$$

$$\tan(\phi - \phi_p) = -\frac{\sin(\gamma/2) \sin \psi}{\sin \theta_p \cos(\gamma/2) + \cos \theta_p \sin(\gamma/2) \cos \psi}, \quad (51)$$

where $\tan \psi = \zeta_p/\xi_p$. We have completely defined the application from $(V, \theta_p, \phi_p, \xi_p, \zeta_p)$, which are provided by the numerical propagation, to $(U, \theta, \phi, \xi, \zeta)$.

3.1.3 Divergence of nearby trajectories in the b-plane

The accuracy in computing the b-plane coordinates of a close encounter strongly influences the quality of the predicted orbit after the encounter. For instance, this aspect becomes especially relevant when considering the case of an asteroid with successive *resonant returns* [100]. The errors in the b-plane coordinates of each encounter propagate to those of the next one. Moreover, the error in ζ , which expresses the time delay between the planet and the asteroid, is amplified by a linear growth in time between two subsequent encounters. Therefore, nearby trajectories in the b-plane diverge. In the following explanation, we shall adopt a system of units in which the Earth's semi-major axis and mean motion, and the Sun's gravitational parameter are all set equal to unity.

Let (ξ, ζ) and (ξ'', ζ'') denote the b-plane coordinates just before the first and second encounters, respectively. For small deviations $(\delta\xi, \delta\zeta)$, we have

$$\begin{pmatrix} \delta\xi'' \\ \delta\zeta'' \end{pmatrix} \approx \begin{bmatrix} \frac{\partial\xi''}{\partial\xi} & \frac{\partial\xi''}{\partial\zeta} \\ \frac{\partial\zeta''}{\partial\xi} & \frac{\partial\zeta''}{\partial\zeta} \end{bmatrix} \begin{pmatrix} \delta\xi \\ \delta\zeta \end{pmatrix}. \quad (52)$$

For small values of the deflection angle of the first encounter (γ), the elements in the first row of the matrix in Equation 52 can be approximated by

$$\frac{\partial\xi''}{\partial\xi} \approx 1 + \mathcal{O}(\gamma), \quad \frac{\partial\xi''}{\partial\zeta} \approx \mathcal{O}(\gamma).$$

It follows that the quantity ξ'' , which represents the MOID for the second encounter, is not significantly affected by the errors in the b-plane coordinates of the first encounter. On the other hand, as regards ζ'' , which is related to the “time-shift” at the second passage, we have [100]

$$\begin{aligned} \frac{\partial\zeta''}{\partial\xi} &\approx -h s(\mathcal{U}', \theta') \frac{4c \xi \zeta}{b^4} \sin \theta + \frac{\partial\zeta'}{\partial\xi}, \\ \frac{\partial\zeta''}{\partial\zeta} &\approx h s(\mathcal{U}', \theta') \frac{2c (\xi^2 - \zeta^2) \sin \theta}{b^4} + \frac{\partial\zeta'}{\partial\zeta}, \end{aligned} \quad (53)$$

where:

$$c = \frac{m_\oplus}{\mathcal{U}^2}, \quad (54)$$

$$s(\mathcal{U}', \theta') = \frac{2\pi [\mathcal{U}' \cos^2 \theta' + \cos \theta' (1 - \mathcal{U}'^2) - 3\mathcal{U}']}{(1 - \mathcal{U}'^2 - 2\mathcal{U}' \cos \theta')^{\frac{5}{2}} \sin \theta'}, \quad (55)$$

$$\partial\zeta'/\partial\xi = \mathcal{O}(1), \quad (56)$$

$$\partial\zeta'/\partial\zeta = 1 + \mathcal{O}(\gamma), \quad (57)$$

and the prime is used to indicate the quantities evaluated after the first encounter. Closed-form expressions of the terms appearing in Equation 53 are reported in Valsecchi et al. [100]. In particular, h is the number of revolutions of the asteroid between the two encounters. Hence, the error in ζ'' grows linearly with respect to the time between subsequent encounters and accumulates in a multiplicative way during a sequence of several resonant returns, eventually leading to chaos [100]. Even if the horizon of predictability of the orbit can still be limited by the onset of chaos, the b-plane coordinates have to be accurately determined to contain the error in time at the next encounter. As a final remark, examination of the first term of Equation 53 reveals that for decreasing b the error increases with $\mathcal{O}(1/b^2)$. This explains why very close encounters produce a large divergence of the post-encounter trajectories, and therefore why they represent very challenging cases for the numerical propagation.

3.2 NUMERICAL EXPERIMENTS FOR THE HYPERBOLIC PHASE

Before considering the propagation of an interplanetary trajectory with close encounters, we start by examining the performance characteristics of several regularized formulations when propagating the encounter phase itself, which is a hyperbolic orbit.

The reference solution for evaluating the performance is always provided by a highly accurate propagation carried out with the unregularized Cowell formulation. We compare the *Dromo* formulation described in [subsection 2.3.2](#) with either the physical time or the constant time element as a state variable, the Kustaanheimo-Stiefel formulation (K-S, described in [subsection 2.3.1](#)), the hyperbolic particularization of Dromo due to Roa and Peláez [85], and the Cowell formulation. The hyperbolic particularization of Dromo is based on Sundman's transformation of first order, consists of 8 first-order ODEs, and is endowed with a time element. We highlight that both Dromo and K-S can take advantage of perturbations arising from a conservative potential [5].

We will refer to the aforementioned formulations as “Dromo”, “Dromo(Cte)”, “HyperDromo”, “K-S”, and “Cowell” in the discussion of the tests.

The code used for the study was implemented in Fortran 2003, and was compiled with Intel Visual Fortran 2013 (14.0.0086.11) in quadruple precision on a 64-bit Windows 8.1 machine.

3.2.1 Parametrization and initial/final conditions

Numerical tests have shown that the performance of regularized formulations is heavily affected by the character of the orbit which is being propagated [5]. This implies that, as to get a complete picture of the computational efficiency in the propagation of hyperbolic orbits, we have to consider several sets of initial conditions. We always start the propagations in the equatorial plane of the Earth ($z_0 = 0, v_{z_0} = 0$), and we compute the initial orbital elements through two parameters:

- $\rho_{\min} = r_{\min}/R_{\oplus}$, the *dimensionless minimum approach distance*, where $r_{\min}$ is the minimum Earth approach distance, and $R_{\oplus} = 6371.22$ km is the Earth's radius.
- e_0 , the *initial eccentricity*, which is related to the minimum approach distance $r_{\min}$ and asymptotic incoming velocity v_{∞_0} through the formula $e_0 = 1 + (v_{\infty_0}^2 r_{\min})/\mu_{\oplus}$. Higher initial eccentricities imply faster encounters, for the same $r_{\min}$.

Once these two parameters are fixed, the geometry of the hyperbolic orbit is completely characterized; only the asteroid's position along the orbit remains to be set. To this end, we can fix, for example, the initial true anomaly ν_0 , whose choice requires more care than in the elliptic case. The asymptotic value of the true anomaly is given by $\nu_{\lim} = \arccos(-1/e_0)$, thus it is reasonable to set $\nu_0 = -\nu_{\lim} + \delta\nu$. The magnitude of $\delta\nu$ is chosen as a compromise: it cannot be too large, since we would propagate a small hyperbolic arc which is not representative of an encounter; on the other hand, it cannot be too small otherwise we would start propagating the trajectory very far away from the Earth while still considering it as the primary body. Analogously, the propagation is stopped when the final true anomaly $\nu_f = \nu_{\lim} - \delta\nu$ is reached.

3.2.2 Mathematical model

We take into account the Earth's oblateness (i.e. the J_2 zonal harmonic) and the third-body gravitational attraction as perturbations.

Table 2: Orbital elements of the Earth and the Moon adopted in the simulations. The values are taken from Vallado and McClain [99].

	Earth	Moon
a (km)	$1.495\,980\,23 \times 10^8$	$3.844\,00 \times 10^5$
i (°)	0	5.1452

For Dromo, Dromo(Cte) and K-S the J_2 perturbation is implemented as the perturbing potential $\mathcal{U}$

$$\mathcal{U} = \frac{\mu_{\oplus} R_{\oplus}^2 J_2}{2r^3} (3 \cos^2 \theta - 1),$$

where r is the distance from the Earth (dimensional), θ is the colatitude and the values for $\mu_{\oplus}$, $R_{\oplus}$, J_2 are taken from Baù, Bombardelli, and Peláez [5]. It is known that the implementation of conservative forces by means of their potentials increases the computational efficiency during the propagation. In fact, since the total energy is an integral of motion when only conservative perturbations are present, we can achieve a better numerical precision if it is employed as a dependent variable [5, 91]. The J_2 perturbation is expressed as an acceleration in the inertial reference frame for the Cowell formulation and HyperDromo:

$$\mathbf{a}_{J_2} = \frac{3}{2} \frac{\mu_{\oplus} R_{\oplus}^2 J_2}{r^5} \left\{ x (5 \cos^2 \theta - 1), y (5 \cos^2 \theta - 1), z (5 \cos^2 \theta - 3) \right\}^T.$$

We only take into account the Sun and the Moon for third-body perturbations. For simplicity, we assume that both the Earth and Moon are on circular, Keplerian orbits with semi-major axes and inclinations displayed in Table 2. We also set $\Omega_{\mathcal{C}} = M_{0\mathcal{C}} = 0$. The values of the gravitational parameters are [99]:

$$\mu_{\odot} = 1.32712428 \times 10^{11} \text{ km}^3/\text{s}^2, \quad \mu_{\mathcal{C}} = 4902.799 \text{ km}^3/\text{s}^2.$$

By denoting with $\mathbf{r}_B$ the position vector of the perturbing body with respect to the Earth, the third-body acceleration takes the expression

$$\mathbf{a}_B = -\mu_B \left(\frac{\mathbf{r} - \mathbf{r}_B}{|\mathbf{r} - \mathbf{r}_B|^3} + \frac{\mathbf{r}_B}{|\mathbf{r}_B|^3} \right).$$

We do not introduce a potential for this perturbation for two reasons. First, the perturbation is time dependent and so we cannot benefit of the conservation of the total energy. Then, in operational applications the ephemerides of the perturbing bodies are usually tabulated or computed from interpolating Chebyshev polynomials, hampering the possibility of deriving the analytical expressions for the time derivative of the potential.

3.2.3 Numerical solver

Our aim is to evaluate which formulations are most efficient for our propagation problem when they are combined with the same numerical solver.

This consists of a multi-step implicit numerical scheme with variable step-size and order, which makes use of the Adams-Moulton and Backwards Differences formulas depending on if the problem is stiff or not [82]. A tolerance on the local truncation error is imposed by the user in order to control the accuracy of the integration. The order can take any value between 1 and 12. The solver is implemented in the LSODAR subroutine of the ODEPACK suite¹. We will expound more on the characteristics of this solver in [chapter 4](#), where we will further investigate the impact of the chosen tolerance (or *accuracy parameter*) on the truncation and round-off errors.

3.2.4 Performance metrics

We have taken as a measure of the required computational effort the total number of *function calls* to the right-hand side of the equations of motion instead of the computational time. In such a way, the performance analysis becomes machine-independent. Moreover, in operational scenarios making use of high-fidelity models the computational time is often dominated by the evaluation of the perturbing forces; therefore the number of function calls is expected to be representative of the computational effort.

The accumulated numerical error, for given values of the absolute and relative tolerance of the integrator, is obtained by propagating the trajectory first forwards (from $\mathbf{v}_0$ to $\mathbf{v}_f$), then backwards (from $\mathbf{v}_f$ to $\mathbf{v}_0$). If $\mathbf{r}_0 = \mathbf{r}(t_0 = 0)$ is the initial position computed from $(\rho_{\min}, e_0)$ and $\mathbf{r}_0^{\text{bwd}} = \mathbf{r}^{\text{bwd}}(t_0 = 0)$ is the initial position at the end of the backwards propagation, we calculate the global relative propagation error as

$$\epsilon = \frac{|\mathbf{r}_0^{\text{bwd}} - \mathbf{r}_0|}{|\mathbf{r}_0|}.$$

We remark that the relative error, rather than the absolute one, allows us to compare the performance between trajectories with different initial conditions.

3.2.5 Performance evaluation

We conducted numerical tests by varying both the initial conditions and the active perturbations, in a similar way as reported in Baù, Bombardelli, and Peláez [5] and Baù and Bombardelli [4]. We integrated each trajectory by decreasing the integrator tolerance from 10^{-7} to 10^{-20} . In the following plots each point is obtained with a tolerance value inside such interval.

[Figure 6](#) shows the results obtained by propagating hyperbolic trajectories perturbed only by J_2 . The minimum approach distance is set equal to the Earth's radius ($\rho_{\min} = 1$) and three initial eccentricities are considered. It can be seen that Dromo(Cte) always exhibits the best performance. This is due to two factors: first, the constant time element performs best when perturbations that do not explicitly depend on time (such as J_2) are present [38]. Then, due to the fact that the J_2 perturbation is conservative, Dromo(Cte), along with Dromo and K-S, which all employ the total energy as a dependent variable, can take advantage of this integral of motion. It is noteworthy that although conservative perturbations are not implemented with a potential in HyperDromo, its performance is comparable to the K-S regularization. [Figure 6b](#), [Figure 6c](#) show that, at high eccentricities, HyperDromo

¹ URL: <https://computation.llnl.gov/casc/odepack/>, last visited 31st May, 2017.

(a) $e_0 = 1.05$ (b) $e_0 = 2$ (c) $e_0 = 5$

Figure 6: Function calls versus position error for a hyperbolic orbit subject to the J_2 perturbation, $\rho_{\min} = 1$, $\delta v = 0.01$ rad. Three different values of the initial eccentricity are considered.

(a) Sun and Moon perturbations

(b) J_2 , Sun and Moon perturbations

Figure 7: Function calls vs. position error for a hyperbolic orbit subject to the J_2 and third-body perturbations, $\rho_{\min} = 1$, $e_0 = 2$, $\delta v = 0.01$ rad.

(a) $\rho_{\min} = 5, \delta\nu = 0.04$ rad (b) $\rho_{\min} = 20, \delta\nu = 0.16$ radFigure 8: Function calls vs. position error for a hyperbolic orbit subject to the J_2 and third-body perturbations, $e_0 = 5$.

is one order of magnitude more accurate than K-S for the same number of function calls. We conceive that the inclusion of a perturbing potential in HyperDromo, as in Baù et al. [3], could improve the method.

The other formulations are not substantially affected by the eccentricity, with the only exception of Dromo. In particular, Dromo's performance is strongly deteriorated at higher eccentricities because the contribution to the numerical error due to the error in time is larger for higher asymptotic velocities.

Figure 7 displays the results achieved with the initial conditions $\rho_{\min} = 1$, $e_0 = 2$ and by considering either only the third-body perturbation or together with the Earth's oblateness. The third-body perturbation has a dramatic impact on the performance of Dromo(Cte). Comparison of Figure 6 and Figure 7 shows that Dromo(Cte), which was the best method previously, now behaves like Dromo and Cowell's formulation. As expected, the introduction of perturbations which explicitly depend on time has a detrimental effect on this kind of time element [4, 38]. When only the third-body perturbations are present, HyperDromo and K-S are the most efficient methods. Their advantage over the other schemes slightly decreases if the Earth's oblateness is also included. We have verified that when the motion is perturbed by just one body between the Moon and the Sun, the performance diagrams do not significantly change with respect to the case in which both bodies are active.

The two panels of Figure 8 show the results obtained by switching on all the perturbations and by setting the initial eccentricity e_0 equal to 5, and the minimum approach distance ($\rho_{\min}$) equal to 5 and 20 Earth radii. By increasing $\rho_{\min}$ the Cowell formulation reduces the gap with respect to the others. The reason of this improvement is due to the fact that the farthest we are from the Earth the strongest is the third-body gravitational attraction. Thus, by fixing the eccentricity and increasing the minimum approach distance we expect that methods based on coordinates will behave better, while element methods, which rely on the underlying assumption of small perturbations, might deteriorate. If the encounter is slow (i.e. with small e_0) and takes place far from the primary, the effect of third-body perturbations on the trajectory will be even more significant. In these cases, taking coordinates rather than elements as state variables could provide the best performance.

In all the propagations we chose δv , the initial angular shift from the asymptote, in such a way that the propagation is started from a geocentric distance of the order of the Earth's sphere of influence. We found that other values of δv do not significantly affect the efficiency unless we are not too close to the incoming asymptote, or, in other terms, the transverse velocity becomes too small (which represents a singular case for Dromo methods). In this case, the performance of Dromo methods can significantly worsen, in particular if third-body perturbations are switched on. One should also consider that different values of δv , and thereby of the length of the integrated hyperbolic trajectory, can produce significant variations in the distance from the trajectory to a perturbing body. As a consequence, the magnitude of the perturbing force and in the end the global propagation error could depend on the choice of δv .

In a general case, in order to obtain the best performance from formulations based on orbital elements, the strength of the perturbations should be minimized. For example, the primary body of attraction should be the one that exerts the greatest gravitational attraction on the particle. An efficient strategy is to switch the primary as soon as the particle is under a

certain distance from the perturbing body. In this way, the magnitude of the perturbations can be kept small enough. This approach ensures that the propagated body is not too far from the new primary, which is desirable for methods (as Dromo) suffering from a singularity for rectilinear motion. This is the basis of the trajectory splitting approach, which will be outlined in more detail in [chapter 4](#).

3.3 NUMERICAL EXPERIMENTS FOR INTERPLANETARY TRAJECTORIES

For the propagation of interplanetary trajectories, we compared the performance of the same formulations considered in the previous section and, in addition, we also consider a linear time element as a state variable in Dromo [4]. This method will be denoted by “Dromo(Lte)”.

We considered the propagation of the trajectory of a fictitious asteroid which undergoes a resonant return with the Earth after a first encounter. As discussed in [subsection 3.1.3](#), trajectories which are initially close in the b-plane of the first encounter diverge, so that the accurate computation of the coordinates in the b-plane of the second encounter is a challenging task. Our model consists of the Earth and Sun in a planar Circular Restricted Three-Body Problem (CR3BP), with the Earth’s orbital radius set equal to 1 au.

3.3.1 Initial conditions and numerical set-up

A resonant return after k years and h asteroid periods can be achieved by setting appropriate initial conditions for both the Earth and the asteroid. These initial conditions are not trivial to find. On the basis of the extended Öpik theory [100], we devised analytical procedures which, given the resonance ratio k/h and the post-encounter heliocentric orbital elements of the asteroid, allow to find an approximate set of initial orbital elements of the asteroid and longitude of the Earth to achieve a resonant return. Then, the true anomaly of the asteroid or the initial longitude of the Earth are corrected by trial-and-error as to obtain the appropriate set of initial conditions that give rise to the desired encounters.

We computed three sets of initial conditions ([Table 3](#)) for three different propagations which have a resonant return 7 years and 8 asteroid periods ($k = 7, h = 8$) after a first encounter. As the orbit eccentricity is one of the main factors that influence the accuracy [4, 5], we propagate orbits with different eccentricities, undergoing similar encounters. The b-plane coordinates obtained from a high-precision propagation are listed in [Table 4](#). In [Table 5](#) we report the post-encounter semi-major axis, eccentricity and inclination of the heliocentric orbit, which are taken one week after the first encounter.

3.3.2 Propagation details

As highlighted in [subsection 2.2.1](#), formulations that make use of orbital elements, such as Dromo, show optimal performance for weakly perturbed motion. However, a planetary close encounter will strongly perturb the trajectory on a short time scale, so that the quick growth of the perturbation as the propagated body approaches the planet irremediably degrades the efficiency of element formulations. The amplification of the global numerical error can be reduced by attempting to minimize the magnitude of the per-

Table 3: Pre-encounter orbital elements of the asteroid and initial longitude of the Earth λ_0 for the interplanetary propagations.

		Case 1	Case 2	Case 3
a_0	(au)	1.02	0.776	0.816
e_0		0.04	0.465	0.67
i_0	(°)	31.7	28.6	39.5
Ω_0	(°)	315	0	0
ω_0	(°)	297	212.6	215
ν_0	(°)	0	0	0
λ_0	(°)	112	287	306

Table 4: Coordinates in the b-planes of the first and second encounters, which are obtained from a reference propagation with a relative tolerance of 10^{-30} .

	Encounter	$\xi (\times 10^{-4} \text{ au})$	$\zeta (\times 10^{-4} \text{ au})$	$b (\times 10^{-4} \text{ au})$
Case 1	1 st	0.878	-0.632	1.82
	2 nd	1.23	4.54	4.71
Case 2	1 st	0.776	0.517	0.933
	2 nd	0.838	3.38	3.45
Case 3	1 st	0.728	0.215	0.759
	2 nd	0.769	3.42	3.51

turbing acceleration. The radius of the Earth's sphere of influence is given by:

$$R_{\text{Sol}} = a_{\oplus} \left(\frac{\mu_{\oplus}}{\mu_{\odot}} \right)^{2/5}.$$

At each integration step, if the distance between the asteroid and the planet is less than R_{Sol} , the Earth is considered as the primary body, otherwise the Sun remains the primary. In this way, the motion is split into a heliocentric phase (elliptic) and a geocentric phase (hyperbolic), both perturbed by a third-body force. The strategy is implemented for all the formulations taken into account with the exception of Cowell, in which always the Sun is always kept as the primary body.

We use the same numerical solver as in the tests presented in the previous section, i.e. the multi-step implicit solver LSODAR. Also, the software implementation is performed using the same configuration.

3.3.3 Performance metrics

As before, we estimate the computational effort required by the propagation with the number of function calls to the right-hand side of the equations of motion. As to evaluate accuracy, we consider the absolute error on the

Table 5: Post-encounter orbital elements of the asteroid for the interplanetary propagations.

	Case 1	Case 2	Case 3
a (au)	0.915	0.846	0.846
e	0.1	0.341	0.6
i (°)	32.5	31.4	43

b-plane coordinates of the second encounter. For this purpose, we obtain the modified b-plane coordinates through a reference propagation with a tolerance of 10^{-30} , and by means of the relations shown in [subsection 3.1.2](#), we compute the reference pre-encounter b-plane coordinates of the second encounter $(\xi''_{\text{ref}}, \zeta''_{\text{ref}})$.

