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Vibrated Fluids in Microgravity

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Vibrated Fluids in Microgravity

Abstract

by José Javier FERNÁNDEZ FRAILE

The aim of this thesis is to understand the potentially complex behaviour of liquids in the presence of vibrations, both with gravity and in weightless environments, and to investigate the extent to which small-amplitude vibrations can be used to influence and control this behaviour. In the presence of gravity, flat interfaces are often selected in the absence of forcing, while this is not so in microgravity environments. In both cases, a variety of instabilities can arise when the fluid is subjected to vibrations.

One of the most prominent surface wave instabilities induced by vibrations produces subharmonic waves called cross-waves, which, in the case of a flat unforced surface (i.e., with gravity), have been described since Jones [J. Fluid Mech. **138**, 53–74 (1984)] using a boundary forced nonlinear Schrodinger equation. We extend this established theory of modulated cross-waves in horizontally vibrated containers by including surface tension and, most importantly, a spatially extended forcing term. The resulting amplitude equations give predictions for onset values, spatial profiles, and temporal modulations that are compared with previous theory and with experimental measurements. The appearance of temporally modulated solutions, confirmed experimentally, are related to the interaction of waves generated at opposite endwalls.

In weightless environments, where surface or contact energy typically dominate, fluids no longer prefer a flat configuration and the static equilibrium surface is generally curved. In such situations, high-frequency vibrations not only drive surface waves but also lead the fluid to take a new quasi-equilibrium (average) shape. This reorientation process is described by vibroequilibria theory, which is based on minimizing an averaged energy functional. Here we present a detailed comparison of the predictions of vibroequilibria theory with the results of direct numerical simulations, finding very good agreement over a range of parameters. The calculations also reveal an important difference in behaviour between small and large fluid volumes in horizontal vibrated containers. With dimensionless volume larger than about 0.36, the symmetric vibroequilibria solution suffers a saddle-node instability prior to contact with the container bottom. An investigation of dynamic effects is also presented, where a strong correlation is found between modulated subharmonic surface waves and the first odd sloshing mode.

Since vibroequilibria theory and simulations predict that, in some cases, the preference of flat interfaces can be restored by high-frequency vibrations, we proposed and developed The Control of Fluids in Microgravity with Vibrations (CFVib) experiment for the 2016 Fly Your Thesis! programme, sponsored by the European Space Agency (ESA), which was selected to be part of the 65th Parabolic Flight Campaign. Piezoelectric materials were used to generate high-frequency vibrations driving surface waves and large-scale reorientation of the interface.

Fluidos Vibrados en Microgravedad

Resumen

por José Javier FERNÁNDEZ FRAILE

El objetivo de esta tesis es comprender el comportamiento de líquidos en presencia de vibraciones tanto en entornos con gravedad como ingravidos, e investigar hasta qué punto se pueden utilizar vibraciones de pequeña amplitud para influir y controlar este comportamiento. En presencia de gravedad, la interfaz fluida es típicamente plana en ausencia de fuerzas, mientras que tiene porque serlo en entornos sin gravedad. En ambos casos, cuando el fluido es sometido a vibraciones, pueden aparecer una gran variedad de inestabilidades.

Una de las inestabilidades de ondas superficiales más destacadas inducidas por vibraciones produce ondas subarmónica conocidas como Cross-waves, las cuales, en el caso de un superficie plana no forzada (i.e, con gravedad), se han descrito desde Jones [J. Fluid Mech., Pág. 138), 53 - 74 (1984)] en base a la ecuación de Schrodinger no lineal forzada en la frontera. Se amplía aquí la teoría de Cross-waves en contenedores vibrados horizontalmente, mediante la inclusión de tensión superficial y, lo que es más importante, una extensión espacial del término forzante. Las ecuaciones de amplitud resultantes predicen valores de comienzo, perfiles espaciales, y modulaciones temporales que son comparados con la anterior teoría y experimentalmente. La aparición de soluciones temporalmente moduladas son relacionadas con la interacción de ondas generadas en las paredes opuestas.

Por otra parte, en ambientes ingravidos, donde la energía superficial y de contact es dominante, los fluidos dejan de preferir la configuración plana, y el equilibrio estático es curvado generalmente. En esta situación, vibraciones de alta frecuencia no solo inducen ondas superficiales sino que además conducen al fluido a tomar una nueva forma de quasi-equilibrio (en media). Este proceso de reorientación es descrito por la teoría de vibroequilibria, la cual está basada en la minimización de la media de un funcional de energía. Aquí se presenta una comparación detallada de las predicciones de la teoría de Vibroequilibria con resultados de simulaciones numéricas directas, encontrando muy buen acuerdo sobre una gama de parámetros. El estudio revela una diferencia importante en el comportamiento para volúmenes pequeños y grandes de fluidos en contenedores vibrados horizontalmente. Con un volumen adimensional por encima de alrededor de 0.36, la solución simétrica de Vibroequilibria sufre una inestabilidad de punto-silla antes de llegar a contactar con el fondo del recipiente. También se presenta un estudio de los efectos dinámicos, donde se encuentra una fuerte correlación entre las ondas subarmónicas superficiales y el primer modo impar de sloshing.

Ya que la teoría y simulaciones de vibroequilibria predice que, en algunos casos, la preferencia de una interfaz plana puede ser restaurada por vibración de alta frecuencia, se ha propuesto y desarrollado el experimento Control de Fluidos en Microgravedad con Vibraciones (CFVib) para el programa 2016 Fly Your Thesis!, patrocinado por la Agencia Espacial Europea, que fue seleccionado para formar parte de la 65th campaña de Vuelos Parabólicos. Se utilizan materiales piezoeléctricos para generar vibraciones de alta frecuencia y, así, inducir ondas superficiales y reorientación a gran escala de la interfaz.

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Para Antonio

1 Introduction

1.1 Important phenomena in vibrated fluids

The fundamental scientific objective of this thesis is to study vibrated fluids, particularly phenomena that will be of increased relevance in microgravity conditions. Although gravity is the weakest of the known natural forces, it has a profound cumulative effect on the distribution of matter and thus on numerous physical processes. The effective absence of the usual gravitational force in space environments such as the International Space Station (ISS) changes many things, creating both problems and opportunities. There are particular challenges with managing fluids as required, for example, in life-support and propulsion systems. In the absence of a constant vertical gravitational force it is essential to understand the effects of the small vibrations — often called g-jitter — that remain. These may be due to on-board machinery, crew movements, orbit or docking manoeuvres, etc., and are problematic for scientific experiments that assume a zero gravity condition (see, e.g., [62] for references). Recent experiments have been conducted on the ISS to understand the impact of dominant capillary forces using small disturbances [72].

The generic response to vibration of a fluid mass with at least one free boundary will include surface motion, but the forcing mechanism and the characteristics of the response vary according to the configuration. A vertically vibrated open container with an initially flat surface (typically in presence of gravity) presents a classic example of a parametric forcing mechanism [8]. The trivial flat state loses stability at a critical forcing value to (typically) subharmonic surface waves; if the size of the container is large compared to the length scale of

the instability, these waves can arrange themselves into a wide variety of patterns, depending on fluid parameters and the characteristics of the forcing function [12]. Any orientation of the forcing other than purely vertical will induce immediate harmonic (synchronous) motion of the surface, thus acting as a direct forcing mechanism. At low frequencies, horizontal forcing tends to generate sloshing motion [34], but may drive both synchronous and subharmonic waves in wavemaker experiments at intermediate frequencies [6] — the subharmonic waves, in fact, are due to a localized parametric forcing mechanism caused by an oscillating pressure gradient in the free surface that is analogous to the vertical forcing case [68].

Subharmonic cross-waves commonly arise in wavemaker experiments when sufficient forcing is applied. So-called because of their crosswise orientation with respect to the applied oscillation and the initial harmonic wave field, they were first described by Faraday, in an experiment utilizing a partially submerged vibrating plate, as forming a series of apparently permanent ridges projecting outward “like the teeth of a coarse comb” [14]. The instability is typically vigorous, with the subharmonic waves quickly overwhelming the initial harmonic wave field, and widespread, occurring not only with plane wavemakers but with virtually any vibrating solid object or boundary, partially or entirely submerged [7, 14, 33, 48, 58, 65, 66]. The parametric nature of the instability, and its origin in the periodic displacement of fluid by a moving boundary, attests to the similarity between cross-waves and the more-studied case of vertically forced subharmonic waves or, simply, “Faraday waves”. The localized nature of the forcing mechanism, however, makes them distinct. In a vertically forced Faraday wave experiment the excitation is homogeneous and the underlying state is trivial (motionless in a comoving frame). The subharmonic instability is the primary one and gives rise to extended patterns, the selection among which is strongly influenced by both temporal and spatial resonances [3, 55, 63, 76, 77]. In a cross-wave experiment, on the other hand, the presence of a directly forced harmonic wave field complicates theoretical analysis of the parametric instability, as does the spatial inhomogeneity of the forcing.

A proper theoretical understanding of cross-waves began with Garrett [19], who showed that the onset of instability could be described by Mathieu's equation and is thus parametric in nature. He found good agreement between his theory and the experiments of Lin and Howard [38], particularly with respect to the rightward shift of the frequency-amplitude curves. Further progress was made, following Jones [28], by considering the limit of weak damping and weak *localized* forcing, where the extent of the evanescent field near the wavemaker is much less than the modulation lengthscale of the subharmonic waves. A multiple-scale technique then accounts for the forcing via the boundary condition for an amplitude equation that describes the slow spatial and temporal evolution of the complex cross-wave amplitude $A(x, t)$. The nonlinear Schrödinger (NLS) equation derived by Jones [28] was subsequently extended by others to include viscous effects [9, 36, 60] and mode interactions [4]. These theoretical models were compared successfully to experiment [31, 35, 60, 69] and capture many features of cross-wave behaviour.

Despite their success, the NLS models are clearly not applicable to all cross-wave experiments. They are limited by the assumptions underlying their derivation including the neglect of surface tension and the assumption of very localized forcing. The latter assumption, in particular, is quite restrictive [54]. It requires either a shallow wavemaker, which would need to be driven at high amplitude to excite cross-waves, or a small dimensionless damping parameter (low viscosity and/or low frequency), to allow the cross-wave field to extend far into the underforced region away from the wavemaker. For typical cross-wave experiments using water [6, 35, 37, 60, 61, 69], where the wavemaker depth is more than 10 cm, the self-consistency of the NLS models requires forcing frequencies on the order of 10 Hz or less [54]. These models cannot be expected to apply to experiments using higher frequencies or larger viscosity fluids and are thus inadequate to describe pattern formation experiments in moderately sized containers, which require higher forcing frequencies (smaller wavelengths). Indeed, several cross-wave experiments in this regime [48, 54, 65] have found interesting rotated patterns that deviate from the assumption of (nearly) crosswise orientation made

in the NLS models.

An extension of cross-wave theory is needed to cover the parameter range beyond traditional (large) wavemaker experiments with low frequencies and small damping. Here, we take a first logical step toward that end, extending the standard analysis that followed Jones [28] and, in particular, that of Bernoff *et al.* [9] by including the effect of surface tension and, most importantly, relaxing the assumption of a narrow forcing region. Other assumptions common to previous cross-wave analyses [9, 28, 31, 36, 46, 60], like that of a nearly crosswise orientation, are maintained here. This is done for the sake of comparison, so that the effect of distributed forcing can be clearly seen without the complications of additional new effects. It follows that the model obtained will not apply to *all* subharmonic patterns excited by horizontal forcing, in particular those that are very rotated with respect to standard cross-waves [50]. It does, however, apply to a wide range of low viscosity experiments in the gravity-capillary regime including some results of Porter *et al.* [54] where typical subharmonic patterns are rotated (i.e., modulated in the downstream direction) but often not excessively so at onset. The model obtained here has the theoretical advantage of allowing direct comparison with the many established results based on NLS models that assume a narrow forcing region.

Vibrations can drive other types of interfacial instabilities in heterogeneous systems, such as a stably stratified layer of immiscible fluids, where horizontal forcing generates a Kelvin-Helmholtz type instability at a critical forcing amplitude, producing a quasi-stationary undulatory surface deformation referred to as a frozen wave [18, 39, 64, 75]. The initial small-amplitude frozen waves can develop nonlinear profiles and more prominent oscillatory behaviour with increasing forcing [26], as well as suffering secondary instabilities like transverse amplitude modulations [25]. Parametric instabilities may occur in such systems as well [30]. Recent experiments in near-critical hydrogen showed that the frozen wave amplitude increases with decreasing gravity level, approaching the length-scale of the container (domain) in microgravity conditions [16].

In addition to surface waves, which oscillate on a fast timescale set by the

forcing, vibrations also induce a time-averaged, slow reorientation of the fluid toward a new quasi-static equilibrium. These states, known as vibroequilibria, were observed by Faraday [14] in his experiments on the flattening of fluid drops suspended beneath a vibrating plate. The theory of vibroequilibria was developed using a time-averaged variational approach [10, 11, 20, 41]. This treatment assumes that the fluid is inviscid and the flow is irrotational so that the Lagrangian (Hamiltonian) variational method can be applied, and that the forcing frequency is much greater than the primary sloshing frequency. The time-averaged surface shape (i.e., the vibroequilibria) can then be found by minimizing an energy functional that includes terms accounting for gravitational energy, surface tension, and vibrational energy [29]. The theory can describe compressible flows, but we concentrate here on the incompressible case, where the forcing frequency is lower than the first acoustic resonance.

Like frozen waves, vibroequilibria are quasi-stationary surface configurations differing from the unforced (capillary) equilibrium and supported by the oscillatory velocity field. The spatial inhomogeneity of this velocity field, and the corresponding oscillatory pressure field, leads to deformation of the surface. In most terrestrial experiments this deformation is negligible compared to gravity and/or surface tension. In microgravity environments, however, vibroequilibria can be important in medium-sized or large systems, particularly in the absence of strong surface tension effects. In the limit of high amplitude, vibroequilibria tend to position the interface perpendicular to the axis of vibration, and this can be viewed as a kind of artificial gravity (a restored preference for flat interfaces). With further understanding of the effects of vibrations on fluid orientation and position, it may be possible to use these effects to advantage, even with small amplitude forcing. In particular, if the container design is appropriately chosen, theory suggests that one could deliberately manipulate and manage fluids with very small amplitude vibrations. The ability of vibrations to mimic the effects of gravity is seen in other situations too. For example, vibrations can induce convection patterns similar to Rayleigh-Bénard (buoyancy driven) convection in a fluid confined between two plates held at different temperatures.

Although there has been general qualitative agreement between the predictions of vibroequilibria theory and experiments [17, 75], there is currently no detailed comparison, that we are aware of, between these predictions and direct numerical simulations. This work fills this gap by comparing the predictions of the time-averaged energy functional approach with numerical simulations of fluid motion in horizontally vibrated two-dimensional rectangular containers. The vibroequilibria calculations can also be compared with those of Gavriluk *et al.* [20] and Lyubimov *et al.* [40], who noted hysteresis and coexistence between singly-connected and disconnected solutions. In this thesis we examine the hysteresis of vibroequilibria states in more detail, finding that, beyond a critical depth, this is related to a saddle-node instability that is present both with and without gravity.

Another potential departure from the predictions of vibroequilibria theory in real fluid systems is due to coupling with dynamic modes like surface waves and sloshing, which are especially important in low viscosity fluids. The effect of these dynamic modes on the averaged vibroequilibria state has apparently not been investigated before. We show that there can be significant interaction among these modes, depending on forcing parameters, viscosity, boundary conditions, and the initial (transient) approach to the vibroequilibria state. In particular, we investigate a strong association between modulated subharmonic surface waves of the type recently studied in horizontally vibrated fluid systems [50, 53, 57, 68] (without the vibroequilibria effect) and the first odd sloshing mode. In some cases, this odd sloshing mode grows large enough to completely disrupt the underlying vibroequilibria state.

Although vibroequilibria phenomena can be observed in the presence of gravity, as shown by Faraday's experiment on the flattening of fluid drops suspended beneath a vibrating plate, the effect is not easy to study on ground. Small fluid masses are dominated by surface tension and large fluid masses are dominated by gravity. A large vibrational amplitude (several g 's or more in typical configurations) is required to observe a significant vibroequilibria effect. There are relatively few microgravity experiments examining this phenomenon (those of [17]

being a notable exception), and it is of interest to test the predictions of vibroequilibria theory directly and to look at potential instabilities of these quasi-stationary states, as has recently been done [15] using numerical simulations.

We proposed, built and operated an experiment to explore the microgravity potential of small-amplitude vibrations for managing fluids. The absence of gravity is essential to permit a large measurable effect with relatively small-amplitude vibrations. The Control of Fluid with Vibration (CFVib) experiment was selected as part of the ESA Education programme FlyYourThesis! 2016, and flew on the 65th ESA Parabolic Flight Campaign. The experiment investigated different fluid configurations confined in containers of two distinct geometries: cylindrical and cuboidal. The experiment used high-frequency piezoelectric-induced vibrations — up to acoustic frequencies — generated with a bimorph actuator, or cantilever, excited at a resonance frequency in order to induce vibrations in the experimental fluid. The CFVib experiment measured the dynamic behaviour of the representative liquid-gas and liquid-liquid combinations confined in moderately small transparent acrylic containers (centimetres) and subjected to periodic vibrations. Frequencies from 20 Hz to 12 kHz were used to drive the reorientation of the fluid interface and explore the potential for controlling it. The use of piezoelectric materials to transmit vibrations to the fluid was effective and required less power than an electromagnetic shaker, making it more conducive to potential microgravity applications. There are challenges, however, for theory and simulations since the excited motion within the container is complicated, nonuniform, and sensitive to resonance conditions. The period of weightlessness (more than 20 s) offered by a parabolic flight campaign was well-suited to the scientific requirements of the CFVib experiment. Furthermore, the g -level cycles resulting from repeated parabolas can be viewed as a natural and variable resetting of the liquid, in keeping with the idea of investigating a robust method for manipulating fluid in microgravity.

1.2 Thesis objectives

The general objective of the proposed thesis is to study vibrated fluids, focusing mainly (but not exclusively) on the phenomena that are of relevance in microgravity conditions. Specific objectives include:

1. A study of the dynamics of Faraday waves and cross-waves, including multiplicity of cross-wave solutions and slow temporal modulation.
2. A theoretical and numerical study of vibroequilibria, including a comparison between theoretical predictions based on an averaged energy functional method and the results of direct numerical simulations. A study of instabilities suffered by the vibroequilibria (average) surface with increasing forcing and their dependence on depth (aspect ratio) and contact angle.
3. A study of the interaction of vibroequilibria with surface waves (Faraday and cross-waves) and with sloshing modes using extensive numerical simulations.
4. A microgravity experiment, CFVIB (Control of Fluids in Microgravity with Vibrations), designed and prepared for the ESA Fly Your Thesis! 2016 campaign. Initial analysis and interpretation of experimental results.

The main goal of the CFVIB experiment is to test theoretical predictions, as much as possible, in a real microgravity environment, and to evaluate the efficacy of using vibrations to control the position of fluid interfaces in microgravity through the vibroequilibria effect.

1.3 Thesis structure

This thesis is organized as follows. First, in Chapter 2, we review the cross-wave theory initiated by Jones [28] and the subsequent extensions that take account of effects like viscosity, discussing its limitations for describing recent cross-waves experiments. The main point of this chapter is to demonstrate that, in the

gravity-capillary parameter regime of interest, distributed forcing is crucial to obtaining predictions consistent with experiment. This is confirmed by a series of new experimental results for the onset of cross-waves in containers of various sizes, and by measurements of the spatial and temporal modulations of the onset patterns. In particular, the interesting quasiperiodic solutions predicted by the model in section 2.4, which cycle between one-sided and two-sided excitation, are located experimentally.

Next, we review results on the hydrostatic solution without gravity in Chapter 3. A familiarity with multiplicity and hysteresis in the hydrostatic (unforced) case will be useful for interpreting subsequent vibroequilibria results. In Chapter 4 we define the mathematical problem considered by vibroequilibria theory. The numerical scheme used to solve the vibroequilibria problem is defined in section 4.1.1 and results are compared with direct numerical simulations of the Navier-Stokes equations in section 4.2. Some detailed vibroequilibria results are presented for two-dimensional rectangular containers with either vertical or oblique forcing in section 4.3. An instability of the vibroequilibria state for sufficiently deep containers is revealed and analysed in Chapter 5, section 5.2, after restricting the minimization problem to specific families of surface shapes. Section 5.3 examines the interaction of (high-frequency) surface waves and sloshing modes with the average vibroequilibria state using direct numerical simulations.

The results of these investigations motivate the experiment described in Chapter 6. The design of experiment subsystems is presented (sections 6.1, 6.2 and 6.3.1) as well as the required ground tests to prepare for the parabolic flight campaign (6.4). Finally, we present a preliminary description of the parabolic flight results in Chapter 7 and draw conclusions in Chapter 8.

2 Surfaces waves: Theory and experiment

2.1 Cross-wave instability

The most influential approach to the theory of cross-waves was initiated by Jones [28], who obtained a cubic NLS equation to describe the slow spatial and temporal evolution of the cross-wave amplitude. Although alternative derivations were presented by other authors that take account of effects like viscosity and mode interactions [4, 9, 36, 60] the underlying assumptions are the same. These are:

1. Damping is small, or absent.
2. Applied forcing is small.
3. Surface tension can be neglected in the free-surface boundary conditions.
4. The downstream variation of the subharmonic waves is slow compared to
 - (a) the crosswise variation (i.e., waves are oriented crosswise, or nearly so)
 - and to (b) the spatial decay of the evanescent harmonic flow field generated by the wavemaker, which provides the forcing mechanism.

The first two assumptions are critical for obtaining analytical results and are still justified over a range of experimental conditions, like those of Porter *et al.* [54] where the NLS models no longer apply. The final two assumptions were motivated by the earlier wavemaker experiments [6, 35, 37, 38, 60, 61, 69] that the NLS models were intended to explain, nearly all of which were performed in large tanks of water (more than a meter in length) and at low frequencies (less than 10 Hz). These assumptions are not essential to the derivation and must be

relaxed if the theory is to be extended to higher frequencies and gravity-capillary waves.

We focus here on the particular model derived by Bernoff *et al.* [9] which includes the leading order effects of viscous damping at the wavemaker, sidewalls, bottom, and free surface. The formulation assumes a planar wavemaker oscillating about $x = 0$ according to $x(t) = a \cos(2\sigma t)$. A fluid with small kinematic viscosity ν is confined in a semi-infinite domain with (crosswise) width W and depth D . The vertical gravitational acceleration is g . The mode number N defines the crosswise wavenumber $k = N\pi/W$. After nondimensionalizing with k and σ the system depends on the small forcing parameter $\epsilon = ak \ll 1$, the Reynolds number $\text{Re} = \sigma/(\nu k^2) \gg 1$, and the nondimensional depth $d = kD$. The damped NLS equation obtained in Bernoff *et al.* [9], after neglecting the inhomogeneous contribution to the detuning due to the harmonic waves, can be written in terms of the original (fast) space and time variables, as

$$iA_t = \frac{S}{4}A_{xx} + (v - i\gamma_0)A + \epsilon^2\beta|A|^2A, \quad (2.1a)$$

$$A_x = -\epsilon R\bar{A} - \gamma_1 A \quad \text{at} \quad x = 0, \quad (2.1b)$$

$$A \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad (2.1c)$$

where $S = 1 + 2d/\sinh(2d)$, $v = (\sigma - \sigma_0)/\sigma_0$ with $\sigma_0^2 = gk \tanh(d)$, $\gamma_0 = 2/\text{Re} + \sqrt{i}[1/\sinh(2d) + (2 - S)/(N\pi)]/\sqrt{\text{Re}}$, $\beta = (16 - 9H^2 - 5/H^2 + 6/H^4)/16$ with $H = 1/\tanh(d)$, $R = [d(H + 3/H) - 2]/S$, and $\gamma_1 = 2/(S\sqrt{i\text{Re}})$.

Note that, as a result of the separation of lengthscales permitted by assumption 4(b), the waves are forced only via the homogeneous boundary condition (2.1b). This contrasts with the case of vertically forced Faraday waves (where the forcing is uniformly distributed) and with the theory of Varas and Vega [70] where the forcing term appears in the governing equation because it varies on the same scale as the modulation. The appearance of the forcing as a boundary condition has significant consequences for the NLS model, particularly with respect to the dependence of the neutral stability curves on forcing frequency σ and viscosity. These neutral stability curves are straightforward to obtain from

the linearized version of Eqs. (2.1) by writing $A(x) = A_0 e^{\kappa x}$. They take the form

$$g_{\text{rel}}^c = \frac{4\sigma^2 |\kappa + \gamma_1|}{gk|R|}, \quad (2.2)$$

with

$$\kappa^2 = \frac{4}{S}(i\gamma_0 - \nu), \quad \text{real}(\kappa) < 0, \quad (2.3)$$

where the critical value is expressed in terms of relative acceleration using $g_{\text{rel}} = a(2\sigma)^2/g$. Equation (2.2) defines a set of neutral stability tongues (indexed by N) whose minima *decrease* with increasing forcing frequency σ (see Fig. 2.1, reproduced from Porter *et al.* [54]). This behaviour is not explained by damping at the

FIGURE 2.1: Neutral stability curves corresponding to Eq.(2.2) with $\nu = 1 \text{ cSt}$, $W = 30 \text{ cm}$ and $D = 15 \text{ cm}$. Note that the minimum values are monotonically decreasing with frequency (mode number). Image reproduced with permission from [54].

container bottom and occurs even if bottom, sidewall, and wavemaker damping are all neglected. It is a result of the separation of lengthscales assumed in the derivation of the NLS models. This assumption is strictly valid only for very small damping (equivalently, very low frequencies) or for very shallow wave-makers. It is doubtful that such a separation of lengthscales holds well enough for a quantitative prediction of the cross-wave threshold, for example, even in some of the experiments to which the NLS models have been applied with some

success.

Consider, for instance, the experiments of Underhill *et al.* [69] where cross-wave modes $N = 5$ and $N = 7$ were measured. The critical acceleration did not decrease with forcing frequency (mode number) as predicted, but increased instead. These experiments were performed with a fluid depth of 25.8 cm, which is greater than the wavelengths of the two modes: 12.36 cm and 8.83 cm, respectively. Damping at the bottom is therefore not important. From Figs. 3 and 7 of that reference, critical acceleration values of 0.064 g for $N = 5$ and 0.074 g for $N = 7$ can be extracted, showing a clear increase with increasing forcing frequency — again, something which cannot be explained by the NLS model.

Neutral stability curves were also measured by Shemer and Lichter [59] for mode numbers 3, 5, 6, and 7 in a wave tank measuring 120 cm across. The corresponding wavelengths for these modes are 80 cm, 48 cm, 40 cm, and 34.3 cm, which should be compared to the depth of 60 cm. From Fig. 2 of that paper, one can obtain estimates for the (minimum) acceleration required to excite these four modes: 0.153 g , 0.137 g , 0.144 g , and 0.167 g , respectively. In dropping from $N = 5$ to $N = 3$ there is a 12% increase in the critical acceleration needed but this can be explained by the fact that the wavelength for $N = 3$ exceeds the container depth, leading to an increase in damping at the bottom. In contrast, when mode number is increased between 5 and 7, where the wavelength is less than the fluid depth, there is a measured increase in critical acceleration, contrary to the NLS model. Although this experiment is a bit harder to interpret than that of Underhill *et al.* [69] because of the segmented wavemaker used, which generates larger dissipation via vortex shedding at the discontinuities [59], the increase in critical acceleration with increasing mode number cannot easily be explained by the NLS model, which suggests that the balance between damping and forcing resulting from the assumed separation of length scales is not entirely justified.

