UNIVERSIDAD POLITÉCNICA DE MADRID
ESCUELA TÉCNICA SUPERIOR DE INGENIEROS AERONÁUTICOS

PROPAGATION AND OPTIMAL CONTROL
OF SPACE TRAJECTORIES USING
PERTURBATION METHODS

TESIS DOCTORAL

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2017

Resumen

El desarrollo de motores espaciales de alto impulso específico y bajo empuje, como los iónicos o los de efecto Hall, ha planteado nuevas posibilidades y retos en el diseño de misiones y maniobras espaciales. Claros ejemplos son misiones recientes como Deep Space 1, SMART-1 o BepiColombo. El alto impulso específico de estos motores permite importantes reducciones en el consumo de propulsante, mientras que su capacidad de operar de forma continuada aporta una mayor flexibilidad y robustez en el diseño. Sin embargo, esta flexibilidad viene a costa de una mayor complejidad de diseño en comparación con los sistemas impulsivos, ya que el control debe expresarse como funciones del tiempo en vez de como un conjunto discreto de maniobras puntuales. El problema se agrava al tener en cuenta que el bajo valor del empuje trae aparejados una menor autoridad del control y tiempos de misión más largos. Para poder afrontar estos desafíos y aprovechar al máximo las ventajas de estos motores, es necesario desarrollar nuevas herramientas matemáticas, tanto numéricas como analíticas. Esto no se limita a definir algoritmos para la resolución computacional de problemas prácticos de complejidad creciente, sino que comprende conseguir una mejor comprensión de la física subyacente y obtener soluciones analíticas aproximadas para diseño preliminar y estimación.

El objetivo fundamental de la tesis es desarrollar métodos y herramientas matemáticas para la propagación y optimización de trayectorias de bajo empuje, tanto ligadas a la Tierra como interplanetarias, y aplicarlos a diferentes problemas de interés práctico. Estas familias de problemas de control óptimo (OCPs, por sus siglas en inglés) se estudiarán tanto de forma numérica, implementando algoritmos que permitan resolverlos de forma precisa y eficiente, como analítica, buscando soluciones aproximadas.

Este trabajo cubre un amplio espectro de técnicas para el estudio de los OCPs, incluyendo tanto métodos directos como indirectos. En los métodos directos, el OCP se transcribe como un problema discreto de programación no lineal (*non-linear programming*, NLP), que se resuelve numéricamente empleando métodos iterativos a partir de una estimación inicial de la solución. En los métodos indirectos la solución se busca imponiendo las condiciones de optimalidad de primer orden, derivadas del principio del máximo de Pontryagin o mediante el cálculo

variacional, lo que da lugar a un problema de condiciones de contorno en dos puntos (*two-point boundary value problem*, TPBVP). Dicho TPBVP se estudia de dos formas diferentes y complementarias. Por un lado, se buscan soluciones analíticas aproximadas mediante métodos de perturbaciones. Por otro, se resuelve numéricamente empleando algoritmos iterativos.

El documento de tesis está estructurado entorno a tres formulaciones orbitales: una nueva formulación para el movimiento relativo en coordenadas curvilíneas; los elementos equinocciales modificados, originalmente introducidos por P. Ce-fola; y Dromo, un propagador orbital basado en elementos orbitales generalizados desarrollado por el Grupo de Dinámica Espacial de la UPM. La sección correspondiente al movimiento relativo se centra en la obtención de soluciones analíticas aproximadas para los problemas de cambio de fase, cambio de radio y cambio de inclinación, identificándose diferentes regímenes de operación. En el estudio en elementos equinocciales se desarrolla una formulación general para optimización indirecta con elementos orbitales, que se aplica al caso concreto del deorbitado de satélites de la constelación Galileo. La última sección estudia la aplicabilidad de Dromo al tratamiento de OCPs, y presenta una solución de escalas múltiples para el problema de empuje radial.

Abstract

The development of low-thrust, high-specific-impulse thrusters, such as ionic and Hall effect ones, has brought new possibilities and challenges into the field of space mission design. This is highlighted by missions such as Deep Space 1, SMART-1 and BepiColombo. The high specific impulse of these thrusters allows for important savings in propellant mass, while their capability to operate continuously confers greater flexibility and robustness to the design process. However, this flexibility comes at the cost of a greater design complexity compared to impulsive thrusters, since the control laws must now be expressed as time functions (instead of as a set of discrete maneuvers). Furthermore, this problem is aggravated by the fact that the low magnitude of the thrust implies a smaller control authority and longer mission times. In order to successfully address this challenges and fully exploit the advantages of low-thrust, high-specific-impulse thrusters, new mathematical tools have to be developed, both numerical and analytical. The aim is not just to be able to computationally solve increasingly complex practical problems, but also to gain a better insight into the underlying physics and to develop approximate analytical solutions for preliminary design and estimation.

The main objective of this thesis is to develop mathematical methods and tools for the propagation and optimization of low-thrust trajectories, both Earth-bound and interplanetary, and apply them to different practical test cases. These families of optimal control problems (OCPs) are studied both numerically, by implementing algorithms for their accurate and efficient resolution, and analytically, by searching for approximate solutions.

This work covers a wide range of techniques for solving OCPs, including both direct and indirect methods. With direct methods, the OCP is transcribed as a discrete nonlinear programming (NLP) problem, which is then solved numerically using iterative algorithms starting from an initial guess of the solution. Conversely, in indirect methods the solution is sought for by imposing the first order optimality conditions, derived from Pontryagin's Maximum Principle or using the calculus of variations, which yields a two-point boundary value problem (TPBVP). Said TPBVP is studied in this thesis in two different, yet complementary ways. On the one hand, approximate analytical solutions are sought for using perturbation

methods. On the other hand, numerical solutions are obtained using iterative algorithms.

The dissertation is structured around three different formulations for orbital dynamics: a novel relative motion formulation in curvilinear coordinates; the modified equinoctial elements, originally introduced by P. Cefola; and Dromo, an element-based orbital propagator developed by the Space Dynamics Group (Technical University of Madrid-UPM). The section devoted to the relative motion formulation focuses on obtaining approximate analytical solutions for the phase change, radius change and inclination change problems, identifying different operation regimes. In the next section a general framework for indirect optimization with element-based formulations is developed, which is then applied, together with the modified equinoctial elements, to the design of end-of-life disposal maneuvers for satellites in the Galileo constellation. The final sections focus on Dromo, studying its applicability to OCPs and presenting a multiple-scales solution to the radial thrust problem.

Acknowledgements

First of all, I would like to thank my supervisors, Claudio Bombardelli and Jesús Peláez. During all these years, Claudio has shown the most impressive capacity of finding new and unexplored roads for my research (many times at the cost of severe sleep deprivation). Jesús has always been a reference on how to handle problems in a calmed and efficient way. Between both of them a fine balance was reached, and I know my PhD years would have been completely different without them.

There is a Spanish saying that reads “De bien nacidos es ser agradecidos”, which can be loosely translated as “those of good breeding are thankful”. I cannot help but feeling deeply thankful to all the people at the Dipartimento di Scienze e Tecnologie Aerospaziali of the Politecnico di Milano, and specially to professors Francesco Topputo, Pierluigi DiLizia and Mauro Massari, for the time I spent there during my PhD research. That was a profoundly rewarding experience both at the personal and formative levels, that I shall always remember with joy.

I am also sincerely grateful to the members of my Committee for agreeing to devote the non-negligible time needed to review and evaluate my dissertation. I hope the quality of this work will be up to their expectations, so that it can be a time well expended.

The decision to willingly spend more than 10 years in a row studying can be difficult to understand for many (and they may be right). That’s why I’m thankful to my family. For not making too many questions about it, for accepting my dysfunctional schedule, and for being ready to handle the occasional sudden crisis of despair. This is certainly a road that should not be walked alone, and they are always there despite not being able to remember the topic of my thesis (true story).

Bureaucracy can be a confusing and frustrating burden. That’s why I am especially grateful to the administrative staff at my department (Mari Carmen), the subdirección de doctorado (María, María Jesús, Pablo), and the vice-rectorate of research (Silvia y Carolina). They have always given their best to help me with the (sometimes bizarre) issues arising from my PhD or my FPU grant, and the only thing I regret is all the time I made them spend on me.

Continuing with official matters, I want to acknowledge the Spanish Ministry of Education, Culture and Sports for my doctoral grant under the “Formación del Profesorado Universitario” (FPU) program. Spain is not an easy country for researchers, so the importance of the FPU program in supporting young researchers during the first steps of their career cannot be downplayed.

Last but not least, I want to thank all my brother and sisters in arms, both current and past. The daily routine wouldn’t be the same without them. By sharing interests, problems, hopes and fears, we find the strength to go on when things look dark (or deadlines approach). There are too many to name, but I cannot end this lines without recalling those who have passed through our office during these years. Hodei, Javier, Davide, Virginia (and sometimes even another Javier),... it has been a good run. But don’t ask me to do it again (you know I’m getting too old for this).

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Chapter 1

Introduction

The development of low-thrust, high-specific-impulse thrusters has brought new challenges and possibilities into the field of space trajectories optimization. In contrast to impulsive propulsion systems, these thrusters operate continuously for long periods of time, so the determination of their control laws takes the form of an optimal control problem (OCP). The fundamental objective of this thesis, entitled *Propagation and Optimal Control of Space Trajectories Using Perturbation Methods*, is **to develop mathematical methods and tools for the propagation and optimization of low-thrust trajectories, both Earth-bound and interplanetary, and apply them to different practical test cases**. To this end, several orbital formulations have been used, as well as direct and indirect optimization methods. From this main objective four concrete objectives have been defined:

- To search for asymptotic solutions for the orbital formulations used, identifying fundamental physical characteristics of the movement and applying them to the posing and resolution of the OCPs.
- To develop methodologies for the study of OCPs with the proposed orbital formulations, applying direct and indirect methods. The direct approach has been focused on the calculation of numerical solutions, while in the indirect case approximate analytical solutions have also been obtained using perturbation methods.
- To obtain initial guesses of the solution, necessary for the numerical resolution of the problem using iterative algorithms.
- To apply the results obtained to several projects under development by the research group, and other problems of practical interest.

Figure 1.1: Schematic representation of the key elements in direct methods.

This thesis covers a wide range of techniques for solving OCPs, including both direct and indirect methods. With direct methods, the OCP is transcribed as a discrete nonlinear programming (NLP) problem, which is then solved numerically using iterative algorithms starting from an initial guess of the solution. Therefore, there are four key elements to consider, as illustrated in Figure 1.1:

- The formulation used for the dynamics.
- The transcription of the original problem into an NLP subproblem.
- The algorithms for the resolution of the NLP subproblem.
- Obtaining an initial guess of the solution good enough to initialize the iterative algorithms.

The third point, the NLP algorithms, constitutes itself a field of study within mathematical programming, and falls outside the scope of this thesis. Therefore, third-party software packages are used for the numerical resolution of the NLP subproblem, particularly, the Global Optimization Toolbox from Matlab and the Ipopt library from the COIN-OR project. The rest of the points have been studied with varying degrees of detail within the thesis.

On the other hand, in indirect methods the solution is sought for by imposing the first order optimality conditions, derived from Pontryagin's Maximum Principle or using the calculus of variations, which yields a two-point boundary value problem (TPBVP). Such TPBVP is studied in this thesis in two different, yet complementary ways:

Figure 1.2: Diagram of the strategies followed for solving OCPs using both direct and indirect methods.

- Obtaining approximate analytical solutions with perturbation methods. These solutions have been applied to the generation of initial guesses for initializing the iterative algorithms used in the numerical resolution of both indirect and direct methods.
- Solving the problem numerically using iterative algorithms. The required initial guesses have been obtained from the above approximate solutions or by using continuation and homotopy methods.

The different strategies followed for solving OCPs using both direct and indirect methods are schematically represented in Figure 1.2.

The dissertation is structured around the different orbital formulations used, which are basically three: a new nonlinear relative motion formulation in curvilinear coordinates in development by the Space Dynamics Group-UPM, to which the PhD candidate belongs; the modified equinoctial elements, originally introduced by P. Cefola; and Dromo, an element-based orbital propagator developed by Professor Jesus Peláez along with other members of the SDG. Following this criteria, the thesis is organized as follows. The first part corresponds to the study in curvilinear coordinates, which consists of three well-defined lines of work. On the one hand, asymptotic solutions of the unperturbed motion have been obtained for moderate inclination and eccentricity, giving rise to expressions in the form of Fourier series of hypergeometric functions. On the other hand, approximate analytical solutions have been obtained using the indirect method for the phase

change, radius change and inclination change OCPs. For the planar cases (phase and radius change), this study begins with the linearization of the equations of motion and a comparative analysis of the orders of magnitude of each term in the equations (typical in fluid mechanics but uncommon in orbital mechanics), which allows one to identify two operating regimes depending on the ratio between the desired displacement and the available thrust. Finally, several cases of interest have been solved numerically using direct and indirect methods, verifying that the previous analytical solutions are a very good approximation to the linear case, and that the introduction of nonlinearities does not substantially modify the qualitative behavior of the solution. Some of these results have been applied to LEOSWEEP, an international project funded by the European Commission under the Seventh Framework Program (FP7) that studies the active removal of space debris using the Ion Beam Shepherd (IBS) concept. The IBS is a concept in which the orbit (or attitude) of an object is modified by having a beam of quasi-neutral plasma impinging against its surface to create a force or a torque, and can be applied for various purposes such as the removal of space debris or the deflection of asteroids.

The next part of the dissertation focuses on the use of the modified equinoctial elements to design end-of-life disposal maneuvers for satellites in the Galileo constellation with low-thrust propulsion. The study is an extension of previous works by other authors, in which impulsive maneuvers are used to place the satellites in orbits that lead to their natural re-entry in less than 100 years due to lunisolar perturbation effects. However, although those results comply with the re-entry condition, the required propellant mass is too high (30 – 60 kg) for practical application. Leveraging the fact that future generations of Galileo satellites are expected to equip electric propulsion systems (e.g. QinetiQ's T6 ionic engine), this new study replaces the impulsive maneuver with a low-thrust, minimum-fuel transfer between the satellite's operational orbit and the re-entry orbit. The results obtained for several re-entry orbits yield fuel consumptions of less than 5 kg for transfer maneuvers of up to a month and a half, demonstrating the practical interest of this approach. The numerical results are obtained using the indirect method only, applying continuation techniques in the maximum thrust to solve the lack of a good initial guess, and an homotopy between the minimum-energy and minimum-fuel problems to reduce the numerical problems derived from the thrust control discontinuity in the minimum-fuel solution (which follows a bang-bang profile). The use of the modified equinoctial elements allows one to express the boundary conditions for orbit-to-orbit transfers in a very simple way, at the cost of increasing the complexity of the equations (both the equations of motion and those derived from the first order optimality conditions) with respect to the Cartesian formulation. The complexity is even greater if the Jacobian is also computed (necessary to propagate the state transition matrix). With this in mind, in the second

half of this part a general mathematical framework for indirect optimization with element-based formulations (e.g. equinoctial, classical, etc.) is developed. Taking advantage of the mathematical structure of the equations, generic matrix expressions are proposed for the equations of motion, costate and their first derivatives, in which the contribution of the formulation used for the dynamics is reduced to a positive semidefinite symmetric matrix and a vector (and their derivatives up to second order). Additionally, it is verified that the contribution of the costates associated with the orbital elements is reduced to a quadratic form defined by the previous matrix.

The final part of the thesis is devoted to the Dromo orbital formulation, and deals with two different topics: obtaining approximate analytical solutions to the radial thrust problem, and studying the applicability of Dromo for the resolution of OCPs. With respect to the former, approximate analytical solutions have been obtained for the two body problem perturbed by a small radial acceleration using perturbation methods, with special focus on the method of multiple scales. The results not only allow one to calculate the position of the secondary with high precision in cases with low thrust and small initial eccentricity, but also reveal fundamental properties of the system dynamics and its time scales. On the other hand, in order to evaluate the applicability of Dromo to OCPs a series of test cases have been studied using different direct transcriptions. The results obtained demonstrate the good performance of Dromo for this type of applications, mainly due to three fundamental characteristics. Firstly, its excellent performances in both precision and computational cost. Secondly, the state of the object is described by a redundant set of generalized orbital elements, whose time scales and magnitude variations are related to those of the control. Finally, it presents a unique formulation for elliptic, hyperbolic and parabolic orbits, allowing optimal trajectories to be obtained that combine different types of orbit (e.g. bi-elliptic transfers), as well as those involving a transition from one type to another.

Part I

Curvilinear Coordinates

Chapter 2

Relative Motion in Curvilinear Coordinates

The first part of the dissertation deals with the study of minimum-time transfers using a relative motion formulation in curvilinear coordinates for dynamics and both indirect and direct methods. This line of work was originally motivated by the need to design small orbit correction maneuvers for Active Debris Removal missions within the LEOSWEEP project. LEOSWEEP (improving Low Earth Orbit Security With Enhanced Electric Propulsion) [83] is an EU-funded project, which ran from 2013 to 2016, aiming at the contactless manipulation of non-cooperative targets using modified electric thrusters [18]. The base LEOSWEEP mission would not only require to transfer from the original orbit, be it the injection one or the final state from a previous debris removal, but also to continuously correct the relative position between debris and satellite (normally subjected to tight operational margins). This kind of precise orbit correction is also needed for other key missions, such as cooperative rendezvous and docking maneuvers, fundamental for the operation of the ISS, and those involving formation flying of satellites. The practical interest of this family of maneuvers is also reflected in the attention they have received in the last decades. A detailed analytical study on the subject, including closed form solutions for some particular conditions, can be found in the book by Marec [64]. A classification of the different regimes for the phasing problem was performed by Hall and Collazo-Perez in [46] using numerical techniques. The more complex problem of designing low-thrust reconfiguration maneuvers for spacecraft in formation flying subjected to multiple operational constraints has been studied in detail by Massari and Bernelli-Zazzera, who propose in [65] an efficient algorithm implementation to cope with the increased computational cost of optimizing the trajectories of several spacecraft simultaneously.

The first results for the rephasing problem were obtained using the Clohessy-Wiltshire equations for relative motion [39], yielding several analytical estimates

for the time of flight and revealing the existence of two limit regimes (corresponding to transfers taking much less or more of one orbital revolution of the reference to complete). Given the great practical interest of the results, the study was extended to coplanar, circular-to-circular transfers (and later on, to inclination change maneuvers). However, the use of the Clohessy-Wiltshire equations still limited the validity of the results to small orbit corrections. Coincidentally, at about that time members of the research group were also working on a new solution for the relative motion problem using a nonlinear formulation in curvilinear coordinates, not affected by this limitation. Synergies soon grew, and all the subsequent works were based on the new formulation [40, 41, 42].

The objective of this chapter is twofold. On the one hand, to introduce the relative motion formulation in curvilinear coordinates used during this part of the dissertation. On the other hand, to present the work related to the derivation of an approximate Fourier series solution for the unperturbed relative motion, carried out by the PhD candidate.

This chapter is organized as follows. The nonlinear relative motion formulation in curvilinear coordinates is introduced first, providing all the relevant assumptions, definitions, reference frames, and transformations. The equations of motion are given both for the exact, nonlinear problem and for the linearized model. The same equations have also been obtained by Alfrend et al. [5] and Geller and Lovell [36] following different approaches. This formulation differs from the classical equations by Clohessy and Wiltshire [27] in two aspects. On the one hand, it introduces curvilinear coordinates to achieve a better description of the natural orbit curvature. On the other hand, it takes into account nonlinear terms to improve accuracy when initial velocity and displacement conditions are not very small. Nevertheless, the limitation of initially circular orbit still applies. Next, an approximate, time-explicit solution is obtained as Fourier series of the mean anomaly, with their coefficients expressed as Taylor expansions of the eccentricity. A quasi-planar and two quadratic solutions (single- and double-frequency) are then derived as especial cases from the Fourier series solution. Finally, several test cases are presented to evaluate the performance of the different approximate solutions.

2.1 Curvilinear Coordinates

Let us consider a chief C (either a real or ‘virtual’ object) describing a circular Keplerian orbit of radius R around a primary O with gravitational constant μ . From now on all equations and variables will be expressed in non-dimensional form taking the chief orbit radius R and inverse mean motion $n^{-1} = \sqrt{R^3/\mu}$ as

Figure 2.1: Schematic representation of the relative motion frames and variables.

units of length and time, respectively. Let us also introduce an inertial reference frame $\mathcal{I} = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and a local-vertical/local-horizontal one $\mathcal{C} = \{C; \mathbf{i}', \mathbf{j}', \mathbf{k}'\}$, as shown in Figure 2.1. Note that $\mathbf{k} = \mathbf{k}'$ is always normal to the orbital plane of C . The relative position and velocity of a follower F (also real or virtual) with respect to the chief can then be expressed as:

$$\begin{aligned} \mathbf{r}' &= x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}', \\ \mathbf{v}' &= \dot{x}\mathbf{i}' + \dot{y}\mathbf{j}' + \dot{z}\mathbf{k}'. \end{aligned} \quad (2.1)$$

The in-plane relative motion of the follower can be better described through two curvilinear coordinates ρ and θ associated to a cylindrical reference frame $\mathcal{F} = \{O; \mathbf{u}_\rho, \mathbf{u}_\theta, \mathbf{k}'\}$, where $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$ are the radial and transversal unitary vectors defined as:

$$\begin{aligned} \mathbf{u}_\rho &= \cos \theta \mathbf{i}' + \sin \theta \mathbf{j}', \\ \mathbf{u}_\theta &= -\sin \theta \mathbf{i}' + \cos \theta \mathbf{j}'. \end{aligned}$$

From the definition of frame $\mathcal{F}$, it follows that ρ is the radial separation from the chief orbit of the projection of point F onto the orbital plane of C , while θ is the angle formed by the projection of the position vector of F onto said orbital plane and the position vector of C . It is then straightforward to convert between (ρ, θ) , the inertial position $\mathbf{r}$ and the relative position $\mathbf{r}'$ using geometric relations:

$$\begin{aligned} \rho &= -1 + \sqrt{(x+1)^2 + y^2}, \quad \theta = \text{atan2}(y, 1+x), \\ x &= -1 + (1+\rho) \cos \theta, \quad y = (1+\rho) \sin \theta, \\ \mathbf{r} &= \mathbf{r}' + \mathbf{r}_C = \mathbf{r}' + \mathbf{i}'. \end{aligned} \quad (2.2)$$

The differential equations describing the evolution of curvilinear coordinates (ρ, θ, z) can be obtained by deriving and projecting the equations of relative motion

in frame $\mathcal{F}$ as done by Bombardelli et al. in [17, 16] (or see [36] for an alternative derivation leading to the same equations), yielding:

$$\begin{aligned}\ddot{\rho} - 2\dot{\theta} - 3\rho &= a_{i\rho} + a_{g\rho} + a_{\rho}^E, \\ \ddot{\theta} + 2\dot{\rho} &= a_{i\theta} + \frac{a_{\theta}^E}{1+\rho}, \\ \ddot{z} + z &= a_{gz} + a_z^E,\end{aligned}\tag{2.3}$$

where ‘ $\cdot$ ’ denotes the derivative with respect to the non-dimensional time τ . Terms $a_{i\rho}$, $a_{g\rho}$, $a_{i\theta}$ and a_{gz} account for the nonlinear contributions in the equations of motion:

$$\begin{aligned}a_{i\rho} &= \dot{\theta}^2 (1 + \rho) + 2\dot{\theta}\dot{\rho}, & a_{i\theta} &= \frac{2\dot{\rho}(\rho - \dot{\theta})}{1 + \rho}, \\ a_{g\rho} &= -2\rho + 1 - \frac{1 + \rho}{[(1 + \rho)^2 + z^2]^{3/2}}, & a_{gz} &= z - \frac{z}{[(1 + \rho)^2 + z^2]^{3/2}},\end{aligned}$$

whereas a_{ρ}^E , a_{θ}^E and a_z^E correspond to additional external perturbations acting on the follower (for instance, the acceleration produced by a thruster).

By introducing state vector $\mathbf{S}$:

$$\mathbf{S} = [\dot{\rho} \quad \dot{\theta} \quad \dot{z} \quad \rho \quad \theta \quad z]^\top,\tag{2.4}$$

the equations of motion can be expressed as a first order system in the form:

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F}(\mathbf{S}; \tau),\tag{2.5}$$

where $\mathbf{F}$ is the right-hand side of the equations:

$$\mathbf{F}(\mathbf{S}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_{i\rho} + a_{g\rho} + a_{\rho}^E \\ -2\dot{\rho} + a_{i\theta} + \frac{a_{\theta}^E}{1+\rho} \\ -z + a_{gz} + a_z^E \\ \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{bmatrix}.\tag{2.6}$$

Neglecting the nonlinear perturbing terms and assuming $\rho \ll 1$, a linear formulation with the same structure as the well-known Clohessy-Wiltshire equations is reached:

$$\begin{aligned}\ddot{\rho} - 2\dot{\theta} - 3\rho &= a_{\rho}^E, \\ \ddot{\theta} + 2\dot{\rho} &= a_{\theta}^E, \\ \ddot{z} + z &= a_z^E,\end{aligned}\tag{2.7}$$

or as a first order system:

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F}^*(\mathbf{S}; \tau), \quad (2.8)$$

with:

$$\mathbf{F}^*(\mathbf{S}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_\rho^E \\ -2\dot{\rho} + a_\theta^E \\ -z + a^T u_z^E \\ \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{S} + \begin{bmatrix} a_\rho^E \\ a_\theta^E \\ a_z^E \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.9)$$

It is important to note that the only coupling between the in-plane and out-of-plane linearized equations could come from the expressions for a_ρ^E , a_θ^E and a_z^E . For the case with no external perturbations ($a_\rho^E = a_\theta^E = a_z^E = 0$), the well-known analytical solution of Eqs. (2.9) is:

$$\begin{bmatrix} \dot{\rho} \\ \dot{\theta} \\ \dot{z} \\ \rho \\ \theta \\ z \end{bmatrix} = \begin{bmatrix} \cos \tau & 2 \sin \tau & 0 & 3 \sin \tau & 0 & 0 \\ -2 \sin \tau & -3 + 4 \cos \tau & 0 & 6(\cos \tau - 1) & 0 & 0 \\ 0 & 0 & \sin \tau & 0 & 0 & \cos \tau \\ \sin \tau & 2(1 - \cos \tau) & 0 & 4 - 3 \cos \tau & 0 & 0 \\ 2(\cos \tau - 1) & -3\tau + 4 \sin \tau & 0 & 6(-\tau + \sin \tau) & 1 & 0 \\ 0 & 0 & \cos \tau & 0 & 0 & -\sin \tau \end{bmatrix} \begin{bmatrix} \dot{\rho}_0 \\ \dot{\theta}_0 \\ \dot{z}_0 \\ \rho_0 \\ \theta_0 \\ z_0 \end{bmatrix} \quad (2.10)$$

where $\dot{\rho}_0$, $\dot{\theta}_0$, ρ_0 , and θ_0 are the initial conditions at $\tau_0 = 0$. For simplicity, the initial time has been set to zero; this is done without loss of generality, since the solution for a general τ_0 can be recovered simply by making the change of variable $\tau^* = \tau + \tau_0$.

2.1.1 Orbital Elements

This subsection presents some relations between curvilinear variables (ρ, θ) and the classic orbital elements for the follower F . The same procedure proposed in [17] is followed, with some steps omitted for brevity.

The inertial position and velocity vectors of the follower can be written as:

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_C + \mathbf{r}', \\ \mathbf{v} &= \mathbf{v}' + \mathbf{v}_C + \boldsymbol{\omega}_C \times \mathbf{r}', \end{aligned}$$

where $\mathbf{r}_C$ and $\mathbf{v}_C$ are, respectively, the inertial position and velocity of the chief:

$$\begin{aligned} \mathbf{r}_C &= \mathbf{i}' = \cos \theta \mathbf{u}_\rho - \sin \theta \mathbf{u}_\theta, \\ \mathbf{v}_C &= \mathbf{j}' = \sin \theta \mathbf{u}_\rho + \cos \theta \mathbf{u}_\theta, \end{aligned}$$

and $\boldsymbol{\omega}_C$ is the angular rate of frame $\mathcal{C}$:

$$\boldsymbol{\omega}_C = \mathbf{k}'.$$

Combining these relations with the expressions for $\mathbf{r}'$ and $\mathbf{v}'$, Eqs. (2.1), it is possible to write:

$$\begin{aligned}\mathbf{r} &= r_\rho \mathbf{u}_\rho + r_z \mathbf{k}', \\ \mathbf{v} &= v_\rho \mathbf{u}_\rho + v_\theta \mathbf{u}_\theta + v_z \mathbf{k}',\end{aligned}$$

with:

$$\begin{aligned}r_\rho &= 1 + \rho, \quad r_z = z, \quad r = \sqrt{r_\rho^2 + r_z^2}, \\ v_\rho &= \dot{\rho}, \quad v_\theta = (1 + \rho) (1 + \dot{\theta}), \quad v_z = \dot{z}, \quad v = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2}.\end{aligned}$$

It is also straightforward to compute the angular momentum of F from $\mathbf{r}$ and $\mathbf{v}$ as:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (h_\rho \cos \theta - h_\theta \sin \theta) \mathbf{i}' + (h_\rho \sin \theta + h_\theta \cos \theta) \mathbf{j}' + h_z \mathbf{k}',$$

with:

$$\begin{aligned}h_\rho &= -z (1 + \rho) (1 + \dot{\theta}), \\ h_\theta &= z \dot{\rho} - (1 + \rho) \dot{z}, \\ h_z &= (1 + \rho)^2 (1 + \dot{\theta}), \\ h &= \sqrt{h_\rho^2 + h_\theta^2 + h_z^2}.\end{aligned}$$

The classic orbital elements for F can now be obtained from $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{h}$, yielding the following expressions for the semi-major axis:

$$a = \frac{r}{2 - rv^2}, \tag{2.11}$$

the eccentricity vector:

$$\mathbf{e} = e_\rho \mathbf{u}_\rho + e_\theta \mathbf{u}_\theta + e_z \mathbf{k}', \tag{2.12}$$

with:

$$\begin{aligned}e_\rho &= (1 + \rho)^3 (1 + \dot{\theta})^2 - z \dot{z} \dot{\rho} + (1 + \rho) \dot{z}^2 - \frac{1 + \rho}{r}, \\ e_\theta &= -(1 + \rho) (1 + \dot{\theta}) [(1 + \rho) \dot{\rho} + z \dot{z}], \\ e_z &= z \dot{\rho}^2 - \dot{\rho} \dot{z} (1 + \rho) + z (1 + \rho)^2 (1 + \dot{\theta})^2 - \frac{z}{r},\end{aligned} \tag{2.13}$$

the inclination:

$$\cos i = \frac{h_z}{h}, \quad (2.14)$$

the right ascension of the ascending node (RAAN):

$$\Omega = \theta + \alpha + \tau, \quad (2.15)$$

with:

$$\sin \alpha = \frac{h_\rho}{\sqrt{h_\rho^2 + h_\theta^2}}, \quad \cos \alpha = -\frac{h_\theta}{\sqrt{h_\rho^2 + h_\theta^2}}, \quad (2.16)$$

the argument of perigee:

$$\sin \omega = \frac{h_\rho (e_z h_\rho - h_z e_\rho) + h_\theta (e_z h_\theta - h_z e_\theta)}{h e \sqrt{h_\rho^2 + h_\theta^2}}, \quad \cos \omega = \frac{e_\theta h_\rho - e_\rho h_\theta}{e \sqrt{h_\rho^2 + h_\theta^2}}, \quad (2.17)$$

and the true anomaly:

$$\nu = \phi - \omega, \quad (2.18)$$

with:

$$\sin \phi = \frac{z (h_\rho^2 + h_\theta^2) - (1 + \rho) h_\rho h_z}{r h \sqrt{h_\rho^2 + h_\theta^2}}, \quad \cos \phi = -\frac{(1 + \rho) h_\theta}{r \sqrt{h_\rho^2 + h_\theta^2}}. \quad (2.19)$$

The eccentric anomaly corresponding to a true anomaly of ν_0 at $\tau = 0$ is then:

$$E_0 = \text{atan2} \left(\sqrt{1 - e^2} \sin \nu_0, e + \cos \nu_0 \right), \quad (2.20)$$

and the time evolution of the mean anomaly of F reads:

$$M = M_0 + n\tau = E_0 - e \sin E_0 + \frac{\tau}{a^{3/2}}. \quad (2.21)$$

For the zero inclination (planar) case Ω becomes undetermined. It is then possible to arbitrary set $\Omega = 0$ (as well as $\mathbf{i} = \mathbf{i}'$ at $\tau = 0$), leading to the following expressions for ω and ν :

$$\begin{aligned} \sin \omega|_{i=0} &= \frac{e_\rho \sin(\theta + \tau) + e_\theta \cos(\theta + \tau)}{\sqrt{e_\rho^2 + e_\theta^2}}, \\ \cos \omega|_{i=0} &= \frac{e_\rho \cos(\theta + \tau) - e_\theta \sin(\theta + \tau)}{\sqrt{e_\rho^2 + e_\theta^2}}, \\ \nu|_{i=0} &= \theta + \tau - \omega. \end{aligned}$$

For the zero eccentricity (circular) case ω becomes undetermined and can be set to zero:

$$\omega|_{e=0} = 0. \quad (2.22)$$

2.2 Fourier Series Solution

It is not possible to find a time-explicit solution of Eqs. (2.3) in closed form, even for the unperturbed case. However, a greater insight on the problem can be achieved by developing approximate analytical solutions. In this section, time-explicit expressions for ρ , θ and z are obtained as Fourier series of the mean anomaly, which allows one to separate the constant, secular and periodic contributions. The coefficients of these Fourier series will themselves be Taylor series expansions of e , implying that their validity is limited to orbits of moderate eccentricity.

Let us begin by expressing the curvilinear coordinates as explicit function of the true anomaly [17]:

$$\begin{aligned}\rho &= -1 + \frac{a(1-e^2)}{1+e\cos\nu} \sqrt{1-\kappa\sin^2(\omega+\nu)} \\ \theta &= -\tau + \Omega + \text{atan2}\left(\sqrt{1-\kappa\sin^2(\omega+\nu)}, \cos(\omega+\nu)\right), \\ z &= \frac{a(1-e^2)}{1+e\cos\nu} \sqrt{\kappa}\sin(\omega+\nu)\end{aligned}\tag{2.23}$$

where:

$$\kappa = \sin^2 i.\tag{2.24}$$

The solution given by Eqs. (2.23) is exact but not time-explicit, since ν has to be obtained numerically from Kepler's equation. The constant, secular and periodic contributions in Eqs. (2.23) can be separated by rewriting them as Fourier series in the mean anomaly:

$$\rho = \rho_{\text{const}} + \sum_{k=1}^{\infty} [\rho_{CkM} \cos(kM) + \rho_{SkM} \sin(kM)] ,\tag{2.25}$$

$$\theta = \theta_{\text{sec}} + \theta_{\text{const}} + \sum_{k=1}^{\infty} [\theta_{CkM} \cos(kM) + \theta_{SkM} \sin(kM)] ,\tag{2.26}$$

$$z = z_{\text{const}} + \sum_{k=1}^{\infty} [z_{CkM} \cos(kM) + z_{SkM} \sin(kM)] .\tag{2.27}$$

Obtaining the coefficients in Eqs. (2.25,2.26) is not straightforward, even with the aid of symbolic manipulators. In the following, the key steps in the development of the Fourier series of ρ and θ are presented in detail (the quite lengthy algebraic manipulations are omitted for clarity and brevity).

First of all, it is important to highlight that the Fourier series will not be computed directly in the mean anomaly M , but in the related angular variable $\phi = \omega + \nu$. The introduction of this intermediate variable yields more compact (and

tractable) expressions, greatly simplifying the related integrals. A final change of independent variable will allow us to obtain the final Fourier series in terms of M .

The Fourier series of ρ is considered first. By expressing the first of Eqs. (2.23) in terms of $\phi = \omega + \nu$, expanding in Taylor series of the eccentricity and performing some trigonometric manipulations, the coefficients for the Fourier series of ρ in ϕ turn out to be a linear combination of integrals of the form:

$$\int_0^{2\pi} \sin^i \phi \cos^j \phi \sqrt{1 - \kappa \sin^2 \phi} d\phi \quad i, j \in \mathbb{N}.$$

For i or j odd these integrals are zero due to symmetries. For i and j even, some additional manipulations and the trigonometric relation $\cos^2 \phi = 1 - \sin^2 \phi$ allow us to write the remaining integrals as:

$$I_q = \int_0^{2\pi} \sin^{2q} \phi \sqrt{1 - \kappa \sin^2 \phi} d\phi \quad q \in \mathbb{N},$$

and leveraging the symmetries of $\sin^2 \phi$:

$$I_q = 4 \int_0^{\pi/2} \sin^{2q} \phi \sqrt{1 - \kappa \sin^2 \phi} d\phi \quad q \in \mathbb{N}.$$

A final change of variable, $w = \sin^2 \phi$, allows us to write I_q in the form:

$$I_q = 4 \int_0^1 \frac{w^{-\frac{1}{2}+q} \sqrt{1 - \kappa w}}{2\sqrt{1-w}} dw \quad q \in \mathbb{N},$$

which can be readily integrated to yield:

$$I_q = \frac{1}{2} \sqrt{\pi} \Gamma\left(\frac{1}{2} + q\right) \tilde{F}\left(-\frac{1}{2}, \frac{1}{2} + q; 1 + q; \kappa\right),$$

where Γ is the gamma function, and $\tilde{F}$ is a regularized hypergeometric function. The latter can be obtained from the corresponding hypergeometric function and an additional gamma function as:

$$\tilde{F}\left(-\frac{1}{2}, \frac{1}{2} + q; 1 + q; \kappa\right) = \frac{F\left(-\frac{1}{2}, \frac{1}{2} + q; 1 + q; \kappa\right)}{\Gamma(1 + q)}.$$

Once the Fourier series in ϕ has been obtained the resulting coefficients can be expressed in terms of the eccentric anomaly E through the definition of $\phi = \omega + \nu$ and the well-known relations:

$$\cos \nu = \frac{\cos E - e}{1 - e \cos E}, \quad \sin \nu = \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E},$$

while E is, in turns, replaced by the mean anomaly through the expansions of $\cos E$ and $\sin E$ using Bessel functions of the first kind J_k (see [7], page 212):

$$\begin{aligned}\cos E &= -\frac{e}{2} + \sum_{k=1}^{\infty} \frac{2}{k^2} \frac{dJ_k(ke)}{de} \cos kM, \\ \sin E &= \frac{2}{e} \sum_{k=1}^{\infty} \frac{1}{k} J_k(ke) \sin kM.\end{aligned}$$

A final Taylor series expansion in e finally yields the Fourier coefficients in terms of the mean anomaly. These coefficients can be written in compact form as:

$$\rho_{\text{const}} = -1 + a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\rho 0}^{pq} h^p \right] e^q = -1 + a \mathbf{h}^{\top} \mathbf{C}_{\rho 0} \mathbf{e}, \quad (2.28)$$

$$\rho_{CkM} = a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\rho k}^{pq} h^p \right] e^q = a \mathbf{h}^{\top} \mathbf{C}_{\rho k} \mathbf{e}, \quad (2.29)$$

$$\rho_{SkM} = a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} s_{\rho k}^{pq} h^p \right] e^q = a \mathbf{h}^{\top} \mathbf{S}_{\rho k} \mathbf{e}, \quad (2.30)$$

where:

$$\mathbf{e} = [1 \quad e \quad e^2 \quad e^3 \quad \dots]^{\top}, \quad \mathbf{h} = [h^0 \quad h^1 \quad h^2 \quad h^3 \quad \dots]^{\top},$$

and h^p are defined in terms of regularized hypergeometric and gamma functions as follows:

$$h^q = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + q\right) \tilde{F}\left(-\frac{1}{2}, \frac{1}{2} + q; 1 + q; \kappa\right).$$

The $\mathbf{C}_{\rho 0}$, $\mathbf{C}_{\rho k}$ and $\mathbf{S}_{\rho k}$ matrices for a Fourier series up to the fifth harmonic (with a Taylor series expansion of the coefficients up to order 5 in e) are reported in Appendix C.

Let us now deal with the Fourier series of θ . First of all, the second of Eqs. (2.23) is rearranged in order to separate the secular and periodic parts in ϕ :

$$\theta = -\tau + \Omega + \phi + [\tan^{-1}(\sqrt{1 - \kappa} \tan \phi) - \phi] = -\tau + \Omega + \phi + f(\phi),$$

where $f(\phi)$ is an odd, π -periodic function of ϕ . Consequently, the Fourier series of $f(\phi)$ will only have terms in $\sin 2k\phi$, with coefficients:

$$I_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(\phi) \sin 2k\phi d\phi = \frac{1}{k} \left(\frac{-1 + \sqrt{1 - \kappa}}{1 + \sqrt{1 - \kappa}} \right)^k \quad k \in \mathbb{N}.$$

It is now straightforward to express the resulting Fourier series in terms of M by applying the same changes of independent variable already presented for ρ . On the other hand, the secular part in ϕ can be written in terms of M as:

$$\phi = \omega + M + g(M) ,$$

where the series expansion of $g(M)$ in the eccentricity up to sixth order can be found in the classic book by Battin (see [7], page 212). Gathering these results the Fourier series for θ finally reads:

$$\theta = \theta_{\text{sec}} + \theta_{\text{const}} + \sum_{k=1}^{\infty} [\theta_{CkM} \cos(kM) + \theta_{SkM} \sin(kM)] ,$$

with:

$$\theta_{\text{sec}} = (n - 1) \tau ,$$

and coefficients:

$$\theta_{\text{const}} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\theta 0}^{pq} s^p \right] e^q = \mathbf{s}^{\top} \mathbf{C}_{\theta 0} \mathbf{e} , \quad (2.31)$$

$$\theta_{CkM} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\theta k}^{pq} s^p \right] e^q = \mathbf{s}^{\top} \mathbf{C}_{\theta k} \mathbf{e} , \quad (2.32)$$

$$\theta_{SkM} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} s_{\theta k}^{pq} c^p \right] e^q = \mathbf{c}^{\top} \mathbf{S}_{\theta k} \mathbf{e} , \quad (2.33)$$

where:

$$\mathbf{s} = \begin{bmatrix} 1 & I_1 \sin 2\omega & \dots & I_k \sin 2k\omega & \dots \end{bmatrix}^{\top} ,$$

$$\mathbf{c} = \begin{bmatrix} 1 & I_1 \cos 2\omega & \dots & I_k \cos 2k\omega & \dots \end{bmatrix}^{\top} ,$$

$$I_k = \frac{(-1 + \sqrt{1 - \kappa^2})^k}{k (1 + \sqrt{1 - \kappa^2})^k} .$$

The $\mathbf{C}_{\theta 0}$, $\mathbf{C}_{\theta k}$ and $\mathbf{S}_{\theta k}$ matrices for a Fourier series up to the fifth harmonic (with a Taylor series expansion of the coefficients up to order 5 in e) are reported in Appendix C.

The procedure for z is much simpler. By performing a change of variables from ν to M , expanding in Taylor series of e and collecting terms in $\cos kM$ and $\sin kM$, the Fourier coefficients are obtained in the form:

$$z_{\text{const}} = -\frac{3}{2} a e \sqrt{\kappa} \sin(\omega) , \quad (2.34)$$

$$z_{CkM} = a\sqrt{\kappa} \sin(\omega) \sum_{p=0}^{\infty} c_{zk}^p e^p = a\sqrt{\kappa} \sin(\omega) \mathbf{e}^\top \mathbf{c}_{zk}, \quad (2.35)$$

$$z_{SkM} = a\sqrt{\kappa} \cos(\omega) \sum_{i=0}^{\infty} s_{zk}^p e^p = a\sqrt{\kappa} \sin(\omega) \mathbf{e}^\top \mathbf{s}_{zk}. \quad (2.36)$$

The $\mathbf{c}_{zk}$, and $\mathbf{s}_{zk}$ 1-D arrays for a Fourier series up to the fifth harmonic (with a Taylor series expansion of the coefficients up to order 5 in e) are reported in Appendix C.

These approximate analytic solutions provide additional insight on the behavior of the dynamics. For instance, by inspecting the expressions in Appendix C it is possible to confirm that a growing eccentricity and inclination in the follower's orbit increase the influence of higher order terms. However, they are too cumbersome to be used in many practical scenarios. More tractable expressions can be obtained by introducing additional simplifications. For the remainder of this section, a quasi-planar and a quadratic solution are derived from the Fourier series solution in Eqs. (2.25-2.27).

2.2.1 Quasi-planar solution

When the initial relative inclination of the follower is small, κ can be set to zero in the expressions for ρ and θ (but not in z). This leads to the following Fourier series up to the third harmonic:

$$\rho \simeq -1 + a \left[1 + \frac{e^2}{2} + \left(-e + \frac{3e^3}{8} \right) \cos M - \frac{e^2}{2} \cos 2M - \frac{3e^3}{8} \cos 3M \right], \quad (2.37)$$

$$\theta \simeq M_0 + \omega + \Omega + \tau(n-1) + \left(2e - \frac{e^3}{4} \right) \sin M + \frac{5}{4}e^2 \sin 2M + \frac{13}{12}e^3 \sin 3M, \quad (2.38)$$

$$\begin{aligned} z \simeq \frac{a\sqrt{\kappa}}{8} \left\{ \sin \omega \left[-12e + (8 - 3e^2) \cos M + \left(4e - \frac{8e^3}{3} \right) \cos 2M + 3e^2 \cos 3M \right] \right. \\ \left. + \cos \omega \left[(8 - 5e^2) \sin M + \left(4e - \frac{10e^3}{3} \right) \sin 2M + 3e^2 \sin 3M \right] \right\}. \end{aligned} \quad (2.39)$$

2.2.2 Quadratic Solution

A quadratic extension of the linearized solution in Eq. (2.10) is now obtained starting from the Fourier series solution in Eqs. (2.25-2.27). For small relative

initial conditions, cylindrical coordinates ρ , θ and z can be expressed as a single-frequency Fourier series of τ in the form:

$$\begin{aligned}\rho(\tau) &\approx \rho_{\text{const}} + \rho_{C\tau} \cos n\tau + \rho_{S\tau} \sin n\tau, \\ \theta(\tau) &\approx \theta_{\text{sec}}(\tau) + \theta_{\text{const}} + \theta_{C\tau} \cos n\tau + \theta_{S\tau} \sin n\tau, \\ z(\tau) &\approx z_{\text{const}} + z_{C\tau} \cos n\tau + z_{S\tau} \sin n\tau,\end{aligned}\tag{2.40}$$

with coefficients given in terms of the initial conditions $(\rho_0, \theta_0, z_0, \dot{\rho}_0, \dot{\theta}_0, \dot{z}_0)$.

The secular term for θ takes the form:

$$\frac{\theta_{\text{sec}}}{\tau} = n - 1 = \frac{1 - a^{3/2}}{a^{3/2}}.$$

Plugging the expression for a in Eq. (2.11) and expanding for small initial conditions yields:

$$\frac{\theta_{\text{sec}}}{\tau} \approx -6\rho_0 - 3\dot{\theta}_0 + \frac{15}{2}\rho_0^2 - \frac{3}{2}\dot{\rho}_0^2 - \frac{3}{2}z_0^2 - \frac{3}{2}\dot{z}_0^2.\tag{2.41}$$

The constant coefficients can be obtained by expanding Eqs. (2.28, 2.31, 2.34) as power series of κ and e up to order 2:

$$\begin{aligned}\rho_{\text{const}} &\approx -1 + a \left(1 + \frac{e^2}{2} - \frac{\kappa}{4} - \frac{3}{64}\kappa^2 \right), \\ \theta_{\text{const}} &\approx M_0 + \Omega + \omega, \\ z_{\text{const}} &\approx -\frac{3}{2}a\sqrt{\kappa}e \sin \omega,\end{aligned}$$

introducing Eqs. (2.11-2.21) to express them in terms of the initial conditions and expanding in Taylor series for small initial conditions. After some algebraic manipulations, the constant coefficients read:

$$\begin{aligned}\rho_{\text{const}} &\approx 4\rho_0 + 2\dot{\theta}_0 + \frac{39}{2}\rho_0^2 + 26\rho_0\dot{\theta}_0 + \frac{3}{2}\dot{\rho}_0^2 + 7\dot{\theta}_0^2 + \frac{3}{4}\dot{z}_0^2 + \frac{3}{4}z_0^2, \\ \theta_{\text{const}} &\approx \theta_0 - 2\dot{\rho}_0 + \frac{1}{2}\rho_0\dot{\rho}_0 + \dot{\rho}_0\dot{\theta}_0 - \frac{3}{2}z_0\dot{z}_0, \\ z_{\text{const}} &\approx \frac{3}{2} \left(\dot{\rho}_0\dot{z}_0 + -2z_0\dot{\theta}_0 - 3\rho_0z_0 \right).\end{aligned}\tag{2.42}$$

Finally, the first-frequency coefficients in τ can be obtained from the Fourier series in M , taking into account that $M = M_0 + n\tau$ and performing some trigonometric manipulations. Taking the coefficients given by Eqs. (2.29, 2.30, 2.32, 2.33, 2.35, 2.36) for $k = 1$, expanding them in Taylor series of e and κ up to order 2, using Eqs. (2.11-2.21) to express them in terms of the initial conditions, and

performing a final Taylor series expansion for small initial conditions, the periodic coefficients in Eqs. (2.40) are reached:

$$\begin{aligned}
\rho_{S\tau} &\approx \dot{\rho}_0 - \dot{\rho}_0 \dot{\theta}_0 + \dot{z}_0 z_0, \\
\rho_{C\tau} &\approx -2\dot{\theta}_0 - 3\rho_0 - 15\rho_0^2 - 20\rho_0 \dot{\theta}_0 - 2\dot{\rho}_0^2 - 5\dot{\theta}_0^2 - \dot{z}_0^2 - \frac{z_0^2}{2}, \\
\theta_{S\tau} &\approx 4\dot{\theta}_0 + 6\rho_0 + 6\rho_0^2 + 12\rho_0 \dot{\theta}_0 + 4\dot{\rho}_0^2 + 2\dot{\theta}_0^2 + 2\dot{z}_0^2 + z_0^2, \\
\theta_{C\tau} &\approx 2\dot{\rho}_0 - 8\rho_0 \dot{\rho}_0 - 6\dot{\rho}_0 \dot{\theta}_0 + 2\dot{z}_0 z_0, \\
z_{S\tau} &\approx \dot{z}_0 + 3\rho_0 \dot{z}_0 + \dot{\rho}_0 z_0 + \dot{\theta}_0 \dot{z}_0, \\
z_{C\tau} &\approx z_0 + 3\rho_0 z_0 - 2\dot{\rho}_0 \dot{z}_0 + 2\dot{\theta}_0 z_0.
\end{aligned} \tag{2.43}$$

2.2.3 Quadratic Solution, Second Order

The previous quadratic solution is limited to periodic terms up to the first harmonic in τ . However, Hernando-Ayuso et al. recently noted in [47] that the double-frequency terms are also of quadratic order. Following a procedure analogous to the one described for the single-frequency quadratic solution, they reach a double-frequency quadratic solution in the form:

$$\begin{aligned}
\rho(\tau) &\approx \rho_{\text{const}} + \rho_{C\tau} \cos n\tau + \rho_{S\tau} \sin n\tau + \rho_{C2\tau} \cos 2n\tau + \rho_{S2\tau} \sin 2n\tau, \\
\theta(\tau) &\approx \theta_{\text{sec}}(\tau) + \theta_{\text{const}} + \theta_{C\tau} \cos n\tau + \theta_{S\tau} \sin n\tau + \theta_{C2\tau} \cos 2n\tau + \theta_{S2\tau} \sin 2n\tau, \\
z(\tau) &\approx z_{\text{const}} + z_{C\tau} \cos n\tau + z_{S\tau} \sin n\tau + z_{C2\tau} \cos 2n\tau + z_{S2\tau} \sin 2n\tau,
\end{aligned} \tag{2.44}$$

where the secular, constant and first order coefficients coincide with the ones for the single-frequency approximation [Eqs. (2.41), (2.42), and (2.43), respectively], whereas the second order coefficients are given by:

$$\begin{aligned}
\rho_{S2\tau} &\approx 3\rho_0 \dot{\rho}_0 - \frac{1}{2} z_0 \dot{z}_0 + 2\dot{\theta}_0 \dot{\rho}_0, \\
\rho_{C2\tau} &\approx -\frac{9}{2} \rho_0^2 - 6\rho_0 \dot{\theta}_0 - \frac{1}{4} z_0^2 + \frac{1}{2} \dot{\rho}_0^2 - 2\dot{\theta}_0^2 + \frac{1}{4} \dot{z}_0^2, \\
\theta_{S2\tau} &\approx \frac{45}{4} \rho_0^2 + 15\rho_0 \dot{\theta}_0 + \frac{1}{4} z_0^2 - \frac{5}{4} \dot{\rho}_0^2 + 5\dot{\theta}_0^2 - \frac{1}{4} \dot{z}_0^2, \\
\theta_{C2\tau} &\approx \frac{15}{2} \rho_0 \dot{\rho}_0 - \frac{1}{2} z_0 \dot{z}_0 + 5\dot{\theta}_0 \dot{\rho}_0, \\
z_{S2\tau} &\approx \frac{3}{2} \dot{z}_0 \rho_0 - \frac{1}{2} z_0 \dot{\rho}_0 + \dot{\theta}_0 \dot{z}_0, \\
z_{C2\tau} &\approx \frac{3}{2} \rho_0 z_0 + z_0 \dot{\theta}_0 + \frac{1}{2} \dot{\rho}_0 \dot{z}_0.
\end{aligned} \tag{2.45}$$

Figure 2.2: Comparisson of different solutions with follower initial conditions $\rho_0 = -0.003$, $\dot{\rho}_0 = 0.1$, $\theta_0 = 0$, $\dot{\theta}_0 = 0.0051$, $z_0 = 0$, $\dot{z}_0 = 0.035$ (corresponding to $e = 0.2$ and $i = 2$ deg).

Figure 2.3: Comparisson of different solutions with follower initial conditions $\rho_0 = -0.020$, $\dot{\rho}_0 = 0$, $\theta_0 = 0.0873$, $\dot{\theta}_0 = 0.0056$, $z_0 = 0$, $\dot{z}_0 = 0.2640$ (corresponding to $e = 0.02$ and $i = 15$ deg).

2.3 Assessment of the Results

Three numerical tests¹ are now presented to check the performance of the different solutions proposed. Particularly, four solutions are compared: the Clohessy-Wiltshire solution given by Eq. (2.10); the quasi-planar solution presented in section 2.2.1, Eqs. (2.37-2.39); the quadratic solution (single-frequency) in section 2.2.2, Eqs. (2.40-2.43); and the n -th order Fourier series solution given by Eqs. (2.23-2.36) and the matrices in Appendix C. These solutions are labeled in the figures as ‘CW’, ‘QP’, ‘quad’ and ‘Fourier(n)’, respectively. In all cases, an

¹Reproduced from [17]

Figure 2.4: Comparisson of different solutions with follower initial conditions $\rho_0 = -0.030$, $\dot{\rho}_0 = 0.350$, $\theta_0 = 0.0873$, $\dot{\theta}_0 = -0.1772$, $z_0 = 0$, $\dot{z}_0 = 0.7982$ (corresponding to $e = 0.45$ and $i = 45$ deg).

‘exact’ reference solution is obtained through high-accuracy numerical propagation of Eqs. (2.3).

The first test case, Figure 2.2, represents a relative motion with small eccentricity and relative inclination, $e = 0.1$ and $i = 2$ deg. The initial conditions in curvilinear coordinates are $\rho_0 = -0.003$, $\dot{\rho}_0 = 0.1$, $\theta_0 = 0$, $\dot{\theta}_0 = 0.0051$, $z_0 = 0$, and $\dot{z}_0 = 0.035$, corresponding to classical elements $e = 0.1$, $i = 2$ deg, $a = 1.0095$, $M_0 = 77.2$ deg, $\omega = 271.4$ deg, and $\Omega = 0$ deg. Both the quadratic and quasi-planar solutions closely follow the reference trajectory, with the latter being nearly indistinguishable from it. On the other hand, the linear Clohessy-Wiltshire approximation clearly separates from the exact solution both for the in-plane and out-of-plane motions.