The propagations were run by varying the solver tolerance from 10^{-8} to 10^{-23} , and the errors on the b-plane coordinates were computed as:

$$\delta\xi'' = \left| \frac{\xi'' - \xi''_{\text{ref}}}{\xi''_{\text{ref}}} \right|, \quad \delta\zeta'' = \left| \frac{\zeta'' - \zeta''_{\text{ref}}}{\zeta''_{\text{ref}}} \right|.$$

These performance metrics were chosen for the following reasons. First, since ζ'' depends mainly on the time of arrival of the asteroid at the second close encounter, any error in time will mainly affect ζ'' and not ξ'' , giving us a way to separate its contribution from the error in position. Second, we are interested in the accurate computation of the b-plane coordinates for predicting resonant returns of NEAs with the Earth.

3.3.4 Performance evaluation

[Figure 9](#) shows the results for the propagation of the trajectories with the initial conditions shown in [Table 3](#). We label as “A+B” a propagation that was performed by combining formulation A for the heliocentric phase and B for the geocentric phase.

The Cowell formulation gives the worst performance. This result confirms the advantage of using regularized formulations over the classical integration of the Newtonian equations of motion. Comparison of [Figure 9a](#) and [Figure 9b](#) reveals that the error in ζ is around 2 orders of magnitude bigger than in ξ . Thus, the overall performance of the propagation is ultimately driven by ζ , which for regularized formulations depends mainly on the error in the integration of time. We note that the addition of a time element to Dromo, which has been shown to improve the accuracy in the time variable [\[4\]](#), should also improve the accuracy in ζ . In fact, in the heliocentric phase, during which the error in time accumulates the most, the time elements always increase the performance of Dromo.

In the low eccentricity case ($e = 0.1$), the best propagation for both coordinates ξ and ζ is achieved by either a combination of Dromo(Lte) in the heliocentric phase and HyperDromo in the geocentric arc, or by the K-S formulation in both phases. In the geocentric hyperbolic motion HyperDromo reaches noticeably higher accuracies, confirming its excellent numerical performance [\[85\]](#).

Figure 9: Errors in the b-plane coordinates of the second encounter for the interplanetary propagations. The values of the post-encounter eccentricity are reported.

With higher eccentricities, element methods endowed with a time element show significant advantages. The combination of Dromo(Cte) and HyperDromo gives the best performance: it is comparable to that of the K-S regularization in the case of moderate eccentricity ($e = 0.34$), while it is superior to all the other combinations for the highest eccentricity case ($e = 0.6$).

Finally, we also considered test cases in which the initial inclination was decreased with the aim of obtaining a slower encounter, and in which the resonance ratio was $k/h = 8/7$, with the effect of changing the initial semi-major axis. The results did not vary significantly from those displayed in Figure 9.

3.4 SUMMARY

In this chapter, we investigated the possibility of reducing the numerical error in the propagation of interplanetary trajectories by making use of regularized formulations of the two-body problem. We focused on the propagation of interplanetary trajectories with at least one close encounter.

Two kinds of numerical experiments were carried out: the first consists in the propagation of geocentric hyperbolic trajectories, and the second one in the propagation of a complete interplanetary trajectory. With respect to the latter, we consider the trajectory of a fictitious asteroid with one resonant encounter with the Earth. This problem is particularly challenging due to the accumulation of the position error deriving from the divergence of initially close trajectories after the encounter.

In the hyperbolic test cases, we estimated the separation with respect to an initial position after a forwards and backwards propagation. The best performance in terms of the relative error versus the number of function calls is shown by Dromo with a constant time element for the case in which the only perturbation is due to the Earth's oblateness, and by the HyperDromo and the Kustaanheimo-Stiefel formulations if the third-body perturbation is also present.

In the interplanetary test case, we computed the numerical error as the difference in the b-plane coordinates of the second encounter with respect to those produced by a reference solution. Moreover, the propagation was carried out by switching the primary body when the propagated body entered inside the Earth's sphere of influence. For low-eccentricity heliocentric orbits, the best methods are either HyperDromo and Dromo with a linear time element for the geocentric and heliocentric propagations, respectively, or the K-S regularization when employed for the propagation of the whole trajectory. For moderate and high eccentricities, the combination of HyperDromo and Dromo with a constant time element for the geocentric and heliocentric propagations gives the best results. We remark that in both the hyperbolic and interplanetary tests with third-body perturbations the best formulations are those based on a Sundman transformation of first order.

As to compute the b-plane coordinates and deriving the initial conditions for the interplanetary test case, the transformation from the Öpik variables in the modified b-plane to those in the b-plane was derived. Moreover, we devised an approximate procedure to provide the initial conditions required to realize a resonant return with the Earth for an asteroid in a circular orbit. The proposed approach can be generalized to the case of an elliptical orbit.

When integrating numerically the equations of motion, planetary close encounters introduce a sudden spike in the magnitude of the right-hand side (i.e. the vector field describing the evolution of the state vector), which may disrupt the integration. Considerable effort has been dedicated to improving the accuracy of the solutions in such cases.

Everhart [34, 35], while working on cometary debris, developed a very efficient single-step integrator, *RADAU*, which has been extensively applied in the integration of planetary close encounters and multiple star systems. Its strength relies in the choice of Gauss-Radau spacings for the substeps, which allow to build a 15th-order numerical scheme with only seven substeps. This integrator has been recently improved by Rein and Spiegel [84] in the *IAS15* code, which achieves outstanding performances in the integration of cometary orbits affected by close encounters with Jupiter.

Dynamical astronomers have classically used symplectic numerical schemes in long-term integrations covering time spans comparable to the age of the Solar System. Due to their Hamiltonian-preserving nature they keep the energy error bounded in many situations, however the symplectic character is lost if the time step of the integration is changed. Nevertheless, several symplectic algorithms capable of efficiently handling close encounters have been developed. In the method of Chambers [20], implemented in the *MERCURY* package, a switch from a completely symplectic to a hybrid regime is applied when two bodies get close enough. A conventional integrator is used for the close encounter terms of the Hamiltonian together with those corresponding to the two-body problems with the Sun, while the other terms are solved symplectically. The method by Wisdom [103], which only treats massless *test particles*, improves on the latter by employing Jacobi coordinates rather than democratic heliocentric ones, and by using a smoother switching function. Duncan, Levison, and Lee [30] devised a variant of mixed-variable symplectic methods which applies different time steps to the individual force components during close encounters. Using universal Kepler solvers [104], Hernandez and Bertschinger [51] proposed a second-order symplectic and time-reversible integrator, whose ability to reproduce close encounters efficiently was demonstrated by Hernandez [50].

Even with non-symplectic methods, collisional problems can still be solved efficiently in the Hamiltonian formalism, especially if other properties such as time-reversibility are exploited. The method by Kvaerno and Leimkuhler [61] is based on the time-reversible integration of Hamiltonian equations. It makes use of a continuous switching function to handle close encounters; the latter are numerically integrated by a high-order scheme using Gauss-Legendre spacings. While the integration is performed with a fixed step-size, the method makes use of a first-order Sundman transformation to achieve automatic step-size regulation. Gonçalves Ferrari, Boekholt, and Portegies Zwart [47] also presented a method for the integration of the N-body problem through a splitting of the Hamiltonian into separate two-body problems. The method is not time-reversible [51], and second-order convergence is achieved by a composition of self-adjoint first-order maps.

Hamiltonian methods are ideal for integrating N-body systems in which most of the bodies are mutually interacting. Symplectic methods enable reliable integrations on time spans comparable to the age of the Solar System and are particularly suited for qualitative studies. The mathematical formulation allows for continuous splitting strategies, in which the terms of the Hamiltonian relative to particles undergoing encounters are interchanged continuously between the main and perturbing parts. However, including dissipative forces requires special care [67, 97], and poses challenges to the accuracy of the methods. Moreover, using variable step-sizes is not straightforward, and high orders of integration are cumbersome to achieve.

Here, we are concerned with formulations of the *perturbed two-body problem*, which is a special case of the N-body. We assume the existence of a primary body that exerts an acceleration significantly larger than those of the remaining $N - 1$, which can be considered as perturbing bodies. This approach accurately models the motion of an asteroid with negligible mass, and is the one commonly used in near-Earth asteroid monitoring activities. The positions and velocities of the $N - 1$ remaining bodies are assumed to be retrieved from tabulated high-precision ephemerides such as the JPL Development Ephemerides [41]. In contrast to Hamiltonian methods, the implementation of both conservative and dissipative perturbations is straightforward, and high-order adaptive solvers can be used without any particular precaution. In this chapter, the regularized formulations of the perturbed two-body problem presented in [subsection 2.3.1](#) and [subsection 2.3.3](#) are used to propagate close encounters with the Earth.

4.1 TRAJECTORY SPLITTING

In planetary close encounters, the fundamental character of the perturbed two-body problem changes on time scales that can be orders of magnitude smaller than the period of the heliocentric orbit. Almost suddenly, the gravitational acceleration from the perturbing body may become larger than that of the primary. Thus, heliocentric orbits experience strong, quasi-impulsive perturbations.

Uncertainties that are small on a Solar System scale have a significant impact on the close encounter, since the latter takes place on much smaller (planetary) time and distance scales. Moreover, trajectories which are close before the encounter diverge afterwards; this divergence can be exponential after subsequent encounters, thereby spawning chaotic dynamics. Therefore, some kind of adaptation of the integration process to the close encounter is mandatory in order to mitigate the propagation error. This can be provided by numerical solvers with variable step-size or order. The sudden growth of the magnitude of $\mathbf{F}$ in [Equation 1](#) is detected by an increase in the local truncation error, which causes the integrator to choose a shorter step-size and a higher order. Adaptive solvers, which are commonly used in Cowell's method, are here applied to regularized formulations. We show that they can successfully accommodate the change in the dynamics brought about by the encounter if the propagation is split in three phases. Far from the perturbing body, the orbit is computed in a heliocentric phase, where the primary body and the origin of the reference system is the Sun. When a certain *switch distance* R_{sw} from a major Solar System body (i.e. a planet) is reached, the integration is stopped, and the position and velocity are computed with respect to the planet. This becomes the new primary body, while the Sun is regarded as a perturber. The new perturbed two-body problem

is numerically solved until the distance R_{sw} is reached again. Then, another switch of reference systems and primary bodies is applied, so that the Sun and the planet come back to play the roles of main attractor and perturber, respectively.

The procedure described above is here called *trajectory splitting*, and it can be automatized and repeated indefinitely for multiple close encounters. Conceptually, it is similar to the analytical technique of patched conics for designing interplanetary trajectories, with the difference that any perturbation can be added in the physical model. The important effect of the proposed approach is to transform a strongly-perturbed two-body problem into a series of less perturbed two-body problems.

Note that when employing the Cowell formulation, this strategy does not bring about evident advantages. In the right-hand side of [Equation 1](#), the primary and the perturber are treated equally. Splitting the trajectory only switches their roles, but the equation formally remains the same. On the other hand, the performance of regularized formulations may be substantially improved. In the K-S formulation, a weaker perturbation produces a smaller rate of variation of the total energy, which results in more stable differential equations for the K-S parameters. Also methods based on orbital elements, like EDromo, become faster and more accurate.

There are some caveats to trajectory splitting. If the independent variable is not the physical time, stopping the propagation requires an iterative algorithm (see [subsection 2.4.1](#)). Moreover, particular care must be taken for the selection of the optimal value of the switch distance.

4.2 NUMERICAL SOLVERS

As to extensively study the performances of regularized sets of equations, we choose two solvers with different numerical schemes. Both of them are adaptive, in the sense that either just the step-size or both the step-size and the order can be changed along the integration. We adjust the accuracy by setting the value of the dimensionless parameter ϵ , which is a threshold on the magnitude of the local truncation error.

The solvers are equipped with event location algorithms, which allow us to stop the integration when a function of the independent variable reaches a desired value. They are necessary to detect close encounters and to assess the value of the physical time, if this is not the independent variable (see [subsection 4.2.3](#)).

The length of the first step of a propagation can be passed to the solver. We discuss this aspect and propose a way to get a reasonable initial step-size in the following.

The numerical schemes used by the solvers in this chapter are not time-reversible. Although time-reversibility is a desirable property for an integrator, its lack does not affect the relative strengths of the methods presented in any way.

4.2.1 LSODAR multi-step solver

Given its reliability and the good performance exhibited in the tests presented in the previous chapter, we consider LSODAR, a multi-step solver with variable step-size and order. LSODAR includes automated root-finding and scheme switching capabilities [82]. It uses the Adams-Bashforth-Moulton (ABM) method, and if the equations become stiff [80] it switches to a Back-

ward Differentiation Formula (BDF) implicit method. The step-size is regulated as to comply with absolute and relative tolerances on the local truncation error. In our simulations, these quantities are set to be both equal to the accuracy parameter ϵ . As to achieve larger step-sizes, the solver is endowed with an order-changing algorithm that selects orders of up to 12 and 5 for the ABM and the BDF methods, respectively.

4.2.2 XRA15 single-step solver

We first implemented the variable step-size 15th-order RADAU solver [35] in the XRA15 Fortran 2003 subroutine. The core of this code is the implicit, single-step numerical scheme with seven stages developed by Everhart [34]. Then, in order to improve the accuracy and robustness of the original solver, we modified the step-size control and predictor-corrector algorithms as proposed by Rein and Spiegel [84].

Everhart's RADAU can integrate either first- or second-order equations. However, Everhart [34] recognizes that "results are up to 10^6 times more accurate at the same number of function evaluations" when the position is obtained from the acceleration instead of the velocity. Therefore, in Cowell's method we directly apply XRA15 to the second-order Equation 1. On the other hand, for the K-S regularization we write a system of ten differential equations of the first order.¹

While LSODAR has been used especially in research areas different from celestial mechanics, like analytical chemistry and computational biology, RADAU was conceived as a powerful tool for orbit propagation in the presence of close encounters.

4.2.3 Event location

An *event* is said to occur at the value $s = s^*$ of the independent variable, if s^* is a root of the *event function* $g(s, \mathbf{y}(s))$, where $\mathbf{y}(s)$ is the state vector. The event location problem consists in finding the roots of g during the propagation.

In our numerical simulations we have to solve this problem for two cases. Time is a function of s in regularized methods, thus stopping the propagation at a prescribed time t_f requires to find the root of the equation

$$t(s) - t_f = 0. \quad (58)$$

Also, if the trajectory splitting is activated (see section 4.1), we want to know for which value of s the distance r from the encountered body is equal to the switch radius R_{sw} , that is

$$r(s) - R_{sw} = 0. \quad (59)$$

LSODAR has a very efficient algorithm to solve the event location problem. When a change of sign in the event function g is detected between two integration steps, LSODAR finds all the roots inside the step in sequence, starting from the leftmost [53]. The value of the root is refined iteratively using a modified *regula falsi* algorithm [23, ch. 6]. This method proved extremely reliable, and we did not find any cases in which the detection of a close encounter failed. The strict tolerances used in our tests imply that the steps taken by the solver are small enough for the event location algorithm to

¹ We did not investigate the possibility of directly integrating Equation 7.

easily locate and converge to the roots of Equation 59. For the XRA15 solver, we implemented the algorithm described in detail in subsection 2.4.1 as to perform event location.

4.2.4 Initial step size selection

In both solvers, the size of the first step of a propagation can be assigned by the user. In general, it will be either reduced or enlarged according to internal checks on the local truncation error and on the numerical stability. Note that when trajectory splitting is applied there are three propagations in the same simulation, each one with its own initial step-size. Its value must be chosen by balancing two factors. On one hand, since the ratio between the lengths of two subsequent steps is bounded, it cannot be too small, as to avoid performing a large number of steps until it stabilizes around a suitable value. On the other hand, if it is too large the solvers will reject many integration steps until it becomes small enough to satisfy the local truncation error check.

We estimate the initial step-size h_0 by means of an empirical formula which takes into account the characteristics of the motion and the accuracy requested by the integration. It is computed as

$$h_0 = -\frac{S}{n \log_{10} \epsilon}, \quad (60)$$

where ϵ is the accuracy parameter and n , S are found as follows. If the osculating orbit at the initial time of the propagation is closed we set $n = 100$, and S equal to the change in the independent variable s that corresponds to one revolution. If it is open, $n = 600$ and S is the increase in s that we have after covering the arc of the orbit contained in a sphere of radius 0.1 au and centred in the primary body.

4.3 PARAMETRIZATION OF PLANAR CLOSE ENCOUNTERS

Most of the tests performed in this chapter rely on massive numerical simulations in the planar, Sun-Earth CR3BP. We introduce a simple parametrization of planar close encounters to determine initial conditions for the integrations. Also, we use the patched conics approach to derive the heliocentric orbital elements from the encounter parameters. These elements allow us to analyse the impact of the dynamics on the amplification of the numerical error.

4.3.1 Close encounter parameters

Let us consider the epoch in which the propagated object reaches the closest point to the Earth's center of mass. A close encounter can be characterized by these three parameters: the minimum approach distance (d); the eccentricity of the geocentric hyperbola ($e > 1$); the angle ϑ between the position vector $\mathbf{d}$ and the Sun-Earth direction (see Figure 10). Two more parameters describing the hyperbolic encounter are of particular interest: the asymptotic velocity U and the angle φ ($\in [0, \pi/2]$) between the incoming (or outgo-

Figure 10: Parametrization of the close encounter through (d, e, ϑ) (section 4.3). The geocentric position vector at the closest approach distance d is in the same direction of the eccentricity vector $\hat{e}$ of the geocentric hyperbola.

ing) asymptote and the apse line of the hyperbola. By the vis-viva and conic equations, we obtain U and φ from d and e as

$$U = \sqrt{\frac{\mu_\oplus}{d} (e - 1)}, \quad (61)$$

$$\cos \varphi = \frac{1}{e}, \quad (62)$$

where $\mu_\oplus$ is the Earth's gravitational parameter. Equation 61 shows that level curves of the Keplerian energy ε are straight lines on the plane (d, e) , with steeper slopes corresponding to higher values of ε . Since $\varepsilon = U^2/2$ we speak equivalently of *high (low)* energy and *fast (slow)* encounters. Note that smaller values of d correspond to higher velocities, while the directions of the asymptotes are not affected. Moreover, higher values of e correspond to faster motion along hyperbolic trajectories that tend to be rectilinear.

4.3.2 From encounter parameters to heliocentric elements

We can gain valuable knowledge about the effect of close encounters in the planar CR₃BP by looking at the variations in the orbital elements of the heliocentric trajectory. These can be estimated with the patched conics approximation, which is based on two hypotheses: the motion is Keplerian, and the close encounter happens instantaneously.

Consider a heliocentric reference system $\{S, x, y\}$ such that the Earth lies on the x axis at 1 au from the Sun, and let $\mathbf{i}, \mathbf{j}$ be the unit vectors of the axes x, y , respectively. We are concerned with computing the energy and eccentricity before and after the encounter, because they strongly affect the numerical performance (see section 4.4.3.1). In the following we use the superscripts “ $-$ ” for the pre-encounter quantities, and “ $+$ ” for the post-encounter ones.

As shown in Figure 10, the orientation of the asymptotic velocity vectors $\mathbf{U}^-, \mathbf{U}^+$ with respect to the Earth's velocity $\mathbf{v}_\oplus$ depends on ϑ and φ . Adding $\mathbf{U}^\mp$ to $\mathbf{v}_\oplus = v_\oplus \mathbf{j}$, we obtain the expressions for the heliocentric velocities,

$$\mathbf{v}^\mp = \mathbf{U}^\mp + \mathbf{v}_\oplus = \pm U \cos(\varphi \pm \vartheta) \mathbf{i} + [U \sin(\varphi \pm \vartheta) + v_\oplus] \mathbf{j}. \quad (63)$$

By employing Equation 63 along with the vis-viva equation we can write the Keplerian energies before and after the encounter as

$$\varepsilon^\mp = \frac{U^2 - v_\oplus^2}{2} + U v_\oplus \sin(\varphi \pm \vartheta). \quad (64)$$

Figure 11: Level curves of $\Delta v(d, e)$ computed through Equation 66. It is expressed in units of Earth orbital velocity $v_\oplus$; the distance d is in Earth radii.

The corresponding eccentricities are readily obtained as

$$e^\mp = \sqrt{1 + 2 \frac{\varepsilon^\mp}{v_\oplus^2} \left(1 + \frac{U}{v_\oplus} \sin(\varphi \pm \vartheta) \right)^2}. \quad (65)$$

From Equation 63 we have that the $\Delta \mathbf{v}$ due to the encounter is

$$\Delta \mathbf{v} = \mathbf{v}^+ - \mathbf{v}^- = -2U \cos \varphi (\cos \vartheta \mathbf{i} + \sin \vartheta \mathbf{j}). \quad (66)$$

Its magnitude $\Delta v = 2U \cos \varphi$ depends only on the shape of the hyperbola. Figure 11 displays level curves of Δv over the set of pairs (d, e) that we considered in our simulations. While ϑ does not influence the rotation of the velocity vector from $\mathbf{U}^-$ to $\mathbf{U}^+$, it has instead a significant impact on the heliocentric dynamics. Indeed, from Equation 64, the energy variation results

$$\Delta \varepsilon = \varepsilon^+ - \varepsilon^- = -2U v_\oplus \sin \vartheta \cos \varphi = \mathbf{v}_\oplus \cdot \Delta \mathbf{v}, \quad (67)$$

and both open and closed orbits may be obtained by changing ϑ for fixed values of d and e .

The analytical variations of the orbital elements are in good agreement with numerical tests, and adequately describe the evolution of the particles in the limits of the patched-conics approach.

4.4 LARGE-SCALE NUMERICAL EXPERIMENTS

The performances of the formulations presented in section 2.3 are analysed in the planar, Sun-Earth CR₃BP by means of massive ensemble integrations, which are carried out with both LSODAR and XRA15 solvers. The main focus of the numerical experiments is to investigate the efficiency of the trajectory splitting strategy (section 4.1) when it is applied with regularized methods. We also study the effect of the switch distance R_{sw} : its choice must be taken with some care to avoid the build-up of truncation error and increases in the computational effort.

In the following the *propagation method* will denote the strategy, i.e. with or without trajectory splitting, the adopted formulation(s), and the solver. First, we expound on the implementation of the numerical experiments and on the quantification of the performance, then we present the results.