2.2 Extension of the nonlinear Schrödinger equation to cross-waves with extended forcing

Here we extend the standard NLS models by deriving an amplitude equation to describe modulated cross-waves that includes the effect of surface tension and does not assume a separation of scales between the downstream modulation of the waves and the extent of the forcing. As in Bernoff *et al.* [9] we first consider a semi-infinite domain: $x \geq 0$, $0 \leq y \leq W$, $-D \leq z \leq 0$ from a coordinate system oscillating periodically along the x axis with frequency 2σ and amplitude a . The fluid has density ρ , viscosity ν , and surface tension Γ . The crosswise wavenumber $k_y = N\pi/W$ is characterized by the mode number N and the natural frequency ω_0 is defined by the inviscid dispersion relation

$$\omega_0^2 = (gk_y + \frac{\Gamma}{\rho}k_y^3) \tanh(k_y D). \quad (2.4)$$

The nondimensional forcing, gravity number, capillary number, and Reynolds number are defined by

$$\epsilon = ak_y, \quad G_0 = \frac{gk_y}{\omega_0^2}, \quad \Gamma_0 = \frac{\Gamma k_y^3}{\rho \omega_0^2}, \quad \text{Re} = \frac{\omega_0}{\nu k_y^2}, \quad (2.5)$$

respectively. Note that $G_0 + \Gamma_0 = 1/\tanh(d) = H$.

The system will be forced near resonance, so we write $\omega \equiv \sigma/\omega_0 = 1 + v$ with v small. After rescaling length and time with k_y and ω_0 , respectively, we have the following equations in the oscillating reference frame:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{v} + 4\omega^2 \epsilon \cos(2\omega t) \mathbf{e}_x, \quad (2.6a)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.6b)$$

where $p - G_0 z$ is the pressure, $\nabla = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y + \mathbf{e}_z \partial_z$, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, and $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors aligned along the coordinate axes. No-slip boundary conditions are imposed on the solid walls. The free surface $z = f(x, y)$ satisfies

the following stress balance and kinematic conditions [49]:

$$p - G_0(\hat{\mathbf{n}} \cdot \mathbf{e}_z)f + 2\Gamma_0 M = \frac{1}{\text{Re}} \hat{\mathbf{n}} \cdot [\nabla \mathbf{v} + \nabla \mathbf{v}^T] \cdot \hat{\mathbf{n}}, \quad (2.7a)$$

$$\hat{\mathbf{n}} \times [\nabla \mathbf{v} + \nabla \mathbf{v}^T] \cdot \hat{\mathbf{n}} = 0, \quad (2.7b)$$

$$(\mathbf{v} - f_t \mathbf{e}_z) \cdot \hat{\mathbf{n}} = 0, \quad (2.7c)$$

where $M = \bar{\nabla} \cdot \hat{\mathbf{n}}/2$ is the mean curvature of the surface, $\bar{\nabla} = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y$ denotes the horizontal gradient operator, and $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{(-\bar{\nabla} f, 1)}{\sqrt{1 + |\bar{\nabla} f|^2}}. \quad (2.8)$$

In the quasipotential approximation the bulk of the fluid is taken to be irrotational and inviscid with $u = \nabla \phi$. The velocity potential ϕ then satisfies the Laplace equation

$$\Delta \phi = 0. \quad (2.9)$$

Boundary layers at the walls and free surface are assumed to be linear. Standard analysis for the boundary layer at the walls provides the velocity, and matching the normal component across the layer leads to the condition [9]:

$$(\hat{\phi}_\Omega)_n = -\frac{1}{\sqrt{i\Omega \text{Re}}} (\hat{\phi}_\Omega)_{nn}, \quad (2.10)$$

where the subscript n indicates a derivative in the normal direction and $\hat{\phi}_\Omega(\mathbf{x})$ denotes the temporal Fourier component of $\phi(\mathbf{x}, t)$ oscillating as $\exp\{i\Omega t\}$; hereafter, the subscript Ω will be suppressed to simplify notation. The boundary layer at the free surface provides linear viscous corrections to both the kinematic compatibility condition and normal stress at the interface. We apply the following

boundary conditions [76], valid to order $O(\epsilon^2 \text{Re}^{-1/2} + \epsilon^3)$

$$f_t - \phi_z = \frac{2}{\text{Re}} \bar{\nabla}^2 f - \bar{\nabla} \phi \cdot \bar{\nabla} f, \quad (2.11a)$$

$$\phi_t + \frac{G_0 f}{\sqrt{1 + |\bar{\nabla} f|^2}} - \Gamma_0 \bar{\nabla} \cdot \left[\frac{\bar{\nabla} f}{\sqrt{1 + |\bar{\nabla} f|^2}} \right] = \frac{2}{\text{Re}} \bar{\nabla}^2 \phi - \frac{1}{2} |\nabla \phi|^2. \quad (2.11b)$$

Expanding about $z = 0$ and retaining terms up to cubic order yields the following system

$$\Delta \phi = 0, \quad (2.12a)$$

with boundary conditions at the surface $z = 0$,

$$f_t - \phi_z = \frac{2}{\text{Re}} \bar{\nabla}^2 f + f \phi_{zz} - \bar{\nabla} \phi \cdot \bar{\nabla} f + \frac{1}{2} \phi_{zzz} f^2 - f \bar{\nabla} \phi_z \cdot \bar{\nabla} f, \quad (2.12b)$$

$$\begin{aligned} \phi_t + G_0 f - \Gamma_0 \Delta f &= 4\epsilon \cos(2\omega t) x - \frac{2}{\text{Re}} \phi_{zz} - \frac{1}{2} |\nabla \phi|^2 - f \phi_{zt} - \frac{1}{2} \phi_{zzt} f^2 \\ &\quad - f \nabla \phi_z \cdot \nabla \phi + \frac{1}{2} G_0 f |\bar{\nabla} f|^2 - \frac{1}{2} \Gamma_0 \bar{\nabla} \cdot [\bar{\nabla} f |\bar{\nabla} f|^2], \end{aligned} \quad (2.12c)$$

where $\Delta = \partial_x^2 + \partial_y^2$, and at the boundaries

$$\hat{\phi}_x = \frac{-1}{\sqrt{i\Omega \text{Re}}} \hat{\phi}_{xx} \quad \text{at } x = 0, \quad (2.12d)$$

$$\phi \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.12e)$$

$$\hat{\phi}_y = \frac{\mp 1}{\sqrt{i\Omega \text{Re}}} \hat{\phi}_{yy} \quad \text{at } y = 0, N\pi, \quad (2.12f)$$

$$\hat{\phi}_z = \frac{-1}{\sqrt{i\Omega \text{Re}}} \hat{\phi}_{zz} \quad \text{at } z = -d. \quad (2.12g)$$

2.2.1 Scaling

We expand [9, 28] dependent variables in powers of ϵ

$$\phi = \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \epsilon^3 \phi^{(3)} + \dots, \quad (2.13)$$

$$f = \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} + \dots, \quad (2.14)$$

$$v = \epsilon^2 v^{(2)}, \quad (2.15)$$

and introduce slow spatial and temporal scales

$$(X, Y) = \epsilon(x, y), \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t. \quad (2.16)$$

We further assume [9] that $\epsilon \text{Re}^{1/2} \equiv \delta^{-1}$ is $O(1)$ and solve the system (2.12) at successive orders in ϵ . Details of this procedure, which leads to a solvability condition that gives the amplitude equations describing subharmonic waves, is described in section 2.2.4.

2.2.2 Leading order solution and harmonic flow

At leading order in ϵ one obtains the inhomogeneous linear problem from Eqs. (2.12):

$$\Delta \phi^{(1)} = 0, \quad (2.17a)$$

with boundary conditions at the surface $z = 0$ given by,

$$f_t^{(1)} - \phi_z^{(1)} = 0, \quad (2.17b)$$

$$\phi_t^{(1)} + G_0 f^{(1)} - \Gamma_0 \Delta f^{(1)} = 4 \cos(2\omega t)x, \quad (2.17c)$$

and at the boundaries

$$\phi_x^{(1)} = 0 \quad \text{at} \quad x = 0, \quad (2.17d)$$

$$\phi^{(1)} \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad (2.17e)$$

$$\phi_y^{(1)} = 0 \quad \text{at} \quad y = 0, N\pi, \quad (2.17f)$$

$$\phi_z^{(1)} = 0 \quad \text{at} \quad z = -d. \quad (2.17g)$$

We distinguish three distinct contributions at this order by writing

$$\phi^{(1)} = [-ix + \phi^H]e^{2i\omega t} + \phi^{SW}e^{i\omega t} + c.c., \quad (2.18)$$

where $\omega = 1 + \epsilon^2 \nu^{(2)}$. The second (harmonic) term can be expressed using cosine transform methods as

$$\phi^H = -\frac{4}{\pi} \int_C \frac{e^{-qx} \cos[q(z+d)]}{q^2 [4 \cos(qd) + (G_0 q - \Gamma_0 q^3) \sin(qd)]} dq, \quad (2.19)$$

where C is a path from $-i\infty$ to $i\infty$ returning clockwise along a semicircle of infinite radius in the right half-plane. This expression, which for simplicity neglects corrections coming from the detuning ν , contains two contributions (see Havelock [23]). The first one is a harmonic traveling wave of the form $\phi^{HW} = B(X, T_1, T_2) \exp\{ikx\} \cosh[k(z+d)]$ corresponding to the complex pole $q = ik$ with k given here by the dispersion relation:

$$(G_0 k + \Gamma_0 k^3) \tanh(kd) = 4. \quad (2.20)$$

In the treatment that follows we will neglect the role of the harmonic waves. The second contribution in Eq. (2.19) is an evanescent field referred to as the oscillating bulk flow [70]. It can be approximated, except in a narrow region near the walls, by

$$\phi^{BF}(x, z) = -\frac{2i}{d} \sum_{n=0}^{\infty} \frac{(-1)^n}{q_n^2} e^{-q_n x} \cos[q_n(z+d)], \quad (2.21)$$

where $q_n d = \pi(n + 1/2)$. This simplified expression will be used in numerical computations. Note that the oscillating bulk flow ϕ^{BF} , which exhibits multi-exponential decay in x , extends over a lengthscale $x \sim d$.

2.2.3 Second and third order solutions

At second order

$$\Delta \phi^{(2)} = -2(\phi_{xX}^{(1)} + \phi_{yY}^{(1)}), \quad (2.22a)$$

with boundary conditions at the surface $z = 0$

$$f_t^{(2)} - \phi_z^{(2)} = -f_{T_1}^{(1)} + f^{(1)}\phi_{zz}^{(1)} - \nabla\phi^{(1)} \cdot \nabla f^{(1)}, \quad (2.22b)$$

$$\phi_t^{(2)} + G_0 f^{(2)} - \Gamma_0 \Delta f^{(2)} = -\phi_{T_2}^{(1)} - \frac{1}{2} |\nabla\phi^{(1)}|^2 + 2\Gamma_0 (f_{xX}^{(1)} + f_{yY}^{(1)}) - f^{(1)}\phi_{zt}^{(1)}, \quad (2.22c)$$

and at the boundaries

$$\hat{\phi}_x^{(2)} = -\hat{\phi}_X^{(1)} - \frac{\delta}{\sqrt{i\Omega}} \hat{\phi}_{xx}^{(1)} \quad \text{at } x = 0, \quad (2.22d)$$

$$\phi^{(2)} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.22e)$$

$$\hat{\phi}_y^{(2)} = -\hat{\phi}_Y^{(1)} \mp \frac{\delta}{\sqrt{i\Omega}} \hat{\phi}_{yy}^{(1)} \quad \text{at } y = 0, N\pi, \quad (2.22f)$$

$$\hat{\phi}_z^{(2)} = -\frac{\delta}{\sqrt{i\Omega}} \hat{\phi}_{zz}^{(1)} \quad \text{at } z = -d. \quad (2.22g)$$

At third order, keeping only the dispersive terms that are required to determine onset, we have

$$\Delta\phi^{(3)} = -\phi_{XX}^{(1)}, \quad (2.23a)$$

with boundary conditions at the surface $z = 0$

$$f_t^{(3)} - \phi_z^{(3)} = 2\delta\Delta f^{(1)} - f_{T_2}^{(1)} - v^{(2)}f^{(1)}, \quad (2.23b)$$

$$\phi_t^{(3)} + G_0 f^{(3)} - \Gamma_0 \Delta f^{(3)} = -\phi_{T_2}^{(1)} + f_{XX}^{(1)} - 2\delta\phi_{zz}^{(1)} - v^{(2)}\phi^{(1)}, \quad (2.23c)$$

and at the boundaries

$$\phi_x^{(3)} = 0 \quad \text{at } x = 0, \quad (2.23d)$$

$$\phi^{(3)} \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.23e)$$

$$\phi_y^{(3)} = 0 \quad \text{at } y = 0, N\pi, \quad (2.23f)$$

$$\phi_z^{(3)} = 0 \quad \text{at } z = -d. \quad (2.23g)$$

Additional terms contributing to the cubic self-interaction coefficient are considered separately in section 2.2.5.

Dissipation at the walls is taken into account [9] by including a localized velocity contribution,

$$\hat{\phi}_z \rightarrow \hat{\phi}_z \left(1 - \sum_{\text{edges}} e^{-\sqrt{i\Omega \text{Re}} \xi_i}\right), \quad (2.24)$$

in the free surface condition. Here ξ_i denotes the perpendicular distance from the wall in question. Note that, although this exponential contribution is formally $O(1)$ it contributes only over an $O(\epsilon)$ region near each wall and is thus taken to be of that order [9].

2.2.4 Solvability condition for cross-waves

The solvability condition for Eqs. (2.22) and Eqs. (2.23) is obtained from Green's identity:

$$\begin{aligned} 0 &= \int_D \Delta \phi^\dagger \phi^{(i)} dx dy dz dt = \int_D \phi^\dagger \Delta \phi^{(i)} dx dy dz dt + \int_{x=0} \phi^\dagger \phi_x^{(i)} dy dz dt \\ &+ \int_{y=0} \phi^\dagger \phi_y^{(i)} dx dz dt - \int_{y=N\pi} \phi^\dagger \phi_y^{(i)} dx dz dt \\ &+ \int_{z=-d} \phi^\dagger \phi_z^{(i)} dx dy dt - \int_{z=0} (\phi^\dagger \phi_z^{(i)} - \phi_z^\dagger \phi^{(i)}) dx dy dt, \end{aligned} \quad (2.25)$$

where D denotes the entire fluid domain and the surface integrals are taken over the specified boundaries. Note the integral over time, which will be used later to select the subharmonic component.

In order to extract terms with subharmonic time dependence, we take $(\phi^\dagger, f^\dagger)$ as solutions of the homogeneous (unforced) version of Eqs. (2.17) (which defines a self-adjoint operator for the standard scalar product) oscillating as $\exp\{-it\}$.

The final term in Eq. (2.25) can be rewritten as

$$\begin{aligned}
\int_{z=0} (\phi^\dagger \phi_z^{(i)} - \phi_z^\dagger \phi^{(i)}) dx dy dt &= \int_{z=0} [\phi^\dagger (\phi_z^{(i)} - f_t^{(i)}) + \phi^\dagger f_t^{(i)} - f_t^\dagger \phi^{(i)}] dx dy dt \\
&= \int_{z=0} [\phi^\dagger (\phi_z^{(i)} - f_t^{(i)}) + f^\dagger (\phi_t^{(i)} + G_0 f^{(i)} - \Gamma_0 \Delta f^{(i)})] dx dy dt \\
&\quad + \Gamma_0 \int_{y=N\pi, z=0} f^\dagger f_y^{(i)} dx dt - \Gamma_0 \int_{y=z=0} f^\dagger f_y^{(i)} dx dt + \Gamma_0 \int_{x=z=0} (f_x^\dagger f^{(i)} - f^\dagger f_x^{(i)}) dy dt, \tag{2.26}
\end{aligned}$$

where periodicity has been used in integrating by parts in time. Finally, the term in Eq. (2.26) with the unknown $f_x(0, y)$ is rewritten as

$$\Gamma_0 \int_{x=z=0} (f_x^\dagger f^{(i)} - f^\dagger f_x^{(i)}) dy dt = i \Gamma_0 \int_{x=z=0} [f^\dagger (f_{xt}^{(i)} - \phi_{xz}^{(i)}) + f^\dagger \phi_{xz}^{(i)}] dy dt. \tag{2.27}$$

Here, assuming continuity, $f_{xt}^{(i)} - \phi_{xz}^{(i)}$ is evaluated by approaching the contact line from the free surface and $\phi_{xz}^{(i)}$ by approaching along the wall. Combining these expressions yields the solvability condition

$$\begin{aligned}
0 &= \int_D \phi^\dagger \Delta \phi^{(i)} dx dy dz dt + \int_{x=0} \phi^\dagger \phi_x^{(i)} dy dz dt + \int_{y=0} \phi^\dagger \phi_y^{(i)} dx dz dt - \int_{y=N\pi} \phi^\dagger \phi_y^{(i)} dx dz dt \\
&\quad + \int_{z=-d} \phi^\dagger \phi_z^{(i)} dx dy dt + \int_{z=0} \phi^\dagger (f_t^{(i)} - \phi_z^{(i)}) - f^\dagger (\phi_t^{(i)} + G_0 f^{(i)} - \Gamma_0 \Delta f^{(i)}) dx dy dt \\
&\quad - \Gamma_0 \int_{y=N\pi, z=0} f^\dagger f_y^{(i)} dx dt + \Gamma_0 \int_{y=z=0} f^\dagger f_y^{(i)} dx dt + i \Gamma_0 \int_{x=z=0} f^\dagger (f_{xt}^{(i)} - \phi_{xz}^{(i)}) + f^\dagger \phi_{xz}^{(i)} dy dt. \tag{2.28}
\end{aligned}$$

The general form of the solvability condition is given in Eq. (2.28). To apply this, we separate vertical and horizontal dependence by writing

$$\phi^{SW} = A(X, T) \frac{\cosh(z+d)}{\cosh(d)} \Psi(x, y), \tag{2.29}$$

$$\tilde{\phi}^\dagger = \frac{\cosh(z+d)}{\cosh(d)} \bar{\Psi}(x, y). \tag{2.30}$$

where $\phi^\dagger = \tilde{\phi}^\dagger \exp\{-it\}$ and $f^\dagger$, which also oscillates as $\exp\{-it\}$, are (resonant subharmonic) solutions of the homogeneous linear problem (2.17) with no forcing. Then, to leading order, at $z = 0$, $f^\dagger = iH^{-1}\phi^\dagger$ and $f^{SW} = -iH^{-1}\phi^{SW}$, where

$f^{SW} \exp\{it\}$ is the corresponding subharmonic component of the free surface motion. By taking into account

$$\int_{-d}^0 \frac{\cosh^2(z+d)}{\cosh^2(d)} dz = \frac{1}{2H} \left(\frac{2d}{\sinh(2d)} + 1 \right),$$

we integrate over the vertical coordinate to obtain the solvability conditions at second order:

$$\begin{aligned} & + \frac{i}{H} \int_{z=0} \tilde{\phi}^\dagger [2\phi_{T_1}^{SW} + iS(\phi_{xX}^{SW} + \phi_{yY}^{SW}) + i\phi^{SW} \sum_{\text{edges}} e^{-\sqrt{i\text{Re}} \xi_i}] dx dy \\ & + \frac{i}{2H} \int_{z=0} \tilde{\phi}^\dagger \{ [2\phi_{zz}^{BF} - (H + 3H^{-1})\phi_z^{BF}] \bar{\phi}^{SW} + H \bar{\nabla} \phi_z^{BF} \cdot \bar{\nabla} \bar{\phi}^{SW} \} dx dy \\ & + \frac{S}{2H} \int_{x=z=0} \tilde{\phi}^\dagger (-\phi_X^{SW} - \frac{\delta}{\sqrt{i}} \phi_{xx}^{SW}) dy + \frac{1}{2} \int_{z=-d} \tilde{\phi}^\dagger (-\frac{\delta}{\sqrt{i}} \phi^{SW}) dx dy \\ & - \frac{S}{2H} \int_{y=N\pi, z=0} \tilde{\phi}^\dagger (-\phi_Y^{SW} + \frac{\delta}{\sqrt{i}} \phi_{yy}^{SW}) dx + \frac{S}{2H} \int_{y=z=0} \tilde{\phi}^\dagger (-\phi_Y^{SW} - \frac{\delta}{\sqrt{i}} \phi_{yy}^{SW}) dx = 0, \quad (2.31) \end{aligned}$$

and at third order:

$$\frac{1}{H} \int_{z=0} \tilde{\phi}^\dagger [2\phi_{T_2}^{SW} + 2iv^{(2)}\phi^{SW} + 4\delta\phi^{SW} + i\frac{S}{2}\phi_{XX}^{SW}] dx dy = 0, \quad (2.32)$$

with

$$S = 1 + \frac{2d}{\sinh(2d)} + 2\Gamma_0 \tanh(d). \quad (2.33)$$

Equation (2.19) must be used to evaluate the terms at $x = z = 0$ to ensure that the terms involving ϕ_{xz} and ϕ_{xzz} vanish (being given by integrals over an odd function of q with no pole at the origin).

To this point, no assumption has been made about the orientation of the subharmonic waves. We now focus on cross-waves of the form $\Psi(x, y) = \cos(y)$, permitting only slow modulation in the x direction. After integrating Eqs. (2.31)

and (2.32) over y and combining the results, we find

$$0 = 2i \int_0^\infty [\epsilon A_{T_1} + \epsilon^2 A_{T_2} + (iv + \gamma_0)A + i\epsilon^2 \frac{S}{4} A_{XX}] dx + \frac{i}{2} \epsilon \bar{A} \int_{z=0} [2\phi_{zz}^{BF} - (H + 3H^{-1})\phi_z^{BF}] dz + \epsilon \frac{S}{2} A_X \Big|_{x=0} + \frac{1}{\sqrt{i\text{Re}}} A \Big|_{x=0}, \quad (2.34)$$

where, as in Eqs. (2.1),

$$\gamma_0 = \frac{2}{\text{Re}} + \sqrt{\frac{i}{\text{Re}}} \left(\frac{1}{\sinh(2d)} + \frac{2-S}{N\pi} \right). \quad (2.35)$$

The terms in Eq. (2.34) must be considered in light of the definition of $X = \epsilon x$. The first integral is formally divergent unless the integrand, which is independent of x , vanishes for all X . This condition provides the main differential equation to be satisfied. The final two terms comes from a boundary term evaluated at $x = 0$ and a rapidly decaying free surface velocity contribution [see Eq. (2.24)], and thus should be interpreted as a boundary condition at $X = 0$. The second term can contribute either to the differential equation or to the boundary condition depending on the assumptions made about the bulk flow ϕ^{BF} . If ϕ^{BF} is assumed to vary rapidly, on the fast lengthscale x , then the second term can also be integrated over x . The result, which is proportional to $\bar{A}$, is added to the boundary condition at $X = 0$ since the principal contribution to the integral comes from $x \sim O(1)$, or $X \ll 1$. In this case, after reconstituting time derivatives with $A_t = \epsilon A_{T_1} + \epsilon^2 A_{T_2}$, returning to the fast lengthscale using $X = \epsilon x$, and including the cubic nonlinearity (see the Appendix) we have the following system to determine the slowly varying amplitude $A(x, t)$:

$$iA_t = \frac{S}{4} A_{xx} + (v - i\gamma_0)A + \epsilon^2 \beta |A|^2 A, \quad (2.36a)$$

$$A_x = -\epsilon R \bar{A} - \gamma_1 A \quad \text{at} \quad x = 0, \quad (2.36b)$$

$$A \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad (2.36c)$$

with

$$R = \frac{i}{S} \int_{z=0} [2\phi_{zz}^{BF} - (H + 3H^{-1})\phi_z^{BF}] dx = \frac{1}{S} [d(H + 3H^{-1}) - 2], \quad (2.37)$$

and

$$\gamma_1 = 2/(S\sqrt{i\text{Re}}). \quad (2.38)$$

This is simply the extension of the amplitude equations (2.1) of Bernoff *et al.* [9] to the gravity-capillary regime, reflected in the definition of S in Eq. (2.33). It applies only if $g(x)$ decays much faster with distance from the endwall than $A(x, t)$, as will be the case, for example, if the fluid (wavemaker in other experimental configurations) is shallow.

On the other hand, if the fluid depth d is larger than or comparable to the extent of the subharmonic field $A(x, t)$ then the bulk flow ϕ^{BF} will extend into the interior and cannot be integrated out [70]. In this more general case, when ϕ^{BF} is taken to vary on the same slow lengthscale as the amplitude A , the second integral should be combined with the first. The combined integrand, which is independent of x , is then set to zero to give the following system with distributed forcing:

$$iA_t = \frac{S}{4} A_{xx} + (v - i\gamma_0)A + \epsilon g(x)\bar{A} + \epsilon^2 \beta |A|^2 A, \quad (2.39a)$$

$$A_x = -\gamma_1 A \quad \text{at} \quad x = 0, \quad (2.39b)$$

$$A \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \quad (2.39c)$$

where

$$g(x) = -\frac{i}{4} \left[2\phi_{zz}^{BF} - (H + 3H^{-1})\phi_z^{BF} \right] \Big|_{z=0}. \quad (2.40)$$

Note that γ_1 , which accounts for damping at the endwall, is due solely to the contribution (2.24).

