Struggling with Riordan involutions formula.

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Outline

1 Previous results:
   - A brief history.
   - Arithmetical triangles.
   - Finite vs infinite Riordan matrices.
   - Riordan involutions formula.

\[ T(f \mid g) = \left( \frac{f}{g}, \frac{g}{f} \right) \]
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1. Previous results:
   - A brief history.
   - Arithmetical triangles.
   - Finite vs infinite Riordan matrices.
   - Riordan involutions formula.

2. Examples of involution formula application
   - Approximations of solutions of Babbage equation.
   - Computation of the A-sequence
   - Self-dual involutions
   - Pseudo-involutions.
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2 Examples of involution formula application
   - Approximations of solutions of Babbage equation.
   - Computation of the A-sequence
   - Self-dual involutions
   - Pseudo-involutions.

3 The group generated by involutions.
   - Products of involutions. Reversible elements.
   - The $\Omega_0$ subgroup.
   - The commutator of the Riordan group.
   - The group generated by Riordan involutions.
A BRIEF HISTORY

THE MASKED RIORDAN GROUP AND SUBGROUPS.

- Jabotinsky matrices and Faber polynomials. (1953)
- Appell polynomials and generalized Appell polynomials. (1880, 1964)
- Sheffer polynomials and Umbral Calculus. Rota (70’s and 80’s)
- Convolution polynomials. Knuth (1992)
- Recursive matrices and Umbral Calculus (1982)
- Self-inverse Sheffer sequences (1976)
- Denh-Sommerville equations (1905-1927)
- and much more....

\[ T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right) \]
A BRIEF HISTORY


\[
\begin{pmatrix}
1 & & & & \\
2 & 1 & & & \\
5 & 4 & 1 & & \\
14 & 14 & 6 & 1 & \\
42 & 48 & 27 & 8 & 1 \\
132 & 165 & 110 & 44 & 10 & 1 \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & &
\end{pmatrix},
\]

In page 89 he asked "(5) Is there a theory of arithmetic triangles where a simple function of the generating function of the first column yield the generation function of the nth column?"

\[
T(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)
\]
A BRIEF HISTORY


RENEWAL ARRAYS AND THE A-SEQUENCE

\[ b_{n,m} = [x^n](B(x))^m, \quad m \geq 1 \quad \text{and} \quad b_{n,0} = [x^n]B(x) \]

\[ b_{n,m} = \sum_{r \leq 0} a_r b_{n-1,m-1+r} \]

with \( b_0 = 1 \) and \( a_0 = 1 \).
A BRIEF HISTORY


**THE RIORDAN GROUP (1991)**

\[ M = (m_{i,j})_{i,j \geq 0} \]

such the jth column is

\[ C_j = g(x)[f(x)]^j \]

where

\[ g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots \]
\[ f(x) = x + f_2 x^2 + f_3 x^3 + \cdots , \]

\[ M = (g(x), f(x)) \] is a Riordan matrix.

\[ Pascal = \left( \frac{1}{1-x}, \frac{x}{1-x} \right) \]

\[ T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right) \]
A BRIEF HISTORY


PROPER RIORDAN ARRAYS (1994)

\[ d_{n,k} = [t^n]d(t)(th(t))^k \]

where \(d_0 \neq 0\) and \(h_0 \neq 0\).

\[ \text{Pascal} = \left( \frac{1}{1-t}, \frac{1}{1-t} \right) \]
Our first triangle

El círculo Mágico

Katherine Neville

Un triángulo mágico:

\[
\begin{aligned}
1 \\
2 & -1 \\
3 & -4 & 1 \\
4 & -1 & 6 & -1 \\
5 & -26 & 23 & -8 & 1 \\
& & & & & \\
& & & & & & Y más
\end{aligned}
\]

Felicitades Mary.

Con cariño.

\[
T(f | g) = \left( \frac{f}{g}, \frac{\bar{g}}{g} \right)
\]
Arithmetical triangles

Our first triangle (1998)

\[
S = \begin{pmatrix}
1 \\
2 & -1 \\
3 & -4 & 1 \\
4 & -11 & 6 & -1 \\
5 & -26 & 23 & -8 & 1 \\
6 & -57 & 72 & -39 & 10 & -1 \\
7 & -120 & 201 & -150 & 59 & -12 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]


\[
S = T \left( \frac{2x - 1}{(1 - x)^2} \right| 2x - 1 \right) \equiv \left( \frac{2x - 1}{(1 - x)^2(2x - 1)}, \frac{x}{2x - 1} \right)
\]

\[
T(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)
\]
Arithmetical triangles (2008)

We find a group, $A(\mathbb{K}[[x]])$ of arithmetical triangles such that

$$T(f \mid g) = (d_{i,j})_{i,j \geq 0} \in A(\mathbb{K}[[x]])$$

where

$$d_{i,j} = [x^i] \frac{x^j f(x)}{g^{j+1}(x)} \quad \text{where} \quad f_0 \neq 0, \quad g_0 \neq 0$$

$$Pascal = T(1 \mid 1 - x)$$

Remark

$$d_{i,j} = [x^i] \frac{f(x)}{g(x)} \left( \frac{x}{g(x)} \right)^j \iff T(f \mid g) = \left( \frac{f(x)}{g(x)}, \frac{x}{g(x)} \right)$$

$$T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right)$$
**Notation**

\[ f \equiv f(x) = \sum_{n \geq 0} f_n x^n, \quad fl \left( \frac{x}{g} \right) \equiv f(x) \cdot l \left( \frac{x}{g(x)} \right) \]

\[
T(f \mid g)T(l \mid m) = T \left( fl \left( \frac{x}{g} \right) \mid gm \left( \frac{x}{g} \right) \right)
\]

\[
T^{-1}(f \mid g) = T \left( \frac{1}{f \left( \left( \frac{x}{g} \right)^{-1} \right)} \left| \left( \frac{1}{g \left( \left( \frac{x}{g} \right)^{-1} \right)} \right) \right. \right)
\]

**Elements in some subgroups**

\[
T(f \mid 1), \quad T(1 \mid g), \quad T(g \mid g), \quad T(g - xg' \mid g),
\]

\[
T \left( \frac{g - xg'}{g} \mid g \right), \quad T \left( \frac{\alpha g}{\alpha \left( \frac{x}{g} \right)} \mid g \right), \quad T \left( \frac{g - x}{1 - x} \mid g \right)
\]
A NEW PROOF OF THE EXISTENCE OF $A$-SEQUENCE I

Observation

The subgroup $\{T(f \mid g) \mid g = 1\}$ is commutative and normal in the Riordan group. If $T(s \mid 1) = (b_{i,j})_{i,j \in \mathbb{N}}$ then $b_{i,j} = s_{i-j}$ for $i \geq j$.

