ON THE HARNACK INEQUALITY FOR QUASILINEAR ELLIPTIC EQUATIONS WITH A FIRST ORDER TERM

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ABSTRACT. We consider weak solutions to

$$-\Delta_p u + a(x, u)|\nabla u|^q = f(x, u),$$

with $p > 1$, $q \geq \max\{p - 1, 1\}$. We exploit the Moser iteration technique to prove a Harnack comparison inequality for C^1 weak solutions. As a consequence we deduce a strong comparison principle.

1. INTRODUCTION

Let us consider positive weak $C_{loc}^1(\Omega)$ solutions to the problem

$$(1.1) \quad -\Delta_p u + a(x, u)|\nabla u|^q = f(x, u) \quad \text{in } \Omega,$$

where $p > 1$, $q \geq \max\{p - 1, 1\}$, Ω is a domain in $\mathbb{R}^N$ and $N \geq 2$. It is well known that the $C_{loc}^{1,\alpha}$ -regularity of the solutions is natural when dealing with such problems (see [4, 12, 13, 14, 15, 22, 23, 24]). The functions $a(x, s)$ and $f(x, s)$ obey to the set of suitable assumption (hp^*) detailed in Section 2. Let us emphasize that we mainly need the source term $f(x, s)$ to be positive in order to apply our technique.

We exploit the Moser iteration technique to derive a Harnack comparison inequality. Actually the method that we use is one developed by Trudinger in [25] to study a degenerate class of operators in weighted Sobolev spaces. We deduce both the weak and the strong Harnack comparison inequality and we may resume our main result in the following

Theorem 1.1. (*Harnack Comparison Inequality*). *Let $p > (2N + 2)/(N + 2)$ and let $u, v \in C_{loc}^1(\Omega)$ with u or v weak solution to (1.1) in Ω . Let $q \geq \max\{p - 1, 1\}$, and assume that $f(x, u), a(x, u)$ fulfill (hp^*) . Suppose that $\overline{B(x, 6\delta)} \subset \Omega' \subset \Omega$ for some $\delta > 0$ and that*

$$u \leq v \quad \text{in } B(x, 6\delta).$$

Then there exists $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')}) > 0$ such that

$$(1.2) \quad \sup_{B(x, \delta)} (v - u) \leq C \inf_{B(x, 2\delta)} (v - u).$$

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Our problem is related to the study of Trudinger [25] mainly because of the fact that it can be studied in weighted Sobolev spaces and the natural weight is the weight $\rho = |\nabla u|^{p-2}$ (or $\rho = (|\nabla u| + |\nabla v|)^{p-2}$) which is degenerate ($p > 2$) or singular ($p < 2$) on the critical set

$$Z_u := \{x \in \Omega \mid \nabla u(x) = 0\}.$$

In the singular case $1 < p < 2$ the condition $p > (2N+2)/(N+2)$ provides integrability properties of the weight (see [2]). It is worth emphasizing that the weight is not in L^1 in general if p is close to one. The problem without first order terms have been studied in [3, 20] and the same result in our case is somehow expected. Some related problems are studied in [9]. Our main effort is to obtain such a Harnack type inequality under suitable general assumptions, having in mind possible applications in the study of qualitative properties of the solutions. An important consequence of the Harnack comparison inequality is in fact the strong comparison principle for (1.1) that we point out in the following

Theorem 1.2. (*Strong Comparison Principle*) *Let $p > (2N+2)/(N+2)$ and $u, v \in C_{loc}^1(\Omega)$ with either u or v weak solution to (1.1). Assume that $q \geq \max\{p-1, 1\}$ and assume that $f(x, u), a(x, u)$ fulfill (hp^*) . Then, if*

$$-\Delta_p u + a(x, u)|\nabla u|^q - f(x, u) \leq -\Delta_p v + a(x, v)|\nabla v|^q - f(x, v), \quad u \leq v \quad \text{in } \Omega$$

in the weak distributional meaning, it follows that

$$u < v \quad \text{in } \Omega$$

unless $u \equiv v$ in Ω .

The strong comparison principle for p -Laplace equations is a very delicate issue and manly still unsolved. Actually it is not hard to derive it far from the critical set, see e.g. [1, 19] but it remains an open problem already for p -harmonic functions (see [8]) near critical points. So our result is crucial in particular to work in regions where the gradient of the solutions vanishes. In particular we are motivated by the possible applications in many issues and in particular in the study of qualitative properties of the solutions. We refer in particular to the papers [5, 11, 16] where it is clear that the strong principle can simplify the proofs and improve the results.