4.4.1 Tests description

Given d, e, ϑ (see Figure 10) we compute the geocentric position and velocity of the particle at the epoch t_{ca} of its closest approach with the Earth. Starting from them, we obtain a *reference* trajectory by integrating forwards and backwards until $t_i = t_{ca} - \Delta t/2$ and $t_f = t_{ca} + \Delta t/2$, respectively. The propagation is done in quadruple precision by applying Cowell's formulation and the LSODAR solver with a very strict accuracy parameter ϵ . The position and velocity relative to the Sun at time t_i are then set as initial conditions for the *test* simulations. These span the interval $[t_i, t_f]$, and are executed in double precision using different propagation methods. Test trajectories are compared to the reference to evaluate their accuracy, as explained in the next section. If trajectory splitting is applied, several switch distances R_{sw} are considered in order to investigate the impact of this parameter on the numerical performance of a propagation method. Our tests are built up by taking thousands of different combinations of (d, e, ϑ) .

The time interval Δt has been set equal to one year, so that we can thoroughly study *only one* close encounter. Much longer propagations would increase the probability of experiencing multiple encounters, and would require prohibitively high computational times. On the other hand, a shorter Δt would not allow an appreciable accumulation of the truncation error.

4.4.1.1 Performance metrics

We quantify the accuracy by means of the relative position and energy error metrics. The former is computed as

$$\delta r = \frac{\|\mathbf{r}_{f,\text{test}} - \mathbf{r}_{f,\text{ref}}\|}{\|\mathbf{r}_{f,\text{ref}}\|}, \quad (68)$$

where $\mathbf{r}_{f,\text{test}} = \mathbf{r}_{\text{test}}(t_f)$ and $\mathbf{r}_{f,\text{ref}} = \mathbf{r}_{\text{ref}}(t_f)$ are the heliocentric positions for the test and reference propagations, respectively, at the final time t_f . The latter is obtained as

$$\delta \varepsilon = \left| \frac{\varepsilon_{f,\text{test}} - \varepsilon_{f,\text{ref}}}{\varepsilon_{f,\text{ref}}} \right|, \quad (69)$$

where ε is the two-body energy of the asteroid with respect to the Sun, and the subscripts have the same meaning as before. The quantity $\delta \varepsilon$ can be understood as a secular drift of the asteroid position in the along-track direction, which in the long run may cause its loss. Thus, in short-term propagations the energy error is more important than the position error.

As to capture the overall behaviour of the error of a propagation method for encounter hyperbolas of different shape we average the quantities δr , $\delta \varepsilon$, over the pairs (d, e) for a fixed ϑ , as follows:

$$\bar{x}(\vartheta) = \left(\prod_{i=1}^n \prod_{j=1}^m x(d_i, e_j, \vartheta) \right)^{\frac{1}{nm}}, \quad (70)$$

where x can be either δr or $\delta \varepsilon$. As it will be shown later, the errors can vary by up to three orders of magnitude depending on the values of (d, e) . By taking the geometric average the influence of large outliers is mitigated, giving a more effective information.

As a metric for the computational cost of each propagation, we use the number of evaluations FE of the right-hand side of the equations of motion. This metric has the advantage of being machine-independent; furthermore,

it is representative of the computational time especially when the evaluation of the perturbing forces is time-consuming. We also average FE according to Equation 70.

4.4.2 Choice of the accuracy parameter

The growth of truncation error during the integration can be contained by using small values of the accuracy parameter ϵ (section 4.2), which lead to shorter integration steps. However, decreasing ϵ also results in more floating-point operations, which generate round-off error. If ϵ is small enough, the global (i.e., accumulated) round-off error will surpass the global truncation error. This situation is to be avoided, since the accuracy is degraded and the numerical scheme might go into instability.

In our simulations ϵ is chosen according to the following criterion. For each propagation method, we computed the lowest possible ϵ such that the numerical error is dominated by the truncation error. This value, which is here called *round-off limit*, is found by an iterative procedure. Each simulation is repeated using decreasing values of ϵ ,

$$\epsilon_1 > \epsilon_2 > \dots > \epsilon_R > \epsilon_{R+1}. \quad (71)$$

The global truncation error, which is measured through the average position error, also decreases until reaching a small enough ϵ ,

$$\overline{\delta r}(\vartheta; \epsilon_1) > \overline{\delta r}(\vartheta; \epsilon_2) > \dots > \overline{\delta r}(\vartheta; \epsilon_R), \quad (72)$$

$$\overline{\delta r}(\vartheta; \epsilon_R) < \overline{\delta r}(\vartheta; \epsilon_{R+1}), \quad (73)$$

for all ϑ . Equation 73 signals the entry into a round-off dominated regime. We define ϵ_R as the round-off limit, that is the value of the accuracy parameter below which the numerical error is dominated by round-off. Different propagation methods are then compared at the round-off limit. We elect the *best method* as the one that reaches the highest accuracy and with the lowest computational cost.

The round-off limit mainly depends on the solver. For LSODAR, it is around or even below machine zero in double precision. For XRA15 it is more difficult to find this value. The very high order of the solver implies that the convergence of the integration is very sensitive to the parameter ϵ : values that are just a few orders of magnitude greater than the round-off limit might lead to a very large global truncation error. Therefore, the XRA15 solver requires some calibration to find the range of ϵ that lead to stable propagations. Also, the round-off limit in XRA15 is noticeably dependent on the type of propagated trajectory, and on the perturbations. While this phenomenon does not strongly affect the numerical performances in our simulations, it should be taken into account to avoid numerical instabilities in practical applications. Algorithms that mitigate the round-off error, as those described in [84], would lower the round-off limit and thereby allow us to choose conservative values for ϵ without the need for calibration. We found a good estimate for the round-off limit to be 10^{-3} for element methods, and between 10^{-6} and 10^{-7} for coordinate-based formulations. These values are quite close to the estimates derived in [84] to reach machine precision in the integration of Newtonian equations. The specific value of ϵ that guarantees the best performance must be chosen on a case-by-case scenario, taking into account the considered perturbations and the propagation span.

The threshold on the local truncation error for most applications will be higher than the round-off limit. However, as long as the round-off error

does not dominate the integration we do not expect the relative strengths of the presented methods to change significantly. Regularized formulations ultimately work by reducing the local truncation error, therefore a gain in performance at the round-off limit ϵ_R will be reflected in propagations at any other $\epsilon > \epsilon_R$.

4.4.3 Simulations without trajectory splitting

We analyse results from large-scale propagations carried out by keeping the Sun as the primary body for the whole integration span. More specifically, we consider one million different combinations of the encounter parameters d, e, ϑ (see Figure 10) uniformly distributed over $[R_\oplus, 20R_\oplus] \times [1.01, 15] \times [0^\circ, 360^\circ]$. Note that most of the values of d and e correspond to deep and slow encounters, which are very challenging to integrate due to larger perturbations.

Our purpose is to assess the performances of regularized formulations compared to Cowell's. This analysis also allows us to gain insight on the dependence of the numerical error on the parameters (d, e, ϑ) , which affect the orbital elements of the heliocentric trajectory and the magnitude of perturbations. Understanding such dependence sheds light on the characteristics of the formulations of dynamics involved in the study, and lets us identify particularly critical close encounters.

We will only show plots of the energy error in this section, since it is the most critical for short-term propagations (section 4.4.1.1). The position error exhibits a similar behaviour.

4.4.3.1 Impact of the dynamics on the performance

The relative energy error $\delta\epsilon$ is plotted in Figures 12 and 13 as a function of d and e for propagations carried out with LSODAR and XRA15, respectively. The upper panels refer to the Cowell's formulation and the lower ones to the Kustaanheimo-Stiefel (K-S). The number of function evaluations FE of all these simulations is shown in Figures 14 and 15. We show the plots for the values $\vartheta = 180^\circ, 270^\circ$, which approximately correspond to extrema of $\delta\epsilon$ with respect to ϑ . The energy error depends on the heliocentric eccentricities before and after the encounter ($e^\pm$, Equation 65) and on the change in velocity imposed to the particle (Δv , Equation 66). The latter is strictly related to the magnitude of the perturbation generated by the close encounter. Comparing the error plots with Figure 11 shows that accuracy degrades where the Δv is larger. Moreover, observe that quasi-parabolic encounters, taking place at $e \approx 1$, imply $\Delta v \approx 0$ in the patched conics approximation, and in fact they are propagated very accurately (this is particularly evident by looking at the blue regions in the lowest part of the plots for $\vartheta = 270^\circ$). The left half of the (d, e) region corresponds to high values of $e^\pm$ and Δv . Both are detrimental to the accuracy of the propagations and increase the computational cost.

Figure 16 displays the behaviour of the average relative errors in energy and position, and of the average number of function evaluations, with respect to ϑ , computed by Equation 70. These plots reveal how the performance is influenced through the angle ϑ by the heliocentric Keplerian energy ($\epsilon^\pm$, Equation 64) and its variation $\Delta\epsilon$ (Equation 67). The former affects the computational cost: more negative values of the energy correspond to orbits with faster time scales and shorter periods. Because the time interval

Figure 12: Relative energy error $\delta\epsilon$ (Equation 69) as a function of the close encounter parameters d , e , for $\vartheta = 180^\circ, 270^\circ$ (see Figure 10). The minimum approach distance d is measured in Earth radii. LSODAR was used with Cowell's (upper panels) and Kustaanheimo-Stiefel (lower panels) formulations. The accuracy parameter is $\epsilon = 10^{-16}$.

of propagation is the same for all the simulations, the number of steps will be higher for these orbits. A faster time scale also implies sharper changes in the perturbing accelerations, which increase the cost. Since for most of close encounters φ takes values close to 90° ($\varphi > 70^\circ$ in 86% of the simulations) we have that variations in $\epsilon^\pm$ are mainly driven by variations of $\cos \vartheta$. In fact, the number of function evaluations in Cowell's formulation rises markedly around $\vartheta = 180^\circ$. The quantity $\Delta\epsilon$ affects the accuracy instead. As we could expect from Equation 67 the error exhibits smaller values when $\sin \vartheta$ is close to 0.

4.4.3.2 Comparison of the propagation methods

The combination of K-S and LSODAR allows one to obtain the best performance. Figure 16 shows that this method is two and four orders of magnitude more accurate than integrating Newtonian equations with LSODAR and XRA15, respectively, for a comparable computational cost.

The remarkable efficiency displayed by the K-S regularization vanishes when the XRA15 solver is used. Figures 13 and 15 show that K-S is computationally more expensive than Cowell for the same level of accuracy. Therefore, quite a large value of ϵ had to be chosen in Figure 16 with energy and position errors that are roughly three orders of magnitude larger. This unusual behaviour of K-S arises because in the initialization phase the XRA15 corrector loop requires many more iterations to converge with respect to Cowell. The resulting number of function evaluation becomes dominant in

Figure 13: Relative energy error $\delta\epsilon$ (Equation 69) as a function of the close encounter parameters d , e , for $\vartheta = 180^\circ, 270^\circ$ (see Figure 10). The minimum approach distance d is measured in Earth radii. `xRA15` was used with Cowell’s (upper panels) and Kustaanheimo-Stiefel (lower panels) formulations. The accuracy parameters are $\epsilon = 10^{-6}$ and $\epsilon = 10^{-7}$, respectively.

one simulation when the time interval of propagation is short, as in this case. We are not aware of any previous works in which the Everhart-Radau numerical scheme has been employed in the integration of regularized formulations, hence this issue might have come unnoticed until now.

The propagation method that, in double precision, reaches the highest accuracy as measured in terms of $\overline{\delta\epsilon}$, $\overline{\delta r}$ is Cowell’s formulation and `xRA15`. It obtains the lowest average energy and position errors, on the order of 10^{-13} and 10^{-12} , respectively. In fact, on the scale of 1 au a relative position error of 10^{-11} corresponds to sub-metric accuracy. However, the computational cost required by the method is very high and makes it uninteresting for this comparison. The combination of K-S and `LSODAR` with $\epsilon = 10^{-16}$ is only slightly less accurate, but the number of function evaluations is 2.5 times smaller.

4.4.3.3 Impact of `LSODAR` order changes on accuracy

We highlight a phenomenon that arises with the `LSODAR` solver when it is used with the Cowell formulation. For values of the accuracy parameter bigger than about 10^{-15} , the numerical error shows an erratic behaviour on the (d, e) region (see Figure 17a). This is due to how `LSODAR` handles very rapid variations in the right-hand side of the differential equations. If the predicted step-size for a given integration order is substantially reduced to satisfy the bound on the local truncation error, then the order is increased so that a larger step-size can be applied. For each simulation of Figure 17a,

Figure 14: Number of function evaluations FE for the same simulations of [Figure 12](#).

we computed the average order $\bar{q}$ over the number of integration steps. By juxtaposing the patterns of the energy error and of the average order as in [Figures 17a](#) and [17b](#), we find out that order changes have a dramatic impact on the accuracy. Apart from these considerations, choosing higher values of ϵ in LSODAR does not bring a qualitative change of the performance with respect to the dynamical characteristics of the trajectories. Finally, we carried out the same type of analysis using K-S. For this formulation the numerical error has a much smoother behaviour, and it is less dependent on $\bar{q}$.

4.4.3.4 Element formulations

We do not show here results obtained with the EDromo formulation; its performance is much less sensitive to the heliocentric eccentricity, but it is heavily affected by the magnitude of Δv . Using EDromo without performing the trajectory splitting is disadvantageous with both solvers as high values of the numerical error and function evaluations are obtained. We also noticed that LSODAR often switches to the BDF method to prevent the arising of stability issues. This signals that the right-hand side of the EDromo equations is not well-behaved if the Sun is kept as the primary body throughout all the integration. The reason is that a fast and strong perturbation on a heliocentric time-scale considerably deforms the trajectory, which cannot be well described by orbital elements. By introducing trajectory splitting it is possible to reduce the magnitude of the perturbations, thus substantially improving the performance of element formulations, as it will be shown in the next section.

Figure 15: Number of function evaluations FE for the same simulations of Figure 13.

4.4.4 Simulations with trajectory splitting

We describe the numerical experiments for the case in which the trajectory is split during the integration of a close encounter according to the procedure detailed in section 4.1.

4.4.4.1 Detection of close encounters

For *test* simulations that apply trajectory splitting we want to know when the distance r between the particle and the Earth is equal to some fixed value R_{sw} (section 4.1). First, in the *reference* integration we solve the event location problem $r(t) - R_{sw} = 0$. Since we only consider one close encounter, there exist two solutions t_1, t_2 , representing the times of beginning and end of the geocentric phase. Then, for each test simulation we solve the event location problem $t(s) - t^* = 0$, where t^* will be t_1, t_2 in sequence. In practice, the switch of the primary is done when the geocentric distance is equal to R_{sw} along the *reference* propagation². The reason for using an event function based on time rather than on distance is that the former, being a strictly increasing function of s , is more convenient for the application of an algorithm for root-finding as the one described in subsection 2.4.1.

4.4.4.2 Impact of the switch distance on the performance

The primary body is switched at a *suitable* geocentric distance R_{sw} along the trajectory. Ideally, we would like R_{sw} to be optimal in the sense of min-

² For a generic test simulation the geocentric distance at $t = t_1, t = t_2$ will differ by some small amount from R_{sw} . This does not affect considerations on the performance of the methods as a function of R_{sw} .

Figure 16: Average relative energy error $\overline{\delta\epsilon}$, relative position error $\overline{\delta r}$, and number of function evaluations $\overline{FE}$ as functions of ϑ (Equation 70). The labels “Cow” and “K-S” refer to the Cowell’s and Kustaanheimo-Stiefel formulations. The letters “L” and “R” stand for the LSODAR and XRA15 numerical solvers. The accuracy parameters ϵ for K-S(R), Cow(L), Cow(R) are chosen so that the computational cost is comparable to the one of K-S(L). We show for this method the best possible performance working in double precision.

Figure 17: Relative energy error $\delta\epsilon$ (Equation 69) and average integration order $\bar{q}$ as a function of the close encounter parameters d , e for $\vartheta = 180^\circ$ (see Figure 10). The minimum approach distance d is measured in Earth radii. LSODAR was used with Cowell’s formulation and the accuracy parameter is $\epsilon = 10^{-8}$.

imizing the numerical error and the computational cost. Finding such a switch distance is not trivial: it should be neither too large nor too small, since strong third-body perturbations would arise in the geocentric and heliocentric propagation phases, respectively. This is the rationale which is often used to justify the change of the origin of the coordinate system at the sphere of influence, or at the Hill sphere, whose radii for the Earth are approximated by R_{SoI} and R_{Hill} , respectively,

$$R_{\text{SoI}} = a_{\oplus} \left(\frac{\mu_{\oplus}}{\mu_{\odot}} \right)^{2/5} \approx 0.0062 \text{ au}, \quad (74)$$

$$R_{\text{Hill}} = a_{\oplus} \left(\frac{\mu_{\oplus}}{3\mu_{\odot}} \right)^{1/3} \approx 0.01 \text{ au}. \quad (75)$$

However, these criteria do not take into account the characteristics of the numerical scheme and of the formulation, hence they might not guarantee the best efficiency. For instance, in regularized schemes the perturbing

Figure 18: Average relative energy error $\overline{\delta\epsilon}$, relative position error $\overline{\delta r}$, and number of function evaluations $\overline{FE}$ as functions of the switch distance R_{sw} for four values of ϑ (Equation 70). The Kustaanheimo-Stiefel formulation is used with LSODAR and the accuracy parameter is $\epsilon = 10^{-15}$. The dashed grey line corresponds to one radius of the sphere of influence.

Figure 19: Same as Figure 18. Here EDromo is used in the heliocentric propagation and Kustaanheimo-Stiefel in the geocentric one. The numerical solver is LSODAR and the accuracy parameter is $\epsilon = 10^{-15}$.

force does not enter the right-hand side of the differential equations alone, because it is multiplied by functions of the state vector.

The optimal switch distance is searched for numerically as follows. We consider the same pairs (d, e) as for the simulations without trajectory splitting. The average errors $\overline{\delta\epsilon}$, $\overline{\delta r}$, and function evaluations $\overline{FE}$ are computed from Equation 70 for $\vartheta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$ and 100 values of R_{sw} in the range $[0.2, 8] \times R_{Sol} \approx [0.0012, 0.049]$ au. Many propagation methods are included in the comparison, with the possibility of adopting distinct formulations for the heliocentric and geocentric phases. Only the performance of the best methods is shown here.

Figure 18 displays results for propagations done with LSODAR and the K-S formulation, while in Figure 19 EDromo is employed (with a constant time element) instead of K-S when the origin of the coordinate system is the Sun. The energy and the position errors are not affected by R_{sw} if this is greater than 0.01 au, while for shorter distances the errors may rise by up to one order of magnitude. On the other hand, with $\vartheta = 90^\circ, 180^\circ, 270^\circ$ the computational cost shows pronounced minima for values of R_{sw} between 0.01 and 0.025 au. The case $\vartheta = 0^\circ$ deserves further attention: in this geometry, most of the particles approach the Earth's orbit nearly tangentially. A larger period of the heliocentric orbit is spent in a highly perturbed region close to the Earth, so the change of primary body is more convenient at a larger distance. In general, switch distances smaller than $1 R_{Sol}$ must be avoided

Figure 20: Same as Figure 18. Here EDromo is used in the heliocentric propagation and Cowell in the geocentric one. The numerical solver is XRA15 and the accuracy parameter is $\epsilon = 10^{-3}$.

since they cause the computational cost to increase of up to 60%, without bringing any improvement of the accuracy.

The best performance with the XRA15 solver is obtained by employing EDromo in the heliocentric phase and Cowell in the geocentric. The behaviours of the errors and computational cost shown in Figure 20 depart from those obtained with LSODAR. The position and energy errors exhibit an evident “plateau” for R_{sw} approximately between 1 and 3 Hill radii; outside of this region, they may increase by up to two orders of magnitude. The number of function evaluations decreases steadily with R_{sw} , only showing signs of growth near the highest values of the considered range. We note that while $\epsilon = 10^{-3}$ is the round-off limit only for EDromo, it does not allow us to get the best accuracy with Cowell’s formulation. This is the reason why for R_{sw} bigger than $3 R_{\text{Hill}}$ the error accumulated in the geocentric propagation starts to deteriorate the accuracy of the whole integration. If a stricter accuracy parameter is used, as $\epsilon = 10^{-5}$, the error plateau is extended but the number of function evaluations, for the same accuracy, becomes comparable to that of Cowell’s method without trajectory splitting.

As a simple rule of thumb, splitting the propagation between 1.2 and 3 Hill radii yields the best numerical performances. On the other hand, by choosing one radius of the sphere of influence we do not achieve, in general, the best performance with regularized methods. Note that although the main features of close encounters are well replicated in the CR3BP, the value of R_{sw} maximizing the performance must be chosen depending on the particular problem.

4.4.4.3 Comparison of the propagation methods

We analyzed the performances of the propagation methods that apply the trajectory splitting in terms of accuracy and computational cost for several values of ϑ in the range $[0^\circ, 360^\circ]$, as previously done in Figure 16. Taking advantage of the results shown in subsection 4.4.4.2 we selected the value 0.015 au for the switch distance because it guarantees an almost optimal efficiency in all the simulations. The best methods at the round-off limit are the pairs EDromo, K-S with LSODAR ($\epsilon = 10^{-15}$), and EDromo, Cowell with XRA15 ($\epsilon = 10^{-3}$). In particular, while the latter is more accurate, the former is computationally faster. In Figures 21 and 22 we separately show their performance together with other propagation methods, which do not split the trajectory and adopt the same solver, for a comparable average number of function evaluations. Concerning the integration of K-S equations with

Figure 21: Performance comparison of several propagation methods with the LSODAR solver in terms of the average relative energy and position errors ($\overline{\delta\epsilon}$, $\overline{\delta r}$) and number of function evaluations ($\overline{FE}$). The curves labeled “Cow” and “K-S” refer to propagations with Cowell’s and Kustaanheimo-Stiefel formulations without performing any change of the primary. In the other two methods that we consider the trajectory is split at $R_{SW} = 0.015$ au with K-S employed in the geocentric propagation and either K-S or EDromo in the heliocentric phase. The accuracy parameters ϵ for Cow, K-S, K-S + K-S are chosen so that the computational cost is comparable to the one of EDr + K-S. We show for this method the best possible performance working in double precision.

Figure 22: Same as Figure 21 but here we consider the xRA15 solver. Cowell’s method with the Sun at the origin of the coordinate system is compared to EDromo and Cowell’s formulations employed in the heliocentric and geocentric phases, respectively, with $R_{SW} = 0.015$ au. We show for EDr + Cow the best possible performance working in double precision.