2.2.5 Self-interaction coefficient

The relevant terms appearing in the free surface boundary conditions (2.12b) and (2.12c) are:

$$f_t^{(3)} - \phi_z^{(3)} = f^{(2)}\phi_{zz}^{(1)} + f^{(1)}\phi_{zz}^{(2)} - \bar{\nabla}\phi^{(2)} \cdot \bar{\nabla}f^{(1)} - \bar{\nabla}\phi^{(1)} \cdot \bar{\nabla}f^{(2)} + \frac{1}{2}\phi_{zzz}^{(1)}f^{(1)}f^{(1)} - f^{(1)}\bar{\nabla}\phi_z^{(1)} \cdot \bar{\nabla}f^{(1)}, \quad (2.41)$$

$$\phi_t^{(3)} + Hf^{(3)} - \Gamma_0\Delta f^{(3)} = -\nabla\phi^{(1)} \cdot \nabla\phi^{(2)} - f^{(2)}\phi_{zt}^{(1)} - f^{(1)}\phi_{zt}^{(2)} - \frac{1}{2}\phi_{zzt}^{(1)}f^{(1)}f^{(1)} - f^{(1)}\nabla\phi_z^{(1)} \cdot \nabla\phi^{(1)} + \frac{1}{2}G_0f^{(1)}|\bar{\nabla}f^{(1)}|^2 - \frac{1}{2}\Gamma_0\bar{\nabla} \cdot [\bar{\nabla}f^{(1)}|\bar{\nabla}f^{(1)}|^2], \quad (2.42)$$

where $\phi^{(2)}$ and $f^{(2)}$ are the nonresonant parts of the solution to Eqs. (2.22):

$$\begin{aligned} \phi^{(2)} &= \frac{1}{8}(iA^2e^{2it} + c.c.) \left\{ 1 + 3H^{-2} \right. \\ &\quad \left. + 3 \frac{(1 + H^{-2})(H - H^{-1} + 2\Gamma_0)}{3\Gamma_0 - H^{-1}} \cos(2y) \frac{\cosh[2(z + d)]}{\cosh(2d)} \right\}, \\ f^{(2)} &= \frac{1}{4}(A^2e^{2it} + c.c.) \frac{3 - H^{-2}}{3\Gamma_0 - H^{-1}} \cos(2y) \\ &\quad + \frac{1}{2}|A|^2 \left[\frac{H^{-2} - 1}{H - \Gamma_0} + \frac{H^{-2} + 1}{H + 3\Gamma_0} \cos(2y) \right], \end{aligned} \quad (2.43)$$

Applying the solvability condition (2.28), the self-interaction coefficient is obtained,

$$\begin{aligned} \beta &= \frac{3}{16}H^{-2}(2H^{-1}\Gamma_0 + 1) + \frac{12H\Gamma_0 + H^{-2} - 10 + 9H^2}{16(3H\Gamma_0 - 1)} \\ &\quad + \frac{2H^{-2} - 3H^{-4} - 3 - (5H^{-1} - 14H^{-3} + 5H^{-5})\Gamma_0}{8(3H^{-1}\Gamma_0 + 1)(H^{-1}\Gamma_0 - 1)}. \end{aligned} \quad (2.44)$$

The first term in Eq. (2.44) comes from the final two terms in Eq. (2.12c), which are cubic corrections due to curvature not included in Bernoff *et al.* [9], while the second and third terms come from harmonic and mean (zero frequency) components of the second order solutions in Eqs. (2.43). The sum of these final two terms evaluated at $\Gamma_0 = 0$ gives the cubic coefficient β in Eqs. (2.1).

There is a singularity in Eq. (2.44) on the allowed interval $0 \leq \Gamma_0 \leq 1$ when $\Gamma_0 = H/3$, which corresponds to the 1:2 resonance when the wavenumber of

harmonic waves, as defined by the inviscid dispersion relation used in Eq. (2.4) with frequency 2σ , is exactly $2k_y$. This leads to a small region of negative β values, for $0.232 \lesssim \Gamma_0 \leq 1/3$ in the limit of large depth ($H = 1$), while outside of this β is positive, as illustrated in Fig. 2.2. This singularity could be removed by a

FIGURE 2.2: Cubic coefficient β as a function of Γ_0 for large depth ($H = 1$).

more careful treatment near the 1:2 resonance including damping effects for the harmonic mode.

2.3 Secondary cross-wave modes

2.3.1 Onset of cross-waves

If the effect of the harmonic surface waves, ϕ^{HW} , is neglected, then cross-waves are driven entirely by the oscillating bulk flow ϕ^{BF} , which itself is generated by the motion of the boundary (wavemaker) at $x = 0$. Whether ϕ^{BF} is assumed to decay rapidly or not in the downstream direction is a critical step in the derivation of the amplitude equations that determines the balance between forcing and damping required for the excitation of subharmonic waves. If ϕ^{BF} decays rapidly then cross-waves are forced via a homogeneous boundary condition as in Eqs. (2.36). If ϕ^{BF} extends into the interior of the domain then a distributed

parametric forcing term $g(x)\bar{A}(x,t)$ appears as in Eqs. (2.39). We examine here the differences between these two systems.

Neutral stability curves can be obtained directly from Eqs. (2.36) by looking for steady solutions of the form $A = A_0 \exp\{\kappa x\}$. These curves are defined by Eq. (2.2) and (2.3) with S given by Eq. (2.33). The distributed forcing model (2.39) must be solved numerically. The neutral stability curves predicted by the two systems are shown together in Fig. 2.3 for parameters appropriate to the experi-

FIGURE 2.3: Comparison of onset values for forcing frequencies up to 80 Hz predicted by the model (2.36) with boundary forcing and the model (2.39) with distributed forcing. Remaining parameters are $\nu = 5 \text{ cSt}$, $\Gamma = 19.7 \text{ dyne/cm}$, $\rho = 0.913 \text{ g/cm}^3$, $g = 980 \text{ cm/s}^2$, $D = 5 \text{ cm}$ and $W = 9 \text{ cm}$.

ments of Porter *et al.*[54]. Note that Eqs. (2.36) give a much lower threshold for the excitation of cross-waves than Eqs. (2.39). With a forcing frequency of 50 Hz, for example, Eqs. (2.36) predict that the $N = 27$ mode will onset first at a critical acceleration of $g_{\text{rel}} \simeq 0.0447$ while Eqs. (2.39) indicate that the $N = 26$ mode emerges first at a critical acceleration of $g_{\text{rel}} \simeq 0.348$.

The two systems are expected to agree only when a separation of lengthscale between the decay of $g(x)$ and the modulation of $A(x,t)$ is justified. This requires either a shallow depth d [recall that d characterizes the extent of $g(x)$], or small viscosity or frequency, where the modulation lengthscale [characterized by κ of

Eq. (2.2)] will generally be larger (depending on detuning) than d . When a separation of lengthscales between $g(x)$ and $A(x, t)$ is not justified, the behaviour of the two systems differs markedly. There is a different balance between forcing and damping and, consequently, a different scaling with respect to forcing frequency and viscosity, as shown in Fig. 2.3 and Fig. 2.4, respectively.

FIGURE 2.4: Scaling of minimum onset value versus viscosity (in cSt) for mode $N = 26$ (which is resonantly forced near 50 Hz) using Eqs. (2.36) and Eqs. (2.39). Remaining parameters are as in Fig. 2.3. The dashed curves show the scalings $g_{\text{rel}} \sim \nu$ (distributed forcing case) and $g_{\text{rel}} \sim \nu^{1/2}$ (boundary forcing case).

2.3.2 Secondary cross-wave modes

The consequences of assuming a separation of scales that places the entire effect of the forcing at the boundary are qualitative as well as quantitative. In the boundary forced system (2.36) the neutral stability curve, with fixed parameters and a given mode number N , is uniquely defined by Eq. (2.2). This is not the case in system (2.39), which allows for multiple solutions at the same parameters and mode number. An example of this multiplicity is shown in Fig. 2.5, which shows the first three neutral stability curves for the $N = 26$ mode ($\omega_0/\pi \approx 50$ Hz) and the corresponding eigenfunction profiles.

FIGURE 2.5: Multiple solutions exist in Eqs. (2.39) with the same crosswise mode number. Right: first three neutral stability curves for $N = 26$. Left: $\text{real}(A(x))$ at the three marked points corresponding to $v = -0.05$ and $g_{\text{rel}} = 0.3844$ (first mode, solid curve), 0.7726 (second mode, dashed curve), and 1.093 (third mode, dash-dotted curve). An asterisk marks the turning point for each mode (discussed in the text), while the dotted curve shows the profile of the forcing function $g(x)$ defined by Eq. (2.40). Remaining parameters are as in Fig. 2.3.

To understand the multiplicity illustrated in Fig. 2.5 it is useful split the domain into a supercritical “standing wave” region, where waves are locally supported by the forcing, and a subcritical region, where they are not. We define the turning point x_{tp} as the place where $\epsilon g(x_{\text{tp}})$ is just enough, supposing it were uniform in space, to excite subharmonic (Faraday) waves. Ignoring spatial modulations this condition becomes $\epsilon g(x_{\text{tp}}) = |v - i\gamma_0|$. Turning points defined in this way are marked with an asterisk in the right panel of Fig. 2.5, which shows the utility of this point in separating the oscillatory part of the solution from the exponentially decaying part. This separation is clearest if the solution is relatively localized, as it is with negative detuning. Solutions with positive detuning tend to oscillate more rapidly in x and to extend over a larger region. Whatever the detuning, there is a family of secondary modes that may be described as having successively more oscillations in the supercritical region.

The existence of secondary cross-wave modes with the same mode number N depends on the finite width of the forced (supercritical) region, which scales as d . Loosely speaking, the different behaviour of $A(x, t)$ in the supercritical region, where energy input maintains something close to a standing wave, and the underforced subcritical region, where $A(x, t)$ resembles a damped traveling wave,

leads to a type of matching condition to be satisfied at the boundary. This can be seen most clearly in a simplified model such as the one proposed by Porter *et al.* [53].

Modes with more oscillations in the supercritical region require greater forcing than the primary mode and are thus expected only to affect dynamics above onset. The more extended the forcing $g(x)$ is, the sooner these secondary modes appear with increasing acceleration and the more likely they are to play a role in selection of the initial cross-wave pattern. This type of complexity cannot be captured by the boundary condition forcing of Eqs. (2.36) that ignores the finite width of $g(x)$.

2.3.3 Dirichlet boundary conditions

Unlike boundary forced NLS models like Eqs. (2.36), the distributed forcing mechanism in Eqs. (2.39) permits a variety of alternative boundary conditions to be sensibly applied, including simple Dirichlet or Neumann conditions. In particular, the application of Dirichlet boundary conditions, $A(x, t) = 0$ at $x = 0$, should be relevant to experiments with fixed contact lines. Thresholds obtained with Dirichlet boundary conditions are not very different, in fact, from those obtained with Eq. (2.39b). As seen in Fig. 2.6, an important qualitative difference is the value of detuning where the minimum occurs. In contrast to Eq. (2.39b), Dirichlet boundary conditions typically lead to a positive detuning value at onset and, thus, an eigenfunction that is modulated more rapidly in x than those found in section 2.3.1. This more rapid modulation is in better agreement with the experimentally observed patterns and, as will be seen in section 2.5, additional evidence suggests that Dirichlet boundary conditions are often closer to the experimental ones.

2.4 Finite-size effects and modulated solutions

In a laboratory experiment the fluid must, of course, be confined to a finite domain. When the length L of the horizontally vibrated container considered is

FIGURE 2.6: Comparison of neutral stability curves for $N = 26$ with Eqs. (2.39) (solid curve) and with simple Dirichlet boundary conditions instead of Eq. (2.39b) (dashed curve). Remaining parameters are as in Fig. 2.3.

large compared to the extent of the forcing, which is on the order of the depth d , and to the extent of the cross-wave field $A(x, t)$, which depends also on damping and detuning, then the excitation of subharmonic cross-waves can be treated separately at each side of the container and the analysis of section 2.2 applies. If this is not the case, because d cannot be considered small compared to L , or because the cross-waves generated at opposing ends extend far enough into the interior to interact with each other, then the full domain must be considered from the beginning.

The solution for the bulk flow when L is not large compared to d is given by

$$\phi^{BF} = \frac{2i}{d} \sum_{n=0}^{\infty} \frac{(-1)^n \sinh(q_n x)}{q_n^2 \cosh(q_n L/2)} \cos[q_n(z + d)], \quad (2.45)$$

which is antisymmetric under reflection $x \rightarrow -x$. No-slip boundary conditions are applied at $x = \pm L/2$ and Eqs. (2.39) are replaced by

$$iA_t = (v - i\gamma_0)A + \frac{S}{4}A_{xx} + \epsilon g(x)\bar{A} + \epsilon^2 \beta |A|^2 A, \quad (2.46a)$$

$$A_x \pm \gamma_1 A = 0 \quad \text{at} \quad x = \pm L/2, \quad (2.46b)$$

where $g(x)$ is again given by Eq. (2.40).

In contrast to the case of a semi-infinite domain, where steady solutions $A(x)$ can be sought (corresponding to $2\pi/\sigma$ -periodic subharmonic waves in the original problem), the interaction of opposing wavetrains in the two-sided problem leads, in general, to modulated (quasiperiodic) solutions [50, 53]. We search for Hopf bifurcations with the ansatz [70],

$$A(x, t) = A^+(x)e^{i\omega t} + A^-(x)e^{-i\omega t}, \quad (2.47)$$

which leads to the following system

$$\frac{S}{4}A_{xx}^\pm + [\pm\omega - i\gamma_0]A^\pm + \epsilon g(x)\bar{A}^\mp = 0, \quad (2.48a)$$

$$A_x^\pm - \gamma_1 A^\pm = 0 \quad \text{at } x = -L/2, \quad (2.48b)$$

$$A_x^\pm + \gamma_1 A^\pm = 0 \quad \text{at } x = L/2, \quad (2.48c)$$

which is equivariant under the symmetry $A^\pm \rightarrow A^\mp, \omega \rightarrow -\omega$. Figure 2.7 shows the neutral stability curve and associated Hopf frequency for the mode $N = 16$ in a square container of length $L = 9$ cm. Note that the effect of the interaction is larger for positive detuning. Here, when (half) the forcing frequency σ is greater than the resonant frequency ω_0 , the pattern compensates with greater modulation in x and penetrates further into the interior, which increases the effect of interaction and, thus, the Hopf frequency. For negative detuning the tendency is for the patterns to remain very localized near the endwalls with fewer oscillations in x . The interaction between the two opposing sides is therefore weaker and the Hopf frequency is smaller.

A modulated solution corresponding to Fig. 2.7 with $v = 0.1$ is illustrated in Fig. 2.8. Note that this quasiperiodic solution switches, on a slow timescale, between one-sided excitation concentrated on the left and one-sided excitation concentrated on the right. Periods of balanced excitation exist between these.

FIGURE 2.7: (a) Neutral stability curve and (b) Hopf frequency ω determined by Eqs. (2.48) for $L = 9$ cm and $N = 16$ as a function of detuning v . For comparison, the dotted curve in (a) shows the neutral stability stability curve for Eqs. (2.39) corresponding to $L = \infty$. Due to the symmetry of Eqs. (2.48) there is an associated solution branch [not shown in (b)] with $\omega \rightarrow -\omega$. Remaining parameters are as in Fig. 2.3.

FIGURE 2.8: Modulated solution of Eqs. (2.48) corresponding to the neutral stability curve in Fig. 2.7 for $L = 9$ cm with $v = 0.1$. Solid lines denote $\text{real}(A(x))$, with A defined by Eq. (2.47), while dashed lines denote $\text{imag}(A(x))$. The modulation period is about 24.3 seconds.

2.5 Comparison with experiments in horizontally vibrated containers

In this section we present experimental measurements of the threshold for subharmonic instability, and of the spatial and (when present) temporal modulations of the corresponding patterns near onset. These are compared with the predictions of section 2.3.1 and section 2.4.

2.5.1 Experimental set-up

The experimental apparatus used to obtain the results reported here is sketched in Fig. 2.9. In this diagram, C denotes the experimental container, which

FIGURE 2.9: Diagram of the experimental set-up. The container of fluid (C) is driven horizontally by the electromagnetic shaker (S1). The vertically oriented shaker (S2) is not used here.

is an open transparent square acrylic box designed to hold a 5 cm deep layer of fluid with an interior cross-section of 9 cm, 15 cm, or 20 cm. A sharp edge (step) at the top of the container is used to pin the contact line when desired, and is coated with an anti-wetting agent (3M Novec EGC-1700). One minor difference between the 9 cm container and the other two is that in this smaller container the step extends around all four sides, while the others have a step only on the endwalls (lateral walls are flat). The container is attached to a horizontal platform that is driven by an electromagnetic shaker S1 (VTS-100). The platform and container slide together in rigid fashion along a dual linear sliding bearing system B2. The

diagram also shows a second electromagnetic shaker S2 (VTS-150) which is used to vibrate a second, heavier platform beneath B2. When activated, this vertically oriented shaker S2 can be used in combination with S1 to study combined vertical and horizontal excitation. The bearing joint B1 is designed to reduce, as much as possible, the coupling between these two modes of excitation. Here we are interested only in cross-waves so only the horizontal shaker S1 is used. The working fluid is DC200 silicone oil with kinematic viscosity of 5 cSt or 10 cSt. At 25°C these two fluids have densities of 0.913 g/cm³ and 0.933 g/cm³, respectively, and characteristic surface tension values of 19.7 dyne/cm and 20.1 dyne/cm. The acceleration experienced by the fluid is measured with an accelerometer (Freescale MMA7361L) mounted on the wall of the container.

Measurements of the surface wave pattern are obtained using a free surface synthetic Schlieren technique described by Moisy *et al.* [47], based on analyzing the refracted image of a random pattern of dots placed at the bottom of the container. A high speed camera (Redlake MotionPro X3) takes 15 snapshots per forcing period (30 images for each period of the subharmonic response); the spatial resolution is 0.129 mm for the 9 cm container and 0.198 mm for the 20 cm container.

The experiments reported here are similar to those described in Porter *et al.* [54] and similar experimental challenges affect the accuracy of the results. Most notable is the difficulty in maintaining uniform boundary conditions over long periods, or from one experiment to the next. The fluid motion is most vigorous at the endwalls and, especially in the case of subcritical bifurcation, there is a rapid growth of the cross-wave excitation that inevitably leads to some degree of splashing or bulging of liquid over the coated step of the container. This unwanted behaviour can be minimized by staying very close to threshold, but it cannot be eliminated entirely. The contact line is therefore not completely fixed, nor is the contact angle always exactly 90°. Since boundary conditions at the contact line are crucial and can contribute significantly to damping [24, 44, 52], this is likely the principal source of experimental error in the threshold values reported below. After repeated measurements, we estimate an uncertainty of 5-10% in the

measured values.

2.5.2 Threshold of subharmonic waves

Figure 2.10(a) shows the measured threshold for cross-waves in a 9 cm square container using 5 cSt silicone oil and forcing frequencies between 50 Hz and

FIGURE 2.10: Instability thresholds in a 9 cm square container with $D = 5$ cm for forcing frequencies σ/π between 50 Hz and 75 Hz. (a) 5 cSt oil with $\Gamma = 19.7$ dyne/cm, $\rho = 0.913$ g/cm³. (b) 10 cSt oil with $\Gamma = 20.1$ dyne/cm, $\rho = 0.933$ g/cm³. Experimentally measured thresholds (shown with estimated error bars) are compared with the predictions of Eqs. (2.46) (open circles) and with Dirichlet boundary conditions instead (open squares).

75 Hz. This is compared with the predictions of Eqs. (2.46) and with calculations using Dirichlet boundary conditions. As seen in the figure, the measured values are about 20% higher than those predicted by Eqs. (2.46) and match quite well with those obtained using Dirichlet boundary conditions. In either case the agreement is dramatically better than it is with the boundary forced model (2.36), which underestimates the cross-wave threshold over this range by an order of magnitude (see Fig. 2.3).

Figure 2.10(b) shows the measured threshold for cross-waves in a 9 cm square container with 10 cSt silicone oil over the same range of forcing frequencies. These values are once more compared with the predictions of Eqs. (2.46) and with calculations using Dirichlet boundary conditions. The measured values are again higher than those predicted by Eqs. (2.46), but by a smaller amount (about 10%)

than in Fig. 2.10(a). The calculations with Dirichlet boundary conditions do no better in this case, overestimating the thresholds to about the same extent.

Figure 2.11 shows the measured threshold for cross-waves in the larger containers, which are 15 cm and 20 cm across, respectively. As with Fig. 2.10, a com-

FIGURE 2.11: Instability thresholds for 5 cSt oil with forcing frequencies σ/π between 50 Hz and 75 Hz in a (a) 15 cm square container and (b) 20 cm square container. Experimentally measured thresholds (shown with estimated error bars) are compared with the predictions of Eqs. (2.46) (open circles) and with Dirichlet boundary conditions instead (open squares). Remaining parameters are as in Fig. 2.3.

parison is made with the predictions of Eqs. (2.46) and with calculations using Dirichlet boundary conditions. In these larger containers, the measured values are somewhere between the two predictions, as with the 10 cSt experiment of Fig. 2.10(b). The difference between the measurements in the 15 cm container and the 20 cm container is very small, as is the difference between the corresponding theoretical predictions.

The main point of Figs. 2.10 and 2.11 is that Eqs. (2.46) with distributed forcing do a much better job of matching the experimentally measured thresholds than the boundary forced model (2.36). As demonstrated in Fig. 2.3, these two models diverge dramatically for forcing frequencies beyond about 10 Hz, with Eqs. (2.36) substantially underestimating the threshold in the gravity-capillary regime explored here. This difference between predicted onset values is huge (roughly a factor of 10 over the frequency range reported here) and directly due to the type of forcing balance (scaling) assumed.

An interesting secondary issue is boundary conditions. As discussed above, the boundary conditions are especially difficult to control experimentally and, consequently, it is not clear if the boundary conditions (2.46b) are the appropriate ones to apply. Using Dirichlet boundary conditions, instead, often gives an equally good [Figs. 2.10(b) and 2.11] or better [Fig. 2.10(a)] comparison with experiment. Taken together, the experiments do not clearly distinguish between the somewhat higher thresholds obtained using Dirichlet boundary conditions and those found with Eqs. (2.46b).

A more quantitative idea of the influence of the boundary conditions can be obtained by investigating the dependence of the threshold on the filling level. If the container is underfilled, the effect of contact line motion increases damping and raises the threshold. If the container is overfilled, the fluid tends to splash over the step and increase damping in that way. The minimum threshold is realized by avoiding these two competing effects. Ideally, this will occur with a perfectly flat surface (90° contact angle) but, in practice, the minimum threshold is usually observed in a slightly underfilled container, which more effectively avoids splashing.

The results of threshold measurements with varying filling levels are shown in Fig. 2.12. This figure shows the relative onset value (measured with respect to that of a flat surface) as a function of the change in filling level (in units of the capillary length). In the 9 cm square container of Fig. 2.12(a) the threshold first decreases as fluid is removed. As the gap between the (interior) fluid level and the step (contact line) increases toward the capillary length the threshold decreases by about 5%. After this, however, the threshold begins rapidly to rise again and, by the time the gap reaches 1.5 capillary lengths, the threshold is nearly 15% higher than its initial value. The explanation for this increase is most likely the motion of the contact line (which is no longer attached to the step). Further removal of fluid continues to raise the threshold, but at a much slower rate. If, on the other hand, fluid is added to the container, there is no immediate change in the threshold. After the increase in height reaches about 0.5 capillary lengths, however, there is a rapid and dramatic increase, evidently triggered by the onset of splashing. This

FIGURE 2.12: Dependence of threshold on filling level for 5 cSt oil with (a) 9 cm square container, (b) 15 cm square container. The threshold is scaled by the initial value g_{rel}^0 corresponding to a flat surface while the change in fluid depth is scaled by the capillary length $\lambda_c = \sqrt{\Gamma/\rho g} \simeq 1.48$ mm. Arrows indicate the hysteretic behaviour in the overfilled (splash prone) region.

transition is hysteretic. If fluid is again removed after splashing has occurred, the threshold maintains a higher value for several steps before rejoining the original curve near the initial point (flat surface).

In the 15 cm square container of Fig. 2.12(b) the same basic scenario takes place as in the smaller container, but with minor differences likely due to the absence of lateral steps. Compared to the 9 cm square container, the variations in onset value are somewhat smoother and less extreme; the increase in threshold with capillary motion and with splashing reaches about 10% rather than 15%. Note also that the hysteresis region associated with overfilling extends beyond the initial point, suggesting that splashing during the course of an experimental run can lead to an error of 5% or more in onset values even if the fluid height is held constant.

2.5.3 Spatial modulation

Both the eigenfunctions predicted by Eqs. (2.46) and those calculated using Dirichlet boundary conditions instead of Eq. (2.46b) are modulated in x . There is, however, a greater similarity between the patterns observed experimentally and those calculated using Dirichlet boundary conditions, as seen in Fig. 2.13. This is because Dirichlet boundary conditions lead to a slightly faster modulation in x that is more consistent with the rotated (oblique) patterns observed in the experiment [54], and because the interior part of the pattern, whose extent is needed to view the modulations clearly, is not overshadowed as much by high amplitude motion at the boundary.

A more quantitative comparison of the predicted spatial modulation with the experimentally measured profile is provided in Fig. 2.14 for the case of a 20 cm square container with 5 cSt oil driven at 50 Hz. The calculated curves correspond to Eqs. (2.46) (solid curves), Dirichlet boundary conditions in place of Eq. (2.46b) (dashed curves), and the boundary forced system (2.36) (dash-dotted curves). The amplitude and temporal phase of the theoretical solutions are matched to the experimental measurements over the region of the exponential tail. Note that all three calculations predict the modulation rate (wavenumber) reasonably well in this far-field region ($x \gtrsim 1.5$), which reflects the diminishing influence of the forcing. It is in the near-field (supercritical) region by the boundary where the models differ most.

The distributed forcing model (2.46) describes the spatial dependence of the amplitude and phase somewhat better with Dirichlet boundary conditions than with the derived boundary conditions (2.46b), although both do fairly well. The boundary forced model (2.36), in contrast, fails to capture the spatial dependence of the phase, as is evident in the right panel.

The spatial dependence of the phase, $\Phi(x)$, is a key difference between the distributed forcing model (2.46) and the boundary forced model (2.36), whose eigenfunction (in the semi-infinite case, a good approximation with these parameters) is a damped traveling wave with constant wavenumber. The inability of such a

FIGURE 2.13: (a) Experimentally observed pattern with 5 cSt oil and 50 Hz forcing. (b) Corresponding eigenfunction obtained using Eqs. (2.46). (c) Alternative eigenfunction obtained with Dirichlet boundary conditions. Remaining parameters are as in Fig. 2.3.

simple function to capture the spatial dependence of the phase is demonstrated in Fig. 2.15. In the experiment (open circles in the figure) and in Eqs. (2.46), whether

FIGURE 2.14: Profiles (open circles) observed just above onset in an experiment in a 20 cm square container (5 cm deep) with 5 cSt oil and 50 Hz forcing at two different temporal phases. In the left figure the waves are passing through a maximum near the endwall, while in the right figure they are passing through a minimum. These measurements are compared to the predicted profiles of Eqs. (2.46) (solid curve), to those calculated with Dirichlet boundary conditions instead of Eq. (2.46b) (dashed curve), and to those of Eqs. (2.36) with boundary forcing (dash-dotted curve); remaining parameters are as in Fig. 2.3.

FIGURE 2.15: The spatial derivative of the phase of $A(x)$, $\partial\Phi(x)/\partial x$, which can be interpreted as a local wavenumber, for the same experiment as Fig. 2.14. Experimental results (open circles) are compared to the predictions of Eqs. (2.46) (solid curve), of calculations using Dirichlet boundary conditions (dashed curve), and of Eqs. (2.36) with boundary forcing (dash-dotted line).

using boundary conditions (2.46b) (solid curve) or Dirichlet boundary conditions (dashed curve), there are spatial variations in the derivative of $\Phi(x)$, which can be considered as a kind of local wavenumber. These variations (oscillations) cannot be captured by a single Fourier component like that predicted by Eqs. (2.36).

The distributed forcing model (2.46), however, describes the measured spatial dependence of the phase $\Phi(x)$ reasonable well, especially when Dirichlet boundary conditions are used.