\[
T(f \mid g) = T(f \mid 1)T(1 \mid g)
\]

\[
T(f \mid g) = T\left(\frac{1}{g} \mid 1\right)T(fg \mid g) = T(fg \mid g)T(s \mid 1)
\]

If $T(f \mid g) = (d_{i,j})_{i,j \in \mathbb{N}}$, then $T(fg \mid g) = (c_{i,j})_{i,j \in \mathbb{N}}$ with $c_{n,0} = f_n$ and $c_{i,j} = d_{i-1,j-1}$ for $i, j \geq 1$. So

\[
d_{i,j} = \sum_{k=j}^{i} c_{i,k}b_{k,j} = \sum_{k=j}^{i} s_{k-j}d_{i-1,k-1}
\]

$T(f \mid g) = \left(\frac{f}{g}, \frac{g}{g} \right)$
A NEW PROOF OF THE EXISTENCE OF $A$-SEQUENCE II

In fact, $s \equiv A$ is the $A$-sequence of $T(f \mid g)$. Then

$$T(A \mid 1) = T^{-1}(f g \mid g) T \left( \frac{1}{g} \mid 1 \right) T(f g \mid g) =$$

$$T^{-1}(1 \mid g) T \left( \frac{1}{f g} \mid 1 \right) T \left( \frac{1}{g} \mid 1 \right) T(f g \mid 1) T(1 \mid g) =$$

$$T \left( \begin{array}{c|c} 1 & \frac{1}{g \left( \left( \frac{x}{g} \right)^{-1} \right)} \\ \hline g \left( \left( \frac{x}{g} \right)^{-1} \right) & \end{array} \right) T \left( \frac{1}{g} \mid g \right) = T \left( \begin{array}{c|c} 1 & g \left( \left( \frac{x}{g} \right)^{-1} \right) \left( \frac{x g}{g} \right) \left( \left( \frac{x}{g} \right)^{-1} \right) \right) \left( \begin{array}{c} \frac{1}{g \left( \left( \frac{x}{g} \right)^{-1} \right)} \\ \hline \end{array} \right) \right),$$

then

$$g \left( \frac{x g \left( \left( \frac{x}{g} \right)^{-1} \right)}{g \left( \left( \frac{x}{g} \right)^{-1} \right)} \right) = 1, \quad \Leftrightarrow \quad g \left( \frac{x g \left( \left( \frac{x}{g} \right)^{-1} \right)}{g \left( \left( \frac{x}{g} \right)^{-1} \right)} \right) = g \left( \left( \frac{x}{g} \right)^{-1} \right)$$

$$T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right).$$
A new proof of the existence of $A$-sequence III

and

$$A = \frac{1}{g \left( \left( \frac{x}{g} \right)^{-1} \right)}, \iff A \left( \frac{x}{g} \right) = \frac{1}{g}, \iff xA \left( \frac{x}{g} \right) = \frac{x}{g}, \iff \left( \frac{x}{g} \right)^{-1} = \frac{x}{A}.$$ 

**Observation**

$$T^{-1}(1 \mid g) = T(1 \mid A) \iff T^{-1}(1 \mid A) = T(1 \mid g)$$

So, if $T(f \mid g)$ is an involution $A = g$
Finite versus infinite matrices I

For every $n \in \mathbb{N}$ consider the general linear group $GL(n + 1, \mathbb{K})$ formed by all $(n + 1) \times (n + 1)$ invertible matrices with coefficients in $\mathbb{K}$. Let $\mathcal{R}$ be the Riordan group. Since every Riordan matrix is lower triangular, we can define a natural homomorphism $\Pi_n : \mathcal{R} \to GL(n + 1, \mathbb{K})$ given by

$$\Pi_n((d_{i,j})_{i,j \in \mathbb{N}}) = (d_{i,j})_{i,j=0,\ldots,n}.$$

$$\mathcal{R}_n = \Pi_n(\mathcal{R})$$

Let $D = (d_{i,j})_{i,j=0,\ldots,n+1} \in \mathcal{R}_{n+1}$. We define $P_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$ by

$$P_n((d_{i,j})_{i,j=0,1,\ldots,n+1}) = (d_{i,j})_{i,j=0,\ldots,n}.$$
Finite versus infinite matrices II

**R as inverse limit**

The Riordan group $\mathcal{R}$ is isomorphic to $\varprojlim \{(\mathcal{R}_n)_{n \in \mathbb{N}}, (P_n)_{n \in \mathbb{N}}\}$.

**Groups of finite Riordan matrices $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$**

$\mathcal{R}_0 = \mathbb{K} \setminus \{0\} \equiv \mathbb{K}^*$

$\mathcal{R}_1 = \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \mid \alpha \neq 0 \text{ and } \beta \neq 0 \right\}$

$\mathcal{R}_2 = \left\{ \begin{pmatrix} r_0 & 0 & 0 \\ \alpha & r_0 r & 0 \\ \beta & \gamma & r_0 r^2 \end{pmatrix} \mid r_0, r \neq 0 \text{ and } \alpha, \beta, \gamma \in \mathbb{K} \right\}$

$\mathcal{R}_3 = \left\{ \begin{pmatrix} r_0 & 0 & 0 & 0 \\ d_0 & r_0 r & 0 & 0 \\ \alpha & r(d_0 + d) & r_0 r^2 & 0 \\ \beta & \gamma & r^2(d_0 + 2d) & r_0 r^3 \end{pmatrix} \mid r_0, r \neq 0 \text{ and } d_0, d, \alpha, \beta, \gamma \in \mathbb{K} \right\}$

$T(f \mid g) = \left( \frac{f}{g}, \frac{\alpha}{g} \right)$
Riordan involutions formula

Suppose \( n \geq 2 \). Let \( D = (d_{i,j}) \in \mathcal{R}_{n-1} \) be an involution and take \( \hat{D} = (d_{i,j}) \in \mathcal{R}_n \) such that \( P_{n-1}(\hat{D}) = D \).