LSODAR, changing the primary improves the accuracy by three orders of magnitude. When K-S is employed only in the geocentric propagation and EDromo in the heliocentric one, the errors are reduced by four orders of magnitude instead. The mitigation of the error in the pre-encounter phase afforded by this element formulation is highly beneficial to numerical efficiency. For the xRA15 solver, using EDromo and Cowell when the Sun and the Earth are, respectively, the primary body enhances the accuracy by two orders of magnitude with respect to Cowell’s formulation without trajectory splitting. The use of K-S instead of Cowell in the geocentric phase does not lead to a further improvement for this solver. It is remarkable that although regularized methods have an overhead due to the event location algorithm implemented in xRA15, EDromo exhibits an excellent numerical performance. Finally, note that for a similar computational effort of the combination EDromo, K-S in Figure 21, Cowell’s method loses up to six more digits in the correct final position (see Figure 22). In fact, the energy error oscillates violently between 10^{-5} and 10^{-10} , indicating that the exact evolu-

tion of some trajectories is not even coarsely reproduced by the numerical integration.

4.4.4.4 Regularized element methods for hyperbolic motion

We also carried out simulations adopting the HDromo scheme, described in [3], in the geocentric phase. However, some trajectories exhibit peculiar dynamical phenomena which complicate its use with trajectory splitting. Therefore, we do not show any results, and we only explain the issues that arise. For slow encounters the Keplerian energy along the geocentric motion may attain a negative value at some point, leading to a temporary capture. In this case the formulation fails. Another problematic event, which may occur in the highly perturbed region far from the Earth, is when the angular momentum vanishes. The method incurs in a singularity and the whole propagation fails. It is always possible to avoid these scenarios by setting a sufficiently small switch radius. However, by proceeding in this way we are not sure to minimize the numerical error, therefore the performance analysis would get more difficult.

4.5 LONG-TERM PROPAGATION OF (99942) APOPHIS

In the previous section we dealt with short-term, bidimensional propagations in the planar CR₃BP. We extend our analysis to the long-term, three-dimensional propagation of a NEA. As a test case of particular interest for impact monitoring activities, we choose the asteroid (99942) Apophis. This NEA will experience a close encounter with the Earth in April 2029 at a minimum distance d of $6 R_{\oplus}$. Such a small value of d corresponds to a very deep encounter, which may be followed by possible resonant returns. We use the same physical model as in the previous section, that is the Sun-Earth CR₃BP. Note that an actual impact risk analysis would require a more refined physical model, as in Farnocchia et al. [36].

4.5.1 Reference trajectory

The *reference* trajectory is obtained with the same procedure described in subsection 4.4.1. The position and velocity at the epoch of the closest approach are given by the nominal solution available in JPL's HORIZONS system³. The total duration of the propagation is 100 years, which is the time interval habitually considered for impact monitoring activities. We checked that the main features of the 2029 encounter are replicated in our propagation. In particular, we reproduce Apophis' jump from the Apollo to the Atens group.

4.5.2 Amplification of round-off error

The amplification of an initial round-off error after the close encounter imposes a limit to the maximum accuracy achievable by the numerical integration. As to quantify this lower bound, we carry out another propagation in which the magnitude of the initial velocity v_i , relative to the Sun, is perturbed as follows:

$$\tilde{v}_i = v_i (1 + p), \quad p = 3\epsilon_{dp}, \quad (76)$$

³ URL: <http://ssd.jpl.nasa.gov/?horizons>, last visited 22 November 2016.

where $\epsilon_{\text{dp}} = 2.22 \times 10^{-16}$ is the double-precision machine zero. The propagation is performed in quadruple precision and with the same formulation, solver and accuracy parameter of the reference one. Hence, the difference between the two trajectories is due to the effect of the encounter on the initial deviation in velocity.

By never performing propagations under the round-off limit defined in [subsection 4.4.2](#), we ensure that the orders of magnitude of the measured errors are not affected by round-off error. However, understanding its behaviour for both solvers may be especially important for highly accurate propagations on time spans longer than a century. We estimated the rate of growth of round-off error by propagating (99942) Apophis in the unperturbed case for up to 10^6 integration steps, and by monitoring the error in the two-body orbital energy. When LSODAR is employed with the coordinate-based formulations Cowell and K-S, the round-off error accumulates linearly in time. If XRA15 is used instead, round-off grows as the square root of time with the Cowell formulation, thus following Brouwer's law [14]. This result is in agreement with that obtained by Rein and Spiegel [84] for the IAS15 solver. Thus, we discover that the achievement of Brouwer's law in IAS15 is not exclusively due to the suppression of round-off error by compensated summation, since this device is not implemented in XRA15. When the K-S formulation is integrated with XRA15, the round-off error again grows linearly.

If the EDromo formulation is employed with a constant time element, there is no round-off accumulation in the unperturbed problem since the right-hand side of Equations 26–32 and of Equation 36 is identically zero. Integrating Equation 35 for the linear time element with LSODAR again gives rise to a linear drift in the orbital energy, but with the XRA15 solver the error is kept within machine precision. The reason for this quite interesting result might be connected to the very high order of the solver, but it is not yet completely understood.

4.5.2.1 Considerations on chaos

Analogously to Equation 76, we have also perturbed the single components of the initial velocity and position for several values of p , with $3\epsilon_{\text{dp}} < p < 10\epsilon_{\text{dp}}$. In all of the tests, the distance δr_f between the perturbed and the reference positions at the end of the propagation does not exceed 10^{-5} au. In both the pre-encounter and post-encounter phases, the distance δr between the perturbed and reference trajectories grows linearly. Moreover, the close encounter amplifies the δr by the same factor for all values of p . Therefore, the magnitude of δr_f is proportional to p . In fact, while the pre-encounter distance is strongly amplified due to the encounter, the regular evolution of the perturbed trajectories indicates that the problem is not chaotic on the considered time span. While a single close encounter results in a linear divergence of initially close trajectories, chaotic dynamics only arise due to sequences of close encounters through the mechanism mentioned in [chapter 3](#). Thus, we do not expect the dynamics to be chaotic on this time span.

4.5.3 Performance analysis

We run in double precision *test* simulations starting 50 years before the 2029 close encounter and covering a time interval of 100 years. Several propagation methods are considered with LSODAR and XRA15. We include in our

comparison Cowell's formulation, keeping the Sun always at the origin of the coordinate system and regularized schemes that are used with trajectory splitting at $R_{sw} = 0.016$ au. More specifically, the Kustaanheimo-Stiefel regularization (K-S) is employed in the geocentric propagation, and either K-S or EDromo in the heliocentric phases. For EDromo the linear time element is implemented rather than the constant one, because it is more efficient for long-term propagations [4].

We measure the accuracy of the propagations by computing the errors in position and velocity with respect to the reference. Moreover, we take into account the relative error on the Jacobi constant J [1, p. 373],

$$\delta J(t) = \left| \frac{J(t) - J_i}{J_i} \right|, \quad (77)$$

where $J_i = J(t_i)$ is the initial value. We estimate the computational cost by measuring the number of function evaluations and the computational time. The latter is computed as the average of 10 identical runs on an Intel i7 quad-core machine at 3.10 GHz, using Linux Mint 18 and the gfortran 5.4.0 compiler without optimizations.

The numerical error and computational cost for each of the tests are displayed in Table 6. At the round-off limit (subsection 4.4.2) the propagation method that reaches the highest accuracy is the pair EDromo, K-S for both numerical solvers. When compared to Cowell's method the position and velocity errors at the end of the propagation are two orders of magnitude smaller. We also report the performance of Cowell's formulation and the pair K-S, K-S for a comparable number of function evaluations to that of the best method. Cowell's final position error is in the order of 1 au, while for the combination EDromo, K-S it can be as low as 100 m. This value is even smaller than the difference between the reference trajectory and the one starting with the perturbed initial velocity. It is also notable that by employing EDromo instead of K-S in the heliocentric propagation the position accuracy is improved by three and six orders of magnitude when using LSODAR and XRA15 respectively.

The errors on position, velocity and Jacobi constant of the propagation methods as a function of time are displayed in Figures 23, 24, and 25, respectively. Since the 2029 close encounter amplifies the error in position and velocity by a factor 10000, the best way to reduce the final error is by improving the accuracy in the computation of the pre-encounter arc of the trajectory. EDromo achieves the lowest pre-encounter errors in position, velocity and on the Jacobi constant. In particular, the position error is kept around 10^{-12} au or 10^{-13} au depending on the numerical solver. It is difficult to find a reliable fit to the errors with a power law in the pre-encounter part of the propagation. This is due to the fact that the asteroid only completes 63 revolutions from 1979 to 2030, thus the fit is not performed on a very large time span.

Contrary to the errors in position and velocity, the error on the Jacobi constant does not show a significant increase after the close encounter. However, in impact monitoring activities it is the position of the asteroid the quantity that must be known most accurately. Therefore, the quality of numerical integrations should be measured by using errors on the fast variables, i.e. position and velocity.

In all the simulations in which trajectory splitting is applied the number of function evaluations relative to the geocentric propagation accounts for less than 3 % of the total. As long as the error accumulated in this portion

Table 6: Numerical error and computational cost for propagations of 100 years of the asteroid (99942) Apophis. The errors δr , δv , δJ are respectively in position, velocity and on the Jacobi constant (the latter is a relative error). The computational cost is shown by the number of function evaluations and the CPU time. The accuracy parameter of EDr + K-S and Cow corresponds to the round-off limit, while for K-S + K-S and Cow* it is chosen to have a computational cost similar to that of EDr + K-S. The last two rows refer to the conservative Bulirsch-Stoer integrator and the hybrid method of the MERCURY software, which are ran at the round-off limit. For these solvers, ϵ is the tolerance of the Bulirsch-Stoer integrator. The time step of the mixed-variable symplectic integrator is 0.1 days.

Solver	Formulation	$\log_{10} \epsilon$	δr (au)	δv (au/day)	δJ	FE	CPU time (s)
LSODAR	EDr + K-S	-16	4.00×10^{-8}	3.15×10^{-11}	1.01×10^{-15}	16980	0.134
	Cow	-15	1.65×10^{-6}	1.29×10^{-9}	1.11×10^{-13}	60881	0.178
	Cow*	-8	0.85	6.10×10^{-4}	1.55×10^{-6}	17364	0.0660
	K-S + K-S	-13	3.21×10^{-5}	2.53×10^{-8}	2.16×10^{-12}	17391	0.0934
XRA15	EDr + K-S	-3	9.61×10^{-10}	7.57×10^{-13}	4.81×10^{-16}	49077	0.159
	Cow	-7	3.53×10^{-8}	2.78×10^{-11}	9.54×10^{-16}	110013	0.251
	Cow*	-1.4	1.01	6.83×10^{-4}	2.19×10^{-5}	50854	0.0856
	K-S + K-S	-1.5	7.47×10^{-4}	5.93×10^{-7}	9.32×10^{-11}	48260	0.128
BS2	Cow	-14	8.29×10^{-9}	1.11×10^{-10}	3.82×10^{-13}	66875	0.333
Hyb	—	-14	1.51×10^{-5}	2.04×10^{-7}	5.63×10^{-12}	—	0.658

Figure 23: Time evolution of the position error of several methods for the long-term propagation of the asteroid (99942) Apophis. The curve “Pert” is obtained in quadruple precision and with a tiny variation of the initial velocity (more details are given in subsection 4.5.2). The other curves refer to the same simulations of Table 6. Cow and EDr + K-S represent the highest accuracy that one can get from Cowell’s formulation (keeping the Sun as primary body) and the combination EDromo, K-S with the latter employed in the geocentric propagation. If we use K-S also in the heliocentric phase instead of EDromo, the resulting error is shown by K-S + K-S for a computational cost similar to that of EDr + K-S. The errors of the K-S + K-S method for the XRA15 solver are multiplied by the factor 10^{-3} .

does not compromise the accuracy of the post-encounter heliocentric phase, the choice of the formulation to be used when the Earth is the primary body is of secondary importance.

4.5.3.1 Propagation efficiency

For all the methods and for both solvers considered, we repeated the propagation of (99942) Apophis with decreasing values of the accuracy parameter ϵ . The final errors and number of function evaluations are displayed in Figures 26, 27 and 28.

Trajectory splitting with EDromo in the heliocentric phase and K-S in the geocentric achieves the highest efficiency. It always exhibits a number of function evaluations from two to three times smaller than the Cowell formulation for both solvers. In particular, for the propagations performed at the round-off limit, the computational cost decreases by the factors 3.6 and 2.2 with LSODAR and XRA15, respectively (Table 6). The advantage from using EDromo is particularly relevant for a solver of very high order, such as XRA15. In fact, the local truncation error is proportional to the $(k + 1)$ -th derivative of the right-hand side of the equations of motion. The smoothing of the right-hand side achieved by using EDromo equations results in a much smaller value of this derivative. The higher the order of the solver k , the more the local truncation error is consequently reduced.

Using the K-S formulation in the heliocentric phase is also advantageous in combination with the LSODAR solver. For coarse accuracies the correspond-

Figure 24: Time evolution of the velocity error of several methods for the long-term propagation of the asteroid (99942) Apophis. The description of the curves is the same as in Figure 23.

ing method shows similar performances to EDromo, however the latter allows one to reach higher accuracies. The K-S formulation is less efficient when used in combination with XRA15, having a number of function evaluations that is intermediate between that of Cowell’s method and EDromo.

Note that the smallest values of ϵ that we considered are smaller than the round-off limit, as to illustrate the entry into a round-off-dominated regime. As outlined in subsection 4.4.2, round-off error starts dominating when the computational cost increases while the accuracy worsens. Using trajectory splitting with the regularized formulations also makes the integration more robust with respect to the accumulation of round-off error. The number of function evaluations after the round-off limit is passed (leftmost points in Figures 26, 27, 28) is not as high as in the case of the Cowell formulation.

4.5.3.2 Comparison with the MERCURY code

Having assessed that the method using EDromo is the most efficient among those examined, we compare it to the Bulirsch-Stoer method for conservative equations (BS) [81, ch. 16] and the hybrid symplectic method of Chambers [20], which are implemented in the MERCURY package⁴. The first method integrates second-order equations in heliocentric Cartesian coordinates using an approach based on Richardson extrapolation. The latter uses a second-order, mixed-variable symplectic scheme to integrate the dynamics with the Sun as a primary body. A close encounter starts when the distance between two bodies decreases below a *changeover radius* R_{crit} , fixed at the beginning of the propagation. During a close encounter, the dynamics are integrated at high precision using the BS method. The switch between the integrator is handled through a continuous function, which ensures the formal symplecticity of the switching algorithm [20, 103]. We propagate at the maximum available accuracy by decreasing the time step and the BS tolerance until the errors in position and velocity start increasing, with a

⁴ URL: <http://star.arm.ac.uk/~jec/mercury/mercury6.tar>, last visited 5 April 2017

Figure 25: Time evolution of the error on the Jacobi constant of several methods for the long-term propagation of the asteroid (99942) Apophis. The description of the curves is the same as in Figure 23. Some of the curves are multiplied by the factors displayed in the plots.

procedure analogous to that of subsection 4.4.2. We find that the round-off limit is reached with a time step of 0.1 days for the mixed-variable symplectic method and a tolerance of 10^{-14} for the BS method. We use the default value of 3 Hill radii for the changeover radius.

The numerical error and computational cost for the tests are reported in Table 6. We do not count the number of function evaluations for the hybrid method, and only report its computational time. In fact there are two choices to consider for the count of the function evaluations in the mixed-variable symplectic scheme, i.e. the evaluation of the Keplerian term and of the interaction term of the Hamiltonian, but choosing either of these does not result in an accurate metric of computational cost.

The BS method is four orders of magnitude more accurate in position and two times less computationally expensive than the hybrid, and appears to be competitive for this problem. However, it does not achieve the performance of the EDromo method, which has a similar accuracy in position with a computational cost that is 30 % lower when used with the XRA15 solver. The latter also requires a computational cost four times lower than the hybrid method while being four orders of magnitude more accurate, and keeps the final error on the Jacobi constant close to machine zero.

Figure 29 shows the time history of the errors. Note that the time span of the integration is probably short enough for the global truncation error of conventional methods (such as BS) to be lower than the one of the symplectic, even if the latter is bounded on the Hamiltonian. The BS method exhibits the smallest error jumps due to the close encounter, although it does not realize the lower final errors. This suggests that it might be applied efficiently to the integration of other formulations; however a full characterization of the BS method is outside the scope of this chapter.

Finally, we emphasize that the integrators in the MERCURY package have been recently shown to be not numerically symplectic [50]. In this case, the

Figure 26: Function evaluations versus final position error for several methods for the long-term propagation of the asteroid (99942) Apophis. The names of the curves refer to simulations performed with Cowell’s formulation (curve “Cow”), K-S in the geocentric phase and either K-S or EDromo in the heliocentric phase (curves “K-S+K-S” and “EDr+K-S”). For the LSODAR solver, the curves were obtained by decreasing ϵ from 10^{-3} to 10^{-19} . For the XRA15 solver, ϵ was decreased from 0.01 to 10^{-8} when using Cowell’s formulation and to 10^{-4} for the others.

lack of symplecticity does not appear to be concerning due to the satisfying accuracy in conserving the Jacobi constant.

4.6 SOFTWARE IMPLEMENTATION

The large-scale numerical experiments were performed with the NAPLES (Numerical Analysis of PLANetary encounterS) Fortran 2003 code. To facilitate the reproducibility of the results, we make it available online as a GitHub repository, <https://github.com/dkamato/NAPLES>. The XRA15 solver and its auxiliary files are also contained in the code.

4.7 SUMMARY

In this chapter, we improved numerical performance in the orbit computation of objects that experience planetary close encounters. We propose a strategy that consists in changing the origin of the coordinate system from the Sun to the encountered planet when the distance between the particle and the planet is smaller than a given value R_{SW} . Therefore we have a sequence of perturbed two-body problems, and each of them is solved by integrating regularized equations of motion with an adaptive solver. We consider the Kustaanheimo-Stiefel regularization (K-S) and EDromo, a formulation based on non-singular orbital elements. The numerical solvers that we employ are a multi-step and a single-step scheme, denominated LSODAR and XRA15, respectively. The latter is an improvement on Everhart’s RADAU integrator of 15th-order [84], and it includes an event location algorithm that we have developed for propagations with regularized formulations.

A systematic study is carried out through ensembles of short-term propagations in the planar Earth-Sun CR3BP. Each propagation is parametrized by three quantities, which univocally define a trajectory. In this way we simulate many different kinds of close encounters. Concerning propagations carried out without splitting the orbit, K-S shows an excellent performance

Figure 27: Function evaluations versus final velocity error for several methods for the long-term propagation of the asteroid (99942) Apophis. The description of the curves is the same as in Figure 26. The points are obtained by varying ϵ .

Figure 28: Function evaluations versus final error on the Jacobi constant for several methods for the long-term propagation of the asteroid (99942) Apophis. The description of the curves is the same as in Figure 26. The points are obtained by varying ϵ .

when the numerical solver is LSODAR. For the same computational cost, this method increases the accuracy by up to four orders of magnitude with respect to the integration in Cartesian coordinates by XRA15. We also note that it is not convenient to combine this solver with the K-S formulation in short-term simulations due to the high number of function evaluations required by the initialization phase. The splitting strategy allows us to obtain the best performances. Use of EDromo in the heliocentric propagation and K-S in the geocentric one, both with LSODAR, is much more accurate than Cowell's method for the same computational effort.

We also studied the impact of the switch radius R_{sw} on the numerical efficiency. With an empirical analysis, we find out that both accuracy and cost of computation are improved if the primary body is changed between 1.2 and 3 Hill radii, rather than at the sphere of influence. The exact value of the switch radius maximizing the performance must be chosen depending on the specific physical problem.

We extended our investigation to the long-term propagation of the asteroid (99942) Apophis, which undergoes a very deep close encounter with the

Figure 29: Time evolution of the errors on position, velocity and Jacobi constant for the long-term propagation of the asteroid (99942) Apophis. The curves “EDr + K-S” are the same as those relative to the `xRA15` solver in Figures 23 to 25. The curves “BS2” and “Hyb” are obtained at the round-off limit with the conservative Bulirsch-Stoer integrator and the hybrid method implemented in the MERCURY code. Spikes in the error on the Jacobi constant take place when the distance from the Earth is less than 0.3 au.

Earth in 2029. The propagation method that in double precision achieves the highest accuracy is given by the combination of EDromo and K-S with the XRA15 solver. The former is employed to propagate the motion when the primary body is the Sun, the latter when it is the Earth. In our simplified model, this method leads to a position error of 100 meters after 100 years. It is notable that the best propagation with Cowell's method is two orders of magnitude less accurate with a computational cost that is 3.6 and 2.2 times higher, depending on the solver. Also, the combination of EDromo and K-S with XRA15 is four orders of magnitude more accurate than the hybrid method implemented in the MERCURY package, with a computational cost four times smaller. We assess that the latter method is better suited for qualitative studies, in which the time spans of integration may be several orders of magnitude larger than the ones considered.

We stress that adaptive solvers represent the best option to integrate Newtonian equations in strongly-perturbed problems. The advantages of regularizations with respect to Cowell's formulation are much greater if fixed step-size, fixed order numerical solvers are used.

Applications beyond the problem of asteroid propagations in the Solar System (e.g. planetary formation) are possible. However, additional work will be needed to address possible limitations of the method under extreme conditions. For instance, in case of very fast encounters due to retrograde objects, one may need to consider a more sophisticated switch criterion. For a forthcoming work, we are investigating the possibility of adopting a smoother criterion based on the estimated local truncation error, rather than on the position. Finally, we expect that integrating regularized formulations with the Bulirsch-Stoer solver would achieve an outstanding numerical performance due to its efficiency. This assertion will be verified through additional numerical tests in a further work.

Part II

EARTH SATELLITE ORBITS

The understanding of the debris environment in near-Earth space requires efficient and scalable orbit propagation methods. In operational debris models developed by space agencies, 10^4 to 10^6 objects are propagated for hundreds of years in hundreds of Monte Carlo simulations, curbing the complexity of the physical model and of the propagation scheme. In specific studies for satellite constellations and in numerical explorations of the near-Earth dynamical environment, the number of objects is reduced but the complexity of the model increases correspondingly. Conventionally, semi-analytical propagators, based on the method of averaging, have been used for these types of simulations.