The second test case, Figure 2.3, is similar to the one reported in [45], and it is characterized by a small eccentricity ($e = 0.02$) but a moderate relative inclination ($i = 15$ deg). The initial conditions in curvilinear coordinates are $\rho_0 = -0.020$, $\dot{\rho}_0 = 0$, $\theta_0 = 0.0873$, $\dot{\theta}_0 = 0.0056$, $z_0 = 0$, and $\dot{z}_0 = 0.2640$, corresponding to classical elements $e = 0.02$, $i = 15$ deg, $a = 1.0$, $M_0 = 0$ deg, $\omega = 0$ deg, and $\Omega = 5$ deg. Regarding the in-plane motion, it is straightforward to check that the relatively high value of the inclination causes the quadratic and quasi-planar solutions to notably separate from the exact solution, failing to reproduce it. The behavior is even worse for the Clohessy-Wiltshire solution, which completely misses the trajectory. Conversely, the first-order Fourier solution is able to closely follow the in-plane reference trajectory. All solutions behave better in the out-of-plane motion, with the Fourier and quasi-planar solutions being very close to the exact solution.

The final test case, Figure 2.4, corresponds to a strongly nonlinear relative motion, with high values of the eccentricity ($e = 0.45$) and the relative inclination ($i = 45$ deg). The initial conditions in curvilinear coordinates are $\rho_0 = -0.030$,

$\dot{\rho}_0 = 0.350$, $\theta_0 = 0.0873$, $\dot{\theta}_0 = -0.1772$, $z_0 = 0$, and $\dot{z}_0 = 0.7982$, corresponding to classical elements $e = 0.45$, $i = 45^\circ$, $a = 1.5$, $M_0 = 22.11^\circ$, $\omega = 301.61^\circ$, and $\Omega = 5^\circ$. For this test case only the Fourier solution (with different orders) is considered, since the others are not valid for high values of the inclination (as seen in the second test case). The first order Fourier solution manages to approximately reproduce the main qualitative characteristics of the reference trajectory, but notably separates from it at some points. Better results can be obtained by increasing the order of the Fourier series, with the fifth order solution closely following the exact solution except for some particularly complex regions.

An additional application example studying the orbit of Mars from a geocentric reference frame can be seen in [17].

2.4 Conclusion

The nonlinear relative motion formulation in curvilinear coordinates that will be used for the first part of the dissertation has been introduced in detail, providing all the relevant coordinate systems, transformations and equations of motion (both nonlinear and linearized). An approximate, time-explicit solution for the relative motion with respect to a circular orbit has been developed in the form of Fourier series in the mean anomaly, with the coefficients expressed as Taylor expansions for small eccentricity. This solution has provided new insight into the problem dynamics, and has served as the basis for the derivation of two other approximate solutions for small eccentricity and inclination: a quadratic solution (both single- and double-frequency) and a quasi-planar solution. The single-frequency quadratic solution can be seen as an extension of the well-known Clohessy-Wiltshire solution for the linearized problem, improving on it by including quadratic terms of the initial conditions. The double-frequency quadratic solution introduces further improvements by retaining the second harmonics of the Fourier series, whose coefficients are also quadratic in the initial conditions.

The performances of the different solutions have been assessed and compared through three numerical test cases, showing that both the quasi-planar and quadratic solutions perform well for problems with small eccentricity and inclination, whereas the Fourier series solution can be used for problems up to moderate eccentricity and inclination by retaining a sufficient number of harmonics.

Chapter 3

Optimal Rephasing Maneuver in Circular Orbit

3.1 Introduction

The work on the optimization of minimum-time transfers using the relative motion formulation introduced in the previous chapter is begun by considering the orbital rephasing problem. An orbital rephasing maneuver is the modification of the angular position, typically measured in terms of true or eccentric anomaly, of a spacecraft along its orbit while leaving all other orbital elements unchanged at the end of the maneuver. Rephasing maneuvers are employed for orbit correction and station-keeping (e.g. in geosynchronous orbits [68]) as well as constellation reconfiguration and maintenance [99]. In addition, they are expected to play an important role in the design of future active debris removal (ADR) missions. Irrespectively of the particular technology employed (e.g. ion beams [83, 18], robotic docking [84]) an ADR mission will always involve a far approach maneuvering phase in which an active spacecraft will need to modify its orbital position to come to the vicinity of a non-cooperative target debris to be removed. Such operation will most probably need to be conducted in a fully autonomous way and the angular rephasing distance will not always be small (the ADR spacecraft could be left far from the target especially when piggy-back launch solutions are adopted).

The rephasing, or, equivalently, the same-orbit rendezvous problem has been studied extensively in the literature under different perspectives. Most of the literature on the subject deals with impulsive control solutions that have been and are still widely used in practical applications (see Luo et al. [63] and references therein). Nevertheless, recent advances in electric propulsion have raised the interest for minimum-time optimal control solutions entirely based on low-thrust continuous control (for instance, see the works by Lawden [62] or Marec [64]). The

use of low-thrust methods, when compatible with the specific spacecraft design considered, can be advantageous not only thanks to the lower propellant consumption but also because it can provide higher maneuvering flexibility in some circumstances.

From the mathematical point of view the minimum-time low-thrust control problem is an optimal control problem (OCP) that is typically solved numerically using direct or indirect methods and leading, in general, to a solution with a complex and highly non-intuitive structure. Nevertheless, it has been recently pointed out that the solution can exhibit some interesting characteristics, including an approximate invariance relating the dimensionless thrust and dimensionless time-of-flight for relatively “fast” maneuvers [46]. A mathematical investigation of these kinds of properties using, as much as possible, analytical models would be extremely relevant for future applications of low-thrust rendezvous techniques and is the main focus of the present work.

One of the key points to facilitate a mathematical investigation of the rephasing OCP is the employment of a convenient formulation of the system dynamics. In this regard, the use of a linear relative motion formulation based on the Clohessy-Wiltshire (CW) equations has proved to be a very convenient approach to tackle the complex optimization problem, as demonstrated by other authors [72, 13]. The use of classical CW equations is however restricted to small separation distances between the different spacecraft and is therefore only applicable to short-range rephasing.

In this chapter, the minimum-time constant-thrust optimum rephasing problem for circular orbits is reformulated and explored in greater details providing key analytical relations that can be used by mission analyst to quickly understand its main physical aspects and estimate key quantities such as thrust orientation and maneuver time.

To begin with, a more general relative motion formulation employing curvilinear coordinates [17, 16, 36] is used in its linear and fully nonlinear form to overcome the limitations of the classical CW formulation. Thanks to the relatively simple mathematical structure of the (dimensionless) linearized equations of motion, the latter can be exploited to find an approximate analytical solution of the OCP using the indirect method in the limit cases of short-maneuver (i.e. taking a small fraction of an orbit) and long-maneuver regime (i.e. taking many orbits) while the complex behavior between these two regimes can still be understood qualitatively. Another key advantage of the proposed formulation is its superior performance when employed as a base for numerical indirect optimization methods especially in the case of multiple revolutions.

The chapter is organized as follows. First, a problem description is given by introducing all relevant assumptions and coordinate systems, as well as the fully

nonlinear and linearized equations of motion. Next, the minimum-time constant-thrust rephasing problem is formulated using an indirect method with the complete nonlinear relative motion equations and later particularized for the linearized case. The linearized formulation, which allows one to integrate the adjoint equations analytically, is employed as a basis for the subsequent analytical investigation. After performing an additional non-dimensionalization a fundamental non-dimensional parameter (χ) is introduced which characterizes the length of the maneuver and identifies three fundamental regimes: short maneuver, intermediate, and long maneuver regime. Approximate solutions for the state and costate in the short and long maneuver regime are developed and validated by the subsequent numerical analysis where the two point boundary value problem is solved with a shooting method. In addition, the numerical analysis is used to characterize the transition regime and the effect of the nonlinearities using a complete model. The last part of the chapter is devoted to compare the performance of the previously studied constant-thrust maneuver with an impulsive one.

3.2 Description of the problem

Consider two objects, a chief C and a follower F , describing a circular orbit of radius R around a primary with gravitational constant μ . Let F be subjected to a planar thrust acceleration of constant magnitude T with the objective of changing its initial angular separation θ_0 from C by a given amount $\Delta\theta$ in minimum time span. Let the mass of the follower (m_F) be considered constant at all times thereby neglecting the expelled propellant mass, for simplicity. The resulting optimal control problem (OCP) is a minimum-time constant-acceleration rephasing problem, which would particularize to an optimal rendezvous problem for the case $\Delta\theta = -\theta_0$. Note that, since the mass of the follower is assumed to be constant, the proposed OCP is also equivalent to minimizing the total impulse required for the maneuver.

A convenient strategy to solve the problem is to use the relative motion formulation in curvilinear coordinates introduced in Chapter 2. From now on all variables are expressed in non-dimensional form, taking R , $\Omega^{-1} = \sqrt{R^3/\mu}$, and m_F as the characteristic magnitudes for length, time and mass, respectively. The relative position of F with respect to C is expressed through two curvilinear coordinates ρ and θ associated to a cylindrical frame $\mathcal{F} = \{0; \mathbf{u}_\rho, \mathbf{u}_\theta, \mathbf{k}'\}$, as shown in Figure 3.1. The relations between these curvilinear coordinates, the relative position vector $\mathbf{r}'$, and the position vector $\mathbf{r}$ are given in Chapter 2, Eqs. (2.2). The evolution of (ρ, θ) is described by differential Eqs. (2.3) particularized for the

Figure 3.1: Schematic representation of the problem.

planar case (i.e. $z = 0$):

$$\begin{aligned}\ddot{\rho} - 2\dot{\theta} - 3\rho &= a_{i\rho} + a_{g\rho} + a_{\rho}^T, \\ \ddot{\theta} + 2\dot{\rho} &= a_{i\theta} + \frac{a_{\theta}^T}{1 + \rho},\end{aligned}$$

where $a_{i\rho}$, $a_{g\rho}$, and $a_{i\theta}$ account for the nonlinear contributions in the equations of motion, while a_{ρ}^T and a_{θ}^T correspond to the action of the continuous-thrust on the follower. The latter can be written as:

$$a_{\rho}^T = \varepsilon \sin \gamma, \quad a_{\theta}^T = \varepsilon \cos \gamma,$$

where the coefficient $\varepsilon = T / (m_F \Omega^2 R) > 0$ is the non-dimensional magnitude of the thrust acceleration, and γ is its orientation with respect to the transversal direction (see Figure 3.1).

It is convenient to express the previous equations as a first order system by introducing the state vector:

$$\mathbf{S} = [\dot{\rho} \quad \dot{\theta} \quad \rho \quad \theta]^T, \quad (3.1)$$

leading to:

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F}(\mathbf{S}; \tau),$$

with right-hand side:

$$\mathbf{F}(\mathbf{S}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_{i\rho} + a_{g\rho} + a_{\rho}^T \\ -2\dot{\rho} + a_{i\theta} + a_{\theta}^T / (1 + \rho) \\ \dot{\rho} \\ \dot{\theta} \end{bmatrix}, \quad (3.2)$$

or, for the linearized problem:

$$\mathbf{F}^*(\mathbf{S}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_\rho^T \\ -2\dot{\rho} + a_\theta^T \\ \dot{\rho} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{S} + \begin{bmatrix} a_\rho^T \\ a_\theta^T \\ 0 \\ 0 \end{bmatrix},$$

where τ is the non-dimensional time.

3.3 Continuous Thrust Maneuver

The minimum-time constant-thrust rephasing problem is now formulated and analyzed using the indirect method. Following Bryson and Ho (see [21]), a cost function corresponding to minimum maneuver time can be defined in the form:

$$J = \phi + \int_0^{\Delta\tau} L \, d\tau = \Delta\tau, \quad \text{with } L = 1, \phi = 0,$$

where L is the Lagrangian, and ϕ would represent the dependence of the cost function with the final state. Since in this case there is no such dependence, the cost function takes the form of a *problem of Lagrange*. The associated Hamiltonian can then be written as:

$$H = \boldsymbol{\lambda}^\top \mathbf{F} + L, \quad (3.3)$$

where:

$$\boldsymbol{\lambda} = [\lambda_u \quad \lambda_v \quad \lambda_\rho \quad \lambda_\theta]^\top, \quad (3.4)$$

is the costate associated to the state $\mathbf{S}$. For clarity, the costates associated to $\dot{\rho}$ and $\dot{\theta}$ are denoted as λ_u and λ_v respectively. For the solution to be optimum, it must fulfill the first order optimality conditions given by the adjoint (or costate) equations:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{S}}, \quad (3.5)$$

the control equation:

$$\frac{\partial H}{\partial \gamma} = 0, \quad (3.6)$$

and the transversality condition imposed by the minimum-time requirement:

$$\left(\frac{\partial \phi}{\partial \tau} + H \right)_{\Delta\tau} = 0 \Rightarrow \boldsymbol{\lambda}^\top \mathbf{F}|_{\Delta\tau} = -1. \quad (3.7)$$

These equations form a differential-algebraic system of eight ODEs, Eqs. (3.2) and (3.5), and two algebraic equations, Eqs. (3.6) and (3.7), to be solved for the state boundary conditions:

$$\mathbf{S}(0) = [0 \ 0 \ 0 \ \theta_0]^\top, \quad \mathbf{S}(\Delta\tau) = [0 \ 0 \ 0 \ \theta_f]^\top. \quad (3.8)$$

Note that, since the state is fully specified at both the initial and final times, there are no boundary conditions for the costate.

After substituting Eqs. (3.2) into Eq. (3.3), the control equation, Eq. (3.6), yields a relation between the thrust orientation angle γ , the radial position ρ , and the costates for $\dot{\rho}$ and $\dot{\theta}$:

$$\lambda_u \cos \gamma - \frac{\lambda_v}{1 + \rho} \sin \gamma = 0, \quad (3.9)$$

showing that, analogously to the Cartesian case, the optimum thrust direction is parallel to a certain *primer vector* $[\lambda_u \ \lambda_v/(1 + \rho)]^\top$ at all times. The main difference between this result and the classic primer vector in Cartesian coordinates is that it depends not only on the costate but also on the state. On the other hand, it is interesting to point out that this result reduces to the classic primer vector for the linearized equations of motion (that is, when $\rho \ll 1$). Through careful manipulation, expressions for $\sin \gamma$ and $\cos \gamma$ which take into account the angular determination can be reached:

$$\sin \gamma = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}, \quad \cos \gamma = -\frac{\frac{\lambda_v}{1+\rho}}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}, \quad (3.10)$$

or in terms of the thrust acceleration:

$$a_\rho^T = -\varepsilon \frac{\lambda_u}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}, \quad a_\theta^T = -\varepsilon \frac{\frac{\lambda_v}{1+\rho}}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}. \quad (3.11)$$

Substituting and operating, the adjoint (costate) equations yield:

$$\begin{aligned} \dot{\lambda}_u &= -\frac{\partial H}{\partial \dot{\rho}} = 2\lambda_v \frac{1 + \dot{\theta}}{1 + \rho} - \lambda_\rho \\ \dot{\lambda}_v &= -\frac{\partial H}{\partial \dot{\theta}} = -2\lambda_u (1 + \rho) (1 + \dot{\theta}) + \lambda_v \frac{2\dot{\rho}}{1 + \rho} - \lambda_\theta \\ \dot{\lambda}_\rho &= -\frac{\partial H}{\partial \rho} = -\lambda_u \left[(1 + \dot{\theta})^2 + \frac{2}{(1 + \rho)^3} \right] - \lambda_v \frac{2\dot{\rho} (1 + \dot{\theta}) + (-2a_\theta^T + (a_\theta^T)^3 / \varepsilon^2)}{(1 + \rho)^2} \\ \dot{\lambda}_\theta &= -\frac{\partial H}{\partial \theta} = 0. \end{aligned}$$

Taking into account the boundary conditions, Eqs. (3.8), the transversality condition, Eq. (3.7), takes the compact form:

$$\lambda_u(\Delta\tau) a_\rho^\top(\Delta\tau) + \lambda_v(\Delta\tau) a_\theta^\top(\Delta\tau) = -\frac{1}{\varepsilon}. \quad (3.12)$$

By squaring and summing up Eqs. (3.9,3.12) one can obtain:

$$\lambda_u^2(\Delta\tau) + \lambda_v^2(\Delta\tau) = \frac{1}{\varepsilon^2}. \quad (3.13)$$

The resulting Two-Point Boundary Value Problem (TPBVP) cannot be solved analytically in closed form. However, more tractable equations can be obtained restricting ourselves to the linear equations of motion. The good agreement between the solutions given by the linearized and full formulations is studied numerically in a later section, and it is discussed from a general perspective by Bombardelli et al. in [17]. The Hamiltonian for the linearized problem takes the form:

$$H^* = \boldsymbol{\lambda}^\top \mathbf{F}^* + L = \lambda_u (2\dot{\theta} + 3\rho + a_\rho^T) + \lambda_v (-2\dot{\rho} + a_\theta^T) + \lambda_\rho \dot{\rho} + \lambda_\theta \dot{\theta} + 1, \quad (3.14)$$

and the adjoint equations are now decoupled from the equations of motion:

$$\begin{aligned} \dot{\lambda}_\rho &= -\frac{\partial H^*}{\partial \dot{\rho}} = 2\lambda_v - \lambda_\rho, \\ \dot{\lambda}_\theta &= -\frac{\partial H^*}{\partial \dot{\theta}} = -2\lambda_u - \lambda_\theta, \\ \dot{\lambda}_\rho &= -\frac{\partial H^*}{\partial \rho} = -3\lambda_u, \\ \dot{\lambda}_\theta &= -\frac{\partial H^*}{\partial \theta} = 0. \end{aligned}$$

These equations are linear and can be directly integrated to yield:

$$\begin{aligned} \lambda_u &= A \sin(\tau + \varphi) - 2B, \\ \lambda_v &= 2A \cos(\tau + \varphi) + 3B\tau + C, \\ \lambda_\rho &= 3A \cos(\tau + \varphi) + 6B\tau + 2C, \\ \lambda_\theta &= B. \end{aligned} \quad (3.15)$$

Since the initial and final costates are unknowns of the problem, the solution depends on four parameters A , B , C , and ϕ which would be determined by solving the TPBVP.

After substituting the equations for λ_u and λ_v into Eqs. (3.11) and, in turns, into Eqs. (3.2) one obtains a TPBVP with four ODEs (the linearized equations of motion) and five unknowns ($\Delta\tau$, A , B , C , and φ), subject to the boundary conditions for the dynamics (Eq. (3.8)) and the transversality condition (Eq. (3.13)).

A recent work by Pontani [75] shows that the linearized minimum-time optimal trajectory presents some symmetry properties, which can be exploited to reduce the dimension of the problem. Particularly, given the previous expressions for λ_u and λ_v it is concluded that they have to be, respectively, even and odd functions with respect to the middle point of the maneuver to preserve its symmetry, which can be expressed as:

$$\dot{\lambda}_u(\Delta\tau/2) = 0,$$

$$\lambda_v(\Delta\tau/2) = 0,$$

yielding:

$$A \cos(\Delta\tau/2 + \varphi) = 0 \quad \Rightarrow \quad \varphi = -\Delta\tau/2 + \pi/2 + n\pi, \quad n \in \mathbb{N}^0,$$

$$2A \cos(\Delta\tau/2 + \varphi) + 3B\Delta\tau/2 + C = 0 \quad \Rightarrow \quad C = -\frac{3B\Delta\tau}{2}. \quad (3.16)$$

Since the choice of n only affects the sign of the trigonometric functions and they are always multiplied by A , it is possible to arbitrarily set $n = 0$ and let the sign of A determine whether λ_u has a maximum or a minimum at $\tau = \Delta\tau/2$. These expressions allows us to eliminate φ and C from the costate solution, leading to a simplified TPBVP with only three unknowns ($\Delta\tau$, A , and B).

The system is still nonlinear due to the contribution of the control and cannot be solved in closed form. However, as it will be shown in the following, approximate analytical solutions can be found under certain conditions providing a very good estimation of the optimum maneuver time to serve as an initial guess for the iterative procedures involved in both the direct and indirect methods. The first step is to introduce a new non-dimensionalization using the absolute value of the desired rephasing angular displacement $\Delta\theta = \theta_f - \theta_0$ and the maneuver time $\Delta\tau$ as characteristic magnitudes for length and time respectively. The choice for the former is made so that the variation of the angular position along the trajectory is equal in magnitude to unity. The non-dimensional curvilinear coordinates and time now read:

$$\hat{\rho} = \frac{\rho}{|\Delta\theta|}, \quad \hat{\theta} = \frac{\theta}{|\Delta\theta|}, \quad \hat{\tau} = \frac{\tau}{\Delta\tau}, \quad (3.17)$$

yielding (' denotes the derivative with respect to $\hat{\tau}$):

$$\begin{aligned} \hat{\rho}'' - 2\hat{\theta}'\Delta\tau - 3\hat{\rho}\Delta\tau^2 &= \frac{\Delta\tau^2}{\chi}u_\rho, \\ \hat{\theta}'' + 2\hat{\rho}'\Delta\tau &= \frac{\Delta\tau^2}{\chi}u_\theta, \end{aligned} \quad (3.18)$$

where $u_\rho^2 + u_\theta^2 = 1$, and the fundamental parameter χ has been introduced as the ratio between the desired angular displacement $\Delta\theta$ and the available thrust ε :

$$\chi = \frac{|\Delta\theta|}{\varepsilon}. \quad (3.19)$$

This non-dimensionalization is also introduced in the expressions for the costate, Eqs. (3.15), along with a scaling of magnitude ε :

$$\hat{\lambda}_u = \varepsilon\lambda_u, \quad \hat{\lambda}_v = \varepsilon\lambda_v, \quad \hat{\lambda}_\rho = \varepsilon\lambda_\rho, \quad \hat{\lambda}_\theta = \varepsilon\lambda_\theta,$$

leading to:

$$\begin{aligned} \hat{\lambda}_u &= \frac{\hat{A}}{\Delta\tau} \sin(\Delta\tau\hat{\tau} + \hat{\varphi}) - 2\frac{\hat{B}}{\Delta\tau}, \\ \hat{\lambda}_v &= 2\frac{\hat{A}}{\Delta\tau} \cos(\Delta\tau\hat{\tau} + \hat{\varphi}) + 3\hat{B}\hat{\tau} + \hat{C}, \\ \hat{\lambda}_\rho &= 3\frac{\hat{A}}{\Delta\tau} \cos(\Delta\tau\hat{\tau} + \hat{\varphi}) + 6\hat{B}\hat{\tau} + 2\hat{C}, \\ \hat{\lambda}_\theta &= \frac{\hat{B}}{\Delta\tau}, \end{aligned} \quad (3.20)$$

where:

$$\hat{A} = \varepsilon\Delta\tau A, \quad \hat{B} = \varepsilon\Delta\tau B, \quad \hat{C} = \varepsilon C, \quad \hat{\varphi} = \varphi. \quad (3.21)$$

Introducing the symmetry conditions, Eqs. (3.16), with $n = 0$ and the convenient change of independent variable $\tilde{\tau} = \hat{\tau} - 1/2$, the non-dimensional expressions for the costate take a very simple form:

$$\begin{aligned} \hat{\lambda}_u &= \frac{\hat{A}}{\Delta\tau} \cos(\Delta\tau\tilde{\tau}) - 2\frac{\hat{B}}{\Delta\tau}, \\ \hat{\lambda}_v &= -2\frac{\hat{A}}{\Delta\tau} \sin(\Delta\tau\tilde{\tau}) + 3\hat{B}\tilde{\tau}, \\ \hat{\lambda}_\rho &= -3\frac{\hat{A}}{\Delta\tau} \sin(\Delta\tau\tilde{\tau}) + 6\hat{B}\tilde{\tau}, \\ \hat{\lambda}_\theta &= \frac{\hat{B}}{\Delta\tau}. \end{aligned} \quad (3.22)$$

The orders of magnitude for the individual terms in the left hand side member of Eqs. (3.18) are:

$$\begin{aligned} \hat{\rho}'' &\sim \frac{\overline{\Delta\rho}}{|\Delta\theta|}, \\ \hat{\theta}'' &\sim 1, \\ \hat{\theta}'\Delta\tau &\sim \Delta\tau, \end{aligned}$$

$$\begin{aligned}\dot{\rho}'\Delta\tau &\sim \frac{\overline{\Delta\rho}}{|\Delta\theta|}\Delta\tau, \\ \hat{\rho}\Delta\tau^2 &\sim \frac{\overline{\Delta\rho}}{|\Delta\theta|}\Delta\tau^2,\end{aligned}$$

where $\overline{\Delta\rho}$ is the characteristic variation of ρ across the time scale determined by $\Delta\tau$. In other words, $\overline{\Delta\rho}$ captures the mean variation of ρ along the maneuver, roughly corresponding to the maximum radial separation.

The following sections study the behavior of the transfer for the limit cases $\chi \ll 1$ and $\chi \gg 1$, corresponding to short and long maneuver times regimes, showing that it undergoes fundamental qualitative changes.

3.3.1 Short maneuver time regime

The regime where the non-dimensional thrust parameter ε is much greater than the rephasing angular distance $\Delta\theta$, corresponding to $\chi \ll 1$ and $\Delta\tau \ll 1$, is called short maneuver time regime and is considered first.

Regarding the transversal equation, it is possible to demonstrate that, in order to minimize the maneuver time, the dominating contribution towards the increase in $\Delta\theta$ is due to the control term with a negligible contribution from the Coriolis term $2\dot{\rho}'\Delta\tau$. Assuming this is the case the order of magnitude analysis of the terms in the second of Eqs. (3.18) leads to a maneuver time of order:

$$\Delta\tau \sim \sqrt{\frac{\chi}{|u_\theta|}}.$$

If this was not the case the increase in $\Delta\theta$ would necessarily come from the coupling of radial and transversal dynamics through the Coriolis term and a control action mainly oriented along the radial direction. In such case the second of Eqs. (3.18) would lead to $\overline{\Delta\rho}/|\Delta\theta| \sim 1/\Delta\tau$, and given that the control contribution in the first of Eqs. (3.18) could not be negligible (since it has been neglected in the second equation) one would obtain $\Delta\tau \sim \sqrt[3]{\chi/|u_\rho|}$. Since $\chi \ll 1$ both scenarios would lead to $\Delta\tau \ll 1$, but the solution for the first one clearly outperforms the other. This implies that $|u_\theta| \sim 1$ and, consequently, $|u_\rho| \ll 1$. Using the latter condition and the $\Delta\tau$ order estimate, it can be shown that the radial acceleration and Coriolis term in the radial equation dominate over the centrifugal and control terms, yielding the characteristic variation of the radial displacement:

$$\overline{\Delta\rho} \sim \Delta\tau |\Delta\theta|.$$

Introducing this result back into the transversal equation confirms that the Coriolis term, of order $(\overline{\Delta\rho}/|\Delta\theta|)\Delta\tau \sim \Delta\tau^2$, is negligible compared to the control

component, of order $\Delta\tau^2 |u_\theta|/\chi \sim 1$, justifying the initial hypothesis. It is interesting to note that the Coriolis term in the radial equation is at least as important as the rest, representing the coupling with the control (which is mostly applied in the transversal direction). One can finally conclude that, in the short maneuver regime, Eqs. (3.18) can be approximated with:

$$\hat{\rho}'' - 2\Delta\tau\hat{\theta}' \approx (\Delta\tau^2/\chi) u_\rho, \quad (3.23)$$

$$\hat{\theta}'' \approx (\Delta\tau^2/\chi) u_\theta. \quad (3.24)$$

Let us now look for an approximate analytical solution of the costate. By expanding in Taylor series around $\tau = \Delta\tau/2$ (i.e. $\tilde{\tau} = 0$) the first two of Eqs. (3.22) for $\tilde{\tau} \ll 1$:

$$\hat{\lambda}_u(\tilde{\tau}) = \frac{\hat{A} - 2\hat{B}}{\Delta\tau} - A\Delta\tau\tilde{\tau}^2/2 + \mathcal{O}(\tilde{\tau}^4),$$

$$\hat{\lambda}_v(\tilde{\tau}) = (3\hat{B} - 2\hat{A})\tilde{\tau} + \mathcal{O}(\tilde{\tau}^3).$$

When $|u_\rho| \ll 1$ and $|u_\theta| \sim 1$ one has $|\hat{\lambda}_u| \ll |\hat{\lambda}_v|$, which from the preceding equations translates into:

$$\hat{A} \approx 2\hat{B},$$

and the transversality condition, Eq. (3.13), takes the form:

$$1 = \hat{\lambda}_u^2(\tilde{\tau} = 1/2) + \hat{\lambda}_v^2(\tilde{\tau} = 1/2) \approx \hat{\lambda}_v^2(\tilde{\tau} = 1/2),$$

leading to:

$$\hat{\lambda}_v(\tilde{\tau} = 1/2) \approx -\frac{\hat{B}}{2} \approx \pm 1,$$

finally providing:

$$\hat{\lambda}_v \approx \pm 2\tilde{\tau} = \pm 2\hat{\tau} \mp 1.$$

Along with Eqs. (3.11) this proves that, in the short maneuver time limit, the thrust control angle has a bang-bang structure with a 180 degrees switch at $\tau = \Delta\tau/2$. The correct sign in the preceding equation for $\Delta\theta > 0$ is easily established from Eq. (3.24) where a positive transversal acceleration (i.e. $\hat{\lambda}_v' < 0$) at $\tau = 0$ is necessary to produce a phase angle increase (the sign reverses for $\Delta\theta < 0$). Therefore:

$$\hat{\lambda}_v \approx \pm_\Delta 2\tilde{\tau} = \pm_\Delta (2\hat{\tau} - 1), \quad (3.25)$$

where:

$$\pm_\Delta = \text{sign}(\Delta\theta), \quad \mp_\Delta = -\text{sign}(\Delta\theta), \quad (3.26)$$

and, for $\Delta\theta > 0$ (the order of the angles is reversed for negative $\Delta\theta$):

$$\gamma \approx \begin{cases} 0, & 0 < \hat{\tau} < 1/2 \\ \pi, & 1/2 < \hat{\tau} < 1 \end{cases}$$

meaning that the thrust is oriented along the direction of the desired displacement in the first half of the maneuver and opposite to it in the second half. Gathering these results, estimates for the coefficients of Eqs. (3.22) describing the linearized costate are also reached:

$$\hat{A} \approx \mp_{\Delta} 4, \quad \hat{B} \approx \mp_{\Delta} 2. \quad (3.27)$$

A better estimate of $\Delta\tau$ can now be obtained by substituting the previous bang-bang control structure into Eq. (3.24) and solving for zero initial conditions (particularly $\hat{\theta}_0 = 0$) to yield:

$$\hat{\theta} \approx \pm_{\Delta} \begin{cases} \frac{\Delta\tau^2}{\chi} \frac{\hat{\tau}^2}{2}, & 0 < \hat{\tau} < 1/2 \\ \frac{\Delta\tau^2}{\chi} \left(-\frac{\hat{\tau}^2}{2} + \hat{\tau} - \frac{1}{4} \right), & 1/2 < \hat{\tau} < 1 \end{cases}$$

By imposing $\hat{\theta} = 1$ for $\hat{\tau} = 1$ it is possible to give an estimate for the time of flight:

$$\Delta\tau \approx 2\sqrt{\chi} = 2\sqrt{\frac{|\Delta\theta|}{\varepsilon}}. \quad (3.28)$$

Regarding the radial motion, Eq. (3.23), the term associated to the thrust acceleration can be at most of the same order as the other two, corresponding to $|u_{\rho}| \sim \Delta\tau \ll 1$. Solving this equation for negligible thrust (i.e. $u_{\rho} \approx 0$) leads to expressions for $\hat{\rho}'$ and $\hat{\rho}$ with deviations in the final boundary conditions of order $\Delta\tau$. This shows that the radial thrust acceleration, albeit small, is not negligible and its magnitude increases with the maneuver time. In other words, the separation between the optimal control law and the simplified bang-bang profile is maximum for χ close to 1 and becomes smaller as χ decreases.

Finally, the total impulse required for this continuous thrust maneuver is of the form:

$$\Delta V_{\text{cont}} = \varepsilon \Delta\tau \approx 2\sqrt{\varepsilon |\Delta\theta|}. \quad (3.29)$$

3.3.2 Long maneuver time regime

In the long maneuver time regime the non-dimensional thrust parameter ε is small compared to the rephasing angular distance $\Delta\theta$, corresponding to $\chi \gg 1$ and $\Delta\tau \gg 1$.

Following a similar reasoning as for the previous case, it is possible to demonstrate that the dominating contribution towards the minimum-time increase in $\Delta\theta$ in Eqs. (3.18) is due to the transversal control term but also to the coupling between the transversal and the radial dynamics through the Coriolis terms $2\hat{\rho}'\Delta\tau$.

As in the previous case, neglecting the transversal thrust effects would yield to $\Delta\rho/|\Delta\theta| \sim 1/\Delta\tau$, and introducing this into the radial equation while retaining the control terms would lead to $\Delta\tau \sim \chi/|u_\rho|$. On the contrary, assuming the transversal thrust acceleration term is at least of the same order as the rest the time of flight estimate is:

$$\Delta\tau \sim \sqrt{\frac{\chi}{|u_\theta|}},$$

outperforming once again the negligible transversal thrust solution, since $\chi \gg 1$. This shows that the maneuver time $\Delta\tau$ is much greater than unity and is minimum when $|u_\theta| \sim 1$ and $|u_\rho| \ll 1$. It follows that the dominating terms in the first of Eqs. (3.18) are the Coriolis ($2\hat{\theta}'\Delta\tau$) and centrifugal ($3\hat{\rho}\Delta\tau^2$) terms, leading to $\overline{\Delta\rho}/|\Delta\theta| \sim 1/\Delta\tau$. Going back to the transversal equation, it is verified that the contribution from the radial velocity is not negligible but of the same order of the thrust acceleration, since $(\overline{\Delta\rho}/|\Delta\theta|)\Delta\tau \sim 1$ is of the same order as $\Delta\tau^2|u_\theta|/\chi \sim 1$. Similarly to the short maneuver regime, the Coriolis term in the radial equation represents the coupling with the control and is at least as important as the rest. All in all, the relative equations of motion in the long maneuver regime can be approximated by:

$$-2\Delta\tau\hat{\theta}' - 3\Delta\tau^2\hat{\rho} \approx 0, \quad (3.30)$$

$$\hat{\theta}'' + 2\Delta\tau\hat{\rho}' \approx (\Delta\tau^2/\chi) u_\theta, \quad (3.31)$$

We can now search for an approximation of the costate. When $|u_\rho| \ll 1$ and $|u_\theta| \sim 1$ we have $|\hat{\lambda}_u| \ll |\hat{\lambda}_v|$, and for not too small $\tilde{\tau}$ (i.e. not too close to the maneuver middle point) the first two of Eqs. (3.22) lead to:

$$|\hat{A}| \ll 3\Delta\tau |\hat{B}| \tilde{\tau},$$

and imposing the transversality condition:

$$\hat{\lambda}_v(\tilde{\tau} = 1/2) \approx \frac{3}{2}\hat{B} \approx \pm 1, \quad (3.32)$$

providing:

$$\hat{\lambda}_v \approx \pm 2\tilde{\tau} = \pm 2\hat{\tau} \mp 1, \quad (3.33)$$

which along with Eqs. (3.11) proves that in the long maneuver time limit the thrust control angle resembles a bang-bang with a 180 degrees switch at $\Delta\tau/2$.

Note that the present approximation does not assure that the bang-bang structure is preserved close to the switching point. The correct sign in the preceding equation can be established by analytically solving Eqs. (3.30,3.31) and choosing the sign corresponding to a positive increase of $\Delta\theta$ to yield:

$$\begin{aligned}\hat{\lambda}_v &\approx \pm_{\Delta} (-2\tilde{\tau}) = \pm_{\Delta} (-2\hat{\tau} + 1), \\ |\hat{A}| &\ll 2\Delta\tau, \quad \hat{B} \approx \mp_{\Delta} \frac{2}{3}, \\ \hat{\theta} &\approx \pm_{\Delta} \begin{cases} \frac{\Delta\tau^2}{\chi} \frac{3}{2} \hat{\tau}^2, & 0 < \hat{\tau} < 1/2 \\ \frac{\Delta\tau^2}{\chi} \left(-\frac{3}{2} \hat{\tau}^2 + 3\hat{\tau} - \frac{3}{4} \right), & 1/2 < \hat{\tau} < 1 \end{cases} \\ \hat{\rho} &\approx \pm_{\Delta} \begin{cases} -2\frac{\Delta\tau}{\chi} \hat{\tau}, & 0 < \hat{\tau} < 1/2 \\ \frac{\Delta\tau}{\chi} (2\hat{\tau} - 2), & 1/2 < \hat{\tau} < 1 \end{cases} \end{aligned} \quad (3.34)$$

This approximate solution aims to reproduce the evolution of the mean value of the state. By imposing $\hat{\theta} = 1$ for $\hat{\tau} = 1$ it is possible to give an estimate for the time of flight:

$$\Delta\tau \approx \sqrt{\frac{4}{3}}\chi = \sqrt{\frac{4}{3}}\frac{|\Delta\theta|}{\varepsilon}, \quad (3.35)$$

In addition, an estimate of the maximum radial displacement is also possible:

$$\rho_{\max} = |\Delta\theta| \hat{\rho}(\hat{\tau} = 1/2) \approx \mp_{\Delta} 2\sqrt{\frac{\varepsilon |\Delta\theta|}{3}}.$$

The preceding relations show that the transversal velocity is obtained through the radial displacement, which in turn is obtained by applying thrust in the direction *opposite* to the one of the desired motion. This represents a qualitative difference with respect to the solution obtained for the short maneuver regime.

The order of differential Eq. (3.30) has been reduced by neglecting the non-dominant terms, and so the boundary conditions for $\hat{\rho}'$ cannot be imposed. Particularly, it presents a constant, non-zero value of magnitude $\Delta\tau/\chi$ during the first half of the maneuver and the same value with opposite sign during the second half. Therefore, a small radial thrust acceleration will be needed to fulfill all the boundary conditions, separating the optimal control law from the simplified bang-bang profile. Interestingly, since $\Delta\tau/\chi$ decreases as χ increases, the separation between both control laws becomes smaller as χ increases.

Regarding the cost of the maneuver, the approximate total impulse can be easily computed for the constant thrust case as:

$$\Delta V_{\text{cont}} = \varepsilon \Delta\tau \approx \sqrt{\frac{4}{3}}\varepsilon |\Delta\theta|. \quad (3.36)$$

3.3.3 Transition zone

The previous analysis shows the existence of a transition zone between the short maneuver and long maneuver regimes for $\chi \sim 1$ (i.e. $\varepsilon \sim |\Delta\theta|$), but provides little information about the behavior of the solution during said transition. The time of flight is now $\Delta\tau \sim 1$, indicating that the transition zone corresponds to rephasing maneuvers which take about one orbit to complete (2π being the orbital period in the non-dimensional variables). Some fundamental qualitative changes have been identified when moving from one regime to the other, the most notable being the reversal of the orientation of thrust. During this transition it is no longer true that $|u_\theta| \sim 1$ and $|u_\rho| \ll 1$, so the control angle γ may take any value instead of remaining close to the transversal direction. Furthermore, all terms in both of Eqs. (3.18) are now of comparable order, so a detailed analytical description of this transition zone would require solving the complete TPBVP associated to the OCP and falls far beyond the scope of this work. Numerical solutions for a wide range of cases within this region are provided in the following section.

3.4 Numerical Tests

This section presents the numerical results obtained for a wide range of $\Delta\theta$ and ε values, calculated using both the linear and nonlinear formulations. While the extended range of this parametric study includes some cases of little practical interest (such as those with a very high thrust acceleration), it allows to reach a deeper understanding of the problem and its characteristics (specially in the far ends of the two limit regimes).

The TPBVP is solved using a basic shooting method, with initial guesses of the unknowns constructed either from the previous approximated solutions or by applying continuation techniques. In order to improve the numerical behavior of the algorithm, it is convenient to use the same non-dimensional variables introduced in Eqs. (3.17) for the analytical study. Setting the initial time and the final angular position to 0, the boundary conditions take the simple form:

$$\begin{aligned} \begin{bmatrix} \hat{\rho}'(0) & \hat{\theta}'(0) & \hat{\rho}(0) & \hat{\theta}(0) \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & \mp\Delta 1 \end{bmatrix}, \\ \begin{bmatrix} \hat{\rho}'(\Delta\tau) & \hat{\theta}'(\Delta\tau) & \hat{\rho}(\Delta\tau) & \hat{\theta}(\Delta\tau) \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (3.37)$$

The linearized ODEs for dynamics in these non-dimensional variables were already given in Eqs. (3.18), while the costate solution corresponds to Eqs. (3.20). On the

other hand, the nonlinear equations of motion can be written as:

$$\begin{aligned}\hat{\rho}'' - 2\Delta\tau\hat{\theta}' - 3\Delta\tau^2\hat{\rho} &= \hat{a}_{i\rho} + \hat{a}_{g\rho} + \frac{\Delta\tau^2}{\chi} \sin \gamma, \\ \hat{\theta}'' + 2\Delta\tau\hat{\rho}' &= \hat{a}_{i\theta} + \frac{\Delta\tau^2}{\chi} \frac{\cos \gamma}{1 + \Delta\theta\hat{\rho}},\end{aligned}$$

with:

$$\begin{aligned}\hat{a}_{i\rho} &= \Delta\theta\hat{\theta}'^2(1 + \Delta\theta\hat{\rho}) + 2\Delta\tau\Delta\theta\hat{\theta}'\hat{\rho}, \quad \hat{a}_{i\theta} = 2\Delta\theta\frac{\hat{\rho}'(\Delta\tau\hat{\rho} - \hat{\theta}')}{1 + \Delta\theta\hat{\rho}}, \\ \hat{a}_{g\rho} &= -\Delta\tau^2\Delta\theta\frac{\hat{\rho}^2(3 + 2\Delta\theta\hat{\rho})}{(1 + \Delta\theta\hat{\rho})^2},\end{aligned}$$

and the nonlinear adjoint equations take the form:

$$\begin{aligned}\hat{\lambda}'_u &= 2\hat{\lambda}_v\frac{\Delta\tau + \Delta\theta\hat{\theta}'}{1 + \Delta\theta\hat{\rho}} - \Delta\tau\hat{\lambda}_\rho, \\ \hat{\lambda}'_v &= -2\hat{\lambda}_u(1 + \Delta\theta\hat{\rho})(\Delta\tau + \Delta\theta\hat{\theta}') + \hat{\lambda}_v\frac{2\Delta\theta\hat{\rho}'}{1 + \Delta\theta\hat{\rho}} - \Delta\tau\hat{\lambda}_\theta, \\ \hat{\lambda}'_\rho &= -\hat{\lambda}_u\left[\Delta\tau\left(1 + (\Delta\theta/\Delta\tau)\hat{\theta}'\right)^2 + \frac{2\Delta\tau}{(1 + \Delta\theta\hat{\rho})^3}\right] \\ &\quad - \hat{\lambda}_v\frac{2\Delta\theta\hat{\rho}'\left(1 + (\Delta\theta/\Delta\tau)\hat{\theta}'\right) + \Delta\tau\left(-2\hat{a}_\theta^T + (\hat{a}_\theta^T)^3/\varepsilon^2\right)}{(1 + \Delta\theta\hat{\rho})^2}, \\ \hat{\lambda}'_\theta &= 0,\end{aligned}\tag{3.38}$$

where:

$$\hat{a}_\theta^T = -\varepsilon\frac{\frac{\hat{\lambda}_v}{1 + \Delta\theta\hat{\rho}}}{\sqrt{\hat{\lambda}_u^2 + \left(\frac{\hat{\lambda}_v}{1 + \Delta\theta\hat{\rho}}\right)^2}}.$$

The control law for thrust orientation γ is expressed through Eqs. (3.10) as a function of the costate, which is in turn given either by Eqs. (3.20) in the linear case or by the numerical integration of Eqs. (3.38) in the nonlinear case. Note that thrust magnitude ε is assumed to be constant during the maneuver.

The TPBVP has five unknowns, $(\Delta\tau, \hat{A}, \hat{B}, \hat{C}, \hat{\varphi})$ in the linear case and $(\Delta\tau, \hat{\lambda}_{u0}, \hat{\lambda}_{v0}, \hat{\lambda}_{\rho0}, \hat{\lambda}_{\theta0})$ in the nonlinear case, to be determined by imposing the boundary conditions for the state, Eqs. (3.37), and the transversality condition, Eq. (3.13). Using a single shooting algorithm the TPBVP can be expressed as a root finding problem with the four final state constraints and the transversality condition as

functions, where each evaluation of these functions requires the propagation of the orbit. The dimension of the linearized problem could be reduced to three by using its symmetry properties, but in order to numerically verify them the complete TPBVP with five unknowns is solved.

The shooting algorithm has to be initialized with an initial guess of the solution, and the quality of this initial guess has a key impact on the convergence of the iterative process. For the linearized problem, initial guesses are generated based on the previous approximate analytical solutions as follows:

$$\left\{ \begin{array}{ll} \Delta\tau^0 = 2\sqrt{\chi}, & \hat{A}^0 = \mp_{\Delta} 4, \quad \hat{B}^0 = \mp_{\Delta} 2, \\ \hat{C}^0 = \pm_{\Delta} 3, & \hat{\varphi}^0 = -\Delta\tau^0 + \pi/2, \end{array} \right. \quad \text{for } \chi < 1$$

$$\left\{ \begin{array}{ll} \Delta\tau^0 = 2\sqrt{\chi/3}, & \hat{A}^0 = 0, \quad \hat{B}^0 = \mp_{\Delta} 2/3, \\ \hat{C}^0 = \pm_{\Delta} 1, & \hat{\varphi}^0 = -\Delta\tau^0 + \pi/2, \end{array} \right. \quad \text{for } \chi > 1$$

where the superscript ‘0’ denotes initial guess. This approach proves to be good enough to solve all the cases in the short maneuver and long maneuver regimes, as well as many in the transition zone. For those cases where the shooting algorithm fails to converge a continuation method is used, employing the solution of the nearest converged case as initial guess (zeroth-order continuation). Regarding the nonlinear problem, the corresponding linear solution is used as a first initial guess, which proves to be good enough for all cases but a few in the transition zone that can be tackled using a continuation method.

The evolution of the time of flight with χ is represented in Figure 3.2, both for the linear and nonlinear cases and limited to $0 < \Delta\theta < \pi$. Note that while the linear solution only depends on χ and all the test points fall into the same curve regardless of the particular values of $\Delta\theta$ and ε , the nonlinear solutions separate for large enough values of χ and ε . This is due to the state variables becoming too large for the nonlinear terms to have a negligible contribution, an effect which is especially important close to the transition zone as shown in Figure 3.3. Therefore, whereas the linear case can be characterized by a single parameter, χ , the nonlinear case requires two, ε and $\Delta\theta$. On the other hand, Figure 3.2 also allows to clearly distinguish the short maneuver and long maneuver regimes, with the transition zone in between. The analytical approximations for $\Delta\tau$, Eqs. (3.28,3.35), fit very well with the numerical results in the two limit regimes, and the transition zone takes place for times of flight of about one orbit as predicted. Going back to Figure 3.3, it is interesting to see how the nonlinear solution for $\varepsilon = 10^{-1}$ and positive $\Delta\theta$ takes notably less time to complete than the linear one for the first part of the transition zone, but becomes slower as the duration of the maneuver increases. In the same figure, the two curves for $\varepsilon = 10^{-2}$ and different signs of $\Delta\theta$ show how the rephasing in the transition zone with positive $\Delta\theta$, which

Figure 3.2: Time of flight as a function of χ for several values of ε and positive $\Delta\theta$ limited to one radian (note that the linear solution does not depend on ε).

will temporarily reduce the orbital radius during the maneuver, benefits from the nonlinear effects while the rephasing with negative $\Delta\theta$, which will temporarily increase the orbital radius, is penalized by them. These differences are reduced and even reversed as the long maneuver regime is approached.

To help illustrate the characteristics of the solutions at each region, three particular cases taken from the short maneuver regime, the long maneuver regime and the transition zone are now considered. These cases have been carefully chosen as representative examples for each region, and the qualitative features identified in them are general for their corresponding regime. A large positive angular displacement of 1 radian is assumed in all cases, in order to better appreciate the effects of the nonlinearity. This also leads to large values of ε in the short maneuver regime and the transition zone, which may not be physically feasible in an Earth-bound mission (although they could be valid for interplanetary missions). Then again, this choice of parameters is oriented towards a better representation of the qualitative differences between the linear and nonlinear solutions, and all these regimes can appear in Earth-bound missions for smaller values of $\Delta\theta$. Figures 3.4 show the thrust control angle, curvilinear position and primer vector for a rephasing in the short maneuver regime, with $\chi = 9.8171 \times 10^{-3}$ and a non-dimensional thrust acceleration of 1.0186×10^2 . The non-dimensional time of flight $\Delta\tau$ is 0.1983 for

Figure 3.3: Detail of the time of flight as a function of χ for several values of ε .

the linear solution, and slightly shorter for the nonlinear case, 0.1969. Therefore, both maneuvers take a little more than one thirtieth of an orbit to complete. It is also observed that the control approaches a bang-bang structure as predicted, with small deviations in the linear solution and much larger ones in the nonlinear case. This is also reflected in the primer vector, with its nonlinear radial component $\hat{\lambda}_u$ clearly separating from its linear counterpart. The effects of the nonlinearity are also easy to spot in the figure for the radial position, with the nonlinear solution for $\hat{\rho}$ reaching relatively large values that justify the influence of the nonlinear terms. Differences are harder to spot for $\hat{\theta}$, explaining the small variation in the time of flight. Finally, it is important to stress that both the linear and nonlinear maneuvers are symmetric with respect to their middle point.

The form of the solution undergoes fundamental changes in the long maneuver regime. The transfers depicted in Figures 3.5, calculated for $\chi = 9.7269 \times 10^2$ and $\varepsilon = 1.0281 \times 10^{-3}$, yield times of flight of 35.9148 and 35.9071 for the linear and nonlinear cases, respectively (corresponding to nearly 6 orbital revolutions). Same as in the previous example, the nonlinear maneuver takes slightly less time to complete than the linear one. The thrust angle profile, while still being close to a bang-bang, has reversed its direction, pointing opposite to the chief during the first half of the maneuver and towards it in the second. The differences between the linear and nonlinear curves for γ , $\hat{\rho}$ and the primer vector are noticeably smaller

Figure 3.4: Optimal rephasing maneuvers for a positive angular displacement of 1 radian and a non-dimensional thrust acceleration of 1.0186×10^2 (corresponding to $\chi = 9.8171 \times 10^{-3}$).

Figure 3.5: Optimal rephasing maneuvers for a positive angular displacement of 1 radian and a non-dimensional thrust acceleration of 1.0281×10^{-3} (corresponding to $\chi = 9.7269 \times 10^2$).

Figure 3.6: Optimal rephasing maneuvers for a positive angular displacement of 1 radian and a non-dimensional thrust acceleration of 1.1485 (corresponding to $\chi = 8.7072 \times 10^{-1}$).

than in the short maneuver regime, due to a smaller influence of the nonlinear terms. All the curves, both linear and nonlinear, are still symmetric, but now the oscillations due to the gravitational effects in the relative motion can be clearly appreciated. Comparing the curve for $\hat{\rho}$ with the approximate analytical solution given in Eqs. (3.34), it is straightforward to check that the latter succeeds in capturing the evolution of the mean value of the radial coordinate but fails to reproduce its oscillatory behavior.

A rephasing in the transition zone with $\chi = 8.7072 \times 10^{-1}$ and $\varepsilon = 1.1485$ is depicted in Figures 3.6, yielding times of flight of 1.9109 (linear) and 1.8084 (nonlinear) roughly corresponding to one third of an orbit. The thrust angle profile no longer resembles a bang-bang control, showing important components in the radial direction. This is reflected in the fact that both components of the primer vector now take similar values. The radial displacement is larger than in the previous examples, leading to an important contribution of the nonlinear

terms which causes the linear and nonlinear curves for $\hat{\theta}$ to separate more than before. Nevertheless, despite this nonlinearity effects the different curves maintain the symmetry properties also observed in the short maneuver and long maneuver regimes.

3.5 Two-Impulse Maneuver

The optimum solution for a multiple-impulse rephasing maneuver in circular orbit is treated in detail in a milestone article by J.B. Jones [55]. Jones shows (as previously demonstrated by Prussing [77]) that a two-impulse solution is optimal for a transfer between zero and approximately 2π radians, while a four-impulse solution is optimal between approximately 2π and 4π radians. Sequences of two and four impulses solutions repeat themselves for increasing transfer angles. Local minima for the two-impulse delta-V solution appear for transfer angles of about 1, 3, 5, ... orbits and have orientations which deviate from the tangential direction by roughly 3, 1.5, 1, ... degrees. The four-impulse solution, when optimal, provides a small improvement with respect to the minima of the two-impulse solution following the addition of two symmetric internal impulses whose position and magnitude depends on the specific transfer angle.

For comparison purposes only, two-impulse maneuvers are briefly considered. The boundary conditions for such maneuvers take the form:

$$\begin{aligned} \rho(0) = 0, \quad \theta(0) = \theta_0, \quad \dot{\rho}(0) = \Delta u_0, \quad \dot{\theta}(0) = \Delta v_0, \\ \rho(\Delta\tau) = 0, \quad \theta(\Delta\tau) = \theta_f, \quad \dot{\rho}(\Delta\tau) = \Delta u_f, \quad \dot{\theta}(\Delta\tau) = \Delta v_f, \end{aligned}$$

where Δu and Δv represent impulses in the radial and transversal directions respectively. The required angular displacement is $\Delta\theta = \theta_f - \theta_0$, positive if it is oriented in the positive direction of $\mathbf{u}_\theta$ and negative otherwise. The initial impulse can also be expressed in terms of its magnitude ΔV_0 and its orientation angle γ_0 as:

$$\Delta u_0 = \Delta V_0 \cos \gamma_0, \quad \Delta v_0 = \Delta V_0 \sin \gamma_0.$$

The boundary conditions show that the extremal values of ρ and θ are fixed, while the corresponding $\dot{\rho}$ and $\dot{\theta}$ represent the unknown impulses. Therefore, it is interesting to rearrange the part of Eqs. (2.10) corresponding to the in-plane motion in the form:

$$\begin{bmatrix} \dot{\rho} & \dot{\theta} & \dot{\rho}_0 & \dot{\theta}_0 \end{bmatrix}^\top = \mathbf{T}(\tau) \begin{bmatrix} \rho & \theta & \rho_0 & \theta_0 \end{bmatrix}^\top, \quad (3.39)$$

where $\mathbf{T}$ is the targeting matrix. Through some algebraic manipulations, the

elements of $\mathbf{T}$ can be obtained as:

$$\mathbf{T} = \begin{bmatrix} T_1 & T_2 & T_3 & -T_2 \\ T_2 - 2 & -T_4 & T_2 & T_4 \\ -T_3 & -T_2 & -T_1 & T_2 \\ T_2 & -T_4 & T_2 - 2 & T_4 \end{bmatrix},$$

with:

$$T_1 = \frac{3\tau \cos \tau - 4 \sin \tau}{8(\cos \tau - 1) + 3\tau \sin \tau},$$

$$T_2 = \frac{2(\cos \tau - 1)}{8(\cos \tau - 1) + 3\tau \sin \tau},$$

$$T_3 = \frac{4 \sin \tau - 3\tau}{8(\cos \tau - 1) + 3\tau \sin \tau},$$

$$T_4 = \frac{\sin \tau}{8(\cos \tau - 1) + 3\tau \sin \tau}.$$

By introducing the boundary conditions in position:

$$[\rho(\Delta\tau) \quad \theta(\Delta\tau) \quad \rho(0) \quad \theta(0)]^\top = [0 \quad \theta_f \quad 0 \quad \theta_0]^\top,$$

into Eqs. (3.39), the required impulses for a given angular displacement $\Delta\theta$ and time of flight $\Delta\tau$ are obtained:

$$\Delta u_f = -\dot{\rho}_f = -T_2 \Delta\theta = -\frac{2\Delta\theta(\cos \Delta\tau - 1)}{8(\cos \Delta\tau - 1) + 3\Delta\tau \sin \Delta\tau},$$

$$\Delta v_f = -\dot{\theta}_f = T_4 \Delta\theta = \frac{\Delta\theta \sin \Delta\tau}{8(\cos \Delta\tau - 1) + 3\Delta\tau \sin \Delta\tau},$$

$$\Delta u_0 = \dot{\rho}_0 = -T_2 \Delta\theta = -\frac{2\Delta\theta(\cos \Delta\tau - 1)}{8(\cos \Delta\tau - 1) + 3\Delta\tau \sin \Delta\tau},$$

$$\Delta v_0 = \dot{\theta}_0 = -T_4 \Delta\theta = -\frac{\Delta\theta \sin \Delta\tau}{8(\cos \Delta\tau - 1) + 3\Delta\tau \sin \Delta\tau}.$$

According to these formulas, for each pair of values $(\Delta\theta, \Delta\tau)$ there is a single pair of impulses that allows performing the two-impulse transfer successfully, though the opposite is not true. Moreover, the linearized solution may become singular for those $\Delta\tau$ which make the denominator zero. There are two of such roots per revolution: the first one is $\Delta\tau = 2n\pi$ with $n \in \mathbb{N}$, corresponding to the transversal impulse case, while the other leads to an infinite discrete set of unfeasible solutions in $\Delta\tau$.