The paper is organized as follows: In Section 2 we recall some preliminary results. The proof of the main result is a consequence of the results in Section 3 where we prove the weak and the strong Harnack Comparison Inequality.

2. PRELIMINARY RESULTS

In this section we start recalling some useful regularity results about solutions to problem (1.1). Through all the paper, generic fixed and numerical constants will be denoted by C (with subscript or superscript in some case) and they will be allowed to vary within a single line or formula. We assume that $a = a(x, u)$ and $f = f(x, u)$ satisfy the following hypotheses (denoted by (hp^*) in the sequel):

- (hp^*) $- a(x, \cdot) \in C^1(\Omega' \times [0, +\infty))$ for any $\Omega' \subset \Omega$.
- $- f(x, \cdot)$ is positive and, more precisely, $f(x, s) > 0$ in Ω' for every $\Omega' \subset \Omega$ and for every $s > 0$.

– $a(x, \cdot)$ and $f(x, \cdot)$ are locally Lipschitz continuous, uniformly w.r.t. x . Namely, for every $\Omega' \subset \Omega$ and for every $M > 0$, there is a positive constant $L = L(M, \Omega')$ such that for every $x \in \Omega'$ and every $u, v \in [0, M]$ we have:

$$|a(x, u) - a(x, v)| \leq L|u - v|, \quad |f(x, u) - f(x, v)| \leq L|u - v|.$$

For the reader's convenience (and since in the sequel we will use the hypotheses (hp^*) also in this form), we remark that the hypotheses (hp^*) imply the following: for every $\Omega' \subset \Omega$ and for every $M > 0$, there exists $K = K(M, \Omega') > 0$, such that for every $x \in \Omega'$ and every $s \in [0, M]$ we have

$$|a(x, s)| \leq K.$$

By standard regularity results, see [4, 10, 24], the solutions to problems involving the $-\Delta_p(\cdot)$ operator, (and under suitable hypotheses) are in general of class $C^{1,\alpha}$. This fact leads to the study of the summability properties of the second derivatives of the solutions that turns out to be crucial in our results. We recall a regularity result in [2, 21], see also [17] for the case of equations with first order terms.

Theorem 2.1. *Let $1 < p < \infty$ and consider $u \in C_{loc}^{1,\alpha}(\Omega)$ a weak solution to (1.1), with $a(x, \cdot)$ and $f(x, \cdot)$ satisfying (hp^*) . Denoting $u_i = \frac{\partial u}{\partial x_i}$, for any $\Omega' \subset \Omega'' \subset \Omega$, we have*

$$(2.1) \quad \int_{\Omega'} \frac{|\nabla u|^{p-2-\beta} |\nabla u_i|^2}{|x - y|^\gamma} dx \leq C \quad \forall i = 1, \dots, N,$$

uniformly for any $y \in \Omega'$, with

$$C := C\left(a, f, p, q, \beta, \gamma, \|u\|_{L^\infty(\Omega'')}, \|\nabla u\|_{L^\infty(\Omega'')}\right),$$

for any $0 \leq \beta < 1$ and $\gamma < (N - 2)$ if $N \geq 3$, or $\gamma = 0$ if $N = 2$.

Moreover, if $f(x, \cdot)$ is positive in Ω'' , then it follows that

$$(2.2) \quad \int_{\Omega'} \frac{1}{|\nabla u|^{r(p-1)}} \frac{1}{|x - y|^\gamma} dx \leq C^*,$$

uniformly for any $y \in \Omega'$, with

$$C^* := C^*\left(a, f, p, q, r, \gamma, \|u\|_{L^\infty(\Omega'')}, \|\nabla u\|_{L^\infty(\Omega'')}\right),$$

for any $r < 1$ and $\gamma < (N - 2)$ if $N \geq 3$, or $\gamma = 0$ if $N = 2$.

We refer to [2, 17] for a detailed proof. Note that, by (2.2), it follows in particular that the critical set Z_u has zero Lebesgue measure provided that $f(x, \cdot)$ is positive.