The method of averaging consists in the elimination of high-frequency components from the equations of motion by averaging over a short time scale. The resulting averaged equations, containing only secular and long-periodic terms, are more amenable to numerical integration. In fact, if averaging is correctly performed, the time scale of the problem changes from the one characteristic of the orbital period to that of the perturbations. The latter can be much slower; for instance, for a MEO satellite the orbital period is in the order of half a day, while the shortest time scale of the perturbations is in the order of the lunar month. Thus, the integrator is able to take step sizes that are about 50 times larger. It is widely acknowledged that this process allows a reduction of about one order of magnitude in computational time.

While being relatively simple conceptually, the method of averaging may require considerable manipulation of the equations of motion. By analysing the frequencies connected with any given perturbation, this has the added benefit of giving insight on the nature of the dynamical system. In fact, a thorough understanding of the dynamical characteristics is required in order to optimize the performance of semi-analytical methods. Care has to be taken in choosing a mathematical model that is able to adequately represent the effect of all the considered perturbations, the more so in particular dynamical configurations such as resonances. The numerical parameters affecting the integration of the equations of motion have to be selected after these considerations.

In this chapter, we will first summarize the theory of averaging in equinoctial elements as presented by Danielson et al. [24]. This theory underlies many of the most widely known semi-analytical propagators such as the Semi-analytical Tool for End-of-life Analysis (STELA) and the Draper Semi-analytical Satellite Theory (DSST). We will carry out a qualitative analysis of the integration error in semi-analytical theories, and highlight the differences with respect to the accumulation of numerical error in non-averaged methods. Finally, we will give some guidelines for optimizing the performance of semi-analytical propagators, which we will use when comparing them to efficient non-averaged techniques in the next chapter.

5.1 THE METHOD OF AVERAGING

5.1.1 Osculating equations of motion

We show the equations of motion for the set of equinoctial elements $\mathbf{E}$ as presented by Giacaglia [42] and Nacozy and Dallas [74], which is expressed in terms of the classical orbital elements as

$$\mathbf{E} = \begin{pmatrix} a \\ h \\ k \\ p \\ Q \\ \lambda \end{pmatrix} = \begin{pmatrix} a \\ e \sin(\omega + \Omega) \\ e \cos(\omega + \Omega) \\ \sin \frac{i}{2} \cos \Omega \\ \sin \frac{i}{2} \sin \Omega \\ M + \omega + \Omega \end{pmatrix} \quad (78)$$

This set only presents a singularity for $i = 180^\circ$, a case that is often not interesting for practical applications. The expression for the osculating equations of motion in Lagrange form is provided by Giacaglia [42]. Letting $\gamma = \sqrt{1 - h^2 - k^2} = \sqrt{1 - e^2}$, the rates of change of the osculating elements are:

$$\begin{aligned} \frac{da}{dt} &= \frac{2}{na} \mathcal{R}_\lambda \\ \frac{dh}{dt} &= -\frac{\gamma}{na^2(1+\gamma)} h \mathcal{R}_\lambda + \frac{\gamma}{na^2} \mathcal{R}_k + \frac{1}{2na^2\gamma} k (P \mathcal{R}_P + Q \mathcal{R}_Q) \\ \frac{dk}{dt} &= -\frac{\gamma}{na^2(1+\gamma)} k \mathcal{R}_\lambda - \frac{\gamma}{na^2} \mathcal{R}_h - \frac{1}{2na^2\gamma} h (P \mathcal{R}_P + Q \mathcal{R}_Q) \\ \frac{dP}{dt} &= -\frac{1}{2na^2\gamma} P \mathcal{R}_\lambda - \frac{1}{4na^2\gamma} \mathcal{R}_Q + \frac{1}{2na^2\gamma} P (h \mathcal{R}_k - k \mathcal{R}_h) \\ \frac{dQ}{dt} &= -\frac{1}{2na^2\gamma} Q \mathcal{R}_\lambda + \frac{1}{4na^2\gamma} \mathcal{R}_P + \frac{1}{2na^2\gamma} Q (h \mathcal{R}_k - k \mathcal{R}_h) \\ \frac{d\lambda}{dt} &= n - \frac{2}{na} \mathcal{R}_a + \frac{\gamma}{2na^2} (k \mathcal{R}_k + h \mathcal{R}_h) + \frac{1}{2na^2\gamma} (P \mathcal{R}_P + Q \mathcal{R}_Q) \end{aligned} \quad (79)$$

Denoting with E_i ($i = 1, \dots, 6$) a generic element, we take into account both conservative and dissipative perturbations in $\mathcal{R}_{E_i}$:

$$\mathcal{R}_{E_i} = \frac{\partial \mathcal{R}}{\partial E_i} + \mathbf{q} \cdot \frac{\partial \mathbf{r}}{\partial E_i}.$$

The disturbing function $\mathcal{R}$ takes into account perturbations due to the non-spherical gravity field of the Earth and to the Moon and the Sun considered as point masses:

$$\mathcal{R} = \mathcal{R}_\oplus + \mathcal{R}_\odot + \mathcal{R}_\zeta.$$

The disturbing function $\mathcal{R}_\oplus$ is expanded in spherical coordinates as [57, p. 31]:

$$\mathcal{R}_\oplus = \sum_{l=1}^{\infty} \sum_{m=0}^l \frac{\mu_\oplus a_\oplus^l}{r^{l+1}} P_{lm}(\sin \phi) (C_{lm} \cos m\vartheta + S_{lm} \sin m\vartheta), \quad (80)$$

where $a_\oplus$ is the mean equatorial radius of the Earth, P_{lm} are the associated Legendre functions of order l and degree m , ϕ and ϑ are respectively the

geographic latitude and longitude, and C_{lm} , S_{lm} are coefficients that are determined empirically. For an axially symmetric body $C_{11} = S_{11} = 0$, therefore we will consider the outer sum to always start from $l = 2$ in the following. The expression for $\mathcal{R}_\oplus$ in equinoctial elements is:

$$\mathcal{R}_\oplus = \sum_{l=2}^{\infty} \sum_{m=0}^l \sum_{p=0}^l \sum_{q=-\infty}^{\infty} \mathcal{R}_{lmpq}. \quad (81)$$

Each of the terms $\mathcal{R}_{lmpq}$ is written as

$$\begin{aligned} \mathcal{R}_{lmpq} = & \frac{\mu_\oplus a_\oplus^l}{a^{l+1}} J_{lmp}(c) K_{lpq}(\gamma) \times \\ & \times [\mathbb{R}_{lmpq}(h, k, P, Q) (A_{lm} \cos \psi_{lmpq} + B_{lm} \sin \psi_{lmpq}) + \\ & + \mathbb{I}_{lmpq}(h, k, P, Q) (A_{lm} \sin \psi_{lmpq} - B_{lm} \cos \psi_{lmpq})], \end{aligned} \quad (82)$$

where

$$A_{lm} = \begin{cases} C_{lm}, & l-m \text{ even} \\ -S_{lm}, & l-m \text{ odd} \end{cases}, \quad B_{lm} = \begin{cases} S_{lm}, & l-m \text{ even} \\ C_{lm}, & l-m \text{ odd} \end{cases}.$$

and $c = \sqrt{1 - P^2 - Q^2} = \cos i/2$. The quantities J_{lmp} and K_{lpq} are, respectively, a polynomial in c and an infinite power series in γ . The functions $\mathbb{R}_{lmpq}$ and $\mathbb{I}_{lmpq}$ are finite power series of their arguments. The angle ψ_{lmpq} ,

$$\psi_{lmpq} = (l - 2p + q)\lambda - m\theta, \quad (83)$$

is a linear combination of the mean longitude λ and the Greenwich Mean Sidereal Time (GMST) θ ; both of them are fast variables.

The disturbing function due to the Moon and the Sun as point masses takes the same form $\mathcal{R}'$ for each of the body:

$$\mathcal{R}' = \sum_{l=2}^{\infty} \sum_{m=0}^l \sum_{p=0}^l \sum_{p'=0}^l \sum_{q=-\infty}^{\infty} \sum_{q'=-\infty}^{\infty} \mathcal{R}'_{lmpqp'q'}. \quad (84)$$

The expression for the terms $\mathcal{R}'_{lmpqp'q'}$ is the following:

$$\begin{aligned} \mathcal{R}'_{lmpqp'q'} = & \mu'(n')^2 \frac{a^l}{(a')^{l-2}} \varepsilon_m \frac{(l-m)!}{(l+m)!} J_{lmp}(c) L_{lpq}(\gamma) F_{lmp'}(i') G_{lp'q'}(e') \times \\ & \times \left(\mathbb{R}_{lmpq} \cos \psi'_{lmpqp'q'} + \mathbb{I}_{lmpq} \sin \psi'_{lmpqp'q'} \right). \end{aligned} \quad (85)$$

In Equation 85, primed quantities pertain to the perturbing body. In particular, μ' is its gravitational parameter, and n' is its mean motion. The eccentricity functions L_{lpq} and $G_{lp'q'}$ are infinite power series, and the inclination function $F_{lmp'}$ is a polynomial in the trigonometric functions of the inclination. The factor ε_m is $\varepsilon_0 = 1$, and $\varepsilon_m = 2$ for $m \neq 0$. The angle $\psi'_{lmpqp'q'}$ is defined as:

$$\begin{aligned} \psi'_{lmpqp'q'} = & (l - 2p + q)\lambda - (l - 2p' + q')\lambda' + q'(\omega' + \Omega') - \\ & - (m + 2p' - l)\Omega'. \end{aligned} \quad (86)$$

All of the primed quantities are slowly-varying functions of time.

5.1.2 Averaged equations of motion

We summarize here the core points of the averaging method as presented in Danielson et al. [24]. Also, we denote osculating orbital elements with a hat in the following. We assume that the osculating elements are related to the mean elements through

$$\hat{E}_i = E_i + \sum_{j=1}^{\infty} \epsilon^j \eta_{i,j} (E, t), \quad (87)$$

where E_i denotes the i -th element, and $i = 1, \dots, 6$. The terms $\epsilon^j \eta_{i,j}$ are small, short-periodic variations from the mean elements E_i . They explicitly depend on time, since perturbations as tesseral harmonics of the Earth's gravity potential and as those due to perturbing bodies are also explicit in time in the perturbed two-body problem. The latter is expressed in osculating equinoctial elements as

$$\frac{d\hat{E}_i}{dt} = n(\hat{a}) \delta_{i,6} + \epsilon F_i(\hat{E}, t), \quad (88)$$

where $\delta_{i,6}$ is a Kronecker delta and $n(\hat{a})$ is the mean motion, which is a function of the osculating semi-major axis $\hat{a}$. We search for a form of the averaged equations of motion of the type:

$$\frac{dE_i}{dt} = n(a) \delta_{i,6} + \sum_{j=1}^{\infty} \epsilon^j A_{i,j}(a, h, k, P, Q, t), \quad (89)$$

where the right-hand side has been expanded as a power series of the perturbation parameter ϵ . Since the high-frequency components have been relegated to the short-periodic terms, the rates of change of the mean elements are small:

$$\begin{aligned} \frac{1}{na} \left| \frac{da}{dt} \right| &\ll 1 \\ \frac{1}{n} \left| \frac{dE_i}{dt} \right| &\ll 1, \quad \text{for } i = 2, \dots, 5 \\ \frac{1}{n} \left| \frac{d\lambda}{dt} - 1 \right| &\ll 1. \end{aligned}$$

As to find a suitable form for the $A_{i,j}$ in Equation 89, we first express $n(\hat{a})$ and $F_i(\hat{E}, t)$ as functions of the mean elements by expanding them in power series of ϵ ,

$$n(\hat{a}) = n(a) + \sum_{j=1}^{\infty} \epsilon^j N_j(a) \quad (90)$$

$$F_i(\hat{E}, t) = F_i(E, t) + \sum_{j=1}^{\infty} \epsilon^j f_{i,j}(E, t). \quad (91)$$

Then, we differentiate Equation 87 and set the resulting expression equal to Equation 88, where we plug in Equations (90)–(91). We obtain the *equations of averaging*,

$$\sum_{j=1}^{\infty} \epsilon^j \left(A_{i,j} + \frac{\partial \eta_{i,j}}{\partial E} \frac{dE}{dt} + \frac{\partial \eta_{i,j}}{\partial t} \right) = \epsilon F_i + \sum_{j=1}^{\infty} \epsilon^j (\delta_{i,6} N_j(a) + \epsilon f_{i,j}). \quad (92)$$

By finding $A_{i,j}$ and $\eta_{i,j}$ such that they satisfy Equation 92 up to the M -th order, we can build an M -th order averaged theory. We will limit ourselves to the first order in the following; more details on how to obtain higher order theories are found in Danielson et al. [24]. We have:

$$A_{i,1} + \frac{\partial \eta_{i,1}}{\partial \lambda} n(a) + \frac{\partial \eta_{i,1}}{\partial t} = F_i + \delta_{i,6} N_1(a). \quad (93)$$

We now define the *single-averaging operator* $\langle \bullet \rangle$:

$$\langle f(a, h, k, P, Q, \lambda, t) \rangle \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a, h, k, P, Q, \lambda, t) d\lambda, \quad (94)$$

and apply it to both sides of Equation 93, yielding:

$$A_{i,1} = \langle F_i(E, t) \rangle. \quad (95)$$

The above equation states that, at first order, the rates of change of the mean elements are the average rates of change of the osculating elements. In the practical calculation of the $A_{i,1}$, we have to take into account their dependence from the total disturbing function $\mathcal{R} = \mathcal{R}_{\oplus} + \mathcal{R}'$. After eliminating the short-periodic terms included in their expansions by setting $q = 2p - l$, we plug them in Equation 95 [42]:

$$A_{i,1} = \left\langle F_i \left(\frac{\partial \mathcal{R}}{\partial \mathbf{E}}, \mathbf{E}, t \right) \right\rangle. \quad (96)$$

We keep the mean elements constant during the averaging operation, thus we can bring the averaging operator inside the parentheses. Thus, the mean element rates are obtained by plugging in the osculating equations of motion the averaged total disturbing function,

$$A_{i,1} = F_i \left(\left\langle \frac{\partial \mathcal{R}}{\partial \mathbf{E}} \right\rangle, \mathbf{E}, t \right) = F_i \left(\frac{\partial}{\partial \mathbf{E}} \langle \mathcal{R} \rangle, \mathbf{E}, t \right). \quad (97)$$

More perturbations can be summed together by considering

$$\begin{aligned} \epsilon A_{i,1} &= \sum_{\alpha} \nu_{\alpha} A_{i,1\alpha} \\ \epsilon F_i &= \sum_{\alpha} \nu_{\alpha} F_{i\alpha}, \end{aligned}$$

where the index α varies over all the perturbations to be considered, and the ν_{α} are the small parameters of each perturbation. Each $A_{i,1\alpha}$ is obtained by averaging the corresponding perturbation,

$$A_{i,1\alpha} = \langle F_{i\alpha}(E, t) \rangle,$$

while coupling terms between different perturbations arise at higher orders.

Once $A_{i,1}$ is obtained through Equation 95, it is plugged in Equation 89, which is integrated with a suitable numerical scheme. At any step, the osculating elements can be recovered from the mean by Equation 87. The short-periodic terms $\eta_{i,1}$ are computed by integrating Equation 93 over the mean longitude λ , while keeping the rest of the mean elements constant.

Until now, we implicitly assumed that the rates $F_i(E, t)$ are slowly-varying functions of time and of the equinoctial elements. Some perturbations are quickly-varying functions of time, but slowly-varying functions of an angle ψ , which is a linear combination of the mean longitude λ and a fast angle ϑ pertaining to the perturbing body. Averaging these perturbations requires special care, and we describe it in the following section.

5.1.3 Resonances

The choice of the averaging operator depends on whether or not the satellite is in a condition of *resonance*. Note that the disturbing functions $\mathcal{R}$ and $\mathcal{R}'$ depend on $\psi_{\text{lm}pq}$ (Equation 83) and $\psi_{\text{lm}pq p'q'}$ (Equation 86) respectively, which are linear combinations of both fast and slow variables. Letting ψ denote either of the angles, we consider the case in which ψ can be written as:

$$\psi = j\lambda - k\vartheta,$$

with j and k mutually prime integers. The angle ϑ is a fast variable corresponding to either θ or λ' , depending on whether we are considering the perturbation from the tesseral harmonics of the Earth's potential or from a perturbing body, respectively. In the latter case, we neglect the slowly-varying orbital elements of the perturbing body in the expression for ψ .

The satellite is in resonance if the inequality

$$\left| \frac{d\psi}{dt} \right| < \frac{2\pi}{\tau} \quad (98)$$

is satisfied, where τ should be several times larger than the integration step and the periods of the mean longitude τ_λ and of the fast angle τ_ϑ , as to avoid including short-periodic variations in the mean elements. Danielson et al. [24] propose

$$\begin{aligned} \tau &\geq 3\tau_\lambda \\ \tau &\geq 3\tau_\vartheta \end{aligned}$$

At the same time, τ must not be so large that long-periodic effects are inadvertently included in the short-periodic terms [73].

In presence of a resonance, long-periodic and secular effects arise due to the commensurability of the two fast frequencies $\dot{\lambda}$ and $\dot{\vartheta}$, which pertain to the satellite and to one of the massive bodies respectively. By keeping ϑ fixed during the application of the single-averaging operator in Equation 94, these effects are missed. Therefore, we also integrate over ϑ by using the *double-averaging operator* $\langle \bullet \rangle$:

$$\begin{aligned} \langle f(a, h, k, P, Q, \lambda, \vartheta, t) \rangle &\triangleq \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, P, Q, \lambda, \vartheta, t) d\lambda d\vartheta + \\ &+ \frac{1}{2\pi^2} \sum_{(j,k) \in \mathcal{B}} \left[\cos \psi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, P, Q, \lambda^*, \vartheta^*, t) d\lambda^* d\vartheta^* + \right. \\ &\left. + \sin \psi \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, P, Q, \lambda^*, \vartheta^*, t) d\lambda^* d\vartheta^* \right], \end{aligned} \quad (99)$$

where $\mathcal{B}$ is the set of all (j, k) for which the inequality (98) is satisfied.

Furthermore, terms depending on λ which are responsible for the resonances have to be retained in the perturbing functions $\mathcal{R}_\oplus$ and $\mathcal{R}'$. For the case of tesseral resonances with the Earth's gravitational potential, and the terms to be retained in $\mathcal{R}_\oplus$ are those satisfying the identities:

$$\begin{aligned} 2p - l &= q - sj \\ m &= sk, \end{aligned}$$

with s integer. On the other hand, for *mean-motion resonances* with the Moon, the terms to be retained in $\mathcal{R}'$ obey:

$$j(l - 2p + q) = k(l' - 2p' + q').$$

Mean-motion resonances with the Sun are negligible due to the extremely high values of the semi-major axis that they require.

5.1.4 Average of the short-periodic terms

In obtaining Equation 95, we assumed that the short-periodic terms $\eta_{i,1}$ average to zero. This is equivalent to requiring that the mean elements are “centered”, that is to say that the short-periodic terms do not contain any long-periodic or secular offsets from the mean elements. If one is only interested in obtaining the mean elements history this hypothesis is superfluous, and it can be overlooked. However, for applications to spacecraft operations and space situational awareness it is necessary to impose that $\langle \eta_{i,j} \rangle = 0$ to avoid divergence of the osculating and mean trajectories in the long term [63].

5.2 ERROR ANALYSIS FOR AVERAGED METHODS

Averaging involves approximations needed to simplify the analytical developments, which would be intractable otherwise. These approximations introduce errors with respect to the real trajectory that have to be considered in addition to the numerical error accumulated during the integration of the mean equations of motion. While the propagation of numerical error in the integration of ordinary differential equations has been studied extensively, we are not aware of any quantitative study of the impact that the approximations involved in averaging methods have on the integration error. In the following, we perform a preliminary analysis of the matter by separating and analysing the contributions to the integration error in averaged methods.

Let δE_i be the total integration error with respect to the i -th osculating equinoctial element, that is the difference between the *computed* osculating equinoctial element $\hat{E}_i$ and its *true* (or *reference*) value $\hat{E}_i^R$,

$$\delta E_i = \hat{E}_i - \hat{E}_i^R.$$

We assume that the reference value $\hat{E}_i^R$ satisfies Equation 87 exactly, while a truncation at order M is introduced for the computed values:

$$\hat{E}_i^R = E_i^R + \sum_{j=1}^{\infty} \epsilon^j \eta_{i,j}(\mathbf{E}^R, t) \quad (100)$$

$$\hat{E}_i = E_i + \sum_{j=1}^M \epsilon^j \eta_{i,j}(\mathbf{E}, t). \quad (101)$$

Subtracting the above equations gives:

$$\delta \hat{E}_i = (E_i - E_i^R) + \sum_{j=1}^M \epsilon^j (\eta_{i,j} - \eta_{i,j}^R) - \sum_{j=M+1}^{\infty} \epsilon^j \eta_{i,j}^R = \delta E_i + \delta \eta_i, \quad (102)$$

where $\eta_{i,j}^R = \eta_{i,j}(\mathbf{E}^R, t)$. The “osculating” integration error $\delta \hat{E}_i$ is the sum of the “mean” integration error δE_i and of the error on the short-periodic terms $\delta \eta_i$. The mean integration error can be further decomposed by considering

$$\delta E_i = \delta E_{i,\text{num}} + \delta E_{i,\text{trunc}} + \delta E_{i,\text{dyn}}, \quad (103)$$

where the first term is the *numerical integration error*, the second is the *model truncation error*, and the last is the *dynamical error*. We explain the nature of these errors, along with the error on the short-periodic terms $\delta\eta_i$, in the following sections.