The total impulse can be expressed as a function of the phasing and the time of flight in the form:

$$\Delta V(\Delta\theta, \Delta\tau) = \Delta V_0 + \Delta V_f = 2\Delta\theta\sqrt{T_2^2(\Delta\tau) + T_4^2(\Delta\tau)}. \quad (3.40)$$

It is straightforward to obtain the minimum ΔV solution for a given $\Delta\theta$ by deriving with respect to $\Delta\tau$ and equating to zero:

$$\frac{d\Delta V^2}{d\Delta\tau} = 0 \Rightarrow -5\cos\frac{\Delta\tau}{2} + 9\cos\frac{3\Delta\tau}{2} + 24\Delta\tau\sin\frac{\Delta\tau}{2} = 0. \quad (3.41)$$

This transcendental equation has infinite roots, one for each 2π period, and they always lie close to $2n\pi$ with $n \in \mathbb{N}$.

The evolution of the state during the maneuver is obtained by introducing the initial conditions into Eqs. (2.10):

$$\begin{aligned} \dot{\rho} &= \Delta u_0 \cos \tau + 2\Delta v_0 \sin \tau, \\ \dot{\theta} &= -2\Delta u_0 \sin \tau + \Delta v_0 (4 \cos \tau - 3), \\ \rho &= \Delta u_0 \sin \tau - 2\Delta v_0 (\cos \tau - 1), \\ \theta &= \theta_0 + 2\Delta u_0 (\cos \tau - 1) + \Delta v_0 (4 \sin \tau - 3\tau). \end{aligned}$$

From now, only two strategies will be considered in further detail: using radial impulses or transversal impulses. In the first case, making $\Delta u_0 = \Delta V_0$ and $\Delta v_0 = 0$:

$$\begin{aligned} \dot{\rho} &= \Delta V_0 \cos \tau, \\ \dot{\theta} &= -2\Delta V_0 \sin \tau, \\ \rho &= \Delta V_0 \sin \tau, \\ \theta &= \theta_0 + 2\Delta V_0 (\cos \tau - 1), \end{aligned}$$

and the time of flight is obtained from imposing $\rho_f = 0$ and $\Delta\theta \neq 0$:

$$\left. \begin{array}{l} \rho_f = 0 \\ \Delta\theta \neq 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \sin \Delta\tau = 0 \\ \cos \Delta\tau \neq 1 \end{array} \right\} \Rightarrow \Delta\tau = \pi + 2n\pi \quad n \in \mathbb{N},$$

that is, the maneuver will take place in half an orbit, plus an arbitrary number of revolutions. The state at this final time is:

$$\dot{\rho}(\Delta\tau) = -\Delta V_0, \quad \dot{\theta}(\Delta\tau) = 0, \quad \rho(\Delta\tau) = 0, \quad \theta_f = \theta_0 - 4\Delta V_0.$$

The results show that the required impulse is directly determined by the transversal displacement to be covered, and a final impulse of magnitude ΔV_0 and the same direction as the first impulse is needed to circularize the orbit:

$$\Delta V_0 = -\frac{\Delta\theta}{4}, \quad \Delta V = 2|\Delta V_0|, \quad \rho_{\max} = \Delta V_0. \quad (3.42)$$

This impulsive ΔV can be compared with the analytical approximations obtained for the continuous-thrust maneuver. In the short maneuver regime, Eq. (3.29) shows that the continuous-thrust ΔV is greater than the impulsive one and both total impulses become comparable in the limit case $\chi \sim 1$ (i.e. $\varepsilon \sim |\Delta\theta|$). For the long maneuver regime, Eq. (3.36), the continuous-thrust maneuver presents a lower ΔV requirement at the cost of a higher time of flight, and once again the cost for both maneuvers becomes comparable as the continuous-thrust one approaches the transition zone at $\chi \sim 1$. In both limit cases the ratio between continuous and impulsive ΔV evolves as the inverse of the time of flight, $\Delta V_{\text{cont}}/\Delta V_{\text{imp}} \propto 1/\Delta\tau_{\text{imp}}$.

Performing the impulse in the transversal direction, $\Delta u_0 = 0$ and $\Delta v_0 = \Delta V_0$, introduces a secular behavior in θ :

$$\begin{aligned}\dot{\rho} &= 2\Delta V_0 \sin \tau, \\ \dot{\theta} &= \Delta V_0 (4 \cos \tau - 3), \\ \rho &= -2\Delta V_0 (\cos \tau - 1), \\ \theta &= \theta_0 + \Delta V_0 (4 \sin \tau - 3\tau).\end{aligned}$$

The time of flight is once again determined by the final condition in the radial position, which being periodic leads to an infinite set of possible solutions:

$$\rho(\Delta\tau) = 0 \quad \Rightarrow \quad \Delta\tau = 2n\pi \quad n \in \mathbb{N}, \quad (3.43)$$

so the maneuver now takes multiple orbits to complete. The rest of the final state variables are:

$$\dot{\rho}(\Delta\tau) = 0, \quad \dot{\theta}(\Delta\tau) = \Delta V_0, \quad \rho(\Delta\tau) = 0, \quad \theta_f = \theta_0 - 3\Delta V_0 \Delta\tau,$$

leading to the expression for the required impulse and the maximum radial displacement:

$$\Delta V_0 = -\frac{\Delta\theta}{3\Delta\tau}, \quad \Delta V = 2|\Delta V_0|, \quad \rho_{\text{max}} = 4\Delta V_0. \quad (3.44)$$

This solution shows some key differences with the radial one. First of all, the distance covered is now time-dependent, which allows to reduce fuel consumption by performing the rendezvous in multiple revolutions. Furthermore, transversal impulses are close to the local minimum of ΔV for a given number of revolutions. On the other hand, this introduces some operational risks, since the spacecraft could be lost if the second impulse is not performed at the proper time. Also noteworthy is that the final instant does not coincide with the maximum approach between chief and target, introducing further operational risks. Finally, regarding the direction of the impulses, the second one must be performed in the direction opposite to the first one, requiring the spacecraft to change attitude or to have

Figure 3.7: In-plane trajectories in non-dimensional variables for several two-impulses and continuous-thrust maneuvers (linearized formulation).

an opposing set of thrusters. Comparing with the continuous-thrust solutions for the short maneuver and long maneuver regimes, Eq. (3.29) and Eq. (3.36) respectively, it is straightforward to check that the ratios of their corresponding total impulses are constant, taking the approximate values $\Delta V_{\text{cont}}/\Delta V_{\text{imp}} \approx 6$ for the short maneuver regime and $\Delta V_{\text{cont}}/\Delta V_{\text{imp}} \approx 2$ for the long maneuver regime. This shows that the two-transversal-impulses and continuous-thrust maneuvers have a comparable ΔV cost, with the former always outperforming the latter. Factoring in for the much higher specific impulse of typical low-continuous-thrust engines (such as ionic or Hall effect) compared to impulsive thrusters, an important propellant mass reduction can be achieved using the continuous-thrust maneuver in many practical scenarios.

Figure 3.7 shows the in-plane trajectories for several two-impulses and continuous-thrust maneuvers, obtained using the linearized formulation of dynamics. This implies that the shape of each maneuver in non-dimensional coordinates does not depend on $\Delta\theta$, only on χ for the continuous thrust ones. Regarding the impulsive cases, it is checked that the maximum radial displacement for both transversal-impulse solutions is smaller than for the radial-impulse one, and decreases with the number of revolutions. On the other hand, the maximum radial displacement for the continuous-thrust maneuvers remains small compared to the impulsive cases for values of χ in the short maneuver and long maneuver regimes (1.0046×10^{-1} and 3.0406×10^2 respectively), while it becomes comparable and even slightly larger

Figure 3.8: Comparison of the minimum total impulse required for a displacement of 0.0001 radians using impulsive and continuous thrust.

in the transition zone ($\chi = 5.1931$). It is also noteworthy that the trajectories for the transition and long maneuver regimes, as well as for the transversal-impulse cases, have a minimum angular distance with respect of the target slightly smaller than the final separation, and that they start moving in the opposite direction at the beginning of the maneuver. The minimum total impulse required to perform a rephasing of $\Delta\theta = 0.0001$ rad using either continuous-thrust or two-impulse maneuvers is compared in Figure 3.8. To construct the impulsive minimum ΔV curve it is important to note that there are cases where the best solution is to delay the maneuver until the previous local optimum, yielding the horizontal sections depicted in the figure. As already predicted based on the the continuous-thrust and transversal-impulse (which is known to be close to the local optimum) ΔV solutions, the impulsive transfer is less costly but both are within the same order of magnitude.

3.6 Conclusion

The minimum time constant-thrust rephasing problem in circular orbit has been analyzed in great detail with the aid of a curvilinear relative motion formulation and using the indirect method. Results show that the structure of the solution undergoes a fundamental qualitative change as the thrust parameter varies, be-

ing possible to distinguish between two different regimes, a short maneuver and a long maneuver regime, separated by a transition zone. Several qualitative features of these regimes have been identified, and three particular solutions have been reported to help illustrate them. Far from the transition zone, in the short maneuver regime, the thrust orientation control law follows a nearly bang-bang structure with the thrust initially oriented along the direction of the desired displacement and reversing itself symmetrically at half the maneuver interval to oppose the spacecraft motion until the end of the maneuver. Conversely, in the long maneuver regime the optimal thrust orientation follows a nearly bang-bang structure that initially opposes the desired motion to exploit a change in semi-major axis towards obtaining the required rephasing and switches direction at half way to accelerate in the direction of motion. Quite accurate analytical formulas describing the state and costate in these two regimes were obtained based on the linearized relative motion equations, which provide a solution structure depending on one parameter only, namely the ratio between the rephasing angle and the dimensionless thrust acceleration. nonlinear effects, which can be safely neglected in most cases of practical interest, would make the solution dependent, generally, on two parameters.

The availability of approximate analytical solutions with a rather accurate estimation of the maneuver time can be quite useful in order to generate initial guesses for obtaining the numerical solution using both the direct or indirect method. This is also true for the less common case of strongly nonlinear problems whose solution can be conveniently obtained starting from its linear counterpart. Dealing with orbital dynamics problem characterized by additional perturbations may be addressed in a similar way using the unperturbed case as a starting point.

Chapter 4

Optimal Radius Change Maneuver in Circular Orbit

4.1 Introduction

Aiming to extend the results obtained for the minimum-time rephasing problem in the previous chapter, the determination of the optimum control law to change the radius of a circular orbit in minimum time is now studied. This is a fundamental problem in astrodynamics and has been dealt with extensively in the past. One of the main contributions is the 1961 milestone paper of Theodore N. Edelbaum [30], which contains an approximate analytical solution for the optimum low thrust transfer between circular orbits including an inclination change maneuver. Edelbaum's solution applies very well to optimization cases where the transfer maneuver takes more than one revolution to be completed. The opposite condition, when the transfer takes less than a full revolution around the primary, has been investigated in other works (see, in particular, [19, 100, 91]). Among the previous references, the work by R. Broucke is especially significant as he provides an analytical solution (as a function of three parameters to be numerically determined) of the costate equations of the linearized problem in polar coordinates, finally leading to a generalization of the classical linear-tangent steering law. Other aspects of the problem have been investigated by different authors. Alfano and Thorne performed in [4] an extensive numerical optimization campaign to study the relation between the magnitude of the available thrust acceleration and the achievable velocity change. Casalino investigated in [23] the limitations of Edelbaum's solution and proposed an approximate analytical method to extend Edelbaum's solution to the short-maneuver regime.

In this chapter, the minimum-time transfer between two coplanar, circular orbits is studied in depth using the relative motion formulation introduced in

Chapter 2 and exploiting linearization as a key element to construct approximate analytical solutions for the transfer time and thrust profile. Following the same approach employed in Chapter 3 (see also [42]), a nonlinear formulation in curvilinear coordinates is used to describe relative dynamics [17], and the optimal control problem in the thrust orientation is posed using the indirect method. While the focus is set to low-thrust transfers, a wide range for the thrust parameter is considered. The formulation used for the relative motion differs from the classical solution by Clohessy and Wiltshire [27] in two aspects. On the one hand, it introduces curvilinear coordinates to achieve a better description of the natural orbit curvature. On the other hand, it takes into account nonlinear terms to improve accuracy when initial velocity and displacement conditions are not very small.

The chapter is organized as follows. First, the problem is described in detail by providing all the relevant assumptions, characteristic magnitudes, coordinate systems and dynamical models. Particular attention is given to the equations of motion and the boundary conditions, both for the exact (nonlinear) and linearized models, as well as to the choice of a reference orbit for the relative motion. Next, the minimum-time transfer problem is posed by applying Pontryagin's Maximum Principle to derive the first order optimality conditions, which take the form of a two-point boundary value problem (TPBVP). A qualitative understanding of the problem is then sought for by linearizing the TPBVP and comparing the orders of magnitude of the different terms in the equations. The latter is a common technique in the field of fluid mechanics, but it is rarely applied to orbital dynamics. This analysis predicts the existence of two different regimes with fundamental qualitative differences depending on the ratio between the desired displacement and the available thrust χ . Approximate expressions relating the time of flight with the displacement-thrust ratio are also obtained, showing that the key difference between both regimes is whether the transfer takes less or more than one revolution of the reference orbit to complete (short- and long-maneuver regimes, respectively). Further developments allow one to find approximate analytical solutions for the linearized TPBVP fulfilling the boundary conditions, both for the short- and long-maneuver regimes. Finally, an extensive set of Earth-bound and interplanetary test cases is solved numerically using a multiple-shooting algorithm, both for the linearized and exact (nonlinear) models. The results confirm the qualitative findings of the analytical study, and allow one to evaluate the influence of the nonlinear effects. Several practical application examples are also provided to ease the physical interpretation of the results.

Figure 4.1: Schematic representation of the problem.

4.2 Problem Statement and Equations of Motion

This work studies the minimum-time transfer of a spacecraft subjected to constant thrust acceleration between two close, coplanar, circular orbits. The propellant mass expelled from the spacecraft m_{prop} is assumed to be negligible compared to its total mass m , so the latter can be taken as constant. As a consequence, this Optimal Control Problem (OCP) would be equivalent to minimizing the total impulse required for the transfer.

The nonlinear relative motion formulation in curvilinear coordinates introduced in Chapter 2 will be used for the dynamics. To this end, let us consider a chief C (either a real or ‘virtual’ object) describing a circular Keplerian orbit of radius R around a primary O with gravitational constant μ . From now on all equations and variables will be expressed in non-dimensional form taking the chief orbit radius R and inverse mean motion $\Omega^{-1} = \sqrt{R^3/\mu}$ as units of length and time, respectively. The relative motion of a follower F (also real or virtual) with respect to the chief can then be expressed through two curvilinear coordinates ρ and θ associated to a cylindrical frame $\mathcal{F} = \{O; \mathbf{u}_\rho, \mathbf{u}_\theta, \mathbf{k}'\}$, as shown in Figure 4.1. The evolution of these curvilinear coordinates in the non-dimensional time τ is given by Eqs. (2.3) particularized for the planar case ($z = 0$):

$$\begin{aligned} \ddot{\rho} - 2\dot{\theta} - 3\rho &= a_{i\rho} + a_{g\rho} + a_\rho^T, \\ \ddot{\theta} + 2\dot{\rho} &= a_{i\theta} + \frac{a_\theta^T}{1 + \rho}. \end{aligned} \tag{4.1}$$

Terms $a_{i\rho}$, $a_{g\rho}$, and $a_{i\theta}$ account for the nonlinear perturbing accelerations, whereas

a_ρ^T, a_θ^T are the components in $\mathcal{F}$ of the thrust acceleration:

$$\begin{bmatrix} a_\rho^T & a_\theta^T \end{bmatrix} = \varepsilon \begin{bmatrix} u_\rho & u_\theta \end{bmatrix} = \varepsilon \mathbf{u},$$

where ε is the non-dimensional thrust acceleration, and $\mathbf{u}$ is the unitary thrust orientation vector. Given that ε is a constant parameter, $\mathbf{u}$ also represents the control vector for the problem.

It is convenient to express Eqs. (4.1) as a first order system:

$$\frac{d\mathbf{S}}{d\tau} = \mathbf{F}(\mathbf{S}, \mathbf{u}, \varepsilon; \tau), \quad (4.2)$$

where $\mathbf{S}$ is a state vector defined as follows:

$$\mathbf{S} = \begin{bmatrix} \dot{\rho} & \dot{\theta} & \rho & \theta \end{bmatrix}^\top, \quad (4.3)$$

and $\mathbf{F}$ is the right-hand side of the equations:

$$\mathbf{F}(\mathbf{S}, \mathbf{u}, \varepsilon; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_{i\rho} + a_{g\rho} + a_\rho^T \\ -2\dot{\rho} + a_{i\theta} + a_\theta^T / (1 + \rho) \\ \dot{\rho} \\ \dot{\theta} \end{bmatrix}. \quad (4.4)$$

By neglecting the nonlinear perturbing terms and assuming $\rho \ll 1$ a linear formulation with the same structure as the well-known Clohessy-Wiltshire equations is reached. The right-hand side of Eq. (4.2) for the linear formulation, denoted as $\mathbf{F}^*$ for clarity, can be written as follows:

$$\mathbf{F}^*(\mathbf{S}, \mathbf{u}, \varepsilon; \tau) \approx \begin{bmatrix} 2\dot{\theta} + 3\rho + a_\rho^T \\ -2\dot{\rho} + a_\theta^T \\ \dot{\rho} \\ \dot{\theta} \end{bmatrix}. \quad (4.5)$$

The initial and final states for the maneuver depend on the choice for departure, arrival and reference (chief) orbit. In all cases, the circular to circular orbit condition implies that the radial velocity will be zero, and the derivative of θ will be the difference between the mean motions of the current and reference orbits. In mathematical form:

$$\begin{aligned} \mathbf{S}(\tau_0) &= \begin{bmatrix} \dot{\rho}_0 & \dot{\theta}_0 & \rho_0 & \theta_0 \end{bmatrix}^\top = \begin{bmatrix} 0 & \sqrt{1/(1+\rho_0)^3} - 1 & \rho_0 & \theta_0 \end{bmatrix}^\top, \\ \mathbf{S}(\tau_f) &= \begin{bmatrix} \dot{\rho}_f & \dot{\theta}_f & \rho_f & \theta_f \end{bmatrix}^\top = \begin{bmatrix} 0 & \sqrt{1/(1+\rho_f)^3} - 1 & \rho_f & \theta_f \end{bmatrix}^\top, \end{aligned} \quad (4.6)$$

where subscripts 0 and f denote initial and final orbit, respectively. The desired radial displacement is $\Delta\rho = \rho_f - \rho_0$, the angular displacement is $\Delta\theta = \theta_f - \theta_0$,

and the time of flight is $\Delta\tau = \tau_f - \tau_0$. When the linearized equations of motion are used the boundary conditions should also be expressed in linear form, leading to:

$$\begin{aligned} \mathbf{S}^*(\tau_0) &= \begin{bmatrix} 0 & -3/2\rho_0 & \rho_0 & \theta_0 \end{bmatrix}^\top, \\ \mathbf{S}^*(\tau_f) &= \begin{bmatrix} 0 & -3/2\rho_f & \rho_f & \theta_f \end{bmatrix}^\top. \end{aligned} \quad (4.7)$$

From now on $\Delta\theta$ will be set free, implying that the phasing problem will be solved by simply selecting the phasing at which the maneuver is started. This is a fuel-efficient solution for LEO orbits, separating it from the radius modification problem. Moreover, it is possible to further simplify the problem by setting $\theta_0 = 0$ and $\tau_0 = 0$, leading to $\theta_f = \Delta\theta$ and $\tau_f = \Delta\tau$. Regarding the choice of reference orbit, two particular cases will be considered henceforth: reference orbit coincident with either the initial or final orbit. The former corresponds to $\rho_0 = 0$, $\rho_f = \Delta\rho$, and the latter to $\rho_0 = -\Delta\rho$, $\rho_f = 0$.

4.3 Maneuver Optimization

The cost function for the minimum-time constant-thrust transfer between two close, coplanar, circular orbits can be expressed in the form (see [21]):

$$J = \int_0^{\Delta\tau} L(\mathbf{S}, \mathbf{u}; \tau) d\tau = \Delta\tau,$$

where $L = 1$ is the Lagrangian. Since the cost function does not depend on the final state, this is a particular case of a Problem of Lagrange. From this objective function a Hamiltonian is defined as:

$$H(\mathbf{S}, \boldsymbol{\lambda}, \mathbf{u}; \tau) = \boldsymbol{\lambda}^\top \mathbf{F} + L, \quad (4.8)$$

where:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_u & \lambda_v & \lambda_\rho & \lambda_\theta \end{bmatrix}^\top, \quad (4.9)$$

is the costate associated to the state $\mathbf{S}$. The equations of motion can then be written in the following form:

$$\dot{\mathbf{S}} = \frac{\partial H}{\partial \boldsymbol{\lambda}}, \quad (4.10)$$

and the evolution of the costate is given by the adjoint (or costate) equations derived from the first order optimality conditions (see [21]):

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{S}}. \quad (4.11)$$

An expression for $\mathbf{u}(\mathbf{S}, \boldsymbol{\lambda}; \tau)$ is now sought for using the Pontryagin Maximum (or minimum) Principle [76], which states that the optimal control for a given optimal trajectory is the one that leads to an extreme value of H over the set of admissible controls. Mathematically (for the minimization case):

$$\mathbf{u}^-(\tau) = \arg \min_{\mathbf{u} \in \mathbf{U}} H(\mathbf{S}^-(\tau), \boldsymbol{\lambda}^-(\tau), \mathbf{u}; \tau) \quad \forall \tau \in [0, \Delta\tau],$$

where the superscript ‘ $-$ ’ denotes the optimal trajectory and $\mathbf{U}$ is the set of valid controls, which in this case corresponds to:

$$\mathbf{U} = \{\mathbf{u} \in \mathbb{R}^2 / \|\mathbf{u}\| = 1\},$$

that is, $\mathbf{u}$ must be a unitary vector. Gathering the terms in H involving the control vector one reaches:

$$\varepsilon u_\rho \lambda_u + \varepsilon \frac{u_\theta}{1+\rho} \lambda_v = \varepsilon \left[\lambda_u \quad \frac{\lambda_v}{1+\rho} \right] \cdot \mathbf{u}.$$

It follows that, for this contribution to minimize H for all τ , the vectors in the previous expression must have opposite directions. Together with the unitary requirement for $\mathbf{u}$ this leads to:

$$u_\rho = -\frac{\lambda_u}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}, \quad u_\theta = -\frac{\frac{\lambda_v}{1+\rho}}{\sqrt{\lambda_u^2 + \left(\frac{\lambda_v}{1+\rho}\right)^2}}. \quad (4.12)$$

Interestingly, for the linear case this expression reduces to $\mathbf{u} = [\lambda_u \lambda_v]$, which has the same expression as the primer vector in Cartesian coordinates [62, 78].

Substituting Eqs. (4.12) into Eqs. (4.11) and operating one reaches:

$$\begin{aligned} \dot{\lambda}_u &= -\frac{\partial H}{\partial \dot{\rho}} = 2\lambda_v \frac{1+\dot{\theta}}{1+\rho} - \lambda_\rho, \\ \dot{\lambda}_v &= -\frac{\partial H}{\partial \dot{\theta}} = -2\lambda_u (1+\rho) (1+\dot{\theta}) + \lambda_v \frac{2\dot{\rho}}{1+\rho} - \lambda_\theta, \\ \dot{\lambda}_\rho &= -\frac{\partial H}{\partial \rho} = -\lambda_u \left[(1+\dot{\theta})^2 + \frac{2}{(1+\rho)^3} \right] - \lambda_v \frac{2\dot{\rho} (1+\dot{\theta}) + \varepsilon (-2u_\theta + u_\theta^3)}{(1+\rho)^2}, \\ \dot{\lambda}_\theta &= -\frac{\partial H}{\partial \theta} = 0. \end{aligned} \quad (4.13)$$

The last equation indicates that the costate associated to the angular coordinate is constant. Moreover, given that at least one of the boundary values (BVs) for θ is free the corresponding costate will be zero.

One last optimality condition is needed to account for the free final time. Imposing J to be stationary with respect to variations in $\Delta\tau$ leads to the following transversality condition [21]:

$$H(\Delta\tau) = 0. \quad (4.14)$$

Substituting the known values of the final state and recalling that λ_θ is zero for all τ , Eq. (4.14) takes a simpler form:

$$\lambda_u(\Delta\tau)^2 + \left[\frac{\lambda_v(\Delta\tau)}{1 + \rho_f} \right]^2 = \frac{1}{\varepsilon^2}. \quad (4.15)$$

Equations (4.2), (4.13) and (4.15) form a differential-algebraic system of eight Ordinary Differential Equations (state and costate equations) and one algebraic equation (transversality condition). The state boundary conditions are given by Eqs. (4.6), where the values of ρ_0 and ρ_f depend on the choice for the reference orbit. All the state BVs are fixed except for the total angular displacement, implying that the corresponding costate BVs will be unknown parameters except for the final λ_θ which will be zero:

$$\begin{aligned} \boldsymbol{\lambda}(0) &= \begin{bmatrix} \lambda_{u0} & \lambda_{v0} & \lambda_{\rho0} & \lambda_{\theta0} \end{bmatrix}^\top, \\ \boldsymbol{\lambda}(\Delta\tau) &= \begin{bmatrix} \lambda_{uf} & \lambda_{vf} & \lambda_{\rho f} & 0 \end{bmatrix}^\top. \end{aligned} \quad (4.16)$$

Gathering these results, the OCP now takes the form of a Two-Point Boundary Value Problem (TPBVP) with 8 differential equations describing the time evolution of state and costate, 9 terminal constraints (7 fixed state BVs, 1 fixed costate BV, and the transversality condition), and 9 unknowns (7 unknown costate BVs, the angular displacement and the time of flight). However, it was already noted that according to the last of Eqs. (4.13) λ_θ is constant, so $\lambda_{\theta0} = \lambda_\theta(\Delta\tau) = 0$ and the TPBVP dimension can be reduced to 8. Moreover, none of the equations depends explicitly on θ , so it is possible to take $\Delta\theta$ and the corresponding equation out of the TPBVP further reducing its dimension to 7.

The full TPBVP cannot be solved analytically for a general case. For the rest of this section the problem will be studied from a qualitative point of view, with the aim of extracting knowledge about the structure of the solution and the evolution of the state. This information can then be used to generate initial guesses for the iterative numerical method presented in the next section. To ease the treatment of the equations, the linearized formulation of the relative motion in curvilinear coordinates, Eqs. (4.5), is used, yielding the Hamiltonian:

$$H^* = \boldsymbol{\lambda}^\top \mathbf{F}^* = \lambda_u \left(2\dot{\theta} + 3\rho + \varepsilon u_\rho \right) + \lambda_v (-2\dot{\rho} + \varepsilon u_\theta) + \lambda_\rho \dot{\rho} + \lambda_\theta \dot{\theta} + 1.$$

This leads to the following set of adjoint equations:

$$\begin{aligned}
\dot{\lambda}_{\dot{\rho}} &= -\frac{\partial H^*}{\partial \dot{\rho}} = -\lambda_{\rho} + 2\lambda_v, \\
\dot{\lambda}_{\dot{\theta}} &= -\frac{\partial H^*}{\partial \dot{\theta}} = -2\lambda_u - \lambda_{\theta}, \\
\dot{\lambda}_{\rho} &= -\frac{\partial H^*}{\partial \rho} = -3\lambda_u, \\
\dot{\lambda}_{\theta} &= -\frac{\partial H^*}{\partial \theta} = 0.
\end{aligned} \tag{4.17}$$

Note that, by neglecting the nonlinear terms in the dynamics, the adjoint equations have been decoupled from the equations of motion. Their solution along with the boundary conditions in Eqs. (4.16) is now straightforward:

$$\begin{aligned}
\lambda_u &= A \sin(\tau + \varphi), \\
\lambda_v &= 2A \cos(\tau + \varphi) + C, \\
\lambda_{\rho} &= 3A \cos(\tau + \varphi) + 2C, \\
\lambda_{\theta} &= 0,
\end{aligned}$$

where the three parameters A , C , and φ are related to the initial value of the costate as follows:

$$\lambda_{u0} = A \sin \varphi, \quad \lambda_{v0} = 2A \cos \varphi + C, \quad \lambda_{\rho0} = 3A \cos \varphi + 2C,$$

$$A = \sqrt{\lambda_{u0}^2 + (2\lambda_{v0} - \lambda_{\rho0})^2}, \quad C = 2\lambda_{\rho0} - 3\lambda_{v0}, \quad \varphi = \text{atan2}(\lambda_{u0}, 2\lambda_{v0} - \lambda_{\rho0}).$$

A further simplification can be achieved by exploiting the symmetry properties of the linearized problem (see Pontani [75]) to express φ as a function of the time of flight, thus removing one unknown from the TPBVP. First of all, note that λ_u and λ_v have opposite behaviors regarding symmetry, one of them will be odd and the other even. However, λ_v cannot be odd for $C \neq 0$, leading to λ_u odd and λ_v even. The symmetry conditions then yield:

$$\sin(\Delta\tau/2 + \varphi) = 0, \quad \cos(\Delta\tau/2 + \varphi) = \pm 1,$$

which amounts to:

$$\varphi = -\Delta\tau/2 + n\pi, \quad n \in \mathbb{N}^0,$$

where the value of n determines whether the cosine has a maximum (n even) or a minimum (n odd) at $\tau = \Delta\tau/2$. Given that the value of n only changes the sign of the trigonometric functions and they are always multiplied by A , it is possible to

arbitrarily set $n = 0$ and let the sign be determined by A . The costate equations for the linearized problem then reduce to:

$$\begin{aligned}\lambda_u &= A \sin(\tau - \Delta\tau/2) , \\ \lambda_v &= 2A \cos(\tau - \Delta\tau/2) + C , \\ \lambda_\rho &= 3A \cos(\tau - \Delta\tau/2) + 2C , \\ \lambda_\theta &= 0 .\end{aligned}\tag{4.18}$$

There is no closed form solution for the linearized equations of motion perturbed by a thrust acceleration, so it is not possible to find a fully analytical solution for the OCP even in the linearized case. Nevertheless, additional information on the solution can be obtained by analyzing the orders of magnitude of the different terms in the equations and locating the dominant ones. To this end, a new non-dimensionalization is introduced taking $\Delta r = |\Delta\rho|$ and $\Delta\tau$ as characteristic displacement and time respectively:

$$\frac{d^2\hat{\rho}}{d\hat{\tau}^2} = 2\Delta\tau \frac{d\hat{\theta}}{d\hat{\tau}} + 3\Delta\tau^2\hat{\rho} + Ku_\rho ,\tag{4.19}$$

$$\frac{d^2\hat{\theta}}{d\hat{\tau}^2} = -2\Delta\tau \frac{d\hat{\rho}}{d\hat{\tau}} + Ku_\theta ,\tag{4.20}$$

with:

$$\hat{\tau} = \frac{\tau}{|\Delta\tau|} , \quad \hat{\rho} = \frac{\rho}{\Delta r} , \quad \hat{\theta} = \frac{\theta}{\Delta r} , \quad K = \frac{\Delta\tau^2}{\chi} ,$$

where:

$$\chi = \frac{\Delta r}{\varepsilon} ,$$

is the ratio between the desired displacement and the available thrust acceleration. The boundary conditions now take the form:

$$\begin{aligned}\hat{\mathbf{S}}(0) &= \begin{bmatrix} 0 & \hat{\theta}'_0 & \hat{\rho}_0 & \hat{\theta}_0 \end{bmatrix}^\top = \begin{bmatrix} 0 & -3/2\Delta\tau\hat{\rho}_0 & \hat{\rho}_0 & 0 \end{bmatrix}^\top , \\ \hat{\mathbf{S}}(1) &= \begin{bmatrix} 0 & \hat{\theta}'_f & \hat{\rho}_f & \hat{\theta}_f \end{bmatrix}^\top = \begin{bmatrix} 0 & -3/2\Delta\tau\hat{\rho}_f & \hat{\rho}_f & \Delta\theta/\Delta r \end{bmatrix}^\top .\end{aligned}\tag{4.21}$$

Introducing the symbols:

$$\pm_\Delta = \text{sign}(\Delta\rho) , \quad \mp_\Delta = -\text{sign}(\Delta\rho) ,$$

the radial boundary values for the case where the initial orbit is taken as reference are:

$$\hat{\rho}_0 = 0 , \quad \hat{\rho}_f = \pm_\Delta 1 ,$$

whereas if the final orbit is used as reference they can be written as:

$$\hat{\rho}_0 = \mp_{\Delta} 1, \quad \hat{\rho}_f = 0,$$

being in all cases $\Delta\hat{\rho} = \hat{\rho}_f - \hat{\rho}_0 = \pm_{\Delta} 1$.

The orders of magnitude for each term in Eqs. (4.19,4.21) are:

$$\begin{aligned} \frac{d^2\hat{\rho}}{d\hat{\tau}^2} \sim 1, \quad \Delta\tau \frac{d\hat{\theta}}{d\hat{\tau}} \sim \Delta\tau \frac{\Delta\theta}{\Delta r}, \quad \Delta\tau^2 \hat{\rho} \sim \Delta\tau^2, \quad Ku_{\rho} \sim \frac{\Delta\tau^2}{\chi} u_{\rho}, \\ \frac{d^2\hat{\theta}}{d\hat{\tau}^2} \sim \frac{\Delta\theta}{\Delta r}, \quad \Delta\tau \frac{d\hat{\rho}}{d\hat{\tau}} \sim \Delta\tau, \quad Ku_{\theta} \sim \frac{\Delta\tau^2}{\chi} u_{\theta}, \end{aligned}$$

where:

$$u_{\rho}^2 + u_{\theta}^2 = 1, \quad |u_{\rho}|, |u_{\theta}| \leq 1.$$

According to Eq. (4.20), the angular displacement can be driven both by gravitational effects and the action of thrust. Making the hypothesis that the latter is at least as important as the former, by comparing the transversal acceleration with the thrust component one reaches:

$$\frac{\Delta\theta}{\Delta r} \sim \frac{\Delta\tau^2}{\chi} u_{\theta}.$$

Introducing this into the orders of magnitude for Eq. (4.19) yields:

$$\frac{d^2\hat{\rho}}{d\hat{\tau}^2} \sim 1, \quad \Delta\tau \frac{d\hat{\theta}}{d\hat{\tau}} \sim \frac{\Delta\tau^3}{\chi} u_{\theta}, \quad \Delta\tau^2 \hat{\rho} \sim \Delta\tau^2, \quad Ku_{\rho} \sim \frac{\Delta\tau^2}{\chi} u_{\rho}.$$

It is now possible to distinguish the limit cases for $\Delta\tau \gg 1$, long maneuver regime, and $\Delta\tau \ll 1$, short maneuver regime. For the long maneuver regime ($\Delta\tau \gg 1$) the second and third terms are dominant, leading to:

$$\Delta\tau \sim \frac{\chi}{|u_{\theta}|},$$

which not only gives a first estimate of the required maneuver time, but also shows that the optimum thrust orientation profile will remain as close as possible to the transversal direction (the need to fulfill all the boundary conditions will of course introduce deviations). For the short maneuver regime ($\Delta\tau \ll 1$) the first and fourth terms become dominant (note that at least one control term must be retained), giving the following estimate for the maneuver time:

$$\Delta\tau \sim \sqrt{\frac{\chi}{|u_{\rho}|}}.$$

The preferred orientation of thrust for minimum time has now changed with respect to the long maneuver regime, from the transversal to the radial direction.

These results have been obtained under the hypothesis that transversal thrust was at least as important as gravitational effects in their respective equation of motion. Therefore, before accepting them as valid we have to check whether this hypothesis holds. The following developments not only will prove this true, but also show that the Coriolis term is non-negligible too. In the long maneuver regime, $\Delta\tau \gg 1$, introducing the estimation for $\Delta\tau$ into the orders of magnitude for Eq. (4.20) yields (note that ‘ $\prime$ ’ denotes derivative with respect to $\hat{\tau}$):

$$\hat{\theta}'' \sim \frac{\Delta\theta}{\Delta r}, \quad \Delta\tau \hat{\rho}' \sim \Delta\tau, \quad Ku_\theta \sim \Delta\tau,$$

showing that both gravitational and thrust effects are of the same order, and giving an estimate for $\Delta\theta$:

$$\Delta\theta \sim \Delta\tau \Delta r.$$

Interestingly, Eq. (4.19) previously showed that the desired displacement in $\hat{\rho}$ was reached through the coupling between both equations due to the Coriolis term, so this regime is driven by gravitational effects. In the short maneuver regime, $\Delta\tau \ll 1$, the orders of magnitude for Eq. (4.20) read:

$$\hat{\theta}'' \sim \frac{\Delta\theta}{\Delta r}, \quad \Delta\tau \hat{\rho}' \sim \Delta\tau, \quad Ku_\theta \sim \frac{u_\theta}{u_\rho}.$$

There are two opposing criteria regarding the magnitude of u_θ . On the one hand, u_θ should be as small as possible since the preferred thrust orientation for minimum time is radial. On the other hand, transversal control needs to be non-negligible in order to ensure the fulfillment of the boundary conditions in $\hat{\theta}'$. Therefore, the transversal acceleration term will be at most of the same order as the other two. Comparing the acceleration and Coriolis terms leads to the same $\Delta\theta$ estimate already obtained for the long maneuver regime, $\Delta\theta \sim \Delta\tau \Delta r$, while comparing the acceleration and Coriolis terms yields $u_\theta \lesssim \Delta\tau \ll 1$ (that is, the shorter the maneuver time the closer the thrust profile will be to the radial direction). Since the displacement in $\hat{\rho}$ is mainly driven by the direct action of thrust in the radial direction, this regime is dominated by thrust effects.

The transition between both regimes takes place for:

$$\Delta\tau \sim \mathcal{O}(1) \Rightarrow \Delta r \sim \varepsilon.$$

In this transition zone, all terms in both equations become of the same order, and no simplifications can be made. The structure of the solution undergoes fundamental qualitative changes as it moves from one regime to the other, transitioning from

nearly radial thrust orientation for the short maneuver regime to nearly tangential for the long maneuver regime. Since $\Delta\tau = 2\pi$ corresponds to one orbital revolution of the leader, it follows that a key physical difference between both regimes is whether the maneuver requires only a small fraction of an orbital revolution to complete (short maneuver regime) or several (long maneuver regime).

For consistency and convenience, in further developments the linear costate solutions, Eqs. (4.18), will also be expressed in non-dimensional form as follows:

$$\begin{aligned}\hat{\lambda}_u &= \varepsilon\lambda_u = \hat{A} \sin(\Delta\tau\tilde{\tau}) , \\ \hat{\lambda}_v &= \varepsilon\lambda_v = 2\hat{A} \cos(\Delta\tau\tilde{\tau}) + \hat{C} , \\ \hat{\lambda}_\rho &= \varepsilon\lambda_\rho = 3\hat{A} \cos(\Delta\tau\tilde{\tau}) + 2\hat{C} , \\ \hat{\lambda}_\theta &= \varepsilon\lambda_\theta = 0 ,\end{aligned}\tag{4.22}$$

with:

$$\hat{A} = \varepsilon A, \quad \hat{C} = \varepsilon C, \quad \tilde{\tau} = \hat{\tau} - 1/2,$$

where a change in the origin of the time variable has been introduced to better reflect the symmetry properties of the linearized solution.

A more accurate description for the short and long maneuver regimes is obtained in the following subsections by solving the approximate problem resulting from neglecting the non-dominant terms in the linearized equations of motion. This kind of approximate analytical analysis cannot be carried out for the transition zone, where all the terms in the equations become of comparable order. In the absence of an approximate analytical solution, this transition zone will be studied numerically.

4.3.1 Short maneuver regime

The previous results for the short maneuver regime already show that $|u_\rho| \sim 1$, but it is possible to obtain additional information on the control using the costate solutions for the linearized problem. Expanding the first two of Eqs. (4.22) in Taylor series around $\tilde{\tau} = 0$ for $\Delta\tau \ll 1$ yields:

$$\begin{aligned}\hat{\lambda}_u &= \hat{A}\Delta\tau\tilde{\tau} + \mathcal{O}(\Delta\tau^3) , \\ \hat{\lambda}_v &= \hat{C} + 2\hat{A} - \hat{A}\Delta\tau^2\tilde{\tau}^2 + \mathcal{O}(\Delta\tau^4) ,\end{aligned}\tag{4.23}$$

and the $|u_\rho| \gg |u_\theta|$ condition implies that:

$$\left| \hat{C} + 2\hat{A} \right| \ll \left| \hat{A}\Delta\tau \right| .\tag{4.24}$$

Introducing the costate expansions into the transversality condition, Eq. (4.15), and retaining only the leading order terms (taking into account the previous inequality) yields:

$$\hat{A} \approx \pm \frac{2}{\Delta\tau} .$$

Plugging this value into Eq. (4.24), an estimation for $\hat{C}$ is also obtained:

$$\hat{C} \approx -2\hat{A} \approx \mp \frac{4}{\Delta\tau}.$$

Gathering these results, the leading component of the primer vector can be approximated for $\Delta\tau \ll 1$ as:

$$\hat{\lambda}_u \approx \pm 2\tilde{\tau}.$$

This expression together with Eqs. (4.12) show that the radial component of thrust has exactly one change of sign at the middle point of the maneuver. Since the control is known to be preferably aligned in the radial direction, this leads to a nearly bang-bang control.

Neglecting small terms in the linearized equations of motion, Eqs. (4.19,4.20), for the short maneuver regime yields:

$$\hat{\rho}'' \approx Ku_\rho, \quad (4.25)$$

$$\hat{\theta}'' + 2\Delta\tau\hat{\rho}' \approx Ku_\theta. \quad (4.26)$$

Knowing that control is nearly bang-bang, it is straightforward to check from Eq. (4.25) that a positive $\Delta\rho$ requires thrust to be initially oriented along the positive radial direction (an vice versa). Along with Eqs. (4.12), this allows to determine the correct signs for the costate:

$$\hat{\lambda}_u \approx \pm_\Delta 2\tilde{\tau} \approx \pm_\Delta (2\hat{\tau} - 1),$$

$$\hat{A} \approx \pm_\Delta \frac{2}{\Delta\tau}, \quad \hat{C} \approx \mp_\Delta \frac{4}{\Delta\tau}.$$

Assuming control is strictly bang-bang and taking the previous signs into account, Eq. (4.25) can be solved as:

$$\hat{\rho} - \hat{\rho}_0 \approx \pm_\Delta \begin{cases} K \frac{\hat{\tau}^2}{2} & 0 < \hat{\tau} < 1/2 \\ K \left(-\frac{\hat{\tau}^2}{2} + \hat{\tau} - \frac{1}{4} \right) & 1/2 < \hat{\tau} < 1 \end{cases}$$

An estimate for the time of flight can now be obtained by imposing the desired radial displacement $\Delta\hat{\rho} = \pm_\Delta 1$:

$$\Delta\tau \approx 2\sqrt{\chi}. \quad (4.27)$$

This solution for $\hat{\rho}$ fulfills the boundary conditions in position and, due to the control symmetry, also in velocity. Building on this result, it would be possible to obtain an expression for $\hat{\theta}$ in the bang-bang control limit by taking $u_\theta = 0$

in Eq. (4.26) and integrating. However, said $\hat{\theta}$ would separate appreciably from the actual optimal maneuver, and particularly it could not fulfill the boundary conditions in velocity since it would take the same value at both extremes. The reason is that the transversal component of thrust, albeit small, can be of the same order as the other terms in the transversal equation and cannot be neglected.

The previous approximation was based on neglecting $\hat{C} + 2\hat{A}$ in the series expansion for the primer vector, supported by the knowledge that it must be preferably oriented along the radial direction. However, a closer look at Eqs. (4.23) reveals that, since $\tilde{\tau}$ crosses 0, no matter how small that term is there will always be a region near the bang-bang switch where its contribution is dominant. In order to reach a better approximation fulfilling all the boundary conditions, this switching region must be preserved. To this end, a change of parameter is proposed for $\hat{C}$ based on Eq. (4.24):

$$\hat{C} + 2\hat{A} = c\hat{A}\Delta\tau^2. \quad (4.28)$$

The series expansion for $[\lambda_u \ \lambda_v]$ can now be written as:

$$\begin{aligned} \hat{\lambda}_u &= \hat{A}\Delta\tau\tilde{\tau} - \frac{1}{6}\hat{A}\Delta\tau^3\tilde{\tau}^3 + \mathcal{O}(\Delta\tau^5), \\ \hat{\lambda}_v &= \hat{A}\Delta\tau^2(c - \tilde{\tau}^2) + \mathcal{O}(\Delta\tau^4), \end{aligned}$$

and the square of its norm:

$$\hat{\lambda}_u^2 + \hat{\lambda}_v^2 = \hat{A}^2\Delta\tau^2 \left[c^2\Delta\tau^2 + (1 - 2c\Delta\tau^2)\tilde{\tau}^2 + \frac{2}{3}\Delta\tau^2\tilde{\tau}^4 \right] + \mathcal{O}(\Delta\tau^6).$$

Because $\Delta\tau \ll 1$ and $|\tilde{\tau}| \leq 1/2$, the leading term of this norm will be the one in $\tilde{\tau}^2$ save for a small region around 0 where the constant part will dominate; it is straightforward to check that both contributions become equal for $\tilde{\tau}^2 = c^2\Delta\tau^2/(1 - 2c\Delta\tau^2)$. Neglecting the term in $\tilde{\tau}^4$ (whose contribution will always be smaller) and knowing from the previous solution that $\hat{A}/|\hat{A}| = \pm_\Delta 1$, it is possible to propose a new control vector in the form:

$$\begin{aligned} u_\rho &\approx \mp_\Delta \frac{\tilde{\tau}}{\sqrt{c^2\Delta\tau^2 + (1 - 2c\Delta\tau^2)\tilde{\tau}^2}}, \\ u_\theta &\approx \mp_\Delta \frac{\Delta\tau(c - \tilde{\tau}^2)}{\sqrt{c^2\Delta\tau^2 + (1 - 2c\Delta\tau^2)\tilde{\tau}^2}}. \end{aligned}$$

Eqs. (4.25,4.26) can be solved for this control vector and the initial conditions in Eqs. (4.21), obtaining the radial solution first and then substituting into the transversal equation. The resulting $\hat{\rho}$ and $\hat{\theta}$ can be expressed in a compact way as:

$$\hat{\rho}' = \pm_\Delta K \frac{K_1 - g}{K_2},$$

$$\begin{aligned}
\hat{\rho} - \hat{\rho}_0 &= \pm_{\Delta} K \left[\frac{K_1 (1 + 4\tilde{\tau}) - 2g\tilde{\tau}}{4K_2} - \frac{c^2 \Delta \tau^2}{2K_2^{3/2} g_2} \right], \\
\hat{\theta}' - \hat{\theta}'_0 &= \pm_{\Delta} \Delta \tau K \left[\frac{-K_1 (1 + 8\tilde{\tau}) + 6g\tilde{\tau}}{4K_2} + \frac{c(-2 + 5c\Delta \tau^2)}{2K_2^{3/2}} g_2 \right], \\
\hat{\theta} - \hat{\theta}_0 &= \hat{\theta}'_0 \left(\tilde{\tau} + \frac{1}{2} \right) \pm_{\Delta} \Delta \tau K \left[\frac{4c(g - K_1) - K_1 \tilde{\tau} + 2(g - 2K_1) \tilde{\tau}^2}{4K_2} \right. \\
&\quad \left. - \frac{c(2 - 5c\Delta \tau^2)}{2K_2^{3/2}} \tilde{\tau} g_2 \right],
\end{aligned}$$

with

$$\begin{aligned}
2K_1 &= \sqrt{1 - 2c(1 - 2c) \Delta \tau^2}, \quad K_2 = 1 - 2c\Delta \tau^2, \\
g &= \sqrt{c^2 \Delta \tau^2 + (1 - 2c\Delta \tau^2) \tilde{\tau}^2}, \quad g_2 = \log \left[\frac{g\sqrt{K_2} + (1 - 2c\Delta \tau^2) \tilde{\tau}}{-1/2 + c\Delta \tau^2 + K_1\sqrt{K_2}} \right].
\end{aligned}$$

The boundary conditions given by Eqs.(4.21) can now be imposed. The radial velocity automatically satisfies them due to symmetries, while for the radial position and transversal velocity one reaches (after several algebraic manipulations):

$$\begin{aligned}
(4 - 7c\Delta \tau^2) &= 2KK_1, \\
\sqrt{K_2} &= 2cK \log \frac{2K_1 + \sqrt{K_2}}{2K_1 - \sqrt{K_2}}.
\end{aligned} \tag{4.29}$$

This nonlinear system can be solved numerically for the values of $\Delta \tau$ and c . Interestingly, by taking the limit $c \rightarrow 0$ it reduces to the results already obtained for the bang-bang control, with the first equation returning the same estimate for $\Delta \tau$ as Eq. (4.27) and the second one becoming incompatible (because it is not possible to fulfill all the boundary conditions without introducing c). The second part of Eqs. (4.29) is not easy to handle (even numerically, since the logarithmic term is close to an indetermination of the type $0 \cdot \infty$ for $\Delta \tau \rightarrow 0$), but the first one turns out to be a polynomial both in c (second degree) and $\Delta \tau$ (sixth degree with only even powers):

$$16\chi^2 - 56c\chi^2 \Delta \tau^2 + (-1 + 49c^2 \chi^2) \Delta \tau^4 + 2c(1 - 2c) \Delta \tau^6 = 0.$$

The roots of this polynomial can be found analytically either for c or $\Delta \tau$. The full solution for $\Delta \tau$ is too lengthy to report here, but by retaining terms up to order 4 in $\Delta \tau$ a first approximation is found in a compact way:

$$\Delta \tau \approx \frac{2\sqrt{\chi}}{\sqrt{1 + 7\chi c}}, \tag{4.30}$$

This shows that the influence of c in the time of flight is small, being scaled by $\chi \ll 1$. On the other hand, the solution for c would take the following form:

$$c = \frac{-28\chi^2 + \Delta\tau^4 + \Delta\tau\sqrt{\chi^2(64 - 7\Delta\tau^2) + \Delta\tau^4(-4 + \Delta\tau^2)}}{-49\chi^2\Delta\tau^2 + 4\Delta\tau^4},$$

which goes to 0 as $\Delta\tau$ approaches $2\sqrt{\chi}$.

Once $\Delta\tau$ and c are determined, the new value for $\hat{A}$ can be recovered by introducing the expansions for $\hat{\lambda}_u$ and $\hat{\lambda}_v$ into the transversality condition, Eq. (4.20), and retaining terms up to order 4 in $\Delta\tau$, yielding:

$$\hat{A} \approx \pm_{\Delta} \frac{2}{\Delta\tau\sqrt{1 + \Delta\tau^2(1/6 - 2c + 4c^2)}},$$

whereas $\hat{C}$ is obtained from the definition of c , Eq. (4.28):

$$\hat{C} \approx \hat{A}(c\Delta\tau^2 - 2).$$

Although this new control improves the solution for Eqs. (4.25) and (4.26), fulfilling all the boundary conditions, its validity as a solution for the linear formulation is still limited by the fact that small terms for $\Delta\tau \ll 1$ have been neglected in the equations of motion. Furthermore, during the definition of the control vector it was assumed that terms of order 4 and higher in $\Delta\tau$ were negligible in the expansion for $\lambda_u^2 + \lambda_v^2$. However, a closer look reveals that the quadratic term is scaled by $K_2 = 1 - 2c\Delta\tau^2$, which goes to zero as χ increases, whereas the coefficient for the quartic term grows with $\Delta\tau^2$. Therefore, the series expansion used to approximate u_ρ and u_θ breaks down for high enough values of χ , further limiting the applicability of this solution for $\chi \sim 1$.

4.3.2 Long maneuver regime

In the long maneuver regime, since $\Delta\tau \gg 1$ from Eqs. (4.22) it follows that:

$$|\hat{\lambda}_u| \leq |\hat{A}|, \quad |\hat{\lambda}_v| \leq 2|\hat{A}| + |\hat{C}|,$$

and the requirement of $|u_\theta| \sim 1$ (i.e. $|\hat{\lambda}_v| \gg |\hat{\lambda}_u|$) leads to:

$$|\hat{C}| \gg |\hat{A}|.$$

Introducing this into the transversality condition, Eq. (4.15), and retaining only the leading terms yields an estimation for $\hat{C}$:

$$\hat{C} \approx \pm 1,$$

while for $\hat{A}$ only the bound:

$$|\hat{A}| \ll 1,$$

is available. Gathering these results, the dominant component of the primer vector can now be approximated as:

$$\hat{\lambda}_v \approx \pm 1,$$

showing that, contrary to the short maneuver regime, there is no reversal in the direction of thrust during the maneuver.

The equations of motion can be approximately solved by neglecting the non-dominant terms and assuming $|u_\theta| = 1$, leading to:

$$\begin{aligned} 2\Delta\tau\hat{\theta}' + 3\Delta\tau^2\hat{\rho} &\approx 0, \\ \hat{\theta}'' + 2\Delta\tau\hat{\rho}' &\approx Ku_\theta. \end{aligned}$$

The first equation provides a relation between the radial displacement and the angular velocity, and substituting it into the second equation leads to a system of first order ODEs:

$$\begin{aligned} \hat{\theta}' &\approx -\frac{3}{2}\Delta\tau\hat{\rho}, \\ \hat{\rho}' &\approx 2\frac{\Delta\tau}{\chi}u_\theta. \end{aligned} \tag{4.31}$$

The second of Eqs. (4.31) shows that u_θ has to be positive for $\Delta\rho > 0$, allowing to determine the signs for the transversal component of the primer vector:

$$\hat{\lambda}_v \approx \mp_\Delta 1, \quad \hat{C} \approx \mp_\Delta 1.$$

Eqs. (4.31) can now be readily integrated to yield:

$$\begin{aligned} \hat{\rho} - \hat{\rho}_0 &\approx \pm_\Delta 2\frac{\Delta\tau}{\chi}\hat{\tau}, \\ \hat{\theta} - \hat{\theta}_0 &\approx \frac{3}{2}\Delta\tau \left(\mp_\Delta \frac{\Delta\tau}{\chi}\hat{\tau}^2 - \hat{\rho}_0\hat{\tau} \right), \end{aligned}$$

and imposing $\Delta\hat{\rho} = \pm_\Delta 1$ gives an estimate for the time of flight:

$$\Delta\tau \approx \frac{\chi}{2}. \tag{4.32}$$

Since the linearized equations of motion have been reduced to a first order system by neglecting small terms, only the initial conditions for $\hat{\rho}$ and $\hat{\theta}$ can be directly imposed. Interestingly, it is straightforward to check that $\hat{\theta}'$ also fulfills its initial condition, whereas $\hat{\rho}'$ is constant and can never meet either of its boundary conditions. This shows that the small components in the linearized equations of

motion, particularly the radial contribution of the thrust acceleration, have to be retained in order to impose all the boundary conditions.