2.5.4 Temporal modulation

Modulated solutions like that shown in Fig. 2.8 are predicted by Eqs. (2.46). These quasiperiodic solutions, however, are very difficult to observed experimentally because, at the frequencies used in section 2.5.2, the waves generated at opposite sides are well-separated at onset and do not interact appreciably. This means that the Hopf (modulation) frequency is very small and that the eigenfunctions can only be expected to persist (if they are stable at all) over a very narrow region of forcing values. Extreme precision and extremely long times would be needed to observe the modulation. To avoid these difficulties, we look for modulated solutions at lower frequency, where the interaction between opposing wavetrains is stronger due to the lower dimensionless damping (larger Re).

Figure 2.16 shows four snapshots of a modulated solution with a period of approximately 45.6 s. It is found immediately after onset in a 20 cm container of 5 cSt oil vibrated at 13.95 Hz. Although the pattern extends far into the container and

FIGURE 2.16: Modulated (quasiperiodic) solution found in a 20 cm square container with 5 cSt oil driven at 13.95 Hz. The crosswise mode number is $N = 11$ and the modulation period is about 45.6 s. The snapshots (a)-(d) are separated by approximately $1/8$ of this period, or 5.7 s.

the interaction of the two sides is therefore not small, the characteristic cycling

illustrated in Fig. 2.8, from waves concentrated on one side (a), to balanced excitation (b), to waves concentrated on the other side (c), back to balanced excitation (d), and so on, can be clearly seen.

This modulated solution persists across a range of nearby forcing frequencies and, consistent with Fig. 2.7(b), its period is observed to depend strongly on frequency (detuning), as shown in Fig. 2.17. This figure also shows the predicted

FIGURE 2.17: Modulation (Hopf) frequency of the solution shown in Fig. 2.16 over an interval of forcing frequencies (in Hz) around the resonance frequency $\omega_0/\pi = 13.52$ Hz. The experimentally measured Hopf frequency ω is compared with that predicted by Eqs. (2.46) with the derived boundary conditions (solid curve) and with Dirichlet boundary conditions (dashed curve); remaining parameters for these calculations are as in Fig. 2.3.

Hopf frequency of Eqs. (2.46) with the boundary conditions (2.46b) (solid curve in the figure) and with Dirichlet boundary conditions (dashed curve). Both the order of magnitude of the modulation frequency and its oscillations with detuning are broadly consistent with the theory. There is particularly good agreement with the calculations using Dirichlet boundary conditions if a leftward shift of 0.15 Hz is applied to the experimental data; such a shift of effective detuning, corresponding to a 1% shift of resonance frequency, is within the level of experimental uncertainty. Figures 2.16 and 2.17 confirm one of the most interesting predictions of Eqs. (2.46), and other models [50, 53] that take account of interaction between the two opposing endwalls.

Although the results in this section seem to suggest that the experiments are

closer to Dirichlet boundary conditions than those of Eq. (2.46b), the correct treatment of boundary conditions is a subtle and difficult problem [24, 51], which is beyond the scope of this work. As already discussed, boundary effects are the principal source of experimental error. We have focused here on extending the established NLS models to the regime of gravity-capillary waves and distributed forcing and thus showing how critical the initial scaling assumptions are. Regardless of the accuracy of the boundary conditions, the difference between boundary forcing and distributed forcing is clear and dramatic. If the depth of the endwall or wavemaker is not small compared to the extent of the cross-wave field — and in many experiments it will not be — then an extended forcing mechanism must be considered.

3 Hysteresis and multiplicity microgravity environments

3.1 Surface tension and non-flat surfaces

In microgravity environments fluid behaves quite differently than it does on earth. Surface tension is no longer masked by gravity and there is no preference for flat surfaces (interfaces). A free-floating mass of liquid will, instead, minimize its surface energy by taking a spherical shape, while contact forces may cause a confined liquid to fully cover (wet) the solid boundaries. In two dimensions, the minimum energy surfaces for confined fluids are generally circular arcs consistent with the given contact angle and fluid volume. Analytical expressions for equilibrium solutions and their energies can be found. Although such solutions are well-known, we present selected results here to motivate the discussion of connected and disconnected vibroequilibria solutions that accompanies the results presented in sections 4.2 and 5.2, and to emphasize the role that fluid volume plays in controlling the multiplicity of possible solutions (or lack thereof); vibrational amplitude will be seen to have a similar effect on multiplicity and hysteresis as decreasing fluid volume.

FIGURE 3.1: Static solutions with no gravity or forcing: (A) Sketch of connected solution for a given fluid volume (area) V , contact angle $\pi/2 - \beta_c$ with $\beta_c < 0$. (B) Sketch of connected solution for a given fluid volume (area) V , contact angle $\pi/2 - \beta_c$ with $\beta_c > 0$.

For a given domain and contact angle, there may be one or more hydrostatic solutions. For deep fluid layers in rectangular domains of length L with dimensionless volume (area) $\mathcal{V} = V/L^2 \gg 1$, there will be a unique, singly-connected, symmetric solution Σ^c that wets the bottom of the container and equal portions of the walls on either side. As volume is reduced, however, families of disconnected solutions arise, characterized both by the number of disconnected fluid volumes and by the relative size of each volume. The connected solutions can be found geometrically (see Fig. 3.1) by obtaining the centre position y_0 and radius R of the corresponding circle:

$$R = \frac{L}{2 \sin \beta_c}, \quad (3.1)$$

$$y_0 = \mathcal{V} + \frac{1}{4 \sin \beta_c} \left[\frac{\beta_c}{\sin \beta_c} + \cos \beta_c \right]. \quad (3.2)$$

After nondimensionalizing with L (the width of the domain), the surface $y = \Sigma^c$ is given by

$$\Sigma^c \equiv \mathcal{V} + \frac{\cos \beta_c}{4 \sin \beta_c} + \frac{\beta_c}{4 \sin^2 \beta_c} + \operatorname{sgn}(\beta_c) \left[-\sqrt{\frac{1}{4 \sin^2 \beta_c} - x^2} \right] : -1/2 > x > 1/2 \quad (3.3)$$

FIGURE 3.2: Sketches of static solutions with no gravity or forcing of disconnected solution: (A) $\beta_c > 45^\circ$, (B) $0 > \beta_c < 45^\circ$.

The simplest family of disconnected solutions Σ_r^d consists of two fluid masses located at the bottom corners of the container. We can parametrize these solutions by the fraction r of the total fluid volume located at the bottom left corner. Solutions can again be obtained geometrically by finding the centre of the circular arc oriented along the 45° line and the corresponding radius (see Fig. 3.2). One of these fluid volumes can be parameterized with the equations (here θ is measured from the horizontal):

$$\Sigma_r^d \equiv \begin{cases} x_i = -\epsilon \sin \beta_c R_i + R_i \cos(\theta) \\ y_i = -\epsilon \sin \beta_c R_i + R_i \sin(\theta) \end{cases}. \quad (3.4)$$

where $\epsilon = \text{sign}(\pi/4 - \beta_c)$ and

$$\psi = \left[\frac{\pi}{4} - \beta_c - \sin \beta_c (\cos \beta_c - \sin \beta_c) \right] \quad (3.5)$$

$$R_i = \sqrt{\frac{r_i \mathcal{V}}{\psi}} \quad (3.6)$$

Figure 3.3 shows several of these solutions for $r = 1$ (single fluid mass in the left corner), $r = 0.75$, and $r = 0.5$ (equal fluid masses) with $\beta_c = \pm 20^\circ$ and $\mathcal{V} = 0.3$. The disconnected solution with equal volumes in each corner is the only one that is symmetric about the midplane $x = 0$ like the singly-connected solution. The case of maximal asymmetry, with a single fluid mass in one corner, is considered as the limiting case of this same family of disconnected solutions.

FIGURE 3.3: Static solutions with no gravity or forcing: symmetric connected solution (Σ^c : solid curve), symmetric disconnected (two mass) solution with equal volumes ($\Sigma_{0.5}^d$: dashed curves), disconnected solution with volume ratio 0.75 ($\Sigma_{0.75}^d$: dot-dashed curve), “disconnected” solution with limiting volume ratio 1 (Σ_1^d : dotted curve). Here $\beta_c = \pm 20^\circ$ (upper and lower panels, respectively) and $\mathcal{V} = 0.3$.

3.2 Hysteresis and multiplicity of solutions

The regions of existence for these classes of solutions are determined by the limiting volumes that are consistent with the assumed geometry. The minimum

volume beyond which concave connected solutions exit, for example, is determined by the height of the midpoint: when the volume is reduced beyond this, the surface touches the bottom and two separate volumes of fluid are created. Singly-connected solutions Σ_c exists beyond the minimum volume:

$$\mathcal{V}_{\min}^c = \begin{cases} \frac{1}{4\sin\beta_c} \left(2 - \cos\beta_c - \frac{\beta_c}{\sin\beta_c} \right) & : \beta_c > 0 \\ \frac{1}{4\sin\beta_c} \left(\cos\beta_c - \frac{\beta_c}{\sin\beta_c} \right) & : \beta_c < 0 \end{cases}. \quad (3.7)$$

Conversely, disconnected solutions exist below a maximum volume defined by

FIGURE 3.4: Coalescence of two fluid drops as the volume is increase: (A) for $\beta_c > 0$ union is formed when the drops reach each other over the wetted bottom, (B) for $\beta_c > 0$ maximum volume is determine when drop get contact when radius is high enough.

the point where the two fluid masses first touch, as shown in Fig. 3.4. Disconnected solutions exist below the maximum volume:

$$\mathcal{V}_{\max}^d = \begin{cases} \frac{\psi}{(\cos \beta_c - \sin \beta_c)^2 (\sqrt{r} + \sqrt{1-r})^2} & : \beta_c > 0 \\ \frac{\psi}{(1 - \sin \beta_c)^2 (\sqrt{r} + \sqrt{1-r})^2} & : \beta_c < 0 \end{cases}. \quad (3.8)$$

where $\epsilon = \text{sign}(\pi/4 - \beta_c)$ and

$$\psi = \left[\frac{\pi}{4} - \beta_c - \sin \beta_c (\cos \beta_c - \sin \beta_c) \right] \quad (3.9)$$

The energy of each solution is the sum of surface and contact energy, and these can be calculated in terms of the reduced volume $\mathcal{V}$ and the contact angle $\pi/2 - \beta_c$. For the symmetric connected solution the dimensionless energy, scaled by ΓL , is

$$E^c = \frac{\cos \beta_c}{2} + \frac{\beta_c}{2 \sin \beta_c} - \sin \beta_c (1 + 2\mathcal{V}). \quad (3.10)$$

The dimensionless energy of the disconnected solutions is given by

$$\epsilon E_r^d = 2\sqrt{\psi} \left(\sqrt{r} + \sqrt{1-r} \right) \sqrt{\mathcal{V}}, \quad (3.11)$$

Figure 3.5 shows the energies of the connected solution Σ^c and the disconnected solutions Σ_r^d , given by Eq. (3.10) and Eq. (3.11), respectively, for $\beta_c = 20^\circ$. The shaded wedge-shaped region is where disconnected solutions exist. The left boundary corresponds to $r = 1/2$ and the right boundary to $r = 0, 1$; the maximum energy is independent of r . The hysteresis illustrated here is typical of most contact angles (the region of hysteresis shrinks as $\beta_c \rightarrow 90^\circ$). For small volumes only the disconnected solutions exist. As $\mathcal{V}$ increases, the singly-connected solution appears, but at a higher energy than (at least some of) the disconnected solutions. In many cases, like $\beta_c = 20^\circ$, this connected solution becomes the global energy minimum at a value of $\mathcal{V}$ where many disconnected solutions still exist. In other cases ($\beta_c < -61.6^\circ$) this connected solution only becomes the global minimum as the last disconnected solution with $r = 0, 1$ ceases to exist at $\mathcal{V}_{\max}^d$ given

FIGURE 3.5: Energy of static solutions with no gravity or forcing and $\beta_c = 20^\circ$: symmetric connected solution (isolated solid curve), disconnected (two mass) solutions (shaded region).

by Eq. (3.8).

The regions where the connected and disconnected solutions exist in the $(\beta_c, \mathcal{V})$ plane are shown in Fig. 3.6, which confirms that the scenario described for $\beta_c = 20^\circ$ is typical for $-61.6^\circ < \beta_c$. For all contact angles, however, there is a region (between the solid and dashed curves) where multiple solutions coexist. Although solutions will be modified when forcing (or gravity) is included, a region of multiplicity will generally remain, and transitions among these solutions can be expected to play an important role in the dynamics of the forced problem.

FIGURE 3.6: Overlapping regions of existence in the (β_c, ν) plane with no gravity or forcing. The symmetric connected solution Σ^c exists above the solid curve. Disconnected (two mass) solutions Σ_r^d exist below the dashed curve. The dotted curve marks the point where the energy E^c of the connected solution becomes less than that of the disconnected solutions, E_r^d , for all r .

4 Vibroequilibria in rectangular containers

The problem of interest is that of an incompressible fluid, bounded by solid boundaries W and a free surface S , and subjected to periodic acceleration $\mathbf{g}(t)$. The velocity field $\mathbf{v}$ is described by the incompressible Navier-Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{g}(t), \quad (4.1a)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (4.1b)$$

where ρ is the fluid density, ν is the kinematic viscosity, ∇ is the gradient operator, and Δ the Laplace operator. The acceleration

$$\mathbf{g}(t) = \varepsilon \omega \cos(\omega t) \mathbf{e}_v - g_0 \mathbf{e}_z, \quad (4.2)$$

includes vertical gravity g_0 (if present) and an applied forcing characterized by an amplitude ε (with units of velocity), frequency ω , and orientation along the unit vector $\mathbf{e}_v$.

No-slip boundary conditions apply on the solid walls while the free surface $z = f(x, y)$ satisfies the following stress balance and kinematic conditions (in a local chart):

$$P + \Gamma \kappa / \rho - \nu \hat{\mathbf{n}} \cdot (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \hat{\mathbf{n}} = 0, \quad (4.3a)$$

$$\hat{\mathbf{n}} \times (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \cdot \hat{\mathbf{n}} = 0, \quad (4.3b)$$

$$\frac{\partial f}{\partial t} + \bar{v} \cdot \bar{\nabla} f - v_z = 0. \quad (4.3c)$$

Here $P = p/\rho$ is the kinematic pressure [determined by Eq. (4.1a)], Γ is the surface tension, $\hat{\mathbf{n}}$ is a unit vector normal to the surface, $\bar{\nabla}$ denotes the horizontal two-dimensional gradient operator, v_z and $\bar{v}$ represent the vertical and horizontal components of velocity, and the mean curvature is given by

$$\kappa = \frac{1}{2} \bar{\nabla} \cdot \left(\frac{\bar{\nabla} f}{\sqrt{1 + |\bar{\nabla} f|^2}} \right). \quad (4.4)$$

4.1 Separation of timescales and the energy functional approach

The mathematical theory of vibroequilibria [10, 11, 20, 41] makes use of the Lagrangian (Hamiltonian) variational approach to fluid mechanics.

Consider, for example, the case of an inviscid fluid with gravity $\mathbf{g} = g_0 \mathbf{e}_z$. If the velocity field is potential so that $\mathbf{v} = \nabla \phi$, its compliance with the Laplace equation

$$\Delta \phi = 0, \quad (4.5a)$$

as well as the evolution of the surface,

$$\frac{\partial f}{\partial t} + \bar{\nabla} \phi \cdot \bar{\nabla} f - \frac{\partial \phi}{\partial z} = 0, \quad (4.5b)$$

$$\frac{\partial \phi}{\partial t} + g_0 f + \frac{1}{2} |\nabla \phi|^2 - 2\kappa \frac{\Gamma}{\rho} = 0, \quad (4.5c)$$

follows [73] from eliminating first order variations in ϕ and Ω of the functional

$$\begin{aligned} \mathcal{L} = & \int_{\Omega} \left(\frac{\partial \phi}{\partial t} + g_0 z + \frac{1}{2} |\nabla \phi|^2 \right) d\Omega \\ & + \frac{\Gamma}{\rho} \int_S dS - \frac{\Gamma}{\rho} \sin \beta_c \int_W dW. \end{aligned} \quad (4.6)$$

The fluid domain Ω is limited by a free surface S and solid walls W (see Fig. 4.1) and the contact angle is $\pi/2 - \beta_c$.

FIGURE 4.1: Sketch of fluid domain and bounding surfaces.

4.1.1 Vibroequilibria

The theory of vibroequilibria deals with the case of high frequency periodic forcing and, in addition to assuming irrotational inviscid flow, relies on a separation of timescales between the forcing and the slower (averaged) response of the fluid. In particular, the period of the forcing should be much smaller than the dissipative timescale [41] and the period of the primary sloshing modes [11, 20]. Incompressibility further requires that the forcing frequency is below the frequency of the first acoustic resonance.

The velocity potential is assumed, at leading order, to vary harmonically in phase with the applied high frequency forcing:

$$\phi(\mathbf{x}) = \varepsilon \tilde{\phi}(\mathbf{x}) \cos(\omega t) + \dots, \quad (4.7)$$

where $\mathbf{x} = (x, y, z)$, while the free surface is split into an average and an oscillatory part

$$f(x, y, t) = F(x, y) + \tilde{f}(x, y) \cos(\omega t) + \dots, \quad (4.8)$$

The vibroequilibria (average fluid position) characterized by $z = F(x, y)$ is dynamically stable if it is a local minimum of the averaged functional [11, 20]

$$\begin{aligned} \mathcal{L} = & \int_{\langle \Omega \rangle} \left(g_0 z + \frac{\varepsilon^2}{4} |\nabla \tilde{\phi}|^2 + P_0 \right) d\Omega \\ & + \frac{\Gamma}{\rho} \int_{\langle S \rangle} dS - \frac{\Gamma}{\rho} \sin \beta_c \int_{\langle W \rangle} dW, \end{aligned} \quad (4.9)$$

where P_0 is a Lagrange multiplier for enforcing conservation of volume. After integration by parts, Eq. (4.9) is equivalent to

$$\begin{aligned} \mathcal{L} = & \int_{\langle \Omega \rangle} (P_0 + g_0 z) d\Omega + \frac{\varepsilon^2}{4} \int_{\langle S \rangle} \tilde{\phi} \frac{\partial \tilde{\phi}}{\partial n} dS \\ & + \frac{\Gamma}{\rho} \int_{\langle S \rangle} dS - \frac{\Gamma}{\rho} \sin \beta_c \int_{\langle W \rangle} dW, \end{aligned} \quad (4.10)$$

provided $\tilde{\phi}$ satisfies the Laplace equation

$$\Delta \tilde{\phi} = 0 \quad \text{in} \quad \langle \Omega \rangle, \quad (4.11a)$$

with boundary conditions (applied in a comoving reference frame)

$$\frac{\partial \tilde{\phi}}{\partial n} = 0 \quad \text{on} \quad \langle W \rangle, \quad (4.11b)$$

$$\tilde{\phi} = \mathbf{e}_v \cdot \mathbf{x} \quad \text{on} \quad \langle S \rangle. \quad (4.11c)$$

Hereafter we omit the brackets, and it should be understood that S , W , and Ω refer to the averaged boundaries and domain.

To obtain the vibroequilibria configuration $F(x, y)$, we must minimize the functional (4.10) over permissible configurations, with $\tilde{\phi}$ satisfying Eqs. (4.11). Since $\tilde{\phi}$ on the free surface S is known from Eq. (4.11c), it is useful to express $q \equiv \partial \tilde{\phi} / \partial n$ on S in terms of $\tilde{\phi}$ using Green's (third) identity:

$$\begin{aligned} \int_S G(\mathbf{x}, \mathbf{x}') q(\mathbf{x}') dS' = & \int_S \tilde{\phi}(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} dS' \\ & + \int_W \tilde{\phi}(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} dW' - \frac{1}{2} \tilde{\phi}(\mathbf{x}), \end{aligned} \quad (4.12)$$

applied to a point $\mathbf{x}$ on the free surface S . The left-hand-side simplifies due to the fact that $q = 0$ on the solid walls W . Using the fact that $\tilde{\phi} = \mathbf{e}_v \cdot \mathbf{x}$ on the free surface, and taking a Green's function satisfying $\partial G(\mathbf{x}, \mathbf{x}') / \partial n' = 0$ on W we obtain

$$\int_S G(\mathbf{x}, \mathbf{x}') q(\mathbf{x}') dS' = -\frac{1}{2} \mathbf{e}_v \cdot \mathbf{x} + \int_S (\mathbf{e}_v \cdot \mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} dS'.$$

With an appropriate Green's function $G(\mathbf{x}, \mathbf{x}')$, this relation allows q to be determined for a given surface S in terms of known functions evaluated on this surface.

Assuming vertical solid walls and integrating the gravity term over z we obtain

$$\int_{\Omega} g z d\Omega = \int_S \frac{g}{2} [F(x, y)^2 - d^2] dx dy, \quad (4.13)$$

where the bottom of the domain is located at $z = -d$ (i.e., d is the depth of the fluid with a flat horizontal surface). The resulting functional involves only the boundaries of the fluid domain, S and W , which are determined by $F(x, y)$:

$$\begin{aligned} \mathcal{L} = & P_0 V + \frac{\Gamma}{\rho} \int_S dS + \frac{\varepsilon^2}{4} \int_S (\mathbf{e}_v \cdot \mathbf{x}) q(\mathbf{x}) dS \\ & - \frac{\Gamma}{\rho} \sin \beta_c \int_W dW + \frac{g_0}{2} \int_S [F(x, y)^2 - d^2] dx dy, \end{aligned} \quad (4.14)$$

where V is the fluid volume.

After nondimensionalizing the integrals in Eq. (4.14) with a characteristic length L (taken to be the width of the domain in what follows) and using the rescaling

$$\mathbf{x} \rightarrow L\mathbf{x}, \quad P_0 \rightarrow \frac{\Gamma}{\rho L} P_0, \quad \tilde{\phi} \rightarrow L\tilde{\phi}, \quad \mathcal{L} \rightarrow \frac{\Gamma L^2}{\rho} \mathcal{L}, \quad (4.15)$$

we obtain

$$\begin{aligned} \mathcal{L} = & P_0 \mathcal{V} + \int_S dS + \eta \int_S (\mathbf{e}_v \cdot \mathbf{x}) q(\mathbf{x}) dS \\ & - \sin \beta_c \int_W dW + \frac{\text{Bo}}{2} \int_S [F(x, y)^2 - d^2] dx dy, \end{aligned} \quad (4.16)$$

where $\mathcal{V}$ is the dimensionless volume, the parameter

$$\eta = \frac{\rho L}{4\Gamma} \varepsilon^2, \quad (4.17)$$

characterizes the ratio of vibrational energy to surface energy, and the Bond number $\text{Bo} = \rho g_0 L^2 / \Gamma$, based on the width L , characterizes the relative importance of gravitational and surface energy terms.

4.1.2 Vibroequilibria problem: numerical solution

Vibroequilibria in the forced problem are found by minimizing the functional (4.16) over permitted surface configurations described by $F(x, y)$. Generally, one must discretize $F(x, y)$ or approximate it with a truncated expansion using suitable basis functions. Here we explore the dependence of vibroequilibria on important parameters by performing a large number of computations and, for the sake of feasibility, only two-dimensional domains are considered. The free surface S is thus described by a curve $z = F(x)$ (possibly composed of disconnected segments) that can be discretized with N elements of length $l_i = \sqrt{(x_i - x_{i-1})^2 + (z_i - z_{i-1})^2}$

The discretized functional (4.16) becomes

$$\begin{aligned} \mathcal{L} \simeq & P_0 \sum_i^N l_i z_i^m + \sum_i^N l_i - \sin \beta_c (z_N + z_1 + 1) \\ & + \eta \sum_i^N \mathbf{e}_v \cdot (x_i^m, z_i^m) l_i q_i + \frac{\text{Bo}}{2} \left[\sum_i^N (x_i - x_{i-1}) \left(\frac{z_i + z_{i-1}}{2} \right)^2 - d^2 \right]. \end{aligned}$$

A superscript m means the quantity is evaluated at the midpoint of the element, and we approximate $V_i \simeq l_i z_i^m$. With the rescaling (4.15), the length of the fixed bottom of the container is unity.

For the boundary element method we assume, for simplicity, that q_i is constant over each element. Green's identity is evaluated at the centre point of each surface element, as is the potential (a first order Gauss quadrature). The problem may be written as a linear system

$$\sum_j K_{ij} q_j = \sum_j H_{ij} (\mathbf{e}_v \cdot \mathbf{x}_j^m), \quad (4.18)$$

where the lowest order quadrature is taken for all but the diagonal elements:

$$K_{ij} \simeq l_j G(\mathbf{x}_i^m, \mathbf{x}_j^m), \quad (4.19)$$

$$H_{ij} \simeq l_j \frac{\partial G(\mathbf{x}_i^m, \mathbf{x}_j^m)}{\partial n_j}, \quad (4.20)$$

for $i \neq j$. The diagonal terms H_{ii} can be obtained from the condition $\sum_j H_{ij} = 0$ (constant potential implies zero normal flow), while the diagonal terms K_{ii} contain singularities that must be removed for each particular Green's function.

For connected solutions we use the Green's function

$$G(w; w') = \frac{1}{4\pi} \ln |w - w'|^2 + \frac{1}{4\pi} \ln |w - \bar{w}'|^2, \\ \text{with } w = \cos(\pi[x - 1/2 + i(y + d)]), \quad (4.21)$$

which fulfills $\partial G / \partial n' = 0$ on all of the fixed walls. After expanding about the singularity,

$$K_{ii} \simeq \frac{l_i}{2\pi} \left[\ln \left(\frac{l_i \pi}{2} \left| \sqrt{1 - (w_i^m)^2} \right| \left| w_i^m - \bar{w}_i^m \right| \right) - 1 \right].$$

Choosing equally spaced, fixed ordinates x_i yields a functional $\mathcal{L}\{z_i\}$ depending only on the vertical displacements, a suitable target for minimization.

For the disconnected solutions, describing separate fluid masses attached to the corners, the following Green's function is used:

$$G(w; w') = \frac{1}{4\pi} \left(\ln |w - w'|^2 + \ln |w - \bar{w}'|^2 + \ln |w + w'|^2 + \ln |w + \bar{w}'|^2 \right),$$

with $w = x + iy$. (4.22)

Discretizing in this manner at discrete points x_i may lead to numerical instability depending on their number. When increasing the dimensionless volume (particularly, for $\mathcal{V} \gtrsim 1.1$), it was found that the number of points needs to be greatly increased for the solution to correctly converge, and this involves a corresponding increase in the number of variables in the functional minimization. An alternative to this discretization, which avoids this numerical difficulty, is to expand the surface in terms of basis functions $P_n(x)$:

$$F(x) = \sum_i a_n P_n(x), \quad (4.23)$$

which yields a functional $\mathcal{L}\{a_n\}$ depending on a finite set of coefficients a_n . This approach, with Legendre polynomials for $P_n(x)$, was found to be both faster and more robust numerically and, thus, is the method used to obtain most of the results reported below. In order to illustrate this issue in the selection of a numerical approach, a set of data for minimum depth $\delta_{\min}$ versus dimensionless volume $\mathcal{V}$, which will be fully discussed in Chapter 5, is shown in Fig. 4.2. Calculations using both discretization schemes can be compared and their divergence for larger volumes is evident; to have the equispaced discretization method converge, it is necessary to triple the number of control points at $\mathcal{V} = 1.5$, requiring a longer computation time. In contrast, using Legendre polynomials yields good results with as few as eight terms in the expansion.