We now recall that, for $\rho \in L^1(\Omega)$ and $1 \leq s < \infty$, the space $H_\rho^{1,s}(\Omega)$ is defined as the completion of $C^1(\Omega)$ (or $C^\infty(\Omega)$) with the norm

$$(2.3) \quad \|v\|_{H_\rho^{1,s}} = \|v\|_{L^s(\Omega)} + \|\nabla v\|_{L^s(\Omega, \rho)},$$

where

$$\|\nabla v\|_{L^s(\Omega, \rho)}^s := \int_\Omega \rho(x) |\nabla v(x)|^s dx.$$

We also have that $H_\rho^{1,s}(\Omega)$ may be equivalently defined as the space of functions with distributional derivatives represented by a function for which the norm defined in (2.3) is bounded. The space

$H_{0,\rho}^{1,s}(\Omega)$ is consequently defined as the closure of $C_c^1(\Omega)$ (or $C_c^\infty(\Omega)$), w.r.t. the norm (2.3). We refer to [2] for more details about weighted Sobolev spaces and also to [8, Chapter 1] and the references therein.

Theorem 2.1 provides the right summability of the weight $|\nabla u(x)|^{p-2}$ in order to obtain a weighted Poincaré-Sobolev type inequality that will be useful in the sequel. For the proof we refer to [2, Section 3].

Theorem 2.2 (Weighted Poincaré-Sobolev type inequality). *Let $p \geq 2$ and let $u \in C_{loc}^{1,\alpha}(\Omega)$ be a solution to (1.1) with $a(x, \cdot)$ and $f(x, \cdot)$ satisfying (hp^*) . For any $\Omega' \subset \Omega$, setting $\rho = |\nabla u|^{p-2}$, we have that $H_0^{1,2}(\Omega', \rho)$ is continuously embedded in $L^q(\Omega')$ for $1 \leq q < \bar{2}_p$, where*

$$\frac{1}{\bar{2}_p} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1} \frac{1}{N}.$$

Consequently, since $\bar{2}_p > 2$, for $w \in H_0^{1,2}(\Omega', \rho)$ we have

$$(2.4) \quad \|w\|_{L^2(\Omega')} \leq C_p \|\nabla w\|_{L^2(\Omega', \rho)} = C_p \left(\int_{\Omega'} \rho |\nabla w|^2 \right)^{\frac{1}{2}},$$

with $C_p = C_p(\Omega') \rightarrow 0$ if $|\Omega'| \rightarrow 0$.

Theorem 2.2 follows from Theorem 2.1, see [2]. The proof it is based on potential estimates, see [7, Lemmas 7.14, 7.16]. Since potential estimates are also available for functions with zero mean, we can prove Theorem 2.4 for functions with zero mean (we will refer to it below in the paper), in particular see [6, Theorem 8, Corollary 2]. Moreover in the paper we will use the fact that, since

$$0 \leq \frac{1}{(|\nabla u| + |\nabla v|)^{r(p-1)}} \frac{1}{|x-y|^\gamma} \leq \frac{1}{|\nabla u|^{r(p-1)}} \frac{1}{|x-y|^\gamma},$$

from (2.2) it readily follows

$$\int_{\Omega} \frac{1}{(|\nabla u| + |\nabla v|)^{r(p-1)}} \frac{1}{|x-y|^\gamma} dx \leq C^*.$$

Therefore, in particular, thanks to Theorem 2.2 we also have a weighted Poincaré-Sobolev inequality with weight $\rho = (|\nabla u| + |\nabla v|)^{p-2}$, for any $v \in C_{loc}^1(\Omega)$. Finally notice that Theorem 2.2 holds for $p \geq 2$. Anyway, if $1 < p < 2$ and $|\nabla u|$ is bounded, the weighted Poincaré-Sobolev inequality (2.4) follows at once by the classic Poincaré-Sobolev inequality.

In the sequel we use the following standard estimates, whose proof can be found e.g. in [1].

Lemma 2.3. *$\forall p > 1$, there exist positive constants $\hat{C}, \check{C}$, depending on p , such that $\forall \eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$*

$$(2.5) \quad [|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] \geq \hat{C}(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2$$

and

$$[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'] \leq \check{C}(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|.$$

3. THE HARNACK COMPARISON INEQUALITY

In this section we show the steps needed to prove a strong Harnack inequality. We start with the following results:

Theorem 3.1. (*Weak Harnack Comparison Inequality*) Let $u, v \in C_{loc}^1(\Omega)$ and assume that either u or v is a weak solution to (1.1), with $q \geq \max\{p-1, 1\}$ and $f(x, u), a(x, u)$ satisfying (hp^*) . Assume that $\overline{B(x, 6\delta)} \subset \Omega' \subset \Omega$ for some $\delta > 0$ and that

$$(3.1) \quad -\Delta_p u + a(x, u)|\nabla u|^q - f(x, u) \leq -\Delta_p v + a(x, v)|\nabla v|^q - f(x, v), \quad u \leq v \quad \text{in } B(x, 6\delta).$$