A more accurate solution is now sought for by defining an approximation for the control vector that includes the small contribution of its radial component. After some manipulations it is possible to write u_ρ and u_θ in the form:

$$u_\rho = \pm_\Delta \frac{\bar{A} \sin(\Delta\tau\tilde{\tau})}{\sqrt{1 + 4\bar{A} \cos \Delta\tau\tilde{\tau} + \frac{1}{2}\bar{A}^2 \cos 2\Delta\tau\tilde{\tau}}},$$

$$u_\theta = \pm_\Delta \frac{1 + 2\bar{A} \cos(\Delta\tau\tilde{\tau})}{\sqrt{1 + 4\bar{A} \cos \Delta\tau\tilde{\tau} + \frac{1}{2}\bar{A}^2 \cos 2\Delta\tau\tilde{\tau}}},$$

where $\bar{A} = \hat{A}/\hat{C}$. Since $|\hat{A}| \ll |\hat{C}|$, it is possible to expand the previous expressions for $|\bar{A}| \ll 1$ to reach:

$$u_\rho = \pm_\Delta \left[\bar{A} \sin(\Delta\tau\tilde{\tau}) - \bar{A}^2 \sin(2\Delta\tau\tilde{\tau}) + \mathcal{O}(\bar{A}^3) \right],$$

$$u_\theta = \pm_\Delta \left[1 - \frac{1}{4}\bar{A}^2 - \frac{1}{4}\bar{A}^2 \cos(2\Delta\tau\tilde{\tau}) + \mathcal{O}(\bar{A}^3) \right].$$

Higher time harmonics appear as more terms in $\bar{A}$ are retained, but their contribution is small because $\bar{A} \ll 1$. From now on only the first harmonic is considered, leading to:

$$u_\rho \approx \pm_\Delta \bar{A} \sin(\Delta\tau\tilde{\tau}) = \pm_\Delta \bar{A} \sin \Delta\tau (\hat{\tau} - 1/2), \quad u_\theta \approx \pm_\Delta \bar{C}, \quad (4.33)$$

where:

$$\bar{C} = 1 - \frac{1}{4}\bar{A}^2, \quad (4.34)$$

and $\bar{A}$ corresponds to the amplitude of the control vector oscillations about the transversal direction.

The full linearized equations of motion, Eqs. (4.5), can be solved analytically for the periodic control vector in Eqs. (4.33) and the initial conditions given by Eqs. (4.21) to reach:

$$\begin{aligned} \hat{\rho}' &= \pm_\Delta 2 \frac{\Delta\tau}{\chi} \left[\left(\Delta\tau \frac{\bar{A}}{4} \sin \Delta\tau (\hat{\tau} - 1/2) \right) \hat{\tau} + \bar{C} (1 - \cos \Delta\tau \hat{\tau}) - \frac{\bar{A}}{4} \sin \frac{\Delta\tau}{2} \sin \Delta\tau \hat{\tau} \right], \\ \hat{\rho} - \hat{\rho}_0 &= \pm_\Delta 2 \frac{\Delta\tau}{\chi} \left[\left(\bar{C} - \frac{\bar{A}}{4} \cos \Delta\tau (\hat{\tau} - 1/2) \right) \hat{\tau} + \left(-\bar{C} + \frac{\bar{A}}{4} \cos \frac{\Delta\tau}{2} \right) \frac{\sin \Delta\tau \hat{\tau}}{\Delta\tau} \right], \\ \hat{\theta}' - \hat{\theta}'_0 &= \pm_\Delta \frac{\Delta\tau}{\chi} \left[\left(4\bar{C} - \bar{A} \cos \frac{\Delta\tau}{2} \right) \sin \Delta\tau \hat{\tau} + \Delta\tau (-3\bar{C} + \bar{A} \cos \Delta\tau (\hat{\tau} - 1/2)) \hat{\tau} \right], \end{aligned}$$

$$\begin{aligned} \hat{\theta} - \hat{\theta}_0 &= \hat{\theta}'_0 \hat{\tau} \pm_{\Delta} \frac{\Delta\tau}{\chi} \left[-\frac{3}{2} \Delta\tau \bar{C} \hat{\tau}^2 + (\bar{A} \sin \Delta\tau (\hat{\tau} - 1/2)) \hat{\tau} \right. \\ &\left. + \left(4 \frac{\bar{C}}{\Delta\tau} - 2 \frac{\bar{A}}{\Delta\tau} \cos \frac{\Delta\tau}{2} \right) (1 - \cos \Delta\tau \hat{\tau}) + \frac{\bar{A}}{\Delta\tau} \sin \frac{\Delta\tau}{2} \sin \Delta\tau \hat{\tau} \right]. \end{aligned}$$

The boundary conditions at $\hat{\tau} = 1$, Eqs. (4.21), can now be imposed:

$$\begin{aligned} \hat{\rho}'_f &= 0 = \pm_{\Delta} 2 \frac{\Delta\tau}{\chi} \left[\bar{C} (1 - \cos \Delta\tau) - \frac{\bar{A}}{4} \sin \frac{\Delta\tau}{2} (\sin \Delta\tau - \Delta\tau) \right], \\ \hat{\rho}_f - \hat{\rho}'_0 &= \pm_{\Delta} 1 = \pm_{\Delta} \left[2 \frac{\Delta\tau}{\chi} \left(\bar{C} - \frac{\bar{A}}{4} \cos \frac{\Delta\tau}{2} \right) \left(1 - \frac{\sin \Delta\tau}{\Delta\tau} \right) \right], \\ \hat{\theta}'_f - \hat{\theta}'_0 &= \mp_{\Delta} \frac{3}{2} \Delta\tau = \pm_{\Delta} \frac{\Delta\tau}{\chi} \left[\bar{C} \Delta\tau - 4 \Delta\tau \left(\bar{C} - \frac{\bar{A}}{4} \cos \frac{\Delta\tau}{2} \right) \left(1 - \frac{\sin \Delta\tau}{\Delta\tau} \right) \right]. \end{aligned}$$

Note that the solution for this system of equations does not depend on the sign of $\Delta\rho$. From the second equation it is possible to obtain an expression for $\bar{A}$ as a function of $\Delta\tau$ and $\bar{C}$:

$$\bar{A} = 8\bar{C} \frac{\sin \frac{\Delta\tau}{2}}{\sin \Delta\tau - \Delta\tau}. \quad (4.35)$$

Introducing Eq. (4.35) into the other two boundary conditions and operating, it turns out that both yield the same relation between $\Delta\tau$ and $\bar{C}$:

$$\Delta\tau = \frac{\chi}{2\bar{C}}. \quad (4.36)$$

Eqs. (4.34), (4.35), and (4.36) form a system of transcendental algebraic equations allowing to determine $\bar{A}$, $\bar{C}$, and $\Delta\tau$ for a given value of χ . It is observed that the time of flight estimate in Eq. (4.36) is a generalization of the one given by Eq. (4.32), which corresponded to $\bar{C} = 1$, showing that the time of flight increases slightly with the amplitude on the control oscillations. Regarding said amplitude, Eq. (4.35) reveals two important characteristics of its evolution with $\Delta\tau$. First of all, its average value decreases with the time of flight, so longer maneuvers have their control vector closer to the transversal direction. Secondly, the amplitude also depends on the time synchronization of the maneuver with respect to the orbital period of the reference orbit through the sine in the numerator. Maneuvers which take nearly a whole number of orbital revolutions of the reference to complete will have a lower amplitude of the control vector oscillation, thus being more efficient. Particularly, Eq. (4.35) predicts no oscillations for $\Delta\tau = 2n\pi$ with $n \in \mathbb{N}$, although this will not happen in the actual trajectory due to the contribution of nonlinear terms.

The values of $\hat{A}$ and $\hat{C}$ can be calculated from $\bar{A}$, $\Delta\tau$ and the transversality condition. Substituting $\hat{A} = \bar{A}\hat{C}$ in Eq. (4.15) and solving for $\hat{C}$ yields:

$$\hat{C} = \mp_{\Delta} \sqrt{\frac{2}{2 + 5\bar{A}^2 + 8\bar{A} \cos \Delta\tau/2 + 3\bar{A}^2 \cos \Delta\tau}},$$

and then $\hat{A}$ can be directly obtained as $\bar{A}\hat{C}$.

4.3.3 Transition Zone

As previously stated, it is not feasible to search for an analytical solution, even approximate, in the region between both limit regimes. In this transition zone the structure of the optimal maneuver undergoes fundamental qualitative and quantitative changes, as it evolves from the nearly bang-bang control profile of the short maneuver regime to the nearly tangential control of the long maneuver regime. It is however possible to predict for which values of the fundamental parameter χ it will developed by comparing the time of flight predictions obtained for the two limit regimes. Equating the time of flight estimates given by Eqs. (4.27) and (4.32) and solving for χ yields:

$$\chi \approx 16,$$

corresponding to a time of flight of

$$\Delta\tau \approx 8,$$

which is very close to the 2π period of the reference orbit. This is consistent with the previous conclusion that all the contributions in the equations of motion become of comparable order for $\Delta\tau \sim \mathcal{O}(1)$, and reinforces the idea that the main difference between both regimes is whether the maneuver takes less or more than one revolution of the reference orbit to complete.

The behavior in this transition zone is studied numerically through a set of test cases in the last section of the chapter.

4.3.4 Nonlinearity effects

The developments in the previous sections are restricted to the linear formulation. Therefore, even the solution for the long maneuver regime, which is feasible and near-optimal, will separate from the actual solution due to the contribution of nonlinear terms.

A time of flight estimate for the long maneuver regime including the nonlinearity effects can be obtained leveraging the result that thrust orientation must

be nearly tangential, and assuming quasi circular orbit during the transfer. Let us consider the vis-viva equation:

$$E = \frac{v^2}{2} - \frac{1}{1+\rho} = -\frac{1}{2a}, \quad (4.37)$$

where E is the specific energy, a is the semi-major axis, and v the velocity (all dimensionless). Deriving with respect to τ a relation between the derivatives of E and a is reached:

$$\frac{dE}{d\tau} = \frac{1}{2a^2} \frac{da}{d\tau}.$$

Because thrust is tangential the derivative of E amounts to εv . Furthermore, the quasi circular condition allows us to write $a \approx 1 + \rho$, and from Eq. (4.37) an expression for v as a function of a is reached:

$$v \approx \frac{1}{\sqrt{a}}.$$

Gathering these results an ODE for the time evolution of a is obtained as follows:

$$\frac{da}{d\tau} \approx 4\sqrt{2}\varepsilon a^{3/2}.$$

Integrating between 0 and $\Delta\tau$ and solving for $\Delta\tau$ finally yields an approximation for the time of flight:

$$\Delta\tau = \pm_{\Delta} \frac{1}{\varepsilon} \frac{\sqrt{a_f} - \sqrt{a_0}}{\sqrt{a_0 a_f}} = \pm_{\Delta} \frac{\chi}{\Delta r} \frac{\sqrt{a_f} - \sqrt{a_0}}{\sqrt{a_0 a_f}}, \quad (4.38)$$

where $a_f = 1 + \rho_f$, $a_0 = 1 + \rho_0$. This result coincides with the classic solution given by Edelbaum [30]. Unlike the solution for the linearized dynamics it is now not possible to explicitly impose the boundary conditions, although the quasi circular hypothesis implies that they will be approximately fulfilled as long as it holds. It is interesting to check that, by expanding for $\rho \ll 1$ and retaining only the leading term, the first linear estimate is recovered:

$$\Delta\tau = \pm_{\Delta} \frac{\chi}{\Delta r} \frac{\rho_f - \rho_0}{2} + \mathcal{O}(\rho^2) \approx \frac{\chi}{2}.$$

A quantitative accounting of the influence of the nonlinearity effects in the optimal maneuver can be seen in the test cases included in the following section.

Figure 4.2: Time of flight as a function of χ for an orbit raise of 200 km departing from GEO (initial orbit is taken as reference).

Figure 4.3: Time of flight as a function of χ for two LEO to LEO transfers (the final orbit is taken as reference). Note that the linear solution is common to all cases.

Figure 4.4: Time of flight as a function of χ for several interplanetary test cases departing from Earth (Earth's orbit is taken as reference). Note that the linear solution is common to all cases.

Figure 4.5: Time of flight as a function of χ for the Earth to Mars test case (Earth's orbit is taken as reference). Curves corresponding to the time of flight estimations for the short and long-maneuver regimes, as well as the classic solution by Edelbaum, are included.

4.4 Test Cases

The proposed minimum-time transfer is now solved numerically for several Earth-bound and interplanetary test cases. Wide ranges are considered for the dimensionless thrust parameter ε and the radial displacement Δr . Although some of these values may lack a practical interest (for instance, having a thrust acceleration too high or requiring an exceedingly long time of flight), they serve to highlight the physical and mathematical characteristics of the problem and the differences between the linear and nonlinear cases. These numerical solutions are obtained solving the TPBVP derived from the indirect method in the previous sections. A different approach using a direct transcription and a large-scale NLP solver can be seen in previous works by Gonzalo et al. [40].

The TPBVP derived from the indirect method is solved using a classic shooting scheme. Keeping the assumption that all values of the initial and final states are fixed except for θ_f , the corresponding shooting function can be written as follows:

$$\mathbf{Z}(\boldsymbol{\lambda}_0, \Delta\tau) = \begin{bmatrix} \dot{\rho}(\Delta\tau) \\ \dot{\theta}(\Delta\tau) - \dot{\theta}_f \\ \rho(\Delta\tau) - \rho_f \\ \lambda_\theta(\Delta\tau) \\ H(\Delta\tau) \end{bmatrix} = \mathbf{0},$$

where $\mathbf{S}(\Delta\tau)$ and $\boldsymbol{\lambda}(\Delta\tau)$ are obtained integrating Eqs. (4.4) and (4.13) (or (4.5) and (4.17) for the linearized problem) with initial conditions $[\mathbf{S}_0 \boldsymbol{\lambda}_0]$. The initial condition for the state, $\mathbf{S}_0$, is given by Eq. (4.6) (or Eq. (4.7) for the linearized case), whereas $\boldsymbol{\lambda}_0$ is an unknown of the shooting algorithm. Continuing with the definition of the shooting algorithm, it is important to note that the previous shooting function corresponds to a single-shooting scheme. The results presented in this work have been obtained using a multiple-shooting algorithm with 32 segments; that requires augmenting the vector of unknowns with the states and costates at the beginning of each intermediate segment as well as adding the equations for the matching of consecutive segments (8 extra unknowns and equations for each additional segment, leading to a total of 261 unknowns). In all cases, the reference orbit is chosen to be coincident with either the initial or final orbit, and the initial angular position is set to zero for simplicity.

Third party algorithms are employed both for integrating the system of ODEs (Matlab's *ode45*) and zero finding (Matlab's *fsolve* and its Trust-Region-Dogleg algorithm). In order to improve the robustness and efficiency of the method, the Jacobian for the shooting function is constructed analytically from the State Transition Matrix (STM) for $[\mathbf{S} \boldsymbol{\lambda}]$, following a procedure analogous to the one proposed by Zhang et al. [101] for the Cartesian case. Computing the STM requires to analytically derive the variational equation and integrating it alongside the ODE

system for state and costate; because the STM is a 8×8 matrix for the planar problem in curvilinear coordinates, this means adding 64 ODEs to the problem (for a total of 72 ODEs). The derivation of the variational equation is relatively simple and can be performed using a symbolic manipulator.

One last aspect to consider is the generation of an initial guess of the solution for the iterative multiple shooting algorithm. For the linear case the refined versions of the analytical approximations are used except for a region around the transition zone corresponding to $\chi \in [6, 16]$, where the underlying hypothesis lose validity (as previously commented) and the first approximation is preferred. This approach proves to be remarkably good, allowing to solve all the linear test cases. For the nonlinear cases the corresponding linear solution is tried first, falling back to a continuation in χ if the shooting algorithm fails to converge. The linear solution proves to be a good enough initial guess for the short-maneuver regime in all the test cases, and also for the transition zone and long-maneuver regime in problems with small Δr (small contribution of the nonlinear terms).

Figure 4.2 shows the evolution of the maneuver time $\Delta\tau$ with χ for an orbit raise of 200 km departing from GEO (42164.1 km). Because the relative displacement is very small, $\Delta r = 4.7433 \times 10^{-3}$, the linear and nonlinear solutions are practically identical, and only the latter is included for clarity. The two different regimes predicted by the qualitative study of the equations can be clearly identified, with the transition between them taking place for values of $\Delta\tau$ around one orbit of the reference, that is, $\Delta\tau/2\pi \approx 1$. This supports the hypothesis that one of the key differences between both regimes is whether the maneuver can be completed in one revolution of the reference orbit or not. The figure also represents the different approximations of $\Delta\tau$ previously obtained for the short- and long-maneuver regimes, showing a great correspondence with the numerical results. Both the first and refined approximations remain very close to the exact solution except for a small region corresponding to the transition zone. It is interesting to note how the refined approximation for the long-maneuver regime holds even for values of χ corresponding to less than one revolution of the reference orbit (inside the transition zone), whereas the refined approximation for the short-maneuver regime diverges rapidly as the transition zone is approached. This breakdown of the refined short-maneuver approximation was already predicted from the equations, because some of the hypothesis used for its derivation lose validity as the time of flight approaches one revolution. Focusing again on the long-maneuver regime, the oscillations in $\Delta\tau$ predicted by the refined approximation can be clearly identified, with the first approximation corresponding to the secular behavior.

The effect of nonlinearities can be clearly appreciated in Figure 4.3, corresponding to two LEO to LEO transfers with the same arrival orbit at an altitude of 650 km. The altitudes of the departure orbits are 850 km and 1400 km, corre-

Table 4.1: Data for the interplanetary test cases. Orbits are assumed to be circular and coplanar, and all transfers depart from Earth’s orbit. Orbital radii are taken from [70] and [69].

Body	Radius [10^6 km]	Δr [–]
Mercury	57.91	0.6130
Venus	108.21	0.2767
Earth	149.60	–
Mars	227.92	0.5235
Ceres	414.09	1.7680
Jupiter	778.57	0.8078
Saturn	1433.53	8.5824
Uranus	2872.46	18.2009

sponding to $\Delta r = 2.7670 \times 10^{-2}$ and 9.6424×10^{-2} , respectively. As expected, the higher the value of Δr the more the exact solution separates from the linear one (common to both cases) due to the nonlinear contributions. However, this separation only appears in the transition zone and the long-maneuver regime, whereas both solutions remain close to the linear case for the short-maneuver regime. This is due to the nonlinear effects not having enough time to fully develop for transfers in the short-maneuver regime, characterized by relatively shorter transfer times.

The previous conclusions about the effect of nonlinearities are reinforced by the results obtained in Figure 4.4 for several interplanetary test cases. All departure and arrival orbits are coplanar and circular, with radii corresponding to the semi-major axis of several planets and asteroids in the Solar System as reported in Table 4.1. In all cases, the departure orbit is the one corresponding to Earth, which is also taken as reference. The results show that the separation between the linear and nonlinear solutions grows with Δr , with negative radial variations increasing the time of flight for a given χ and positive radial variations decreasing it. Additionally, the oscillations in $\Delta \tau$ for the long-maneuver regime decrease as Δr increases, completely disappearing in the Saturn and Uranus examples. Furthermore, it is also observed that those cases present a larger transition zone until the long-maneuver asymptotic evolution of $\Delta \tau$ is reached.

More detailed information for the Earth-Mars case including the time estimates is given in Figure 4.5. The refined long-maneuver approximation perfectly reproduces the time of flight for the linear solution, but both separate appreciably from the exact solution due to the effect of the nonlinearities. On the other hand, the

Figure 4.6: Earth to Mars transfer for a non-dimensional thrust acceleration of 2.1764 (corresponding to $\chi = 0.2405$).

Figure 4.7: Earth to Mars transfer for a non-dimensional thrust acceleration of 4.0680×10^{-3} (corresponding to $\chi = 1.2869 \times 10^2$).

Figure 4.8: Earth to Mars transfer for a non-dimensional thrust acceleration of 3.2684×10^{-2} (corresponding to $\chi = 1.6018 \times 10^1$).

Table 4.2: Data for three selected Earth-to-Mars test cases, with $\Delta r = 5.2353 \times 10^{-1}$.

	Short-maneuver	Transition	Long-maneuver
$\chi [-]$	0.2405	1.6018×10^1	1.2869×10^2
$\varepsilon [-]$	2.1765	3.2684×10^{-2}	4.0680×10^{-3}
$\Delta\tau_{\text{linear}} [-]$	0.9644	9.1327	6.4481×10^1
$\Delta\tau_{\text{nonlinear}} [-]$	0.9709	6.9437	4.7314×10^1
$\Delta\tau_{\text{linear}}/2\pi [-]$	0.1535	1.4535	1.0262×10^1
$\Delta\tau_{\text{nonlinear}}/2\pi [-]$	0.1545	1.1051	7.5302

$\Delta\tau$ estimate by Edelbaum, which is based in the nonlinear equations, properly captures the secular behavior but fails to reproduce the small oscillations. Interestingly, the first long-maneuver approximation and the Edelbaum solution are parallel lines in logarithmic scale.

Figures 4.6-4.6and Table 4.2 present three representative cases taken from both regimes and the transition zone, for the Earth-Mars test case. The qualitative features shown for each regime are common to all test cases, keeping in mind that the separation between the linear and nonlinear solution increases with Δr . Figure 4.6 shows the thrust orientation angle, curvilinear position and costate of the velocity for a transfer in the short-maneuver regime, with $\chi = 0.2405$ and a dimensionless thrust acceleration of 2.1764. Times of flight are 0.9644 for the linear solution and 0.9619 for the exact one (corresponding to 0.1534 and 0.1530 revolutions of the reference orbit, respectively). The control variable γ resembles a bang-bang control as predicted, with the switch located at the middle of the maneuver. Thrust is oriented close to the positive radial direction during the first half of the maneuver, and in the second half it reverses to point opposite to it. Interestingly, the linear and nonlinear γ profiles are very similar at the beginning but appreciably separate towards the end of the transfer. This is also reflected in the curves for the costate elements involved in the control equation, $\hat{\lambda}_u$ and $\hat{\lambda}_v$. However, the most notable differences are observed in the final values of the angular position. The main reason behind this is the different value of the boundary condition for the final angular velocity for the linear and nonlinear problems. On the other hand, the profiles for the radial position are almost identical in both cases. In general, the linear solution for γ , $\hat{\rho}$ and the costates fit very well with the behaviors predicted by the approximate analytic solutions. Also noteworthy is that the symmetry properties of the linear solution are not preserved for the complete nonlinear model.

The solution has a completely different structure in the long-maneuver regime,

Table 4.3: Data for three selected GEO to GEO plus 200km test cases, with $\Delta r = 4.7434 \times 10^{-3}$.

	Short-maneuver	Transition	Long-maneuver
$\chi [-]$	0.2405	1.6018×10^1	1.2869×10^2
$\varepsilon [-]$	1.9719×10^{-2}	2.9613×10^{-4}	3.6858×10^{-5}
$\Delta \tau_{\text{linear}} [-]$	0.9644	9.1327	6.4481×10^1
$\Delta \tau_{\text{nonlinear}} [-]$	0.9645	9.1062	6.4191×10^1
$\Delta \tau_{\text{linear}} / 2\pi [-]$	0.1535	1.4535	1.0262×10^1
$\Delta \tau_{\text{nonlinear}} / 2\pi [-]$	0.1535	1.4493	1.0216×10^1

as shown in Figure 4.7 for $\chi = 1.2869 \times 10^2$ and a dimensionless thrust acceleration of 4.0680×10^{-3} . Time of flight is 64.4812 for the linear case and 47.3139 for the nonlinear one (10.2625 and 7.5302 revolutions of the reference orbit, respectively). Contrary to the short-maneuver example, the separation between the linear and complete models due to the effect of the nonlinearities is clearly observed in all variables. The thrust control angle is now oriented along the transversal direction, describing small oscillations about it both in the linear and nonlinear cases. However, the amplitude of the oscillations in the linear solution is constant, whereas for the exact model it increases with time. Similar oscillatory behaviors can also be found for the rest of variables, except for $\hat{\theta}$. Interestingly, the secular evolution of $\hat{\rho}$ has changed from quadratic to linear, justifying the different slopes showed in Figure 4.5 for $\Delta \tau(\chi)$. Same as with the short-maneuver example, the nonlinear solution does not preserve the symmetry properties of the linearized model.

Finally, Figure 4.8 corresponds to a transfer maneuver in the transition zone, with $\chi = 1.6018 \times 10^1$ and dimensionless thrust acceleration 3.2684×10^{-2} . Resulting times of flight are 9.1327 for the linearized maneuver and 6.9437 for the complete equations (corresponding to 1.4535 and 1.1051 revolutions of the reference orbit, respectively). The γ profile reflects the transition between both limit cases, showing an approximately one-period oscillation with large amplitude and characteristic features about the middle point of the maneuver. The radial distance profile is still similar to the short-maneuver case, but its secular behavior is transitioning from quadratic to linear. The curves for $\hat{\lambda}_u$ and $\hat{\lambda}_v$ also present clear differences from the short-maneuver regime, beginning to develop the oscillatory behavior which characterizes the long-maneuver regime. Interestingly, the features observed for the linearized solution are more similar to the ones of the long-maneuver limit case, indicating that the nonlinearities increase the size of the

Figure 4.9: GEO to GEO plus 200 km transfer for a non-dimensional thrust acceleration of 1.9719×10^{-2} (corresponding to $\chi = 0.2405$).

Figure 4.10: GEO to GEO plus 200 km transfer for a non-dimensional thrust acceleration of 3.6858×10^{-5} (corresponding to $\chi = 1.2869 \times 10^2$).

Table 4.4: Data for three selected Earth-to-Venus test cases, with $\Delta r = 2.7667 \times 10^{-1}$.

	Short-maneuver	Transition	Long-maneuver
$\chi [-]$	0.2405	1.6018×10^1	1.2869×10^2
$\varepsilon [-]$	1.1502	1.7273×10^{-2}	2.1498×10^{-3}
$\Delta\tau_{\text{linear}} [-]$	0.9644	9.1327	6.4481×10^1
$\Delta\tau_{\text{nonlinear}} [-]$	0.9548	1.0541×10^1	8.1811×10^1
$\Delta\tau_{\text{linear}}/2\pi [-]$	0.1535	1.4535	1.0262×10^1
$\Delta\tau_{\text{nonlinear}}/2\pi [-]$	0.1520	1.6777	1.3021×10^1

transition zone in χ .

The strong relation between Δr and nonlinear effects can be seen by comparing the previous Earth-to-Mars examples with the ones shown in Figures 4.9-4.11, corresponding to three samples of the GEO to GEO plus 200 km test case with the same values of χ but $\Delta r = 4.7434 \times 10^{-3}$. The resulting thrust acceleration parameters and times of flight are reported in Table 4.3. It is straightforward to check that the linear solutions are identical to the ones for the Earth-to-Mars test cases, both in terms of the time of flight and the evolution of state and costate. This is consistent with the results from the analytical study in section 4.3, which showed that the solution for the linearized, dimensionless model only depended on the fundamental parameter χ . On the other hand, the nonlinear solutions, which do depend on the total radial displacement, are now much closer to their linear counterparts due to the smaller value of Δr . These results show that the linearized model can be a good approximation of the exact solution in any regime for small enough values of the radial displacement Δr .

The previous test cases corresponded to maneuvers with positive $\Delta\rho$, that is, where the final radius is greater than the initial one. For completeness, a set of solutions corresponding to the Earth-to-Venus test case, with $\Delta\rho = -2.7667 \times 10^{-1}$ (that is, $\Delta r = |\Delta\rho| = 2.7667 \times 10^{-1}$), are represented in Figures 4.12-4.14. The values for χ (equal to previous test cases), ε , and $\Delta\tau$ are compiled in Table 4.4. The foremost difference is that the signs for state and costate have reversed, as predicted by the analytical solutions. Moreover, the control angle has rotated 180 deg in all three regimes, and its amplitude in the long-maneuver regime now decreases with time instead of increasing. Regarding $\Delta\tau$, the nonlinear solution is faster than the linear one in the short-maneuver regime but slower in the long-maneuver one, opposite to the behavior observed for $\Delta\rho > 0$. Finally, the fulfillment of the terminal condition for the angular velocity in the short-maneuver example causes γ to separate from the linear solution towards the end of the maneuver.

Figure 4.11: GEO to GEO plus 200 km transfer for a non-dimensional thrust acceleration of 2.9613×10^{-4} (corresponding to $\chi = 1.6018 \times 10^1$).

Figure 4.12: Earth to Venus transfer for a non-dimensional thrust acceleration of 1.1502 (corresponding to $\chi = 0.2405$).

Figure 4.13: Earth to Venus transfer for a non-dimensional thrust acceleration of 2.1498×10^{-3} (corresponding to $\chi = 1.2869 \times 10^2$).

Figure 4.14: Earth to Venus transfer for a non-dimensional thrust acceleration of 1.7273×10^{-2} (corresponding to $\chi = 1.6018 \times 10^1$).

4.5 Conclusion

The minimum-time constant-thrust acceleration transfer between two close, coplanar, circular orbits has been studied in great detail, using a relative motion formulation in curvilinear coordinates and the indirect method. Results show that the time-optimal transfer undergoes fundamental qualitative changes with the required time of flight, being possible to identify two limit cases, a short-maneuver and a long-maneuver regime, separated by a transition zone. The short- and long-maneuver limit regimes correspond, respectively, to transfers requiring much less or more than one revolution of the reference orbit to complete, whereas transfers in the transition zone present times of flight of approximately one period of the reference orbit. The qualitative characteristics of each region have been studied in detail using analytical and numerical techniques, both for the linearized and complete equations of motion, and three representative examples have been presented to illustrate them. For transfers in the short-maneuver regime the thrust orientation control law is nearly bang-bang along the radial direction, being initially oriented along the desired radial displacement and reversing itself at half the maneuver. Conversely, in the long-maneuver regime the optimal thrust orientation oscillates with small amplitude about the transversal direction, oriented along it for positive radial displacements and opposite to it for negative radial displacements. The effect of the nonlinearities depends mainly on the required radial displacement, and it is especially important in the transition zone and the long-maneuver regime.

The ratio of the required radial displacement and the available thrust acceleration is identified as the main parameter of the problem and a clear relation between this parameter and the time of flight is observed for both regimes. An approximate analytical study of the first order optimality conditions for the linearized equations of motion leads to explicit expressions for the time of flight depending only on the displacement-thrust ratio, as well as other estimations and qualitative information about the evolution of state and costate. Particularly, a suboptimal analytical solution fulfilling all the linearized differential equations and constraints is reached for the long-maneuver regime. For the complete equations of motion, nonlinear effects make the solution dependent on two parameters (the radial displacement and the displacement-thrust ratio).

These approximate analytical solutions can be very useful to construct the initial guesses required by iterative numerical solvers for both the direct and indirect methods. Particularly, all the test cases for the linearized equations of motion were successfully solved using their approximate analytical solution as initial guess. Furthermore, the linearized solution can be used as initial guess for the complete problem in many practical cases, especially for the short-maneuver regime or when the required radial displacement is small.

Chapter 5

Optimal 3D Orbit Corrections

5.1 Introduction

This chapter concludes the part of the dissertation devoted to the optimization of minimum-time transfers using relative motion formulations by exploring maneuvers including a change of orbital plane. It can be seen as the finalization of the previous studies on the phasing and radius change problems, and as such it has a special focus on the out-of-plane component of the motion. Once again, the relative motion formulation in curvilinear coordinates introduced in Chapter 2 is used to for the dynamics, which yields a very compact set of equations having as variables the radial separation, the phasing and the out-of-plane distance. Same as before, this formulation proves to be very convenient for both the numerical and analytical analysis.

The optimal control problem in the thrust orientation is treated numerically but starting from key analytical relations to support the generation of a suitable initial guess. The first analytical relation is concerned with the minimum-time change of inclination with unaltered orbit radius. An exact optimal solution in closed analytical form is obtained starting from the linearized equations of motion and therefore is valid for small inclination changes. The solution is employed to generate the initial guess for a sequence of optimization problems in which a gradual change in orbit radius is imposed together with the fixed inclination change. The second analytical relation is concerned with the minimum-time change of orbital radius and was obtained in Chapter 4. The structure of the solution when going from an inclination-only problem towards an increasingly radial-change-characterized problem is studied with an emphasis on the evolution of the minimum maneuver time. The relation between the latter and the sum of the maneuver times required for the inclination-only and radial-only subproblems is investigated.

5.2 Problem Statement and Equations of Motion

The objective is to study the minimum-time orbit corrections between two close, non-coplanar orbits. In the most general case, this implies modifying the radius, the inclination and the phasing of the orbit. To model the problem, let us consider two objects, a chief C and a follower F , describing two close circular orbits of radii R and R_C around a primary O with gravitational constant μ . The follower is the controllable spacecraft whose orbit will be modified, and the chief may represent a real object (e.g. another spacecraft, a space debris) or a virtual one (such as the nominal final position of the spacecraft). All the equations and variables will be expressed in non-dimensional form, taking R , $\Omega^{-1} = \sqrt{R^3/\mu}$, and m_F as characteristic magnitudes for length, time, and mass, respectively. Since we are studying small orbit modifications the required propellant mass m_{prop} can be assumed to be negligible compared to the mass of the follower m_F , and the latter is taken as a constant.

The dynamics are described using the nonlinear relative motion formulation in curvilinear coordinates introduced in Chapter 2. The relative position of F with respect to C is then expressed through cylindrical coordinates (ρ, θ, z) , whose evolution is governed by differential Eqs. (2.3) (reproduced here for clarity):

$$\begin{aligned}\ddot{\rho} - 2\dot{\theta} - 3\rho &= a_{i\rho} + a_{g\rho} + a_\rho^T, \\ \ddot{\theta} + 2\dot{\rho} &= a_{i\theta} + \frac{a_\theta^T}{1 + \rho}, \\ \ddot{z} + z &= a_{gz} + a_z^T.\end{aligned}\tag{5.1}$$

Terms $a_{i\rho}$, $a_{g\rho}$, $a_{i\theta}$ and a_{gz} account for the nonlinear perturbation effects, while a_ρ^T , a_θ^T and a_z^T correspond to the actions of the continuous-thrust engine. The thruster's acceleration can also be expressed through the total magnitude of the acceleration a^T and a unitary vector $\mathbf{u}^T$:

$$\begin{aligned}a^T &= \sqrt{(a_\rho^T)^2 + (a_\theta^T)^2 + (a_z^T)^2}, \quad \mathbf{u}^T = \begin{bmatrix} u_\rho^T & u_\theta^T & u_z^T \end{bmatrix}^\top, \quad \|\mathbf{u}^T\| = 1, \\ a_\rho^T &= a^T u_\rho^T, \quad a_\theta^T = a^T u_\theta^T, \quad a_z^T = a^T u_z^T.\end{aligned}$$

Although this representation of control is redundant, introducing one additional variable and an algebraic restriction, it will be preferred for the numerical treatment of the problem due to its better behavior for small values of a^T .

Introducing the state vector:

$$\mathbf{S} = \begin{bmatrix} \dot{\rho} & \dot{\theta} & \dot{z} & \rho & \theta & z \end{bmatrix}^\top,$$

as well as the control vector:

$$\mathbf{U} = \begin{bmatrix} u_\rho^T & u_\theta^T & u_z^T & a^T \end{bmatrix}^\top,$$

Eqs. (5.1) can be reduced to a first order system with right-hand side:

$$\mathbf{F}(\mathbf{S}, \mathbf{U}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a_{i\rho} + a_{g\rho} + a^T u_\rho^T / (1 + \rho) \\ -2\dot{\rho} + a_{i\theta} + a^T u_\theta^T \\ -z + a_{gz} + a^T u_z^T \\ \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{bmatrix}, \quad (5.2)$$

where τ is the non-dimensional time. An additional equation must be imposed to the components of the control vector to reflect the physical limit of the maximum available thrust, depending on the representation used for control either:

$$0 < a^T < \varepsilon, \quad (5.3)$$

or:

$$(a_\rho^T)^2 + (a_\theta^T)^2 + (a_z^T)^2 \leq \varepsilon^2,$$

where ε is the non-dimensional nominal thruster acceleration:

$$\varepsilon = \frac{T}{m_C R \Omega^2},$$

being T the nominal thrust. The linearized formulation, used in some parts of the study, is similar to the well-known Clohessy-Wiltshire equations:

$$\mathbf{F}^*(\mathbf{S}, \mathbf{U}; \tau) = \begin{bmatrix} 2\dot{\theta} + 3\rho + a^T u_\rho^T \\ -2\dot{\rho} + a^T u_\theta^T \\ -z + a^T u_z^T \\ \dot{\rho} \\ \dot{\theta} \\ \dot{z} \end{bmatrix}. \quad (5.4)$$

It is important to note that the only coupling between the in-plane and out-of-plane linearized equations comes from the thrust restriction expressed through Eq. (5.3).

The final state of the follower is determined from the state of the chief. In general, one can consider any arbitrary phasing $\Delta\theta$ between chief and follower in the final state, but the values of ρ , z and all the derivatives should be zero so that the orbits of follower and chief coincide:

$$\mathbf{S}(\tau_f) = [\dot{\rho}_f \quad \dot{\theta}_f \quad \dot{z}_f \quad \rho_f \quad \theta_f \quad z_f]^\top = [0 \quad 0 \quad 0 \quad 0 \quad \Delta\theta \quad 0]^\top. \quad (5.5)$$

The initial state corresponds to the original orbit of the follower. Said orbit can be characterized by three parameters: its inclination with respect to the chief orbit

Δi , the difference in radius with respect to the chief orbit $\Delta r = (R - R_C)/R$, and the time since the follower crossed the orbit plane of the chief at the ascending node τ_0 . Note that the sign of Δr indicates whether the initial orbit is above the reference orbit, negative Δr , or below it, positive Δr . An additional simplification has been made in the equations by imposing that the chief is placed along the line of nodes at τ_0 ; this can be done without loss of generality by defining a new virtual chief and including its phasing with respect to the real one as part of θ_f . From these parameters, the initial state can be computed as:

$$\mathbf{S}(\tau_0) = [\dot{\rho}_0 \quad \dot{\theta}_0 \quad \dot{z}_0 \quad \rho_0 \quad \theta_0 \quad z_0]^\top, \quad (5.6)$$

with:

$$\begin{aligned} \dot{\rho}_0 &= \frac{\kappa \sin(2n_0\tau_0)}{2\sqrt{(1-\Delta\rho)[1-\kappa\sin^2(n_0\tau_0)]}}, \\ \dot{\theta}_0 &= -1 + \frac{n_0\sqrt{1-\kappa}}{(1-\kappa)\sin^2(n_0\tau_0) + \cos^2(n_0\tau_0)}, \\ \dot{z}_0 &= \frac{\sqrt{\kappa}\cos(n_0\tau_0)}{\sqrt{1-\Delta r}}, \\ \rho_0 &= -1 + (1-\Delta r)\sqrt{1-\kappa\sin^2(n_0\tau_0)}, \\ \theta_0 &= -\tau_0 + \tan^{-1}2\left(\sqrt{1-\kappa}\sin(n_0\tau_0), \cos(n_0\tau_0)\right), \\ z_0 &= (1-\Delta r)\sqrt{\kappa}\sin(n_0\tau_0), \end{aligned}$$

where:

$$\kappa = \sin^2 \Delta i, \quad n_0 = \frac{1}{\sqrt{(1-\Delta r)^3}}.$$

5.3 Inclination Change Subproblem

When $\Delta r = 0$ and no phasing is imposed the only orbit correction to be performed is an inclination change. However, the coupling between Eqs. (5.2) and the dependence of $\dot{\rho}_0$, $\dot{\theta}_0$, ρ_0 and θ_0 with Δi implies that some degree of in-plane control will be needed even in this simplified maneuver. In cases where Δi is small the linearized model given by Eqs. (5.4) can be used, uncoupling the in-plane and out-of-plane equations; the only coupling then comes from the maximum thrust restriction. Relaxing this condition (i.e. neglecting the in-plane thrust requirements), the out-of-plane dynamics in the linear approximation is described as:

$$\ddot{z} + z = a_z^T, \quad (5.7)$$

$$|a_z^T| \leq \varepsilon ,$$

with initial conditions at $0 < \tau_0 < 2\pi$:

$$z(\tau_0) = z_0 = \sqrt{\kappa} \sin \tau_0 ,$$

$$\dot{z}(\tau_0) = \dot{z}_0 = \sqrt{\kappa} \cos \tau_0 .$$

Note that Eq. (5.7) corresponds to a forced linear oscillator. The minimum-time strategy to drive the out-of-plane velocity to zero is to have:

$$a_z^T = -\varepsilon \text{sign}(\dot{z}) ,$$

but in general this does not assure that z will also reach zero along with $\dot{z}$. Nevertheless, it is possible to force this condition by choosing an adequate time τ_0 to start the maneuver.

The solution of Eq. (5.7) for the proposed control between the initial time τ_0 and the first control switch (zero velocity) time τ_1 is:

$$\begin{aligned} z(\tau_0 < \tau < \tau_1) &= \dot{z}_0 \sin(\tau - \tau_0) + (z_0 - \varepsilon) \cos(\tau - \tau_0) - \varepsilon , \\ \dot{z}(\tau_0 < \tau < \tau_1) &= \dot{z}_0 \cos(\tau - \tau_0) - (z_0 - \varepsilon) \sin(\tau - \tau_0) , \end{aligned}$$

where the switching time τ_1 obeys:

$$\tau_1 - \tau_0 = \begin{cases} \tau^* & \text{for } \dot{z}_0(z_0 - \varepsilon) > 0 \\ \pi/2 & \text{for } z_0 = \varepsilon \\ \tau^* + \pi & \text{for } \dot{z}_0(z_0 - \varepsilon) < 0 \end{cases}$$

$$\tau^* = \tan^{-1} \left(\frac{\dot{z}_0}{z_0 - \varepsilon} \right) ,$$

and the coordinate at the switching point reads:

$$z_1 = z(\tau_1) = \varepsilon \pm \sqrt{\dot{z}_0^2 + (z_0 - \varepsilon)^2} ,$$

where the positive sign corresponds to the case $\dot{z}_0(z_0 - \varepsilon) > 0$.

For the successive sections between consecutive switching points ($n = 2, \dots, N$) the solution is:

$$\begin{aligned} z_n(\tau_{n-1} < \tau < \tau_n) &= -\varepsilon \\ &\pm \frac{[\dot{z}_0 \sin(\tau - \tau_0) + (z_0 - \varepsilon) \cos(\tau - \tau_0)] \left[\pm \sqrt{\dot{z}_0^2 + (z_0 - \varepsilon)^2} + (-1)^n (2n - 1) \varepsilon \right]}{\sqrt{\dot{z}_0^2 + (z_0 - \varepsilon)^2}} \end{aligned}$$

with:

$$\tau_n = \tau_1 + (n - 1) \pi ,$$

and:

$$z_n = z(\tau_n) = (-1)^n \left[- (2n - 1) \varepsilon \mp \sqrt{z_0^2 + (z_0 - \varepsilon)^2} \right] .$$

By imposing $z_n = 0$ one can now derive the optimum maneuver start time τ_0 to have the forced oscillator arrive at $z = 0$ with zero velocity:

$$\sin \tau_0 = - \left[\frac{2N(N-1)}{\Theta} - \frac{\Theta}{2} \right] , \quad \cos \tau_0 = -\sqrt{1 - \sin^2 \tau_0} , \quad (5.8)$$

where:

$$\Theta = \frac{\sin \Delta i}{\varepsilon} > 0 , \quad (5.9)$$

is a non-dimensional parameter quantifying the duration of the inclination change maneuver. The total number of required cycles is:

$$N = 1 + \left\lfloor \frac{\Theta}{2} \right\rfloor ,$$

where $\lfloor \cdot \rfloor$ represents the floor operator, and the total maneuver time finally yields:

$$\Delta \tau = \pi (N - 1) + \tau_1 - \tau_0 = \pi \left\lfloor \frac{\Theta}{2} \right\rfloor + \tau_1 - \tau_0 . \quad (5.10)$$

From the definition of $\tau_1 - \tau_0$ it is straightforward to check that its value is bounded and always lies between 0 and π . Consequently, for $\Theta < 2$ the required maneuver time equals $\tau_1 - \tau_0$, while for $\Theta \gg 1$ it mainly depends on Θ . It is key to highlight that Θ represents the ratio between the desired displacement and the available thrust. For a small inclination variation Δi it is possible to write Eq. (5.10) as:

$$\Delta \tau \simeq \pi \left\lfloor \frac{\Delta i}{2\varepsilon} \right\rfloor + \tau_1 - \tau_0 .$$

This behavior shows important qualitative similarities with the same-orbit rephasing and the planar radius change studied in previous chapters and in [39, 40, 42]. In both cases, simple expressions for the time of flight as a function of the ratio between the desired displacement and the available thrust parameter were found, using the linearized equations of relative motion and the Euler-Lagrange first order optimality conditions. For the phasing problem, two different regimes were found:

$$\Delta \tau \simeq \sqrt{\frac{4}{3}} \frac{|\Delta \theta|}{\varepsilon} \quad \text{for } |\Delta \theta| \gg \varepsilon ,$$

$$\Delta\tau \simeq 2\sqrt{\frac{|\Delta\theta|}{\varepsilon}} \quad \text{for } |\Delta\theta| \ll \varepsilon,$$

and also for the radius change maneuver:

$$\Delta\tau \simeq \frac{|\Delta r|}{2\varepsilon} \quad \text{for } |\Delta r| \gg \varepsilon,$$

$$\Delta\tau \simeq 2\sqrt{\frac{|\Delta r|}{\varepsilon}} \quad \text{for } |\Delta r| \ll \varepsilon.$$

The key difference between both regimes was whether the maneuver could be completed within one orbit or not. Furthermore, the corresponding structures of the control law and the evolution of the state also showed fundamental qualitative differences between regimes. In the inclination change subproblem there is no qualitative change in the control profile between both regimes, but the main contribution to the maneuver time switches from $\tau_1 - \tau_0$ in the less-than-one-orbit case to $\pi[\Theta/2]$ in the multiple-revolutions scenario.

These approximate solutions for the radius-only, phasing-only and inclination-only problems can be used to generate reasonably accurate initial estimations of a maneuver. For any given case, the profile corresponding to the most time-demanding individual subproblem should be used as a base. Since ε is the same for all the expressions, this provides a decision criterion based on the relative values of Δi , Δr and $\Delta\theta$. Furthermore, a fast estimate for the time of flight can be obtained by adding the $\Delta\tau$ estimates for each individual maneuver; the results in the following section will show that this estimate is conservative.

5.4 Numerical Solution With Direct Methods

The proposed minimum-time orbit correction between two close, non-coplanar, circular orbits is now solved numerically for the continuous, limited thrust case, using a direct transcription method [12, 92]. While the control is not restricted to constant thrust, the results show that it will naturally tend to use all the available thrust as it is common with minimum-time problems. Furthermore, since the mass of the maneuvering spacecraft has been assumed to be constant this problem is equivalent to minimizing the total impulse for the transfer.

The optimal control problem (OCP) will be solved for several values of inclination and radial displacement. While the algorithm also allows for a fixed value of θ , in order to better study the interaction between radius and inclination change the final phasing will be set free:

$$-\infty < \theta_f < \infty.$$

For simplicity the initial time τ_0 will be fixed, taking the value which optimizes the inclination-only linearized subproblem, Eqs. (5.8).

In order to transcribe the original, continuous problem into a discrete Non-Linear Programming (NLP) one the trapezoidal discretization and a uniformly spaced grid of M nodes are used. The optimization variable $\mathbf{x}$ is then formed by the values of state $\mathbf{S}$ and control $\mathbf{U}$ at each node of the grid, together with the free final time τ_f . The grid is constructed using a normalized ‘time’ variable t defined in the range $[0, 1]$, so the physical non-dimensional time can be expressed as $\tau = \tau_0 + t(\tau_f - \tau_0)$. This allows for the node positions in t to be constant while τ_f varies as part of the optimization variable. Furthermore, the objective function J simply takes the form:

$$J = \tau_f - \tau_0,$$

where τ_0 is constant.

The equations of motion, Eqs. (5.2), are expressed as a set of non-linear equality constraints called defect constraints using the Trapezoidal method, a 2-stages, 3rd order Implicit Runge-Kutta scheme. The equation for the Trapezoifal method can be seen in Appendix B, section B.3. The selection of an IRK algorithm is based on their good performance for this kind of applications [12, 37]. This yields a total of $6(M - 1)$ defect constraints, each one involving only two adjacent nodes. An additional set of non-linear equality constraints results from imposing the thrust orientation vector $\mathbf{u}^T$ to be unitary, producing M extra equations each one involving the three components of $\mathbf{u}^T$ at the corresponding node. The Jacobian for these defect and control constraints is constructed analytically, exploiting its strong sparsity. Since each defect constraint only depends on τ_f and two neighboring nodes, and each control constraint only on one, an adequate ordering of the constraints and the optimization variable allows to place all the non-zero elements of the Jacobian either in the first column or in a narrow band along the main diagonal. The constraints for the OCP are finally completed with the simple bound constraints coming from the known values of the state at τ_0 and τ_f , Eqs. (5.6) and (5.5) respectively, and the maximum nominal value for the thrust acceleration, Eq. (5.3).

The NLP subproblem is solved numerically using *Ipop*t (Interior Point OPTimizer), a third-party software package for large-scale nonlinear optimization [96]. It is distributed as open source code under the Eclipse Public License (EPL), and available from the COIN-OR initiative¹. The analytical expressions for the objective function, the non-linear constraints and their corresponding derivatives are fed to the algorithm as required inputs, while a choice is made for the Hessian. For small grids ($M < 300$) a user-supplied Hessian is calculated numerically using finite differences of the gradients [28], while the build-in L-BFGS update

¹<http://www.coin-or.org/>

Figure 5.1: Time of flight as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.01.

is preferred for larger grids. This decision is based on computational cost considerations: although the former way of computing the Hessian is more precise, requiring a smaller number of iterations to converge, the approximation given by the latter is faster to compute and factorize. From a computational cost point of view the advantages of the L-BFGS update outweigh those of the finite differences approximation as the number of nodes increases, justifying a change in strategy.

The proposed transcription is probably one of the simplest that can be considered for numerically solving an OCP using direct methods. Although this could be seen as a drawback at first, it is actually an advantage for this study. On the one hand, the focus is in the use of a new formulation for dynamics and the characterization of the maneuver, not the transcription itself. On the other hand, the good results obtained using a simple method seem to indicate that further advantages could be obtained in the future by considering more elaborated transcriptions.

The minimum-time transfer is now computed for several values of the inclination Δi , the radial displacement Δr and the thrust parameter ε . In all cases the final, or reference, orbit is defined as:

$$R = 7000 \text{ km}, \quad T_{\text{orb}} = 97.142 \text{ min}, \quad \Omega = 1.078 \cdot 10^{-3} \text{ rad/s},$$

where T_{orb} is the orbital period of said orbit. Several inclinations between 0 and 2 degrees are considered, while the radial displacement varies in the 0 to 100000 meters range (corresponding to $0 \leq \Delta r \leq 1.429 \cdot 10^{-2}$). The NLP problem is solved iteratively, starting with a fast approximation computed for a coarse grid (50 to 300 nodes) and an initial guess constructed from the analytical results previously

Figure 5.2: Time of flight as a function of the inclination change for several radial displacements and a non-dimensional thrust parameter of 0.01.

Figure 5.3: Time of flight as a function of the radial displacement for several inclination changes and a non-dimensional thrust parameter of 0.01.

Figure 5.4: Synergy coefficient as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.01.

presented. This first approximation is then used as initial guess for a finer grid, and the process continues until an accurate enough optimum is obtained.

Figures 5.1-5.4 show the results obtained for a thrust parameter of $\varepsilon = 10^{-2}$, which corresponds to a thrust acceleration of one hundred of the local gravity at the final orbit (for a 1 kg spacecraft, this amounts to 81.35 mN). While this value is high from a physical point of view, so are the required displacements in inclination and radius. In all cases, this allows to perform the maneuver within one orbital revolution. The time of flight as a function of the radial displacement and the inclination change is represented in Figures 5.1-5.3. Considering the evolution of $\Delta\tau$ with Δr first, Figure 5.3, it is observed that the slope with Δr decreases as Δi increases. Likewise, Figure 5.2 reveals that the curves of $\Delta\tau$ as a function of Δi for different values of Δr get closer as the inclination change increases. This supports the intuitive idea that the simultaneous change of orbital radius and inclination presents some synergy, with the available thrust being distributed according to which are the more efficient parts of the orbit to modify each element. To quantify this effect, a synergy coefficient σ is proposed as follows:

$$\Delta\tau = \frac{\Delta\tau_r + \Delta\tau_i}{1 + \sigma} \quad \Rightarrow \quad \sigma = \frac{\Delta\tau_r + \Delta\tau_i}{\Delta\tau} - 1, \quad (5.11)$$

where $\Delta\tau_r$ is the time needed for the radius-only ($\Delta i = 0$) maneuver and $\Delta\tau_i$ is the

Figure 5.5: Thrust profiles for two optimal orbit corrections with $\varepsilon = 0.01$ and $\Delta R = 29.7$ km; left figure corresponds to $\Delta i = 2$ deg and right figure to $\Delta i = 0$ deg.

Figure 5.6: Radial, angular and out-of-plane curvilinear coordinates and their derivatives for two optimal orbit corrections with $\varepsilon = 0.01$ and $\Delta R = 29.7$ km; left figure corresponds to $\Delta i = 2^\circ$ and right figure to $\Delta i = 0^\circ$.

Figure 5.7: Thrust control law and state evolution for an orbit correction with $\varepsilon = 0.01$, $\Delta R = 29.7$ km and $\Delta i = 0.7$ deg.

time for the inclination-only ($\Delta r = 0$) maneuver. Logic dictates that σ should lie between 0 and 1. Negative values of σ would indicate that the combined maneuver take more time than performing two dedicated maneuvers for Δr and Δi , while values of σ greater than 1 would lead to the combined maneuver being shorter than the dedicated maneuvers. Figure 5.4 shows the evolution of σ with Δi and Δr for the considered value of ε . It is straightforward to check that σ goes to zero for the inclination-only and radius only maneuvers, while it presents its maximum value for a ridge along the cases where $\Delta\tau_i$ and $\Delta\tau_r$ are equal (marked with a black line in the figures). In other words, as the relative weights of the radius and the inclination changes in the total maneuver become more similar, the synergy factor increases.

The maximum σ ridge in the previous plots also serves to separate the radius-dominated cases from the inclination-dominated ones. Figures 5.5-5.6 illustrate the differences in the trajectory and control laws for two transfers situated at different sides of said ridge. Both cases correspond to a radial displacement of 29.7 km, but while the first one is coplanar the second includes a 2 degrees inclination change. The out-of-plane thrust profiles, Figure 5.5, show that the zero-inclination maneuver devotes only a small fraction of the available thrust to inclination modifications, while the high-inclination maneuver clearly resembles the bang-bang structure predicted by the inclination-only analytical solution. The behavior is reversed for the in-plane thrust components a_ρ^T and a_θ^T , also in Figure 5.5, with a_ρ^T being dominant compared to the other two for the zero-inclination maneuver. Similar trends are observed in the evolutions of the state, represented in Figure 5.6. Finally, it is also interesting to consider a case in the maximum σ region. Figures 5.7 show the minimum-time transfer for $\Delta R = 29.7$ km and $\Delta i = 0.7$ deg. Contrary to the previous examples, the available thrust is more evenly distributed between its in-plane and out-of-plane components, although their qualitative behaviors loosely resemble their dominant-case profiles. The variations of the different elements of the state also show comparable orders of magnitude along the transfer.

Moving towards smaller, and more typical, values of the thrust parameter, Figures 5.9-5.12 show the times of flight for $\varepsilon = 0.001$ and $\varepsilon = 0.0001$ respectively. The same ranges of radial displacement and inclination change studied so far are used for the former, while smaller values of the inclination are considered for the latter. The overall behavior is similar to the $\varepsilon = 0.01$ case, although the evolution of time with Δi is smoother, without the steep increase shown by that family of maneuvers around $\Delta i = 1.1$ deg. Furthermore, most of the transfers now take more

Figure 5.8: Thrust control law and state evolution for an orbit correction with $\varepsilon = 0.001$, $\Delta R = 100.0 \text{ km}$ and $\Delta i = 1.0 \text{ deg}$.

Figure 5.9: Time of flight as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.001.

Figure 5.10: Time of flight as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.0001.

Figure 5.11: Synergy coefficient as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.001.

Figure 5.12: Synergy factor as a function of the radial displacement and the inclination change for a non-dimensional thrust parameter of 0.0001.

than one orbit to complete. The synergy coefficient, Figures 5.11-5.12, also behaves similarly, with the minimum σ rift displaced towards higher values of the radial displacement. Finally, Figures 5.8 depict a multi-revolution transfer for $\varepsilon = 0.001$, $\Delta R = 100$ km and $\Delta i = 1.0$ deg. Clearly repetitive patterns are observed for the state and control, which show the same structures previously discussed. It is interesting to highlight that the maximum values of the in-plane components of thrust concentrate around the zero-crossings of the nearly-bang-bang out-of-plane thrust profile, illustrating the thrust-distribution strategies which lead to good synergy coefficients (in this case $\sigma = 0.2609$).