(i) If \( n \) is even, \( \hat{D} \) is an involution if and only if

\[
d_{n,1} \text{ is arbitrary and } d_{n,0} = -\frac{1}{2d_{0,0}} \sum_{k=1}^{n-1} d_{k,0}d_{n,k}
\]

(ii) If \( n \) is odd, \( \hat{D} \) is an involution if and only if

\[
d_{n,0} \text{ is arbitrary and } d_{n,1} = -\frac{1}{2d_{1,1}} \sum_{k=2}^{n-1} d_{k,1}d_{n,k}
\]

Moreover,

\[
a_{n-1} = \frac{1}{d_{n-1,n-1}} \left( d_{n,1} - \sum_{j=0}^{n-2} a_{j}d_{n-1,j} \right)
\]

\([j \mid g] = \left( \frac{g}{j}, \frac{\tilde{g}}{j} \right)\)
General forms of non trivial involutions in the lower sizes: I

**Involutions in $\mathcal{R}_0$**

$$(d_{0,0}) \in \mathcal{R}_0 \quad \text{with} \quad d_{0,0} = \pm 1$$

**Involutions in $\mathcal{R}_1$**

$$\begin{pmatrix} d_{0,0} & -d_{0,0} \\ d_{1,0} & -d_{0,0} \end{pmatrix} \quad \text{with} \quad d_{0,0} = \pm 1, \quad \text{and} \quad d_{1,0} \in \mathbb{K}$$

**Involutions in $\mathcal{R}_2$**

$$\begin{pmatrix} d_{0,0} & -d_{0,0} & -d_{1,0}d_{2,1} \\ d_{1,0} & -d_{0,0} & d_{2,1} \\ -\frac{d_{1,0}d_{2,1}}{2d_{0,0}} & d_{2,1} & d_{0,0} \end{pmatrix} \quad d_{0,0} = \pm 1, \quad d_{1,0}, d_{2,1} \in \mathbb{K}$$

$$T(f \ | \ g) = \left( \frac{f}{g}, \frac{g}{f} \right)$$
General forms of non trivial involutions in the lower sizes: II

**Involutions in** $\mathcal{R}_3$

\[
\begin{pmatrix}
  d_{0,0} & d_{1,0} & -d_{0,0} \\
  d_{1,0} & -\frac{d_{1,0}d_{2,1}}{2d_{0,0}} & d_{2,1} \\
 -\frac{d_{1,0}d_{2,1}}{2d_{0,0}} & -\frac{d_{1,0}d_{2,1}+2d_{2,1}^2}{2d_{0,0}} & -(d_{1,0} + 2d_{2,1}) \\
 d_{3,0} & -\frac{d_{1,0}d_{2,1}+2d_{2,1}^2}{2d_{0,0}} & -d_{0,0}
\end{pmatrix}
\]

$d_{0,0} = \pm 1, \quad d_{1,0}, d_{2,1}, d_{3,0} \in \mathbb{K}$

**Corollary**

Any Riordan involution $D_n \in \mathcal{R}_n$ can be extended to a Riordan involution $D_{n+1} \in \mathcal{R}_{n+1}$, i.e. $P_n(D_{n+1}) = D_n$.

Equivalently,

For any finite Riordan involution $D_n \in \mathcal{R}_n$ there is an infinite Riordan involution $D \in \mathcal{R}$ such that $\Pi_n(D) = D_n$. 

$T(f \mid g) = \left( \frac{f}{g}, \frac{f}{g} \right)$
CONSEQUENCES OF INVOLUTIONS FORMULA

**Corollary**

Let $\alpha = \sum_{i \in \mathbb{N}} \alpha_i x^i$ be an arbitrary formal power series then

1. There is an unique nontrivial involution $D = (d_{i,j})_{i,j \in \mathbb{N}}$ such that

   $$d_{0,0} = 1, \quad d_{2i+1,0} = \alpha_{2i} \quad \text{and} \quad d_{2i+2,1} = \alpha_{2i+1} \quad \text{for} \quad i = 0, 1, \ldots$$

   we denote it by $I_+^\alpha$.

2. There is an unique nontrivial involution $D = (d_{i,j})_{i,j \in \mathbb{N}}$ such that

   $$d_{0,0} = -1, \quad d_{2i+1,0} = \alpha_{2i} \quad \text{and} \quad d_{2i+2,1} = \alpha_{2i+1} \quad \text{for} \quad i = 0, 1, \ldots$$

   we denote it by $I_-^\alpha$.

Moreover, any nontrivial Riordan involution can be constructed by this way.

**Corollary**

We can construct nontrivial involutions $D = (d_{i,j})_{i,j \in \mathbb{N}}$ with $A$-sequence $A = \sum_{i \in \mathbb{N}} a_i x^i$ such that

$$d_{2i+1,0} \quad \text{and} \quad a_{2i+1} \quad \text{are arbitrary.}$$
Non trivial involutions $\mathcal{I}_\alpha^+$ and $\mathcal{I}_\alpha^-$

$$\mathcal{I}_\alpha^+ = \begin{pmatrix}
1 \\
\alpha_0 & -1 \\
d_{2,0} & \alpha_1 & 1 \\
\alpha_2 & d_{3,1} & d_{3,2} & -1 \\
d_{4,0} & \alpha_3 & d_{4,2} & d_{4,3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$$\mathcal{I}_\alpha^- = \begin{pmatrix}
-1 \\
\alpha_0 & 1 \\
d_{2,0} & \alpha_1 & -1 \\
\alpha_2 & d_{3,1} & d_{3,2} & 1 \\
d_{4,0} & \alpha_3 & d_{4,2} & d_{4,3} & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

$T(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)$
Non trivial diagonal involutions

\[ \mathcal{I}^+_0 = (1, -x) = T(-1 | -1) \quad \text{and} \quad \mathcal{I}^-_0 = (-1, -x) = T(1 | -1) \]

\[ \mathcal{I}^-_0 = -\mathcal{I}^+_0 \]

But, in general

\[ -\mathcal{I}^+_\alpha = \mathcal{I}^-_{-\alpha} \]

Proposition

\[ \mathcal{I}^\pm_\alpha = \mathcal{I}^+_0 \mathcal{I}^\mp_\alpha \mathcal{I}^-_0 \]

Consequently,

If \[ \mathcal{I}^+_\alpha = T(f \mid g) \quad \Rightarrow \quad \mathcal{I}^-_\alpha = T(-f(-x) \mid g(-x)) \]

If \[ \mathcal{I}^+_\alpha = (d, h) \quad \Rightarrow \quad \mathcal{I}^-_\alpha = (-d(-x), -h(-x)) \]
Finite Riordan involutions: \((D^2 = I)\)

\[
\begin{pmatrix}
1 & -1 \\
1 & 1 \\
-\frac{1}{2} & 1 \\
1 & -\frac{3}{2} \\
-\frac{9}{8} & \frac{15}{2} \\
1 & \frac{17}{8} \\
\end{pmatrix}
\]

\(T_+^{\frac{1}{1-x}}, \ g_1(x) = A_1(x) = -1 - 2x + 0x^2 + x^3 - 2x^4\)

\[
\begin{pmatrix}
1 & -2 & 1 \\
0 & 4 & -1 \\
0 & 12 & -1 \\
0 & 32 & 1 \\
0 & 8 & -1 \\
\end{pmatrix}
\]