We distinguish the two cases:

- Case (a) : $p \geq 2$. Define

$$\frac{1}{2_p} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1} \frac{1}{N}.$$

Then, for every

$$0 < s < \frac{2_p}{2},$$

there exists $C > 0$ such that

$$(3.2) \quad \|(v - u)\|_{L^s(B(x, 2\delta))} \leq C \inf_{B(x, \delta)} (v - u)$$

where $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')})$.

- Case (b) : $(2N+2)/(N+2) < p < 2$. Define

$$\frac{1}{t^\sharp} = \frac{2p-3}{2(p-1)}$$

and let 2^* be the classical Sobolev exponent $2^* = 2N/(N-2)$. Then, for every

$$0 < s < \frac{2^*}{t^\sharp},$$

there exists $C > 0$ such that

$$(3.3) \quad \|(v - u)\|_{L^s(B(x, 2\delta))} \leq C \inf_{B(x, \delta)} (v - u)$$

where $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')})$.

The proof is based on the Moser iteration scheme, see [18]. Actually we exploit here the improved technique due to Trudinger [25] which is only based on weighted Sobolev inequalities and avoid the use of John-Nirenberg's Lemma.

Proof. Let us first note that, since we are assuming that $u, v \in C_{loc}^1(\Omega)$, we need to work in a sub-domain Ω' . To simplify the notation we relabel it as Ω in all the proof. Let us consider the function $w_\tau = v - u + \tau$, where $\tau > 0$ in Ω (at the end we will let $\tau \rightarrow 0$). Moreover let us define the function $\phi_\tau = \eta^2 w_\tau^\beta$ with $\beta < 0$ and $\eta \in C_0^1(B(x, 6\delta))$. Then it follows

$$(3.4) \quad \nabla \phi_\tau = 2\eta w_\tau^\beta \nabla \eta + \beta \eta^2 w_\tau^{\beta-1} \nabla w_\tau.$$

Using ϕ_τ positive as a test function in (3.1), it follows

$$(3.5) \quad \begin{aligned} & \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \phi_\tau) dx \\ & \leq \int_{\Omega} (a(x, v) |\nabla v|^q - a(x, u) |\nabla u|^q) \phi_\tau dx + \int_{\Omega} (f(x, u) - f(x, v)) \phi_\tau dx. \end{aligned}$$

Case (a): $p \geq 2$. Taking into account (3.4), the first term in (3.5) can be written as

$$\begin{aligned} & \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \phi_\tau) dx \\ & = \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla w_\tau) \beta \eta^2 w_\tau^{\beta-1} dx \\ & + 2 \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \eta) \eta w_\tau^\beta dx. \end{aligned}$$

Using the inequalities (2.5), from (3.5) it follows

$$(3.6) \quad \begin{aligned} & \hat{C}|\beta| \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla(u-v)|^2 \eta^2 w_\tau^{\beta-1} dx \\ & \leq \int_{\Omega} (a(x, v) |\nabla v|^q - a(x, u) |\nabla u|^q) \phi_\tau dx + \int_{\Omega} (f(x, u) - f(x, v)) \phi_\tau dx \\ & + 2 \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \eta) \eta w_\tau^\beta dx. \end{aligned}$$

Applying the second inequality of (2.5) and Young's inequality in the last term of the previous expression, we obtain

$$(3.7) \quad \begin{aligned} & 2 \left| \int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla \eta) \eta w_\tau^\beta dx \right| \\ & \leq 2\check{C} \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla(u-v)| |\nabla \eta| \eta w_\tau^\beta dx \\ & \leq 2\check{C} \int_{\Omega} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} |\nabla(u-v)| |\nabla \eta| \eta w_\tau^{\frac{\beta-1}{2}} w_\tau^{\frac{\beta+1}{2}} dx \\ & \leq 2\check{C}\varepsilon \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla(u-v)|^2 \eta^2 w_\tau^{\beta-1} + \frac{2\check{C}}{\varepsilon} \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla \eta|^2 w_\tau^{\beta+1} dx. \end{aligned}$$