5.5 Conclusion

The minimum-time continuous-thrust orbit correction between two close, non-coplanar, circular orbits has been studied, using a nonlinear relative motion formulation in curvilinear coordinates and the direct method. An analytical solution for the minimum-time change of inclination with unaltered orbit radius has been obtained in closed form for the linearized problem, providing an expression of the required maneuver time and qualitative insight about the optimal thrust profile. The ratio of the required inclination change and the available thrust has been identified as the main driver of the maneuver time, and the thrust profile shows a bang-bang structure. These results complement the similar studies performed for the phasing and radius change problems. The complete optimal control problem has been numerically solved for several values of inclination and radial displacement, validating the previous time estimates and showing their suitability as initial guesses generators. A synergistic behavior between inclination and radius change has been observed and quantified by comparing the total maneuver time with the sum of the times required for the inclination-only and radius-only maneuvers, showing that this synergy reaches its maximum when the times for the individual maneuvers are equal.

Part II

Modified Equinoctial Elements

Chapter 6

Low-Thrust End-of-Life Disposal for Galileo Constellation

6.1 Introduction

This chapter begins the part of the dissertation devoted to the indirect optimization of space trajectories using the modified equinoctial elements. The study focuses on the practical case of designing low-thrust end-of-life disposal trajectories for satellites in the Galileo constellation, leveraging the fact that future generations of Galileo satellites are expected to equip fully electric propulsion. The difficulties found during the manipulation of the state and costate equations (and their derivatives) prompted the development of a general mathematical framework for the indirect optimization of trajectories using element-based formulations (not necessarily MEEs), introduced in Chapter 7. The majority of this work was carried out during my PhD stay at the Dipartimento di Scienze e Tecnologie Aerospaziali of the Politecnico di Milano, under the supervision of professor Francesco Topputo, and has been recently presented in an international conference [44, 43].

The commercial use of space is expected to rapidly grow in the coming years, including satellite constellations for diverse applications such as navigation, communications, Earth imaging or resource mapping. Given the increasing number of objects launched into orbit, a sustained, secure and profitable exploitation of space requires end-of-life (EoL) disposal strategies to be applied. In the case of Medium Earth Orbit (MEO) regions two options are available: 1) move the satellites into graveyard orbits, 2) insert them into Earth re-entry paths. The latter shall be preferred as it physically removes the satellite, however it is generally associated with high Δv and re-entry times (~ 100 m/s for a re-entry in 100 years). With current chemical propulsion systems this solution is not practical, but the scenario will change when fully-electric platforms will be utilized. This can be the case of

the next generation satellites in the Galileo constellation [66, 32].

The EoL disposal of satellites in MEO has been extensively studied by several authors in the last decades. References [25, 54, 88, 82, 29, 73, 81] deal with the stability of disposal orbits for GNSS, and the associated long-term risks for operational satellites. The design of low-cost EoL re-entry strategies is studied in [53, 87, 3, 2], leveraging perturbation forces naturally present in the MEO region. However, Armellin et al. recently noted (see [66, 6]) that most of these works focus on analyzing the evolution of the orbits for different initial conditions, without actually taking into account the maneuver needed to give the spacecraft said initial conditions. They propose an optimization problem to design single-impulse disposal maneuvers leading to the re-entry of the satellite within 100 years thanks to lunisolar perturbation effects. Several re-entry time- Δv Pareto fronts for the different operational planes of Galileo are obtained, with minimum values of $\Delta v \sim 100 \text{ m/s}$.

The low-thrust disposal maneuver proposed in this dissertation aims to move the satellite from its original orbit into a new one leading to its re-entry, but the particular positions at which the transfer starts and ends need not to be fixed. Therefore, this optimal control problem is more naturally expressed using an orbital elements-based formulation, such as the modified equinoctial elements [24]. However, this choice introduces additional complications compared to the classic Cartesian coordinates when solving the problem using the indirect method. The well-known result of the primer vector is no longer valid, and the straightforward analytical derivation of the variational equations, required to efficiently solve the associated two-point boundary value problem, becomes too cumbersome to be practical (even with symbolic manipulators). A careful treatment of the state and costate equations is then needed to derive variational equations and to develop an efficient and robust solver.

In this chapter, the design of low thrust EoL disposal trajectories for satellites in the Galileo constellation is studied, using the modified equinoctial elements to model dynamics and the indirect method to solve the associated optimal control problem. A set of previously computed orbits that exploit the lunisolar perturbation forces in the MEO region to naturally achieve re-entry times within 100 years [66, 6] are considered as candidate arrival conditions, while the departure ones are given by the actual orbits of the Galileo constellation. The use of the modified equinoctial elements allows us to express the boundary conditions in a simple way, where all the initial and final elements are fixed except for the true longitude (determining the particular points in the initial and final orbits where the transfer maneuver starts and ends respectively). This easier treatment of the boundary conditions comes at the cost of more complex expressions for the two-point boundary value problem (TPBVP) arising from the first order optimality conditions in

the indirect method. A careful study of these equations and their behavior is then performed to implement an efficient and robust solver for the TPBVP following the structure in [101], including the integration of the analytically-derived variational equations to obtain the State Transition Matrix and the accurate detection and treatment of thruster-switching events. Finally, a representative set of test cases for different initial and final orbits is studied using this solver.

6.2 Problem Statement

6.2.1 Galileo disposal

Galileo is the Global Navigation Satellite System currently being developed by the European Union and the European Space Agency [33]. Once completed, it will consist of 24 satellites plus 6 spares, located in three MEO planes at an altitude of 23,222 km and an orbital inclination of 56 deg.

Recent works by Armellin et al. [6, 66] have studied the possibility of leveraging lunisolar perturbation effects to achieve the EoL disposal of Galileo satellites through a single impulsive maneuver placing the satellite in a new orbit that leads to its natural re-entry within a 100 years timespan. Although effective, this strategy has the drawback of requiring Δv levels impractically high for the impulsive thrusters currently used in Galileo. However, this limitation may be removed in the future generations of Galileo satellites, which are expected to feature electric thrusters [32] such as the T6 from QinetiQ [90, 51, 26].

This work extends the previous Galileo disposal studies by replacing the impulsive maneuver with a minimum-fuel low thrust transfer between the operational and disposal orbits, for values of $T_{\max}$ and I_{sp} typical of the T6 ion engine [51, 26]. Note that only the departure and arrival orbits need to be imposed, not the actual position of the spacecraft on them. For simplicity, the departure position and mass will be considered fixed from now on, whereas their final values are set free. Because the duration of the maneuver is very short compared to the re-entry time (tens of days versus a hundred years), a model including only Earth's central attraction and the thrust acceleration is used.

6.2.2 Equations of motion

The dynamics are formulated using the modified equinoctial elements (MEEs) [24, 20, 98, 97]. Compared to Cartesian coordinates, these have the key advantage of allowing to express the orbit-to-orbit boundary conditions as fixed values for some elements of the initial and final states, rather than as nonlinear functions of them. For a direct orbit, the MEEs can be related with the classical orbital elements as

follows:

$$\begin{aligned}
 p &= a (1 - e^2) , \\
 e_x &= e \cos (\omega + \Omega) , \\
 e_y &= e \sin (\omega + \Omega) , \\
 h_x &= \tan (i/2) \cos (\Omega) , \\
 h_y &= \tan (i/2) \sin (\Omega) , \\
 L &= \omega + \Omega + \theta ,
 \end{aligned} \tag{6.1}$$

where a is the semi-major axis, e is the eccentricity, ω is the argument of perigee, Ω is the right ascension of the ascending node, i is the orbital inclination, θ is the true anomaly, p is the semilatus rectum, and L is the true longitude. It is straightforward to check that p , e_x , e_y , h_x and h_y define the orbit, whereas L indicates the position of the spacecraft inside it. By introducing the state vector:

$$\mathbf{s} = [p \ e_x \ e_y \ h_x \ h_y \ L]^\top ,$$

as well as:

$$\mathbf{x} = [\mathbf{s} \ m] ,$$

and the control vector:

$$\mathbf{u} = [f \ \boldsymbol{\alpha}]^\top ,$$

where m is the mass of the spacecraft, $f \in [0, 1]$ is the thrust throttling parameter, and $\boldsymbol{\alpha}$ the thrust pointing unitary vector, the equations of motion, including the mass equation, take the form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}, t) := \begin{bmatrix} \mathbf{B} f \frac{T_{\max}}{m} \boldsymbol{\alpha} + \mathbf{A} \\ -\frac{T_{\max}}{c} f \end{bmatrix} , \tag{6.2}$$

where:

$$c = I_{\text{sp}} g_0 ,$$

$$\begin{aligned}
 \mathbf{A} &= [0 \ 0 \ 0 \ 0 \ 0 \ W^2 \sqrt{\mu/p^3}]^\top , \\
 \mathbf{B} &= \sqrt{\frac{p}{\mu}} \begin{bmatrix} 0 & \frac{2p}{W} & 0 \\ \sin L & \frac{1}{W} [(W+1) \cos L + e_x] & -\frac{e_y}{W} K \\ -\cos L & \frac{1}{W} [(W+1) \sin L + e_y] & \frac{e_x}{W} K \\ 0 & 0 & \frac{s^2 \cos L}{2W} \\ 0 & 0 & \frac{s^2 \sin L}{2W} \\ 0 & 0 & \frac{2W}{K} \end{bmatrix} ,
 \end{aligned}$$

with:

$$\begin{aligned} W &= 1 + e_x \cos L + e_y \sin L, \\ K &= h_x \sin L - h_y \cos L, \\ s^2 &= 1 + h_x^2 + h_y^2, \end{aligned}$$

being $T_{\max}$ the maximum available thrust, I_{sp} the specific impulse, g_0 the gravity acceleration at sea level, and μ Earth's gravitational parameter.

6.2.3 Indirect optimization method

Indirect optimization methods are based on deriving the first order optimality conditions for the problem using the calculus of variations. This leads to a TPBVP in the state and its associated costates, which normally has to be solved using numerical methods.

Although the aim of the study is to design minimum fuel transfers, this task cannot be tackled directly. First of all, a minimum-time solution is needed to determine the feasible region for the time of flight. The objective function for this minimum-time problem takes the form:

$$\hat{J} = \int_{t_i}^{t_f} 1 \, dt = \int_{t_i}^{t_f} \hat{\mathcal{L}} \, dt, \quad (6.3)$$

where subscripts i and f denote, respectively, initial and final point. Secondly, the bang-bang structure of minimum-fuel solutions introduces discontinuities that severely affect the performance of numerical solvers. This issue can be addressed by following a homotopic approach [10, 101]. To this end, an objective function J dependent on an homotopy parameter ε is defined as follows:

$$J = \frac{T_{\max}}{c} \int_{t_i}^{t_f} [f - \varepsilon f(1 - f)] \, dt = \int_{t_i}^{t_f} \mathcal{L} \, dt, \quad \varepsilon \in [0, 1]. \quad (6.4)$$

Note that the two limit values 0 and 1 of the homotopy parameter correspond, respectively, to the minimum-fuel and minimum-energy. Then, the homotopic approach would consist on solving the (easier) minimum-energy problem first, and then performing a continuation in ε until the minimum-fuel solution is reached.

The Hamiltonian for the minimum-energy/fuel problems is (see [21]):

$$H = \boldsymbol{\lambda} \cdot \mathbf{F} + \mathcal{L}, \quad (6.5)$$

where $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_s \, \lambda_m]$ is the vector of costates, with $\boldsymbol{\lambda}_s = [\lambda_p \, \lambda_{e_x} \, \lambda_{e_y} \, \lambda_{h_x} \, \lambda_{h_y} \, \lambda_L]$. Note that the Hamiltonian for the minimum-time problem will have the same structure, simply replacing the Lagrangian $\mathcal{L}$ with $\hat{\mathcal{L}}$. The differential equations governing

the evolution of the costates can be obtained by imposing the first order optimality conditions derived from the calculus of variations [21], leading to:

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}. \quad (6.6)$$

The boundary conditions for the costate depend on the boundary conditions imposed to the state. In particular, the initial (or final) value of a costate component will be zero if the corresponding initial (or final) state is free, and an unknown parameter otherwise [21]. Since only the final true longitude and mass are free, the costates associated to L and m will be zero at t_f :

$$\lambda_L(t_f) = 0, \quad \lambda_m(t_f) = 0,$$

whereas the rest will be unknowns parameters of the TPBVP. Note that, same as with the state, the use of MEEs allows to express the boundary conditions in a simple way.

Control variables f and $\boldsymbol{\alpha}$ can be expressed as functions of the state and costate by applying the Pontryagin Maximum Principle [76], which states that the optimal control for a given optimal trajectory is the one that leads to an extreme value (in this case a minimum) of H over the set of admissible controls. Gathering the terms involving control variables in Eq. (6.5) one reaches:

$$H = \boldsymbol{\lambda}_s \cdot \mathbf{A} + f \frac{T_{\max}}{c} \left[\frac{c}{m} \boldsymbol{\lambda}_s \mathbf{B} \boldsymbol{\alpha} - \lambda_m + 1 - \varepsilon (1 - f) \right]. \quad (6.7)$$

Because $fT_{\max}/c$ is always semi-positive, minimizing the Hamiltonian with respect to $\boldsymbol{\alpha}$ amounts to choosing its value so that the first term inside the brackets is as small as possible. This leads to:

$$\boldsymbol{\alpha}^* = -\frac{\mathbf{B}^\top \cdot \boldsymbol{\lambda}_s}{\|\mathbf{B}^\top \cdot \boldsymbol{\lambda}_s\|}. \quad (6.8)$$

This expression for the thrust orientation is notably more complex than the primer vector in Cartesian coordinates [78], and constitutes one of the main drawbacks of using the MEEs. Substituting $\boldsymbol{\alpha}^*$ back into Eq. (6.7) yields:

$$H = \boldsymbol{\lambda}_s \cdot \mathbf{A} + f \frac{T_{\max}}{c} [S - \varepsilon + \varepsilon f], \quad (6.9)$$

where a switching function S has been introduced as:

$$S = -\frac{c}{m} \frac{\boldsymbol{\lambda}_s^\top \mathbf{B} \mathbf{B}^\top \boldsymbol{\lambda}_s}{\|\mathbf{B}^\top \cdot \boldsymbol{\lambda}_s\|} - \lambda_m + 1. \quad (6.10)$$

It is now straightforward to analytically find the minimum of H with respect to f by deriving and equating to zero, which leads to $f = (\varepsilon - S)/2\varepsilon$. However, because f is limited to the range $[0, 1]$, the optimal throttling takes the form of a piecewise function:

$$f^* = \begin{cases} 0 & \text{for } S > \varepsilon \\ (\varepsilon - S)/2\varepsilon & \text{for } -\varepsilon \leq S \leq \varepsilon \\ 1 & \text{for } S < -\varepsilon \end{cases} \quad (6.11)$$

where the value of the switching function allows determining the operation mode for the thruster: off, intermediate, or full on. Note that in the minimum-fuel case ε is zero and the intermediate region vanishes, so leading to a bang-bang profile.

Regarding the minimum-time problem, the expression for $\boldsymbol{\alpha}^*$ in Eq. (6.8) still holds, and the Hamiltonian takes the form:

$$\hat{H} = \boldsymbol{\lambda}_s \cdot \mathbf{A} + f \frac{T_{\max}}{c} \hat{S} + 1,$$

with switching function:

$$\hat{S} = -\frac{c}{m} \frac{\boldsymbol{\lambda}_s^\top \mathbf{B} \mathbf{B}^\top \boldsymbol{\lambda}_s}{\|\mathbf{B}^\top \cdot \boldsymbol{\lambda}_s\|} - \lambda_m. \quad (6.12)$$

Therefore, throttling factor should be 1 (full on) for $\hat{S} < 0$, and 0 (off) for $\hat{S} > 0$. Moreover, it is possible to check from Eq. (6.6) that $\lambda_m \geq 0$ for the minimum-time problem with free final mass (because $\lambda_m = -\partial_m \hat{H} < 0$ and the free final mass condition implies $\lambda_m(t_f) = 0$), leading to $\hat{S} < 0$ for $t \in [t_i, t_f]$. As a consequence, f will always be 1 for minimum-time problems. Additionally, the extra degree of freedom in the minimum-time problem (t_f is unknown) requires the introduction of a transversality condition setting $\hat{H}(t_f) = 0$ (see [21]).

Gathering all the previous developments, the TPBVP derived from the first order optimality conditions for the minimum-energy/fuel problems can be stated as follows: to find the $\boldsymbol{\lambda}_i$ that satisfies the shooting function:

$$\mathbf{Z}(\boldsymbol{\lambda}_i) = \begin{bmatrix} p(t_f) - p_f \\ e_x(t_f) - e_{xf} \\ e_y(t_f) - e_{yf} \\ h_x(t_f) - h_{xf} \\ h_y(t_f) - h_{yf} \\ \lambda_L(t_f) \\ \lambda_m(t_f) \end{bmatrix} = \mathbf{0}, \quad (6.13)$$

where the mappings $\mathbf{x}(t_f; \mathbf{x}_i, \boldsymbol{\lambda}_i, t_i)$ and $\boldsymbol{\lambda}(t_f; \mathbf{x}_i, \boldsymbol{\lambda}_i, t_i)$ come from the integration of Eqs. (6.2) and (6.6) with the controls $\mathbf{u}^* = [f^* \boldsymbol{\alpha}^*]$ derived from the PMP in Eqs. (6.8) and (6.11). The minimum-time problem presents the same structure, introducing an additional unknown, t_f , and associated algebraic condition, $\hat{H}(t_f) = 0$, and setting $f^* = 1$ for all t .

6.2.4 Algorithmic implementation

The previous TPBVP is to be solved numerically using an adequate scheme. In this work a classic multiple shooting algorithm is employed, with a third-party root finder (Matlab's *fsolve* and its Trust-Region-Dogleg algorithm) and a custom Runge-Kutta Fehlberg 7(8) propagator [34, 35]. The proposed algorithm was implemented as an extension to the indirect optimization software being developed by Prof. Topputo's group at the Politecnico di Milano, being the new contributions focused on (but not limited to) the improvement of the event detection algorithm and the treatment of non-Cartesian formulations (in particular MEEs). These two contributions are presented in detail in the following.

One key aspect regarding the propagator is the need to accurately locate the thrust switching events, that is, when the thruster should change its operation mode according to Eq. (6.11). Event detection is a highly desirable characteristic for any numerical propagator, normally associated to the capability of producing a dense output [50]. Although the latter is not strictly necessary, it dramatically decreases the computational cost by preventing the need to take additional steps inside the zero-finding algorithm. Regretfully, the classic RKF 7(8) scheme, while very efficient for orbital propagation, lacks such a dense output [31, 89]. In general, all high order RK algorithms either lack dense output or have to introduce a large number of additional stages to support it. Since each RKF 7(8) step has a non negligible computational cost, involving 13 additional evaluations of the functions, it is important to define an efficient event detection algorithm requiring as little additional steps as possible. To do this, we first search for any change in the operation mode associated to the value of S at both ends of a successful integration step. If a change is detected it means a switching event takes place at an intermediate point inside the step, and the zero t_{switch} for the switching function is sought for using Newton's method. This choice of algorithm is based on its quadratic convergence rate, and the relatively low cost of computing the derivative of S (requiring just one additional evaluation of the differential equations). In order to fully exploit the high convergence rate of Newton's method, a good enough initial guess is needed for t_{switch} . This initial guess is obtained using the values of S and its first derivative with respect to t at both endpoints of the step to construct a cubic Hermite interpolant, which is then solved analytically. Note that the information required to construct the interpolant is already provided by the RKF 7(8) step except for the derivative of S at the end of the interval, so the added computational cost is small. The initial guess obtained from the interpolant is normally very close to the actual root, and the Newton's method takes just 2-3 iterations in most cases. Because the convergence of Newton's method inside the interval cannot be guaranteed in general, a fall-back algorithm based on a combination of modified *regula falsi* (Anderson-Björk version) and bisection is

Algorithm 1 Thrust switching detection algorithm

```

1: Take integration step from  $t_k$  to  $t_{k+1}$ 
2: if operation modes at  $t_k$  and  $t_{k+1}$  are different then
3:   Obtain analytical guess for  $t_{\text{switch}}$  with a cubic Hermite interpolant (from
      $S$  and  $\dot{S}$  at  $t_k$  and  $t_{k+1}$ )
4:   Search for  $t_{\text{switch}}$  using Newton's method and the previous guess (max. 10
     iterations)
5:   if fails to converge in  $]t_k, t_{k+1}[$  then
6:     Search for  $t_{\text{switch}}$  using regula falsi (Anderson-Björk)/bisection fallback
7:   end if
8:    $t_{k+1} := t_{\text{switch}}$ 
9:   Retake integration step
10: end if

```

also implemented. Once the switching time has been determined, one RKF fixed step is taken up to t_{switch} , and the propagation continues for the new value of f^* . This procedure is summarized in Algorithm 1.

The Trust-Region-Dogleg algorithm employed to solve the TPBVP requires the Jacobian of the shooting function (or a suitable approximation). Although it is possible to approximate it numerically using finite differences and BFGS updates, a more efficient and robust algorithm can be obtained by constructing it analytically from the State Transition Matrix (STM) [101]. The STM Φ maps small variations in the initial conditions $[\delta \mathbf{x}_i \ \delta \boldsymbol{\lambda}_i]$ to variations $[\delta \mathbf{x} \ \delta \boldsymbol{\lambda}]$ at a given time t in the form:

$$\begin{bmatrix} \delta \mathbf{x} \\ \delta \boldsymbol{\lambda} \end{bmatrix} = \Phi(t_i, t) \begin{bmatrix} \delta \mathbf{x}_i \\ \delta \boldsymbol{\lambda}_i \end{bmatrix},$$

and it can be calculated through the integration of the variational equation:

$$\dot{\Phi}(t_i, t) = \mathbf{J} \Phi(t_i, t), \quad \Phi(t_i, t_i) = \mathbf{I}_{14 \times 14},$$

where $\mathbf{J}$ is the Jacobian for the ODE system describing the evolution of state and costate, Eqs. (6.2) and (6.6). Because the STM is a 14x14 matrix, 196 additional differential equations have to be added, for a total of 210 equations. However, the straightforward derivation of the variational equation for this problem is not an easy task. Even though the procedure is conceptually simple, the resulting expressions turn out to be too cumbersome to handle even with the use of symbolic manipulators. This issue can be circumvented by taking advantage of the structure of the equations, obtaining general expressions for the variational equations in terms of $\mathbf{B}\mathbf{B}^\top$, $\mathbf{A}$ and their derivatives up to order 2 [43]. The analytical derivatives for $\mathbf{B}\mathbf{B}^\top$ and $\mathbf{A}$ are then easy to determine with the help of a symbolic manipulator.

Table 6.1: Environmental, thruster and spacecraft parameters.

Parameter	Value
μ [km ³ /s ²]	398600.433
g_0 [m/s ²]	9.807
I_{sp} [s]	4000
m_i [kg]	675

The derivation of such a mathematical framework is dealt with in great detail in Chapter 7, for a generic formulation of the dynamics. One last aspect to consider regarding the STM is how to treat the discontinuities that appear in bang-bang control profiles. In this work, the strategy proposed by Russell in [85] to obtain the composite STM for a trajectory with several bang-bang points is used.

Finally, the lack of information about the solution complicates the task of finding an initial guess for λ_i falling inside the convergence region of the solver. This can be addressed by solving the problem for a higher value of $T_{\max}$ first, and then performing a continuation in thrust until the desired nominal thrust is reached. The higher control authority increases the convergence region, allowing to find a valid initial guess by testing random values for λ_i .

Now that a robust and efficient solver for the TPBVP has been defined, the minimum-fuel problem for given $T_{\max}$ and boundary conditions can be solved according to the following scheme

1. The minimum-time problem is solved performing a continuation in thrust, starting from a high enough value to ease the initial convergence of the TPBVP.
2. The minimum-energy problem is solved performing a continuation in thrust, for a final time taken longer than the time of flight for the minimum-time problem.
3. The minimum-fuel problem is solved performing a continuation in ε (starting from the minimum-energy solution), for the desired values of $T_{\max}$ and time of flight.

Under some circumstances, the continuation process may stall. In those cases, the freedom in the choice of θ_i is leveraged to restart it by updating the starting point within the departure orbit.

Table 6.2: Departure orbital elements for all test cases.

Orb. Elems.	Departure
a [km]	29598.896
e [—]	0.000173
i [deg]	54.982
Ω [deg]	203.549
ω [deg]	272.857
θ [deg]	166.269

Table 6.3: Arrival orbital elements for each test case.

Orb. Elems.	Case 1	Case 2	Case 3
a [km]	31862.568	33006.338	33249.803
e [—]	0.071201	0.103189	0.109737
i [deg]	54.993	54.986	55.267
Ω [deg]	203.568	203.552	203.585
ω [deg]	256.033	39.376	30.013
θ [deg]	—	—	—
Δv_{imp} [m/s]	128.152	185.147	197.477
$t_{\text{re-entry}}$ [years]	98.403	76.003	68.408

6.3 Test Cases

The proposed Galileo low-thrust EoL disposal is now studied for three different test cases taken from the previous works by Armellin et al. [6]. The departure orbital elements, common to all test cases, are compiled in Table 6.2, and the arrival orbital elements are shown in Table 6.3. The latter table also includes the re-entry time for each disposal orbit, as well as the Δv needed to reach it with an impulsive maneuver. Initial spacecraft mass is set to 675 kg, to be consistent with the dry mass of about 670 kg assumed by Armellin et al. [6] when obtaining the disposal orbits. Two different thrust levels are considered: 150 mN and 230 mN. The former is close to the 145 mN qualified by ESA for the BepiColombo [26], while the latter corresponds to the maximum achievable thrust according to QnetiQ [51]. In both cases, a conservative value for I_{sp} of 4000 s is used. All the other relevant parameters are summarized in Table 6.1.

Figure 6.1 shows the minimum-fuel maneuver m_f and Δv for the three test cases at a maximum thrust level of 230 mN and increasing times of flight. As

Figure 6.1: Minimum-fuel m_f and Δv for the three test cases, for $T_{\max} = 230$ mN and increasing times of flight Δt . Top figure corresponds to the physical values, whereas bottom figure is normalized with the minimum-time solution.

Figure 6.2: Minimum-fuel m_f and Δv for the three test cases, for $T_{\max} = 150$ mN and increasing times of flight Δt . Top figure corresponds to the physical values, whereas bottom figure is normalized with the minimum-time solution.

Figure 6.3: Minimum-fuel trajectory for test case 1 with $T_{\max} = 150 \text{ mN}$ and $t_f = 19.009 \text{ days}$ (1.5 times the t_f for the minimum-time transfer), in inertial reference frame. Coasting arcs are represented in gray, and propelled arcs in black.

expected, the values for Δv are appreciably higher than for the impulsive maneuver (especially for times of flight close to the minimum-time solution), but the high I_{sp} of electric thrusters leads to small fuel consumptions. It is observed that fuel consumption, corresponding to $m_f - m_i$, is less than 5 kg in most cases for times of flight up to two half months, showing the practical feasibility of this EoL disposal approach. Additionally, increasing the time of flight with respect of the minimum-time maneuver up to a factor of 3.5 can provide reductions in the required Δv of around 45%. Considering even longer times of flight does not provide noteworthy reduction in Δv , as the curves approach a horizontal asymptotic behavior.

Similar results are obtained for a maximum thrust of 150 mN, as shown in Figure 6.2. Times of flight experiment a notable increase, whereas the ranges for Δv and m_f remain mostly unchanged. An asymptotic behavior is again observed for increasing values of the time of flight.

Figure 6.4: Minimum-fuel throttling profile and mass for test case 1 with $T_{\max} = 150$ mN and $t_f = 19.009$ days (1.5 times the t_f for the minimum-time transfer).

Figure 6.5: Minimum-fuel throttling profile and mass for test case 2 with $T_{\max} = 150$ mN and $t_f = 27.309$ days (1.5 times the t_f for the minimum-time transfer).

The trajectory corresponding to a case 1 minimum-fuel transfer with $T_{\max} = 150$ mN and $t_f = 19.009$ days (1.5 times the t_f for the minimum-time transfer) is represented in Figure 6.3. It is straightforward to check that each revolution consists on one or two propelled arcs taking place around the centers of the orbit, with coasting arc in between. This is because the imposed orbital maneuver changes the argument of perigee, not the orbit geometry. Additional information on the control strategy is provided by Figures 6.4-6.6, where the throttling profiles and mass curves for a minimum-fuel transfer with $T_{\max} = 150$ mN and a time of flight 1.5 times the t_f for the minimum-time transfer are represented for the three test cases. It is checked that the second propelled arc only appears during the last orbits, and its width grows with time. As expected, mass only decreases during

Figure 6.6: Minimum-fuel throttling profile and mass for test case 3 with $T_{\max} = 150$ mN and $t_f = 29.111$ days (1.5 times the t_f for the minimum-time transfer).

the propelled parts of the maneuver.

Finally, Figures 6.7-6.9 depict the evolution of the classical orbital elements for those same three minimum-fuel transfers. Figures 6.7 correspond to the case 1 minimum fuel transfer with $T_{\max} = 150$ mN and $t_f = 19.009$ days. Most of the elements evolve steadily during the propelled arcs (one or two per orbital revolution), with the notable exception of the argument of perigee ω . This angle experiments large variations during each arc, but the net difference between the values at the beginning and the end of the arc are small. Both the amplitude of the oscillations during the arc and the final variation decrease with the time of flight. It is also noteworthy that the slope for the secular evolution of Ω increases slightly coinciding with the inclusion of the second propelled arc per revolution. The results for case 2, with $T_{\max} = 150$ mN and $t_f = 27.309$ days, are represented in Figures 6.8. The evolution of the orbital elements is very similar to case 1, with the slope change due to the second propelled arc being slightly more evident (now it can also be observed for a and i). Conversely, case 3 ($T_{\max} = 150$ mN and $t_f = 29.111$ days) experiments a strong change in the behavior of Ω during the last part of the maneuver, as shown in Figures 6.9.

Figure 6.7: Minimum-fuel orbital elements for test case 1 with $T_{\max} = 150$ mN and $t_f = 19.009$ days (1.5 times the t_f for the minimum-time transfer).

Figure 6.8: Minimum-fuel orbital elements for test case 2 with $T_{\max} = 150$ mN and $t_f = 27.309$ days (1.5 times the t_f for the minimum-time transfer).

Figure 6.9: Minimum-fuel orbital elements for test case 3 with $T_{\max} = 150$ mN and $t_f = 29.111$ days (1.5 times the t_f for the minimum-time transfer).

6.4 Conclusion

The EoL disposal of satellites in the Galileo constellation using low thrust propulsion has been studied, leveraging previously computed orbits leading to the natural re-entry of the satellite due to lunisolar perturbation effects within a 100 years timespan. The low thrust transfers between the operational and disposal orbits have been designed using indirect optimization methods and a MEE formulation for the dynamics. A special focus has been placed in applying different techniques to increase the robustness and efficiency of the indirect optimization algorithm, such as continuation, analytical derivation of the variational equations to propagate the STM, and precise and fast determination of thruster-switching events during the propagation. The use of MEE has allowed to express the orbit-to-orbit boundary conditions in a simple way, at the cost of more complex state and costate equations.

Numerical results for several representative test cases show fuel requirements of less than 5 kg for transfer maneuvers of up to one and a half months, proving the practical interest of this approach for the EoL disposal of future generations of Galileo satellites featuring electric propulsion. Furthermore, this strategy could be extended in future works to other constellations in the MEO region.

Chapter 7

Unified Formulation for Element-Based Indirect Optimization

7.1 Introduction

One of the greatest challenges when designing optimal trajectories using indirect methods is the treatment of the costate. These variables, introduced by the TP-BVP derived from the first-order necessary optimality conditions and linked to their corresponding state variables, lack a physical interpretation in general. This complicates the task of finding an initial guess for the solution of the TPBVP, required if it is to be solved using iterative numerical algorithms. Furthermore, the introduction of the costate into the state equations through the control term increases their complexity and makes it more difficult to derive variational equations for the accurate integration of the state transition matrix, essential to increase the performance of numerical shooting methods. When dynamics are formulated using Cartesian coordinates, these issues are alleviated by the well known result of Lawden's primer vector theory [62]. By applying Pontryagin Maximum Principle (PMP), it is possible to verify that the direction of thrust is given by a vector whose components are the costate variables associated to velocity. This leads to a fairly simple expression of the control involving only the costate, while also providing some physical sense for at least part of the costate.

However, there are practical cases where formulations for the dynamics other than Cartesian may be more convenient. For instance, using orbital elements allows one to express constraints in a natural way as fixed initial and final values of the state when moving from one orbit to another without specifying the specific departure and arrival points. This was the case for the Galileo disposal study

presented in Chapter 6, where MEEs were used for the dynamics. Unfortunately, when the system dynamics are parametrized through orbital elements the classical primer vector result is not preserved and the optimal thrust orientation becomes a function of both state and costate. This fact greatly complicates the computation of partial derivatives and the derivation of an STM in analytical form as required in order to improve the convergence of the associated TPBVP [101, 44].

The systematic treatment of indirect optimization problems using element-based formulations is scarcely covered by the existing literature, normally focused on the Cartesian coordinates. Two decades ago Kechichian published several papers on the formulation of OCPs using equinoctial elements [58, 60, 59, 57], providing the complete state and costate equations as well as some derivatives. The resulting expressions, however, were cumbersome to handle and restricted to the equinoctial elements only. A more recent work by Campagnola and Russell [22] tackles the problem from a more general point of view, considering a generic formulation of the Gauss equations. They provide formulation-independent expressions for the state and costate equations, as well as their derivatives, using tensor notation and logarithmic differentiation, and identify a generalization of the primer vector given by state-dependent feed-back laws with the costates as weights.

In this chapter, it is shown that by carefully manipulating the equations derived from the first order optimality conditions it is possible to write them in a compact and more convenient way, partially isolating the different contributions of state and costate and leading to a unified, efficient formulation of the indirect equations for trajectory optimization that can be applied to any set of generalized orbital elements or orbital parametrization. The proposed approach has some points in common with the work of Campagnola and Russell (see [22]), but also key differences. This effort was initially motivated by the shortcomings found while working with the variational equation in MEEs for the Galileo disposal study in Chapter 6, but the resulting expressions are not limited to any particular formulation. In the first section the necessary optimality conditions of a generic optimization problem are rewritten to highlight the existence of a quadratic form of the costate whose partial derivatives directly provide the contribution of the costate to all the OCP equations (including the variational ones). The computation of the partial derivatives is simplified thanks to the separation of the state and costate contributions in the quadratic form. In the subsequent section we proceed to the derivation of the state transition matrix terms applicable to a generic formulation. Finally, we derive the partial derivatives of the Lagrangian and thrust throttling factor corresponding to minimum-time, minimum-energy and minimum-fuel problems.

The mathematical framework proposed in this chapter has been used as basis for the low-thrust end-of-life Galileo disposal study in Chapter 6, as well as in other recent works by Gonzalo et al. [44]. A first computational implementation

restricted to MEEs has been developed by the PhD candidate, and incorporated to the indirect optimization software by Prof. Topputo's group at the Politecnico di Milano.

7.2 Indirect Optimal Control Problem Formulation

A generic formulation for the orbital motion of a spacecraft of mass m with an n -dimensional state vector $\mathbf{s}$ and independent variable t (from now on considered a "generalized time") can be expressed in the form:

$$\frac{d\mathbf{s}}{dt} = \mathbf{F}(\mathbf{s}, m, \mathbf{u}, t) = \mathbf{B}(\mathbf{s}, t) \mathbf{p}(\mathbf{s}, m, \mathbf{u}, t) + \mathbf{A}(\mathbf{s}, t), \quad (7.1)$$

together with the mass equation:

$$\frac{dm}{dt} = F_m(\mathbf{u}, t) = -\frac{T_{\max}}{c} f, \quad (7.2)$$

where $\mathbf{u} = [f \boldsymbol{\alpha}]$ is the control vector formed by the thruster throttling parameter f and pointing unity vector $\boldsymbol{\alpha}$, $T_{\max}$ and c are the thruster maximum thrust and exhaust velocity, $\mathbf{p}$ is the vector of perturbing accelerations, $\mathbf{A}$ is an $n \times 1$ vector and $\mathbf{B}$ a $n \times 3$ matrix. Without loss of generality, it is possible to assume that the thrust acceleration $\mathbf{f}$ is the only perturbing action:

$$\mathbf{p} = \mathbf{f} = f \frac{T_{\max}}{m} \boldsymbol{\alpha},$$

leading to:

$$\mathbf{F} = f \frac{T_{\max}}{c} \mathbf{B} \cdot \boldsymbol{\alpha} + \mathbf{A}. \quad (7.3)$$

Note that this expression holds even if there are additional perturbing accelerations, simply by redefining $\mathbf{A}$ as $\mathbf{A} + \mathbf{B}(\mathbf{p} - \mathbf{f})$.

In general, an OCP can be defined as minimizing (or maximizing) an objective function of the form (see [21]):

$$J = \phi(\mathbf{x}_f, t_f) + \int_{t_i}^{t_f} \mathcal{L}(\mathbf{x}, \mathbf{u}, t) dt, \quad (7.4)$$

where $\mathbf{x} = [\mathbf{s} \ m]$, for a given set of constraints of the state and control imposed at the terminal points t_i and t_f :

$$\begin{aligned} \psi_{il} &\leq \psi_i(\mathbf{x}_i, \mathbf{u}_i, t_i) \leq \psi_{iu}, \\ \psi_{fl} &\leq \psi_f(\mathbf{x}_f, \mathbf{u}_f, t_f) \leq \psi_{fu}, \end{aligned}$$

and along the trajectory (also known as path constraints):

$$\begin{aligned}\mathbf{g}_l &\leq \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)) \leq \mathbf{g}_u, \\ \mathbf{x}_l &\leq \mathbf{x}(t) \leq \mathbf{x}_u, \\ \mathbf{u}_l &\leq \mathbf{u}(t) \leq \mathbf{u}_u.\end{aligned}$$

Indirect optimization methods make use of the calculus of variations to derive the first order optimality conditions corresponding to this OCP [21], leading to a TPBVP which is normally solved with numerical algorithms such as shooting methods. The number of equations (both differential and algebraic) and unknowns will depend on the number and nature of the constraints, but in all cases a set of costate variables $\boldsymbol{\lambda}$ and λ_m associated to $\mathbf{s}$ and m are introduced (along with the differential equations describing their evolution). A detailed treatment of the different types of constraints falls beyond the scope of this work. For simplicity, in the following it is assumed that there are no path constraints, and that the terminal constraints can be expressed as fixed or free values of $\mathbf{x}$.

The Hamiltonian associated to the OCP can be written as:

$$H = \boldsymbol{\lambda}^\top \cdot \mathbf{F} + \lambda_m F_m + \mathcal{L}, \quad (7.5)$$

where the evolution of the costates is given by the Euler-Lagrange equations derived from the first order optimality conditions using the calculus of variations [21]:

$$\dot{\boldsymbol{\lambda}} = \mathbf{G} = -\frac{\partial H}{\partial \mathbf{s}}, \quad (7.6)$$

$$\dot{\lambda}_m = G_m = -\frac{\partial H}{\partial m}. \quad (7.7)$$

An expression for the optimal control $\mathbf{u}^* = [f^* \boldsymbol{\alpha}^*]$ as a function of $\mathbf{x}$, $\boldsymbol{\lambda}$, λ_m and t can be obtained by applying the PMP [76], which states that the optimal control for a given optimal trajectory is the one that leads to an extreme value of H over the set of admissible controls. For the minimization case, the PMP can be written in mathematical form as follows:

$$\mathbf{u}^* = \arg \min_{\mathbf{u} \in \mathbf{U}} H(\mathbf{x}^*, \boldsymbol{\lambda}^*, \lambda_m^*, \mathbf{u}; t) \quad \forall t \in [t_i, t_f],$$

where superscript $*$ denotes the optimal solution, and:

$$\mathbf{U} = \{\boldsymbol{\alpha} \in \mathbb{R}^3 / \|\boldsymbol{\alpha}\| = 1\} \cup \{f \in \mathbb{R} / f \in [0, 1]\},$$

is the allowable set for the control. Substituting Eqs. (7.2) and (7.3) into Eq. (7.5), rearranging terms and **assuming that $\mathcal{L}$ may depend of the control magnitude but not on its orientation** one reaches:

$$H = \frac{T_{\max}}{m} f [\boldsymbol{\lambda}^\top \cdot \mathbf{B} \cdot \boldsymbol{\alpha}] + \boldsymbol{\lambda}^\top \cdot \mathbf{A} - \lambda_m f \frac{T_{\max}}{c} + \mathcal{L}.$$

Because $(T_{\max}/m)f$ is semi-positive the term in brackets should be negative and as large (in absolute value) as possible, leading to an expression for $\boldsymbol{\alpha}^*$:

$$\boldsymbol{\alpha}^* = -\frac{\mathbf{B}^\top \cdot \boldsymbol{\lambda}}{||\mathbf{B}^\top \cdot \boldsymbol{\lambda}||}. \quad (7.8)$$

This expression is a generalization of the well-known result of the primer vector in Cartesian coordinates by Lawden [62], which takes the simple form $\boldsymbol{\alpha}^* = -\boldsymbol{\lambda}_v/\lambda_v$. However, for an arbitrary formulation of the dynamics $\boldsymbol{\alpha}^*$ can be a complex expression of both the state and the costate; this is one of the main drawbacks of using element formulations to solve OCPs with indirect methods. On the other hand, the application of the PMP to determine f will depend on the particular expression of $\mathcal{L}$ for each OCP, so a general solution cannot be derived. In the following, it is assumed that f^* is a known function of state, mass and costate.

Plugging the expression for $\boldsymbol{\alpha}^*$, Eq. (7.8), back into Eq. (7.3) yields:

$$\mathbf{F} = -f^* \frac{T_{\max}}{m} \frac{\mathcal{B} \cdot \boldsymbol{\lambda}}{\sqrt{\boldsymbol{\lambda}^\top \cdot \mathcal{B} \cdot \boldsymbol{\lambda}}} + \mathbf{A}, \quad (7.9)$$

where $\mathcal{B} = \mathbf{B}\mathbf{B}^\top$ is a square matrix of dimension $n \times n$. This matrix is, by construction, symmetric and positive (semi-)definite. Consequently, $\boldsymbol{\lambda}^\top \cdot \mathcal{B} \cdot \boldsymbol{\lambda}$ represents a quadratic form of $\boldsymbol{\lambda}$, where the coefficients of the quadratic form are in turn functions of $\mathbf{s}$. Introducing:

$$\Psi = \sqrt{\boldsymbol{\lambda}^\top \cdot \mathcal{B} \cdot \boldsymbol{\lambda}}, \quad (7.10)$$

as the square root of the quadratic form, it is possible to express $\mathbf{F}$ in a more compact way:

$$\mathbf{F} = -f^* \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \boldsymbol{\lambda}} + \mathbf{A}, \quad (7.11)$$

as well as the Hamiltonian:

$$H = -f^* \frac{T_{\max}}{m} \Psi + \boldsymbol{\lambda}^\top \cdot \mathbf{A} - \lambda_m f^* \frac{T_{\max}}{c} + \mathcal{L}. \quad (7.12)$$

Substituting H into Eqs. (7.6) and (7.7) and operating, the ordinary differential equations (ODEs) describing the time evolution of $\boldsymbol{\lambda}$ and λ_m are reached (for clarity, the equations are written using Einstein notation):

$$\begin{aligned} \left(\dot{\boldsymbol{\lambda}}\right)_l = \dot{\lambda}_l = (\mathbf{G})_l &= \frac{\partial f^*}{\partial s_l} \frac{T_{\max}}{m} \Psi + f^* \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial s_l} \\ &\quad - \lambda_j \frac{\partial A_j}{\partial s_l} + \frac{\partial f^*}{\partial s_l} \lambda_m \frac{T_{\max}}{c} + \frac{\partial \mathcal{L}}{\partial s_l}, \end{aligned} \quad (7.13)$$

$$\dot{\lambda}_m = G_m = \frac{\partial f^*}{\partial m} \frac{T_{\max}}{m} \Psi - f^* \frac{T_{\max}}{m^2} \Psi + \frac{\partial f^*}{\partial m} \lambda_m \frac{T_{\max}}{c} - \frac{\partial \mathcal{L}}{\partial m}, \quad (7.14)$$

with:

$$\frac{\partial \Psi}{\partial s_l} = \frac{1}{2} \frac{\lambda_j \frac{\partial B_{jk}}{\partial s_l} \lambda_k}{\Psi}, \quad (7.15)$$

being A_j and B_{jk} the elements of $\mathbf{A}$ and $\mathbf{B}$, respectively.

Gathering the previous results, the ODE system describing the evolution of $\mathbf{s}$, m , $\boldsymbol{\lambda}$ and λ_m can be written in a compact form as:

$$\frac{d}{dt} \begin{bmatrix} \mathbf{s} \\ m \\ \boldsymbol{\lambda} \\ \lambda_m \end{bmatrix} = \frac{d}{dt} \mathbf{y} = \mathbf{R} = \begin{bmatrix} \mathbf{F} \\ F_m \\ \mathbf{G} \\ G_m \end{bmatrix}, \quad (7.16)$$

where the expressions for $\mathbf{F}$, F_m , $\mathbf{G}$ and G_m are given in Eqs. (7.11), (7.2), (7.13) and (7.14), respectively.

7.2.1 Variational equation and STM

A typical approach for solving the TPBVP associated to the OCP is using shooting methods, which require the Jacobian of the shooting function (or a suitable approximation). Although it can be approximated numerically using finite differences and BFGS updates, more efficiency and robustness can be achieved by constructing it analytically from the State Transition Matrix (STM) of the problem [101].

The STM maps small variations of the initial conditions, $\delta \mathbf{y}_i = [\delta \mathbf{s}_i \ \delta m_i \ \delta \boldsymbol{\lambda}_i \ \delta \lambda_{mi}]$, into small variations of $\mathbf{y}$ at a given time t , $\delta \mathbf{y} = [\delta \mathbf{s} \ \delta m \ \delta \boldsymbol{\lambda} \ \delta \lambda_m]$:

$$\delta \mathbf{y} = \boldsymbol{\Phi}(t_i, t) \delta \mathbf{y}_i.$$

The STM is calculated through the integration of the variational equation:

$$\dot{\boldsymbol{\Phi}}(t_i, t) = \mathbf{D}_{\mathbf{y}} \mathbf{R} \boldsymbol{\Phi}(t_i, t),$$

where $\mathbf{D}_{\mathbf{y}} \mathbf{R}$ is the Jacobian of $\mathbf{R}$, and the initial value for $\boldsymbol{\Phi}$ corresponds to the identity matrix of dimension $2(n+1) \times 2(n+1)$.

Although computing the Jacobian for $\mathbf{R}$ entails no intrinsic challenges from a conceptual point of view, its practical implementation may become rather burdensome. As previously indicated, $\boldsymbol{\alpha}^*$ can be a complex function involving both the state and the costate, and the derivatives of $\mathbf{R}$ may rapidly become too cumbersome to handle even with symbolic manipulators. However, it is relatively simple to derive general expressions for any formulation leveraging the structures

previously revealed in the equations. The Jacobian will then be expressed as a set of problem-independent algebraic relations involving $\mathcal{B}$, $\mathbf{A}$, f^* , $\mathcal{L}$ and their derivatives, which will be problem-dependent.

Following this approach, the Jacobian can be written as:

$$\mathbf{D}_y \mathbf{R} = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{s}} & \frac{\partial \mathbf{F}}{\partial m} & \frac{\partial \mathbf{F}}{\partial \boldsymbol{\lambda}} & \frac{\partial \mathbf{F}}{\partial \lambda_m} \\ \frac{\partial F_m}{\partial \mathbf{s}} & \frac{\partial F_m}{\partial m} & \frac{\partial F_m}{\partial \boldsymbol{\lambda}} & \frac{\partial F_m}{\partial \lambda_m} \\ \frac{\partial \mathbf{G}}{\partial \mathbf{s}} & \frac{\partial \mathbf{G}}{\partial m} & \frac{\partial \mathbf{G}}{\partial \boldsymbol{\lambda}} & \frac{\partial \mathbf{G}}{\partial \lambda_m} \\ \frac{\partial G_m}{\partial \mathbf{s}} & \frac{\partial G_m}{\partial m} & \frac{\partial G_m}{\partial \boldsymbol{\lambda}} & \frac{\partial G_m}{\partial \lambda_m} \end{bmatrix},$$

with:

$$\begin{aligned} \left(\frac{\partial \mathbf{F}}{\partial \mathbf{s}} \right)_{jk} &= \frac{\partial F_j}{\partial s_k} = -\frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_j} \frac{\partial f^*}{\partial s_k} - f^* \frac{T_{\max}}{m} \frac{\partial^2 \Psi}{\partial s_k \partial \lambda_j} + \frac{\partial A_j}{\partial s_k}, \\ \left(\frac{\partial \mathbf{F}}{\partial m} \right)_j &= \frac{\partial F_j}{\partial m} = -\frac{\partial f^*}{\partial m} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_j} + f^* \frac{T_{\max}}{m^2} \frac{\partial \Psi}{\partial \lambda_j}, \\ \left(\frac{\partial \mathbf{F}}{\partial \boldsymbol{\lambda}} \right)_{jk} &= \frac{\partial F_j}{\partial \lambda_k} = -\frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_j} \frac{\partial f^*}{\partial \lambda_k} - f^* \frac{T_{\max}}{m} \frac{\partial^2 \Psi}{\partial \lambda_j \partial \lambda_k}, \\ \left(\frac{\partial \mathbf{F}}{\partial \lambda_m} \right)_j &= \frac{\partial F_j}{\partial \lambda_m} = -\frac{\partial f^*}{\partial \lambda_m} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_j}, \\ \left(\frac{\partial F_m}{\partial \mathbf{s}} \right)_j &= \frac{\partial F_m}{\partial s_j} = -\frac{\partial f^*}{\partial s_j} \frac{T_{\max}}{c}, \\ \frac{\partial F_m}{\partial m} &= -\frac{\partial f^*}{\partial m} \frac{T_{\max}}{c}, \\ \left(\frac{\partial F_m}{\partial \boldsymbol{\lambda}} \right)_j &= \frac{\partial F_m}{\partial \lambda_j} = -\frac{\partial f^*}{\partial \lambda_j} \frac{T_{\max}}{c}, \\ \frac{\partial F_m}{\partial \lambda_m} &= -\frac{\partial f^*}{\partial \lambda_m} \frac{T_{\max}}{c}, \\ \left(\frac{\partial \mathbf{G}}{\partial \mathbf{s}} \right)_{jk} &= \frac{\partial G_j}{\partial s_k} = \frac{\partial^2 f^*}{\partial s_j \partial s_k} \frac{T_{\max}}{m} \Psi + \frac{T_{\max}}{m} \frac{\partial f^*}{\partial s_j} \frac{\partial \Psi}{\partial s_k} + \\ &\quad \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial s_j} \frac{\partial f^*}{\partial s_k} + f^* \frac{T_{\max}}{m} \frac{\partial^2 \Psi}{\partial s_j \partial s_k} - \\ &\quad \lambda_l \frac{\partial^2 A_l}{\partial s_j \partial s_k} - \frac{\partial^2 \mathcal{L}}{\partial s_j \partial s_k} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial s_j \partial s_k}, \end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial \mathbf{G}}{\partial m}\right)_j &= \frac{\partial G_j}{\partial m} = \frac{\partial^2 f^*}{\partial s_j \partial m} \frac{T_{\max}}{m} \Psi - \frac{\partial f^*}{\partial s_j} \frac{T_{\max}}{m^2} \Psi + \\
&\frac{\partial f^*}{\partial m} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial s_j} - f^* \frac{T_{\max}}{m^2} \frac{\partial \Psi}{\partial s_j} - \frac{\partial^2 \mathcal{L}}{\partial s_j \partial m} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial s_j \partial m}, \\
\left(\frac{\partial \mathbf{G}}{\partial \boldsymbol{\lambda}}\right)_{jk} &= \frac{\partial G_j}{\partial \lambda_k} = \frac{\partial^2 f^*}{\partial s_j \partial \lambda_k} \frac{T_{\max}}{m} \Psi + \frac{\partial f^*}{\partial s_j} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_k} + \\
&\frac{T_{\max}}{m} \frac{\Psi}{\partial s_j} \frac{\partial f}{\partial \lambda_k} + f^* \frac{T_{\max}}{m} \frac{\partial^2 \Psi}{\partial s_j \partial \lambda_k} - \frac{\partial A_k}{\partial s_j} - \\
&\frac{\partial^2 \mathcal{L}}{\partial s_j \partial \lambda_k} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial s_j \partial \lambda_k}, \\
\left(\frac{\partial \mathbf{G}}{\partial \lambda_m}\right)_j &= \frac{\partial G_j}{\partial \lambda_m} = \frac{\partial f^*}{\partial s_j \partial \lambda_m} \frac{T_{\max}}{m} \Psi + \frac{\partial f^*}{\partial \lambda_m} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial s_j} - \\
&\frac{\partial^2 \mathcal{L}}{\partial s_j \partial \lambda_m} + \frac{T_{\max}}{c} \frac{\partial f^*}{\partial s_j} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial s_j \partial \lambda_m}, \\
\left(\frac{\partial G_m}{\partial \mathbf{s}}\right)_j &= \left(\frac{\partial \mathbf{G}}{\partial m}\right)_j, \\
\frac{\partial G_m}{\partial m} &= \frac{\partial^2 f^*}{\partial m^2} \frac{T_{\max}}{m} \Psi - 2 \frac{\partial f^*}{\partial m} \frac{T_{\max}}{m^2} \Psi + f^* \frac{2T_{\max}}{m^3} \Psi - \\
&\frac{\partial^2 \mathcal{L}}{\partial m^2} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial m^2}, \\
\left(\frac{\partial G_m}{\partial \boldsymbol{\lambda}}\right)_j &= \frac{\partial G_m}{\partial \lambda_j} = \frac{\partial^2 f^*}{\partial m \partial \lambda_j} \frac{T_{\max}}{m} \Psi + \frac{\partial f^*}{\partial m} \frac{T_{\max}}{m} \frac{\partial \Psi}{\partial \lambda_j} - \\
&\frac{\partial f^*}{\partial \lambda_j} \frac{T_{\max}}{m^2} \Psi - f^* \frac{T_{\max}}{m^2} \frac{\partial \Psi}{\partial \lambda_j} - \frac{\partial^2 \mathcal{L}}{\partial m \partial \lambda_j} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial m \partial \lambda_j}, \\
\frac{\partial G_m}{\partial \lambda_m} &= \frac{\partial^2 f^*}{\partial m \partial \lambda_m} \frac{T_{\max}}{m} \Psi - \frac{\partial f^*}{\partial \lambda_m} \frac{T_{\max}}{m} \Psi - \frac{\partial^2 \mathcal{L}}{\partial m \partial \lambda_m} + \\
&\frac{T_{\max}}{c} \frac{\partial f^*}{\partial m} + \lambda_m \frac{T_{\max}}{c} \frac{\partial^2 f^*}{\partial m \partial \lambda_m}.
\end{aligned}$$

These expressions involve the partial derivatives of Ψ with respect to $\mathbf{s}$ and $\boldsymbol{\lambda}$ up to order two. The first order derivative of Ψ with respect to $\mathbf{s}$ was already given in Eq. (7.15), and the rest can be calculated as follows:

$$\begin{aligned}
\frac{\partial \Psi}{\partial \lambda_l} &= \frac{B_{lj} \lambda_j}{\Psi}, \\
\frac{\partial^2 \Psi}{\partial s_j \partial s_k} &= \frac{1}{2} \frac{\lambda_l \frac{\partial^2 B_{ln}}{\partial s_j \partial s_k} \lambda_n}{\Psi} - \frac{1}{4} \frac{\left(\lambda_l \frac{\partial B_{ln}}{\partial s_j} \lambda_n\right) \left(\lambda_l \frac{\partial B_{ln}}{\partial s_k} \lambda_n\right)}{\Psi^3},
\end{aligned}$$

$$\frac{\partial^2 \Psi}{\partial \lambda_j \partial \lambda_k} = \frac{B_{jk}}{\Psi} - \frac{(B_{jl} \lambda_l)(B_{kl} \lambda_l)}{\Psi^3},$$

$$\frac{\partial^2 \Psi}{\partial s_j \partial \lambda_k} = -\frac{1}{2} \frac{\left(\lambda_l \frac{\partial B_{ln}}{\partial s_j} \lambda_n \right) (B_{kl} \lambda_l)}{\Psi^3} + \frac{\frac{\partial B_{kl}}{\partial s_j} \lambda_l}{\Psi}.$$

7.2.2 Minimum-time, fuel and energy problems

Three especially interesting OCPs are the minimum-time, minimum-energy and minimum-fuel problems. In this subsection, the values of $\mathcal{L}$, f^* and their derivatives for these families of OCPs are presented.