\(T_+^{\frac{-2x}{1-4x^2}}, \ g_2(x) = A_2(x) = -1 + 2x + 0x^2 + 0x^3 + 0x^4\)

\[T(f \mid g) = \left( \frac{f}{g}, \frac{f^2}{g^2} \right)\]
Finite Riordan involutions: \((D^2 = I)\)

\[
\begin{pmatrix}
1 & -1 \\
0 & 1 \\
-1 & 1 \\
1 & -3 \\
-9 & 1 \\
2 & 1 \\
1 & -4 \\
1 & -8 \\
1 & -16
\end{pmatrix}
\]

\(\mathcal{T}^+_{\frac{1}{1-x}}\), \(g_1(x) = A_1(x) = -1 - 2x + 0x^2 + x^3 - 2x^4\)

\[
\begin{pmatrix}
1 & -1 \\
0 & 2 \\
0 & -4 \\
0 & 12 \\
0 & 32 \\
0 & -6 \\
0 & 8 \\
0 & -24 \\
0 & 8
\end{pmatrix}
\]

\(\mathcal{T}^+_{\frac{1}{(1-x)^2}}\), \(g_2(x) = A_2(x) = -1 + 2x + 0x^2 + 0x^3 + 0x^4\)

\[
\begin{pmatrix}
1 & -1 \\
-\frac{1}{2} & 1 \\
2 & -3 \\
-45 & 5 \\
12 & -131 \\
0 & 1 \\
0 & -3 \\
0 & 5 \\
0 & -7
\end{pmatrix}
\]

\(\mathcal{T}^+_{\frac{1}{1-4x^2}}\), \(g_3(x) = A_3(x) = -1 - 3x + 0x^2 + 8x^3 - 24x^4\)

\[
\begin{pmatrix}
1 & -1 \\
0 & 2 \\
0 & -5 \\
0 & 3 \\
0 & 1 \\
0 & 8 \\
0 & 20 \\
0 & 24 \\
0 & 8
\end{pmatrix}
\]

\(\mathcal{T}^+_{\frac{1}{C(x)}}\), \(g_4(x) = A_4(x) = -1 - 2x + 0x^2 - 4x^3 + 8x^4\)

\[
T(f \mid g) = \left( \frac{f}{g}, \frac{f}{g} \right)
\]
Solutions of Babbage’s equations. I

An approximation of solution of Babbage’s equation.

The Taylor polynomial of order 10 of any nontrivial solution, \( \omega \in K[[x]] \), of Babbage’s equation is

\[
T_{10}(\omega) = -x + \beta_0 x^2 - \beta_0^2 x^3 + \beta_1 x^4 + (2\beta_0^4 - 3\beta_0 \beta_1) x^5 + \beta_2 x^6 + \\
(-13\beta_0^6 + 18\beta_0^3 \beta_1 - 4\beta_0 \beta_2 - 2\beta_1^2) x^7 + \beta_3 x^8 + \\
(145\beta_0^8 - 221\beta_0^5 \beta_1 + 35\beta_0^3 \beta_2 + 50\beta_0^2 \beta_1^2 - 5\beta_0 \beta_3 - 5\beta_1 \beta_2) x^9 + \beta_4 x^{10}
\]

where \( \beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \in K \).
Solutions of Babbage’s equations. II

Taking $\alpha$ an odd formal power series and $\alpha_{2i+1} = \beta_i$,

$$
I_\alpha^+ = \begin{pmatrix}
1 & 0 & 0 & 0 & \beta_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\beta_0^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta_0 & 1 & 0 & -2\beta_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3\beta_0^2 & 1 & 3\beta_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\beta_0^4 - 3\beta_0\beta_1 & -2\beta_0^3 - 2\beta_1 & -6\beta_0^2 & -4\beta_0 & -1 & 0 & 0 & 0 & 0 \\
0 & \beta_2 & -3\beta_0^4 + 8\beta_0\beta_1 & 7\beta_0^3 + 3\beta_1 & 10\beta_0^2 & 5\beta_0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

$$
\{I_\alpha^+, \alpha \in \mathbb{K}[x] \text{ odd}\} = \{T(g \mid g) \in \mathcal{R} \mid g_0 = -1, g(x)g\left(\frac{x}{g}\right) = 1\}
$$

If $\omega = \frac{x}{g}$, $I_\alpha^+ = T\left(\frac{x}{\omega} \mid \frac{x}{\omega}\right)$ and

$$
g(x)g\left(\frac{x}{g}\right) = 1 \iff \omega(\omega(x)) = x
$$

so

$$
\omega = \sum_{n \geq 1} d_{n,1} x^n
$$

$$
T(f \mid g) = \left(\frac{f}{g}, \frac{x}{g}\right)
$$
Computing the $A$-sequence

Let $\alpha$ be an even formal power series and $I^{-\alpha} = (d_{i,j})_{i,j \in \mathbb{N}}$. Then

$$A(x) = \sum_{n=0}^{\infty} d_{n,0} x^n$$

is the $A$-sequence of the involution $I^{-\alpha}$.

If $\alpha$ is even $d_{2i+1,1} = \alpha_{2i+1} = 0$, then $d_{n,1} = 0$ for all $n \geq 2$. By induction for $n = 2$

$$a_1 = \frac{1}{d_{11}} (d_{2,1} - a_0 d_{1,0}) = d_{1,0}$$

Suppose true to $n$. What happens in $n + 1$? Taking into account that $d_{n+1,1} = 0$, by induction hypothesis $a_k = d_{k,0}$ for all $k \leq n - 1$ and using again the formula

$$a_n = \frac{1}{d_{n,n}} \left( d_{n+1,1} - \sum_{j=0}^{n-1} a_j d_{n,j} \right) = \frac{1}{d_{n,n}} \left( - \sum_{j=0}^{n-1} d_{n,j} d_{j,0} \right)$$

$$r(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)$$
As $I^-_{\alpha}$ is an involution, the product of its n-row by its 0-column is 0, then

$$
\sum_{j=0}^{n} d_{n,j} d_{j,0} = 0 \iff - \sum_{j=0}^{n-1} d_{n,j} d_{j,0} = d_{n,n} d_{n,0}.
$$

so

$$
a_n = \frac{1}{d_{n,n}} \left( - \sum_{j=0}^{n-1} d_{n,j} d_{j,0} \right) = \frac{1}{d_{n,n}} (d_{n,n} d_{n,0}) = d_{n,0}
$$
**Self-dual involutions. I**

In *Some inverse limit approaches to the Riordan group*. Linear Algebra Appl. 491 (2016) 239-262.