4.2 Comparison with Navier-Stokes simulations

Here we present results obtained using the vibroequilibria potential theory outlined above, and compare some of these with the results of direct numerical

FIGURE 4.2: Minimum depth $\delta_{\min}$ of the midpoint of the connected solution versus dimensionless volume $\mathcal{V}$ for $\text{Bo} = 0$ and $\beta_c = 45^\circ$ using Legendre polynomials (black dots) and discretization with 60 points (gray dots). Triangles show the effect of increasing the number of points x_i from 60 to 210 for $\mathcal{V} = 1.5$.

simulations of the Navier-Stokes equations. This is an important test because viscosity and damping, neglected in the vibroequilibria theory, are known to be important in determining streaming flow [49] and they should therefore affect the (quasi-stationary) equilibrium reached by the fluid.

The vibroequilibria are calculated, using either the discretized surface points $\mathbf{x}_i$ or the Legendre polynomial expansion of Eq. (4.23), in Matlab [1] with the subroutine *fmincon*. The $P_0 V$ term in Eq. (4.14) is implemented as a constraint to maintain constant volume and is not directly included in the minimization process.

To initialize the vibroequilibria calculations we use the following procedure. Symmetric connected solutions are located by allowing an initially flat (horizontal) surface to relax to the hydrostatic solution of section 3.2, then turning on the vibrational forcing and minimizing the functional (4.16). Disconnected solutions are found using an analogous scheme, but beginning from a linear surface $F(x)$ of slope one prescribing a triangular fluid domain located in (one of) the corners

of the container. The initial minimization process takes only a few seconds on a typical PC, depending on the resolution.

COMSOL Multiphysics [2] is used for the direct numerical simulations with a weak, Arbitrary Lagrangian Eulerian (ALE) formulation of the Navier-Stokes equations. These simulations begin from a flat (horizontal) surface.

Figure 4.3 shows that the surface shape calculated using vibroequilibria theory and that found in the numerical simulation agree very well for moderate forcing

FIGURE 4.3: Comparison of fluid surface obtained from vibroequilibria formulation (solid dots) and Navier-Stokes simulations (continuous curve) without gravity. Parameters are: $V = 0.8 \text{ cm}^2$, $\beta_c = 60^\circ$, $L = 2 \text{ cm}$, $\rho = 1 \text{ g/cm}^2$, $\Gamma = 20 \text{ dyne}$, $\nu = 200 \text{ cSt}$, $\varepsilon = 3.97 \text{ cm/s}$, $\omega/2\pi = 50 \text{ Hz}$, with the corresponding dimensionless parameters $\mathcal{V} = 0.2$, $\eta = 0.394$, and $\text{Bo} = 0$.

amplitude. Here the Navier-Stokes simulations are performed with a fluid volume (area) $V = 0.8 \text{ cm}^2$ in a domain of width $L = 2 \text{ cm}$ vibrated at 50 Hz with amplitude $\varepsilon = 3.97 \text{ cm/s}$. The fluid parameters used were: density $\rho = 1 \text{ g/cm}^2$, (one-dimensional) surface tension $\Gamma = 20 \text{ dyne}$ (characteristic of silicone oils), and viscosity $\nu = 200 \text{ cSt}$. The dimensionless volume and forcing were $\mathcal{V} = 0.2$ and $\eta = 0.394$; gravity was not considered ($\text{Bo} = 0$). The use of the relatively high kinematic viscosity of 200 cSt in the simulation of Fig. 4.3 means that surface wave phenomena, which will be examined later, are minimal here, although small harmonic surface waves are indeed present near the boundaries.

When the forcing amplitude is varied, with the remaining parameters as in Fig. 4.3, the excellent agreement between vibroequilibria theory and numerical

simulations holds over the entire range of forcing values considered, as seen in Fig. 4.4. At a forcing of $\eta \simeq 2.75$, when the dimensionless height of the midpoint

FIGURE 4.4: Comparison of the (averaged) contact point and midpoint of the surface obtained from vibroequilibria formulation (solid dots) and Navier-Stokes simulations (continuous curve) without gravity as a function of the vibrational forcing amplitude η . Remaining parameters are as in Fig. 4.3.

is about 0.02, the numerical simulation fails as the fluid nears the lower boundary. Vibroequilibria theory, however, indicates that the fluid smoothly reaches the bottom of the container at $\eta = 3.30$. The results of the numerical simulations were obtained by linearly increasing the forcing velocity ε at a rate of 0.031 cm/s^2 .

4.3 Vibroequilibria effect in rectangular containers

FIGURE 4.5: Surfaces calculated using vibroequilibria theory with vertical forcing. Parameters are: $\mathcal{V} = 0.4$, (a) $\beta_c = -75^\circ$ and (b) $\beta_c = 75^\circ$, and $Bo = 0$, with forcing values of $\eta = 0$ hydrostatic solution (solid curves), $\eta = 1.01$ (short dashed curves), $\eta = 3$ (dotted curves), $\eta = 5$ (long dashed curves), $\eta = 10.16$ (dash-dotted curves) and $\eta = 50.42$ (medium length dashed curves).

As discussed in Chapter 3, in microgravity fluids generally avoid the flat surfaces familiar in Earth's gravity, while spherical shapes, instead, are favored due to surface tension. The vibroequilibria effect can suggest artificial gravity since, as forcing amplitude increases, the interface evolves toward greater alignment perpendicular to the axis of vibration — in this sense, a restored preference for flat interfaces. Recovering a flat surface, with a desired spatial orientation, could thus be viewed as a potentially useful way to manage and control confined fluids via the vibroequilibria effect. Some representative results are presented below

for vertical and oblique forcing. More detailed results for horizontal forcing are given in Chapter 4.

4.3.1 Vertically vibrated containers

FIGURE 4.6: (A) Contact point (dash-dotted curve) and midpoint (solid curve) of the surface obtained from vibroequilibria theory without gravity as a function of the vibrational forcing amplitude η . Parameters are: $\mathcal{V} = 0.4$, $\beta_c = -75^\circ$. (B) Local curvature of surfaces calculated from vibroequilibria theory with vertical forcing. Parameters are: $\mathcal{V} = 0.4$, $\beta_c = -75^\circ$, and $\text{Bo} = 0$, with forcing values of $\eta = 0$ (hydrostatic solution, solid curves), $\eta = 1.01$ (short dashed curves), $\eta = 3$ (dotted curves), $\eta = 5$ (long dashed curves), $\eta = 10.16$ (dash-dotted curves) and $\eta = 50.42$ (medium length dashed curves).

In the presence of vertical oscillations, the surfaces calculated by minimizing the energy functional (4.16) are as shown in Fig. 4.5 for different forcing values η . This figure shows how vibrations alone can flatten the surface (perpendicular to the axis of vibration) as if gravity were present. This same flattening process is further illustrated in Fig. 4.6 (A), where the fluid depth at the midpoint and the height of the contact points are shown versus the vibrational forcing amplitude η . Both values tend asymptotically to the same dimensionless height of 0.4 (corresponding to $\mathcal{V} = 0.4$), which would be the height of the hydrostatic surface in a strong-gravity environment. Local curvature can be calculated from the expression $y = f(x)$ for the free surface and, as seen in Fig. 4.6 (B), this curvature tends to zero except for a small (meniscus) region near the boundaries.

4.3.2 Obliquely vibrated containers

FIGURE 4.7: Surfaces calculated using vibroequilibria theory with oblique forcing at 35° . Parameters are: $\mathcal{V} = 0.4$, (a) $\beta_c = -45^\circ$ and $\text{Bo} = 0$, with forcing values of $\eta = 0$ (hydrostatic solution, solid curves), $\eta = 1.02$ (short dashed curves), $\eta = 3.03$ (dash-dotted curves), $\eta = 7.23$ (short dashed curves) and (b) $\beta_c = 45^\circ$, with forcing values of $\eta = 0$ (hydrostatic solution, solid curves), $\eta = 1.02$ (long dashed curves), $\eta = 2.50$ (dash-dotted curves), $\eta = 4.7$ (short dashed curves).

In the case of oblique (combined vertical and horizontal) forcing, the vibroequilibria solutions derived from the energy functional (4.16) are as shown in Fig. 4.7 for different forcing values η and a forcing angle of 35° with respect to the horizontal. It can be seen how the oblique forcing also increasingly flattens the surface (as a gravitational force along the axis of vibration would). This tendency is further illustrated in Fig. 4.8 (B), where a central portion of the surface has been approximated by a straight line and the inclination of the corresponding

normal vector shown. The flattening of the surface perpendicular to the excitation direction can be seen in the approach of this inclination to 35° . Figure 4.8 (A) shows the height of both contact points versus forcing amplitude η ; note how the left contact point moves up while the right one goes down.

FIGURE 4.8: (A) Left (dash-dotted curve) and right (solid curve) Contact points of the surface obtained from vibroequilibria formulation without gravity as a function of the vibrational forcing amplitude η . (B) Inclination of the surface normal vector of the centre region obtained from vibroequilibria formulation without gravity as a function of the vibrational forcing amplitude η . Parameters are: $\mathcal{V} = 0.4$, $\beta_c = 45^\circ$.

5 Vibroequilibria phenomena in horizontally vibrated containers

Here we present results in horizontally vibrated containers using the vibroequilibria theory described above. As shown in the previous chapter for vertical and obliquely excited containers, the fluids tends to position themselves with the interface perpendicular to the vibration. For the case of horizontal vibration, this same process can lead the fluid to split into two separate masses. Figure 5.1 shows how horizontal vibrations can cause the surface to deform smoothly until reaching the bottom of the container. This is not always the case. If the container is deep enough, there is a critical vibrational amplitude where the surface suffers an instability before reaching the bottom. At this point the (central part of the) surface plunges toward the bottom of the container and the fluid forcefully splits into two (or more) disconnected volumes. This instability is reflected in the eigenvalues of the Hessian of the functional (4.16), the smallest of which tends to zero as this point is approached. Figure 5.2 shows the behavior of this eigenvalue (calculated numerically) and the critical surface at the point of instability.

Figure 5.3 illustrates the dependence of this instability on $\mathcal{V}$ by recording the fluid depth (height above the bottom of the container) at the midpoint of the symmetric connected solution at the highest forcing value for which a corresponding minimum of Eq. (4.16) can be found, a quantity denoted by $\delta_{\min}$. At this point, the symmetric connected solution either touches the bottom, or the calculation fails due to the instability shown in Fig. 5.2. The dependence of $\delta_{\min}$ on $\mathcal{V}$ reveals two regimes: for $\mathcal{V} \lesssim 0.37$, the surface of the connected vibroequilibria solution smoothly dips toward the bottom and makes contact (i.e., $\delta_{\min} = 0$), while for

FIGURE 5.1: Surfaces calculated using vibroequilibria theory. At all three forcing values both (a) connected and (b) disconnected solutions exist simultaneously. Parameters are: $\mathcal{V} = 0.265$, $\beta_c = 45^\circ$, and $\text{Bo} = 0$, with forcing values of $\eta = 2.19$ (solid curves), $\eta = 8.76$ (dashed curves) and $\eta = 19.71$ (dash-dotted curves).

$\mathcal{V} \gtrsim 0.37$ the solution is disrupted prior to this by an instability. The plot of minimum fluid depth $\delta_{\min}$ at the midpoint of the container at this instability shows the characteristic square root behavior of saddle-node or pitchfork bifurcations [22].

The dependence of this instability on $\mathcal{V}$ varies weakly with contact angle, as demonstrated in Fig. 5.4. Selecting β_c values between 0° and 75° only alters the boundary of the instability regime by about 4%; the critical value of $\mathcal{V}$ above which the instability occurs is about 0.36 for $\beta_c = 0^\circ$ and 0.375 for $\beta_c = 75^\circ$.

5.1 Multiplicity of solutions

When the symmetric connected solution either loses stability (in sufficiently deep containers) or is destroyed by contact with the container bottom (in shallow

FIGURE 5.2: With $\mathcal{V} = 1$ there is an instability at $\eta = 28.3$. (a) Behavior of the most dangerous eigenvalue of the Hessian. (b) Critical surface just prior to instability (solid curve) and the disconnected solution (dashed curve) that also exists at this forcing value. Remaining parameters are: $\beta_c = 45^\circ$ and $\text{Bo} = 0$.

containers) the fluid selects a distinct configuration that is, in general, disconnected. As in the hydrostatic case, there are an infinite number of disconnected solutions available, characterized by the number and volume of the separate fluid masses. Each of these disconnected solutions will exist for forcing values beyond a certain lower bound (possibly zero) where two (or more) fluid masses make contact with each other.

Figure 5.5 shows that, at the point of instability or contact with the container bottom, there are typically an infinite number of asymmetric disconnected solutions available at lower energy (cf. Fig. 3.5). Just as in the hydrostatic case with decreasing volume, the fluid, which is gradually separated and “pushed” toward

FIGURE 5.3: Minimum depth $\delta_{\min}$ of the midpoint of the connected solution versus $\mathcal{V}$. For $\mathcal{V} \lesssim 0.37$ the surface deforms smoothly until reaching the bottom, while for $\mathcal{V} \gtrsim 0.37$ an instability disrupts the connected solution prior to contact with the container bottom. Remaining parameters are: $\beta_c = 45^\circ$ and $\text{Bo} = 0$.

the lateral walls by increasing horizontal vibration, must select one of these disconnected solutions. The variational approach does not provide selection criteria for this process, which is generally dynamic.

If the container is not too deep, as in Fig. 5.5(a), then the surface of the symmetric connected solution smoothly approaches the container bottom (at the midpoint) with increasing forcing. After contact is made (tangentially), there must be an adjustment of the fluid surfaces to conform with the given contact angle, and this can be associated with a sudden separation of the fluid (rapid motion of the contact points along the container bottom). Splashing may occur but, if not, it can be expected that the fluid will separate into two volumes of equal size and shape. These fluid volumes press more closely against the horizontal walls as vibration amplitude is increased. Despite its likely selection in the case of an orderly separation process, this symmetric disconnected solution is of higher energy than the asymmetric solutions that exist simultaneously.

If the container is deep enough, as in Fig. 5.5(b) and (c), then the symmetric connected solution loses stability prior to contact with the container bottom. Again, there will be a rapid (nonlinear) selection process as the fluid moves toward a disconnected solution, and this can be expected to be more violent and disordered than in the case of gradual contact. Splashing and sloshing are more

FIGURE 5.4: Minimum depth $\delta_{\min}$ of the midpoint of the connected solution versus ν for $Bo = 0$ and distinct values of β_c (moving upward through the curves for $\nu \gtrsim 0.45$): $0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$.

likely to occur due to the velocity acquired by the fluid as the surface plunges toward the container bottom and crashes against it. The disordered selection process that follows is likely to amplify existing asymmetries and to increase the likelihood of asymmetric disconnected solutions being found as the final configuration. At least in the case of two disconnected fluid volumes, the asymmetric solutions are of lower energy than the symmetric one.

FIGURE 5.5: Dimensionless energies of the symmetric connected solution (E^c , solid curve) and of two disconnected fluid masses (E_r^d , shaded region) parameterized by the ratio r of their respective fluid volumes. The vibroequilibria calculations are done with $\beta_c = 30^\circ$ and $\text{Bo} = 0$. For (a) $\mathcal{V} = 0.25$ the fluid smoothly touches the bottom at $\eta = 5.79$ while for (b) $\mathcal{V} = 0.5$ and (c) $\mathcal{V} = 0.75$ the symmetric connected solution first suffers an instability at $\eta = 10.24$ and $\eta = 18.7$, respectively.

5.2 Instability of vibroequilibria in deep containers

The results in the previous section show that increasing horizontal vibration, which acts to push fluid toward the vertical walls and out of the middle of the

container, is analogous to decreasing volume in the static case. Eventually the continuous symmetric solution is destroyed and the fluid must separate; the simplest separation creates two (symmetric) fluid masses located at opposite corners of the recipient.

In the hydrostatic case of section 3.2, the destruction of the symmetric connected solution as volume is decreased is always via contact with the bottom of the container. Until this breaking point is reached, the minimum height of the fluid, measured at the midpoint of the container for $\beta_c > 0$, decreases smoothly (linearly, in fact) with decreasing volume.

In the case of a vibrated fluid, the destruction of the symmetric connected solution as forcing amplitude increases may be either smooth or sudden, depending on fluid volume. For $\mathcal{V} \lesssim 0.36$ the fluid surface smoothly approaches the bottom of the container but for larger $\mathcal{V}$ the symmetric connected solution suffers an instability prior to contact. As discussed above, at this point the vibroequilibria surface plunges toward the bottom, which suggests a more violent and unpredictable separation process than in the hydrostatic case.

We analyze the nature of this instability here by first simplifying the variational problem. Rather than minimizing the functional $\mathcal{L}$ of Eq. (4.16) over all possible surface configurations, we select two families of surfaces $z = h(x)$ that are qualitatively similar to the calculated surfaces [see Figs. 5.1(a) and 5.2(b)]. We take, for simplicity, the case of $\beta_c = 0^\circ$ (contact angle 90°). Each of the surfaces, shown in Fig. 5.6, has zero mean (to preserve volume), and is parameterized by a single amplitude α .

5.2.1 Without gravity

With a one-parameter family of surfaces like those shown in Fig. 5.6, the variational problem of Eq. (4.16) with $\text{Bo} = 0$ simplifies to that of finding critical points of the function

$$\mathcal{L}(\alpha; \eta) = P_0 \mathcal{V} + E_{\text{sur}}(\alpha) + \eta E_{\text{vib}}(\alpha), \quad (5.1)$$

FIGURE 5.6: Solution families used to approximate the vibroequilibria: a simple cosine function $z = d + \alpha \cos(2\pi x + \pi)$ (solid curve) and a hyperbolic tangent profile $z = d + \alpha \tanh[3 \cos(2\pi x + \pi)]$ (dashed curve) describing a steeper dip in the middle, as observed with higher forcing values. Here d is the dimensionless depth of the undisturbed surface (equal to $\mathcal{V}$ for rectangular containers).

where α is the amplitude of the surface, E_{sur} denotes the surface energy (including contact energy) and E_{vib} is the (normalized) vibrational energy:

$$E_{\text{sur}} = \int_S dS - \sin \beta_c \int_W dW, \quad (5.2)$$

$$E_{\text{vib}} = \int_S (\mathbf{e}_v \cdot \mathbf{x}) q(\mathbf{x}) dS. \quad (5.3)$$

The selected surface is an extremum of $\mathcal{L}$ with respect to α so

$$E'_{\text{sur}}(\alpha) = -\eta E'_{\text{vib}}(\alpha), \quad (5.4)$$

where the prime denotes a derivative in α . In other words, the variation with α of the surface energy and vibrational energy must cancel for the total energy $E = E_{\text{sur}} + \eta E_{\text{vib}}$ to have a critical point. It is easy to visualize the critical surface amplitude by plotting these different energies.

For deep containers, both families of surfaces (see Fig. 5.6) exhibit two critical points up to a certain forcing value. The critical point at smaller α is an energy minimum (stable), while the critical point at larger α is an energy maximum (unstable). As forcing η is increased, these two critical points approach each other

and merge in a *saddle-node bifurcation* that eliminates the symmetric connected solution. Figure 5.7 illustrates this process for the cosine surface profile, and Fig. 5.8 for the hyperbolic tangent surface profile.

FIGURE 5.7: Surface (E_{sur} , dashed curve), vibrational (ηE_{vib} , dash-dotted curve), and total energies (E , solid curve) for the cosine family of surfaces with $\mathcal{V} = 1.35$ (deep container) as a function of α . (a) Two critical points (shown with solid dots) with $\eta = 44.3$, one stable at $\alpha = 0.642$ and one unstable at $\alpha = 1.305$. (b) Saddle-node bifurcation at $\eta = 76.7$ where the two critical points merge at $\alpha = 1.025$.

The symmetric connected solution reaches its minimum depth δ at the midpoint of the surface and we use this quantity to characterize the critical points in Fig. 5.9. For both the cosine and hyperbolic tangent surface families the qualitative picture is the same. For shallow containers (like $\mathcal{V} = 0.1, 0.25$) the minimum depth δ decreases smoothly to zero as forcing η is increased (indicating contact with the container bottom). For deeper containers there is a turning point where the (local) minimum energy solution at smaller α joins with a (local) maximum energy solution at larger α in a saddle-node bifurcation. Beyond this forcing

FIGURE 5.8: Surface (E_{sur} , dashed curve), vibrational (ηE_{vib} , dash-dotted curve), and total energies (E , solid curve) for the hyperbolic tangent family of surfaces with $\mathcal{V} = 1.35$ (deep container) as a function of α . (a) Two critical points (shown with solid dots) with $\eta = 44.3$, one stable at $\alpha = 0.463$ and one unstable at $\alpha = 1.066$. (b) Saddle-node bifurcation at $\eta = 72.9$ where the two critical points merge at $\alpha = 0.783$.

value, which increases with $\mathcal{V}$, the symmetric connected solution vanishes and the fluid must find a disconnected vibroequilibria solution.

We confirm that this scenario of a stable symmetric connected solution colliding with an unstable (larger amplitude) solution and disappearing in a saddle-node bifurcation is correct for sufficiently large $\mathcal{V}$ by comparing the calculations based on the simple one-parameter family of surfaces with the vibroequilibria calculations that permit any (smooth) surface. Figure 5.10 shows that the approximation based on the cosine family of surfaces is surprisingly good at capturing the dependence of δ on η and $\mathcal{V}$. As one can see in Fig. 5.11, the cosine family of surfaces overestimates the lower bound of the saddle-node bifurcation somewhat, placing it at $\mathcal{V} = 0.455$, while the hyperbolic tangent family underestimates

FIGURE 5.9: Minimum depth δ of the critical surfaces as a function of η for the cosine family of surfaces. The curves shown are for volumes $\mathcal{V} = 0.1, 0.25, \dots, 2.35, 2.5$, increasing outward in steps of 0.15. For $\mathcal{V}$ values with a saddle-node bifurcation (marked with solid dots), the upper solutions (solid curves) are stable while the lower solutions (dashed curves) are unstable.

it, at $\mathcal{V} = 0.305$. Despite such differences, both one-parameter families clearly predict the same qualitative dependence of $\delta_{\min}$ on $\mathcal{V}$ seen in section 2.5. This strongly suggests that such behavior is generic and not dependent on the details of the approximation used here to describe the vibroequilibria surface.

5.2.2 With gravity

With gravity present, the variational problem of Eq. (4.16), restricted to a one-parameter family of surfaces as in Fig. 5.6, simplifies to that of locating extrema of the function

$$\mathcal{L}(\alpha; \eta) = P_0 \mathcal{V} + E_{\text{sur}}(\alpha) + \eta E_{\text{vib}}(\alpha) + E_g(\alpha), \quad (5.5)$$

FIGURE 5.10: Comparison of δ for the critical surfaces of the cosine solution family (solid curves stable, dashed curves unstable) with that obtained from the full vibroequilibria calculation (open circles) on the stable branch. The volumes (equivalently, depths) used are $\mathcal{V} = 0.1, 0.4, 0.7, 1, 1.6, 2.5$ (increasing outward from the origin).

where E_{sur} and E_{vib} are again given by Eq. (5.2) and Eq. (5.3), respectively, and the gravitational energy is

$$E_g = \frac{\text{Bo}}{2} \int_S [F(x, \alpha)^2 - d^2] dx. \quad (5.6)$$

The derivative of $\mathcal{L}$ with respect to α must vanish for a critical surface:

$$0 = E'_{\text{sur}}(\alpha) + \eta E'_{\text{vib}}(\alpha) + E'_g(\alpha), \quad (5.7)$$

meaning that the variations of surface, vibrational, and gravitational energy cancel out.

Figure 5.12 illustrates how these three types of energy vary with the “vibroequilibria amplitude” α . For deep enough containers the same scenario occurs as in section 5.2.1. Initially, with very small forcing η , there is a unique stable connected solution that deviates very little from the hydrostatic surface. As forcing is increased, this solution grows in amplitude, dipping further in the center of

FIGURE 5.11: Comparison of $\delta_{\min}$ (value of δ at the saddle-node bifurcation) obtained with the full vibroequilibria calculation (solid curve), the cosine family (dashed curve), and the hyperbolic tangent family (dash-dotted curve). The lower bound of the saddle-node bifurcation is $\mathcal{V} = 0.36, 0.455, 0.305$ in these three cases, respectively.

the container and moving upward along the vertical walls. At a certain value of η , a second unstable solution arises; its surface just reaches the bottom of the container and it is characterized by a larger amplitude α . With increasing forcing, this second unstable solution diminishes in size, moving upward in the center and downward along the vertical walls. The two solutions, the first a local energy minimum and the second a local energy maximum, move towards each other as η is further increased and finally merge in a saddle-node bifurcation. Beyond this saddle-node point, there are no symmetric connected solutions.

Note that, although gravitational energy typically dominates over surface energy for moderately sized containers ($\text{Bo} = 196$ in the calculations of Fig. 5.12), it also increases with amplitude α and so plays a similar role. The main effect of adding gravity, compared to the case of $\text{Bo} = 0$ in Fig. 5.7, is that much larger forcing values are required to see the equivalent vibroequilibria phenomena when Bo is large.

Figure 5.13 shows the depth δ at the midpoint of each critical surface for the cosine family. As in Fig. 5.9, there is a minimum volume ($\mathcal{V} \sim 0.6$) beyond which

FIGURE 5.12: Surface (E_{sur} , dashed curve), gravitational (E_g , dotted curve), vibrational (ηE_{vib} , dash-dotted curve), and total energies (E , solid curve) for the cosine family of surfaces with $\mathcal{V} = 1.35$ (deep container) as a function of α . (a) Two critical points (shown with solid dots) with $\eta = 1557$, one stable at $\alpha = 0.858$ and one unstable at $\alpha = 1.295$. (b) Saddle-node bifurcation at $\eta = 2150$ where the two critical points merge at $\alpha = 1.106$. The Bond number $Bo = 196$.

a turning point exists that joins the upper stable branch with a lower unstable branch. As noted above, gravity does not change this basic scenario, but does delay it until larger forcing values.

With gravity, the simple cosine and hyperbolic tangent families of surfaces provide a less accurate quantitative comparison with the full vibroequilibria calculations. In particular, they tend to overestimate the forcing value of the saddle-node bifurcation, as seen in Fig. 5.14 for the cosine surface family. Despite this, it is clear that the basic qualitative scenario is still captured and, thus, that a saddle-node bifurcation is implicated both with and without gravity.

FIGURE 5.13: Minimum depth δ of the critical solutions as a function of η for the cosine family of surfaces. The curves shown are for volumes $\mathcal{V} = 0.1, 0.25, \dots, 2.35, 2.5$, increasing outward in steps of 0.15. For $\mathcal{V}$ values with a saddle-node bifurcation, the upper (lower) solution on each curve is stable (unstable). The Bond number $\text{Bo} = 196$.

5.3 Dynamical effects: Surface waves, sloshing modes, instability and hysteresis

The numerical simulations of the Navier-Stokes equations used in Figs. 4.3 and 4.4 of section 2.5 were performed with high viscosity fluid (200 cSt) in order to suppress surface waves and their effects. In this section we look at the case where larger amplitude surface waves are present and investigate the interaction between them and the vibroequilibria (average) surface.