The objective function for minimum-time (understood as the independent variable, or ‘generalized time’) problems takes the form:

$$J = \int_{t_i}^{t_f} 1 \, dt = \int_{t_i}^{t_f} \mathcal{L} \, dt,$$

leading to a constant Lagrangian of value 1. Furthermore, it is possible to prove that f^* is also constant and equal to 1. Consequently, all the derivatives of $\mathcal{L}$ and f^* become zero, greatly simplifying the previous expressions for the Jacobian of $\mathbf{R}$.

Minimum-fuel problems have a great practical interest for mission design, but the discontinuities introduced by the characteristic bang-bang structure of f^* greatly difficult their numerical treatment. One way to address this issue is to apply homotopy techniques [10], solving an easier problem such as minimum-energy first and performing a continuation in the objective function until the original minimum-fuel problem is reached. To this end, a modified cost function J depending on an homotopy parameter ε is introduced as done by Zhang et al. in [101]:

$$J = \frac{T_{\max}}{c} \int_{t_i}^{t_f} [f - \varepsilon f(1 - f)] \, dt = \int_{t_i}^{t_f} \mathcal{L} \, dt, \quad \varepsilon \in [0, 1].$$

The two limit cases in ε correspond to minimum-fuel for $\varepsilon = 0$ and minimum-energy for $\varepsilon = 1$. Substituting $\mathcal{L}$ into the Hamiltonian and rearranging terms one reaches [101]:

$$H = \boldsymbol{\lambda}^\top \cdot \mathbf{A} + f^* \frac{T_{\max}}{c} [S - \varepsilon + \varepsilon f],$$

where a switching function S has been defined as:

$$S = -\frac{c}{m} \Psi - \lambda_m + 1. \quad (7.17)$$

This S is a generalization of the expression obtained by Zhang et al. in [101] for the CR3BP in Cartesian coordinates, or the one reached in Chapter 6 for the orbit-to-orbit transfer in MEEs. Same as in those cases, by deriving H with respect to f and equating to zero a minimum for f is found in the form $f = (\varepsilon - S)/2\varepsilon$. Restricting this solution to the allowable set for the control, $f \in [0, 1]$, a piecewise function for f^* is finally obtained:

$$f^* = \begin{cases} 0 & \text{for } S > \varepsilon \\ (\varepsilon - S)/2\varepsilon & \text{for } -\varepsilon \leq S \leq \varepsilon \\ 1 & \text{for } S < -\varepsilon \end{cases} \quad (7.18)$$

This expression for f^* coincides both with the one given by Zhang et al. [101] using Cartesian coordinates and with the one in Chapter 6 for the MEEs, only changing the definition of S . Furthermore, it is straightforward to check that for the minimum-fuel case ($\varepsilon = 0$) a bang-bang profile is obtained.

The derivatives of f^* will be different from zero only when the thruster is operating in the intermediate regime, that is, for $|S| \leq \varepsilon$. In that case, the derivatives of S with respect to $\mathbf{y}$ up to order 2 can be expressed as functions of the known derivatives of Ψ as follows (for brevity, only the derivatives needed to compute $\mathbf{G}$, G_m and the variational equation are reported):

$$\begin{aligned} \left(\frac{\partial f^*}{\partial \mathbf{s}} \right)_j &= \frac{\partial f^*}{\partial s_j} = \frac{1}{2\varepsilon} \frac{c}{m} \frac{\partial \Psi}{\partial s_j}, \\ \frac{\partial f^*}{\partial m} &= -\frac{1}{2\varepsilon} \frac{c}{m^2} \Psi, \\ \left(\frac{\partial f^*}{\partial \boldsymbol{\lambda}} \right)_j &= \frac{\partial f^*}{\partial \lambda_j} = \frac{1}{2\varepsilon} \frac{c}{m} \frac{\partial \Psi}{\partial \lambda_j}, \\ \frac{\partial f^*}{\partial \lambda_m} &= \frac{1}{2\varepsilon}, \\ \left(\frac{\partial^2 f^*}{\partial \mathbf{s} \partial \mathbf{s}} \right)_{jk} &= \frac{\partial^2 f^*}{\partial s_j \partial s_k} = \frac{1}{2\varepsilon} \frac{c}{m} \frac{\partial^2 \Psi}{\partial s_j \partial s_k}, \\ \left(\frac{\partial^2 f^*}{\partial \mathbf{s} \partial m} \right)_j &= \frac{\partial^2 f^*}{\partial s_j \partial m} = -\frac{1}{m} \frac{\partial f^*}{\partial s_j}, \\ \left(\frac{\partial^2 f^*}{\partial \mathbf{s} \partial \boldsymbol{\lambda}} \right)_{jk} &= \frac{\partial^2 f^*}{\partial s_j \partial \lambda_k} = \frac{1}{2\varepsilon} \frac{c}{m} \frac{\partial \Psi}{\partial s_j \partial \lambda_k}, \\ \left(\frac{\partial^2 f^*}{\partial \mathbf{s} \partial \lambda_m} \right)_j &= \frac{\partial^2 f^*}{\partial s_j \partial \lambda_m} = 0, \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f^*}{\partial m^2} &= \frac{1}{\varepsilon} \frac{c}{m^3} \Psi, \\ \left(\frac{\partial^2 f^*}{\partial m \partial \lambda} \right)_j &= \frac{\partial^2 f^*}{\partial m \partial \lambda_j} = -\frac{1}{m} \frac{\partial f^*}{\partial \lambda_j}, \\ \frac{\partial^2 f^*}{\partial m \partial \lambda_m} &= 0.\end{aligned}$$

The derivatives of $\mathcal{L}$ with respect to $\mathbf{y}$ up to order 2 can now be expressed in a compact way in terms of the derivatives of f^* (again, only the terms required to evaluate $\mathbf{G}$, G_m and $D_{\mathbf{y}}\mathbf{R}$ are included):

$$\begin{aligned}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{s}} \right)_j &= \frac{\partial \mathcal{L}}{\partial s_j} = \frac{T_{\max}}{c} \frac{\partial f^*}{\partial s_j} K, \\ \frac{\partial \mathcal{L}}{\partial m} &= \frac{T_{\max}}{c} \frac{\partial f^*}{\partial m} K, \\ \left(\frac{\partial^2 \mathcal{L}}{\partial \mathbf{s} \partial \mathbf{s}} \right)_{jk} &= \frac{\partial^2 \mathcal{L}}{\partial s_j \partial s_k} = \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial s_j \partial s_k} K + 2\varepsilon \frac{\partial f^*}{\partial s_j} \frac{\partial f^*}{\partial s_k} \right], \\ \left(\frac{\partial^2 \mathcal{L}}{\partial \mathbf{s} \partial m} \right)_j &= \frac{\partial^2 \mathcal{L}}{\partial s_j \partial m} = \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial s_j \partial m} K + 2\varepsilon \frac{\partial f^*}{\partial s_j} \frac{\partial f^*}{\partial m} \right], \\ \left(\frac{\partial^2 \mathcal{L}}{\partial \mathbf{s} \partial \lambda} \right)_{jk} &= \frac{\partial^2 \mathcal{L}}{\partial s_j \partial \lambda_k} = \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial s_j \partial \lambda_k} K + 2\varepsilon \frac{\partial f^*}{\partial s_j} \frac{\partial f^*}{\partial \lambda_k} \right], \\ \left(\frac{\partial^2 \mathcal{L}}{\partial \mathbf{s} \partial \lambda_m} \right)_j &= \frac{\partial^2 \mathcal{L}}{\partial s_j \partial \lambda_m} = \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial s_j \partial \lambda_m} K + 2\varepsilon \frac{\partial f^*}{\partial s_j} \frac{\partial f^*}{\partial \lambda_m} \right], \\ \frac{\partial^2 \mathcal{L}}{\partial m^2} &= \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial m^2} K + 2\varepsilon \left(\frac{\partial f^*}{\partial m} \right)^2 \right], \\ \left(\frac{\partial^2 \mathcal{L}}{\partial m \partial \lambda} \right)_j &= \frac{\partial^2 \mathcal{L}}{\partial m \partial \lambda_j} = \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial m \partial \lambda_j} K + 2\varepsilon \frac{\partial f^*}{\partial m} \frac{\partial f^*}{\partial \lambda_j} \right], \\ \frac{\partial^2 \mathcal{L}}{\partial m \partial \lambda_m} &= \frac{T_{\max}}{c} \left[\frac{\partial^2 f^*}{\partial m \partial \lambda_m} K + 2\varepsilon \frac{\partial f^*}{\partial m} \frac{\partial f^*}{\partial \lambda_m} \right],\end{aligned}$$

with:

$$K = (1 - \varepsilon + 2\varepsilon f^*) .$$

Figure 7.1: Block diagram of the different components in the unified formulation and their functional dependencies.

7.2.3 Framework scheme

By close inspection of the previous results, it is possible to isolate the formulation-dependent, objective-function-dependent and problem-independent components of the element-based indirect optimization problem. Figure 7.1 shows a schematic representation of these main blocks and the functional relations between them. It is straightforward to see that the proposed framework is very well suited for its algorithmic implementation, being possible to define interchangeable modules for different formulations and objectives functions (maybe in the form of a library). A first version restricted to MEEs and the minimum-time, energy, and fuel problems has been implemented and tested as an extension to the indirect optimization software being developed by Prof. Topputo's group at the Politecnico di Milano.

7.3 Conclusion

A mathematical framework for the formulation of OCPs using indirect methods and a generic formulation for the dynamics has been proposed. The mathematical structure of the equations derived from the first order optimality conditions has been leveraged to identify and isolate the formulation-dependent, objective-function-dependent and problem-independent components of the OCP, proposing problem-independent formulas involving a set of problem-dependent matrices and their derivatives with respect to the state and mass. In particular, all the con-

tributions of the costate elements associated to the state (excluding mass) have been reduced to a quadratic form of the costate and its derivatives, and a generalization of the primer vector has been identified. Relatively simple expressions have also been derived for the variational equation needed to compute the STM, key to improve the numerical behavior of solvers for the TPBVP associated to the OCP. Formulation-independent expressions for the Lagrangian and thruster throttling parameter have been provided for the minimum-time, minimum-energy and minimum-fuel problems.

Part III

Dromo

Chapter 8

Multiple Scales Asymptotic Solution For The Radial Thrust Problem

8.1 Introduction

The method of multiple scales (see [67, 71, 49]) is a powerful perturbation technique that can be employed to study the behavior of complex dynamical systems. It is particularly effective when secular terms are introduced in the dynamics causing classical perturbation methods to diverge. As a consequence, the approximate analytical solution is only valid within a small interval which decreases as the small perturbing parameter (ε) grows.

Multiple scales methods have been widely employed in almost all branches of physics, ranging from molecular dynamics to computational fluid mechanics. However, their use in orbital mechanics is relatively scarce and limited to a few references; see for instance [61], page 396 of 391-461, or [56].

One interesting orbital mechanics problem case that can be tackled with the multiple scales perturbation technique is the problem of a body orbiting a primary and perturbed by a constant radial acceleration. The problem was first investigated by Tsien in [93]. In the classic astrodynamics book of Battin [7] an exact solution of the problem for the case of initially circular orbit is reached in terms of elliptic integrals. Several authors have dealt with the problem and its nuances [80, 1, 79, 86]. Very recently, an exact solution for the radial problem in the general case has been obtained by Izzo et al. [52], in terms of a fictitious time introduced with a Sundman transformation and the Weierstrass elliptic functions. However, the undoubtedly elegant solution requires the numerical computation of elliptic functions, which is not straightforward.

In this chapter the use of the multiple scales perturbation method is exploited in order to obtain an approximate solution with a simpler analytical representation but retaining high accuracy across a reasonably wide range of initial conditions and acceleration magnitudes. As it will be shown in the results the approximate solution obtained can be expressed in terms of simpler functions providing more insight about the underlying physics of the problem. In particular, relatively compact expressions are provided to quickly evaluate the main frequencies of the dynamics which are not straightforward to compute even using numerical integration. Last but not least, a relatively simple example of the application of the multiple scale technique to an example of perturbed two-body problem is provided.

One fundamental aspect of the analytical procedure employed is the use of a convenient orbital motion formulation to ease the analytical treatment as much as possible. The so-called Dromo formulation initially introduced by Peláez et al. in [74] and complemented by other authors (see [9, 8]) is chosen here due to its suitability when dealing with a constant radial thrust problem. This is because it is intrinsically based on a local-vertical/local-horizontal orbiting frame where a purely radial acceleration component appears as a simple perturbing term in the equations of motions. The same formulation has proven to be an excellent propagation tool, and its suitability for the formulation of low thrust optimal control problems is studied in the next chapter. Also, an asymptotic solution for low tangential acceleration was obtained by [15] based on the same formulation and using a regular expansion in the non-dimensional thrust.

The chapter is structured as follows. First, the equation of motions for the constant radial thrust problem are written in Dromo variables and an asymptotic solution for the problem is obtained using a regular expansion in the small perturbing acceleration. It is seen that this solution breaks for large values of the independent variable, suggesting the use of more complex perturbation techniques. The following section deals with the solution of the problem using the method of multiple scales. Finally, the multiple scales solution is compared with high precision numerical propagations for several cases, finding a good agreement between them. The asymptotic solution obtained through the regular expansion is also included in these comparisons, highlighting the great gains in accuracy and validity range of the asymptotic solution achieved through the method of multiple scales.

8.2 Equations of Motion

Let us consider a particle orbiting around a primary of gravitational constant μ and perturbed by a radial acceleration of constant magnitude A , leading to a planar motion with constant angular momentum. Let R_0 denote the initial distance

between the particle and the center of the primary and ν_0 the initial value of the true anomaly. All equations and variables considered hereafter are expressed in non-dimensional form taking R_0 as characteristic length and $1/n_0$ as characteristic time, where $n_0 = (\mu/R_0)^{1/2}$ is the angular frequency of the circular orbit of radius R_0 around the primary.

To describe the motion of the body, the Dromo orbital formulation developed by Peláez et al. [74] is used. In this formulation, a fictitious time θ is introduced through a change of independent variable given by a Sundman's transformation. Then, the variation of parameters technique is applied to obtain seven generalized orbital parameters $\mathbf{q}$ which, along the non-dimensional time t , describe the state of the particle. These orbital parameters are constant in the unperturbed problem, but evolve in presence of perturbing forces; this property is very convenient for the mathematical developments in this study. Moreover, three of these parameters describe the geometry of the orbit in its plane, while the other four are related to the orientation of said plane. Therefore, the latter are constant for the planar case, and the motion of the particle can be described by a 4-dimensional state vector:

$$(t, q_1, q_2, q_3), \quad (8.1)$$

whose evolution for the radial thrust problem is given by the following system of four differential equations:

$$\frac{dt}{d\theta} = \frac{1}{q_3 s^2}, \quad (8.2)$$

$$\frac{d}{d\theta} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{\varepsilon}{q_3 s^2} \begin{bmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{bmatrix}, \quad (8.3)$$

with initial conditions:

$$t(\theta_0) = t_i, \quad q_1(\theta_0) = q_{1i}, \quad q_2(\theta_0) = q_{2i}, \quad q_3(\theta_0) = q_{3i},$$

where ε is the non-dimensional acceleration parameter:

$$\varepsilon = \frac{A}{R_0 n_0^2} = \frac{A}{\mu/R_0^2}, \quad (8.4)$$

and s corresponds to the non-dimensional velocity in the transversal direction:

$$s = q_3 + q_1 \cos \theta + q_2 \sin \theta. \quad (8.5)$$

The non-dimensional orbital distance can be related to previous quantities by the relation:

$$r = \frac{1}{q_3 s}. \quad (8.6)$$

It is also possible to establish several relations between the generalized orbital parameters and the classical orbital elements, as developed in [94] and [9] (additional relations can be found in Appendix A):

$$q_1 = \frac{e}{h} \cos \gamma, \quad q_2 = \frac{e}{h} \sin \gamma, \quad q_3 = \frac{1}{h}, \quad (8.7)$$

$$e = \frac{\sqrt{q_1^2 + q_2^2}}{q_3}, \quad \gamma = \tan^{-1} \left(\frac{q_2}{q_1} \right), \quad h = \frac{1}{q_3}, \quad (8.8)$$

$$a = \frac{1}{q_3^2 - q_1^2 - q_2^2}, \quad E = \frac{q_1^2 + q_2^2 - q_3^2}{2}, \quad E_T = E - \varepsilon r. \quad (8.9)$$

In the above expressions, e is the eccentricity, h is the non-dimensional angular momentum, a is the non-dimensional semi-major axis, E is the non-dimensional Keplerian energy, E_T is the non-dimensional total energy, and angle γ is the difference between the variations of the fictitious time and the true anomaly (see [9]):

$$\Delta\theta = \Delta\nu + \gamma. \quad (8.10)$$

In the planar case, γ coincides with the angle between the eccentricity vectors of the initial and of the osculating orbit, and θ becomes the angular position of the particle measured from the eccentricity vector of the initial orbit.

To close the mathematical formulation of the dynamics, the initial conditions are obtained from Eqs. (8.7), taking into account that $\theta_0 = \nu_0$, $\gamma_0 = 0$:

$$q_{1i} = \frac{e_0}{h_0}, \quad q_{2i} = 0, \quad q_{3i} = \frac{1}{h_0}, \quad (8.11)$$

with $h_0 = \sqrt{1 + e_0 \cos \nu_0}$. For simplicity and clarity, in the following developments the initial value of the independent variable is assumed to be zero, $\theta_0 = 0$. The results can be generalized for an arbitrary value of θ_0 by introducing the corresponding integration constants.

While the generalized orbital element q_3 is constant as a consequence of the conservation of angular momentum the other two vary according to Eqs. (8.3) whose solution will be approached in the remainder of the chapter.

More information on the Dromo formulation, its definition and main characteristics is included in Appendix A.

8.3 Regular expansion

8.3.1 Generalized orbital elements q_1, q_2

An asymptotic solution for the low-thrust two body problem defined by Eqs. (8.3, 8.11) is now sought for, in the form of a regular expansion¹ in the non-dimensional thrust parameter ε . To this end, the state is expanded in power series of $\varepsilon \ll 1$ as follows:

$$\begin{aligned}\hat{q}_1(\theta; \varepsilon) &= \hat{q}_{10}(\theta) + \varepsilon \hat{q}_{11}(\theta) + \varepsilon^2 \hat{q}_{12}(\theta) + \mathcal{O}(\varepsilon^3), \\ \hat{q}_2(\theta; \varepsilon) &= \hat{q}_{20}(\theta) + \varepsilon \hat{q}_{21}(\theta) + \varepsilon^2 \hat{q}_{22}(\theta) + \mathcal{O}(\varepsilon^3).\end{aligned}$$

Expanding also the initial conditions, with $\theta_0 = 0$, and identifying terms of equal power of ε :

$$\begin{aligned}\hat{q}_{10}(0) &= q_{1i}, & \hat{q}_{11}(0) &= \hat{q}_{12}(0) = 0, \\ \hat{q}_{20}(0) &= \hat{q}_{21}(0) = \hat{q}_{22}(0) = 0.\end{aligned}\tag{8.12}$$

For compactness, the power series expansion of $\hat{s}$ is also defined as:

$$\hat{s}(\theta) = \hat{s}_0(\theta) + \varepsilon \hat{s}_1(\theta) + \mathcal{O}(\varepsilon^2),$$

with:

$$\begin{aligned}\hat{s}_0(\theta) &= q_{3i} + \hat{q}_{10} \cos \theta + \hat{q}_{20} \sin \theta, \\ \hat{s}_1(\theta) &= \hat{q}_{11} \cos \theta + \hat{q}_{21} \sin \theta.\end{aligned}$$

Introducing the expansion of the state into the first two of Eqs. (8.3), expanding in Taylor series of ε and retaining the leading order terms yields:

$$\frac{d}{d\theta} \begin{bmatrix} \hat{q}_{10} \\ \hat{q}_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{q}_{10} \\ \hat{q}_{20} \end{bmatrix} = \begin{bmatrix} q_{1i} \\ 0 \end{bmatrix}.$$

That is, the zeroth order terms of the asymptotic solution are constant and equal to their initial values; this result was expected, since the limit $\varepsilon = 0$ corresponds to the unperturbed orbit, for which Dromo orbital parameters remain constant. Additionally, the expression for $\hat{s}_0$ now takes the simpler form:

$$\hat{s}_0 = q_{3i} + q_{1i} \cos \theta.\tag{8.13}$$

To retain the effect of the thrust, it is necessary to consider the first order terms of the asymptotic solution. The differential equations which describe their evolution with θ are obtained canceling terms of $\mathcal{O}(\varepsilon)$ in the expansion of Eqs. (8.3):

$$\frac{d}{d\theta} \begin{bmatrix} \hat{q}_{11} \\ \hat{q}_{21} \end{bmatrix} = \frac{1}{q_{3i} (\hat{s}_0)^2} \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}.\tag{8.14}$$

¹For clarity, the symbol $\hat{\cdot}$ will be used to denote all the variables related to this asymptotic solution.

Since the previous equations are uncoupled, $\hat{q}_{11}(\theta)$ and $\hat{q}_{21}(\theta)$ can be obtained independently as quadratures. Introducing the known results for $\hat{q}_{10}$ and $\hat{q}_{20}$, and taking into account the initial conditions given by Eqs. (8.12), the solution for $\hat{q}_{11}(\theta)$ is straightforward:

$$\hat{q}_{11}(\theta) = \frac{1 - \cos \theta}{q_{3i}(q_{3i} + q_{1i})\hat{s}_0}. \quad (8.15)$$

The solution for $\hat{q}_{21}(\theta)$ is more complex. Assuming $q_{3i} > q_{1i}^2$ and integrating yields:

$$\hat{q}_{21}(\theta) = \frac{-\sin \theta}{(q_{3i}^2 - q_{1i}^2)\hat{s}_0} + \frac{2q_{1i}}{q_{3i}(q_{3i}^2 - q_{1i}^2)^{3/2}} \left(\frac{\theta}{2} + \tan^{-1} \hat{\mathcal{K}} \right), \quad (8.16)$$

with :

$$\hat{\mathcal{K}} = -\frac{\sin \theta \left(-q_{3i} + q_{1i} + \sqrt{q_{3i}^2 - q_{1i}^2} \right)}{(1 + \cos \theta)\sqrt{q_{3i}^2 - q_{1i}^2} + (1 - \cos \theta)(q_{3i} - q_{1i})}.$$

The second term of the expression for $\hat{q}_{21}$ introduces a secular behavior in θ ; as a result, $\varepsilon \hat{q}_{21}$ becomes of order one for $\theta \sim 1/\varepsilon$, and the regular asymptotic expansion breaks down. This poses a clear limitation to the applicability of this solution for long term propagation, and suggests the convenience of resorting to more complex formulations.

8.3.2 Physical Time

The previous results are given in the fictitious time introduced by the Sundman transformation, which in this case coincides with the angular position measured from the initial eccentricity vector. Therefore, it is interesting to establish a relation between this fictitious time θ and the non-dimensional physical time $\hat{t}$. However, instead of working directly with the physical time and Eq. (8.2), the time element for the Dromo formulation recently proposed by Baù et al. [8] is preferred. Particularizing Baù's equations for the constant radial thrust case, the algebraic relation between the physical time and the time element ζ_0 takes the form:

$$\zeta_0 = t - \frac{u}{2Eq_3s} - \frac{1}{E\sqrt{-2E}} \arctan \left(\frac{u}{s + \sqrt{-2E}} \right), \quad (8.17)$$

while the differential equation describing the evolution of said time element with θ is:

$$\frac{d\zeta_0}{d\theta} = a^{3/2} \left[1 + \frac{\varepsilon u}{q_3 s^2} \left(6a \arctan \left(\frac{u}{f + w} \right) + k_1 \right) + \frac{\varepsilon}{q_3 s} k_2 \right], \quad (8.18)$$

²This assumption is valid as long as $e_0 < 1$, since $q_{1i} = e_0 q_{3i}$.

where:

$$k_1 = \frac{\sqrt{a}u}{s^2} \left(\frac{q_3 + s}{f} + \frac{2w}{q_3} + 1 \right), \quad k_2 = \frac{1}{s^2} \left(\frac{f}{q_3} + \frac{w}{f} + \frac{u^2}{fs} \right),$$

$$u = q_1 \sin \theta - q_2 \cos \theta, \quad w = q_1 \cos \theta + q_2 \sin \theta,$$

$$f = q_3 + \sqrt{-2E}.$$

Calling $\hat{\zeta}_0$ to the time element for the regular asymptotic solution, and expanding it in power series of ε :

$$\hat{\zeta}_0(\theta) = \hat{\zeta}_{00}(\theta) + \varepsilon \hat{\zeta}_{01}(\theta) + \varepsilon^2 \hat{\zeta}_{02}(\theta) + \mathcal{O}(\varepsilon^3).$$

The zeroth-order term $\hat{\zeta}_{00}$ can be obtained by introducing this expression and the known solutions for $\hat{q}_1$ and $\hat{q}_2$ into Eq. (8.18), expanding in power series of ε up to the leading order terms and integrating:

$$\hat{\zeta}_{00} = \frac{1}{(q_{3i}^2 - q_{1i}^2)^{3/2}} \theta.$$

The resulting expression for $\hat{t}$ includes both secular and oscillatory terms in θ . Unlike what happened with $\hat{q}_2$, the presence of a secular term was expected and necessary, since time must be monotonically increasing with θ .

8.3.3 Orbital Elements

The previous asymptotic solution is given in terms of Dromo generalized parameters, but it is not straightforward to give physical interpretations for all of them. Therefore, it is convenient to express it in terms of more familiar orbital elements, by substituting $\hat{q}_1(\theta; \varepsilon)$ and $\hat{q}_2(\theta; \varepsilon)$ into Eqs. (8.8-8.9) and expanding in power series of ε up to first order. Proceeding in this manner, the following expression for the eccentricity $\hat{e}$ is obtained:

$$\hat{e}(\theta; \varepsilon) = e_0 + \varepsilon \frac{1 - \cos \theta}{q_{3i}^4 (1 + e_0)(1 + e_0 \cos \theta)} + \mathcal{O}(\varepsilon^2). \quad (8.19)$$

As a first approximation, the eccentricity oscillates between e_0 and $e_0 + 2\varepsilon q_{3i}^{-4} (1 - e_0^2)^{-1}$, with a period of 2π in the fictitious time θ . Note that for $e_0/\varepsilon \gtrsim 1$ the preceding expansion ceases to be valid, and should be replaced by the full Eq. (8.8) or by the expression for initially circular or quasi-circular orbit derived later in this chapter.

Likewise, the expansion for the orbit radius (valid regardless of the value of e_0) yields:

$$\hat{r}(\theta; \varepsilon) = \frac{1}{q_{3i}\hat{s}_0} + \varepsilon \frac{-q_{1i} \sin \theta (\theta + 2 \arctan \mathcal{K}) + (1 - \cos \theta) \sqrt{q_{3i}^2 - q_{1i}^2}}{q_{3i}^2 (q_{3i}^2 - q_{1i}^2)^{3/2} (\hat{s}_0)^2} + \mathcal{O}(\varepsilon^2). \quad (8.20)$$

From a physical point of view, the presence of the secular term in θ would imply that the orbital radius is unbounded for any value of e_0 and ε , which is in contradiction with the results given by Battin [7]. Moreover, the mathematical validity of the expansion breaks for $\theta \sim 1/\varepsilon$. On the other hand, the leading order term of $\hat{r}$ turns out to be 2π periodic in θ , retaining the same period as the unperturbed motion.

The Keplerian energy expansion yields:

$$\hat{E}(\theta; \varepsilon) = \frac{q_{1i}^2 - q_{3i}^2}{2} + \varepsilon \frac{q_{1i} (1 - \cos \theta)}{q_{3i}(q_{3i} + q_{1i}) (q_{3i} + q_{1i} \cos \theta)} + \mathcal{O}(\varepsilon^2), \quad (8.21)$$

corresponding to small oscillations about the initial value. On the other hand, the total energy $\hat{E}_T$ expansion yields the constant value (satisfying energy conservation):

$$\hat{E}_T(\theta; \varepsilon) = \frac{q_{1i}^2 - q_{3i}^2}{2} - \varepsilon \frac{1}{q_{3i} (q_{3i} + q_{1i})} + \mathcal{O}(\varepsilon^2). \quad (8.22)$$

Finally, it is interesting to consider the evolution of the angle $\hat{\gamma}$ formed by the initial and osculating eccentricity vectors, which is also the angular drift between the true anomaly ν and the fictitious time θ . Following Eq. (8.8), the regular expansion of $\hat{\gamma}$ up to $\mathcal{O}(\varepsilon)$ can be given as:

$$\hat{\gamma}(\theta; \varepsilon) = -\varepsilon \left(\frac{\sin \theta}{q_{1i}(q_{3i}^2 - q_{1i}^2) (q_{3i} + q_{1i} \cos \theta)} - \frac{\theta + 2 \arctan \hat{\mathcal{K}}}{q_{3i}(q_{3i}^2 - q_{1i}^2)^{3/2}} \right) + \mathcal{O}(\varepsilon^2). \quad (8.23)$$

Same as with $\hat{r}$, this expression includes a secular behavior in θ . However, the lack of a zeroth order term introduces the reasonable doubt of whether this is just a mathematical artifact or it actually represents a physical characteristic of the solution. This question shall be addressed in the following section, where a more accurate asymptotic solution is reached. Aside from this, the previous expansion is not valid for the case of initially circular or quasi-circular orbits; in those cases, the general expression in Eqs. (8.8) must be used.

8.3.4 Initially circular orbit

For the particular case of an orbit with $e_0 = 0$, the regular asymptotic solution presented in this section takes a simpler form:

$$\hat{q}_1(\theta; \varepsilon) = \varepsilon \frac{1 - \cos \theta}{q_{3i}^3} + \mathcal{O}(\varepsilon^2), \quad \hat{q}_2(\theta; \varepsilon) = \varepsilon \frac{-\sin \theta}{q_{3i}^3} + \mathcal{O}(\varepsilon^2),$$

and from it:

$$\hat{e}(\theta; \varepsilon) = \varepsilon \frac{\sqrt{2}}{q_{3i}^4} \sqrt{1 - \cos \theta} + \mathcal{O}(\varepsilon^2), \quad \hat{r}(\theta; \varepsilon) = \frac{1}{q_{3i}^2} + \varepsilon \frac{1 - \cos \theta}{q_{3i}^6} + \mathcal{O}(\varepsilon^2),$$

$$\hat{E}(\theta; \varepsilon) = -\frac{q_{3i}^2}{2} + \mathcal{O}(\varepsilon^2), \quad \hat{E}_T(\theta; \varepsilon) = -\frac{q_{3i}^2}{2} - \varepsilon \frac{1}{q_{3i}^2} + \mathcal{O}(\varepsilon^2),$$

$$\hat{\gamma}(\theta; \varepsilon) = \frac{\theta - \pi}{2} + \mathcal{O}(\varepsilon^2).$$

The secular term in θ has vanished from the solutions for $\hat{q}_2$ and $\hat{r}$, but this does not imply a good behavior for large values of the independent variable θ . Certainly, the numerical results displayed in later sections show that the approximation is still bad for $\theta \sim 1/\varepsilon$. It is also observed that the energy varies comparatively less than the other elements, no longer containing a term of $\mathcal{O}(\varepsilon)$.

8.4 Multiple Scales

8.4.1 Generalized orbital elements q_1, q_2

The breakdown of the regular expansion for $\theta \sim 1/\varepsilon$ suggests the existence of a slow ‘time’ scale in the independent variable θ . This hypothesis is further supported by the perturbation model for the tangential case presented in [15], and the solutions in terms of elliptic equations given for the radial problem by Battin [7]. All those cases are characterized by a fast, periodic evolution associated to the orbital period, and a slow, secular behavior whose characteristic period depends on the magnitude of the thrust. Furthermore, previous works by the PhD candidate (see [38]), confirm that the mathematical structure of the multiple scales problem supports exactly two independent ‘time’ scales, with additional slower scales being a correction of the slow one. Therefore, a high-order multiple scales solution is proposed by introducing a fast ‘time’ scale τ and a ‘slow’ time scale T in the form:

$$\tau = \theta, \quad T = \Omega(\varepsilon) \theta,$$

where the unknown function $\Omega(\varepsilon)$ represents a coordinate strain in the slow ‘time’ scale T ; note that it must fulfill $\Omega(0) = 0$. Using the chain rule, the derivative operator can be rewritten in terms of the new independent variables as:

$$\frac{d}{d\theta} = \frac{\partial}{\partial \tau} \frac{d\tau}{d\theta} + \frac{\partial}{\partial T} \frac{dT}{d\theta} = \frac{\partial}{\partial \tau} + \Omega(\varepsilon) \frac{\partial}{\partial T},$$

and introducing it into Eqs. (8.3) yields:

$$\frac{\partial}{\partial \tau} \begin{bmatrix} q_1(\tau, T) \\ q_2(\tau, T) \end{bmatrix} + \Omega(\varepsilon) \frac{\partial}{\partial T} \begin{bmatrix} q_1(\tau, T) \\ q_2(\tau, T) \end{bmatrix} = \frac{\varepsilon}{q_3 s^2} \begin{bmatrix} \sin \tau \\ -\cos \tau \end{bmatrix}. \quad (8.24)$$

The unknown function giving the coordinate strain in the slow scale will be approximated through its power series expansion in ε :

$$\Omega(\varepsilon) = \Omega_1 \varepsilon (1 + \Omega_2 \varepsilon) + \mathcal{O}(\varepsilon^3). \quad (8.25)$$

Note that coefficient Ω_1 is not determined by the secularity conditions in the multiple scales problem; it has been included as an additional degree of freedom and its value will be chosen as to simplify the expressions obtained for the zeroth order solution.

The series expansions for $q_1(\tau, T)$ and $q_2(\tau, T)$ in $\varepsilon \ll 1$ up to second order terms are now of the form:

$$\begin{aligned} q_1(\tau, T; \varepsilon) &= q_{10}(\tau, T) + \varepsilon q_{11}(\tau, T) + \varepsilon^2 q_{12}(\tau, T) + \mathcal{O}(\varepsilon^3), \\ q_2(\tau, T; \varepsilon) &= q_{20}(\tau, T) + \varepsilon q_{21}(\tau, T) + \varepsilon^2 q_{22}(\tau, T) + \mathcal{O}(\varepsilon^3), \end{aligned} \quad (8.26)$$

with initial conditions:

$$\begin{aligned} q_{10}(0, 0) &= q_{1i}, & q_{11}(0, 0) &= q_{12}(0, 0) = 0, \\ q_{20}(0, 0) &= q_{21}(0, 0) &= q_{22}(0, 0) &= 0. \end{aligned} \quad (8.27)$$

For compactness we also define:

$$s(\tau, T; \varepsilon) = s_0(\tau, T) + \varepsilon s_1(\tau, T) + \mathcal{O}(\varepsilon^2), \quad (8.28)$$

with:

$$\begin{aligned} s_0(\tau, T) &= q_{3i} + q_{10} \cos \tau + q_{20} \sin \tau, \\ s_1(\tau, T) &= q_{11} \cos \tau + q_{21} \sin \tau. \end{aligned}$$

Substituting Eqs. (8.26) into Eqs. (8.24), expanding in power series of the small parameter ε and retaining terms of order $\mathcal{O}(\varepsilon^0)$ yields:

$$\frac{\partial}{\partial \tau} \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} q_1(\tau, T) \\ q_2(\tau, T) \end{bmatrix} = \begin{bmatrix} q_{10}(T) \\ q_{20}(T) \end{bmatrix}, \quad (8.29)$$

with:

$$q_{10}(0) = q_{1i}, \quad q_{20}(0) = 0.$$

Hence, q_{10} and q_{20} are no longer constants, unlike their regular expansion counterparts $\hat{q}_{10}$ and $\hat{q}_{20}$, but rather functions of the slow scale T . Moreover, $q_{20}(T) \neq 0$ in general, and the corresponding simplifications introduced in the derivation of the regular solution cannot be made.

The zeroth order equations do not provide enough information to fully determine q_{10} and q_{20} . This is the expected behavior when applying the method of multiple scales. As shown in many classic perturbation texts (see [71, 49, 67]), the equations to close the zeroth order solution will be obtained from the cancellation of the secular terms in the subsequent order solution (also known as secularity condition). Collecting terms of order $\mathcal{O}(\varepsilon)$ in the expansion of Eqs. (8.24) yields:

$$\frac{\partial}{\partial \tau} \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} + \Omega_1 \frac{\partial}{\partial T} \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} = \frac{1}{q_{3i} s_0^2} \begin{bmatrix} \sin \tau \\ \cos \tau \end{bmatrix}. \quad (8.30)$$

Rearranging terms and integrating with respect to τ , the first order solutions q_{11} and q_{21} are reached:

$$\begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \end{bmatrix} + \left(\frac{\tau}{2} + \tan^{-1} \mathcal{K} \right) \begin{bmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{bmatrix} - \tau \Omega_1 \frac{\partial}{\partial T} \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \quad (8.31)$$

The functions $\mathcal{P}_1(\tau, T)$, $\mathcal{P}_2(\tau, T)$, $\mathcal{K}(\tau, T)$, $\mathcal{S}_1(T)$ and $\mathcal{S}_2(T)$ in the above expressions are reported in Appendix D.1. Suffice to say that they are all periodic functions of the fast scale τ while their dependence on the slow time scale T only appears through the zeroth order terms q_{10} and q_{20} . In addition, two unknown functions of the slow scale, $g_1(T)$ and $g_2(T)$, appear as constants of integration.

Imposing the cancellation of the τ secular terms in Eqs. (8.31) yields the following ODE system for q_{10} and q_{20} :

$$\Omega_1 \frac{dq_{10}}{dT} = - \frac{q_{20}}{q_{3i} (q_{3i}^2 - q_{10}^2 - q_{20}^2)^{3/2}}, \quad (8.32)$$

$$\Omega_1 \frac{dq_{20}}{dT} = \frac{q_{10}}{q_{3i} (q_{3i}^2 - q_{10}^2 - q_{20}^2)^{3/2}}, \quad (8.33)$$

$$q_{10}(0) = q_{1i}, \quad q_{20}(0) = 0. \quad (8.34)$$

Dividing Eq. (8.32) by Eq. (8.33) and integrating, a first integral is found in the form:

$$q_{10}^2(T) + q_{20}^2(T) = q_{1i}^2. \quad (8.35)$$

Introducing the first integral into Eqs. (8.32,8.33) and by further simplifying them with the choice of:

$$\Omega_1 = \frac{1}{q_{3i} (q_{3i}^2 - q_{1i}^2)^{3/2}},$$

they can be solved to yield:

$$q_{10}(T) = q_{1i} \cos T, \quad q_{20}(T) = q_{1i} \sin T. \quad (8.36)$$

What remains to be determined for a complete solution of the first order terms q_{11} and q_{21} are the unknown functions g_1 and g_2 , which can be obtained by looking at the $\mathcal{O}(\varepsilon^2)$ terms in the expansion of Eq. (8.24), from which one obtains the differential equation:

$$\frac{\partial}{\partial \tau} \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = -\Omega_1 \frac{\partial}{\partial T} \left(\begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} + \Omega_2 \begin{bmatrix} q_{10} \\ q_{20} \end{bmatrix} \right) + \frac{2s_1}{q_{3i}s_0^3} \begin{bmatrix} -\sin \tau \\ \cos \tau \end{bmatrix}. \quad (8.37)$$

The functions g_1 and g_2 as well as the slower scale frequency Ω_2 can, in principle, be determined by integrating the second member with respect to τ and imposing the cancellation of secular terms in τ . Unfortunately the integration cannot be completed in close analytical form. An approximate solution for the case of small initial eccentricity is however possible and is developed in Appendix D.2.

8.4.2 Physical Time

A multiple scales expression for the non-dimensional physical time can be obtained using the previous results and the time element presented in Eqs. (8.17,8.18). Introducing the new independent variables τ and T and expanding in power series of ε , the time element takes the form:

$$\zeta_0(\tau, T) = \zeta_{00}(\tau, T) + \varepsilon \zeta_{01}(\tau, T) + \mathcal{O}(\varepsilon^2),$$

The zeroth order term ζ_{00} is given by the differential equation in τ which results from setting $\varepsilon = 0$ in Eq. (8.18). After some manipulation, this yields:

$$\zeta_{00}(\tau, T) = q_{3i} \Omega_1 \tau + g_t(T), \quad (8.38)$$

where the additive function g_t comes from imposing the secularity condition to the solution for ζ_{01} . An approximation for small initial eccentricity is given in Appendix D.3, in the form of a power series expansion in q_{1i} ; interestingly, this solution turns out to be a linear function in T .

The introduction of the time element has allowed to simplify the multiple scales expression for the physical time by separating its secular and oscillatory components, with the former corresponding entirely to the time element. Moreover, the

Figure 8.1: Comparison of the non-dimensional time obtained for $e_0 = 0.4$ and $\varepsilon = 0.01$.

slope of the secular part, which takes the form of a straight line, differs from the one obtained for the regular expansion due to the contribution of the secular terms in the slow scale T coming from g_t , which represent a small correction depending on the thrust parameter ε .

Figures 8.1 and 8.2 compare this solution for the physical time with those obtained by the regular expansion and a high precision numerical propagator. It is straightforward to check the good agreement between the numeric (exact) solution and the multiple scales asymptotic solution.

8.4.3 Orbital Elements

The multiple scales solution for the Dromo generalized orbital parameters presented so far can be used to calculate the classical orbital elements by direct application of Eqs. (8.8,8.9). Nevertheless, it is interesting to also express them as a power series expansion in ε , to better appreciate their physical behavior. Introducing the expressions for $q_1(\tau, T)$ and $q_2(\tau, T)$ into the first of Eqs. (8.8) and expanding in power series of ε up to order one, the eccentricity can be given as follows:

$$e(\tau, T; \varepsilon) = e_0 + \frac{\varepsilon}{q_{1i}q_{3i}} (q_{10}q_{11} + q_{20}q_{21}) + \mathcal{O}(\varepsilon^2) . \quad (8.39)$$

Figure 8.2: Comparisson of the non-dimensional time obtained for $e_0 = 0.2$ and $\varepsilon = 0.005$.

For this expression to be valid, the term in ε must be a small correction to the initial eccentricity e_0 ; otherwise, the first of Eqs. (8.8) must be used. In both cases, the effect on the eccentricity of the low radial thrust is a bounded variation around its initial value e_0 , with an amplitude modulated by ε .

The non-dimensional orbital radius takes the form:

$$r(\tau, T; \varepsilon) = \frac{1}{q_{3i}s_0} - \varepsilon \frac{s_1}{q_{3i}s_0^2} + \mathcal{O}(\varepsilon^2). \quad (8.40)$$

This expression for r is bounded, as it should (see [7]), and the leading order term is oscillatory with period $2\pi/(1 - \Omega)$.

The Keplerian energy presents small oscillations about the constant zeroth-order term:

$$E(\tau, T; \varepsilon) = -\frac{q_{3i}^2 - q_{1i}^2}{2} + \varepsilon (q_{10}q_{11} + q_{20}q_{21}) + \mathcal{O}(\varepsilon^2), \quad (8.41)$$

while total energy takes the form:

$$E_T(\tau, T; \varepsilon) = -\frac{q_{3i}^2 - q_{1i}^2}{2} + \varepsilon \left(q_{10}q_{11} + q_{20}q_{21} - \frac{1}{q_{3i}s_0} \right) + \mathcal{O}(\varepsilon^2).$$

The conservation of E_T up to first order requires the second term to be constant. After some manipulations, this term can be written as:

$$q_{10}g_1 + q_{20}g_2 - \frac{1}{q_{3i}(q_{3i} - q_{10})},$$

which is a function of only the slow scale T . Imposing the conservation of E_T in T finally yields:

$$q_{10}g_1 + q_{20}g_2 = \frac{1}{q_{3i}(q_{3i} - q_{10})} - \frac{1}{q_{3i}(q_{3i} + q_{1i})}. \quad (8.42)$$

By introducing the expressions for g_1 and g_2 given in Appendix D.2 and expanding in power series of q_{1i} , it is checked that the conservation of energy is fulfilled for all the computed orders of g_1 and g_2 , leaving a residual of $\mathcal{O}(q_{1i}^{l+1})$ where l is the highest order of the expansions of g_1 and g_2 . Moreover, this result can be used to improve the accuracy of the expressions for e and E by eliminating the direct dependence with g_1 and g_2 :

$$q_{10}q_{11} + q_{20}q_{21} = \frac{1}{q_{3i}s_0} - \frac{1}{q_{3i}q_{1i}(q_{3i} + q_{1i})}. \quad (8.43)$$

The angle γ between the initial and osculating eccentricity vector can be given as:

$$\gamma(\tau, T; \varepsilon) = T + \varepsilon \left[\frac{1}{q_{1i}^2} [q_{10}(\mathcal{P}_2 + g_2) - q_{20}(\mathcal{P}_1 + g_1)] + 2\Omega_1 \arctan \mathcal{K} \right] + \mathcal{O}(\varepsilon^2). \quad (8.44)$$

The leading order term, which was absent from the regular expansion, shows a secular behavior in the slow scale. This can be compared with the regular expansion having a secular component in θ inside its first order term. On the other hand, the first order term is now periodic and bounded. Note that this expansion is not valid for initially circular or quasi-circular orbits.

8.4.4 Initially circular orbit

The particular case of initially circular orbit is specially interesting for this multiple scales solution, since the expressions given for g_1 and g_2 are then exact. Particularizing the previous results for $e_0 = 0$ yields:

$$q_1(\tau, T; \varepsilon) = \frac{\varepsilon}{q_{3i}^3} \left(\cos \frac{T}{q_{3i}^4} - \cos \tau \right) + \mathcal{O}(\varepsilon^2),$$

$$q_2(\tau, T; \varepsilon) = \frac{\varepsilon}{q_{3i}^3} \left(\sin \frac{T}{q_{3i}^4} - \sin \tau \right) + \mathcal{O}(\varepsilon^2).$$

Although q_{10} and q_{20} no longer appear, the variation with the slow scale is retained through the contributions from g_1 and g_2 . Comparing these expressions with those obtained for the regular expansion, it is straightforward to check that the latter coincide with the former for $T = 0$.

From this solution, it is possible to derive:

$$e(\tau, T; \varepsilon) = \varepsilon \frac{\sqrt{2}}{q_{3i}^4} \left[1 - \cos \left(\tau - \frac{T}{q_{3i}^4} \right) \right]^{1/2} + \mathcal{O}(\varepsilon^2),$$

$$r(\tau, T; \varepsilon) = \frac{1}{q_{3i}^2} + \varepsilon \frac{1}{q_{3i}^6} \left[1 - \cos \left(\tau - \frac{T}{q_{3i}^4} \right) \right] + \mathcal{O}(\varepsilon^2),$$

$$E(\tau, T; \varepsilon) = -\frac{q_{3i}^2}{2} + \mathcal{O}(\varepsilon^2),$$

$$E_T(\tau, T; \varepsilon) = -\frac{q_{3i}^2}{2} - \varepsilon \frac{1}{q_{3i}^2} + \mathcal{O}(\varepsilon^2),$$

$$\gamma(\tau, T; \varepsilon) = \arctan \frac{\sin(T/q_{3i}^4) - \sin \tau}{\cos(T/q_{3i}^4) - \cos \tau} + \mathcal{O}(\varepsilon).$$

Note that the leading order terms of e and r now show a new period $2\pi/(1 - \Omega/q_{3i}^4)$ in θ , resulting from the combination of the periods for the fast and slow ‘time’ scales.

8.4.5 Eccentricity vector rotational frequency

The number of orbits it takes for the eccentricity vector to describe a whole turn can be approximated through the leading order term in Eq. (8.44), yielding:

$$\frac{\theta^*}{2\pi} = \frac{1}{\Omega}.$$

It is interesting to express this result in terms of the physical time t . From Eq. (8.17), it follows that the only contributions to the secular evolution of t are those coming from the time element ζ ; then, using Eq. (8.38), it is possible to write:

$$t^* = (q_{3i}\Omega_1 + D_t\Omega) \theta^* = 2\pi \left(\frac{q_{3i}}{\varepsilon(1 + \Omega_2\varepsilon)} + D_t \right). \quad (8.45)$$

Figure 8.3 shows the error committed by the previous expression compared with a high-precision numerical propagation. As expected, the accuracy of the approximation degrades as ε and e_0 increase.

Figure 8.3: Error in the time needed for the eccentricity vector to complete a whole revolution. The error is defined as $err_{t^*} = |t^{\text{ms}}/t^{\text{num}} - 1|100$, where t^{ms} is the multiple scales solution, and t^{num} is a high-precision numerical solution. The curves are truncated at the values of ε and e_0 for which escape conditions are reached.

Figure 8.4: Evolution of parameter Ω_1 with the initial eccentricity e_0 . Starting from 1 for initially circular orbits, it grows rapidly with the ellipticity of the initial orbit.

Figure 8.5: Evolution of coefficient $q_{1i}\Omega_1^{1/4}$ with the initial eccentricity e_0 .

8.4.6 Validity of the expansion

The validity of the multiple scales solution depends on both the non-dimensional thrust parameter ε and the initial eccentricity e_0 . Whereas the influence of the former is easy to evaluate, the latter requires a more careful study. Two different aspects have to be considered, namely the expanding interval of the fictitious time in which the results are valid and the asymptoticness of the additional expansions for g_1 , g_2 and Ω_2 .

Before going into deeper detail, it is convenient to take a look at the evolution of Ω_1 with e_0 . As the ellipticity of the initial orbit is increased, the values of q_{3i} and q_{1i} approach and Ω_1 grows rapidly; this behavior can be seen in Figure 8.4. It is also important to study the evolution of the factor $q_{1i}\Omega_1^{1/4}$, since it plays an important part in the expansions for g_1 , g_2 and Ω_2 . As shown in Figure 8.5, it also grows with the initial eccentricity, although the values reached are noticeably smaller.

According to the theoretical basis of the multiple scales method, and neglecting for the time being the influence of the extra expansion performed in q_{1i} , the results should be valid for an expanding interval of the fictitious time of length at least $\mathcal{O}(1/\Omega_1\varepsilon)$. Since a higher order formulation is used, a longer interval of $\mathcal{O}(1/(\Omega_1\varepsilon)^2)$ could be expected, but there are no mathematical warranties for this (see [67]). Comparing the numerical results given in Section 8.5 with those obtained in [38] using a first order multiple scales formulation, a clear improvement in the expanding interval is observed for the higher order solution. In both cases, the expanding interval length depends on the value of $\varepsilon\Omega_1$, placing a limit for the

values of θ for which the expansion remains valid as a function of ε and e_0 .

The additional regular expansion performed in q_{1i} for the calculation of g_1 , g_2 and Ω_2 introduces further considerations regarding the validity of the solution with the initial eccentricity e_0 . By careful examination of the results in Appendix D.2, it is checked that the k -th term of the expansion for each of these functions is multiplied by:

$$\Omega_1 \left(q_{1i} \Omega_1^{1/4} \right)^k .$$

Consequently, the behavior of these expansions will depend on the coefficient $q_{1i} \Omega_1^{1/4}$, whose evolution was represented in Figure 8.5. Considering only the influence of this coefficient, the validity of the expansions should break for values of e_0 around 0.74, for which it becomes greater than one. However, the expansions also contain numerical coefficients which decrease as the order increases, offering an appreciable improvement. In any case, it is concluded that the quality of the solution degrades rapidly for nearly parabolic initial orbits.

8.5 Numerical Evaluation of the Results

In this section, the quality of the asymptotic solutions obtained so far is evaluated by comparing them with reference numerical solutions, computed using a high-precision integrator. It will be seen that the use of multiple scales not only improves the quality of the results, but also provides interesting information about the physics of the problem.

Figures 8.6-8.8 show the evolution of Dromo parameters (q_1, q_2) , eccentricity e , non-dimensional orbital radius r , angular displacement of the eccentricity vector γ and non-dimensional semi-major axis a for several values of the initial eccentricity e_0 and the non-dimensional thrust acceleration parameter ε . The values of e , r , γ and a have been calculated using Eqs. (8.8-8.9) and (8.6). The first set of results, Figures 8.6, corresponds to the case of $\varepsilon = 0.005$ and $e_0 = 0.2$; using Eqs. (8.11), the initial values of the Dromo variables associated to this e_0 are $q_{1i} = 0.1826$ and $q_{3i} = 0.9129$. The first conclusion is that the regular expansion fails very soon for $q_1(\theta)$; the mean value remains constant, so it cannot reproduce the evolution in the slow scale. Its behavior is better for $q_2(\theta)$, since it contains a secular term that approximately reproduces the sinusoidal slow scale evolution for small values of θ . It is interesting to remember that the secular components in the regular solution correspond to the first terms of the Taylor expansions of the expressions for $q_{10}(T)$ and $q_{20}(T)$ given by the multiple scales method. The multiple scales

Figure 8.6: Comparison of the solution obtained for q_1 , q_2 , e , r , γ and a in the radial thrust case, for $e_0 = 0.2$ and $\varepsilon = 0.005$

Figure 8.7: Comparison of the solution obtained for q_1 , q_2 , e , r , γ and a in the radial thrust case, for $e_0 = 0.1$ and $\varepsilon = 0.02$

Figure 8.8: Comparison of the solution obtained for q_1 , q_2 , e , r , γ and a in the radial thrust case, for $e_0 = 0$ and $\varepsilon = 0.02$

Figure 8.9: RMS errors in the non-dimensional orbital distance r of the regular and multiple scales asymptotic solutions, as a function of e_0 and ε . The thin black line marks the 5% error. The region to the left of the thick black line corresponds to orbits which escape before completing one revolution. For easier comparison, the color scale is the same in all the figures; the white areas correspond to relative errors lower than 10^{-6} .

solution turns out to be remarkably good, slowly separating from the real one as θ grows. The results for e , a and r inherit the properties from $q_1(\theta)$ and $q_2(\theta)$; since the reference solutions for e , a and r oscillate between fixed values, the evolutions in the slow scale of $q_1(\theta)$ and $q_2(\theta)$ must compensate each other to obtain a good approximation. This is not possible for the regular expansion, which only contains a secular term in $q_2(\theta)$. As a consequence, a spurious secular evolution appears for e and a , separating them from the reference solution very soon; while the amplitude of r increases with θ instead of remaining constant. On the other hand, the multiple scales formulation faithfully represents the real solution for the range of θ shown in the figures. Finally, a good agreement is observed between the reference and multiple scales solutions for γ , while the regular expansion slowly diverges from them.

Figures 8.7 correspond to an orbital propagation with $\varepsilon = 0.02$ and $e_0 = 0.1$ ($q_{1i} = 0.0935$, $q_{3i} = 0.9535$). Most of the comments made for the previous case still hold, only now the separation between the reference and the multiple scales solutions grows faster with θ . It is checked that the greater errors for the multiple scales solution come from the evolution of the mean values in the slow scale, not the amplitude or period of the oscillations in the fast scale. Therefore, the agreement with the reference solution is still very good for e , r and a , since in those cases the secular evolutions of the mean values cancel. On the other hand, the values of γ and a given by the regular expansion not only separate very soon from the reference solution, but are also incapable of reproducing the amplitude of the oscillations. The figure for $q_2(\theta)$ is particularly interesting, clearly showing the sinusoidal evolution of the mean value this parameter in the slow scale.

A case of initially circular orbit is considered in Figures 8.8, for a dimensionless perturbing acceleration of $\varepsilon = 0.02$. This example is of special interest for the study of the multiple scales solution, since the expressions used for g_1 and g_2 are now exact. It is observed that the amplitudes of the oscillations in both the regular and multiple scales solutions are slightly smaller than the amplitudes for the reference solution; this error is due to the order of the asymptotic approximation, and could be reduced by retaining terms of greater order in the expansion. Regarding the period of the oscillations, the results for initially circular orbits developed in the previous section suggested that said period is a combination of the characteristic periods for the fast and slow scales. This behavior is confirmed by the excellent agreement between the oscillation periods of both the reference and the multiple scales solutions, with a small drift driven by the slower time scales not included in the formulation. Meanwhile, the regular expansion, which does not take into account the slow time scale, shows a slightly shorter period than the order two

The particular cases considered so far have shown that the multiple scales solution behaves much better than the regular expansion, giving a more accurate

Figure 8.10: Evolution of an orbit with $e_0 = 0.7$ and $\varepsilon = 0.001$. The solid line corresponds to the reference numerical solution, while the dashed orbit is computed using the multiple scales solution.

description of the physics of the problem. Nevertheless, a systematic evaluation of the error of both methods is advisable. To this end, the RMS error in the non-dimensional orbital position has been calculated for a significant range in both the initial eccentricity and the non-dimensional thrust parameter [15]. The results are displayed in Figure 8.9, for several values of the final independent variable θ_f . In those cases where the escape is reached, the error is calculated at the value of θ for which the energy becomes zero in the numerical integration; a thick black line delimits the area corresponding to this group of orbits, associated with high values of e_0 and ε . In general, the errors are notably smaller for the multiple scales solution, and the difference between both asymptotic solutions increase with θ . However, the multiple scales solution presents large areas of bad behavior for high values of e_0 ; this is due to the approximations taken for the calculation of g_1 and g_2 , which are only valid for low values of the initial eccentricity, and the singularity at $e_0 = 1$.