**Self-dual Riordan matrices.**

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the solutions of $D = D^\diamond$ are the Riordan matrices $T(f \mid g)$ such that

$$A(x) = g(x), \quad f(x) = \lambda \sqrt{g(x)(g(x) - xg'(x))} e^{\phi(x, \frac{x}{g(x)})}$$

with $\lambda \in \mathbb{K}^*$ and $\phi(x, z)$ is a symmetric bivariate power series with $\phi(0, 0) = 0$. If in addition $g(0) = 1$, then $T(f \mid g)$ is a Toeplitz matrix.

In other words, the Riordan array $R(d(x), h(x))$ is self-dual if and only if $h$ is self inverse for the composition operation, $h(h(x)) = x$ and

$$d(x) = \lambda \sqrt{x \frac{h'(x)}{h(x)}} e^{\phi(x, h(x))}$$

for $\lambda$, and $\phi$ as above. Moreover, if $h'(0) = 1$ then $h(x) = x$. 

$$T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right)$$
**Self-dual involutions. II**

**Example**

Take $g(x) = 2x - 1$ and $\phi(x, z) = 0$. $B(\sqrt{1 - 2x} \mid 2x - 1)$ is then odd-symmetric. Below we write

$$\gamma_3(B(\sqrt{1 - 2x} \mid 2x - 1)) = B_3(\sqrt{1 - 2x} \mid 2x - 1) = T_6(\sqrt{1 - 2x}(2x - 1)^3 \mid 2x - 1)$$

$$D = D^\diamond = D^{-1}$$

$$T(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)$$
Suppose $T(f \mid g) \in \mathcal{R}$ and consider $h = \frac{x}{g}$. Then, $T(f \mid g)$ is a self-dual involution if and only if

$$h(h(x)) = x \quad \text{and} \quad f(x) = \pm \sqrt{g(x)(g(x) - xg'(x))}$$

If $\checkmark$

Now, if $T(f \mid g)$ is self-dual involution then, from the above Theorem, we get $h(h(x)) = x$ and

$$f(x) = \lambda \sqrt{g(x)(g(x) - xg'(x))} e^{\phi(x, \frac{x}{g(x)})}$$

where $\phi$ is a symmetric bivariate formal power series. Since $T(f \mid g)$ is an involution then $f(x)f\left(\frac{x}{g}\right) = 1$. Consequently, $\lambda^2 e^{2\phi(x, \frac{x}{g(x)})} = 1$. So we can choose $\lambda = \pm 1$ and $\phi \equiv 0$ to get all of them.

$$T(f \mid g) = \left(\frac{f}{g}, \frac{x}{g}\right)$$
Self-dual involutions. IV

**Constructing a self-dual involution.**

\[ d_{0,0} = -1, \quad \begin{pmatrix} 1 & \gamma_0 & -1 \\ \gamma_0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ d_{1,-1} & \gamma_0 & 1 \\ d_{0,-2} & \gamma_0 & -1 \\ \gamma_1 & d_{1,-1} & \gamma_0 & -1 \\ d_{2,-2} & \gamma_1 & d_{2,0} & d_{2,1} & -1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ d_{-2,-3} & \gamma_1 & d_{1,-1} & \gamma_0 & -1 \\ d_{-1,-3} & d_{1,-2} & \gamma_0 & 1 \\ d_{0,-3} & d_{0,-2} & \gamma_0 & -1 \\ d_{1,-3} & \gamma_1 & d_{1,-1} & \gamma_0 & 1 \\ \gamma_2 & d_{2,-2} & \gamma_1 & d_{2,0} & d_{2,1} & -1 \\ d_{3,-3} & \gamma_2 & d_{1,3} & d_{3,0} & d_{3,1} & d_{3,2} & 1 \end{pmatrix} \]

with \( d_{-j,-i} = d_{i,j} \) and \( \gamma_i \in \mathbb{K} \) arbitrary.

\[ T(f \mid g) = \begin{pmatrix} f \\ g \end{pmatrix} \]
**Pseudo-involutions. I**

**Pseudo-involutions**

$D \in \mathcal{R}$ is a pseudo-involution if and only if $D I_0^+$ is an involution.

**Remark**

Note that once obtained an involution using the formula, to get the corresponding pseudo-involution we have only to change signs suitably.

**Pseudo-involutions in the Appell subgroup ($\mathcal{R}_A$).**

Let $\alpha$ a formal power series such that $\alpha_{2i+1} = -\alpha_{2i}$. Then $I_\alpha^+ I_0^+$ and $I_\alpha^- I_0^+$ are pseudo-involutions in the Appell subgroup. Moreover, any pseudo-involution in the Appell subgroup can be obtained by this way.

For $n = 2$ we get

$$I_\alpha^+ I_0^+ = \begin{pmatrix} 1 & -1 \\ \alpha_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \alpha_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ d_{2,0} & \alpha_0 \end{pmatrix} \in \mathcal{R}_A$$

$t(f \mid g) = \left( \frac{f}{g}, \frac{\overline{g}}{g} \right)$
PSEUDO-INVERSIONS. II

By induction, we suppose that is true for every \( k \leq n \), this means that \( a_0 = -1 \) and \( a_k = 0 \) for \( 1 \leq k \leq n - 1 \), then, en particular \( d_{k+1,1} = -d_{k,0} \) for \( 1 \leq k \leq n - 1 \). Once again by Riordan involution formula

\[
a_n = (-1)^n(d_{n+1,1} + d_{n,0}) = 0
\]

The product

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 \\
\alpha_0 & -\alpha_0 & \alpha_0 & 0 & \cdots & 0 \\
\frac{1}{2}\alpha_0 & -\frac{1}{2}\alpha_0^2 & -\alpha_2 & \alpha_0 & \cdots & 0 \\
\alpha_2 & -\alpha_2 & -\frac{1}{2}\alpha_0^2 & \alpha_0 & \cdots & 0 \\
\alpha_0\alpha_2 - \frac{1}{8}\alpha_0^4 & -\alpha_2 & \frac{1}{2}\alpha_0^2 & -\alpha_0 & \cdots & 0 \\
\alpha_4 & -\frac{1}{8}\alpha_0^4 + \alpha_0\alpha_2 & -\frac{1}{2}\alpha_0^2 & \alpha_0 & \cdots & 1
\end{pmatrix} I_0^+ \in \mathcal{R}_A
\]

In Riordan group involutions. Linear Algebra Appl. 428 (2008) 941-952. G.-S. Cheon, H. Kim, L.W. Shapiro treat, from a different point of view, this kind of involutions.
If $M$ is an involution then $M = M^{-1}$.

A generalization can be $M = PM^{-1}P^{-1}$, that is, $M$ is not equal to $M^{-1}$ but $M$ is similar to $M^{-1}$.

**Reversible elements**

If an element $M$ of a group is the product of two involutions then is similar to $M^{-1}$. These kind of elements are called *reversible elements*.