With low-viscosity fluids, surface waves are supported with less forcing amplitude, and penetrate further into the interior of the container. Figure 5.15 shows a comparison of the surface predicted by vibroequilibria theory with that obtained by direct numerical simulation using the same forcing amplitude and viscosities of 5 cSt and 10 cSt. The simulations are done in a container of length 9 cm with a fluid volume of 45 cm² and surface tension appropriate for silicone oil, parameters that correspond to recent surface wave experiments [54, 68]. Although the (vibroequilibria) surface here is not flat, similarities to the surface wave dynamics of the flat case may be anticipated, including temporal modulation of the subharmonic waves [53, 57, 68].

FIGURE 5.14: Comparison of δ for the critical surfaces of the cosine solution family (solid curves) with that obtained from the full vibroequilibria calculation (open circles) on the stable branch. The volumes used are $\mathcal{V} = 0.4, 0.7, 1, 1.6, 2$ (increasing outward from the origin) and $Bo = 196$.

With viscosity $\nu = 10$ cSt the surface waves seen in the simulations are of modest amplitude and mainly localized near the boundaries of the domain. At this amplitude ($\varepsilon = 4.05$ cm/s) they are predominantly subharmonic in time, as seen from the two snapshots separated by one forcing period. The average surface does not differ substantially from that predicted by the vibroequilibria theory, being only slightly more deformed (higher at the contact points and lower in the center).

With viscosity $\nu = 5$ cSt, the surface waves at the same forcing amplitude are larger and penetrate further into the interior of the domain. At the same time, there is a clear difference in the averaged surface, which is noticeably asymmetric. We show below that this can be attributed mainly to the excitation of an odd sloshing mode that seems to be sustained by coupling to the subharmonic surface waves.

Results presented in this section suggest that the quasi-static surfaces predicted by vibroequilibria theory can be destabilized by sloshing modes. However, sloshing itself is neither surprising nor incompatible with vibroequilibria

FIGURE 5.15: Comparison of predicted vibroequilibria surface (solid black curve) with surfaces obtained by direct numerical simulation (grey curves) using viscosities of 5 cSt and 10 cSt. Two surfaces are shown from each simulation, obtained after 67 s and separated in time by one forcing period (0.02 s). Remaining parameters are: $V = 45 \text{ cm}^2$, $\beta_c = 0^\circ$, $L = 9 \text{ cm}$, $\rho = 1 \text{ g/cm}^2$, $\Gamma = 20 \text{ dyne}$, $\varepsilon = 4.05 \text{ cm/s}$, $\omega/2\pi = 50 \text{ Hz}$, with the corresponding dimensionless parameters $\mathcal{V} = 5/9$, $\eta = 1.85$, and $\text{Bo} = 0$.

theory as long as it is transient. Vibroequilibria theory requires that the final state is quasi-steady [41] (i.e., steady after averaging over the fast timescale of the periodic forcing). Depending on how the forcing is initiated and on the initial configuration of the surface, various transient dynamics can be observed including sloshing, as illustrated in Fig. 5.16.

If the forcing is turned on quickly with an initially flat surface, the fluid responds rapidly and tends to “overshoot” the vibroequilibria solution. Relaxation toward the final state can take the form of damped low-frequency oscillations (sloshing). On the other hand, if the forcing is applied gradually enough, the fluid may adiabatically follow an “instantaneous” vibroequilibria state, reducing the excitation of transient modes. The manner in which we initialize the simulations here, from an initially flat surface, benefits from the use of an exponential prefactor $[1 - \exp(-t/T_0)]$ to avoid abrupt motion and numerical instability. This type of initialization naturally excites an even sloshing mode (symmetric about the midplane $x = 0$) if T_0 is smaller than its decay time.

FIGURE 5.16: Filtered height of the two endpoints and the midpoint over time t (in s), after the forcing is turned on with an initially flat surface. Two simulations are shown, both with a prefactor $[1 - \exp(-t/T_0)]$ to smooth the transition to the vibroequilibria. $T_0 = 1$ in the more oscillatory case (solid black curves) and $T_0 = 5$ in the other (solid grey curves). The heights predicted by vibroequilibria theory are shown as dashed lines. Parameters are as in Fig. 5.15 with $\nu = 10$ cSt. Most of the rapid harmonic and subharmonic oscillations have been filtered out.

A key finding of this section is that surface waves, and particularly subharmonic surface waves, can destabilize the underlying vibroequilibria state by driving a type of sloshing motion that is sustained rather than transient. If this happens, the system no longer exhibits a vibroequilibria solution since averaging over the fast timescale of the forcing leaves a modulated time-dependent state. While it may be possible to average again over the longer timescale of the sloshing and recover a quasi-steady surface, this is not the usual meaning of “vibroequilibria” and we will not refer to these slowly oscillating surfaces as such.

The fact that sloshing modes may be easily excited is not surprising in light of Fig. 5.7. The profile of the total energy E (the sum of vibrational and surface energies) is often relatively flat (when considering variations in surface amplitude) near the vibroequilibria state. This is especially true as one approaches the saddle-node instability where the second derivative of E also vanishes. A slow

drift among surfaces having nearly the same energy but different amplitude is possible but, due to viscosity, such amplitude modulations will normally decay over time, as in Fig. 5.16.

The results presented below suggest that high frequency surface waves can drive these low frequency sloshing modes and support them despite damping. Such persistent sloshing can sometimes be observed when the subharmonic surface waves are large enough to penetrate into the interior of the domain. It is known that (for a nearly flat surface) this situation leads to temporal modulation of the subharmonic waves [50, 53, 57, 68] and that this modulation frequency can be very small if the coupling between surface waves emanating from opposite endwalls is weak. We speculate that these modulated subharmonic surface wave states, which introduce a slow frequency to the surface wave dynamics, are the key to effective coupling between the high-frequency surface waves and the low-frequency sloshing modes that can destabilize vibroequilibria states.

Figure 5.17 shows the endpoints and midpoint height for a simulation with 5 cSt fluid at a fixed forcing amplitude of $\varepsilon = 4.05$ cm/s (the same simulation as Fig. 5.15b). Here, most of the rapid harmonic and subharmonic waves are filtered out in order to emphasis the sloshing motion. A spectrogram of the full time series of the right contact point is shown alongside this to illustrate the changing frequency content of, in particular, the (modulated) subharmonic waves. Initially, after the forcing is switched on with $T_0 = 5$, the subharmonic waves are modulated with a frequency between 2 and 3 Hz. At the same time, an odd sloshing mode with frequency of about 0.085 Hz grows in amplitude, reaching a maximum after about 45 s. By this point, the modulation of the subharmonic surface waves, initially quite evident, has almost ceased. The amplitude of the odd sloshing mode decays as the simulation continues, leaving only a small level of the even sloshing mode after very long times (approximately 0.003 peak-to-peak amplitude at the midpoint of the container).

This scenario varies with forcing parameters. Figure 5.18 shows the same basic simulation as Fig. 5.17 but with the forcing turned on more rapidly using $T_0 = 1$. The more rapid approach to the vibroequilibria state excites more of the even

FIGURE 5.17: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint over time t (in s), after forcing of amplitude $\varepsilon = 4.05$ cm/s is turned on using the prefactor $[1 - \exp(-t/5)]$. Dashed lines show the prediction of vibroequilibria calculations. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated until $t \simeq 40$ s; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\nu = 5$ cSt.

sloshing mode, a greater level of excitation that seems to persist for very long times (approximately 0.011 peak-to-peak amplitude at the midpoint after 200 s). The correlation between the modulation of the subharmonic surface waves and the odd sloshing mode is notable in this case as well. The modulation, which is present for the first 25 s or so of the simulation coincides with the growth of the odd sloshing mode. After the modulation decreases to a negligible level the amplitude of the odd sloshing mode decays. In contrast, the even sloshing mode appears to be largely independent of whether the subharmonic surfaces waves are modulated or not.

The asymmetry in the average position of the two endpoints that is visible in the two previous plots is not emphasized here because of the likelihood that it is

FIGURE 5.18: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint over time t (in s), after forcing of amplitude $\varepsilon = 4.05$ cm/s is turned on using the prefactor $[1 - \exp(-t)]$. Dashed lines show the prediction of vibroequilibria calculations. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated until $t \simeq 25$ s; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\nu = 5$ cSt.

at least partly numerical. Asymmetry like that in Figs. 5.17 and 5.18 sometimes develops during the simulations, particularly with low viscosity and $\beta_c = 0^\circ$, but not always in a consistent way. This asymmetry can be affected by resolution and other numerical parameters. Furthermore, we find no evidence from the simplified surface families of section 5.2, when considering various asymmetric perturbations, that a real pitchfork bifurcation (to asymmetric vibroequilibria) exists in this problem.

Figure 5.19 shows the results of a simulation with a lower forcing amplitude of $\varepsilon = 3.7$ cm/s. Again, most of the harmonic and subharmonic content is filtered out of the times series for the endpoints and midpoint to emphasize the sloshing

FIGURE 5.19: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint over time t (in s), after forcing of amplitude $\varepsilon = 3.7$ cm/s is turned on using the prefactor $[1 - \exp(-t/5)]$. Dashed lines show the prediction of vibroequilibria calculations. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated until about $t = 75$ s; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\nu = 5$ cSt.

motion. A spectrogram of the full time series shows the changing frequency content of the (modulated) subharmonic waves. In this case the modulation, which has a frequency of about 2 Hz, persists until $t \simeq 75$ s. An odd sloshing mode with frequency of about 0.095 Hz is excited during this time and continues to grow until reaching an amplitude of approximately 0.67 (peak-to-peak) at the endpoints. This odd sloshing mode persists for a relatively long time after this, even without modulated subharmonic surface waves, although its gradual decay is apparent. The slow decay of the odd sloshing mode in this case contrasts with the more rapid decay seen in Figs. 5.17 and 5.18.

The results of a simulation with slightly modified parameters is shown in Fig. 5.20, this time with $\varepsilon = 4.05$ cm/s and $\beta_c = 15^\circ$. With these parameters

FIGURE 5.20: (a) Filtered height, with $\beta_c = 15^\circ$, of the two endpoints (left in black, right in gray) and the midpoint over time t (in s), after forcing of amplitude $\varepsilon = 4.05$ cm/s is turned on using the prefactor $[1 - \exp(-t/5)]$. Dashed lines show the prediction of vibroequilibria calculations. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated throughout the simulation; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\nu = 5$ cSt.

the modulation of the subharmonic waves, which is irregular but concentrated between 2 and 3 Hz, persists for a couple of minutes. This appears to drive the odd sloshing mode, with a frequency of about 0.091 Hz, to very large amplitude (approximately 2.2 at the endpoints, peak-to-peak) before the simulation fails at $t = 121$ s. In this case the growth of the odd sloshing mode apparently leads to the complete destruction of the vibroequilibria state (i.e., not even a modulated vibroequilibria remains).

Figure 5.21 shows the endpoints and midpoint height for a simulation with 5 cSt fluid and $\beta_c = 15^\circ$ at a fixed forcing amplitude of $\varepsilon = 3.25$ cm/s. Once again, most of the rapid harmonic and subharmonic waves are filtered out in order to emphasis the sloshing motion. A spectrogram of the full time series of

FIGURE 5.21: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint over time t (in s), after forcing of amplitude $\varepsilon = 3.25$ cm/s is turned on using the prefactor $[1 - \exp(-t/5)]$. Dashed lines show the prediction of vibroequilibria calculations. (b) Spectrogram of right endpoint showing the presence of harmonic waves at 50 Hz and modulated subharmonic waves centered at 25 Hz; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\beta_c = 15^\circ$ and $\nu = 5$ cSt.

the right contact point is shown below this to illustrate the frequency content of, in particular, the (modulated) subharmonic waves. After an initial transient as the forcing is switched on with $T_0 = 5$, the subharmonic waves appear at 25 Hz and are modulated with a frequency of about 1.5 Hz. As the even sloshing mode decays, an odd sloshing mode with frequency of about 0.1 Hz grows in amplitude, reaching an apparently stable amplitude of approximately 0.01 (peak-to-peak amplitude at the endpoints).

Figure 5.22(a) shows the results of three more simulations like that of Fig. 5.21 but with increasing amplitudes of $\varepsilon = 3.5$ cm/s, 3.75 cm/s, and 4 cm/s. In each of these simulations the subharmonic waves are modulated and in each simulation,

FIGURE 5.22: Filtered height of the two endpoints (left in black, right in gray) over time t (in s) with forcing turned on at $t = 0$ using the prefactor $[1 - \exp(-t/5)]$. Dashed lines show the predictions of vibroequilibria calculations. Parameters are as in Fig. 5.15 with $\beta_c = 15^\circ$, $\nu = 5 \text{ cSt}$ and (a) $\varepsilon = 3.5 \text{ cm/s}$, (b) $\varepsilon = 3.75 \text{ cm/s}$, (c) $\varepsilon = 4 \text{ cm/s}$. In each simulation the subharmonic waves are modulated.

after an initial transient and the decay of the even sloshing mode, the odd sloshing mode grows. In the final case with $\varepsilon = 4 \text{ cm/s}$ [Fig. 5.22(c)] this growth does not saturate and the numerical simulation eventually fails. Thus, with sufficiently large forcing, we observe again the complete destruction of the vibroequilibria state by the odd sloshing mode.

The simulations presented above (as well as others that were not included) reveal a complex interaction between transient motion, surface waves, and sloshing motion. For low viscosity fluids, where surface waves are more easily excited, the final state is varied and depends on forcing parameters, contact angle, and the

initial (transient) approach to the (modulated) vibroequilibria state. At sufficient amplitude both harmonic and subharmonic waves will be present, but it is the subharmonic waves that seem to influence the underlying vibroequilibria state the most, particularly when these waves are modulated (at typical frequencies of 1-3 Hz). Both odd and even sloshing modes may be present. For some parameters, the surface appears to be rendered completely unstable by the growth of the odd sloshing motion.

The most important relationship suggested by the dynamics found in these numerical simulations is the correlation between the growth of the odd sloshing mode and modulated subharmonic surface waves. If modulated subharmonic waves are present, they drive the growth of the odd sloshing mode; this growth may or may not saturate, depending on parameters. The correlation of these two dynamic modes is further demonstrated in Figs. 5.23, 5.24 and 5.25 which shows the results of a simulation made by linearly increasing ε at a rate of 0.0156 cm/s^2 (except for a short initial period of several seconds).

FIGURE 5.23: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint as a function of amplitude ε , increasing linearly at a rate of 0.0156 cm/s^2 except for the prefactor $[1 - \exp(-t)]$. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz), which are initially modulated, then monochromatic, then modulated more strongly beyond $\varepsilon \simeq 3.2 \text{ cm/s}$; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\beta_c = 15^\circ$ and $\nu = 5 \text{ cSt}$.

Figure 5.23(b) shows that the subharmonic waves appear at $\varepsilon \simeq 2 \text{ cm/s}$, and are modulated initially (until $\varepsilon \simeq 2.18 \text{ cm/s}$). The onset of modulated subharmonic waves coincides with the rapid growth of the odd sloshing mode. As the (subharmonic) modulation disappears, this odd sloshing motion appears to (slowly) decay. At $\varepsilon \simeq 3.28 \text{ cm/s}$ the subharmonic modulation returns more vigorously, and with a broader bandwidth than before. This is accompanied by renewed growth of the odd sloshing mode, which reaches a large enough amplitude to destroy the vibroequilibria state.

Figure 5.24, with $\nu = 5 \text{ cSt}$ and $\beta_c = 0^\circ$, present similar behaviour: the onset of subharmonic waves is at $\varepsilon \simeq 2.1 \text{ cm/s}$, while the modulation of these waves

FIGURE 5.24: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint as a function of amplitude ε , increasing linearly at a rate of 0.0156 cm/s^2 except for the prefactor $[1 - \exp(-t)]$. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated for a range beginning at $\varepsilon \simeq 3.36 \text{ cm/s}$; sloshing frequencies cannot be distinguished at this scale in the transform. Remaining parameters are as in Fig. 5.15 with $\nu = 5 \text{ cSt}$.

appears at $\varepsilon \simeq 3.36 \text{ cm/s}$, which coincides with the first (sustained) excitation of the odd sloshing mode. In Fig. 5.25, with $\nu = 10 \text{ cSt}$ and $\beta_c = 0^\circ$, the appearance of modulated subharmonic surface waves is again accompanied by an excitation of the odd sloshing mode, in this case at a significantly lower amplitude due to the increased damping.

We remark that, although the transitions seen in Figs. 5.23, 5.24 and 5.25 are likely delayed due to the drift in forcing amplitude [5, 42], the observation that initial subharmonic modulations disappear with increasing forcing (before returning again at still higher amplitude) is consistent with the scenario presented

FIGURE 5.25: (a) Filtered height of the two endpoints (left in black, right in gray) and the midpoint as a function of amplitude ε , increasing linearly at a rate of 0.0156 cm/s^2 except for the prefactor $[1 - \exp(-t)]$. (b) Spectrogram of right endpoint showing the presence of harmonic waves (50 Hz) and subharmonic waves (25 Hz) that are modulated beyond $\varepsilon \simeq 5 \text{ cm/s}$; sloshing frequencies cannot be distinguished at this scale in the transform.

Remaining parameters are as in Fig. 5.15 with $\nu = 10 \text{ cSt}$.

in Salgado *et al.* [57] for a horizontally vibrated fluid with no vibroequilibria effect, where the modulated subharmonic surface waves are destroyed in a saddle-node heteroclinic bifurcation and replaced by unmodulated subharmonic surface waves. The current case, complicated both by the vibroequilibria effect and by susceptibility to sloshing modes, is more varied and complex — primary evidence being the reappearance of (irregular) modulations at $\varepsilon \simeq 3.28 \text{ cm/s}$ in Fig. 5.23 — but may still reflect several of the key transitions found for a simple pair of coupled parametrically forced subharmonic modes [57].

6 Preparation of Fly Your Thesis!

experiment: CFVib (Control of Fluids in Microgravity with Vibrations)

6.1 Piezoelectric actuators and resonances

In most vibrated fluid experiments the forcing that supports surface instabilities is delivered by relatively large electromagnetic shakers or mechanical wave-makers ([13, 54, 68]); a motor-driven cam mechanism, as another example, is used in [16] to drive frozen waves. Instead of using such devices, we propose in the CFVib experiment the use of a comparatively low power and lightweight piezoelectric device to introduce the vibrational forcing. Piezoelectric materials are commonly used as sensors to measure forces since mechanical deformation induces a voltage difference. Conversely, applying an oscillatory electric field will produce mechanical deformation, which can be used as a source of small-amplitude high-frequency vibrations. The CFVib experiment makes use of this mechanism to generate a vibrational field in the experimental fluids.

FIGURE 6.1: Piezoelectric ceramics are placed on the two sides of the bend-der plate (aluminium beam) to induce flexural motion.

Vibrations produced by a piezoelectric material excited at its resonance frequency (corresponding to thickness modulations) can be amplified by means of a transducer — a piece of metal used to increase the amplitude of vibrations. As

shown by [56], a Langevin-type piezoelectric transducer can be used to increase the amplitude of the response. However, this kind of system operates at ultrasonic frequencies (tens of kHz). In contrast, the vibrations in the CFVib experiment depend on the vibrational modes of the cantilever beam (known as a bender plate), excited by means of two piezoelectric ceramics driven in counter-phase; see Fig. 6.1. The ceramics are placed on opposite horizontal faces of the beam to induce flexural motion. Both the transducer and the cantilever beam configurations require mechanical coupling for the vibrations to reach the fluid. In the transducer case, the mechanical connection is made by attaching the device directly to one surface of the fluid container. In the bender plate configuration, the vessel is attached at a certain point on the cantilever beam. We selected the bender plate configuration to alleviate the need to operate the piezoelectric at its own (ultrasonic) resonance, among the other considerations listed in Table 6.1. The CFVib bender plate configuration is sketched in Fig. 6.2 for the cuboid sample that will be described in section 6.2.

FIGURE 6.2: Midplane cut showing the cuboid assembly. Piezoelectric ceramics are placed on the two sides of the bender plate to induce vibrations in the container, which is attached with a bolt to the center of the beam.

Directly placed piezoelectric	Bender beam
Expensive amplifier.	Acoustic amplifier: lower cost, easier to tune-up.
Transducer needed.	Bender beam acts as a transducer.
Thicker, more expensive PZTs.	Thinner, less expensive PZTs.
Easy to increase the number of excited container faces in the assembly.	Hard to increase the number of excited container faces.
Numerical issues: more difficult to predict assembly resonances.	Numerical modeling easier: analytical solutions available to compare with.
Higher natural frequencies (over the envelope of the amplifier).	Several resonances in the range of interest, up to 20 kHz

TABLE 6.1: Main advantages and disadvantages of the two excitation schemes.

6.1.1 Frequency response of a bender plate in cantilever configuration

With a cantilever configuration and anti-phase excitation of the piezoelectric ceramics, the longitudinal deformation of the ceramics is transmitted to the beam as shear stress of opposite sign on each side of the beam, which induces a linear shear stress profile across the beam. As predicted by Euler-Bernoulli theory, in the limit of a slender (linear) beam, a linear shear stress distribution, ε_x , leads to a flexural deformation, $zw''(x)$ (no longitudinal motion is generated because the corresponding integral vanishes due to symmetry). The linear constitutive equation for a piezoelectric deformed along its axis in a bender configuration can be written as:

$$\sigma_x = Y_{\text{pzt}}\varepsilon_x - e_{31}E_3, \quad (6.1)$$

$$D_3 = e_{31}\varepsilon_x + \kappa_{33}E_3, \quad (6.2)$$

where Y_{pzt} , e_{31} and κ_{33} are the Young's modulus of the piezoelectric, a constant measuring coupling to the electrical field and the ceramic deformation in the longitudinal direction, and the dielectric constant respectively. Neglecting the effects of rotational inertia, the bender can be treated as an Euler-Bernoulli beam. The contribution of the piezoelectric can thus be seen as flexural momentum of the

beam:

$$m(x) = \int_{\text{pzt}} \sigma_x z dy dz = \frac{2Y_{\text{pzt}} b_{\text{pzt}}}{3} \left[(h/2 + h_{\text{pzt}})^3 - (h/2)^3 \right] w''(x). \quad (6.3)$$

This approach leads to an eigenfunction/eigenvalue problem after assuming that the response of the linear system to a sinusoidal input is also sinusoidal with the same frequency. As shown in references such as [32, 67], this assumption allows one to find the natural frequencies of the beam. These predictions are in good agreement with the experimental data. The resonant frequencies of the flexural modes of the cantilever can be used to most effectively excite the fluid mass. In Fig. 6.3 the resonances over a range from 100 Hz to 14 kHz are shown for the beam used in the experiment.

FIGURE 6.3: Resonances for a 100 mm long cantilever beam excited by piezoelectric ceramics that cover 20% of the beam. The beam thickness is 3 mm and the cross length is 20 mm. The vertical displacement is measured at the midpoint of the beam.

6.2 Container design, experimental configuration

The main experiment configuration includes the following items:

- (i) Three different fluids: tap water, silicon oil (Dow Corning 5 cSt), and vacuum oil (Krytox 1506).
- (ii) Twelve experimental containers: six rectangular cuboids and six cylinders, half excited by vibration and half passive.

- (iii) Twelve piezoelectric devices, two for each of the six excited containers, to be attached on opposite faces of the vibrating (bender) beam.

6.2.1 Fluid cell design, manufacturing and filling

Sample	Fluid volume per cell (ml)	Air volume per cell (ml)	Weight
Cuboid: water (x2 cells)	18	18	≈ 140 gr
Cuboid: Silicone oil (x2 cells)	18	18	≈ 140 gr
Cuboid: Mixture, silicone oil + Krytox (x2 cells)	18 + 18	0	≈ 17 gr
Cylinder: Water (x2 cells)	21.25	21.25	≈ 170 gr
Cylinder: Silicone oil (x2 cells)	21.25	21.25	≈ 170 gr
Cylinder: Mixture, silicone oil + Krytox (x2 cells)	21.25 + 21.25	0	≈ 200 gr

TABLE 6.2: Summary of CFVib experiment cells

Cuboidal sample description

- Manufacturing: Regular cuboids made of methacrylate are used as cell samples and hold the test fluids inside. To ensure no fluid leakage, they are built using two pieces, a body and lid. The lid and the body are sealed against each other with an O-ring and pressed together by two plates held by bolts.
- Dimension and geometry: the regular cuboids made of methacrylate have an interior dimension of 3x3x4 cm and a wall thickness of 0.5 cm. They are approximately half-filled in the case of a single liquid, and full in case of the two fluid mixture. The total volume of fluid is about 70 ml.

Cylindrical sample description

- Manufacturing: Cylinders made of methacrylate are used as cell samples and hold the test fluids inside. To ensure no fluid leakage, they are built using two pieces, a body and lid. The lid and the body are sealed against each other with an O-ring and pressed together by two plates held by bolts.
- Dimension: The cylinders are 6 cm in height and 3 cm in base diameter and the thickness of the wall is 0.5 cm. They are approximately half filled in the case of one liquid, and full in case of the two fluid mixture. The total volume of fluid is about 85 ml.

The six cylindrical containers have the same shape and dimension, as do the six regular cuboid containers, respectively. Three of each type are active, with the shaking device attached, while the other three are passive, providing an unforced reference configuration. The passive cells are attached to a similar bender beam as the exited ones but without piezoelectric material.

6.2.2 Experiment fluids

	ρ (gm/cm ³)	Γ (dyne/cm)	γ (cSt)
Water	1.0	72 *	1
Silicon oil	0.9	19	50
Krytox 1506	1.9	19	250

TABLE 6.3: CFVib experiment fluids properties at 20° C. Note (*): the reported Γ value is for pure water, which can decrease with contamination.

The experiment investigates different liquid-gas and liquid-liquid configurations in relatively small, closed, transparent acrylic containers. The experiment cells include the case of a wetting fluid like silicon oil and a non-wetting fluid like water. The experiment fluids should be representative of common space applications and satisfy the safety requirements of the parabolic flight environment.

As mentioned before, one goal of the CFVib experiment is to evaluate the effect of surface tension and contact angle in the selection of the vibroequilibria solution. In the CFE experiment [27, 72] the importance of the boundary condition in selecting the preferred hydrostatic configuration was shown. The CFVib

experiment takes this into account, using different representative fluids with distinct surface and contact properties. The experiment uses tap water, a relatively low viscosity silicon oil, and a mixture of the same silicon oil and Krytox 1506 vacuum oil, which has a nearly 2:1 density ratio. The important physical properties of each experimental fluid are given in Table 6.3. One container from each set of three is filled with silicon oil, one with water, and one with a two-liquid mixture composed of silicon oil and Krytox vacuum oil. A summary of the number of cells used, and the cell volume and weight when filled, is found in Table 6.2

Pure distilled water is used, a low viscosity fluid with high surface tension that fulfills the parabolic flight safety requirements. In addition, it is a non-wetting fluid which allows us to explore a fixed contact angle boundary condition.