Figure 8.10 shows an orbit with $e_0 = 0.7$ and $\varepsilon = 0.001$, for both the reference and the multiple scales solutions. The asymptotic solution remains close to the exact solution during the first orbital revolutions, but slowly separates for larger values of θ . The rotation of the eccentricity vector, given by γ , can be clearly appreciated.

Finally, Figures 8.11 and 8.12 compare the characteristic frequency of the long-

Figure 8.11: Comparison of the characteristic frequency of the long-term evolution of the orbit, obtained using high-precision numerical propagations and the multiple scales asymptotic solution.

Figure 8.12: Error in the value of the characteristic frequency of the long-term evolution of the orbit given by the multiple scales solution, using a high-precision numerical propagation as reference.

term evolution given by the multiple scales approximation with a high-precision numerical solution, for different values of e_0 and ε . As expected, the asymptotic solution is very close to the reference value for small e_0 and ε , and slowly separates from it as the initial eccentricity and the thrust acceleration increase.

8.6 Conclusion

Two asymptotic solutions for the two body problem perturbed by a small, constant acceleration oriented along the radial direction have been obtained, using regular expansions and the method of multiple scales. The equations of motion have been defined using Dromo orbital formulation, which has proven very adequate for this purpose. Some important conclusions are reached from studying the structure and behavior of these solutions:

- The regular expansion fails for $\theta \sim 1/\varepsilon$, where θ is Dromo independent variable, related with the true anomaly, and ε is the non-dimensional perturbing acceleration. Consequently, it cannot be used to propagate orbits for long periods of time, unless a reinitialization process like the one proposed by Bombardelli in [15] is included.
- The method of multiple scales reveals that the problem has two fundamental scales. The first one is responsible of the 2π -periodic oscillations along each orbit; while the second one, with a period depending on ε and e_0 , drives the long term variations of the mean values and the amplitudes of the oscillations. New periods appear in some cases as the combination of the two fundamental ones. As expected, this solution behaves noticeably better than the regular expansion in most cases.

Finally, to close the multiple scales solution for the terms of $\mathcal{O}(\varepsilon)$, an additional expansion in the initial eccentricity e_0 has been introduced.

Chapter 9

Low-Thrust Trajectory Optimization with Dromo

9.1 Introduction

The Dromo orbital propagator was initially introduced by Peláez et al. [74] and has been under active development since then [9, 8]. In addition to its excellent performance in terms of propagation speed and accuracy, the regularized formulation of Dromo has proven to be particularly advantageous to find approximate solutions of basic astrodynamics problems [15, 14, 38], such as the radial thrust problem studied in the previous chapter. Recently, the Dromo formulation has been employed as a basis for the solution of multi-revolution low-thrust optimization problems in Earth orbit [37] showing dramatic improvements when compared to a Cartesian based formulation. The reason behind this comes from three important characteristics of the method. First of all, its good performance in both precision and computational cost. Secondly, the state of the object is described by a set of redundant generalized orbital parameters, whose variations are related in magnitude and time scale with that of the control; this greatly improves the numerical behavior compared to a Cowell formulation, for which the variations of the state due to the acceleration exerted by the low-thrust engine are normally very small compared with the characteristic variations along one orbit. Finally, the same formulation is valid for elliptic, hyperbolic and parabolic orbits, making it applicable to many different scenarios and easily allowing for optimal trajectories that combine several types of orbit. Other non-singular element methods have been successfully used by other authors [12] with positive results.

In this chapter, a formulation of the OCP based on a direct transcription method and the Dromo orbital propagator is used to solve two types of Earth-bound problems: LEO to GEO transfers and escapes from Earth orbit. The

Figure 9.1: Schematic representation of the problem.

proposed formulation has a strong focus on efficiency, taking advantage of the sparsity of the discretized dynamic problem and using analytic expressions for the gradients of the equations. Approximated analytic results for the case of very low tangential thrust are used (see [15]) to construct an initial guess as close to the true optimum as possible.

The structure of the chapter is as follows. First of all, the model employed for the optimization problem with all its simplifying assumptions is defined. The equations of motion in Dromo variables are then briefly introduced and employed to formulate the planar, constant thrust control problem. Next, the optimal control problem solution strategy using a non-linear programming approach is described. Finally, numerical results for the optimization of a number of LEO to GEO and LEO to escape trajectories are presented and discussed.

9.2 Description of the problem

Let us consider a spacecraft S of mass m orbiting around a primary E of gravitational parameter μ . Its initial radial position is R_0 , measured from the center of the primary, and its initial angular position is ν_0 , measured from the initial eccentricity vector (i.e. ν_0 is the initial true anomaly). The only forces acting on the spacecraft are the attraction of the primary, which from now on is assumed to be the Earth, and the constant, in-plane thrust produced by its propulsion system F_t . The mass of the propellant is considered to be negligible compared to the initial mass of the spacecraft, so the latter can be taken as constant as a first approximation. Under these conditions, the OCP of determining the thrust orientation profile α to perform a series of Earth-bound, planar, optimum orbital maneuvers is studied.

All equations and variables considered hereafter are expressed in dimensionless

form: to this end, three characteristic magnitudes are introduced for length (R_0), mass (m) and time (n_0^{-1}), where n_0 is the angular frequency of the circular orbit of radius R_0 around the primary, $n_0 = \sqrt{\mu/R_0^3}$. The remaining magnitudes are derived from them, as shown in Appendix A, Table A.1.

9.3 Equations of Motion. Dromo orbital formulation

The dynamics is formulated using the Dromo orbital propagator, introduced in detail in Appendix A. For the planar problem, the motion of the particle is described by a 4-dimensional state vector:

$$(\tau, q_1, q_2, q_3), \quad (9.1)$$

where τ is the physical time, and q_1, q_2, q_3 are generalized orbital elements. The evolution of $\mathbf{q}$ is given by Eqs. (A.4) and (A.5), reproduced here for clarity:

$$\frac{d\tau}{d\theta} = \frac{1}{q_3 s^2} = m, \quad (9.2)$$

$$\frac{d}{d\theta} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{q_3 s^3} \begin{bmatrix} s \sin \theta & (s + q_3) \cos \theta \\ -s \cos \theta & (s + q_3) \sin \theta \\ 0 & -q_3 \end{bmatrix} \begin{bmatrix} \mathbf{f}_p \cdot \mathbf{i} \\ \mathbf{f}_p \cdot \mathbf{j} \end{bmatrix} = \mathbf{F}, \quad (9.3)$$

with:

$$s = q_3 + q_1 \cos \theta + q_2 \sin \theta, \quad (9.4)$$

$$r = \frac{1}{q_3 s}, \quad (9.5)$$

$$u = \frac{dr}{d\tau} = q_1 \sin \theta - q_2 \cos \theta, \quad (9.6)$$

where θ is a fictitious time introduced through a Sundman transformation, $\mathbf{f}_p$ is the total perturbing force, r is the orbital radius, u is the radial velocity, and s is the transversal velocity, all of them in non-dimensional form. Unity vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the radial and transversal directions respectively, and they define a local-vertical/local-horizontal (LVLH) reference frame.

A set of relations between Dromo parameters and the classical orbital elements is given in Appendix A, section A.3.1. It is important to highlight that, for planar orbits, θ becomes the inertial angular position of the particle measured from the initial eccentricity vector.

To conclude this brief introduction to Dromo, it is interesting to point out some interesting properties regarding its applicability for studying OCPs using direct methods:

- The same formulation is valid for any kind of orbit, allowing to solve a large variety of optimization problems.
- Speed and precision. This is capital, since a large number of orbits have to be computed while solving the NLP subproblem obtained from the discretization of the continuous OCP.
- Good behavior when integrated with fixed-step routines, which are normally used when solving OCPs using the direct method.
- The state varies in the same time scale as the perturbation. The main perturbation term is the thrust of the engine, whose orientation is the control law to be optimized; therefore, state and control vary in the same time scale.

9.3.1 Planar, Constant Thrust Problem

The motion of the particle in the planar case is governed by the ODE system formed by Eqs. (9.2) and (9.3), where (τ, q_1, q_2, q_3) is the state vector. To close the problem, it only remains to define the perturbing force, $\mathbf{f}_p$. The only perturbation considered in this work is the thrust produced by the propulsion system, whose orientation along the orbit will be the control function in the family of OCPs studied in the following section. The magnitude of the thrust is assumed to be constant, and it is defined in its non-dimensional form as:

$$\varepsilon = \frac{F_t}{m\mu/R_0^2} = \frac{A_t}{\mu/R_0^2}, \quad (9.7)$$

where F_t is the thrust exerted by the propulsion system, and $A_t = F_t/m$ is the corresponding acceleration. The study is centered in propulsion systems of high specific impulse I_{sp} , such as ionic, plasma or nuclear thermal thrusters, so ε and the specific fuel consumption take very small values. Taking the latter into account, for a first approximation the mass of the vehicle is assumed to be constant.

Projecting $\mathbf{f}_p$ into the LVLH frame yields:

$$\begin{bmatrix} \mathbf{f}_p \cdot \mathbf{i} \\ \mathbf{f}_p \cdot \mathbf{j} \end{bmatrix} = \begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \varepsilon \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix}, \quad \varepsilon = \sqrt{a_r^2 + a_\theta^2}, \quad (9.8)$$

where a_r is the radial component of the perturbing acceleration, a_θ is the transversal component of the perturbing acceleration, and α is the angle formed by the acceleration vector and the transversal direction. While expressing the perturbing forces in terms of α is convenient, normally the angle β formed by the acceleration

and the instantaneous velocity vector is of greater interest. The relation between α and β is given by:

$$\alpha = \beta + \arctan \left(\frac{e \sin \nu}{1 + e \cos \nu} \right). \quad (9.9)$$

Therefore, the problem can be posed in terms of α , which yields simpler equations than β , and then the latter can be recovered by applying Eq. (9.9).

9.4 Direct Transcription Method

OCPs can be solved using either direct or indirect methods; for this study, only the former are considered. In direct methods, the original, continuous OCP must be transcribed into a discrete Non-Linear Programming (NLP) problem, which can then be solved using a suitable algorithm. This implies approximating the continuous functions and equations involved by a set of discrete parameters and constraints, as well as defining an adequate objective function to minimize. To this end, a grid of M uniformly spaced nodes is defined for the independent variable. To ease the construction of the grid for free-final-time problems, a new change of independent variable is introduced:

$$\theta = \theta_0 + (\theta_f - \theta_0)t, \quad t \in [0, 1], \quad (9.10)$$

so that the position of the grid nodes in t is constant regardless of the values of θ_0 and θ_f . Then, the discrete values for Dromo fictitious time, state and control at the k -th node can be written as:

$$t^k = \frac{k-1}{M-1}, \quad \theta^k = \theta_0 + (\theta_f - \theta_0)t^k, \quad \tau^k = \tau(\theta^k), \quad \mathbf{q}^k = \mathbf{q}(\theta^k), \quad \alpha^k = \alpha(\theta^k).$$

The equations of motion are enforced by transcribing them into a set of defect constraints, using a suitable method from the Implicit Runge-Kutta family of algorithms for the integration of initial value problems. The advantages of IRK algorithms for the direct transcription of OCPs as well as the equations corresponding to the Trapezoidal and Hermite-Simpson Separated methods are given in Appendix B. Note that, since the equations for the physical time and the generalized orbital elements are uncoupled, only the former are considered for the definition of the defect constraints. In those cases where physical time plays a role, be it as part of the objective function or as a constraint of the OCP, the corresponding equations will be added.

The OCP may introduce additional constraints apart from those associated with dynamics. For the purposes of this study, it is important to consider the fixed final energy constraint, whose equation in Dromo is:

$$g^E(\mathbf{x}) \equiv \frac{1}{2} [(y_1^M)^2 + (y_2^M)^2 - (y_3^M)^2] - E_f^* = 0, \quad (9.11)$$

where $\mathbf{x}$ is the optimization variable, and E_f^* is the desired final energy; note that the particular case of escape can be obtained by setting $E_f^* = 0$. It is also interesting to consider the fixed final time constraint:

$$g^\tau(\mathbf{x}) \equiv \tau_f^* - \tau_f = 0, \quad (9.12)$$

where τ_f^* is the desired final time, and τ_f can be determined using the equations presented in the next paragraph for the minimum-time objective function.

To close the transcription of the OCP, a suitable expression must be given for the objective function J . Multiple objective functions can be defined depending on the problem; for this study, only the cases of maximum final energy and minimum total time are considered. The maximum energy case is a problem of Mayer, since the objective function only contains terms depending on the final value of the state:

$$J(\mathbf{x}) = -E = -\frac{1}{2} (q_1^2 + q_2^2 - q_3^2) \big|_{\theta=\theta_f} = -\frac{1}{2} [(q_1^M)^2 + (q_2^M)^2 - (q_3^M)^2]. \quad (9.13)$$

On the other hand, the minimum-time transfer is a problem of Lagrange, since it is expressed as a quadrature. Note that the physical time is a dependent variable in Dromo formulation. The quadrature for the final dimensionless time τ_F can be obtained from Eq. (9.2), yielding:

$$J = \tau_F = \int_{\theta_0}^{\theta_F} \frac{1}{q_3 s^2} d\theta = \int_{\theta_0}^{\theta_F} m(q_1, q_2, q_3, \theta) d\theta. \quad (9.14)$$

To transcribe Eq. (9.14), a numerical scheme consistent with the one used for the dynamics is recommended. Therefore, a trapezoidal quadrature is used for the Trapezoidal transcription:

$$J(\mathbf{x}) = \sum_{k=1}^{M-1} \frac{h_k}{2} (m^k + m^{k+1}) = \frac{h_1}{2} m^1 + \frac{1}{2} \sum_{k=2}^{M-1} (h_k + h_{k-1}) m^k + \frac{h_{M-1}}{2} m^M,$$

while a Simpson scheme is preferred for the HSS transcription:

$$J(\mathbf{x}) = \sum_{k=1}^{M-1} \frac{h_k}{6} (m^k + 4\bar{m}^{k+1} + m^{k+1}),$$

where m^k is the value of the integrand at the k -th node, $\bar{m}^k$ is the value of the integrand at the midpoint between the k -th and the $(k+1)$ -th nodes, and $h_k = \theta^{k+1} - \theta^k$.

To finalize this section on the direct transcription of the OCP, there are some issues regarding computational efficiency and memory usage worth noting. NLP

algorithms normally require to compute the Jacobian and Hessian of the constraints and the objective function (or a suitable approximation). Since the size of the grid needed to adequately transcribe most real life problems is large, so will be the resulting Jacobian. However, constraints associated with dynamics normally depend on a small number of adjacent nodes (particularly, two for the Trapezoidal and HSS methods), so their gradient matrix is sparse. Taking advantage of this can dramatically reduce the memory footprint and the computational effort required for both calculating the gradients and solving each NLP step (as briefly studied in Appendix B). Furthermore, the gradient matrix of the defect constraints can be constructed analytically using the gradients of the planar Dromo equations compiled in Appendix A, Eqs. (A.23-A.23). Note that the derivatives with respect to the control are not provided in Appendix A, since they are problem-dependent. For the OCP proposed in this chapter, the gradient with respect to the control $\mathbf{u} = \alpha$ takes the form:

$$\nabla_{\mathbf{u}} \mathbf{F} = \frac{\partial}{\partial \alpha} \mathbf{F} = \mathbf{R} \cdot \frac{\partial}{\partial \alpha} \begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \mathbf{R} \cdot \varepsilon \begin{bmatrix} \cos \alpha \\ -\sin \alpha \end{bmatrix}.$$

The gradient of the RHS of Eq. (9.2) is also needed to analytically construct the Jacobian for certain objective functions (particularly, minimum-time problems):

$$\nabla_{\mathbf{q}} m = -\frac{1}{q_3 s^3} \begin{bmatrix} 2 \cos \theta \\ 2 \sin \theta \\ 2 + s/q_3 \end{bmatrix},$$

$$\nabla_u m = 0,$$

$$\frac{\partial}{\partial \theta} m = \frac{2}{q_3 s^3} (q_1 \sin \theta - q_2 \cos \theta).$$

Analogously, the contribution of the defect constraints to the Hessian matrix shows a strong sparsity. Since the analytical expressions for the gradients of the constraints are known, the Hessian can be computed numerically using the method of finite differences of the gradients. By following the procedure detailed in Appendix B, only 6 evaluations of the gradients of the defect constraints are needed to compute a forward differences estimate of the Hessian (10 for a centered differences estimate).

Figure 9.2: LEO-escape maneuver, for $e_0 = 0$ and $\varepsilon = 10^{-3}$.

Figure 9.3: LEO-escape maneuver, for $e_0 = 0.72$, $\varepsilon = 10^{-4}$.

Figure 9.4: LEO to GEO transfer, for $e_0 = 0$ and $\varepsilon = 10^{-1}$.

Figure 9.5: LEO to GEO transfer, for $e_0 = 0.2$ and $\varepsilon = 10^{-3}$.

Figure 9.6: LEO to GEO transfer, for $e_0 = 0.5$ and $\varepsilon = 10^{-4}$.

Figure 9.7: Maximum energy increase from a GTO in 1 day, for $\varepsilon = 10^{-3}$

9.5 Numerical Experiments

The problem of finding the optimum thrust orientation control law in the two-body planar, constant thrust scenario previously presented is formulated, using Dromo to propagate the orbit and the direct method to transcribe the OCP as a NLP problem. A third-party software package is used to numerically solve this NLP problem, particularly Matlab's global optimization toolbox and its *interior point* algorithm. The algorithm selection is based on its support for sparse, large scale problems, a highly desirable property when dealing with problems coming from the discretization of dynamical systems, as already discussed.

One key aspect not yet considered is the construction of a good enough initial guess for the iterative NLP solver. The quality of said initial guess not only affects the number of iterations needed to reach the solution, but it is also a deciding factor for the convergence of the NLP algorithm. For the purposes of this study, a simple yet good enough initial guess can be constructed knowing that, for the two-body problem with constant thrust, the maximum instantaneous energy raise is achieved using tangential thrust [15]. Furthermore, this is also the asymptotic solution when $\varepsilon \rightarrow 0$ for maximizing the orbital energy increase during a fixed period of time. Taking these results into account, an initial guess is constructed propagating the orbit for the tangential thrust case until a certain stopping criterion is reached. Said stopping criterion will depend on the characteristics of the OCP: for escape problems, it is good enough to propagate the tangential thrust orbit until escape is reached; while for orbit transfers various criteria involving q_3 (which is linked to the angular momentum) or the orbital energy can be considered.

All the numerical experiments were run in a computer with an Intel® Core™ i5-3450 processor with 4 physical cores at 3.10 GHz, and 8 GB of DDR3 RAM at 1600 MHz. Parallel computation was not enabled. The function and constraints tolerances for the *Interior-point* algorithm were set to 10^{-10} .

The escape from an equatorial LEO is now considered, for different values of the initial eccentricity e_0 and the non-dimensional thrust parameter ε . In all cases, the maneuver starts at the perigee of the orbit ($\nu_0 = 0$), which is placed at an altitude of $r_{p0} = 300$ km. The numerical results have been computed using the Trapezoidal discretization, with a number of grid nodes depending on the value of ε (the lower the thrust the larger the number of revolutions needed to escape, thus requiring more nodes to get a fine enough grid). The first example, Figures 9.2, corresponds to an initially circular orbit and a thrust parameter of $\varepsilon = 10^{-3}$ ($a = 8.1346 \times 10^{-3} \text{ m/s}^2$), and is calculated using a grid of $M = 700$ nodes. The NLP algorithm only took 5 iterations to converge, for a total computational time of 1.1126 s. The thrust orientation remains close to the tangential direction during most of the maneuver, oscillating around it. This supports the already introduced notion that the tangential control law is the low-thrust asymptotic solution for

maximizing the orbital energy increase in a given time. The orbit shows the typical spiraling shape of low-thrust maneuvers, while the figures for the eccentricity and the semi-major axis confirm the similarities between the optimum and tangential control laws. A closer look at the eccentricity curves reveals that escaping using tangential thrust requires less orbits but more time than the optimum solution: this implies that the vehicle moves faster in the optimum case. Also noteworthy is that, while both the eccentricity and the amplitude of β increase with time, said variations tend to concentrate at the end of the maneuver. A much more eccentric initial orbit is considered in Figures 9.3, calculated for $e_0 = 0.72$, $\varepsilon = 10^{-4}$ ($a = 8.1346 \times 10^{-4} \text{ m/s}^2$) and $M = 5000$. In this case the convergence of the NLP algorithm was slower, with 42 iterations and 417.8463 s of computational time. A specially low value for thrust has been selected to obtain more interesting results, since the high initial eccentricity reduces the number of revolutions needed to escape. Thrust orientation angle β still oscillates around the tangential direction, but it no longer resembles a sinusoidal. It presents rapid variations near the apogee of the orbit, while the evolution in the vicinity of the perigee becomes smoother. There is also a qualitative change in the evolution with time of the eccentricity and the amplitude of β . A tendency to circularize the orbit reducing both magnitudes appears during most of the maneuver, followed by a strong increase near the end. The differences between the optimum and tangential thrust control laws grow bigger towards the end of the trajectory, still verifying that the optimum escape takes place in less time but with more orbits than the tangential thrust escape.

The next set of OCPs corresponds to three LEO to GEO transfers. Same as before, the maneuver starts at the perigee ($\nu_0 = 0$), which takes place at an altitude of $r_{p0} = 300 \text{ km}$, and the discretization is performed using the Trapezoidal method. The results for the first case, calculated with $e_0 = 0$, $\varepsilon = 10^{-1}$ ($a = 8.1346 \times 10^{-1} \text{ m/s}^2$) and $M = 300$, are represented in Figures 9.4. The convergence of the NLP algorithm was reached in 18 iterations and 3.2362 s. The value chosen for ε is rather large, not having a low-thrust application, but it serves to illustrate one of the benefits of the Dromo formulation. The curves for the eccentricity reveal that the orbit, which is elliptic during most of the maneuver, briefly becomes hyperbolic towards the end. Such a solution is possible because a formulation valid for all kind of orbits is used, offering a higher flexibility for the optimization process. Aside from this, the figures for e and a also include the curves corresponding to the tangential thrust control law: both cases are similar during the first part of the maneuver, but they separate as the optimum control law circularizes the orbit to fulfill the constraints. The next test problem, corresponding to $e_0 = 0.2$, $\varepsilon = 10^{-3}$ ($a = 8.1346 \times 10^{-3} \text{ m/s}^2$) and $M = 700$, is shown in Figures 9.5. This trajectory also showed a good behavior of the NLP algorithm, which required 18 iterations and 8.065 s to converge. The lower thrust leads to a

typical spiraling orbit, with β oscillating around a mean value slightly separated from the tangential direction. This mean value is negative, indicating that thrust shows a small bias towards the inner part of the orbit. It is also worth noting that the amplitude of β decreases with time, while the oscillations in the eccentricity grow larger. The last LEO to GEO transfer, Figures 9.6, is calculated for $e_0 = 0.5$, $\varepsilon = 10^{-4}$ ($a = 8.1346 \times 10^{-3} \text{ m/s}^2$) and $M = 5000$. Unlike the previous cases, the convergence of this test problem presented some issues, requiring 200 iterations and 401.7085 s. The most notable characteristic of this transfer is that thrust actually points in the direction opposite to velocity during some parts of the orbit, near the end of the maneuver. This decelerating regions take place around the perigee, and are associated to the need of circularizing the orbit to meet the final constraints. A closer inspection of the curves for the eccentricity and the thrust orientation angle reveals that, instead of gradually reducing the eccentricity along the maneuver, it is more efficient to follow a nearly tangential thrust profile during most of it, and circularize the orbit at the end.

Finally, the problem of determining the thrust orientation control law that maximizes the final energy E_f reached in a given time departing from a GTO ($e_0 = 0.72$) is considered. The maneuver starts at the perigee, at an altitude of $r_{p0} = 300 \text{ km}$, lasts one day and is performed with $\varepsilon = 10^{-3}$ ($a = 8.1346 \times 10^{-3} \text{ m/s}^2$). The OCP is discretized using the HSS method and a 300 points grid; under these conditions, the NLP algorithm took 14 iterations and 5.8399 s to converge. The results, represented in Figures 9.7, show that the thrust orientation angle remains close to the tangential direction, but with a bias towards the inner part of the orbit due to its high eccentricity. It is checked that β becomes zero at the perigee, and experiments the greatest variations near the apogee. Furthermore, its value is negative for the first half of the orbit, when moving from the perigee to the apogee, and positive during the other half. Regarding the eccentricity and the semi-major axis, both combine a secular and an oscillatory term, the latter having the same period of the orbit. The secular component for the eccentricity is decreasing, indicating that the orbit is circularized as the energy increases, while for the semi-major axis it is increasing. The improvement with respect to the tangential thrust case is small, as depicted in the curve for the energy.

9.6 Conclusion

A new formulation for solving optimal control problems of space trajectories, restricted to the planar, two bodies, low constant thrust case, has been defined using Dromo orbital propagator and a direct transcription method. The formulation has been developed with computational efficiency as a key aspect, obtaining analytical expressions for the gradients of the constraints coming from dynamics and exploit-

ing the sparsity of the involved Jacobian and Hessian matrices. The availability of a good initial guess based on the tangential thrust problem has allowed for fast and stable solutions.

The results obtained have confirmed Dromo as a great orbital propagator for this type of problems. This is due to several interesting properties. First of all, the same formulation is valid for all kinds of orbits. Secondly, it shows very good speed and precision, and behaves well when integrated with fixed-step routines (used in the transcription process). Finally, the state varies in the same time scale as the perturbation, which in this case is also the control.

The numerical experiments studied with the new formulation have yielded some interesting conclusions. In escape and energy increase problems, the optimum thrust profile approaches the tangential thrust case for small values of the non-dimensional thrust, and separates as it grows. Depending on the mission, the eccentricity of the initial orbit and the thrust, the differences between both control laws can be quite important or negligible. Regarding LEO to GEO transfers, it has been observed that the eccentricity reduction needed to reach the final circular orbit tends to concentrate at the final part of the maneuver.

Chapter 10

Conclusion and Future Works

Using orbital formulations other than the Cartesian coordinates can bring many advantages to the study of OCPs. Likewise, applying asymptotic techniques and perturbations methods (both conventional and non-conventional) can lead to a wealth of information in terms of approximate analytical solutions and a deeper understanding of the underlying physics. In this thesis, several OCPs have been studied in depth using two element-based and one relative motion formulations for dynamics, as well as a wide range of direct and indirect optimization methods. In all cases, a profound synergy has been identified between the exact, numerical solutions and the approximate, analytical expressions. The former benefit from the latter through the availability of highly-accurate initial guesses of the solution, needed to start the iterative algorithms involved. Conversely, the qualitative behaviors predicted by the latter can be verified through the results given by the former. On the other hand, the use of non-Cartesian formulation has also proven to have its drawbacks, increasing the complexity of the equations specially when using indirect methods.

The minimum-time, constant-thrust-acceleration phase change, radius change and inclination change problems have been studied in great detail, using a relative motion formulation in curvilinear coordinates and both the indirect and direct methods. In all cases, results show that the time of flight strongly depends on the ratio between the desired displacement and the available thrust acceleration, being possible to find simple-yet-accurate analytical expressions relating both magnitudes for the linearized problem. For the planar cases (phase change and radius change), by linearizing both the state and costate equations and performing an analysis of their relative orders of magnitude, it is possible to identify two limit cases with fundamental qualitative differences, called short-maneuver and long-maneuver regimes, separated by a transition zone. These short- and long-maneuver limit regimes correspond, respectively, to transfers requiring much less or more than one revolution of the reference orbit to complete, whereas transfers

in the transition zone present times of flight of approximately one period of the reference orbit. Approximate analytical solutions have been obtained for both limit regimes, highlighting the key characteristics of the optimal thrust control profiles. The thrust orientation control law for the rephasing maneuver in the short-maneuver regime follows a nearly bang-bang structure with the thrust initially oriented along the direction of the desired displacement and reversing itself symmetrically at half the maneuver interval to oppose the spacecraft motion until the end of the maneuver. Conversely, in the long-maneuver regime the optimal thrust orientation follows a nearly bang-bang structure that initially opposes the desired motion to exploit a change in semi-major axis towards obtaining the required rephasing and switches direction at half way to accelerate in the direction of motion. Regarding radius change maneuvers, in the short-maneuver regime the thrust orientation control law is nearly bang-bang along the radial direction, being initially oriented along the desired radial displacement and reversing itself at half the maneuver. Conversely, in the long-maneuver regime the optimal thrust orientation oscillates with small amplitude about the transversal direction, oriented along it for positive radial displacements and opposite to it for negative radial displacements. For the combined inclination and radius change problem, a synergy between both individual maneuvers has been found depending on the relative values of the corresponding displacement-thrust ratios. These approximate analytical solutions can be very useful to construct the initial guesses required by iterative numerical solvers for both the direct and indirect methods. Finally, extensive sets of numerical results have been presented to validate the analytical expressions and to assert the influence on the nonlinear terms.

The modified equinoctial elements have been used to study the end-of-life disposal of satellites in the Galileo constellation with low thrust propulsion, leveraging previously computed orbits leading to the natural re-entry of the satellite due to lunisolar perturbation effects within a 100 years timespan. The low-thrust transfers between the operational and disposal orbits have been designed using indirect optimization methods, putting a special focus in applying techniques to increase the robustness and efficiency of the optimization algorithm (such as continuation, propagation of the STM, and precise and fast determination of thruster-switching events). Numerical results for several representative test cases have shown the practical interest of this approach for the EoL disposal of future generations of Galileo satellites.

The use of MEE in the Galileo disposal study allowed to express the orbit-to-orbit boundary conditions in a simple way, at the cost of more complex state and costate equations. To alleviate this issue, a mathematical framework for the formulation of OCPs using indirect methods and a generic formulation for the dynamics has been proposed. The mathematical structure of the equations derived

from the first order optimality conditions has been leveraged to propose problem-independent formulas involving a set of problem-dependent matrices and their derivatives with respect to the state and mass. In particular, all the contributions of the costate elements associated to the state (excluding mass) have been reduced to a quadratic form of the costate and its derivatives. Relatively simple expressions have also been derived for the variational equation needed to compute the STM.

The last part of the thesis has dealt with the Dromo propagator. First of all, two asymptotic solutions for the two body problem perturbed by a small, constant acceleration oriented along the radial direction have been obtained, using regular expansions and the method of multiple scales. The equations of motion have been defined using Dromo orbital formulation, which has proven very adequate for this purpose. Applying the method of multiple scales has allowed to notably increase the accuracy and validity range of the solution compared to the regular expansion, and it has revealed that the problem has exactly two fundamental time scales (one linked to the periodic oscillations along each orbit, and the other to the long-term variations due to the small perturbing acceleration). Finally, the applicability of Dromo for the resolution of OCPs using the direct method has been evaluated by solving a series of planar test cases. The results obtained have confirmed Dromo as a great orbital propagator for this type of problems, based on several interesting properties. Firstly, the same formulation is valid for all kinds of orbits. Secondly, it shows very good speed and precision, and behaves well when integrated with fixed-step routines (used in the transcription process). Finally, the state varies in the same time scale as the perturbation, which in this case is also the control.

This thesis dissertation has tried to cover a wide range of mathematical and numerical techniques, orbital formulations and optimization methods. As a consequence, it has not been possible to fully explore all of the research lines. This yields a wealth of future works to be considered. Amongst others, it is possible to highlight:

- Obtaining solutions for the relative motion problem in curvilinear coordinates taking into account additional perturbations.
- Expanding the rephasing problem in curvilinear coordinates to include an improved analytical solution such as the one for the radius change problem.
- Developing new criteria for the selection of the reference orbit in radius change problems, in order to minimize the impact of the nonlinearities in the numerical convergence.
- Tackling radius change problems with eccentric arrival orbit using curvilinear coordinates.

- Improving the characterization of the synergy coefficient for the 3D orbit corrections.
- Adding new perturbations to the rephasing, radius change and inclination change problems in curvilinear coordinates.
- Applying the Galileo EoL disposal study to other constellations in MEO.
- Including new types of boundary conditions in the unified formulation for element-based indirect optimization, and developing a reference implementation (preferably with a modular structure).
- Applying the multiple scales technique to other fundamental problems in orbital mechanics (transversal thrust, tangential thrust, fundamental satellite problem, etc.).
- Continuing the direct optimization study in Dromo by including non-planar maneuvers.
- Comparing the numerical performance of the different formulations, both for direct and indirect methods and a varied and representative set of test cases.

Appendix A

Dromo Orbital Formulation

A.1 Introduction

In this appendix, the Dromo orbital formulation is briefly introduced. The derivation of Dromo is not reproduced here, but its main characteristics and some properties that are relevant to its application in an Optimal Control scenario are commented. After that, the equations of the method for the planar case are presented; this is the formulation used for the Optimal Control Problem proposed in Chapter 9 and the asymptotic solution for the radial thrust problem in Chapter 8.

A more detailed description of the method, along with some alternative formulations and several auxiliary expressions, can be seen in [74, 9, 94].

A.2 Dromo Formulation.

The special perturbation method known as Dromo was developed in the *Grupo de Dinámica de Tethers* (now *Space Dynamics Group*), of the Technical University of Madrid (UPM), for the dynamical simulation of tethers, but it can be used to propagate the orbit of any material particle. Its main characteristics are simplicity of computer implementation, speed and precision. It is appropriate for any kind of orbit, and works well with different numerical integration procedures (see [74, 15, 94]).

The method focuses on the evolution of several generalized orbital parameters based on Euler parameters; they are constants in the unperturbed problem, but evolve in the same time scale as the perturbation in the perturbed problem. To derive the method, a fictitious time θ is introduced through a Sundman transformation:

$$\frac{d\theta}{d\tau} = \frac{r^2}{h}, \quad (\text{A.1})$$

where r , τ , and h are the non-dimensional orbital distance, time and angular momentum respectively; this is the first step in the regularization of the equations of motion. Then, the variation of parameters technique is used to develop expressions for the derivatives of seven orbital elements for the general case, with an additional equation based on Eq. (A.1) for the time. Consequently, the state of the particle in the most general case is given by eight variables. A comprehensive derivation of the method can be found in [74], with a revisited formulation being proposed in [94]. It is also interesting the modified formulation developed in [9], for the case of perturbing forces that are partially or totally derivable from a potential.

Let us consider a particle O of mass m orbiting around a primary of gravitational parameter μ . The initial radial position is R_0 , measured from the center of the primary, and the initial angular position is ν_0 , measured from the initial eccentricity vector (i.e. ν_0 is the initial true anomaly). All equations and variables considered hereafter are expressed in non-dimensional form: to this end, three characteristic magnitudes are introduced for length (R_0), mass (m) and time (n_0^{-1}), where n_0 is the angular frequency of the circular orbit of radius R_0 around the primary, $n_0 = \sqrt{\mu/R_0^3}$. The remaining magnitudes are derived from them, as shown in Table A.1.

The particle is subjected to the action of the perturbation force $\mathbf{f}_p$, defined as the vector resultant of all forces except the main term of the gravitational potential; if $\mathbf{f}_p = \mathbf{0}$, the motion of the particle is Keplerian. An inertial frame $\mathcal{I} = \{O_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1\}$ is defined with its origin O_1 at the mass center of the primary, and its axis oriented along fixed directions, fulfilling the right-handed condition ($\mathbf{k}_1 = \mathbf{i}_1 \times \mathbf{j}_1$). Then, the position vector of the particle relative to $\mathcal{I}$ is $\mathbf{x} = \mathbf{O}_1\mathbf{O}$, its velocity $\mathbf{v} = \dot{\mathbf{x}}$, and its angular momentum $\mathbf{h} = \mathbf{x} \times \mathbf{v}$.

An orbital frame associated with the particle motion, $\mathcal{R} = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, is also defined. Its origin O is at the particle, and its unit vectors are :

$$\mathbf{i} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{R}, \quad \mathbf{j} = \mathbf{k} \times \mathbf{i}, \quad \mathbf{k} = \frac{\mathbf{h}}{h}, \quad (\text{A.2})$$

where $R(t)$ is the orbital distance. Therefore, $\mathcal{R}$ coincides with the local-vertical/local-horizontal (LVLH) reference system. It must be noted that this definition of

Table A.1: Characteristic values used to introduce non-dimensional variables.

Length	Time	Mass	Velocity	Acceleration	Force
R_0	$n_0^{-1} = \sqrt{R_0^3/\mu}$	m	$R_0 n_0$	$R_0 n_0^2$	$m R_0 n_0^2$

$\mathcal{R}$ coincides with the one given in [9] and [94], but differs slightly from the original one used in [74].

Dromo requires the introduction of an additional reference frame $\mathcal{R}_0$, named departure frame. Its attitude is represented by its Euler-Rodrigues parameters ε_1^0 , ε_2^0 , ε_3^0 and η^0 ; said parameters are part of the 8-dimensional state variable (for clarity, they are referred to as q_4 , q_5 , q_6 and q_7 hereafter). In the unperturbed problem $\mathcal{R}_0$ is fixed and defined as frame $\mathcal{R}$ at the initial time; while in the perturbed problem it represents the departure frame at the epoch, and changes in a time scale defined by the out-of-plane component of the perturbation.

The 8-dimensional state vector describing the motion of the particle can then be written as:

$$(\tau, q_1, q_2, q_3, q_4, q_5, q_6, q_7), \quad (\text{A.3})$$

where (q_4, q_5, q_6, q_7) are the Euler-Rodriguez parameters of $\mathcal{R}_0$, and (q_1, q_2, q_3) describe the shape of the orbit in its plane. The evolution of the state vector is given by the following system of eight differential equations:

$$\frac{d\tau}{d\theta} = \frac{1}{q_3 s^2}, \quad (\text{A.4})$$

$$\frac{d}{d\theta} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{q_3 s^3} \begin{bmatrix} s \sin \theta & (s + q_3) \cos \theta \\ -s \cos \theta & (s + q_3) \sin \theta \\ 0 & -q_3 \end{bmatrix} \begin{bmatrix} \mathbf{f}_p \cdot \mathbf{i} \\ \mathbf{f}_p \cdot \mathbf{j} \end{bmatrix}, \quad (\text{A.5})$$

$$\frac{d}{d\theta} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \end{bmatrix} = \frac{\mathbf{f}_p \cdot \mathbf{k}}{2q_3 s^3} \begin{bmatrix} 0 & 0 & -\sin \Delta\theta & \cos \Delta\theta \\ 0 & 0 & \cos \Delta\theta & \sin \Delta\theta \\ \sin \Delta\theta & -\cos \Delta\theta & 0 & 0 \\ -\cos \Delta\theta & -\sin \Delta\theta & 0 & 0 \end{bmatrix} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \\ q_7 \end{bmatrix}, \quad (\text{A.6})$$

where:

$$s = q_3 + q_1 \cos \theta + q_2 \sin \theta, \quad (\text{A.7})$$

$$r = \frac{1}{q_3 s}, \quad (\text{A.8})$$

$$u = \frac{dr}{d\tau} = q_1 \sin \theta - q_2 \cos \theta, \quad (\text{A.9})$$

$$\Delta\theta = \theta - \theta_0. \quad (\text{A.10})$$

There are several aspects worth noting about this formulation:

- Equation (A.4) is uncoupled from Eqs. (A.5)-(A.6), so the problem can be solved for $(q_1, q_2, q_3, q_4, q_5, q_6, q_7)$ without calculating the dimensionless time τ .

- Although parameters (q_4, q_5, q_6, q_7) do not appear directly in equations (A.5), they are needed to determine the instantaneous orientation of frame $\mathcal{R}$, in which $\mathbf{f}_p$ must be projected. Therefore, Eqs. (A.5) and (A.6) are normally coupled. Note that $\mathcal{R}$ and $\mathcal{R}_0$ are always related through a rotation of angle $\Delta\theta = \theta - \theta_0$ around $\mathbf{k}$.
- For the unperturbed problem, θ is the true anomaly of the Keplerian orbit. Therefore, the initial condition for θ in both perturbed and unperturbed problems is $\theta_0 = \nu_0$.
- The method presents two singular cases: one when the trajectory reaches a point at the infinite ($r = \infty$), and other when the angular momentum vanishes ($h = 0$). Nevertheless, these situations are uncommon.
- In the planar case, and without loss of generality, the inertial frame $\mathcal{I}$ can be defined with its axis parallel to the axis of $\mathcal{R}_0$; then, $i = \Omega = \omega = 0$, $\varepsilon_1^0 = \varepsilon_2^0 = \varepsilon_3^0 = 0$, and $\eta^0 = 1$.

A list of advantages and drawbacks of the method, as well as a series of benchmarks comparing it with other orbital propagators, are included in [74]. Some of them are specially interesting regarding its use in an OCP formulation:

- It has a unique formulation for the three types of orbits: elliptic, parabolic and hyperbolic. Therefore, a single formulation can be used with no or little modification to solve a great variety of problems.
- It works well with fixed step schemes for solving ODEs. This is specially useful when formulating the OCP through the direct method, since fixed step schemes are very common in direct transcriptions.
- It shows a great overall performance, understood as the relation between precision and computational time, when compared to other methods. Since the orbit is to be propagated many times while solving the OCP, this is very convenient.

A.2.1 Physical Interpretations

To improve the understanding of the method, it is interesting to consider the physical interpretation of some of the parameters involved.

A physical interpretation for independent variable θ is found by integrating the Sundman transformation, Eq. (A.1), to obtain:

$$\Delta\theta = \Delta\nu + \Delta\gamma, \quad (\text{A.11})$$

which underlines that the variation of θ differs from the variation of the true anomaly, ν , by an angular drift $\Delta\gamma$, due to the action of the orbital perturbations. Therefore, θ can be referred to as the pseudo true anomaly of the perturbed motion. According to [9], $\Delta\gamma$ takes the form:

$$\Delta\gamma = \Delta\omega + \int_{\Omega_0}^{\Omega} \cos i d\Omega, \quad (\text{A.12})$$

where $\Delta\omega$ is the argument of periapsis variation with respect to the initial orbit, Ω is the longitude of ascending node, and i is the orbital inclination. This shows that $\Delta\gamma$ is of the same order as, but generally not equal to (depending on the orbit evolution), the angle between the eccentricity vector of the initial and of the osculating orbits. In the particular case in which the orbit plane is invariant ($\Delta i = 0$), $\Delta\gamma$ coincides with such angle and θ becomes the inertial angular position of the particle measured from the eccentricity vector of the initial orbit.

There are some other variables and parameters with an interesting physical meaning. Variable q_3 is, by definition, the inverse of the specific angular momentum h . Radial and transverse velocities are, respectively, u and s , given by Eqs. (A.9) and (A.7). Distance is equal to $(q_3 s)^{-1}$, according to Eq. (A.8).

A.3 Planar case

Equations (A.6) show that the evolution of $\mathcal{R}_0$ is linked to the out-of-plane component of $\mathbf{f}_p$. If there is no such component, $\mathcal{R}_0$ is fixed and its Euler-Rodrigues parameters are constant. This corresponds to the planar case, in which the motion of the particle is described by a 4-dimensional state vector:

$$(\tau, q_1, q_2, q_3), \quad (\text{A.13})$$

whose evolution is given by Eqs. (A.4)-(A.5).

A.3.1 Some Useful Relations

In this subsection, some useful relations between the generalized orbital elements, the classical orbital elements and the Cartesian coordinates are shown. They have been compiled from [74] and [9], and particularized for the planar case.

Total energy is expressed in terms of the generalized orbital elements as:

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{R} = \frac{m\mu}{2R_0} [q_1^2 + q_2^2 - q_3^2]. \quad (\text{A.14})$$

Position and velocity in $\mathcal{R}$ are respectively $(r, 0, 0)$ and $(u, s, 0)$. The inertial position and velocity can then be obtained with a change of reference frame, yielding:

$$x = \frac{1}{q_3 s} \cos \Delta\theta, \quad y = \frac{1}{q_3 s} \sin \Delta\theta, \quad z = 0, \quad (\text{A.15})$$

$$\dot{x} = u \cos \Delta\theta - s \sin \Delta\theta, \quad \dot{y} = u \sin \Delta\theta + s \cos \Delta\theta, \quad \dot{z} = 0. \quad (\text{A.16})$$

The following equations allow one to calculate the generalized orbital elements from the classical orbital elements:

$$q_1 = \frac{e}{h} \cos \Delta\gamma, \quad q_2 = \frac{e}{h} \sin \Delta\gamma, \quad q_3 = \frac{1}{h}. \quad (\text{A.17})$$

It is also useful to derive the expression of the initial dimensionless angular momentum:

$$h_0 = \frac{\dot{\nu}_0 r_0^2}{n_0 r_0^2} = \sqrt{1 + e_0 \cos \nu_0}, \quad (\text{A.18})$$

and the initial value of the generalized orbital elements q_1 , q_2 and q_3 :

$$q_{10} = \frac{e_0}{\sqrt{1 + e_0 \cos \nu_0}}, \quad q_{20} = 0, \quad q_{30} = \frac{1}{\sqrt{1 + e_0 \cos \nu_0}}. \quad (\text{A.19})$$

Using Eqs. (A.17), the orbit eccentricity, the angular drift, the non-dimensional angular momentum and the true anomaly can be obtained as functions of the generalized orbital elements q_1 , q_2 and q_3 :

$$e = \frac{\sqrt{q_1^2 + q_2^2}}{q_3}, \quad \Delta\gamma = \tan^{-1} \left(\frac{q_2}{q_1} \right), \quad h = \frac{1}{q_3}, \quad (\text{A.20})$$

$$\nu = \tan^{-1} \left(\frac{q_1 \sin \theta - q_2 \cos \theta}{q_1 \cos \theta + q_2 \sin \theta} \right). \quad (\text{A.21})$$

Finally, the next set of equations allows one to obtain the generalized orbital elements from the inertial position and velocity:

$$\begin{aligned} q_1 &= \left(\frac{h}{r} - \frac{1}{h} \right) \cos \theta + u \sin \theta, \\ q_2 &= \left(\frac{h}{r} - \frac{1}{h} \right) \sin \theta - u \cos \theta, \quad q_3 = \frac{1}{h}, \end{aligned} \quad (\text{A.22})$$

where:

$$r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}, \quad u = \mathbf{v} \cdot \mathbf{i} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r}, \quad h = |\mathbf{x} \times \mathbf{v}|.$$

A.3.2 Analytical gradients

The gradients of the planar Dromo equations can be computed analytically in a quite straightforward manner. Let us consider:

$$\mathbf{F} = \mathbf{R} \cdot \begin{bmatrix} a_r \\ a_\theta \end{bmatrix},$$

where $\mathbf{F}$ is the RHS of Eqs. (A.5). Then, the gradient of $\mathbf{F}$ with respect to the reduced state vector $\mathbf{q}^T = (q_1, q_2, q_3)$ takes the form:

$$\nabla_{\mathbf{q}} \mathbf{F}^T = \left(\frac{\partial}{\partial q_1} \mathbf{F} \quad \frac{\partial}{\partial q_2} \mathbf{F} \quad \frac{\partial}{\partial q_3} \mathbf{F} \right), \quad (\text{A.23})$$

where:

$$\frac{\partial}{\partial q_i} \mathbf{F} = \frac{\partial}{\partial q_i} \mathbf{R} \cdot \begin{bmatrix} a_r \\ a_\theta \end{bmatrix}, \quad (\text{A.24})$$

assuming that the perturbing acceleration vector does not depend on the state vector. Performing the partial derivatives and grouping terms:

$$\frac{\partial}{\partial q_1} \mathbf{R} = \frac{\cos \theta}{q_3 s^4} \begin{bmatrix} -2 \sin \theta s & -\cos \theta (2s + 3q_3) \\ 2 \cos \theta s & -\sin \theta (2s + 3q_3) \\ 0 & 3q_3 \end{bmatrix}, \quad (\text{A.25})$$

$$\frac{\partial}{\partial q_2} \mathbf{R} = \frac{\sin \theta}{q_3 s^4} \begin{bmatrix} -2 \sin \theta s & -\cos \theta (2s + 3q_3) \\ 2 \cos \theta s & -\sin \theta (2s + 3q_3) \\ 0 & 3q_3 \end{bmatrix}, \quad (\text{A.26})$$

$$\frac{\partial}{\partial q_3} \mathbf{R} = \frac{1}{q_3^2 s^4} \begin{bmatrix} -\sin \theta s(s + 2q_3) & -\cos \theta (2q_3^2 + (s + q_3)^2) \\ \cos \theta s(s + 2q_3) & -\sin \theta (2q_3^2 + (s + q_3)^2) \\ 0 & 3q_3^2 \end{bmatrix}. \quad (\text{A.27})$$

Finally, the derivative with respect to the independent variable θ is given by:

$$\frac{\partial}{\partial \theta} \mathbf{F} = \frac{\partial}{\partial \theta} \mathbf{R} \cdot \begin{bmatrix} a_r \\ a_\theta \end{bmatrix} = \frac{1}{q_3 s^4} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix} \cdot \begin{bmatrix} a_r \\ a_\theta \end{bmatrix}, \quad (\text{A.28})$$

where:

$$\begin{aligned} D_{11} &= s [2q_1 - (-s + 2q_1 \cos \theta + 2q_2 \sin \theta) \cos \theta], \\ D_{12} &= \cos \theta (2s + 3q_3) (q_1 \sin \theta - q_2 \cos \theta) - s \sin \theta (s + q_3), \\ D_{21} &= -2 \cos \theta s (q_1 \sin \theta - q_2 \cos \theta) + s^2 \sin \theta, \\ D_{22} &= \sin \theta (q_1 \sin \theta - q_2 \cos \theta) (2s + 3q_3) + s \cos \theta (s + q_3), \\ D_{31} &= 0, \\ D_{32} &= -3q_3 (q_1 \sin \theta - q_2 \cos \theta). \end{aligned} \quad (\text{A.29})$$

A.4 Conclusion

The special perturbation method for orbital dynamics known as Dromo has been briefly presented, highlighting its main characteristics and advantages. Some of them are specially useful in an OCP context, mainly:

- The same formulation is valid for any kind of orbit, allowing to solve a large variety of optimization problems.
- Simplicity of computational implementation. It is possible to derivate the analytic gradient for some cases of interest, see [37].
- Speed and precision. This is of great importance, since a large number of orbits have to be computed while solving the OCP.
- Good behavior when integrated with fixed-step routines, which are normally used in direct methods when transcribing the OCP into a NLP subproblem.
- The state varies in the same time scale as the perturbation. If the main perturbation term is the acceleration produced by the thruster, state and control will vary in the same time scale.

In the second part of the appendix, the equations of the method have been particularized for the planar case. Useful relations between the Dromo elements, the classical orbital elements and the Cartesian coordinates have been provided, and the analytical gradients for the planar Dromo equations have been obtained.

Appendix B

Direct Transcription for Optimal Control Problems

B.1 Introduction

Optimal Control Problems normally involve continuous functions for the state and control, while in NLP problems the optimization vector is formed by a finite set of variables. The process of converting the continuous problem into a finite-dimensional approximation is called transcription, and can be performed in a number of different ways (collocation methods, spectral methods, etc.) [11, 92].

One key issue when transcribing an OCP is how to enforce the system of differential equations that describes the dynamics. The most typical approach [12] is to define a set of defect constraints derived from a suitable numerical method for solving ODE initial value problems. In this appendix, three common ways to define said defect constraints, all based in the family of Implicit Runge-Kutta (IRK) methods, are presented.

Other capital aspect is the efficient computation of the gradients of the objective function and the nonlinear constraints. In most real life problems the size of the grid needed to transcribe the problem is very large, and so is the nonlinear constraints gradients matrix. However, constraints associated to dynamics normally depend on a small number of adjacent nodes, so their gradient matrix is sparse: taking advantage of this can dramatically reduce the computational effort required for both calculating the gradients and solving each NLP step. Techniques to construct the defect constraints gradients matrix for two different transcriptions (trapezoidal and Hermite-Simpson Separated) are presented.

Analogously, the contribution of the defect constraints to the Hessian matrix shows strong sparsity. If the analytical expressions for the gradients are known, the method of finite differences of the gradients can be used to efficiently compute

the Hessian.

B.2 Some Previous Concepts

When facing the election of a scheme for solving an ODE initial value problem, three big questions arise: Single-step or multiple-step? Fixed-step or variable-step? Implicit or explicit? The answers are very important, and in a NLP context they may differ from the usual ones.

Regarding the single-or-multiple step issue, single step is preferred. These algorithms respond better to fast variations in the problem (see [48]), which normally appear at intermediate iterations (or even at the final result, e.g. for bang-bang control laws), and behave better with relative large steps. Therefore, the Runge-Kutta family of schemes is selected.

The fixed-or-variable step question is trickier. Normally, variable step algorithms present better performance and are used for most applications; therefore, it is tempting to use them here too. However, the behavior of the NLP problem quickly degrades when using variable steps algorithms, both in computational time and amount of failed cases. The reasons behind this are not obvious, and correspond to what Betts [12] classifies as noise associated to a non-coherent discretization: since the step size changes each time, an additional source of nonlinearity is introduced when numerically evaluating the gradients. Therefore, fixed-step schemes are selected.

Implicit methods are normally preferred over explicit methods when performing direct transcriptions. The reason behind this is twofold:

- The main advantages of implicit algorithms over explicit ones are better stability properties, and greater order for the same number of stages. Both properties are convenient in a NLP formulation.
- The drawback of implicit methods is their higher computational cost, but this behaves differently when used inside a NLP problem. Let us consider the following generic formulation for a step in an implicit Runge-Kutta algorithm:

$$\mathbf{y}^{k+1} = \mathbf{f}(\mathbf{y}^{k+1}, \mathbf{y}^k, t_k), \quad (\text{B.1})$$

where $\mathbf{f}$ is a function that depends on the scheme, $\mathbf{y}^k$ is the dependent variable at the k -th step, $\mathbf{y}^{k+1}$ is the dependent variable at the $(k+1)$ -th step, and t_k is the independent variable at the k -th step. Equation (B.1) defines a nonlinear system of equations which has to be solved for $\mathbf{y}^{k+1}$ (for instance, using a Predictor-Corrector method), which requires an important computational effort. When used inside a NLP problem, however, that equation is

expressed as a nonlinear equality constraint:

$$\mathbf{0} = \mathbf{y}^{k+1} - \mathbf{f}(\mathbf{y}^{k+1}, \mathbf{y}^k, t_k), \quad (\text{B.2})$$

where $\mathbf{y}^{k+1}$ and $\mathbf{y}^k$ are included in the optimization variable $\mathbf{x}$. When the NLP algorithm tries to enforce Eq. (B.2) it does so modifying simultaneously $\mathbf{y}^{k+1}$ and $\mathbf{y}^k$, effectively acting as a Predictor-Corrector algorithm and alleviating the computational cost issues.

In conclusion, fixed step Implicit Runge-Kutta (IRK) methods are a very suitable choice for NLP problems. Three IRK schemes for the direct transcription of OCPs are now presented, assuming a generic dynamical system in the form:

$$\frac{d\mathbf{y}}{d\theta} = \mathbf{F}(\mathbf{y}, \mathbf{u}; \tau), \quad (\text{B.3})$$

where $\mathbf{F}$ is the right-hand side of the ODE system that describes the dynamics, $\mathbf{y}$ is the m -th dimensional state vector, $\mathbf{u}$ is a l -th dimensional control vector, and τ is the independent variable or “time”¹.

Before commencing the description of the different methods, three important remainders:

1. A grid of fixed nodes, either uniformly spaced or not, is always used to transcribe the continuous functions of the problem into a set of discrete values, that form the optimization variable, $\mathbf{x}$.
2. When using IRK schemes, the state at the nodes of the grid is part of the optimization variable. In some cases, the control or even the state at the middlepoints are also part of said variable.
3. The equations of the IRK method are enforced through a set of nonlinear equality constraints, called the *defect constraints*.