If $M = I_1I_2$, then $I_1M_1 = I_2$, $I_1MI_1 = I_2I_1$ and $M^{-1} = I_2I_1$, so $M^{-1} = I_1MI_1$. 

$$T(f | g) = \left( \frac{f}{g}, \frac{g}{g} \right)$$
**Remark**

The pseudo-involutions are a particular case of reversible elements.


Manuel’s talk.

**Theorem Gustafson-Halmos-Radjavi (1976) LAA 13, 157-162**

Every square matrix over a field, with determinant ±1, is the product of not more than four involutions.
THE SUBGROUP $\Omega_0$. I

**Proposition**

If $T(f \mid g)$ is a Riordan involution then $g_2 = 0$, where $g = \sum_{n \geq 0} g_n x^n$.

Let $T(f \mid g)$ be a non-trivial Riordan involution, then $g = A$. Where $A$ is the $A$-sequence of $T(f \mid g)$.

Hence $a_0 = -1$, using the involutions formula and the construction of Riordan matrices we get,

$$a_1 = -\frac{d_{1,0} + d_{2,1}}{d_{0,0}}, \quad d_{2,0} = -\frac{d_{1,0}d_{2,1}}{2d_{0,0}}, \quad d_{3,2} = -d_{1,0} - 2d_{2,1},$$

$$d_{3,1} = -\frac{(d_{1,0} + 2d_{2,1})d_{2,1}}{2d_{0,0}}$$

then

$$a_2 = \frac{1}{d_{2,2}} \left( d_{3,1} - a_0d_{2,0} - a_1d_{2,1} \right) =$$

$$T(f \mid g) = \left( \frac{f}{g}, \frac{a}{g} \right)$$
The subgroup $\Omega_0$. II

$$\frac{1}{d_{0,0}} \left( \frac{(d_{1,0} + 2d_{2,1})d_{2,1}}{2d_{0,0}} - \frac{d_{1,0}d_{2,1}}{2d_{0,0}} + \frac{d_{1,0} + d_{2,1}}{d_{0,0}}d_{2,1} \right) = 0$$

In fact, all Riordan matrices with this condition form a subgroup.

The subgroup $\Omega_0$

If $\Omega_0 = \{ T(f \mid g) \in \mathcal{R}, \mid g_2 = 0 \}$, then $\Omega_0$ is a subgroup of $\mathcal{R}$.

Consider $T(f \mid g) \in \Omega_0$, thus if $g = \sum_{n \geq 0} g_n x^n$, then $g_2 = 0$. If $A = \sum_{n \geq 0} a_n x^n$ is the $A$-sequence of $T(f \mid g)$, we get.

$$T^{-1}(f \mid g)T \left( \frac{1}{f \left( \frac{x}{A} \right)|A} \right) = T(1 \mid 1)$$

then

$$g(x)A \left( \frac{x}{g(x)} \right) = 1$$

$$g(x)A \left( \frac{x}{g(x)} \right) = g(x) \left( a_0 + a_1 \frac{x}{g(x)} + a_2 \left( \frac{x}{g(x)} \right)^2 + O(x^3) \right) = T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right)$$
The subgroup $\Omega_0$. III

$$a_0 g(x) + a_1 x + a_2 \frac{x^2}{g(x)} + O(x^3)$$

then

$$[x^2]g(x)A \left( \frac{x}{g(x)} \right) = \frac{a_2}{g_0}$$

so $a_2 = 0$ and then $T^{-1}(f \mid g) \in \Omega_0$. Suppose that $T(f \mid g), T(l \mid m) \in \Omega_0$. If $g = \sum_{n \geq 0} g_n x^n$ and $m = \sum_{n \geq 0} m_n x^n$, then $g_2 = m_2 = 0$. Since

$$T(f \mid g)T(l \mid m) = T \left( fl \left( \frac{x}{g} \right) \mid gm \left( \frac{x}{g} \right) \right)$$

$$g(x)m \left( \frac{x}{g(x)} \right) = g(x)(m_0 + m_1 \frac{x}{g(x)} + O(x^3)) = m_0 g(x) + m_1 x + O(x^3)$$

then $[x^2]g(x)m \left( \frac{x}{g(x)} \right) = 0$ and $T(f \mid g)T(l \mid m) \in \Omega_0$.

$$T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right)$$
The subgroup $\Omega_0$. IV

Remarks

The group $\Omega_0$ above can be described as the set of Riordan matrices whose $A$-sequences have null quadratic coefficient.

In the usual notation for a Riordan matrix $(d, h)$ with $d_0 \neq 0$, $h_0 = 0$, and $h_1 \neq 0$, the condition $g_2 = 0$ is equivalent to the equality

$$h_2^2 = h_1 h_3$$

where $h = \sum_{n \geq 1} h_n x^n$.

$\Omega_0$ is not a normal subgroup.
Other subgroups.

**Proposition**

- The set \( \{ T(f \mid g) \in \mathcal{R} \mid g_1 = 0 \} \) is a subgroup. \((g_1 = 0 = a_1)\)
- Given \( m \geq 3, \ m \in \mathbb{N} \), the set \( \{ T(f \mid g) \in \mathcal{R} \mid g_m = 0 \} \) is not a subgroup.
- Given \( k \in \mathbb{N}, \ k \geq 1 \), the set \( \{ T(f \mid g) \in \mathcal{R} \mid g_1 = g_2 = \cdots g_k = 0 \} \) is a normal subgroup.

**The group \( \mathcal{K} \)**

\[ \mathcal{K} = \{ I, -I, \mathcal{I}_0^+, \mathcal{I}_0^- \} \]
The commutator of Riordan group I

Observation

\[(d, h) \in \mathcal{R}, \quad (d, h) = \left( \frac{d}{d_0}, \frac{h}{h_1} \right) (d_0, h_1x)\]

The commutator of Riordan group

\[\mathcal{R}, \mathcal{R} = \{(d, h) \in \mathcal{R}, \ / \ d_0 = 1, \ h_1 = 1\}\]

Moreover, every element in \[\mathcal{R}, \mathcal{R}\] is a commutator.

An alternating Lecco’s proof.

\[C = \{(d, h) \in \mathcal{R}, \ / \ d_0 = 1, \ h_1 = 1\}\]

\[\mathcal{R}, \mathcal{R} \subseteq C?\]

\[T(f \mid g) = \left( \frac{f}{g}, \frac{g}{g} \right)\]
If $D \in [\mathcal{R}, \mathcal{R}] \Rightarrow \exists C_1, C_2, \ldots, C_k$ commutators such that

$$D = C_1C_2 \cdots C_k.$$ 

But $C_i$ commutator and triangular then $C_i \in C$ so $D \in C$. Consequently $[\mathcal{R}, \mathcal{R}] \subseteq C$. 