5.2.1 Dynamical error $\delta E_{i,\text{dyn}}$

For a first-order theory, the second term in Equation 89 is truncated at $j = 1$ yielding:

$$\frac{dE_i}{dt} = n(a)\delta_{i,6} + \epsilon A_{i,1}. \quad (104)$$

The mean rate $A_{i,1}$ is obtained by Equation 96, in which the mean elements are kept constant during the averaging operation. We define the reference mean rates $A_{i,1}^R$ as those that would be obtained by computing exactly the definite integrals in Equation 94 and Equation 99, i.e. by taking into account the time dependence of the mean orbital elements over the period of integration. Plugging in Equation 89 the exact mean rates yields:

$$\frac{d\tilde{E}_i}{dt} = n(a)\delta_{i,6} + \sum_{j=1}^{\infty} \epsilon^j A_{i,j}^R, \quad (105)$$

which, when subtracted from (104), gives the error on the total mean rates of change:

$$\delta\dot{E}_i = \epsilon \left(A_{i,1} - A_{i,1}^R \right) + \sum_{j=2}^{\infty} \epsilon^j A_{i,j} = \epsilon \delta A_{i,1} + \mathcal{O}(\epsilon^2). \quad (106)$$

The error on the mean rates of change $\delta\dot{E}_i$ is the sum of a term of order ϵ , due to the inexact result from the application of the averaging operator, and a term of order ϵ^2 due to neglecting the higher-order $A_{i,j}$. At each integration step, we obtain the *dynamical error* δE_i as:

$$\delta E_{i,\text{dyn}} = \delta\dot{E}_i \Delta t.$$

From the standpoint of numerical integration, $\delta E_{i,\text{dyn}}$ propagates during the integration in a way similar to the local truncation error. However, the dynamical error cannot be reduced by choosing a smaller time step or a higher integration order because the mean equations of motion describe only an approximation of the full physical model. Situations in which the rate of change of the orbital elements is not small compared to the mean motion and effects of higher-order are non-negligible give rise to a large $\delta E_{i,\text{dyn}}$.

In Equation 106, we have only considered first-order mean rates of a single perturbation. The development carried out here can be extended to higher orders and applied to several perturbations, which will give rise to further terms in the definition of the dynamical error.

5.2.2 Model truncation error $\delta E_{i,\text{trunc}}$

As to perform an analytical averaging of conservative perturbations, the expressions for the perturbing functions in Equations (81) and (84) and the eccentricity functions K_{lpq} , L_{lpq} and $G_{lp'q'}$ have to be truncated at given values q_M and l_M . In this way, only the lowest frequencies governing the

motion are taken into account, reducing computational complexity and cost. By defining $\mathcal{R}_\oplus^T$ and $\mathcal{R}'^T$ as the truncated perturbing functions,

$$\mathcal{R}_\oplus^T = \sum_{l=2}^{l_M} \sum_{m=0}^l \sum_{p=0}^l \sum_{q=-q_M}^{q_M} \mathcal{R}_{lmpq} \quad (107)$$

$$\mathcal{R}'^T = \sum_{l=2}^{l_M} \sum_{m=0}^l \sum_{p=0}^l \sum_{p'=0}^l \sum_{q=-q_M}^{q_M} \sum_{q'=-q'_M}^{q'_M} \mathcal{R}'_{lmpqp'q'}, \quad (108)$$

we find the *model truncation error* $\delta\mathcal{R}$ as:

$$\delta\mathcal{R} = (\mathcal{R}_\oplus^T - \mathcal{R}_\oplus) + (\mathcal{R}'^T - \mathcal{R}').$$

Since the mean rate $A_{i,1}$ is obtained by averaging of $\mathcal{R}^T$, the error $\delta\mathcal{R}$ will be added to the error on the mean rate $\delta A_{i,1}$, and will likewise propagate during the integration of the equations of motion. However, we consider this contribution separately from the dynamical error because their origins are distinct: the dynamical error is caused by neglecting higher orders in ϵ for the mean rates, while the model truncation error is caused by neglecting higher-order terms in the perturbing functions. In fact, one may build high-order theories (in the mean rates) while keeping only a limited number of the most relevant terms in the perturbing functions.

Note that the gravitational potential of the geoid is always truncated even for non-averaged methods, since a closed form does not exist. However, the Sun and the Moon are usually considered as point masses, an approximation that holds well for Earth satellite orbits. Thus, their gravitational potential can be expressed in a closed form that can be evaluated efficiently by non-averaged methods [1, section 8.4].

5.2.3 Numerical error $\delta E_{i,num}$

The mean elements at each step of the numerical integration of Equation 89 are affected by truncation and round-off errors. Dropping the subscript i for ease of notation, we denote them with $\delta E_{num,T}$ and $\delta E_{num,R}$ respectively,

$$\delta E_{num} = \delta E_{num,T} + \delta E_{num,R}.$$

If the numerical scheme is of p -th order, the local truncation error accumulated with a step size Δt is proportional to the $(p+1)$ -th derivative of the mean element:

$$\delta E_{num,T} \sim \frac{d^{p+1}E}{dt^{p+1}} \Delta t^{p+1}.$$

The round-off error $\delta E_{num,R}$ grows with the number of floating-point operations performed during the integration. Thus, it is proportional to the total number of integration steps and inversely proportional to the step size. The accumulated round-off error can be estimated through statistical laws [14, 69]. Note that in non-averaged methods the total integration error (Equation 102) is only due to the numerical error, $\delta \hat{E} = \delta \hat{E}_{num}$.

5.2.4 Error on the short-periodic terms $\delta \eta_i$

Following Equation 102, the error on the short-periodic terms can be written as:

$$\delta \eta_i = \sum_{j=1}^M e^j (\eta_{i,j} - \eta_{i,j}^R) - \sum_{j=M+1}^{\infty} e^j \eta_{i,j}^R. \quad (109)$$

We expand $(\eta_{i,j} - \eta_{i,j}^R)$ as:

$$\eta_{i,j}(\mathbf{E}, t) - \eta_{i,j}(\mathbf{E}^R, t) \approx \frac{\partial \eta_{i,j}}{\partial \mathbf{E}_i} \delta \mathbf{E}_i.$$

Substituting in Equation 109 we obtain:

$$\delta \eta_i = \sum_{j=1}^M \epsilon^j \frac{\partial \eta_{i,j}}{\partial \mathbf{E}_i} \delta \mathbf{E}_i - \sum_{j=M+1}^{\infty} \epsilon^j \eta_{i,j}^R = \delta \eta_{i,M} + \delta \eta_{i,HO}. \quad (110)$$

The above equation states that the error on the short-periodic terms is due to two contributions. The first is due to the mean integration error $\delta \mathbf{E}_i$. In fact, the short-periodic terms are computed from values of the mean elements that are affected by the mean integration error, generating an error on the short-periodic terms up to the M -th order. The second term is due to the truncation to the M -th order that is performed in Equation 101. The first term is proportional to ϵ , and will be negligible in the expression of the total integration error for most situations.

5.2.5 Error budget

It is desirable that the dynamical error $\delta \mathbf{E}_{i,dyn}$ dominates over the remaining terms. This is because both the model truncation error $\delta \mathbf{E}_{i,trunc}$ and the numerical error $\delta \mathbf{E}_{i,num}$ can be driven down, within certain limits, with a careful choice of the integration parameters.

Assuming that the analytical expressions for the expansions are available, the model truncation error for the lunar and solar perturbing functions can be reduced by choosing higher truncation indices in Equations (107)-(108).

In practical situations, the numerical error $\delta \mathbf{E}_{i,num}$ will be of secondary importance with respect to the other terms. In fact, it is possible to reduce $\delta \mathbf{E}_{i,num}$ below a dimensionless value of 10^{-6} quite easily by using an efficient numerical scheme between those available in the literature. This value is roughly comparable to the dynamical and model truncation errors resulting from neglecting second-order terms in l in Equation 108. These considerations affect both the choice of the numerical scheme and of the step size of any particular integration. Both have to be only accurate enough to drive the magnitude $\delta \mathbf{E}_{i,num}$ below that of the other error components. A more accurate solution of the averaged equations will not result in a smaller total integration error due to the presence of the dynamical, model truncation and short-periodic errors in Equation 102. Situations in which the numerical error dominates could only arise in extremely long propagations, for which round-off error could play a significant role.

5.3 SUMMARY

In this chapter, we presented a summary of the theory of averaging for equinoctial elements as developed by [24]; the employment of this theory is widespread in orbit propagation programs for long-term propagations of Earth satellite orbits. We focus on the analytical developments for first-order terms, which are the ones mostly affecting the dynamics. Particular attention is devoted to the averaging of perturbations in the presence of resonances, which are potentially detrimental to the quality of the solution. We develop a qualitative analysis of the integration error in semi-analytical techniques on the basis of the aforementioned theory. The integration error (the

difference between the computed and exact osculating trajectories) is broke down in several contributions, each stemming either from approximations in the analytical expressions of the perturbations or from numerical factors. We perform a total error budget with the aim of identifying the most important contributions to the integration error. This allows one to fine-tune the parameters of the mathematical model and of the numerical solvers as to achieve the best possible performance with any semi-analytical propagator.

NON-AVERAGED METHODS FOR EARTH SATELLITE ORBITS

The method of averaging presented in the previous chapter is particularly effective when some assumptions on the dynamical system are satisfied. The most important of these regards the ratio between the mean motion of the satellite and the characteristic frequencies of the considered perturbations. The higher this ratio (i.e. the “slower” the perturbations), the more effective will be the elimination of the higher frequencies in the increase of the step-size of the numerical integration. Another assumption regards the presence of resonances, in which case two or more of the frequencies present in the motion are commensurable. For these cases, particular care has to be taken in the averaging as not to neglect long-periodic or secular motions that arise from the interaction of two fast variables. This is particularly critical for mean-motion resonances with the Moon.

In this chapter, we will explore the limits of the averaging technique by comparing an averaged method based on equinoctial elements to the integration of the non-averaged Cowell and EDromo formulations (section 2.3) with an efficient solver. We will perform several numerical test cases corresponding to the MEO and HEO orbital regimes, which are particularly important for space debris consideration due to the absence of natural “sinks” as the atmospheric drag. The relative strengths of the methods will be quantified, and the best-performing techniques for each orbital regime will be highlighted.

6.1 ORBIT PROPAGATION SOFTWARE

In the following, we outline the characteristics of the orbit propagation software used to perform the numerical tests in this chapter.

6.1.1 STELA

The STELA (Semi-analytical Tool for End of Life Analysis) orbit propagator has been developed by the French space agency CNES to assess compliance with spacecraft re-entry requirements imposed by the French Space Act. The software is publicly available as a Java executable¹.

STELA integrates averaged equations of motion for the equinoctial element set $\mathbf{E} = (\alpha, h, k, P, Q, \lambda)$, shown in subsection 5.1.1, with a fixed step size Runge Kutta integrator. The order of integration can be chosen between four and six; we use the latter value for all of our simulations. We assign the initial conditions for the propagations in osculating orbital elements. STELA automatically removes the short-periodic terms from the initial osculating elements to obtain the mean initial conditions for the propagation.

¹ URL: <https://logiciels.cnes.fr/content/STELA>, last visited 8th May 2017.

6.1.2 STELA mathematical model

STELA allows the user to choose which perturbations to include among several sophisticated models. For the conservative perturbations, we take into account a 2×2 geopotential corresponding to the EGM-GOC-2 model [25], and a simplified version of the analytical solar and lunar ephemerides by Meeus, J. [65]².

By default, STELA truncates the expansion in q of the geopotential (Equation 107) at a value $q_M = 2$. Also, the expansion in l of the solar and lunar potentials (Equation 108) is truncated at a value l_M that can be varied from 2 to 5. The short-periodic terms are computed at first order for all the conservative perturbations except the second zonal harmonic, whose terms are computed at second order. Resonant tesseral harmonics are retained in the geopotential according to their period, that is $2\pi/\psi_{lmpq}$ with ψ_{lmpq} defined according to Equation 83. If the period of a tesseral harmonic is greater than a customizable multiple of the integration step n_{tess} , it is retained in the geopotential [73]. Dissipative perturbations are averaged numerically by a Simpson quadrature [90]. While STELA considers precessional and nutational movements of the rotational axis of the Earth, we disabled these effects as they are not relevant for the evaluation of the numerical performance.

6.1.3 THALASSA

The Cowell and EDromo formulations have been implemented in the THALASSA orbit propagation code. The non-averaged equations of motion are integrated with the LSODAR solver (subsection 4.2.1) in double or quadruple precision. We use the latter, together with a very strict tolerance on the local truncation error, to build a *reference* trajectory for each of the test cases considered.

6.1.3.1 THALASSA mathematical model

The perturbations included in THALASSA are the same as those taken into account in STELA, and they are taken from the same models. All the constants involved in the computations are identical, as to exclude any difference in the computed trajectories due to different perturbation models.

6.1.3.2 THALASSA implementation

THALASSA is implemented in Fortran 2003, and is available in a Gitlab repository³. For the purposes of comparing the computational time with respect to STELA, we emphasize that Fortran codes exhibit lower computational times than Java. The exact amount by which the computational time is reduced with respect to Java varies depending on multiple factors such as the machine architecture, JVM/JDK versions, and the Fortran compiler version. Benchmarks for scientific computing [15] and astrodynamics applications [32] show that performances obtained with Java codes can be anything from equally fast as Fortran codes to one order of magnitude slower. Therefore, computational times obtained with STELA are affected by a “penalty factor” of about 5.

² The subroutine for the computation of the ephemerides was kindly provided by Florent Deleflie.

³ URL: <https://gitlab.com/souvlaki/thalassa>.

6.2 NUMERICAL TESTS

We test the non-averaged methods implemented in the THALASSA code and the averaged method implemented in STELA in several orbital regimes of operational interest. For each of the methods, we check that the orbits are qualitatively reproduced. If so, we compute the errors in osculating quantities with respect to a reference solution computed using a strict tolerance in quadruple precision with the Cowell formulation. We devote particular attention to the analysis in terms of the error breakdown developed in [section 5.2](#).

The THALASSA code is compiled by using gfortran 4.8.4 with the optimization flag `-O3`; all tests are performed on an Intel i7-3612QM quad-core machine at 2.10 GHz running Linux Mint 18. For tests performed with THALASSA, we take the average CPU time out of three runs for each simulation. We never run more than three simulations at once as not to incur in bottlenecks that would skew the values of the CPU time.

The STELA propagator allows the user to fine-tune several numerical and dynamical parameters to increase the propagation efficiency. We are particularly concerned with the maximum order of the lunar and solar potentials l_M , the threshold n_{tess} on long-periodic effects due to tesseral harmonics, and the time step Δt of the Runge-Kutta integrator. For each test case, we perform several propagations for different combinations of the above parameters. We find by trial and error the values of $(l_M, n_{\text{tess}}, \Delta t)$ giving rise to the trajectory exhibiting the lowest osculating error on the orbital elements. The latter is denominated the STELA *baseline trajectory*, which we use to establish the comparison with respect to the non-averaged methods.

6.2.1 MEO test case

As a test case for a MEO orbit, we take initial conditions corresponding to a Galileo-like satellite, which are displayed in [Table 7](#). We propagate the orbit for 200 years in the presence of perturbations from the geopotential and the Sun and Moon, according to the mathematical model specified in [subsection 6.1.2](#). The evolutions of the reference osculating orbital elements and of the osculating and mean orbital elements from the STELA baseline propagation are displayed in [Figure 30](#). The latter is obtained by setting $l_M = 4$, $n_{\text{tess}} = 3.2$, and $\Delta t = 120$ h. Further increasing l_M and further decreasing Δt and n_{tess} does not bring any reduction in the errors on the osculating orbital elements. A value of $n_{\text{tess}} = 3.2$ implies that only tesseral harmonics with a period greater than $n_{\text{tess}} \times \Delta t = 16$ d are retained in the geopotential.

STELA reproduces well the qualitative characteristics of the orbit, which is quasi circular during the whole integration span. The osculating orbital elements obtained with the reference and the STELA baseline propagation overlap.

6.2.1.1 STELA performance

The errors in the osculating orbital elements corresponding to the STELA baseline trajectory are displayed in [Figure 31](#).

In the initial part of the propagation, the errors $\delta\hat{a}$, $\delta\hat{e}$, $\delta\hat{\omega}$ and $\delta\hat{M}$ are constant, and they start to grow linearly only after 10^3 days. This is because the errors in the short-periodic terms prevail at the beginning, when the dy-

Figure 30: Orbital elements as a function of time for the propagation of the Galileo initial conditions in Table 7 for 200 years. The curves “Ref” and “STELA (osc)” refer to the osculating orbital elements obtained in the reference propagation and in the STELA baseline propagation ($l_M = 4$, $n_{\text{tess}} = 3.2$, and $\Delta t = 120$ h), and exhibit a significant overlap. The curve “STELA (mean)” refers to the mean orbital elements obtained from the STELA baseline propagation. The semi-major axis is plotted as a difference from the initial osculating value.

Table 7: Initial Modified Julian Day (MJD) and orbital elements for the Galileo test case.

MJD	58 474.7433
a (km)	29 601.310 447 014 60
e	10^{-4}
i ($^\circ$)	56
Ω ($^\circ$)	116.640 939 804 248 0
ω ($^\circ$)	0
M ($^\circ$)	0

namical error still has not accumulated significantly. The latter produces a small shift in the frequency of both the short- and long-periodic oscillations, which generates the linear growth in the errors on all of the osculating orbital elements. If the initial dynamical error is of the same order of magnitude of the short-periodic, the total osculating error exhibits a linear trend since the beginning of the propagation, as is the case with $\delta\hat{i}$ and $\delta\hat{\Omega}$.

The model truncation error does not appear to be relevant in this case, since increasing l_M to a value of 5 does not affect the error magnitudes.

Since the orbit is quasi-circular, the argument of perigee and the mean anomaly vary rather abruptly along the propagation, and the corresponding errors exhibit larger magnitudes.

6.2.1.2 Performance of non-averaged methods

We propagated the initial conditions in Table 7 for values of the solver tolerance from 10^{-4} to 10^{-15} with the formulations implemented in THALASSA. The CPU time as a function of the errors on the orbital elements is displayed in Figure 32. When the EDromo formulation is used, we propagate the physical time by integrating the Sundman transformation, the constant, and the linear time elements.

EDromo shows superior performance with respect to the Cowell formulation in the propagation of all the orbital elements. The final accuracy in the osculating semi-major axis and eccentricity is particularly important, since they dictate the radius of perigee and thus the possibility of a re-entry. EDromo with a constant time element allows one to obtain values of these elements which are four orders more accurate than Cowell, for the same computational cost.

Regions where the distribution of the points is aligned vertically signal a bad convergence of the propagations. This can take place for two reasons: for high error values the accumulation of truncation error makes the integration enter into numerical instability and the geometry of the orbit is lost, while for low error values the accumulated round-off error becomes of the same order of the global truncation error. In the latter case, we say that the accuracy of the integration is *round-off limited*. Note that the numerical instability can only be avoided by using strict tolerances with the Cowell formulation, which lead to high computational times. Also, EDromo with a constant time element exhibits accuracies up to two orders of magnitude higher; this is particularly evident in the case of the mean anomaly. In fact, using a con-

Figure 31: Errors on the osculating orbital elements as a function of time for the STELA baseline trajectory, obtained with $l_M = 4$, $n_{\text{tess}} = 3.2$, and $\Delta t = 120 \text{ h}$. The CPU time corresponding to the propagation is 2 s.

stant time element greatly reduces the local truncation error with respect to a linear one, but the truncation error growth rate becomes quadratic rather than linear [75]. However, the integration span is short enough to let the reduced magnitude of the local truncation error prevail over the increased growth rate. The solver takes larger steps, which lead to a delayed entry into the round-off limited regime.

We show the CPU time as a function of the errors in position and velocity⁴ in Figure 33. For almost all the propagations with the Cowell formulation, errors in position of the order of the semi-major axis signal that the final position along the orbit is completely lost. In comparison, EDromo with a constant time element shows a positional accuracy of less than 100 m. This is a consequence of the very low errors obtained on the value of the mean anomaly.

The same performance is obtained by integrating the linear time element and the Sundman transformation with EDromo. We highlight that the constant time element always keeps a value close to the initial one, which is set to zero in our propagations. Therefore, we only assigned an absolute tolerance on the local truncation for this quantity. In fact, a relative tolerance on a near-zero quantity is ill-defined.

6.2.2 HEO test case

As to provide a challenging HEO test case, we consider the orbit of Simbol-X, an X-ray telescope that was proposed by ASI and CNES [39]. The initial orbital elements [62] are provided in Table 8, and correspond to a highly elliptical orbit with a large semi-major axis, corresponding to a 7 : 1 mean motion resonance with the Moon. As in the Galileo test case, we only consider the conservative perturbations specified in subsection 6.1.2. For this case, we expect lunisolar perturbations to play a rather important role.

Figure 34 displays the history of the orbital elements for the reference propagation and for two STELA baseline propagations performed by truncating the lunar and solar potentials at $l_M = 4, 5$. All the propagations are stopped when detecting a re-entry, which is defined to take place when the height h of the spacecraft is $h < 80$ km. Re-entries take place after 80 years for the reference propagation and the STELA propagation with $l_M = 4$, and after 82 years for the one with $l_M = 5$. Note that since we don't take into account atmospheric drag the behaviour of the trajectory could be unphysical when attaining very high eccentricities; also the re-entry date could be incorrect. However, this does not affect the evaluation of numerical performances since drag would affect the trajectory only during very limited spans in a realistic case.

6.2.2.1 Characteristics of the averaged propagation

The propagation of the dynamical error is evident in STELA propagations by looking at Figure 34. The oscillations in the orbital elements displayed by the STELA trajectories display a different period than the ones of the reference trajectory. In fact, we expect the dynamical error to be quite relevant for this propagation because of the relatively large value of the orbital period of 4 solar days. This implies that the approximation of keeping the Moon's orbital elements constant during an orbital period is poor. The situation is made worse because of the 7 : 1 resonance with the Moon, which generates

⁴ Position and velocity are obtained from the osculating orbital elements.

Figure 32: CPU time as a function of the errors on the osculating orbital elements for the Galileo test case. The points are obtained with the formulations included in the THALASSA code. The label “Cow” refers to the Cowell formulation, while the labels “EDr(t)”, “EDr(c)” and “EDr(l)” refer to the EDromo formulation in which the physical time is integrated through the Sundman transformation, a constant time element, and a linear time element, respectively.

Figure 33: CPU time as a function of the errors in position and velocity for the Galileo test case. The description of the labels is the same as in Figure 32.

Table 8: Initial Modified Julian Day (MJD) and orbital elements for the Simbol-X test case.

MJD	56 664.863 368 054 20
a (km)	106 247.136 454 000 0
e	0.751 73
i ($^\circ$)	5.2789
Ω ($^\circ$)	49.351
ω ($^\circ$)	180
M ($^\circ$)	0

long-periodic and secular behaviours depending on the linear combination of the spacecraft's and the Moon's mean anomalies through the resonant angle $\psi_{l_{\text{mpp}}'qq'}$. These effects are not taken into account, since the lunar perturbing function is not double-averaged in the approach used by STELA. An accurate double-averaged approach would need to consider the short-periodic variations in the orbit of the Moon, which generate a considerable error if neglected [24].