50 cSt Dow Corning silicone oil is selected for the experiment (Sigma Aldrich reference 378356 Aldrich). In addition, 5 cSt Dow Corning silicone oil is selected as a backup (Sigma Aldrich reference 317667 Aldrich). Both are not hazardous substances or mixtures according to Regulation (EC) No. 1272/2008 and are not classified as dangerous according to Directive 67/548/EEC. More information can be found in the material SDS.

Krytox vacuum oil is selected to be used in combination with silicone oil for the two-fluid experiments. The greater density contrast (with respect to oil/water) and smaller interfacial surface tension is expected to strengthen the vibroequilibria phenomenon. This is not regulated as a hazardous material by DOT, IMO, or IATA. More information can be found in the material SDS.

FIGURE 6.4: Prototypes for (A) rectangular cuboidal recipient and (B) cylindrical recipient.

6.2.3 Bubble-free filling procedure for two-fluid cells

The procedure below is executed to avoid air bubbles in the cell containing a two-fluid mixture. The steps are as follows:

- Fill the recipient halfway with the heavier fluid.
- Continue filling with the other fluid, up to the top of the large methacrylate piece.
- Fit the O-ring on the dedicated channel of the methacrylate lid.
- Place the aluminium covers and tighten by hand the four screws/bolts at the corners.
- Using a syringe placed between the O-ring and the cell, complete the filling until no air is observed.
- Finish by tightening the four screws/bolts to prevent leakage.

It is found that no bubbles form in the interior for about two weeks after the filling. Subsequently, a small bubble (or more) appears. It is straightforward to repeat steps 5-6 and remove the bubble(s).

6.2.4 Shaking device

Shaking device (piezoelectric + beam) assemblies: As described in section 6.1, the selected forcing device consists in a pair of piezoelectric ceramic materials and a plate forming a bimorph or bender actuator, which is attached to a selected cell face. The piezoelectric ceramics are thick (see main characteristics in table 6.4) and excite a 3 mm thick aluminium plate with a surface area of $100 \times 20 \text{ mm}^2$. The plate is fixed in cantilever configuration using an aluminium piece that is 6 mm in height with a surface area of $20 \times 10 \text{ mm}^2$. The union between the cantilever plate and the sample cell is via a methacrylate piece. The methacrylate piece is glued using a welding solution for methacrylate (chloroform based solution for PMMA) and fixed to the plate by a bolt, as shown in Fig. 6.4.

name	description	value
e_{31}	Piezoelectric charge coefficients	$170 \cdot 10^{-12} \text{ C/N}$
s_{33}	Elastic compliances	$12 \cdot 10^{-12} \text{ m}^2/\text{N}$
κ_{33}	Coupling factors	0.7

TABLE 6.4: Main piezoelectric properties. Ceramic model Pz27 from Ferroperm Piezoceramics.

6.2.5 Cell assembly in the experiment rack

The cuboidal cells are arranged at two different levels to optimize space on the base plate, while the cylindrical cells are on the same level. The cuboidal cells are arranged at these two levels using an aluminium 6082 structure. The support plate for the upper level cuboid is fixed to the structure by 3xM4 steel bolts. The aluminium structure also supports two mirrors to allow visualization of the upper cuboid face. The beams for the cylindrical cells are directly attached by screws to the base plate. A side-view mirror is held using a 3D printed structure

and also attached to the base plate by screws. Figure 6.5 shows the assembly of both cylindrical and cuboidal recipients.

FIGURE 6.5: Assembly for the experiment rack: (A) rectangular cuboidal recipients and (B) cylindrical recipients.

6.3 Electronic set-up and data acquisition system

The electronic and data acquisition system is composed of the following items.

- (i) Four cameras, twelve mirrors and an LED light strip for the video acquisition set-up.
- (ii) Six accelerometers to measure the vibration of each excited cell.
- (iii) Three wave generators, six acoustic amplifiers and an electronics board (including a DC Power Supply) to drive the piezoelectric devices.
- (iv) One Laptop to operate and control the experiment.

6.3.1 Power unit and electrical board

The power supply and the electrical system, detailed below, are constituted by the following.

- Laptop computer: A commercial high performance MSI laptop and its battery is included to control the experiment and the actuators. The power requirement of the computer is 120 W and it supplies lines for the cameras and signal generators via USB.
- Amplifiers: Considering the system to be excited and the maximum desired amplitude, six independent amplifiers are needed. For the frequency range considered, the maximum voltage of the signal (on the order of 100 V and maximum of 200 V) and characteristics of the piezoelectric devices lead to a top power level of 17 W per amplifier (or amplifier channel), and thus a total maximum power of about 102 W. The selected amplifier model is a PDm200B from PiezoDrive, a high performance amplifier with the following features:
 - (a) Overload protection for current (0.5 A) and temperature (343 K).
 - (b) Input voltage of ± 17 V with a maximum input current of 0.5 A.
 - (c) Maximum output voltage/current: 200 V/300 A.
 - (d) Dimensions: 80 x 39 x 40 mm.
 - (e) Weight: 70 gr.

The amplifier is placed in an open box for cooling purposes. The box is attached to the base plate using TY-rap ties.

- Signal generators: The Analog Discovery 2 is the device selected to drive the piezoelectric ceramics. Since it has two channels for signal generation and two for analog-to-digital conversion, it is used for two shakers and to collect data from two accelerometers. Given the weight of the element, it is attached to the base plate using TY-rap ties.
- Amplifier power supply: One "TENMA 72-10505 Power supply, 3CH, 30V, 3A, Adjustable" is used for providing the required bipolar voltage to the amplifier (± 17 V). In order to drive the amplifiers, a power connection box

is used to get the six feeding lines. Seven protective fuses are used: one after the power supply and one fuse for each of the six amplifier lines.

- Three different types of cabling and electrical joints are used, depending on the current and power transmitted:
 - (a) Amplifiers: power supply and input signal
 - (b) Amplifiers: output signal
 - (c) Accelerometer signals

FIGURE 6.6: Electrical power system: distribution of all powered elements of the CFVib experiment. Main input is provided by the Novespace electrical power block at 230 V (protection fuse of 3 A). Lines show the voltage, maximum current and, in case it's needed, the current limit of the protection fuse.

The piezoelectric material requires an electrical signal to produce vibration. As noted above, the equipment used is a wave generator, which produces a sinusoidal signal at the desired frequency. This signal is amplified to the target level by the amplifier. Both systems are controlled and synchronized in time with a real-time clock board that communicates with the computer. The (maximum)

Power supply	Input/Output voltage	Supplying equipment	Peak power	Protection fuse
PS-01	230 V-AC/12 V-DC	Laptop	140 W	Not needed
PS-02	230-AC/ ± 17 V-DC	Six piezoelectric amplifiers using a power distribution box	102 W	3 A fuse for feeding line (x2)
PS-03	230 V-AC/12 V-DC	LED array	24 W	Not needed

TABLE 6.5: Electrical power system: summary of peak power and supplying elements. PS-0X (with $X = 1, \dots, 3$) correspond to the power supply in Fig. 6.6.

power consumption is estimated to be about 17 W per pair of piezoelectric materials in each bender assembly. An electrical diagram showing all this equipment is presented in Fig. 6.6 while Fig. 6.7 shows in more detail the connections of the amplifier, signal generator (Analog Discovery), bender actuator and power supply. A summary of all power supplies is given in Table 6.5

6.3.2 Data acquisition systems

The data acquisition system is configured around the camera and mirror system, accelerometers and the illumination system. As sketched in Fig. 6.7, the cameras, which are described in more detail below, are powered via USB connection from the Laptop and record two views for each cell by using a mirror. These videos are used to see to the evolution of the surface. In addition, accelerometers are used to measure accelerations (and thus velocities). These accelerometers, which are described as well below, are controlled and powered by the Analog Discovery devices. Further details of each element are given below.

- Video cameras: The selected video camera is the ThorLabs DCC1645C, a CMOS camera which records video with a resolution of 1280×1024 pixels at 25 fps. Also, the MVL6WA camera lens is selected from ThorLabs for use with the camera. A high speed camera is not required since the objective is to observe the mean (averaged) position of the interfaces. We have selected a camera and lens pair that have the same format ($1/2''$) to avoid cropping and vignetting of the images and set the camera field of view (FOV) to the

FIGURE 6.7: Block diagram of the optical and measurement system. Cameras are controlled and powered via USB connection to the Laptop as well as the data image acquisition and can view several recipients and mirrors at the same time. Accelerometers are connected to the Analog Discovery device which provides power and collects acceleration measurements. The LED array is directly connected to the power distribution box.

maximum allowed by the lens. The lens has a depth of field (DOF) of around 50 mm, which allows us to observe the interior volume of the cells.

- **Accelerometers:** Analog device ADXL001-500BEZ accelerometers are used to measure the vibration induced by the bender plates on the cells. The accelerometer is connected to one of the channels of the Analog Discovery 2, receiving the required voltage (5 V) and providing the measurement signal.
- **Laptop computer:** The same Laptop described in section 6.3 controls the collection of all the data produced by the cameras and accelerometers.

	Format	FOV	Length to the object (mm)	Focal Length	Pixels/mm (horizontal)	Pixels/mm (vertical)
Cuboid	1/2"	69.4	200	6"	9.2	7.4
Cylinder	1/2"	69.4	250	6"	7.4	5.9

TABLE 6.6: Summary of camera and lens performance for the given object distance.

6.4 Ground testing

The main objectives of the ground campaign were as follows:

1. Calibrate and test the reliability of the accelerometer measurements against a vibrometer.
2. Locate different frequency ranges of interest, where the most vigorous motion of the fluid is generated and vibroequilibria effects are more likely to be dominant (i.e., locate resonances).
3. Find different phenomena related to (high frequency) vibrations in fluids. These could be surface waves such as cross-waves and Faraday waves, which are typically found in vibrated containers. The instability threshold provides an estimate of the velocities reached at the surface.
4. Analyze the structural coupling between experiment samples to avoid undesired interactions.
5. Test the structural integrity of the recipient when subjected to long excitation times up to three times the flight duration.
6. Determine the location and different settings of an optical acquisition system.
7. Test the software/user interface with a flight experiment procedure and design the nominal and off-nominal operations (in case of an unexpected issue). Test the correct functioning of the different items after integration: laptop, amplifiers, accelerometer. Previous stand-alone tests are completed prior to this.

In order to fulfill each objective, a particular test (or series of tests) is conducted.

6.4.1 Accelerometer calibration

The test is conducted using a vibrometer, measuring the velocity of one point of the beam, with the accelerometer attached to the beam. A frequency sweep

over the range of interest is conducted, measuring the vertical velocity of the beam with the vibrometer and vertical acceleration using the accelerometer.

FIGURE 6.8: Accelerometer calibration: Vertical velocity (mm/s) of the cantilever beam in the frequency (Hz) range of interest as measured by the vibrometer (dashed curve) and the accelerometer (black curve). (A) With no filtering and (B) filtered.

In Fig. 6.8(A) the measured vertical velocity for both measurement instruments is given, showing great similarity at low and medium frequencies, except for the resonance frequency shifts due to the added mass of the accelerometer, which likely affects the measured velocity peak. For high frequencies, a resonance of the accelerometer occurs (as documented in the datasheet of the device) and the signal needs to be processed with a low pass filter.

Two parameters are involved in the filter transfer function definition: the cut-off frequency and the quality factor. A numerical search based on minimizing the difference of the integrals of each curve has been done using an optimization algorithm to determine the values of the filter parameters.

The filtered curve is included in Fig. 6.8(B). Hereafter, the tests are conducted with the accelerometer and filtered accordingly. We note that, due to the filtering, we only expect good measurements up to approximately 20 kHz.

6.4.2 Full system resonances

The resonances of the experiment and actuator assembly are found by performing a frequency sweep over the region of interest: 70 Hz to 12 kHz. The accelerometer is placed at the furthest wall from the support and the vertical displacement of the assembly is measured. There are five modes suitable for vibroequilibria investigation, as shown in Fig. 6.9, where the sweep is performed for the cuboid filled with silicone oil. This test locates the resonances of the system used during the parabolic flight campaign.

As one can see, the most energetic mode (in terms of velocity) is around 80 Hz for the three cuboidal cells. With that mode (and with the second mode at 250 Hz) the excitation can drive surface waves in gravity in low viscous fluids — water and 5 cSt oil.

6.4.3 Relationship of actuator excitation to input voltage

When the accelerometer is calibrated and resonances are found, the transfer function that relates excitation output (velocity) and input (voltage) is of interest. It can be calculated experimentally using the output signal of the accelerometer after sweeping over an interval of voltage for each frequency. Figure 6.10 shows the experiment data points for the first three resonant modes. The transfer functions are the following:

FIGURE 6.9: Vertical velocity measured at the midpoint of the right end-wall of the water cuboid for a frequency sweep (70 Hz - 12 kHz).

$$\text{First mode at 81 Hz: } |v_1| = 0.6271V_{pp}, \quad (6.4)$$

$$\text{Second mode at 217 Hz: } |v_2| = 0.2832V_{pp}, \quad (6.5)$$

$$\text{Third mode at 841 Hz: } |v_3| = 0.2898V_{pp}, \quad (6.6)$$

6.4.4 Image acquisition system set-up

Visualization of the fluid cell was the main object of the experiment measurements. Before purchasing a set of four cameras, one was acquired and tested to verify its properties. Depth of field and field of view at the desired object distance were tested to ensure clear visualization and to verify distortion levels of the images. As shown Fig. 6.11, the camera performance is good enough for both

FIGURE 6.10: Amplitude of vertical velocity measured at the midpoint of the right endwall of the cuboidal cell with water for different excitation voltages (V_{pp}), given by the peak-to-peak value, at the first three natural frequencies: 81 Hz (circles), 217 Hz (squares) and 841 Hz (triangles).

qualitative and quantitative measurement of the interface and no critical distortions are found, even in the worst case (the outside cylinder visualization).

The locations of the mirrors that allow upper and side views of the cuboids and two side views of the cylinders were also tested. To determine the final location of the cells with respect to the camera, several positions were tested before making the final assembly. In Fig. 6.12 the three cylinders observed by one camera are shown, including the location of the mirrors. Illumination is provided by an LED strip which extends around the Zarges box.

6.4.5 Experiment operation: Onboard procedure and software control

The experiment procedure is organized in accord with the sequence of separate microgravity periods presented by the parabolic flight plan. Each microgravity period is used to measure one experimental data point. The natural variables of the experiment include the container geometry (shape, size, aspect ratio), the fluid properties (viscosity, surface tension, density) and the excitation parameters

FIGURE 6.11: Visualization of three experimental cells at the cuboid distance (upper figure) and the cylinder distance (bottom figure).

(frequencies, amplitudes). The first two of these are constant between runs — different geometries (cuboids and cylinders) and different fluid configurations are considered simultaneously during the experiment.

The control variables of the experiment are the excitation parameters: frequency, f , and peak-to-peak voltage, V_{pp} . The frequency selects the vibrational mode of the experimental assembly and, thus, the (qualitative shape of the) vibroequilibria solution reached, while, in the absence of instabilities or transitions, the voltage selects its amplitude. The movement of the interface is recorded with a video camera.

The procedure for each run is as follows:

1. Video acquisition is started several seconds before injection (i.e., several seconds prior to microgravity).
2. A wait of 5 s (maximum) may be implemented after the start of the microgravity phase while the static equilibrium is reached.

FIGURE 6.12: Two side views of a cylinder using the Thorlabs camera in the final experiment assembly

3. Forcing is applied (with fixed f and V_{pp}) and accelerometer measurements are recorded for at least 25 s; a minimum of 15 s of data is obtained during the microgravity period. The behaviour of the fluids in hypergravity may also be observed.
4. The forcing and data acquisition (video and accelerometers) are stopped.

The applied vibration is different for different sets of runs in order to explore distinct modes of excitation occurring at different (resonance) frequencies. These resonances can be seen in Fig. 6.9 for the case of 50 cSt silicone oil in a cuboidal container. Since the resonance frequencies vary a bit depending on the assembly procedure (bolt tightness, e.g., being one variable) the forcing frequencies were determined on site (i.e., after the experiment was fully assembled). More than seven frequencies were explored over the first sixty parabolas. Roughly, these frequencies may be classified as follows:

- (i) 50-100 Hz: near the first resonance.
- (ii) 150-250 Hz: near the second resonance.
- (iii) 500-1000 Hz: near the third resonance.
- (iv) 2-5 kHz: near the fourth resonance.

- (v) ~ 8 kHz: contact line displacement (deformation) observed here, even with normal gravity.
- (vi) 10-12 kHz: strongly acoustic regime.

Except for the first two resonances, where surface waves were seen during preliminary ground experiments and scientific interest is assured, the remaining frequencies were selected after the first flight (30 parabolas) — this was the only flight with a fully predefined experimental procedure. This flight was divided into two sets of ten runs and five sets of two runs. Over the two first sets of ten runs, the excitation amplitude was increased linearly from an initially small value at a fixed frequency. These runs coincide with the first two resonances of the experimental assembly. For the remaining five sets of two runs each, various resonances over the range of interest were tested using intermediate V_{pp} values; depending on the results, they were selected (or not) for further study on subsequent flights.

To implement the experimental procedure, the CFVib team developed a software-based control system to start and stop the vibration and acquisitions systems independently (or simultaneously). This software has a user interface that allows communication with the data acquisition system (video cameras and Analog Discovery 2 devices, based on a C++ library). The software accepts a predefined parameter selection, a manual parameter selection, or a combination of the two. Among other settings, the operator can adjust:

1. The parameters of the video cameras (for each device independently): exposure time and frame rate.
2. The amplitude of the applied vibration (independently for each excited container).

These changes are updated and saved as defaults for the next parabola. In addition, the CFVib application arranges all acquired data in appropriate folders and allows the operator to select for the repetition of possibly aborted parabolas during the flight with no detrimental effects. Furthermore, a general abort of the

program is permitted without turning off the Analog Discovery devices (which is prohibited by the operational requirements of the amplifiers). A flow chart in Fig. 6.13 summarizes the experimental procedure, including use of the CFVib software.

The final scientific data is contained in a compilation of four high quality videos (no compression) per parabola, six time series for the vertical acceleration (one per excited cell) and the accelerometer records of the airplane. This data was processed in a preliminary fashion between flights in order to decide on the selection of frequencies for subsequent flights. This processing includes:

- (i) Analysis of the videos for these experimental runs, evaluating the degree of surface motion (compared with those of the corresponding unforced experiment cell).
- (ii) Analysis of the accelerometer data to estimate the fluid velocities induced.

Additionally, minor changes in the experimental procedure can be implemented by adjusting the vibration start/stop time, for example, or the set of runs (number of V_{pp} amplitudes measured for each frequency). Among other things, such possibilities were allowed for because the location of the resonances can shift due to the capillary driven relocation of the fluid and starting the vibration before injection (microgravity phase), when the interface is still relatively flat due to hypergravity, may help mitigate this effect.

6.5 Safety Analysis

Novespace, which manages the parabolic flight campaigns, imposes a number of requirements for the purpose of ensuring safe operation of the experiment and compatibility with the airplane systems. Guidelines are provided to help with the task of meeting these requirements, the most significant of which are outlined below.

- Safety during the parabolic flight is strictly enforced since the laboratory space is shared by all experiment teams on the plane. All potential hazards

FIGURE 6.13: CFVib experiment operational concept and software schematic flow-chart.

are required to be classified and analyzed, with appropriate safety barriers used to contain the associated risks. The most relevant safety concerns for the CFVib experiment are summarized in Table 6.7.

- The design of the electronics system is required to ensure protection for operating personnel, the equipment itself, and the aircraft interfaces. All electrical equipment and conducting surfaces are required to be grounded, high voltage electrical connections isolated (not exposed), and fuses included to protect other circuit elements in the event of short circuits. Additionally, a main fuse is required on (each of) the principal power lines.
- Liquids, powders and particles are required to be safe from the risk of release into the airplane cabin, which can pose a threat to the experiment operators. This contamination risk is reduced by a double containment (or, two levels of containment) requirement for such materials.
- The payload structure must be able to withstand emergency loads, which imposes conditions on the cabin rail attachments, including the number of attachment points. The material of the payload structure and the attachments between this structure and experiment elements have load requirements as well.

6.6 Review of final configuration

The final configuration of the CFVib experiment follows from integrating the subsystems described above, as well as additional hardware and material related to safety requirements. The heart of the experiment is a set of six cuboidal and six cylindrical methacrylate cells. Half of the experimental cells are vibrated using a PZT piezoelectric material attached via an aluminum bender. The other half of the cells are for reference and are not forced; although there is no piezoelectric actuator, the rest of the mechanical assembly is (nearly) identical to that of the

Hazard Group	Hazardous conditions and risks	Classification	Risk Management
Fire	– Short circuit	Catastrophic	– Placement of fuses in electrical design
Electrical shock, Static discharge	– Contact with 230 V or 12 V – Possible burns or death	Catastrophic	– Fuses placed on high voltage lines – Experiment and metal surfaces grounded – Ground fault interrupter on main line
Structural failure	– Failure of Racks #1, #2 during emergency landing	Catastrophic	– Use of aluminium 6082 and steel bolts – Attachment points withstand specified loads
Toxic materials, Contamination	– Cell leakage – Shatter of cells (PMMA) – Shatter of mirrors – Cabin contamination by liquid	Critical	– Cells sealed with O-rings – Shatterable material and cells within leak-proof Zarges box

TABLE 6.7: Risk analysis for CFVib experiment: four groups of hazards are identified.

forced cells. Each specific fluid configuration used, such as water/air for example, is found in a total of four experimental cells: two forced cells of different shape (cuboidal and cylindrical) and two corresponding unforced cells.

Because each excited fluid cell may have (slightly) different resonance frequencies when fully assembled, each piezoelectric actuator is operated independently. Specially designed software allows for independent frequency control while switching all piezoelectrics on (and off) simultaneously. The experimental containers (both excited and reference) are observed using four cameras (controlled by the software), twelve mirrors and an LED illumination system.

All of the hardware described, as well as the cabling, USB hubs, and the structures supporting the mirrors, was placed inside a Zarges box and attached to a base plate in the airplane. This Zarges box, referred to as Rack #1, contains the experimental cells and related hardware. A picture of this assembly is shown in Fig. 6.15; this is the final onboard configuration, which includes some minor changes to satisfy integration requirements that arose during preparation of the

experiment in the Novespace facilities, such as the inclusion of an acoustic isolator on the Zarges box faces to reduce the noise caused by the high frequency vibrations.

Rack #1 is connected by a series of cables to the electronics and control assembly, placed on Rack #2. Six power cables transmit the forcing signal, three Ethernet cables connect to the accelerometers (two for each cable), USB cables allow communication with the four cameras, and an additional power cable supplies the LED lighting. Each cable is passed through one of four water proof glands and attached to the interior base plate to avoid inadvertent contact with the experimental hardware.

The base of Rack #2 is a large aluminum plate where the different items of equipment are attached; this plate was directly joined to the airplane. Rack #2 includes the Laptop, two electrical boxes containing the six amplifiers, a power distribution box, three Analog Discovery devices, one Power Supply Unit, one USB Hub, the LED lighting remote control, and the electrical power strip. The connections are as sketched in Fig. 6.6. A picture of Rack #2 is shown in Fig. 6.16.

The interfaces between both Racks and the A310 ZeroG were specified by Novespace and fall into two categories:

- (i) The mechanical interface, determined by the railings in the interior of the plane
- (ii) The electrical interface, determined by a power block that converts the electrical power supplied by the airplane to more standard power settings.

The positions of the Racks were selected to permit comfortable operation of the experiment tasks. Two restraining straps, located by Novespace staff, prevented the experiment operators from floating in an uncontrolled fashion during the microgravity period.

FIGURE 6.16: Rack #2: Electronics and control systems.

7 Results of ESA parabolic flight campaign

7.1 Experimental observations

A typical profile of gravity levels during a full parabola maneuver on the A310 ZeroG airplane is shown in Fig. 7.1. As intended, the maneuver causes the apparent gravity level inside the airplane to vary. The pull-up phase, which comes before the zero-g period, can experience up to about 2 g's in the z direction and 0.2 g's in the x direction. The “zero-g” phase shows residual g-jitter in all three directions, with a particularly significant (roughly) linear variation in the x direction — decreasing steadily from positive to negative g's during the microgravity period. The pull-out phase can exhibit gravity levels of up to 1.8 g's.

FIGURE 7.1: (A) Typical gravity levels on the A310 ZeroG during a parabola maneuver: in the z direction (perpendicular to the plane floor), in the x direction (towards the cabin) and in the y direction (across the plane). (B) Typical microgravity levels on the A310 ZeroG during the zero-g portion of the parabola maneuver.

7.1.1 Effects of gravity profile on reference cells

Water is a non-wetting fluid with high surface tension. A series of snapshots of the behaviour of the water in the cuboidal container during the microgravity period is shown in Fig. 7.2. The water inside the cuboidal cell occupies a static equilibrium position near the last configuration prior to entering the microgravity phase. As described above, typical gravity levels (see Fig. 7.1) during the pull-up phase involve about 1.8 g's in the z direction and 0.2 g's in the x direction. This force causes an asymmetry in the equilibrium configuration as seen in Fig. 7.2 (a). Within a few seconds (about 3 s) the equilibrium position is reached (Fig. 7.2 (b) and (c)). This hydrostatic surface is then affected by the pull-out maneuver when gravity levels suddenly increase in the x direction (changing sign so that the fluid is displaced toward the opposite wall). As seen later in this section, the vibrations of this cell during the first and second of the selected modes (80 Hz and 215 Hz) have qualitatively appreciable effects.

FIGURE 7.2: Four snapshots from the reference cuboidal container with water: (a) at the beginning of microgravity, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-out.

Figure 7.3 shows four time snapshots of the cuboidal cell with 50 cSt silicone oil. In this case, we also observe a significant effect from the residual gravity in the x direction. Since silicone is a more wetting fluid with low contact energy, we see that within a few seconds the fluid manages to wet the entire cell — the distribution is asymmetric as with the cuboidal cells (see Fig. 7.7 (b) and (c)). Still, the silicone oil takes longer to reach the equilibrium configuration and, although its viscosity is higher than that of water, the equilibrium is strongly affected by g -jitter. For the volume fraction used (one half), a circular bubble forms, as expected. The effects of residual acceleration or g -jitter in the x direction are noticeable in this cell as well, as the air bubble moves toward one of the walls of the cuboidal cell.

FIGURE 7.3: Four snapshots from the reference experiment with 50 cSt silicone oil in a cuboidal container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity begins, (c) about 9 s after microgravity, (d) just before pull-out.

The use of low viscosity 5 cSt silicone oil was not originally planned because we expected that it would be more affected by the g-jitter and sloshing. After the first two flights, in which the vibroequilibria effects were less than anticipated, the decision was made to try this less viscous fluid. We thus observed its behavior under the different gravity levels experienced during the flight. It was decided to use a volume fraction of less than 50% (about 30%) in the hope that this would allow more vibrational energy to be transmitted to the fluid and encourage the vibroequilibria effect. As discussed later in this section, the excited cell using low viscosity oil did not exhibit a large vibroequilibria effect. In the case presented in Fig. 7.8, which shows the unexcited cuboidal cell, the fluid can be seen to have the same behavior as in the corresponding 50 cSt oil case. Specifically, it appears to be affected by the g-jitter in the same way: it wets the entire wall of the cuboid

within a few seconds, and its stable shape is (close to) a spherical bubble of air surrounded by fluid, even though a lower volume fraction is used.