$C \subseteq [\mathcal{R}, \mathcal{R}]$? If $D \in C$ then $D = (d, h)$ with $d_0 = 1$ and $h_1 = 1$. Let $r \in \mathbb{K}$ such that $r \neq 0, r^n \neq 1, \forall n \geq 1$. Exist $(l, m) \in \mathcal{R}$ such that

$$(d, h) = (1, rx)(l, m) \left(1, \frac{x}{r}\right) \left(\frac{1}{l(m^{-1})}, m^{-1}\right)$$

$$(d, h) = \left(\frac{l(rx)}{l \left(m^{-1} \left(\frac{m(rx)}{r}\right)\right)}, m^{-1} \left(\frac{m(rx)}{r}\right)\right)$$

$$h = m^{-1} \left(\frac{m(rx)}{r}\right) \iff m(h) = \frac{m(rx)}{r} \iff (1, h)m = \left(\frac{1}{r}, rx\right)m$$

$$T(f \mid g) = \left(\frac{f}{g}, \frac{f}{g}\right)$$
The commutator of Riordan group III

Consequently for $n \geq 2$

$$m_n = \frac{1}{rn-1 - 1} \sum_{k=1}^{n-1} [x^n] h^k m_k$$

with $m_1 \neq 0$ arbitrary

$$d = \frac{l(rx)}{l\left(\frac{m(rx)}{r}\right)} \quad \Leftrightarrow \quad d = \frac{l(rx)}{l(h)} \quad \Leftrightarrow$$

$$\Leftrightarrow dl(h) = l(rx) \quad \Leftrightarrow \quad (d, h)l = (1, rx)l$$

then

$$l_n = \frac{1}{rn-1} \sum_{k=0}^{n-1} d_{n,k} l_k$$

Hence

$$D \in [\mathcal{R}, \mathcal{R}]$$

Moreover $D$ is a commutator.
Remark

The group generated by involutions in any group is a normal subgroup of such group.

The group $[\mathcal{R}, \mathcal{R}]_0$ is defined as $[\mathcal{R}, \mathcal{R}]_0 = \Omega_0 \cap [\mathcal{R}, \mathcal{R}]$.

$\mathcal{I} \equiv$ the set of all Riordan involutions.

$< \mathcal{I} > \equiv$ the group generated by Riordan involutions.

$T(f \mid g) = \left( \frac{f}{g}, \frac{g}{f} \right)$
The group generated by Riordan involutions II

**Main Theorem**

Every element in the group generated by Riordan involutions is the product of not more than four Riordan involutions.

Note that if

\[ \tilde{D} \in \langle \mathcal{J} \rangle \quad \Rightarrow \quad \tilde{D} = DK \]

where \( D \in [\mathcal{R}, \mathcal{R}]_0 \) and \( K \in \mathcal{K} \).

Suppose now that

\[ D = I_\alpha^+ I_\beta^+ I_\gamma^+ I_0^+ \quad \Rightarrow \quad \tilde{D} = I_\alpha^+ I_\beta^+ I_\gamma^+ \tilde{K} \]

where \( \tilde{K} = I_0^+ K \in \mathcal{K} \).

\[ T(f \mid g) = \left( \frac{f}{g}, \frac{a}{g} \right) \]
The group generated by Riordan involutions

III

In fact,

**Theorem. Actually, lemma**

If \( D \in [\mathcal{R}, \mathcal{R}]_0 \), there are three Riordan involutions, \( I_\alpha^+, I_\beta^+, I_\gamma^+ \) such that

\[
D = I_\alpha^+ I_\beta^+ I_\gamma^+ I_0^+
\]

\[
D = I_\alpha^+ I_\beta^+ I_\gamma^+ I_0^+ \iff DI_0^+ = I_\alpha^+ I_\beta^+ I_\gamma^+
\]

If

\[
DI_0^+ = (d, h), \quad I_\alpha^+ = (\delta_1, \omega_1) \quad I_\beta^+ = (\delta_2, \omega_2) \quad I_\gamma^+ = (\delta_3, \omega_3)
\]

where \( d_0 = 1 \) and \( h_1 = -1 \).

\[
DI_0^+ = I_\alpha^+ I_\beta^+ I_\gamma^+ \iff (d, h) = (\delta_1, \omega_1)(\delta_2, \omega_2)(\delta_3, \omega_3)
\]

\[
T(f \mid g) = \left(\frac{f}{g}, \frac{\omega}{g}\right)
\]
The group generated by Riordan involutions IV

The problem is, given \(d\) and \(h\), are there three involutions such that
\[(d, h) = (\delta_1, \omega_1)(\delta_2, \omega_2)(\delta_3, \omega_3)?\]
\[(d, h) = (\delta_1, \omega_1)(\delta_2, \omega_2)(\delta_3, \omega_3) \iff \begin{cases} 
\delta_1(x)\delta_2(\omega_1(x))\delta_3(\omega_2(\omega_1(x))) = d(x) \\
\omega_3(\omega_2(\omega_1(x))) = h(x) 
\end{cases}\]

First, we solve \(\omega_3(\omega_2(\omega_1(x))) = h(x),\)

Note that if \((\delta_i, \omega_i)\) is an involution, \((1, \omega_i)\) too.
The group generated by Riordan involutions V

\[ \omega_3(\omega_2(\omega_1(x))) = h(x) \iff (1, \omega_1)(1, \omega_2)(1, \omega_3) = h(x) \]
\[ (1, \omega_1)(1, h) = (1, \omega_2)(1, \omega_3) \iff (1, \omega_1)h = (1, \omega_2)\omega_3 \]

Suppose now

\[ (1, \omega_1) = (a_{i,j})_{i,j \in \mathbb{N}}, \quad (1, \omega_2) = (b_{i,j})_{i,j \in \mathbb{N}}, \quad (1, \omega_3) = (c_{i,j})_{i,j \in \mathbb{N}} \]

then \( \omega_3 = \sum_{n \geq 1} c_{n,1}x^n \).