The model truncation error also has an important impact on the accuracy. Figure 34 shows that decreasing l_M from 5 to 4 changes the qualitative evolution of the trajectory. Further tests made by progressively decreasing l_M to a value of 2 demonstrate that the STELA trajectories shift even further away from the reference.

Also, note that the contribution of the short-periodic terms is almost unnoticeable by inspection of Figure 34. The amplitudes of the short-periodic terms due to the gravity potential of the Earth are very small due to the large geocentric distance attained by the spacecraft.

Ultimately, the most important factor affecting the trajectory is the large value of the ratio of the semi-major axis to the semi-major axis of the Moon (a/a_ζ) ≈ 0.3 . This implies that the the magnitude of the fast time scale, which is proportional to the orbital period, is relatively close to that of the perturbations. Moreover, the expansions of the perturbing functions of the Moon and the Sun have to be truncated at higher degrees l_M to achieve a satisfactory accuracy.

Figure 34: Orbital elements as a function of time for the propagation of the Simbol-X initial conditions in Table 8 for 83 years. The curve “Ref” refers to the osculating orbital elements obtained in the reference propagation. The curves “ $l_M = 4$ (mean)”, “ $l_M = 4$ (osc)” and “ $l_M = 5$ (mean)” refer to the mean (respectively, osculating) orbital elements obtained from STELA propagations with two values of the lunar and solar potential truncation order and with $\Delta t = 48$ h, $n_{\text{tess}} = 20$. The re-entry dates of the STELA propagations differ with respect to the reference by up to two years.

6.2.3 Comparison of the methods

Tables 9 and 10 display the errors in the osculating orbital elements for STELA and the non-averaged methods for the Galileo and Simbol-X propagations, respectively.

STELA is quite efficient in the propagation of the Galileo test case; it achieves a computational time that is one order of magnitude lower than the non-averaged methods for the same values of the errors. This is due to the fact that Galileo is on a circular orbit that is only moderately perturbed by the Sun and the Moon; these factors imply a fast convergence of the perturbing functions and a smooth averaged right-hand side. However, in the case of Simbol-X its performance degrades markedly. While the computational time is still low, the accumulation of the dynamical and truncation errors causes large errors in all the orbital elements except the semi-major axis, as its mean rate of change is very small. Moreover, increasing the order of truncation of the lunar and solar potentials does not improve the results, suggesting that values $l_M > 5$ are necessary to achieve convergence of the expansions. This is not possible in the current version of STELA, and the substantial difference in computational time between the cases $l_M = 4$ and $l_M = 5$ suggests that an even bigger value would result in a dramatic increase. For Simbol-X, non-averaged methods are more performing; for instance, EDromo achieves a similar accuracy in a computational time of 1.6 s, which is comparable to the lowest one obtained with STELA. It is possible to obtain accuracies that are quite higher simply by decreasing the integrator tolerance, as shown in Figures 32 and 33.

6.3 SUMMARY

We compared the non-averaged methods based on the Cowell and EDromo formulations, implemented in the THALASSA code, to the semi-analytical technique based on equinoctial elements implemented in the STELA orbit propagator. We consider two test cases, corresponding to a Galileo-like MEO orbit and to the Simbol-X HEO. In the first case, STELA shows a marked advantage. With proper optimization of its numerical and dynamical parameters, it achieves a computational time that is one order of magnitude lower than the non-averaged methods for about the same value of the final errors, which are in the order of 1 km in the semi-major axis. The situation is reversed for the second case, describing a challenging HEO orbit. The accumulation of the integration error, in which the truncation error of the lunar perturbing potential is dominant, causes large errors in the orbital elements obtained with the STELA propagator. Ultimately, the large value of the Simbol-X semi-major axis involves a relatively low ratio between its mean motion and that of the Moon, which entails the largest perturbations. In this situation, non-averaged methods show superior performance.

Table 9: CPU time and final errors on the orbital elements for STELA and the non-averaged methods for the propagation of the Galileo initial conditions in Table 7. The results with non-averaged methods are obtained with a tolerance for the LSODAR solver such that the final errors in semi-major axis and eccentricity are roughly the same as those of the STELA baseline propagation. We only display the results for EDromo with a constant time element; the performances obtained with a linear or with no time element are very similar.

	STELA	Cowell	EDromo
CPU time (s)	2	71	24
$\delta \hat{a}$ (km)	0.874	0.935	0.577
$\delta \hat{e}$	5.15×10^{-6}	3.70×10^{-6}	1.88×10^{-6}
$\delta \hat{i}$ (°)	0.10	4.63×10^{-4}	0.0102
$\delta \hat{\Omega}$ (°)	1.22	4.06×10^{-4}	0.0974
$\delta \hat{\omega}$ (°)	172	119	165
$\delta \hat{M}$ (°)	173	136	101

Table 10: CPU time and final errors on the orbital elements for STELA and the non-averaged methods for the propagation of the Simbol-X initial conditions in Table 8. The results with non-averaged methods are obtained with a tolerance for the LSODAR solver such that the final errors in semi-major axis are roughly the same as those of the STELA propagations. The results for STELA are relative to the two baseline propagations with $l_M = 4$ and $l_M = 5$. We only display the results for EDromo with a constant time element; the performances obtained with a linear or with no time element are very similar.

	STELA, $l_M = 4$	STELA, $l_M = 5$	Cowell	EDromo(c)
CPU time (s)	2	14	9.7	1.6
$\delta \hat{a}$ (km)	45.0	43.0	65.5	12.9
$\delta \hat{e}$	0.0716	0.15	2.01×10^{-3}	5.06×10^{-3}
$\delta \hat{i}$ (°)	0.0254	0.535	0.0423	0.254
$\delta \hat{\Omega}$ (°)	340	340	0.656	0.162
$\delta \hat{\omega}$ (°)	14.1	2.37	1.56	2.25
$\delta \hat{M}$ (°)	248	234.4	103	122

Part III

ASTEROID DEFLECTION

A hypothetical asteroid impact scenario was presented at the 2015 Planetary Defense Conference (PDC) in Frascati, Italy, with the aim of supporting an emergency response exercise conducted during the conference and stimulating the discussion on several aspects of asteroid threat mitigation. The fictitious 150 m to 400 m diameter asteroid, discovered on April 13, 2015 and named 2015PDC, was found to have several potential impacts with the Earth, the earliest and most likely on September 3, 2022. The impact probability, estimated in mid-June 2015, would reach 1% and would continue to rise with the rest of the scenario to be played out at the conference.

Starting from ephemeris data provided by the conference organizers and propagating forward until impact with the Earth one finds a nominal impact point in the South China Sea roughly 550 km off the Vietnam coast at around 03:52:10 UT, with an impact velocity of about 16 km s^{-1} at roughly 56° from the surface. However, by considering a major line of variation error in the initial asteroid orbital determination one would obtain a path of risk stretching from eastern Turkey until the middle of the Pacific Ocean (almost 2000 km off the coast of Mexico) and passing through heavily populated areas such as Northern India, and major cities like New Delhi, Tehran and Dhaka. Owing to the asteroid orbit geometry with respect to the Earth, there appear to be no significant opportunities to increase the accuracy of the asteroid orbit by ground-based observations before the impact.

The chapter deals with the outcomes of a deflection action of 2015PDC carried out with a contactless technology, the *Ion Beam Shepherd* [11, 12]. The concept has the key advantage of exploiting ionic thrusters of high specific impulse as an efficient means to both transfer the spacecraft to a rendezvous trajectory with the asteroid and, if required, to produce an accurate deflection that can be tailored according to specific requirements. Three worst-case scenarios are considered, in which the asteroid is predicted to impact in the vicinity of New Delhi (India), Dhaka (Bangladesh) and Tehran (Iran). The work carried out in this thesis has been concerned with the impact damage mitigation assessment for these scenarios.

7.1 IMPACT AND DEFLECTION ASSESSMENTS

Given the potentially devastating outcome of this scenario the first action to be taken is, arguably, to send a spacecraft to rendezvous with the asteroid and drastically reduce the uncertainty of its orbit by collaborative orbit determination [68]. In the case an impact with the Earth is predicted to occur, the first thing to do will be to carefully assess its consequences.

An impact in the open sea, for instance, might not be critical enough to warrant deflection as asteroids smaller than 500 m lack the capability of generating tsunami-like waves when the asteroid impacts as a monolithic body [44]. The generation of what is referred to as a “meteotsunami” [72], as a consequence of the coupling between the explosion-induced atmospheric waves and the ocean surface, is considered a rare event [43]. On the other hand, impacts to within about 100 km from coastlines or densely populated area would have catastrophic consequences [44], and the expected infras-

Table 11: Nominal ephemerides of fictitious asteroid 2015PDC at the epoch April 13, 2015, 00:00:00 UT.

Epoch (MJD)	57 125
a (au)	1.775 998 173 759 480
e	0.490 405 466 821 784 6
i ($^{\circ}$)	5.346 959 360 109 084
Ω ($^{\circ}$)	340.404 944 894 090 4
ω ($^{\circ}$)	313.438 355 343 508 1
M ($^{\circ}$)	330.162 703 310 612 8

Table 12: Modified reference epoch of Table 11 leading to worst-case impacts.

Epoch (MJD)	Impact point coordinates	Nearest city
57 124.998 402 7	28.8°N, 77.2°E	New Delhi
57 124.998 831 0	23.8°N, 90.3°E	Dhaka
57 124.997 978 0	35.9°N, 51.4°E	Tehran

structure damage including massive evacuation will most definitely dwarf the cost of a large space deflection mission.

The reference orbital elements of fictitious asteroid 2015PDC are provided in Table 11 and are relative to the epoch of April 13, 2015 at 00:00:00 UT. The elements are referred to a heliocentric J2000 ecliptic reference frame. The asteroid orbit is propagated up to the instant of intersection with the Earth ellipsoid using a simplified solar system model including gravitational perturbations from the planets, the Moon (through the JPL DE405 ephemerides) and the three largest asteroids¹ (Ceres, Vesta and Pallas) as well as first-order relativistic corrections.

While the propagation of the nominal asteroid ephemerides would lead to an impact in the South China Sea (113.9°E, 13.8°N), a preliminary estimate of the path of risk can be obtained by propagation starting from a set of slightly modified reference epochs. This would approximate a Line of Variation (LOV) sampling of the asteroid close to the direction of greatest orbit determination uncertainty [70].

The path of risk obtained following the above procedure is plotted in Figure 35, and stretches from Eastern Turkey (37.9°N, 37.3°E) up to the South Pacific (6.7°N, −118.9°E) passing through heavily populated regions of the Earth, including Northern India.

In particular, the three heavily populated capitals of New Delhi, Dhaka and Tehran happen to lay very close to the path, as shown in Figure 36. The initial conditions of orbits leading to an impact close to these cities are reported in Table 12 and were used as reference worst-case scenarios for the deflection mission.

¹ Note that, due to the eccentric orbit of 2015PDC, the perturbation from the largest asteroid belt objects can have an important effect. For high fidelity propagation it is recommended to employ as many as the largest objects as possible.

Figure 35: Path of risk of the fictitious asteroid 2015PDC. Credits: Hodei Urrutxua and Claudio Bombardelli.

7.2 RENDEZVOUS MISSION DESIGN AND IMPACT UNCERTAINTY COMPUTATION

Given the quite high collision probability estimated in June 2015 and the lack of significant observation opportunities until prior to the impact date it is highly recommended to design and launch a space mission to rendezvous with 2015PDC as soon as possible, track its trajectory and determine its size. The mission architecture that is proposed here is based on a rendezvous by a contactless Ion Beam Shepherd (IBS) [11], that will be sized as to make the deflection of a medium-sized asteroid possible.

The most favourable trajectory found for the deflection consists of a launch on May 28th, 2017 and arrival at the asteroid on September 30th, 2019 which requires 200 kg of fuel. The transfer trajectory consists of a thrust-coast-thrust structure 2.34 years long. From the moment of arrival, the visibility conditions from the Earth are good until the impact, with increasing Sun elongation.

An additional advantage of a rendezvous mission consists in the possibility of performing collaborative orbit determination to reduce the uncertainties associated with the observations of 2015PDC. Without performing a detailed analysis and assuming an accuracy in position and velocity similar to the one available for the Itokawa asteroid at the time of rendezvous with the *Hayabusa* spacecraft, it is possible to construct a covariance matrix for 2015PDC under conservative assumptions. Propagating the covariance matrix up to the impact and projecting the uncertainty ellipsoid on the Earth surface gives a 1-sigma ellipse of 120 km × 6 km, and a 5-sigma ellipse of 600 km × 30 km. Thanks to the continuous orbit determination campaign this uncertainty will be continuously reduced as the impact date approaches, however even the initial uncertainty is small enough to reasonably estimate the area affected by the impact.

We refer to [10] for further details on the design of the rendezvous mission and on the estimation of the impact uncertainty.

7.3 MITIGATION OF IMPACT DAMAGE

Owing to the relatively short warning time and the complexity of the interplanetary trajectory to reach 2015PDC it will be difficult to obtain, using a non-nuclear method, a total deflection sufficient to have the asteroid missing

Figure 36: Path of risk of fictitious asteroid 2015PDC near New Delhi, Dhaka and Tehran. Credits: Hodei Urrutxua and Claudio Bombardelli.

the Earth by a safe amount. This is especially true if the asteroid diameter is confirmed to be in the range of 200 m to 250 m or larger. On top of that, the risk of having the asteroid striking a different part of the planet as a result of a mission failure or an unexpected deflection outcome may in the end play against a full-scale deflection mission. Such a mission should be designed with a safety factor large enough to make an unwanted impact totally unlikely. This may be very complicated, the more so the larger the asteroid.

An alternative mitigation action is to shift the predicted impact point of a small amount, just enough to have the asteroid strike a deserted or extremely low populated area, as to drastically reduce any expected damage or casualty risk. This approach is analysed in detail in the following.

7.3.1 *Damage indices*

A land impact of a 150 m to 400 m diameter asteroid may cause a significant amount of human casualties and damage to infrastructures, the magnitude of such effects depending on whether the impact point will be located close to a populated or urbanized area. Estimating such damage and its variation as a function of the impact point displacement is paramount in order to construct an effective mitigation action. Assuming that the asteroid is accurately and continuously tracked by the rendezvous spacecraft during the whole deflection mission, a rather accurate estimation, possibly within a few kilometres, of the impact point location will be available a few months before the impact. If the asteroid impact point is retargeted towards a min-

imum population density region there will still be the need to evacuate a sufficiently large area falling within the expected radius of action of the asteroid impact. The evacuation procedure will have a cost depending on the total number of inhabitants to be displaced. In addition, all infrastructures located within the radius of action of the impact may suffer irreversible damage that will need to be quantified.

In light of the above considerations, two complementary mitigation indexes are considered. The first, denominated *Evacuation Cost Index* (ECI) is here defined as the ratio between the value P of the estimated population potentially affected by the impact in a generic location and the value P_0 computed for the initial, undeflected impact location,

$$ECI = \frac{P}{P_0}.$$

The values for P and P_0 can be obtained from the 2015 projections of global population density data [19].

Note that the approach followed here differs from the one used by other authors (see for instance Rumpf [89]) as no convolution with the impact position uncertainty distribution is applied. This is justified by the assumption that the impact uncertainty is much smaller than the critical radius of action of the asteroid impact damage.

The second mitigation index is the *Infrastructure Damage Index* (IDI), defined as the ratio of integrated night-time light intensity over the asteroid impact damage area computed for the deflected and undeflected impact point:

$$IDI = \frac{\int_A s dA}{\int_{A_0} s dA}.$$

Here s is the light intensity per unit area quantized in 101 grayscale levels in a high-resolution, night-time image of the Earth [55]. The IDI index is important as areas with similar population density can result in very different estimated damage figures, depending on the presence of valuable infrastructures whose damage can never be completely eliminated.

7.4 DEFLECTION SCENARIOS

A simulation campaign to evaluate the IBS deflection capability of asteroid 2015PDC has been conducted for each of the three worst-case scenarios of Table 12 using the same high-fidelity orbit propagation model previously described. A continuous low-thrust deflection force is transmitted to the asteroid along or opposite to the instantaneous velocity vector. The thrust is applied starting in November 2019 (one month after rendezvous) and for a duration between one and 33 months obtaining a series of displaced impact points according to the duration of the deflection. The scenario assumes a 250 m diameter asteroid with 2 g cm^{-3} density has been perturbed with a 185 mN deflection force at 1 au, with a varying magnitude inversely proportional to the distance from the Sun elevated to the 1.7 power. The ECI and IDI are computed as a function of the shifted impact point during the whole deflection period. In this way it is possible to estimate the duration of deflection which minimizes evacuation costs and infrastructural damage during the whole deflection period. The estimated population within the asteroid impact radius of action has also been computed.

Figure 37: IBS deflection track starting from a predicted impact in New Delhi. A cross marks the initial impact point.

Figure 38: Same as Figure 37 but visualizing a population density map. The circle represents a 100 km radius area around the minimum damage impact point.

7.4.1 New Delhi impact

A predicted impact of 2015PDC near New Delhi would prompt an immediate deflection action. By looking at Figures 37 and 38 one immediately realizes that the nearest unpopulated region where the asteroid could be sent is central Afghanistan. Of course, such a decision would be extremely delicate from a political point of view: the asteroid would need to travel across a highly populated region in Pakistan, who may strongly oppose such a move. However, political negotiations could finally lead to a solution in which the threatened country (India) decides to pay Pakistan a toll for the right of having the asteroid impact point passing over its territory and to pay a fee to Afghanistan for using its unpopulated territory to “absorb” the asteroid impact threat. India may decide to conduct the deflection mission on its own or have to pay another country or space agency for doing that. The deflecting country/agency may need to accept liability for any failure to accomplish its mission, which would contribute to the price to be paid in exchange for the deflection.

In Figures 37 and 38, the displaced impact points corresponding to each month of deflection are denoted with circles. The achieved shift is larger in the first half of the deflection period, and gets significantly smaller in the last months, when the applied deflection becomes less efficient due to the time before impact decreasing. Figure 39 highlights the benefit of such a

Figure 39: Evolution of the ECI and IDI indexes during the deflection, starting from a predicted impact in New Delhi.

deflection. It is possible to reduce the estimated casualties by two orders of magnitude and to virtually eliminate any infrastructural damage after 22 months of applied deflection (from that point on the integrated illumination map becomes zero). This corresponds to relocating the impact point in the sparsely populated Paktika province of Afghanistan, close to the border with Pakistan. There appear to be virtually no modern infrastructures in the area according to the night light intensity map, which means that the estimated damage would be limited to the cost of a coordinated evacuation of a relatively low number of inhabitants (below 500 000 according to Figure 46) and possible compensation for minor damages. For shorter deflections (of less than 15 months) it is still possible to reduce casualties and infrastructural damage by roughly one order of magnitude and have the impact point move to the Punjab region of Pakistan. This would have a reduced benefit in terms of infrastructural damage as reflected by a somewhat higher level of night-time illumination compared to the secluded Paktika region.

7.4.2 Dhaka impact

A deflection action to avert an impact in Dhaka would be directed at displacing the impact point towards Myanmar, also crossing the Siliguri corridor belonging to India. Figures 40 and 41 show the displaced impact points for each additional month of deflection duration. The optimal deflection, which reduces casualties and infrastructure damage by two orders of magnitude, is achieved after 13 months (see Figure 42). This corresponds to shifting the impact point to the mountainous and scarcely inhabited Chin State in Myanmar. A considerably large number of people (about 1.2 million) would still fall in the estimated 100 km radius of action of the asteroid damage. However, one should also consider the benefit of having the asteroid land in a mountainous region with higher damage attenuation capability. As before, this would imply negotiations in which Bangladesh agrees to pay an impact fee to Myanmar for the right to displace the impact point to its territory, which would also cover the cost of evacuating the affected zones. Since some of the impact points would affect both Bangladesh and India, this could lead to an internationally coordinated action in which both states cooperate in the deflection.

Figure 40: IBS deflection track starting from a predicted impact in Dhaka. A cross marks the initial impact point.

Figure 41: Same as Figure 40 but visualizing a population density map. The circle represents a 100 km radius area around the minimum damage impact point.

7.4.3 Tehran impact

A predicted impact in Tehran presents the easiest challenge, since it is possible to cut potential casualties by more than two orders of magnitude after between one and two months of deflection. Figures 43 and 44 reveal that the impact points which allow to minimal damage are still contained in Iran, thanks to the presence of large swaths of deserted land that could accommodate the impact. This would greatly simplify political issues. However, the mission budget could still require a degree of international cooperation. In addition, international coordination would greatly benefit the transparency of the deflection action and provide opportunities for exchanging technical and scientific knowledge derived from the mission. As shown in Figure 45, after about 5 months of deflection both the ECI and IDI indexes start to rise again, with the IDI reaching a peak around month 8 due to the path of the deflected impact points passing through the city of Gonabad. This demonstrates the importance of accurately planning the deflection maneuver as to avoid causing collateral damage. Between months 12 and 16 the IDI becomes zero as a consequence of the asteroid landing in a completely dark zone, which would be ideal for the impact. The estimated evacuation action in that case would involve less than 170 000 inhabitants (Figure 46).

Figure 42: Evolution of the ECI and IDI indexes during the deflection, starting from a predicted impact in Dhaka.

Figure 43: IBS deflection track starting from a predicted impact in Tehran. A cross marks the initial impact point.

7.5 SUMMARY

A preliminary design and analysis of a slow push deflection mission of fictitious asteroid 2015PDC has been conducted making use of an ion beam shepherd (IBS) spacecraft. The solution preferred here is to retarget the asteroid impact point to a low populated region and the spacecraft mass required to effectively do so is computed starting from three worst-case scenarios involving an impact in highly populated cities. A spacecraft would be sent to rendezvous with the asteroid and perform an indispensable refinement of its orbit accuracy to confirm or rule out an impact. In case an impact near a major city is predicted an impact point retargeting operation would follow, which greatly reduces mission costs when compared to full deflection. Two indices are introduced to quantify the benefit of such action by looking at evacuation costs and infrastructure damage. The approach could also be feasible from the political point of view as long as mission costs, liabilities and compensations are agreed upon at international level (still, an important open question is how long it would take to reach such an agreement).