FIGURE 7.4: Four snapshots from the reference experiment with 5 cSt silicone oil in a cuboidal container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-out.

The evolution of the interface for the fluid mixture in the cuboidal recipient during the microgravity period is shown in Fig. 7.5. The lower density fluid (silicone oil) is surrounded by the denser one after about 3 s of microgravity. This equilibrium is again asymmetric due to residual inertial forces produced during the maneuver; the heavier fluid acts here as the silicone oil does in the air/oil experiments. We observe that within a few seconds the equilibrium state is reached (see Fig. 7.5 (b) and (c)).

FIGURE 7.5: Four snapshots from the reference experiment with mixture of 50 cSt silicone oil Krytox in a cuboidal container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-out.

Figure 7.6 shows four snapshots of the unexcited water in a cylindrical cell. In this case, we observe that g-jitter does not appreciably affect the free surface. With the resolution of the camera, we are able to see little more than some small-amplitude sloshing and minor curvature of the surface.

FIGURE 7.6: Four snapshots from the reference experiment with water in a cylindrical container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-put.

Figure 7.7 shows the evolution of the unexcited cylinder with 50 cSt silicone oil due to g-jitter. In contrast to the 50 cSt cuboidal reference cell, where the fluid completely wets the wall and an air bubble forms in the interior, the fluid in the cylindrical recipient wets the wall with a small contact angle, but is unable to completely climb the cylinder (i.e., it does not wet the entire surface). This fact suggests that the surface conditions may play an important role in selecting the position of the fluid (NASA, for instance, is conducting research with coating films and containers subjected to controlled sloshing [43]). Here, the difference in surface properties of the containers may stem from the manufacture process. Cuboidal cells are hollowed out of a solid block with a milling machine, then polished by hand with a rotating tool, leaving a surface of uncertain smoothness on the edges. The cylinders are manufactured with a lathe and the polished surface is more uniform since there are no corners. Such manufacturing differences may help explain the qualitatively distinct equilibrium surfaces obtained for the same fluid (and the same volume fraction) in the two containers.

FIGURE 7.7: Four snapshots from the reference experiment with 50 cSt silicone oil in a cylindrical container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-out.

Figure 7.8 shows the evolution of the unexcited cylinder with 5 cSt silicone oil due to g-jitter. The behaviour of the lower viscosity silicone oil is similar to that of the 50 cSt oil, both in terms of the time needed to reach equilibrium and the fluid response to residual gravity.

FIGURE 7.8: Four snapshots from the reference experiment with 5 cSt silicone oil in a cylindrical container: (a) at the beginning of the microgravity phase, (b) about 3 s after microgravity starts, (c) about 9 s after microgravity starts, (d) just before pull-out.

7.1.2 Effects in excited experimental containers

As mentioned above, 5 cSt silicone oil was used in the third flight. Its low viscosity means it is susceptible to developing surface wave instabilities (analogous to Faraday waves), which were seen clearly on ground. With the first resonant mode of the cantilever, droplets are ejected from the fluid surface. This phenomena masks the vibroequilibria effect, as shown in Fig. 7.9. In Fig. 7.9 (a) one sees that droplets are ejected from the free surface with forcing of 1.8 g's and, in (b) and (c), the droplets fill the cell. As will be seen in Fig. 7.18 of the next section, amplitudes on the order of 10 cm/s are needed to significantly deform the equilibrium state with 50 cSt silicone oil. Assuming that same forcing amplitude is needed for lower viscosity oil, we may conclude that observing vibroequilibria at low frequency can be masked by the interaction with energetic surface waves and droplets.

FIGURE 7.9: Three snapshots from the experiment with 5 cSt silicone oil vibrated at 81.1 Hz with a forcing amplitude of 13.6 cm/s in a cuboidal container: (a) in the hypergravity phase, (b) about 2 s after the microgravity phase starts, (c) about 8 s after microgravity.

Figure 7.10 shows the vibroequilibria observed with 50 cSt silicone cell in a cylindrical cell excited at the first resonance of the assembly (frequency 59.3 Hz). In this run, the forcing is initiated about two seconds after the microgravity phase begins. The sequence shows how the free surface evolves from hypergravity, in Fig. 7.10(a), to microgravity when the forcing is initiated, in Fig. 7.10(b). After two seconds of excitation, the fluid climbs the furthest cylinder wall. This movement is explained by the fact that the excitation in the first resonant mode is not

purely vertical but involves rotational motion, which results in an obliquely oriented forcing. The free surface is, thus, positioned roughly perpendicular to this forcing, as seen in Fig. 7.10(c) after almost two seconds. Since the rotation is more prominent at the top of the container, making vibration there more vigorous, when the fluid touches the top lid, it continues accumulating there.

FIGURE 7.10: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container: (a) at the beginning of the microgravity phase, (b) just before forcing is applied, (c) about 2 s after forcing of amplitude 8.7 cm/s is initiated, (d) about 4 s after forcing is initiated.

Figure 7.11 shows four snapshots of the same experiment as in Fig. 7.10 (same forcing amplitude and frequency) performed in a different parabola, illustrating fluid accumulation near the top lid. The four snapshots are separated by two seconds in time.

FIGURE 7.11: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container at two second intervals showing fluid accumulation near the top lid. This is the second repetition of the experiment shown in Fig. 7.10

Additionally, we find that the time taken to reach the top lid, observed with 50 cSt silicone cell, depends on the forcing amplitude applied. Figure 7.12 shows the moment when the fluid reaches the upper lid in four experiments performed with different forcing amplitudes. Note that as the forcing amplitude increases, the fluid takes less time to climb to the upper lid. This trend reverses for the highest amplitude, likely due to the microgravity disturbances experienced during this parabola; higher g -levels in the z direction were experienced compared with the other ones.

FIGURE 7.12: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container for different forcing amplitudes selected at the time the fluid reached the top lid: (a) about 9 s with 7 cm/s forcing, (b) about 6 s with 7.5 cm/s forcing, (c) about 4 s with 8 cm/s forcing, (d) about 7 s with 8.5 cm/s forcing.

In addition to the vibroequilibria phenomenon, we observe the development of surface waves with 50 cSt silicone oil in a cylindrical cell. They are seen when the fluid reaches the top lid (where the most vigorous vibration is found). Figure 7.13 shows the waves generated during the experiment with a forcing amplitude of 8 cm/s.

FIGURE 7.13: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container with a forcing amplitude of 8 cm/s: (a) about 7 s after the microgravity phase starts, (b) 300 ms after the previous image, (c) 1 s after the previous image.

7.1.3 Comparison between reference and vibrated cells

As mention above, the first vibrational mode of the cantilever-cell assembly generates an obliquely oriented forcing. In the cuboidal cells, this first resonance mode is excited at a frequency of about 80 Hz. Figure 7.14 shows three snapshots taken at different phases of the parabola, comparing the cuboidal cell with water excited at 11.7 cm/s with the corresponding reference cell. During the hypergravity phase [Fig. 7.14(a)] we observe that surface waves develop at this amplitude and droplets are ejected from the interface. When the microgravity phase starts, fluid is pushed against rear wall (also seen in the experiment with 50 cSt silicone oil in a cylindrical recipient). This effect is shown in Fig. 7.14 (b) and (c) for two different times during the microgravity phase.

FIGURE 7.14: Three snapshots from the experiment with water vibrated at 79.8 Hz in a cuboidal container (left) and the corresponding reference cell (right) with a forcing amplitude of 11.7 cm/s: (a) during the hypergravity phase, (b) after 2 s of microgravity, (c) after 7 s of microgravity.

The movement of fluid toward the rear wall noted in the experiment with

water in a cuboidal cell is also seen in the cylindrical cell when the water is excited in the first vibrational mode of the cantilever-cell assembly at 56.5 Hz (previously described for silicone oil). Figure 7.15 shows both the excited and unforced water experiment. In contrast with the case of silicone oil, which has less surface tension and surface energy, the time to reach the vibroequilibria state is greater — on the order of 12 s.

FIGURE 7.15: Four snapshots from the experiment with water vibrated at 56.5 Hz in a cylindrical container (left) and the corresponding reference cell (right) for forcing amplitude of 7.1 cm/s which starts during hypergravity phase: (a) Excited cell at the beginning of microgravity, (b) after 10 second of microgravity,

The second vibrational mode of the cantilever-cell assembly produces purely vertical displacement of the cell, and occurs at a frequency of about 215 Hz for the cuboidal cells. To illustrate this vertical motion of the cell, we select three frames from both excited and unexcited cuboidal cells, which are presented in Fig. 7.16. In (a) we see small surface waves generated by this mode in the hypergravity phase. The temporal resolution of the camera does not permit a determination but they are likely subharmonic (Faraday) waves. Figure 7.16 (b) and (c)

show two effects, a combination of small-amplitude surface waves and surface flattening due to the vibroequilibria effect. This effect is visible when comparing with the reference cell, where the interface clearly has more curvature, but the experiment design makes a quantitative measurement difficult.

FIGURE 7.16: Three snapshots from the experiment with water vibrated at 215 Hz in a cuboidal container (left) and the corresponding reference cell (right) with a forcing amplitude of 5.5 cm/s: (a) during the hypergravity (2 g) phase, (b) after 12 s of microgravity, (c) just before the parabola pull-out begins.

This effect is seen as well in the experiment with 50 cSt silicone oil vibrating with the second mode at a frequency of 213 Hz. Figure 7.17 shows two snapshots during the microgravity phase for both the reference and excited cells. The

curvature of the free surface is reduced (flattened) by the vibroequilibria effect.

FIGURE 7.17: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 213 Hz in a cylindrical container for forcing amplitude of 5 cm/s which starts during hypergravity phase: (a) Excited cell at the beginning of microgravity, (b) Excited cell after 5 second of microgravity, (c) Reference cell at the beginning of microgravity, (d) Reference cell after 5 second of microgravity

Figure 7.18 shows an image (via a mirror) of the 50 cSt silicone oil experiment in a cuboidal container at two times during the microgravity phase. Note that vibration deforms the air bubble surrounded by silicone oil, which changes from a spherical shape to an ellipsoidal one.

FIGURE 7.18: Top view of the experiment with 50 cSt silicone oil vibrated at 81 Hz in a cuboidal container (left) and in the corresponding reference cell (right) with a forcing amplitude of 11.8 cm/s: (a) after 3 s of microgravity, (c) after 12 s of microgravity.

7.2 Comparison with theoretical expectations

The image processing of the experiment data is, in some ways, made challenging by the low contrast of the images and by the design itself. Images can be processed in a rudimentary way by manually selecting the location of the contact lines. In order to compare the experimental results with vibroequilibria predictions, we measure the contact points on the rear and front walls and in the midplane as shown in Fig. 7.19. We can compare the vibroequilibria contact point heights shown in Fig. 4.8 if we assume that the midplane of the experiment has a reduced volume (area) $\mathcal{V} = 0.4$, the contact angle is around 45° , and the midplane behaves as a two-dimensional fluid. The first assumption is motivated by

the fact that the fluid is concentrated near the corners in the experiment (moving out of the middle). Although these assumptions are not vigorously justified, Fig. 7.20 shows that the experiment data follows the vibroequilibria prediction fairly well. A linear fitting of the experiment data is used to verify this tendency. Even though there is considerable variance at low amplitudes, the general behavior of the fluid is consistent with the predictions of vibroequilibria theory.

FIGURE 7.19: Contact point height used to process the cuboidal cell images for the first resonance mode.

FIGURE 7.20: The vibroequilibria solution shows good agreement with the real experiment. Experiment data points are taken in the midplane of the cuboid, and the excitation for the vibroequilibria calculation is along the vector $(0.81; 0.58)$. Parameters are $\mathcal{V} = 0.4$, $\Sigma = 70$ dyne, $\beta_c = 45^\circ$.

8 Conclusions and direction for future work

8.1 Conclusions

In this thesis a review was made of previous theory describing weakly modulated subharmonic cross-waves in horizontally vibrated containers and it was shown how one can include the effects of surface tension and distributed forcing to allow more accurate predictions for experiments where significant spatial modulation is present — typical of high-frequency forcing. The general theory of vibroequilibria was summarized and its effects in microgravity environments were discussed. Detailed results were obtained for containers of rectangular shape, including excitation of various orientations. A comparison with direct numerical simulations of the Navier-Stoke equations was made. A saddle-node instability was discovered in the case of purely horizontal vibrations when the reduced volume $\mathcal{V}$ is larger than a critical value, and this instability was studied in depth. Additionally, the effects of surface wave instabilities were investigated, where it was found that modulated subharmonic waves, in particular, could drive sloshing motion and lead to the destruction of the underlying vibroequilibria state. Finally, the Control of Fluids in Microgravity (CFVib) experiment, which flew on the 65th ESA Parabolic Flight campaign, was presented. The design of the experiment for exciting various fluid configurations at high frequencies was described, as well as preliminary results. Some of those results show good agreement with vibroequilibria theory, and indicate the potential for vibrations in managing confined liquids, while some fluid configurations did not behave as expected. In any case, the experiment serves as an interesting and useful step

toward a more systematic investigation of vibroequilibria phenomena in micro-gravity. More specific conclusions for each of these topics is discussed below.

Cross-waves: We extended the standard theory describing weakly modulated subharmonic cross-waves in horizontally vibrated containers to include the effects of surface tension and distributed forcing, and compared the predictions of this theory with experimental results. Prior cross-wave analyses based on the work of Jones [28] assumed a separation of scales between the slow spatial modulation of the cross-wave pattern and the fast spatial decay of the localized harmonic mode (oscillating bulk flow) attached to the endwalls, which provides the forcing mechanism for exciting subharmonic waves. This assumption led to the forcing appearing solely in the boundary condition for an NLS equation describing the slowly varying amplitude $A(x, t)$. While this approach is justified for cross-wave experiments where viscosity is small enough that the spatial modulation of the pattern is very slow [6, 35, 37, 60, 61, 69], or where the wavemaker is very shallow, it does not apply to more recent experiments performed with higher frequencies [48, 54, 65] where significant spatial modulation is observed. In the experiments reported here, and those of Porter *et al.* [54], the oscillating bulk flow extends throughout the container and must be modeled by a distributed forcing term.

The damped NLS equation with distributed forcing derived here is a straightforward extension of the standard theoretical approach and, in particular, care was taken to follow the derivation of Bernoff *et al.* [9] as closely as possible. Eqs. (2.39) reduce to the model obtained in that reference if surface tension is set to zero (and the remaining nonlinear curvature term is neglected in Eq. (2.12c) as well) and the oscillating bulk flow is assumed to decay on a fast lengthscale compared to the modulation of $A(x, t)$.

A quantitative comparison was made between the boundary forced model (2.36), which is similar to others in the literature [28, 31, 36, 46, 60], and the distributed forcing model (2.39). Apart from the low frequency limit, these two models diverge dramatically in their predictions. The scaling with viscosity ν is different — critical acceleration is roughly proportional to ν in the case of

distributed forcing and to $\sqrt{\nu}$ in the case of boundary forcing (see Fig. 2.4) — as are the predicted thresholds (see Fig. 2.3). In general, boundary forced models tend to overestimate the efficacy of the forcing since an integral over the entire oscillating bulk flow is used and this will include a region where the forcing is not strong enough to support waves. With distributed forcing there is a natural distinction between supercritical regions, where the forcing is sufficient to locally excite subharmonic waves and underforced subcritical regions, where the waves lose energy and decay.

An interesting feature of distributed forcing models is the multiplicity of cross-waves solutions that may be excited with the same crosswise mode number N . These secondary solutions are distinguished by the number of spatial oscillations they undergo in the supercritical region and cannot be found in boundary forced models. The importance of these secondary modes increases with the width of the forcing mechanism (the depth of the container or wavemaker).

Boundary conditions at the endwall (wavemaker) play a crucial role, both in theory and experiment [54], affecting not only the minimum onset value but the leftward or rightward shift of the neutral stability curve (i.e., the detuning value at the minimum). The influence of boundary conditions, which are the principal source of uncertainty for the experiments presented here, can be quantitatively seen in Fig. 2.12. Because the distributed forcing term appears directly in the amplitude equation, alternative boundary conditions can be sensibly applied without disrupting the essential balance between forcing and damping. In fact, simple Dirichlet boundary conditions were seen to work quite well, leading both to a more accurate prediction of threshold acceleration in some cases [see Fig. 2.10(a)] and to a more rapidly modulated pattern consistent with experiment (see Figs. 2.13-2.15).

Finally, it was seen how horizontal excitation in a finite-sized container, with two opposing out-of-phase wavetrains, naturally leads to a Hopf bifurcation and thus to temporally quasiperiodic eigenfunctions that cycle slowly between two-sided and one-sided patterns [50, 53]. If the overlap of the subharmonic waves originating at opposite endwalls is small, then the Hopf frequency is also small

and a small interval of forcing values where quasiperiodic solutions can be found. In section 2.5.4 the interaction strength was increased by using lower forcing frequencies, near 13.5 Hz, allowing the existence of quasiperiodic solutions to be experimentally confirmed. The slow modulation frequency was consistent with the prediction of Eqs. (2.46).

The results presented demonstrate the important qualitative and quantitative consequences of assuming a separation of lengthscales between the forcing mechanism and the cross-wave modulation. Avoiding such an assumption and the resulting boundary forcing term is essential for any theory aiming to capture the thresholds and features of the more complicated patterns observed in the gravity-capillary regime [48, 54, 65]. When compared with boundary forced models like Eqs. (2.36), the success of the distributed forcing model (2.46) in capturing the experimentally measured thresholds in the gravity-capillary regime, the observed spatial modulation of onset patterns, and the slow temporal modulation observed with lower frequencies, is considerable. The distributed forcing model (2.46) describes the experiments reported here quite well, and many of those in Porter *et al.* [54], despite the significant spatial modulation (rotation) of the patterns. It should be remembered, though, that Eqs. (2.46) were derived with the assumption of (weakly) modulated cross-waves and thus cannot describe subharmonic patterns that depart too much from a crosswise orientation. An alternative framework that permits more general oblique patterns is considered in [50].

Vibroequilibria: We have examined the predictions of vibroequilibria theory by comparing them with the results of direct numerical simulations of the Navier-Stokes equations. We reviewed selected hydrostatic results in section 3.2, emphasizing the multiple branches of solutions available and, in particular, the transition as volume is decreased from a unique symmetric connected solution to one of an infinite number of disconnected, generally asymmetric, solutions. It was argued that an analogous transition occurs with vibroequilibria solutions in horizontally vibrated rectangular containers as the forcing, which acts to separate the fluid by pushing it out of the centre of the container and toward the two vibrating

endwalls, is increased. In many configurations, variations in forcing amplitude will drive hysteretic transitions among distinct vibroequilibria solutions.

The results of combined vibroequilibria calculations and numerical simulations presented in Chapter 4 and 5 established two main points. First, the theory of vibroequilibria, based on an averaged variational principle, succeeds in describing the average fluid surface over a wide range of forcing values and directions. Second, beyond a minimum (dimensionless) volume $\mathcal{V} \sim 0.36$ of fluid, the symmetric vibroequilibria solution, presented in horizontally vibrated containers, is destroyed not by gradual contact with the container bottom (i.e, contact presaged by gradual downward movement of the midpoint), as it is for smaller volumes, but by an instability that occurs prior to this. After this instability the (central dip in the) fluid plunges rapidly toward the bottom, initiating a more violent and unpredictable transition to one of the disconnected vibroequilibria solutions.

This instability was examined in detail in section 5.2 after restricting the vibroequilibria problem to relevant one-parameter families of surfaces. This simplification allows unstable solutions to be followed and for simple energy profiles to be drawn, revealing the instability as a saddle-node bifurcation where the stable vibroequilibria state collides with an unstable vibroequilibria state of larger amplitude (greater separation along the vibrating endwalls). This instability does not occur in the hydrostatic case, but does in horizontally vibrated rectangular containers of fluid, even when gravity is included. Compelling evidence that this saddle-node instability occurs in the full problem, not just when restricted to particular surface families, was provided by comparing predictions for the dependence of the minimum depth δ (at the centre of the symmetric connected solution) as a function of forcing η and volume $\mathcal{V}$.

Hysteresis due to sloshing and other nonlinear effects is expected to be greatly enhanced in sufficiently large fluid volumes where this saddle-node instability occurs. The transition to a disconnected vibroequilibria solution branch is further complicated by dynamic effects related to surface waves and sloshing modes,

which were considered in section 5.3. Such effects are most relevant for low viscosity fluids, like water, and when surface waves are excited across the interior of the surface, not just in narrow regions near the endwalls. Simulations show that the final state varies with forcing parameters and boundary conditions, and can also depend on the initial (transient) approach to the vibroequilibria state. The final fluid configuration may resemble the vibroequilibria surface predicted by theory, but with harmonic and subharmonic surface waves superimposed on it, or it may deviate from this prediction (see Fig. 5.15) due to excitation of the first odd and even sloshing modes, which are at much lower frequency than the vibrational forcing.

The most salient conclusion of section 5.3 is that the interaction between vibroequilibria (i.e., the average surface) and high-frequency surface waves is significantly enhanced by the presence of modulated subharmonic waves. These quasiperiodic solutions, which are typical as well in (nearly) flat horizontally forced systems [50, 53, 57, 68], appear to be crucial in driving the odd sloshing mode because of their slow modulation frequency. If modulated subharmonic waves persist for long enough, they can drive the odd sloshing mode to such an amplitude that the (modulated) vibroequilibria solution is completely destroyed. We may suppose that the subsequent transition process, which, because of the odd sloshing, is likely to begin from very asymmetric conditions, will lead to increased variation and asymmetry in the final state and, perhaps, for certain parameters, even to repeated cycles of instability, separation/splashing, coalescing, instability, and so on. It will be an interesting direction for further work to examine the interaction between surface waves and average vibroequilibria more carefully, including a more realistic treatment of contact line dynamics and the extension of the vibroequilibria calculations and numerical simulations to the three-dimensional case.

CFVib Experiment and results: The experiment focuses on the vibroequilibria phenomenon. The fact that the selected vibroequilibria surfaces are increasingly flattened (perpendicular to the axis of vibration) by the applied periodic forcing

means that this effect can be considered as a type of artificial gravity in some situations, and as a potential means to control and manipulate fluids in microgravity.

The experimental set-up is designed to record the behaviour of several different fluid configurations in six cuboidal and six cylindrical cells, only half of which are deliberately excited by vibrations. The forcing is delivered through a cantilever beam driven by a pair of piezoelectric PZT ceramics subjected to a counter-phase electrical signal. Locating the resonances of this piezoelectric actuator (where the most energy can be transferred to the fluid) is a critical part of the preparation of the experiment. Changes in vibrational energy associated with different interface configurations must be comparable to those due to contact and surface tension terms. The piezoelectric actuator used in the experiment allows for a wide range of frequencies to be investigated and the corresponding modes and vibroequilibria solutions include both hydrodynamic and acoustic cases.

The experiment flew in the 65th ESA Parabolic Flight Campaign as part of the 2016 Fly Your Thesis! programme. Although not all of the fluid configurations behaved as expected, we can summarize the main preliminary conclusions:

- (i) The power required to clearly observe the vibroequilibria effect was underestimated somewhat, perhaps due to the details of the boundary layer dynamics and an attenuation of the velocity transmitted to the fluid bulk. In some cases, the required power was roughly twice that originally estimated.
- (ii) No clear vibroequilibria effects were observed in the liquid-liquid mixtures, perhaps due to the greater load of these heavier experimental cells. This configuration was abandoned on the third and final flight.
- (iii) The length of the cables introduced additional noise in the signals and reduced the operational envelope of the amplifiers for the high frequency forcing cases. This noise affected the accelerometer measurements as well, so filtering is required during post-processing.
- (iv) For silicone oil (a wetting liquid), a larger vibroequilibria effect was observed in the cylindrical containers compared to the cuboidal ones. This experiment was repeated twice, with similar results (see Fig. 7.10 and

Fig. 7.11). There is likely a strong dependence on the residual gravity, as suggested by Fig. 7.12 where the time taken to reach the top lid is given. It is notable that, with the highest forcing amplitude, the time increases from the previous (lower-amplitude) case, contrary to the tendency for this climbing time to decrease with forcing amplitude seen in the other data points.

- (v) In the experiments with water, vibroequilibria effects were seen for both cuboidal and cylindrical recipients. In contrast with the silicone oil experiment in a cylinder (with similar forcing amplitude), the water does not reach the top lid. This is explained by the fact that water has higher contact energy than silicone oil.
- (vi) Vibroequilibria observed in the experiment with water in a cuboidal container allow us to verify the predictions of vibroequilibria theory in rectangular (2D) containers. Figure 7.20 shows that the displacement of the front and rear contact points follow the values predicted by theory.
- (vii) In the last flight, we observe with low viscosity, low surface energy fluid in a cuboidal recipient that surface wave instabilities driven by vibration at frequencies on the order of tens of Hz may mask the development of new quasi-equilibrium states; see Fig. 7.9.
- (viii) Experiments with vertical vibration show that such forcing is effective in flattening the surface, both for water in a cuboidal cell and for 50 cSt silicone oil in a cylindrical cell (see Fig. 7.16 and Fig. 7.17).
- (ix) The piezoelectric actuators were generally effective, allowing controlled excitation over a wide range of frequencies. A variety of modes can be selected (even more if additional bender configurations are used) and, thus, a variety of vibrational motion and fluid response investigated. Signal generation and amplification worked well over this large range of frequencies.

Overall, the experiment was successful, providing a lot of useful data to compare with the predictions of vibroequilibria theory. As a full analysis of the results

is completed, more specific modifications and improvements of both theoretical (numerical) models and microgravity experiment design will be suggested.

8.2 Expectations and future work

The CFVib project was a first step for the design and development of a microgravity experiment. Many of the conclusions presented above suggest improvements to both theoretical models and to the design of a future microgravity experiment.

Regarding theoretical work, we note the following issues:

- There is a need to better understand the influence of the contact line, its dynamics and energy, as well as its influence on the vibroequilibria states. The fact that no large vibroequilibria effects are seen for the wetting fluid when it covers all container walls (the oil experiments in cuboidal recipients) needs further explanation.
- The use of higher frequencies (in the range of kHz) also needs to be investigated further, since minimal effects were seen during the experiment. New models are needed to understand its potential use for interface positioning.

In terms of hardware design, we note the following observations:

- The cable length should be as short as possible to avoid losses when using a high-frequency signal. Although not entirely unexpected, we underestimated the importance of this effect.
- The space needed to allocate the experiment should be optimized. We selected a small experimental rack (Zarges box) in the beginning and, at the end, there were some difficulties fitting the experiment inside it.
- To obtain more quantitative results, alternative optical and illumination techniques should be explored. Although it was not the main concern of the CFVib experiment, it will be important for future experiments.

- In order to better understand vibrational phenomena, it would be an advantage to have a good technique for measuring the induced bulk flow on ground. This would help to refine the numerical models.

In addition to the things mentioned above, the use of piezoelectric materials to induce vibrations in a confined fluid should provide a good means of exploring surface waves instabilities with a small, low-cost, but effective, system. Such experiments could be used, for example, to test coupled oscillator models like those in [57].

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