B.3 Trapezoidal Method

This is a 2-stages, 3rd order, IRK scheme, given by equation:

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \frac{h_k}{2}(\mathbf{F}_{k+1} + \mathbf{F}_k), \quad (\text{B.4})$$

¹Depending on the dynamical model, independent variable τ may not correspond to a physical time. However, in the optimal control literature it is common to refer to the independent variable as time. From now on, quotation marks are used to denote this generalized use of the term “time”.

where:

$$\mathbf{F}_k = \mathbf{F}(\mathbf{y}^k, \mathbf{u}^k, \tau_k), \quad (\text{B.5})$$

being $\mathbf{y}^k$, $\mathbf{u}^k$ and τ_k the state, control and “time” at the k -th node, respectively. Expressing it as a defect constraint yields:

$$\boldsymbol{\xi}_k(\mathbf{x}) \equiv \mathbf{y}^{k+1} - \mathbf{y}^k - \frac{h_k}{2}(\mathbf{F}_{k+1} + \mathbf{F}_k) = \mathbf{0} \quad \text{for } k = 1, \dots, M-1. \quad (\text{B.6})$$

The optimization variable $\mathbf{x}$ is now defined as a vector of length $n = (m+l)M+1$ in the form:

$$\mathbf{x}^T = (\tau_F, \mathbf{y}^1, \mathbf{u}^1, \dots, \mathbf{y}^k, \mathbf{u}^k, \dots, \mathbf{y}^M, \mathbf{u}^M), \quad (\text{B.7})$$

being M the number of nodes. To account for the possibility of free final “time” τ_F , a new independent variable t is introduced in the form:

$$\begin{aligned} \tau_k &= t_k \tau_F, \\ 0 &= t_1 < t_2 < \dots < t_k < \dots < t_M = 1, \end{aligned} \quad (\text{B.8})$$

where t_k is the new independent variable at the k -th node. Note that the values of t_k are fixed, defined by the grid spacing, while those of τ_k vary with τ_F . This way, the range of τ can vary without affecting the grid spacing.

Imposing the defect constraints at all nodes except the last, the resulting $m(M-1)$ nonlinear constraints derived from dynamics can be written as:

$$\mathbf{c}(\mathbf{x}) = \begin{bmatrix} \boldsymbol{\xi}_1(\mathbf{x}) \\ \boldsymbol{\xi}_2(\mathbf{x}) \\ \vdots \\ \boldsymbol{\xi}_{M-1}(\mathbf{x}) \end{bmatrix}. \quad (\text{B.9})$$

To exploit sparsity, it must be noted that, for each defect constraint, only two consecutive nodes appear. Grouping the linear terms and considering $h_k = \tau_{k+1} - \tau_k = (t_{k+1} - t_k)\tau_F = \Delta t_k \tau_F$, Eq. (B.9) can be written in matrix form as:

$$\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{p}(\mathbf{x}), \quad (\text{B.10})$$

where matrices $\mathbf{A}$ and $\mathbf{C}$ are sparse and constant, and vector $\mathbf{p}(\mathbf{x})$ contains the nonlinear terms:

$$\mathbf{p}(\mathbf{x}) = \begin{bmatrix} \tau_F \mathbf{F}_1 \\ \tau_F \mathbf{F}_2 \\ \vdots \\ \tau_F \mathbf{F}_M \end{bmatrix}. \quad (\text{B.11})$$

For a general case, matrices $\mathbf{A}$ and $\mathbf{C}$ have the following expressions:

$$\mathbf{A} = \begin{bmatrix} 0 & -\mathbf{I}_m & \mathbf{0}_{m,l} & \mathbf{I}_m & \mathbf{0}_{m,l} & & & \\ & & & -\mathbf{I}_m & \mathbf{0}_{m,l} & \mathbf{I}_m & \mathbf{0}_{m,l} & \\ \vdots & & & & & \ddots & & \\ & & & & & & -\mathbf{I}_m & \mathbf{0}_{m,l} & \mathbf{I}_m & \mathbf{0}_{m,l} \end{bmatrix}, \quad (\text{B.12})$$

$$\mathbf{C} = -\frac{1}{2} \begin{bmatrix} \Delta t_1 \mathbf{I}_m & \Delta t_1 \mathbf{I}_m & & & & \\ & \Delta t_2 \mathbf{I}_m & \Delta t_2 \mathbf{I}_m & & & \\ & & & \ddots & & \\ & & & & \Delta t_{M-1} \mathbf{I}_m & \Delta t_{M-1} \mathbf{I}_m \end{bmatrix}, \quad (\text{B.13})$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix, and $\mathbf{0}_{m,l}$ is the $m \times l$ zero matrix. It is interesting to note that $\mathbf{A}$ is a $m(M-1) \times n$ matrix and $\mathbf{C}$ a $m(M-1) \times mM$ matrix.

B.3.1 Gradient of the constraints

Considering only Eq. (B.10), the gradient of the constraints is:

$$\mathbf{G} \equiv \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \equiv \mathbf{A} + \mathbf{C}\mathbf{P}, \quad (\text{B.14})$$

where:

$$\mathbf{P} \equiv \frac{\partial \mathbf{p}}{\partial \mathbf{x}} \equiv \begin{bmatrix} \mathbf{P}_{10} & \mathbf{P}_{11} & & & \\ \mathbf{P}_{20} & & \mathbf{P}_{22} & & \\ \vdots & & & \ddots & \\ \mathbf{P}_{M0} & & & & \mathbf{P}_{MM} \end{bmatrix}, \quad (\text{B.15})$$

$$\mathbf{P}_{k0} = \mathbf{F}_k + \tau_F \frac{\partial \mathbf{F}_k}{\partial \tau_F}, \quad \mathbf{P}_{kk} = \tau_F \frac{\partial \mathbf{F}_k}{\partial \mathbf{x}^k},$$

with $\mathbf{x}^k = (\mathbf{y}^k, \mathbf{u}^k)$.

B.4 Hermite-Simpson Methods

The Hermite-Simpson method is a 3-stages, 4th order IRK scheme. Its main equations are:

$$\mathbf{0} = \bar{\mathbf{y}}^{k+1} - \frac{1}{2}(\mathbf{y}^{k+1} + \mathbf{y}^k) - \frac{h_k}{8}(\mathbf{F}_{k+1} - \mathbf{F}_k) \quad \text{Hermite interpolant} \quad (\text{B.16})$$

$$\mathbf{0} = \mathbf{y}^{k+1} - \mathbf{y}^k - \frac{h_k}{6}(\mathbf{F}_{k+1} + 4\bar{\mathbf{F}}_{k+1} + \mathbf{F}_k) \quad \text{Simpson quadrature} \quad (\text{B.17})$$

for $k = 1, \dots, M - 1$, with:

$$\bar{\mathbf{F}}_{k+1} = \mathbf{F}(\bar{\mathbf{y}}^{k+1}, \bar{\mathbf{u}}^{k+1}, t_k \tau_F + h_k/2),$$

where $\bar{\mathbf{y}}^{k+1}$ and $\bar{\mathbf{u}}^{k+1}$ are, respectively, the state and the control at the middle point between nodes k and $k + 1$. Two different formulations arise depending on whether the state at the middle points, $\bar{\mathbf{y}}^{k+1}$, is included in the optimization variable or not: Hermite-Simpson Separated (HSS) and Hermite-Simpson Compressed (HSC), respectively. The latter is briefly presented now, while the former is studied in section B.5.

B.4.1 Hermite-Simpson Compressed

In the HSC scheme, the state at the middle points is eliminated from the Simpson quadrature, Eq. (B.17), using the Hermite interpolant, Eq. (B.16). Consequently, the optimization variable includes the control at the middle points, but not the state:

$$\mathbf{x}^T = (\tau_F, \mathbf{y}^1, \mathbf{u}^1, \bar{\mathbf{u}}^2, \mathbf{y}^2, \mathbf{u}^2, \dots, \bar{\mathbf{u}}^M, \mathbf{y}^M, \mathbf{u}^M), \quad (\text{B.18})$$

and the defect constraints come from imposing the Simpson quadrature:

$$\boldsymbol{\xi}_k(\mathbf{x}) = \mathbf{y}^{k+1} - \mathbf{y}^k - \frac{h_k}{6} [\mathbf{F}_{k+1} + 4\bar{\mathbf{F}}_{k+1} + \mathbf{F}_k] \quad \text{for } k = 1, \dots, M - 1. \quad (\text{B.19})$$

This yields a total of $m(M - 1)$ defect constraints. Although it is the same amount of constraints as for the trapezoidal case, now the RHS function $\mathbf{F}$ has to be evaluated at the nodes and at the middle points, a total of $2M - 1$ evaluations versus the M evaluations of the trapezoidal case; therefore, the computational cost of evaluating the constraints is almost doubled.

The HSC formulation requires less variables and constraints than the HSS formulation, but also shows worse sparsity properties and more complex expression for constructing the gradients from the analytical derivatives of $\mathbf{F}$. Therefore, HSS will be preferred in this dissertation for solving large scale problems, and the analytical, sparse expressions of $\mathbf{G}$ and ∇J for the HSC method are neither developed nor used here.

B.5 Hermite-Simpson Separated

In the HSS scheme, both state and control at the middle points of the grid are introduced in the optimization variable:

$$\mathbf{x}^T = (\tau_F, \mathbf{y}^1, \mathbf{u}^1, \bar{\mathbf{y}}^2, \bar{\mathbf{u}}^2, \mathbf{y}^2, \mathbf{u}^2, \dots, \bar{\mathbf{y}}^M, \bar{\mathbf{u}}^M, \mathbf{y}^M, \mathbf{u}^M), \quad (\text{B.20})$$

whose length is $n = (m + l)(2M - 1) + 1$.

Defect constraints are formed by both Eqs. (B.16) and (B.17):

$$\boldsymbol{\xi}_k(\mathbf{x}) = \begin{bmatrix} \bar{\mathbf{y}}^{k+1} - \frac{1}{2}(\mathbf{y}^{k+1} + \mathbf{y}^k) - \frac{h_k}{8}(\mathbf{F}_{k+1} - \mathbf{F}_k) \\ \mathbf{y}^{k+1} - \mathbf{y}^k - \frac{h_k}{6}(\mathbf{F}_{k+1} + 4\bar{\mathbf{F}}_{k+1} + \mathbf{F}_k) \end{bmatrix} \quad \text{for } k = 1, \dots, M-1. \quad (\text{B.21})$$

Then, the $2m(M-1)$ -th dimensional column vector of nonlinear constraints derived from the dynamics is:

$$\mathbf{c}(\mathbf{x})^T = \begin{bmatrix} \boldsymbol{\xi}_1(\mathbf{x}) & \boldsymbol{\xi}_1(\mathbf{x}) & \dots & \boldsymbol{\xi}_{M-1}(\mathbf{x}) \end{bmatrix}. \quad (\text{B.22})$$

As in the trapezoidal case, Eq. (B.22) can be written in matrix form as :

$$\mathbf{c}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{p}(\mathbf{x}), \quad (\text{B.23})$$

where matrices $\mathbf{A}$ and $\mathbf{C}$ are constant, and:

$$\mathbf{p}(\mathbf{x})^T = \begin{bmatrix} \tau_F \mathbf{F}_1 & \tau_F \bar{\mathbf{F}}_2 & \tau_F \mathbf{F}_2 & \dots & \tau_F \bar{\mathbf{F}}_M & \tau_F \mathbf{F}_M \end{bmatrix}. \quad (\text{B.24})$$

For a general case, $\mathbf{A}$ and $\mathbf{C}$ have the following expressions:

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & & & \\ 0 & & & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \\ & & & & \ddots & & \\ 0 & & & & & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix}, \quad (\text{B.25})$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} & \mathbf{C}_{1,3} & & & \\ & \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \mathbf{C}_{2,3} & & \\ & & \ddots & & & \\ & & & \mathbf{C}_{M-1,1} & \mathbf{C}_{M-1,2} & \mathbf{C}_{M-1,3} \end{bmatrix}, \quad (\text{B.26})$$

where $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are $2m \times (m + l)$ matrices:

$$\mathbf{A}_1 = \begin{bmatrix} -\frac{1}{2}\mathbf{I}_m & \mathbf{0}_{m,l} \\ -\mathbf{I}_m & \mathbf{0}_{m,l} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{I}_m & \mathbf{0}_{m,l} \\ \mathbf{0}_{m,m} & \mathbf{0}_{m,l} \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} -\frac{1}{2}\mathbf{I}_m & \mathbf{0}_{m,l} \\ \mathbf{I}_m & \mathbf{0}_{m,l} \end{bmatrix},$$

and $\mathbf{C}_{k,1}$, $\mathbf{C}_{k,2}$ and $\mathbf{C}_{k,3}$ are $2m \times m$ matrices:

$$\mathbf{C}_{k,1} = \Delta t_k \begin{bmatrix} \frac{1}{8}\mathbf{I}_m \\ -\frac{1}{6}\mathbf{I}_m \end{bmatrix}, \quad \mathbf{C}_{k,2} = \Delta t_k \begin{bmatrix} \mathbf{0}_{m,m} \\ -\frac{2}{3}\mathbf{I}_m \end{bmatrix}, \quad \mathbf{C}_{k,3} = \Delta t_k \begin{bmatrix} -\frac{1}{8}\mathbf{I}_m \\ -\frac{1}{6}\mathbf{I}_m \end{bmatrix}.$$

B.5.1 Gradient of the constraints

The gradient of the constraints is:

$$\mathbf{G} \equiv \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \equiv \mathbf{A} + \mathbf{C}\mathbf{P}, \quad (\text{B.27})$$

where:

$$\mathbf{P} \equiv \frac{\partial \mathbf{p}}{\partial \mathbf{x}} \equiv \begin{bmatrix} \mathbf{P}_{10} & \mathbf{P}_{11} & & & & \\ \bar{\mathbf{P}}_{20} & & \bar{\mathbf{P}}_{22} & & & \\ \mathbf{P}_{20} & & & \mathbf{P}_{22} & & \\ \vdots & & & & \ddots & \\ \bar{\mathbf{P}}_{M0} & & & & & \bar{\mathbf{P}}_{MM} \\ \mathbf{P}_{M0} & & & & & & \mathbf{P}_{MM} \end{bmatrix}, \quad (\text{B.28})$$

$$\begin{aligned} \mathbf{P}_{k0} &= \mathbf{F}_k + \tau_F \frac{\partial \mathbf{F}_k}{\partial \tau_F}, & \mathbf{P}_{kk} &= \tau_F \frac{\partial \mathbf{F}_k}{\partial \mathbf{x}^k}, \\ \bar{\mathbf{P}}_{k0} &= \bar{\mathbf{F}}_k + \tau_F \frac{\partial \bar{\mathbf{F}}_k}{\partial \tau_F}, & \bar{\mathbf{P}}_{kk} &= \tau_F \frac{\partial \bar{\mathbf{F}}_k}{\partial \bar{\mathbf{x}}^k}. \end{aligned}$$

B.6 Sparse Hessian Computed by Finite Differences of the Gradients

If the analytical expressions for the gradients of the objective function and the nonlinear constraints are known, Hessian information can be calculated in an efficient way by using the finite differences of those gradients. Moreover, taking advantage of the problem's sparsity allows to construct the sparse Hessian matrix by evaluating the gradients for a small number of perturbed vectors. This is proposed by Coleman, Garbow and Moré in [28].

Let $\mathbf{H}$ be the Hessian of the Lagrangian of the NLP problem:

$$\mathbf{H} = \Delta_{\mathbf{xx}}^2 L(\mathbf{x}, \boldsymbol{\lambda}) = \Delta_{\mathbf{xx}}^2 J(\mathbf{x}) - \sum_{i=1}^m \lambda_i \Delta_{\mathbf{xx}}^2 c_i(\mathbf{x}), \quad (\text{B.29})$$

where $L(\mathbf{x}, \boldsymbol{\lambda}) = J(\mathbf{x}) - \sum_{i=1}^m \lambda_i c_i(\mathbf{x})$ is the Lagrangian, $J(\mathbf{x})$ is the objective function, $c_i(\mathbf{x})$ are the nonlinear constraints, $\lambda_i = (\boldsymbol{\lambda})_i$ are the Lagrange multipliers² and $\mathbf{x}$ is the optimization variable. The gradient of the nonlinear constraints is matrix $\mathbf{G}$, whose calculation has been dealt with in previous sections. It is also

²Some authors define $L(\mathbf{x}, \boldsymbol{\lambda}) = J(\mathbf{x}) + \sum_{i=1}^m \omega_i c_i(\mathbf{x})$, where $\boldsymbol{\omega} = -\boldsymbol{\lambda}$; such is the case of the formulation used by Matlab's *fmincon*. All the formulas developed here are valid for both formulations, introducing the corresponding changes in the multipliers' signs.

convenient to consider the gradient of the Lagrangian, whose expression in terms of ∇J and $\mathbf{G}$ is straightforward:

$$\nabla L(\mathbf{x}) = \nabla J(\mathbf{x}) - \mathbf{G}^T \boldsymbol{\lambda}.$$

Given a vector $\boldsymbol{\Delta}$, the value of the vector Hessian times $\boldsymbol{\Delta}$ can be readily computed using a forward differences estimate as:

$$\mathbf{H}\boldsymbol{\Delta} = \nabla L(\mathbf{x} + \boldsymbol{\Delta}) - \nabla L(\mathbf{x}),$$

or with a central differences estimate:

$$\mathbf{H}\boldsymbol{\Delta} = \frac{1}{2} [\nabla L(\mathbf{x} + \boldsymbol{\Delta}) - \nabla L(\mathbf{x} - \boldsymbol{\Delta})]. \quad (\text{B.30})$$

Since the Hessian $\mathbf{H}$ is a symmetric, sparse matrix, the goal is to find a small set of vectors $\boldsymbol{\Delta}^1, \boldsymbol{\Delta}^2, \dots, \boldsymbol{\Delta}^\gamma$ such that the Hessian is uniquely determined by the products $\mathbf{H}\boldsymbol{\Delta}^1, \mathbf{H}\boldsymbol{\Delta}^2, \dots, \mathbf{H}\boldsymbol{\Delta}^\gamma$. In [28] two different techniques for automatically constructing those sets of vectors in a general case are proposed; however, in this appendix the set of vectors will be picked by hand. When the dynamics of the system are discretized using the Trapezoidal or the Hermite-Simpson Separated methods, the sparsity structure of the Hessian is as shown on Figure B.1. Therefore, the following set of $1 + m + l$ n -dimensional vectors is enough to construct the whole Hessian:

$$(\boldsymbol{\Delta}^1)_k = \begin{cases} \delta^1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\boldsymbol{\Delta}^j)_k = \begin{cases} \delta^j & \text{if } k = j + \kappa(m + l) \\ 0 & \text{otherwise} \end{cases}$$

where:

$$k = 1, 2, \dots, n$$

$$j = 2, 3, \dots, 1 + m + l$$

$$\kappa = 0, 1, \dots, M - 1.$$

Consequently, only $2 + m + l$ evaluations of $\nabla L(\mathbf{x})$ are needed to compute $\mathbf{H}$ using a forward differences estimate, or $2(1 + m + l)$ for a centered differences estimate. Constructing the sparse Hessian from vectors $\mathbf{H}\boldsymbol{\Delta}^j$ is not complicated but lengthy; therefore, the process will not be described here. For simplicity sake, all the δ^j are given the same value:

$$\delta^1 = \delta^2 = \dots = \delta^{1+m+l} = \delta. \quad (\text{B.31})$$

Compromise values for δ are 10^{-8} when the forward differences estimation is used, and 10^{-5} for the centered differences estimation case. Those values have been

selected with the criteria of getting truncation and round-off errors of similar order.

It must be noted that the computation of the contributions to the Hessian of the objective function and the constraints could be separated, since $\nabla_{\mathbf{x}\mathbf{x}}^2$ is a linear operator. This can be useful to exploit particular properties of the objective function: for instance, if the objective function $J(\mathbf{x})$ does not depend on the control vector l evaluations can be avoided. However, associated reductions in computational cost are very small, and so it can be convenient to always use the formulation above for generality.

Finally, if constraints other than the ones considered in this appendix are used, it would be enough to include them in $\mathbf{G}$, provided the sparsity structure is not changed. If the new constraints do modify the sparsity of $\mathbf{H}$, a straightforward solution would be to compute their contribution to $\mathbf{H}$ separately.

B.7 Computational Cost and Memory Usage

All the developments in the preceding sections have been centered in defining an efficient transcription of the Optimal Control Problem into a NLP one, making use of the problem's sparsity and the availability of analytical expressions for the RHS of the ODE system that describes the dynamics. Although the potential gains in computational cost and memory usage have been presented in a qualitative manner, it is interesting to perform a preliminary quantitative estimation of said gains.

The density of a sparse matrix is the ratio between its non-zero elements and its size: the lower the density, the more efficient the sparse algebra is both in memory usage and computational time. The densities of the gradient of the defect constraints and the Hessian for the trapezoidal and HSS transcriptions can be calculated as follows:

$$\begin{aligned} d_{\text{TRAPZ}}^{\mathbf{G}} &= \frac{2(m+l)+1}{(m+l)M+1}, \\ d_{\text{TRAPZ}}^{\mathbf{H}} &= \frac{2n + M(m+l)^2 - 1}{n^2}, \\ d_{\text{HSS}}^{\mathbf{G}} &= \frac{5(m+l)+3}{2[(m+l)(2M-1)+1]}, \\ d_{\text{HSS}}^{\mathbf{H}} &= \frac{2n + (2M-1)(m+l)^2 - 1}{n^2}. \end{aligned}$$

Figure B.1: Sparsity structures of the gradient and Hessian matrices, with $M = 3$ nodes, a state vector of dimension $m = 3$, and a control vector of dimension $l = 1$.

For $M \gg 1$ they can be approximated as:

$$d_{\text{TRAPZ}}^{\mathbf{G}} \approx \frac{2(m+l)+1}{(m+l)M}, \quad d_{\text{TRAPZ}}^{\mathbf{H}} = \frac{(m+l+2)}{(m+l)M},$$

$$d_{\text{HSS}}^{\mathbf{G}} = \frac{5(m+l)+3}{4(m+l)M}, \quad d_{\text{HSS}}^{\mathbf{H}} = \frac{(m+l+2)}{2(m+l)M},$$

showing that the memory efficiency of using sparse matrices increases inversely to the size of the problem (the memory needed to store the full matrices grows quadratically with M , while for the sparse case the dependence with M is linear). Figure B.1 shows the sparsity structures of the gradient of the defect constraints and the Hessian for a problem with a 3-dimensional state vector and a 1-dimensional control vector.

On the other hand, computing the gradient matrix numerically for the trapezoidal discretization requires n evaluations of $\mathbf{c}(\mathbf{x})$, each one involving M evalu-

ations of $\mathbf{F}(\mathbf{x})$; a total of $n \times M \approx (m + l)M^2$ evaluations of $\mathbf{F}(\mathbf{x})$ ($4(m + l)M^2$ evaluations for the HSS discretization). Meanwhile, the analytical method proposed here requires just M evaluations ($2M - 1$ for HSS) of $\mathbf{F}(\mathbf{x})$ and its analytical gradient; therefore, computational cost grows linearly with the size of the problem, instead of quadratically.

B.8 Conclusion

In this appendix, three common schemes for the direct transcription of OCPs into NLP problems have been presented, with an especial emphasis in taking advantage of the sparsity of the associated matrices and the availability of analytical gradients for the equations describing the dynamics. The gains in memory usage and computational cost stemming from the sparsity have been quantitatively evaluated.

Appendix C

Fourier Series Solution for Relative Motion

This appendix summarizes the Fourier series solution for the relative motion in curvilinear coordinates developed in Chapter 2, section 2.2. The coefficient matrices for expansions up to order 5 are reported.

C.1 Fourier series expansion for ρ

Given the Fourier series expansion for the ρ coordinate:

$$\rho = \rho_{\text{const}} + \sum_{k=1}^{\infty} [\rho_{CkM} \cos(kM) + \rho_{SkM} \sin(kM)],$$

the coefficients ρ_{const} , ρ_{CkM} , ρ_{SkM} can be written in compact matrix form as:

$$\rho_{\text{const}} = -1 + a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\rho 0}^{pq} h^p \right] e^q = -1 + a \mathbf{h}^{\top} \mathbf{C}_{\rho 0} \mathbf{e}, \quad (\text{C.1})$$

$$\rho_{CkM} = a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\rho k}^{pq} h^p \right] e^q = a \mathbf{h}^{\top} \mathbf{C}_{\rho k} \mathbf{e}, \quad (\text{C.2})$$

$$\rho_{SkM} = a \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} s_{\rho k}^{pq} h^p \right] e^q = a \mathbf{h}^{\top} \mathbf{S}_{\rho k} \mathbf{e}, \quad (\text{C.3})$$

with:

$$\mathbf{e} = [1 \quad e \quad e^2 \quad e^3 \quad \dots]^{\top}, \quad \mathbf{h} = [h^0 \quad h^1 \quad h^2 \quad h^3 \quad \dots]^{\top},$$

where the coefficients h^q are defined in terms of gamma (Γ) and hypergeometric (F) functions involving the orbit inclination parameter $\kappa = \sin^2 i$:

$$h^q = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + q\right) \frac{F\left(-\frac{1}{2}, \frac{1}{2} + q; 1 + q; \kappa\right)}{\Gamma(1 + q)}.$$

The matrices appearing in Eqs. (C.1-C.3) are reported below up to order five, defining:

$$c_{n\omega} = \cos(n\omega), \quad s_{n\omega} = \sin(n\omega),$$

for compactness:

$$\mathbf{C}_{\rho 0} = [c_{\rho 0}^{pq}] = \begin{bmatrix} 1 & 0 & \frac{1}{2} + 3c_{2\omega} & 0 & \frac{15c_{4\omega}}{8} & 0 \\ 0 & 0 & -6c_{2\omega} & 0 & -15c_{4\omega} & 0 \\ 0 & 0 & 0 & 0 & 15c_{4\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\rho 1} = [c_{\rho 1}^{pq}] = \begin{bmatrix} 0 & -1-5c_{2\omega} & 0 & \frac{1}{24}(9+100c_{2\omega}-229c_{4\omega}) & 0 & \frac{-50-935c_{2\omega}+13826c_{4\omega}-9805c_{6\omega}}{1920} \\ 0 & 10c_{2\omega} & 0 & \frac{1}{24}(-200c_{2\omega}+1832c_{4\omega}) & 0 & \frac{1870c_{2\omega}-110608c_{4\omega}+176490c_{6\omega}}{1920} \\ 0 & 0 & 0 & -\frac{229c_{4\omega}}{3} & 0 & \frac{110608c_{4\omega}-470640c_{6\omega}}{1920} \\ 0 & 0 & 0 & 0 & 0 & \frac{1961c_{6\omega}}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\rho 1} = [s_{\rho 1}^{pq}] = \begin{bmatrix} 0 & 5s_{2\omega} & 0 & \frac{1}{24}(-98s_{2\omega}+229s_{4\omega}) & 0 & \frac{525s_{2\omega}-13524s_{4\omega}+9805s_{6\omega}}{1920} \\ 0 & -10s_{2\omega} & 0 & \frac{1}{24}(196s_{2\omega}-1832s_{4\omega}) & 0 & \frac{-1050s_{2\omega}+108192s_{4\omega}-176490s_{6\omega}}{1920} \\ 0 & 0 & 0 & \frac{229s_{4\omega}}{3} & 0 & \frac{-108192s_{4\omega}+470640s_{6\omega}}{1920} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1961s_{6\omega}}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\rho 2} = [c_{\rho 2}^{pq}] = \begin{bmatrix} 2c_{2\omega} & 0 & \frac{1}{2}(-1-14c_{2\omega}+29c_{4\omega}) & 0 & \frac{1}{24}(8+142c_{2\omega}-808c_{4\omega}+277c_{6\omega}) & 0 \\ -4c_{2\omega} & 0 & 14c_{2\omega}-116c_{4\omega} & 0 & \frac{1}{24}(-284c_{2\omega}+6464c_{4\omega}-4986c_{6\omega}) & 0 \\ 0 & 0 & 116c_{4\omega} & 0 & \frac{1}{24}(-6464c_{4\omega}+13296c_{6\omega}) & 0 \\ 0 & 0 & 0 & 0 & -\frac{1108c_{6\omega}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\rho 2} = [s_{\rho 2}^{pq}] = \begin{bmatrix} -2s_{2\omega} & 0 & 7s_{2\omega}-\frac{29s_{4\omega}}{2} & 0 & \frac{1}{24}(-140s_{2\omega}+808s_{4\omega}-277s_{6\omega}) & 0 \\ 4s_{2\omega} & 0 & -14s_{2\omega}+116s_{4\omega} & 0 & \frac{1}{24}(280s_{2\omega}-6464s_{4\omega}+4986s_{6\omega}) & 0 \\ 0 & 0 & -116s_{4\omega} & 0 & \frac{1}{24}(6464s_{4\omega}-13296s_{6\omega}) & 0 \\ 0 & 0 & 0 & 0 & \frac{1108s_{6\omega}}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\rho 3} = [c_{\rho 3}^{pq}] = \begin{bmatrix} 0 & 3c_{2\omega}-9c_{4\omega} & 0 & -\frac{3}{8}(1+25c_{2\omega}-133c_{4\omega}+31c_{6\omega}) & 0 & \frac{1}{640}(5751c_{2\omega}-5(-45+9758c_{4\omega}-7372c_{6\omega}+372c_{8\omega})) \\ 0 & -6c_{2\omega}+72c_{4\omega} & 0 & -\frac{3}{8}(-50c_{2\omega}+1064c_{4\omega}-558c_{6\omega}) & 0 & -\frac{5751c_{2\omega}}{320}+\frac{4879c_{4\omega}}{8}-\frac{16587c_{6\omega}}{16}+93c_{8\omega} \\ 0 & -72c_{4\omega} & 0 & -\frac{3}{8}(-1064c_{4\omega}+1488c_{6\omega}) & 0 & -\frac{4879c_{4\omega}}{8}+\frac{5529c_{6\omega}}{2}-465c_{8\omega} \\ 0 & 0 & 0 & 372c_{6\omega} & 0 & -1843c_{6\omega}+744c_{8\omega} \\ 0 & 0 & 0 & 0 & 0 & -372c_{8\omega} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\rho 3} = [s_{\rho 3}^{pq}] = \begin{bmatrix} 0 & -3(s_{2\omega}-3s_{4\omega}) & 0 & \frac{3}{8}(25s_{2\omega}-133s_{4\omega}+31s_{6\omega}) & 0 & \frac{1}{640}(-5709s_{2\omega}+48790s_{4\omega}-36860s_{6\omega}+1860s_{8\omega}) \\ 0 & 6(s_{2\omega}-12s_{4\omega}) & 0 & \frac{3}{8}(-50s_{2\omega}+1064s_{4\omega}-558s_{6\omega}) & 0 & \frac{5709s_{2\omega}}{320}-\frac{4879s_{4\omega}}{8}+\frac{16587s_{6\omega}}{16}-93s_{8\omega} \\ 0 & 72s_{4\omega} & 0 & \frac{3}{8}(-1064s_{4\omega}+1488s_{6\omega}) & 0 & \frac{4879s_{4\omega}}{8}-\frac{5529s_{6\omega}}{2}+465s_{8\omega} \\ 0 & 0 & 0 & -372s_{6\omega} & 0 & 1843s_{6\omega}-744s_{8\omega} \\ 0 & 0 & 0 & 0 & 0 & 372s_{8\omega} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\rho 4} = [c_{\rho 4}^{pq}] = \begin{bmatrix} 2c_{4\omega} & 0 & 4c_{2\omega} - 31c_{4\omega} + \frac{11c_{6\omega}}{2} & 0 & \frac{1}{24}(-8 - 304c_{2\omega} + 2697c_{4\omega} - 1380c_{6\omega} + 33c_{8\omega}) & 0 \\ -16c_{4\omega} & 0 & -8c_{2\omega} + 248c_{4\omega} - 99c_{6\omega} & 0 & \frac{76c_{2\omega}}{3} - 899c_{4\omega} + 1035c_{6\omega} - 44c_{8\omega} & 0 \\ 16c_{4\omega} & 0 & -248c_{4\omega} + 264c_{6\omega} & 0 & 899c_{4\omega} - 2760c_{6\omega} + 220c_{8\omega} & 0 \\ 0 & 0 & -176c_{6\omega} & 0 & 1840c_{6\omega} - 352c_{8\omega} & 0 \\ 0 & 0 & 0 & 0 & 176c_{8\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\rho 4} = [s_{\rho 4}^{pq}] = \begin{bmatrix} -2s_{4\omega} & 0 & -4s_{2\omega} + 31s_{4\omega} - \frac{11s_{6\omega}}{2} & 0 & \frac{1}{24}(304s_{2\omega} - 2697s_{4\omega} + 1380s_{6\omega} - 33s_{8\omega}) & 0 \\ 16s_{4\omega} & 0 & 8s_{2\omega} - 248s_{4\omega} + 99s_{6\omega} & 0 & -\frac{76s_{2\omega}}{3} + 899s_{4\omega} - 1035s_{6\omega} + 44s_{8\omega} & 0 \\ -16s_{4\omega} & 0 & 248s_{4\omega} - 264s_{6\omega} & 0 & -899s_{4\omega} + 2760s_{6\omega} - 220s_{8\omega} & 0 \\ 0 & 0 & 176s_{6\omega} & 0 & -1840s_{6\omega} + 352s_{8\omega} & 0 \\ 0 & 0 & 0 & 0 & -176s_{8\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\rho 5} = [c_{\rho 5}^{pq}] = \begin{bmatrix} 0 & 7c_{4\omega} - c_{6\omega} & 0 & \frac{125c_{2\omega}}{24} - \frac{561c_{4\omega}}{8} + \frac{109c_{6\omega}}{4} - \frac{c_{8\omega}}{4} & 0 & \frac{1}{384}(-125 - 6337c_{2\omega} + 80242c_{4\omega} - 69774c_{6\omega} + 2568c_{8\omega} - 24c_{10\omega}) \\ 0 & -56c_{4\omega} + 18c_{6\omega} & 0 & -\frac{125c_{2\omega}}{12} + 561c_{4\omega} - \frac{981c_{6\omega}}{2} + 8c_{8\omega} & 0 & \frac{1}{384}(12674c_{2\omega} - 641936c_{4\omega} + 1255932c_{6\omega} - 82176c_{8\omega} + 1200c_{10\omega}) \\ 0 & 56c_{4\omega} - 48c_{6\omega} & 0 & -561c_{4\omega} + 1308c_{6\omega} - 40c_{8\omega} & 0 & \frac{1}{384}(641936c_{4\omega} - 3349152c_{6\omega} + 410880c_{8\omega} - 9600c_{10\omega}) \\ 0 & 32c_{6\omega} & 0 & -872c_{6\omega} + 64c_{8\omega} & 0 & \frac{1}{384}(2232768c_{6\omega} - 657408c_{8\omega} + 26880c_{10\omega}) \\ 0 & 0 & 0 & -32c_{8\omega} & 0 & \frac{1}{384}(328704c_{8\omega} - 30720c_{10\omega}) \\ 0 & 0 & 0 & 0 & 0 & 32c_{10\omega} \end{bmatrix},$$

$$\mathbf{S}_{\rho 5} = [s_{\rho 5}^{pq}] = \begin{bmatrix} 0 & -7s_{4\omega} + s_{6\omega} & 0 & \frac{1}{24}(-125s_{2\omega} + 1683s_{4\omega} - 654s_{6\omega} + 6s_{8\omega}) & 0 & \frac{1}{384}(6337s_{2\omega} - 80242s_{4\omega} + 69774s_{6\omega} - 2568s_{8\omega} + 24s_{10\omega}) \\ 0 & 2(28s_{4\omega} - 9s_{6\omega}) & 0 & \frac{125s_{2\omega}}{12} - 561s_{4\omega} + \frac{981s_{6\omega}}{2} - 8s_{8\omega} & 0 & \frac{1}{192}(-6337s_{2\omega} + 320968s_{4\omega} - 627966s_{6\omega} + 41088s_{8\omega} - 600s_{10\omega}) \\ 0 & -56s_{4\omega} + 48s_{6\omega} & 0 & 561s_{4\omega} - 1308s_{6\omega} + 40s_{8\omega} & 0 & -\frac{40121s_{4\omega}}{24} + \frac{34887s_{6\omega}}{4} - 1070s_{8\omega} + 25s_{10\omega} \\ 0 & -32s_{6\omega} & 0 & 872s_{6\omega} - 64s_{8\omega} & 0 & -\frac{11629s_{6\omega}}{2} + 1712s_{8\omega} - 70s_{10\omega} \\ 0 & 0 & 0 & 32s_{8\omega} & 0 & -856s_{8\omega} + 80s_{10\omega} \\ 0 & 0 & 0 & 0 & 0 & -32s_{10\omega} \end{bmatrix}.$$

C.2 Fourier series expansion for θ

Given the Fourier series expansion for the θ coordinate:

$$\theta - \theta_{\text{sec}} = \theta_{\text{const}} + \sum_{k=1}^{\infty} [\theta_{CkM} \cos(kM) + \theta_{SkM} \sin(kM)] ,$$

with:

$$\theta_{\text{sec}} = (n - 1) \tau ,$$

the coefficients θ_{const} , θ_{CkM} , θ_{SkM} can be written in compact form as:

$$\theta_{\text{const}} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} \rho_0^{pq} s^p \right] e^q = \mathbf{s}^\top \mathbf{C}_{\theta 0} \mathbf{e} , \quad (\text{C.4})$$

$$\theta_{CkM} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} c_{\theta k}^{pq} s^p \right] e^q = \mathbf{s}^\top \mathbf{C}_{\theta k} \mathbf{e} , \quad (\text{C.5})$$

$$\theta_{SkM} = \sum_{q=0}^{\infty} \left[\sum_{p=0}^{\infty} s_{\theta k}^{pq} c^p \right] e^q = \mathbf{c}^\top \mathbf{S}_{\theta k} \mathbf{e} , \quad (\text{C.6})$$

where:

$$\mathbf{s} = \begin{bmatrix} 1 & I_1 s_{2\omega} & \dots & I_k s_{2k\omega} & \dots \end{bmatrix}^\top , \quad \mathbf{c} = \begin{bmatrix} 1 & I_1 c_{2\omega} & \dots & I_k \cos c_{2k\omega} & \dots \end{bmatrix}^\top ,$$

$$I_k = \frac{(-1 + \sqrt{1 - \kappa})^k}{k (1 + \sqrt{1 - \kappa})^k} .$$

The matrices appearing in Eqs. (C.4-C.6) are reported below up to order five:

$$\mathbf{C}_{\theta 0} = [c_{\theta 0}^{pq}] = \begin{bmatrix} M_0 + \Omega + \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\mathbf{C}_{\theta 1} = [c_{\theta 1}^{pq}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & \frac{11}{6} & 0 & -\frac{5}{192} \\ 0 & 0 & 0 & -\frac{17}{6} & 0 & \frac{203}{120} \\ 0 & 0 & 0 & 0 & 0 & -\frac{773}{320} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

$$\mathbf{S}_{\theta 1} = [s_{\theta 1}^{pq}] = \begin{bmatrix} 0 & 2 & 0 & -\frac{1}{4} & 0 & \frac{5}{96} \\ 0 & -2 & 0 & \frac{5}{3} & 0 & -\frac{5}{64} \\ 0 & 0 & 0 & -\frac{17}{6} & 0 & \frac{133}{80} \\ 0 & 0 & 0 & 0 & 0 & -\frac{773}{320} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\theta 2} = [c_{\theta 2}^{pq}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 0 & \frac{167}{48} & 0 \\ 0 & 0 & \frac{11}{2} & 0 & -\frac{149}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{157}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\theta 2} = [s_{\theta 2}^{pq}] = \begin{bmatrix} 0 & 0 & \frac{5}{4} & 0 & -\frac{11}{24} & 0 \\ 1 & 0 & -4 & 0 & \frac{163}{48} & 0 \\ 0 & 0 & \frac{11}{2} & 0 & -\frac{149}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{157}{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\theta 3} = [c_{\theta 3}^{pq}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -\frac{27}{4} & 0 & \frac{2079}{320} \\ 0 & -4 & 0 & \frac{45}{2} & 0 & -\frac{591}{16} \\ 0 & 0 & 0 & -\frac{67}{4} & 0 & \frac{4053}{64} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2691}{80} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{S}_{\theta 3} = [s_{\theta 3}^{pq}] = \begin{bmatrix} 0 & 0 & 0 & \frac{13}{12} & 0 & -\frac{43}{64} \\ 0 & 2 & 0 & -\frac{27}{4} & 0 & \frac{2061}{320} \\ 0 & -4 & 0 & \frac{45}{2} & 0 & -\frac{591}{16} \\ 0 & 0 & 0 & -\frac{67}{4} & 0 & \frac{4053}{64} \\ 0 & 0 & 0 & 0 & 0 & -\frac{2691}{80} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{C}_{\theta 4} = [c_{\theta 4}^{pq}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13}{4} & 0 & -\frac{259}{24} & 0 \\ 1 & 0 & -16 & 0 & \frac{247}{4} & 0 \\ 0 & 0 & \frac{57}{4} & 0 & -\frac{829}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1285}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned}
\mathbf{S}_{\theta 4} = [s_{\theta 4}^{pq}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{103}{96} & 0 \\ 0 & 0 & \frac{13}{4} & 0 & -\frac{259}{24} & 0 \\ 1 & 0 & -16 & 0 & \frac{247}{4} & 0 \\ 0 & 0 & \frac{57}{4} & 0 & -\frac{829}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1285}{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{C}_{\theta 5} = [c_{\theta 5}^{pq}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{59}{12} & 0 & -\frac{3221}{192} \\ 0 & 4 & 0 & -\frac{85}{2} & 0 & \frac{6845}{48} \\ 0 & -6 & 0 & \frac{345}{4} & 0 & -\frac{12605}{32} \\ 0 & 0 & 0 & -\frac{149}{3} & 0 & \frac{21281}{48} \\ 0 & 0 & 0 & 0 & 0 & -\frac{33797}{192} \end{bmatrix}, \\
\mathbf{S}_{\theta 5} = [s_{\theta 5}^{pq}] &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1097}{960} \\ 0 & 0 & 0 & \frac{59}{12} & 0 & -\frac{3221}{192} \\ 0 & 4 & 0 & -\frac{85}{2} & 0 & \frac{6845}{48} \\ 0 & -6 & 0 & \frac{345}{4} & 0 & -\frac{12605}{32} \\ 0 & 0 & 0 & -\frac{149}{3} & 0 & \frac{21281}{48} \\ 0 & 0 & 0 & 0 & 0 & -\frac{33797}{192} \end{bmatrix}.
\end{aligned}$$

C.3 Fourier series expansion for z

Given the Fourier series expansion for z :

$$z = z_{\text{const}} + \sum_{k=1}^{\infty} [z_{CkM} \cos(kM) + z_{SkM} \sin(kM)],$$

the coefficients z_{const} , z_{CkM} , z_{SkM} can be written in compact form as:

$$z_{\text{const}} = -\frac{3}{2}ae\sqrt{\kappa} \sin(\omega), \quad (\text{C.7})$$

$$z_{CkM} = a\sqrt{\kappa} \sin(\omega) \sum_{p=0}^{\infty} c_{zk}^p e^p = a\sqrt{\kappa} \sin(\omega) \mathbf{e}^{\top} \mathbf{c}_{zk}, \quad (\text{C.8})$$

$$z_{SkM} = a\sqrt{\kappa} \cos(\omega) \sum_{i=0}^{\infty} s_{zk}^p e^p = a\sqrt{\kappa} \cos(\omega) \mathbf{e}^{\top} \mathbf{s}_{zk}, \quad (\text{C.9})$$

where the vectors $\mathbf{c}_{zk}$, $\mathbf{s}_{zk}$ are reported below up to order five:

$$\mathbf{c}_{z1} = [c_{z1}^p] = \begin{bmatrix} 1 & 0 & -\frac{3}{8} & 0 & \frac{5}{192} & 0 \end{bmatrix}^{\top},$$

$$\mathbf{s}_{z1} = [s_{z1}^p] = \begin{bmatrix} 1 & 0 & -\frac{5}{8} & 0 & -\frac{11}{192} & 0 \end{bmatrix}^{\top},$$

$$\mathbf{c}_{z2} = [c_{z2}^p] = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{3} & 0 & \frac{1}{16} \end{bmatrix}^\top,$$

$$\mathbf{s}_{z2} = [s_{z2}^p] = \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{5}{12} & 0 & \frac{1}{24} \end{bmatrix}^\top,$$

$$\mathbf{c}_{z3} = [c_{z3}^p] = \begin{bmatrix} 0 & 0 & \frac{3}{8} & 0 & -\frac{45}{128} & 0 \end{bmatrix}^\top,$$

$$\mathbf{s}_{z3} = [s_{z3}^p] = \begin{bmatrix} 0 & 0 & \frac{3}{8} & 0 & -\frac{51}{128} & 0 \end{bmatrix}^\top,$$

$$\mathbf{c}_{z4} = [c_{z4}^p] = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{2}{5} \end{bmatrix}^\top,$$

$$\mathbf{s}_{z4} = [s_{z4}^p] = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{13}{30} \end{bmatrix}^\top,$$

$$\mathbf{c}_{z5} = [c_{z5}^p] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{125}{384} & 0 \end{bmatrix}^\top,$$

$$\mathbf{s}_{z5} = [s_{z5}^p] = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{125}{384} & 0 \end{bmatrix}^\top.$$

Appendix D

Equations for the Multiple-Scales Radial Thrust Solution

This appendix contains several functions involved in the multiple scales solution for the two-body, low radial thrust problem developed in Chapter 8.

D.1 First order terms

The first order terms of the multiple scales solution for the two-body, low radial constant thrust problem are given as a combination of the following functions:

$$\begin{aligned}\mathcal{P}_1(\tau, T) &= -\frac{(q_{10} + q_{3i})(1 + \cos \tau) + q_{20} \sin \tau}{q_{3i}(q_{3i}^2 - q_{10}^2 - q_{20}^2) s_0}, \\ \mathcal{S}_1(T) &= -\frac{2q_{20}}{q_{3i}(q_{3i}^2 - q_{10}^2 - q_{20}^2)^{3/2}}, \\ \mathcal{P}_2(\tau, T) &= \frac{q_{10}q_{20}(1 + \cos \tau) + (-q_{3i}^2 + q_{20}^2 + q_{3i}q_{10}) \sin \tau}{q_{3i}(q_{3i} - q_{10})(q_{3i}^2 - q_{10}^2 - q_{20}^2) s_0}, \\ \mathcal{S}_2(T) &= \frac{2q_{10}}{q_{3i}(q_{3i}^2 - q_{10}^2 - q_{20}^2)^{3/2}}, \\ \mathcal{K}(\tau, T) &= -\frac{\left(\sqrt{q_{3i}^2 - q_{10}^2 - q_{20}^2} - q_{3i} + q_{10}\right) \sin \tau - q_{20}(1 + \cos \tau)}{s_0 - q_{10} - q_{3i} \cos \tau + (1 + \cos \tau)\sqrt{q_{3i}^2 - q_{10}^2 - q_{20}^2}},\end{aligned}$$

where q_{10} and q_{20} are functions only of the slow ‘time’ scale T .

Introducing the first integral $q_{10}^2 + q_{20}^2 = q_{1i}^2$, Eq. (8.35), these expressions take simpler forms:

$$\mathcal{P}_1(\tau, T) = -\frac{(q_{10} + q_{3i})(1 + \cos \tau) + q_{20} \sin \tau}{q_{3i}(q_{3i}^2 - q_{1i}^2) s_0}, \quad (\text{D.1})$$

$$\mathcal{S}_1(T) = -\frac{2q_{20}}{q_{3i}(q_{3i}^2 - q_{1i}^2)^{3/2}} = -2\Omega_1 q_{20}, \quad (\text{D.2})$$

$$\mathcal{P}_2(\tau, T) = \frac{q_{10}q_{20}(1 + \cos \tau) + (-q_{3i}^2 + q_{20}^2 + q_{3i}q_{10}) \sin \tau}{q_{3i}(q_{3i} - q_{10})(q_{3i}^2 - q_{1i}^2)s_0}, \quad (\text{D.3})$$

$$\mathcal{S}_2(\tau, T) = \frac{2q_{10}}{q_{3i}(q_{3i}^2 - q_{1i}^2)^{3/2}} = 2\Omega_1 q_{10}, \quad (\text{D.4})$$

$$\mathcal{K}(\tau, T) = -\frac{\left(\sqrt{q_{3i}^2 - q_{1i}^2} - q_{3i} + q_{10}\right) \sin \tau - q_{20}(1 + \cos \tau)}{s_0 - q_{10} - q_{3i} \cos \tau + (1 + \cos \tau)\sqrt{q_{3i}^2 - q_{1i}^2}}. \quad (\text{D.5})$$

D.2 Expressions for Ω_2 , $g_1(T)$ and $g_2(T)$

As stated in section 8.4, the direct application of the secularity condition which determines the values of Ω_2 , $g_1(T)$ and $g_2(T)$ is not feasible due to the complexity of the equations involved. To address this issue, an approximate solution is searched for in the form of an additional series expansion in the initial value of q_1 , related to the initial eccentricity e_0 . Then, for $q_{1i} \ll 1$ it is possible to write:

$$\Omega_2 = \Omega_2^{(0)} + q_{1i}\Omega_2^{(1)} + q_{1i}^2\Omega_2^{(2)} + \dots,$$

$$g_l(T; q_{1i}) = g_l^{(0)}(T) + q_{1i}g_l^{(1)}(T) + q_{1i}^2g_l^{(2)}(T) + \dots, \quad l = 1, 2.$$

The initial conditions for g_1 and g_2 are obtained from Eqs. (8.24) knowing that $q_{11}(0) = q_{21}(0) = 0$:

$$g_1(0) = \frac{2}{q_{3i}(q_{3i}^2 - q_{1i}^2)} = g_{1i}, \quad g_2(0) = 0.$$

To achieve a better reproduction of the physics of the problem, Ω_1 is retained as a fixed parameter instead of q_{3i} . The latter is then expanded as a power series of q_{1i} using the definition of Ω_1 , Eq. (8.36):

$$q_{3i} = \Omega_1^{-1/4} + q_{1i}^2 \frac{3}{8} \Omega_1^{1/4} - q_{1i}^4 \frac{3}{128} \Omega_1^{3/4} - q_{1i}^6 \frac{7}{1024} \Omega_1^{5/4} + q_{1i}^8 \frac{195}{32768} \Omega_1^{7/4} + \mathcal{O}(q_{1i}^{10}).$$

Expanding also g_{1i} :

$$g_{1i} = 2\Omega_1^{3/4} - q_{1i}^2 \frac{1}{4} \Omega_1^{5/4} + q_{1i}^4 \frac{5}{64} \Omega_1^{7/4} - q_{1i}^6 \frac{11}{512} \Omega_1^{9/4} + q_{1i}^8 \frac{51}{16384} \Omega_1^{11/4} + \mathcal{O}(q_{1i}^{10}).$$

Introducing the previous expressions into Eqs. (8.37), expanding in power series of q_{1i} , and imposing the secularity condition to the ODE systems obtained from canceling the terms of equal powers of q_{1i} , a sequence of straightforward problems

for determining $\Omega_2^{(k)}$, $g_1^{(k)}$ and $g_2^{(k)}$ is reached. Note that, due to the structure of these problems, the asymptotic solution for Ω_2 will have one order less than those for g_1 and g_2 . The coefficients obtained for Ω_2 up to order 7 are:

$$\left\{ \Omega_2^{(k)} \right\} = \left\{ \frac{7}{2}\Omega_1, 3\Omega_1^{5/4}, \frac{9}{4}\Omega_1^{3/2}, \frac{3}{8}\Omega_1^{7/4}, 0, -\frac{9}{128}\Omega_1^{9/4}, -\frac{9}{128}\Omega_1^{5/2}, \frac{9}{1024}\Omega_1^{11/4} \right\}$$

$$k = 0, \dots, 7.$$

On the other hand, the first two terms of g_1 and g_2 take the form:

$$g_1^{(0)}(T) = \Omega_1^{3/4} (1 + \cos T), \quad g_2^{(0)}(T) = \Omega_1^{3/4} \sin T,$$

$$g_1^{(1)}(T) = 0, \quad g_2^{(1)}(T) = -\Omega_1 \sin T.$$

For the rest of terms, at least up to order eight, a simple functional expression can be found:

$$g_1^{(k)}(T) = \Omega_1^{\frac{3+k}{4}} \left[C_0^{(k)} + \sum_{l=1}^k C_l^{(k)} \cos lT \right],$$

$$g_2^{(k)}(T) = -\frac{dg_1^{(k)}}{dT} = \Omega_1^{\frac{3+k}{4}} \sum_{l=1}^k l C_l^{(k)} \sin lT,$$

with coefficients:

$$\mathbf{C}^{(2)} = -\frac{1}{8} \begin{bmatrix} -3 & 1 & 4 \end{bmatrix},$$

$$\mathbf{C}^{(3)} = \frac{1}{8} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix},$$

$$\mathbf{C}^{(4)} = \frac{1}{384} \begin{bmatrix} -9 & 15 & 40 & 0 & -16 \end{bmatrix},$$

$$\mathbf{C}^{(5)} = \frac{1}{64} \begin{bmatrix} 0 & -2 & 0 & 3 & 0 & -1 \end{bmatrix},$$

$$\mathbf{C}^{(6)} = \frac{1}{5120} \begin{bmatrix} -35 & -55 & -100 & 0 & 112 & 0 & -32 \end{bmatrix},$$

$$\mathbf{C}^{(7)} = \frac{1}{768} \begin{bmatrix} 0 & 3 & 0 & -9 & 0 & 8 & 0 & -2 \end{bmatrix},$$

$$\mathbf{C}^{(8)} = \frac{1}{229376} \begin{bmatrix} 1365 & 357 & -336 & 0 & -1568 & 0 & 1152 & 0 & -256 \end{bmatrix}.$$

This solution is obtained by expanding the initial condition for g_1 as a power series in q_{1i} . Alternatively, g_{1i} could be retained as a parameter, leading to a

solution which keeps the same structure and most of the coefficients. Particularly, for Ω_2 only the odd terms are modified:

$$\begin{aligned}\Omega_2^{(1)} &\rightarrow 3\Omega_1^{1/4} g_{1i} - \Omega_2^{(1)}, & \Omega_2^{(3)} &\rightarrow \frac{3}{4}\Omega_1^{3/4} g_{1i} - \Omega_2^{(3)}, \\ \Omega_2^{(5)} &\rightarrow \frac{3}{32}\Omega_1^{3/4} g_{1i} - \Omega_2^{(5)}, & \Omega_2^{(7)} &\rightarrow -\Omega_2^{(7)}.\end{aligned}$$

Similarly, for g_1 and g_2 the only differences are in the zeroth order terms of the expansion:

$$g_1 = \left(g_{1i} - \Omega_1^{3/4}\right) \cos T + \Omega_1^{3/4}, \quad g_2 = \left(g_{1i} - \Omega_1^{3/4}\right) \sin T, \quad (\text{D.6})$$

and the coefficients for the sines and cosines of T in the rest of even terms:

$$C_1^{(2)} \rightarrow -C_1^{(2)}, \quad C_1^{(4)} \rightarrow -C_1^{(4)}, \quad C_1^{(6)} \rightarrow -C_1^{(6)}, \quad C_1^{(8)} \rightarrow -C_1^{(8)}. \quad (\text{D.7})$$

D.3 Function $g_t(T)$

The additive function in $\zeta_{00}(\tau, T)$ arising from the secularity condition is approximated through its power series expansion in q_{1i} , valid for small initial eccentricity:

$$g_t(T) = g_t^{(0)}(T) + q_{1i}g_t^{(1)}(T) + q_{1i}^2g_t^{(2)}(T) + \dots$$

All the resulting terms are monomials of T , that is, $g_t^{(k)} = D^{(k)}T$, so the expansion can be expressed in a more convenient form as:

$$g_t(T) = (D^{(0)} + q_{1i}D^{(1)} + q_{1i}^2D^{(2)} + \dots) T = D_t T$$

with coefficients up to order seven

$$\begin{aligned}\{D^{(k)}\} &= \left\{ 2\Omega_1^{3/4}, 3\Omega_1, \frac{15}{4}\Omega_1^{5/4}, \frac{3}{2}\Omega_1^{3/2}, \frac{45}{64}\Omega_1^{7/4}, 0, -\frac{67}{512}\Omega_1^{9/4}, -\frac{3}{64}\Omega_1^{5/2} \right\} \\ &k = 0, \dots, 7.\end{aligned}$$

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