Figure 44: Same as Figure 43 but visualizing a population density map. The circle represents a 100 km radius area around the minimum damage impact point.

Figure 45: Evolution of the ECI and IDI indexes during the deflection, starting from a predicted impact in Tehran.

Figure 46: Total population to be evacuated in case of an impact as a function of the deflection duration for the three worst-case scenarios.

Following the success of the exercise on a hypothetical impact developed in the 2015 Planetary Defense Conference, which was described in the previous chapter, another fictitious impact scenario was proposed in the 2017 Planetary Defense Conference in Tokyo, Japan.

The fictitious asteroid 2017PDC is discovered on March 6, 2017 at a distance of approximately 0.36 au, and it is determined that it will exhibit a close encounter with the Earth on July 21, 2027 at a distance that may be less than 0.05 au. The impact probability for this encounter is estimated to be around 0.0025 % at discovery, rising in May 2017 to 1 %. It has an absolute magnitude of $H = 21.9 \pm 0.4$ but its albedo is unknown, indicating a diameter in the range 100 m to 300 m. Considering an average density of 2500 kg m^{-3} , its mass is estimated between 10^9 kg and 10^{11} kg . These figures correspond to a destructive potential on a regional scale. The path of risk computed from the orbital data starts from the North Pacific ocean, includes several heavily populated regions in Asia and Europe, and terminates in the North Atlantic ocean.

In the work of Hernando-Ayuso, Amato, and Bombardelli [52], a deflection mission through a stand-off nuclear explosion is proposed. A low-thrust trajectory solution is found, allowing a rendezvous with the asteroid about 3.5 years before impact, on November 30, 2023. The deflection impulse is minimized through a fast semi-analytical algorithm under a constraint on the b-plane coordinates, allowing one to choose between several deflection strategies. As a contribution stemming from this thesis, the resonant returns of 2017PDC after the deflection manoeuvre were analysed as to estimate the probability of a subsequent short-term impact. The chapter gives an outline of the proposed deflection mission and shows in detail the results from the analysis of the resonant returns.

8.1 RESONANT RETURNS AND KEYHOLES

In [chapter 3](#), we gave a review of the b-plane reference frame and described its importance in the analysis of close encounters. We recall here some concepts essential to the analysis of the resonant returns and expand the theoretical background, as originally developed by Valsecchi et al. [100]. The reader is invited to consult the aforementioned reference for a detailed presentation of the analytical theory of resonant returns.

As in [subsection 3.1.3](#), we assume the Earth's orbit to be circular and reference units such that the Earth's distance from the Sun is unitary, its orbital period is 2π and the mass of the Sun is also unitary.

8.1.1 Effect of the encounter

We recall that the components of the incoming (pre-encounter) asymptotic velocity $\mathbf{U} = \mathbf{v}_{\text{Ast}} - \mathbf{v}_{\oplus}$ in the (X, Y, Z) frame defined in [subsection 3.1.1](#) are defined by the quantities U, θ, ϕ :

$$\begin{Bmatrix} U_X \\ U_Y \\ U_Z \end{Bmatrix} = \begin{Bmatrix} U \sin \theta \sin \phi \\ U \cos \theta \\ U \sin \theta \cos \phi \end{Bmatrix}. \quad (43 \text{ revisited})$$

The asteroid travels in a uniform rectilinear motion with velocity $\mathbf{U}$ until crossing the b -plane. At the time of crossing, its coordinates in the b -plane frame are $(\xi, 0, \zeta)$ and the asteroid reaches the minimum distance from the Earth $b = \sqrt{\xi^2 + \zeta^2}$. We assume that no impact takes place, that is to say $b > b_{\oplus}$ where $b_{\oplus}$ is the Earth radius augmented by the gravitational focussing,

$$b_{\oplus} = R_{\oplus} \sqrt{1 + 2 \frac{1}{R_{\oplus}} \frac{\mu_{\oplus}}{\|\mathbf{v}_{\oplus} - \mathbf{v}_{\text{Ast}}\|^2}}. \quad (111)$$

The effect of the encounter is to rotate the velocity $\mathbf{U}$ by the angle γ , computed from [Equation 48](#), along the angular momentum vector $\mathbf{h} = \mathbf{b} \times \mathbf{U}$. Thus the post-encounter velocity magnitude U' is left unchanged, while $\mathbf{U}'$ is found by:

$$\mathbf{U}' = \mathbf{R}_{\mathbf{h}}(\gamma) \mathbf{U},$$

where $\mathbf{R}_{\mathbf{h}}(\gamma)$ is the matrix expressing the counter-clockwise rotation by the angle γ around the angular momentum $\mathbf{h}$. The post-encounter angles are obtained from the components of $\mathbf{U}$ through:

$$\cos \theta' = U'_Y / U \quad (112)$$

$$\tan \varphi' = \text{atan2}(U'_X, U'_Z). \quad (113)$$

Since the post-encounter b -plane is perpendicular to $\mathbf{U}'$, the post-encounter b -plane coordinates also change. We can find them as a function of the pre-encounter parameters:

$$\xi' = \frac{(b^2 + c^2) \xi \sin \theta}{\sqrt{[(b^2 - c^2) \sin \theta - 2c\zeta \cos \theta]^2 + 4c^2 \xi^2}} \quad (114)$$

$$\zeta' = \frac{(b^2 - c^2) \zeta \sin \theta - 2b^2 c \cos \theta}{\sqrt{[(b^2 - c^2) \sin \theta - 2c\zeta \cos \theta]^2 + 4c^2 \xi^2}}. \quad (115)$$

It is also useful to express the post-encounter semi-major axis of the asteroid's heliocentric orbit as:

$$a' = \frac{b^2 + c^2}{(b^2 + c^2)(1 - U^2) - 2U[(b^2 - c^2) \cos \theta + 2c\zeta \sin \theta]}. \quad (116)$$

8.1.2 Resonant returns

The value of the post-encounter semi-major axis a' may be such that the post-encounter orbital periods of the asteroid and of the Earth are commensurable. In this case, after k revolutions of the Earth and h revolutions of the asteroid, the bodies will be in the same geometric configuration, giving

rise to a *resonant return*. The post-encounter semi-major axis that verifies this condition is:

$$a'_R = \sqrt[3]{\frac{k^2}{h^2}}. \quad (117)$$

It can be shown [100] that the locus of points on the b-plane for which the post-encounter semi-major axis satisfies equation (117) is a *resonant circle* of radius R centered in $(0, D)$, where:

$$R = \left| \frac{c \sin \theta'_R}{\cos \theta'_R - \cos \theta} \right| \quad (118)$$

$$D = \frac{c \sin \theta}{\cos \theta'_R - \cos \theta}, \quad (119)$$

and

$$\cos \theta'_R = \frac{1 - U^2 - 1/a'_R}{2U}. \quad (120)$$

Assuming unperturbed Keplerian motion between the encounters, the geometry of the next encounter will be identical. Thus the pre-encounter asymptotic velocity at the second encounter, defined by $(U'', \theta'', \varphi'')$ coincides with the post-encounter (outgoing) velocity at the first,

$$U'' = U' = U, \quad \theta'' = \theta', \quad \varphi'' = \varphi'. \quad (121)$$

and since also the MOID of the two orbits is unchanged $\xi'' = \xi'$. The coordinate ζ'' for the second encounter is found as:

$$\zeta'' = \zeta' - \left[\text{mod} \left(h \cdot 2\pi a'^{3/2} + \pi, 2\pi \right) - \pi \right] \sin \theta'. \quad (122)$$

Note that plugging a'_R (Equation 117) in the above equation results in $\zeta'' = \zeta'$. In fact, the ζ'' coordinate only changes if the encounter does not take place after exactly h asteroid periods.

8.1.3 Keyhole calculation

Even if an Earth impact does not take place in the first close encounter, there is the possibility that it could take place in a resonant return. For this reason, it is interesting to understand which are the regions of the first b-plane through which the asteroid has to pass in order to have an impact in the subsequent resonant return. The projection of the Earth on the b-plane of the second encounter is simply a circle of radius $b_\oplus$, with its center in the origin. A *keyhole* is a region in the b-plane of the first encounter giving rise to an impact in the second, and it is a pre-image of this circle. Keyholes are localized in the vicinity of resonant circles, since a resonant return is a necessary condition for them to arise.

For each $k : h$ resonant circle, which is found through Equations (117)-(120), the associated keyholes are computed through an iterative procedure. Let (ξ_K, ζ_K) denote a point on the circle; since we assume that the MOID is unchanged by the Keplerian propagation we have that a necessary condition for (ξ_K, ζ_K) to belong to a keyhole is that:

$$\xi''_K = \xi'_K = \xi'(U, \theta, \xi_K, \zeta_K) \leq b_\oplus, \quad (123)$$

where ξ'_K is computed by Equation 114. If the above inequality is satisfied, we use a root-solving procedure (such as *regula falsi*) to find the value $\zeta_{K,0}$ leading to a direct impact with the Earth,

$$\zeta''_{K,0} = \zeta''(U, \theta, \xi_K, \zeta_{K,0}) = 0, \quad (124)$$

where $\zeta''_{K,0}$ is found by using Equations (115), (116) and (122) in succession. By approximating ζ'' as

$$\zeta'' \approx \left. \frac{\partial \zeta''}{\partial \zeta} \right|_{\zeta=\zeta_{K,0}} (\zeta - \zeta_{K,0}), \quad (125)$$

we obtain the upper and lower bounds of the keyhole $(\zeta_{K,u}, \zeta_{K,d})$ as:

$$\begin{aligned} \zeta_{K,u} &= \zeta_{K,0} + \frac{\sqrt{b_{\oplus}^2 - \xi_K''}}{(\partial \zeta'' / \partial \zeta)_{\zeta=\zeta_{K,0}}} \\ \zeta_{K,d} &= \zeta_{K,0} - \frac{\sqrt{b_{\oplus}^2 - \xi_K''}}{(\partial \zeta'' / \partial \zeta)_{\zeta=\zeta_{K,0}}}. \end{aligned} \quad (126)$$

The derivative $\partial \zeta'' / \partial \zeta$ can be written as:

$$\frac{\partial \zeta''}{\partial \zeta} = h s(u', \theta') \frac{\partial \cos \theta'}{\partial \zeta} + \frac{\partial \zeta'}{\partial \zeta}, \quad (127)$$

where $s(u', \theta')$ is as in Equation 57 and the full expression for $\partial \zeta' / \partial \zeta$ is presented in [100]. We construct the keyhole by repeating this procedure for all the (ξ_K, ζ_K) on the $k : h$ resonant circle.

Since initially close trajectories diverge strongly after the encounter due to different values of α' , the derivative $\partial \zeta'' / \partial \zeta$ takes very large values. Therefore, the image of the Earth in the second b-plane is compressed into a thickened arc in the vicinity of the $k : h$ resonant circle.

We emphasize that various approximations are involved in the computation of the keyhole. In reality, the MOID is a slowly-varying function of time due to lunar and planetary perturbations. The accuracy of the computation can be increased by considering ξ'' to be linear in time, with its rate of change computed by either an analytical theory or a numerical propagation. Also, corrections can be applied to take into account the eccentricity of the Earth and small terms due to the non-zero miss distance b .

8.2 DEFLECTION MANOEUVRE

The impulsive deflection can be linearized around the unperturbed in-plane motion in a compact way by expressing the resulting variation in *Dromo* orbital elements [9]. Assume that the manoeuvre is performed Δf radians before impact, including multiple revolutions. The resulting variation in b-plane coordinates can be written as:

$$\mathbf{b} = \mathbf{b}_0 + \sqrt{\frac{r_c^3}{\mu_{\odot}}} \mathbf{R} \mathbf{K} \mathbf{D} \Delta \mathbf{v}, \quad (128)$$

where $\mathbf{R}$, $\mathbf{K}$, $\mathbf{D}$ are the Rotation, Kinematics, and Dynamics matrices respectively, and $\Delta \mathbf{v}$ is the deflection impulse in the radial, tangential and out-of-plane directions. The elements of the $\mathbf{D}$ matrix depend on the angular anticipation Δf ; the full expressions for all the matrices are presented in [52].

Equation 128 allows one to write the square of the miss distance as a quadratic form of the post-deflection b-plane coordinates, which are a function of the optimization variable $\Delta \mathbf{v}$. The minimum magnitude $\Delta \mathbf{v}$ that is necessary to achieve a fixed miss distance at collision for a given manoeuvre epoch (i.e. given the angular anticipation Δf) can be obtained by solving a convex optimization problem.

Table 13: Asymptotic velocity U , angles θ and ϕ , and b-plane coordinates for the nominal impact of 2017PDC on 2027 July 21, 08:17 UTC.

U (km s $^{-1}$)	θ (°)	ϕ (°)	ξ ($R_{\oplus}$)	ζ ($R_{\oplus}$)
13.210	65.73	71.41	−0.8682	0.141

Hernando-Ayuso, Amato, and Bombardelli [52] show that assuming a 1200 kg-spacecraft equipped with an electric propulsion system with a 3500 s specific impulse and 400 mN thrust at 1 au, it is possible to find an “early-launch” solution departing on June 27, 2021 and arriving at the asteroid on November 30, 2023. This is the preferred solution for the deflection, since it is possible to carry out an orbit determination campaign lasting two months before executing a deflection manoeuvre around January 29, 2024. This epoch corresponds to a minimum in the required Δv , which is as low as 0.0211 m s $^{-1}$ for a miss distance of 3 $R_{\oplus}$. As the asteroid properties are mostly uncertain and the execution of the deflection manoeuvre is affected by non-negligible error, a sensitivity analysis is performed for the “early-launch” manoeuvre assuming a 3-sigma uncertainty in the orientation angles of 10° and a $\pm 25\%$ variation in the impulse magnitude. The errors translate in a thin dispersion ellipsoid of the b-plane intersection points aligned with the ζ axis. The minimum miss distance considering the uncertainties is about 2 $R_{\oplus}$.

Finally, also a “late-launch” solution has been found, departing on August 2, 2025 and arriving at the asteroid on June 4, 2027. This requires an initial payload mass of 1695 kg to achieve the deflection at 3 $R_{\oplus}$, rather than the 1400 kg obtained with the “early-launch” solution.

8.3 KEYHOLE ANALYSIS

Using the ephemerides published by the NASA Center for Near Earth Object Studies¹, we computed the quantities defining the pre-encounter velocity vector and the b-plane coordinates of the nominal impact point for the 2027 impact of 2017PDC (Table 13).

As outlined above, the outcome of an optimal deflection manoeuvre is to displace the b-plane crossing point towards a circle of radius 3 $R_{\oplus}$. However, the manoeuvre has to be planned in order to avoid pushing the asteroid into a keyhole, since this situation would lead to a subsequent impact in a resonant return. Low-order resonances which lead to short-term impacts are of particular concern. Thus, we only consider values $k \leq 20$ corresponding to resonant returns in at most 20 years. The analysis can be extended in a straightforward way to higher values of k .

We compute the resonant circles on the b-plane of the 2027 encounter using the procedures detailed in subsection 8.1.2 and show them in Figure 47a, where the inner dashed circle has a radius of 1 $R_{\oplus}$ and the solid black circle is the image of the Earth; its radius $b_{\oplus} \approx 1.3 R_{\oplus}$ is augmented due to the gravitational focusing (Equation 111). The nominal impact point and the crossing points following the optimal deflection are marked with a red cross and with red dots, respectively. To guide the eye, we also plot the circle of

¹ URL: <https://cneos.jpl.nasa.gov/pd/cs/pdc17/>, last visited May 5, 2017.

Figure 47: Resonant circles and keyholes on the b-plane of the July 2027 encounter of 2017PDC. The inner dashed circle has a radius of $1 R_{\oplus}$ and the solid black circle is the image of the Earth; its radius $b_{\oplus} \approx 1.3 R_{\oplus}$ is augmented due to the gravitational focusing. The nominal impact point is indicated by a red cross and the crossing points consequent to the deflection for varying manoeuvre anticipation times are displayed with red dots.

Figure 48: Zoomed views of the crossing points close to the 2 : 1 resonant circle and in the upper left part of the b-plane. The colored lines refer to the keyholes displayed in Figure 47b.

radius $3 R_{\oplus}$ with a dashed line. The color of the resonant circles corresponds to k , that is the number of years after which a resonant return takes place. For deflections during the asteroid's last orbit, the crossing point moves towards a region spanned by several resonant circles. However, reaching this region requires an excessive Δv and we will not consider it in this analysis. For manoeuvres other than last-minute, the deflection lead to points close to resonant returns with $\xi < b_{\oplus}$. In particular, some of the crossing points amass close to the 2 : 1 resonant circle (dark blue circle with radius of approximately $1.5 R_{\oplus}$ in the lower half of the b-plane), leading to a possible impact after two years. In these cases, a deflection error causing the crossing points to disperse in the b-plane might push the asteroid into a resonant return with a possible impact.

Figure 47b displays the keyholes relative to the resonant circles, which are computed with the procedure in subsection 8.1.3. By choosing a distance of $3 R_{\oplus}$ for the deflection it is possible to avoid the most dangerous regions of the b-plane, where the keyholes are densest. Examination of the regions where the crossing points concentrate, displayed in Figures 48a and 48b, re-

Figure 49: Maximum keyhole width as a function of k .

veal that none of these cases is particularly critical. In both of the regions, the minimum distance from the closest keyholes is in the order of several hundreds of kilometers. However, Hernando-Ayuso, Amato, and Bombardelli [52] show that the dispersion of points on the b -plane consequent to a deflection error during the “early-launch” manoeuvre could indeed span several keyholes.

8.3.1 Probability of a subsequent impact

We display the maximum width of the keyholes as a function of k in Figure 49. The largest keyholes correspond to the smallest values of k , since in this case the reduced interval of time between the encounters constrains the divergence of the trajectory. Note that, due to the derivative $\partial\zeta''/\partial\zeta$ attaining values in the order of 10^4 , the widths of the keyholes are extremely small compared to the Earth’s radius. While the largest keyhole is 4 kilometres wide at maximum, most of the keyholes only have a width of several hundred meters.

Knowing the keyhole widths, we can make a rough calculation of the probability of causing an impact in a resonant return due to the deflection error of the “early-launch” manoeuvre. We approximate the dispersion on the b -plane due to the uncertainties in asteroid properties and manoeuvre execution with a vertical segment centered on $(\xi, \zeta) = (-0.86 R_\oplus, 2.87 R_\oplus)$ spanning a length of $1.43 R_\oplus$, corresponding to 3-sigma ζ values. The impact probability for a subsequent resonant return, estimated by dividing the length of the segment by the sum of the maximum widths of the intersected keyholes, is of 0.01 %. While this is not a negligible value, it is enough to assert the “early-launch” option as a viable deflection strategy. Moreover, the value of 0.01 % has to be taken as an upper bound which can be refined by removing the approximations in the computation of the keyholes (subsection 8.1.3), integrating the probability density function in the b -plane over the keyholes, and by Monte Carlo simulations exploiting more sophisticated physical models.

8.4 SUMMARY

We explored the possibility of averting an impact by the fictitious asteroid 2017PDC on July 21, 2027 through a nuclear deflection mission. Using electric propulsion allows a spacecraft to arrive at the asteroid sufficiently early as to deliver an impulsive Δv large enough to increase the miss distance to 3 Earth radii. The Δv required magnitude and orientation angles are opti-

mized through a semi-analytical procedure. We carried out an analysis of the keyholes consequent to the deflection manoeuvre for resonant returns in less than 20 years after the 2027 encounter. Due to the small width of the keyholes (a few kilometres at most) the probability of inserting the asteroid into a resonant return is low, around 0.01 %, confirming the validity to the proposed deflection manoeuvre.

CONCLUSIONS

This thesis focuses on novel applications of advanced orbit propagation methods to intriguing open problems in the astrodynamics of asteroids and space debris. Furthermore, we present two studies in the deflection of fictitious impacting asteroids. We provide a summary of the obtained results, with concluding remarks, in the following.

We analysed asteroid close encounters by comparing the performance of regularized and Cowell formulations in the propagation of hyperbolic orbits and asteroids exhibiting resonant returns. The initial conditions for the latter were generated by an approximate procedure making use of the Öpik theory. In particular, we measured the errors on the b-plane coordinates of resonant returns, a choice that has direct implications for impact monitoring activities. A novel trajectory splitting algorithm, akin to the patched conics technique, has been presented with the aim of improving performance of regularized formulations. Regularized formulations used in combination with a trajectory splitting algorithm exhibit the best performance in all the numerical experiments. A remarkable result is that the error in position on the b-plane is abated by using regularized formulations rather than Cowell.

By introducing a bidimensional parametrization of close encounters in the planar, circular, restricted three-body problem, we have expanded the results through large-scale numerical simulations with both multi-step and single-step integrators. While confirming the excellent performance obtained with regularized formulations, we also study the criterion for the switch of primary bodies in the trajectory splitting algorithm. We find that the switch distance that minimizes the numerical effort is between 1.2 and 3 Hill radii, i.e. significantly larger than the sphere of influence radius. In the propagation of the asteroid (99942) Apophis, we compare our methods to those implemented in the MERCURY code and find out that the best performance is achieved by integrating EDromo regularized equations with the single-step XRA15 solver. The latter is based on an improved Everhart-Radau numerical scheme together with an algorithm for the output at prescribed values of physical time for regularized formulations, and it has been developed specifically for the study.

In the long-term propagation of Earth satellite orbits, we take into account semi-analytical propagators as a benchmark. We carry out a qualitative analysis of the integration error for averaged methods, presenting a breakdown in components that can be attributed to different approximations in the averaging process. We compare regularized formulations to the semi-analytical propagator STELA, highlighting some limits of the averaged equations. Specifically, regularized element methods display an excellent performance for the propagation of challenging HEO orbits. This is of particular relevance for end-of-life studies, since the HEO regime lacks a natural removal mechanism such as the atmosphere, and operational orbits have to be designed for demise.

The improvement of orbit propagation methods for asteroid close encounters plays an important role in the prevention of destructive impacts. We present two studies of deflection of the fictitious impacting asteroids 2015PDC and 2017PDC. We introduce damage indices measuring the repercussions of the 2015PDC asteroid impact, which can be minimized by ex-

ecuting a retargeting of the impact point to a deserted area. Moreover, we carry out a detailed analysis of the short-term resonant returns after the deflection of 2017PDC through a nuclear device, and we estimate the probability that the deflection manoeuvre could originate an impact in a following return, taking into account uncertainties.

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