In \( \mathcal{R}_2 \) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & a_{2,1} & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
h_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & b_{2,1} & 1 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
c_{2,1} \\
\end{pmatrix}
\]

Equivalently, \( a_{2,1} - b_{2,1} + c_{2,1} = h_2 \) that has solutions.

\[ T(f \mid g) = (\frac{f}{g}, \frac{a}{g}) \]
The group generated by Riordan involutions

The equation in $\mathcal{R}_3$ is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & a_{2,1} & 1 & 0 \\
0 & -a_{2,1}^2 & -2a_{2,1} & -1
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
h_2 \\
h_3
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & b_{2,1} & 1 & 0 \\
0 & -b_{2,1}^2 & -2b_{2,1} & -1
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
c_{2,1} \\
-c_{2,1}^2
\end{pmatrix}
\]

The above equality is equivalent to the system

\[
\begin{cases}
a_{2,1} - h_2 = b_{2,1} - c_{2,1} \\
a_{2,1}^2 - 2h_2a_{2,1} - h_3 = (b_{2,1} - c_{2,1})^2.
\end{cases}
\]

Since $(1, h) \in \Omega_0$ and $h_1 = -1$ $h_2^2 = h_1h_3 = -h_3$ the system above reduces to unique linear equation

\[a_{2,1} - b_{2,1} + c_{2,1} = h_2\]

which, obviously, has solutions.

\[T(f \mid g) = \left( \frac{f}{g}, \frac{a}{g} \right)\]
THE GROUP GENERATED BY RIORDAN INVOLUTIONS

VII

By induction, suppose now that the equation in \( R_n \) has solution. In \( R_{n+1} \) only a new equation appears

\[
\sum_{k=1}^{n+1} a_{n+1,k} h_k = \sum_{k=1}^{n+1} b_{n+1,k} c_{k,1}
\]

(2)

In the case of \( n \) is odd (2) can be written as

\[
a_{n+1,1} - b_{n+1,1} + c_{n+1,1} = h_{n+1} + \sum_{k=2}^{n} (a_{n+1,k} h_k - b_{n+1,k} c_{k,1})
\]

(3)

By induction hypothesis we have solutions for the case \( n \). Once we fix one of them and using the construction by rows of a Riordan matrix, the right side of equation (3) is known. Moreover, by Involutions Formula \( a_{n+1,1}, b_{n+1,1}, c_{n+1,1} \) are arbitrary to construct the involutions, so (3) has solutions.

In the case of \( n \) is even, to construct the involutions the coefficients \( a_{n+1,1}, b_{n+1,1}, c_{n+1,1} \) are not arbitrary, they depend, in particular, on \( a_{n,1}, b_{n,1} \) and \( c_{n,1} \).

\[
T(f \mid g) = \left( \frac{f}{g}, \frac{g}{f} \right)
\]
So, to be sure of the existence of solutions for \( \mathcal{R}_{n+1} \), assuming that in \( \mathcal{R}_n \), we have to study the compatibility of the system

\[
\begin{align*}
\left\{ \begin{array}{l}
a_{n,1} - b_{n,1} + c_{n,1} = h_n + \sum_{k=2}^{n-1} (a_{n,k} h_k - b_{n,k} c_{k,1}) \\
\sum_{k=1}^{n+1} a_{n+1,k} h_k = \sum_{k=1}^{n+1} b_{n+1,k} c_{k,1}
\end{array} \right.
\end{align*}
\]

where the unknown variables are \( a_{n,1}, b_{n,1} \) and \( c_{n,1} \).

We can reduce the last expression to

\[
\left( 2h_2 - \left( \frac{n}{2} + 1 \right) a_{2,1} \right) a_{n,1} + \left( \left( \frac{n}{2} + 1 \right) b_{2,1} - 2c_{2,1} \right) b_{n,1} + \left( \left( \frac{n}{2} + 1 \right) c_{2,1} - nb_{2,1} \right) c_{n,1} = K
\]

together all equations the system has solution if

\[
a_{2,1} - b_{2,1} + c_{2,1} = h_2 \quad \text{and} \quad a_{2,1} \neq b_{2,1}
\]

Finally, to finish the proof we must find solutions to

\[
\delta_1(x) \delta_2(\omega_1(x)) \delta_3(\omega_2(\omega_1(x))) = d(x)
\]

\[
T(f \mid g) = \left( \frac{f}{g}, \frac{x}{g} \right)
\]
The group generated by Riordan involutions IX

First, some observations: The A-sequence of \((1, \omega_i)\) is the same that \((\delta_i, \omega_i)\). We use that we have solved the above problem and it means that we have \(\omega_1, \omega_2\) and \(\omega_3\) fixed. Moreover, after intensive inspection we conclude that we can choose \(\delta_3 \equiv 1\), so we are going to prove that we can find \(\delta_1\) and \(\delta_2\) such that

\[
\delta_1(x)\delta_2(\omega_1(x)) = d(x)
\]

has solutions. But this equation is equivalent to

\[
(\delta_1, \omega_1)(\delta_2, \omega_2)(1, \omega_3) = d(x) \iff (\delta_2, \omega_2)(1, \omega_3) = (\delta_1, \omega_1)d(x)
\]

Let \((\delta_1, \omega_1) = (u_{i,j})\) and \((\delta_2, \omega_2) = (v_{i,j})\)

To determine \(\delta_i\) it is enough to solve the system for the 0-column. So in \(R_1\) we have

\[
\begin{pmatrix}
1 & 0 \\
u_{1,0} & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
d_{1,0}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
v_{1,0} & -1
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

\[
u_{1,0} - v_{1,0} = d_{1,0}
\]

\[
T(f | g) = \left(\frac{f}{g}, \frac{\omega}{g}\right)
\]
The group generated by Riordan involutions $X$

has solutions. As in an involution the entry $(2,0)$ is given by the formula, in $R_2$ we must check that the following system has solutions. Using the formula and that the $A$-sequences are the same we get

$$
\begin{pmatrix}
\frac{1}{u_{1,0}} & -1 & 1 \\
\frac{v_{1,0}^2 - a_{2,1} u_{1,0}}{2} & a_{2,1} - u_{1,0} & 1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{d_{1,0}} \\
\frac{v_{1,0}^2 - b_{2,1} v_{1,0}}{2} & b_{2,1} - v_{1,0} & 1 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
\end{pmatrix}
$$

this is equivalent to

$$
\begin{cases}
u_{1,0} - v_{1,0} = d_{1,0} \\
-b_{2,1} u_{1,0} + (2b_{2,1} - a_{2,1})v_{1,0} = 2d_{2,0} - d_{1,0}^2
\end{cases}
$$

that has solution if $a_{2,1} \neq b_{2,1}$.

Then by induction, if $n$ is even the entries in the place $(n + 1, 0)$ is arbitrary so

$$u_{n+1,0} - v_{n+1,0} = -\sum_{k=1}^{n} u_{n+1,k} d_{k,0} - d_{n+1,0}$$

$T(f \mid g) = \left(\frac{f}{g}, \frac{x}{g}\right)$
The group generated by Riordan involutions

XI

has solutions.
If \( n \) is odd, in a similar development than before, we get solutions.

**Remark**

\([\mathcal{R}, \mathcal{R}]_0\) is normal in the group generated by involutions.

**Corollary**

\( \langle \mathcal{I} \rangle \simeq [\mathcal{R}, \mathcal{R}]_0 \rtimes \mathcal{K} \)
Thank you!

See you in Madrid next RART!