Therefore using (3.7), setting $\rho = (|\nabla u| + |\nabla v|)^{p-2}$ and choosing $\varepsilon = \hat{C}|\beta|/4\check{C}$, (3.7) becomes

$$(3.8) \quad \begin{aligned} & \frac{\hat{C}|\beta|}{2} \int_{\Omega} \rho |\nabla(u-v)|^2 \eta^2 w_\tau^{\beta-1} dx \\ & \leq \int_{\Omega} (a(x, v) |\nabla v|^q - a(x, u) |\nabla u|^q) \phi_\tau dx + \int_{\Omega} (f(x, u) - f(x, v)) \phi_\tau dx \\ & + \frac{8\check{C}^2}{\hat{C}|\beta|} \int_{\Omega} \rho |\nabla \eta|^2 w_\tau^{\beta+1} dx. \end{aligned}$$

We estimate now the first two terms on the right-hand side of (3.8). In particular, using hypotheses (hp^*) and the mean value theorem, for the first term we have that

$$\begin{aligned}
(3.9) \quad & \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, u)|\nabla u|^q) \phi_{\tau} dx = \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, u)|\nabla u|^q) \eta^2 w_{\tau}^{\beta} dx \\
& = \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, v)|\nabla u|^q) \phi_{\tau} dx + \int_{\Omega} (a(x, v) - a(x, u)) |\nabla u|^q \phi_{\tau} dx \\
& \leq K(\|v\|_{L^{\infty}(\Omega)}) \int_{\Omega} (|\nabla v|^q - |\nabla u|^q) \eta^2 w_{\tau}^{\beta} dx + \int_{\Omega} (a(x, v) - a(x, u)) |\nabla u|^q \phi_{\tau} dx \\
& \leq q \int_{\Omega} (|\nabla u| + |\nabla v|)^{q-1} |\nabla(v-u)| \eta^2 w_{\tau}^{\beta} dx + L(\|v\|_{L^{\infty}(\Omega)}) \cdot \|\nabla u\|_{L^{\infty}(\Omega)}^q \int_{\Omega} (v-u) \phi_{\tau} dx \\
& \leq q \int_{\Omega} (|\nabla u| + |\nabla v|)^{\frac{p-2}{2}} |\nabla(v-u)| \eta w_{\tau}^{\frac{\beta-1}{2}} \eta w_{\tau}^{\frac{\beta+1}{2}} \frac{(|\nabla u| + |\nabla v|)^{q-1}}{(|\nabla u| + |\nabla v|)^{\frac{p-2}{2}}} dx \\
& \quad + L(\|v\|_{L^{\infty}(\Omega)}) \cdot \|\nabla u\|_{L^{\infty}(\Omega)}^q \int_{\Omega} (v-u + \tau) \phi_{\tau} dx \\
& \leq \varepsilon q \int_{\Omega} \rho |\nabla(v-u)|^2 \eta^2 w_{\tau}^{\beta-1} dx + \frac{C(p, q, \|\nabla u\|_{L^{\infty}(\Omega)}, \|\nabla v\|_{L^{\infty}(\Omega)})}{\varepsilon} \int_{\Omega} \eta^2 w_{\tau}^{\beta+1} dx \\
& \quad + C(L, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}) \int_{\Omega} \eta^2 w_{\tau}^{\beta+1} dx,
\end{aligned}$$

where in the last line we applied the Young's inequality and we used the fact that $q \geq p/2$. For the second term on the right-hand side of (3.8), using (hp^*) we obtain that

$$\begin{aligned}
(3.10) \quad & \int_{\Omega} (f(x, u) - f(x, v)) \phi_{\tau} dx \\
& \leq \int_{\Omega} |(f(x, u) - f(x, v))| \phi_{\tau} dx \leq L(\|v\|_{L^{\infty}(\Omega)}) \int_{\Omega} (v-u + \tau) \phi_{\tau} dx \\
& \leq L(\|v\|_{L^{\infty}(\Omega)}) \int_{\Omega} \eta^2 w_{\tau}^{\beta+1} dx.
\end{aligned}$$

Choosing $\varepsilon = \frac{\hat{C}|\beta|}{4q}$ and using (3.9) and (3.10), from (3.8) we have

$$\begin{aligned}
& \frac{\hat{C}|\beta|}{4} \int_{\Omega} \rho |\nabla(v-u)|^2 \eta^2 w_{\tau}^{\beta-1} dx \\
& \leq \frac{C(p, q, \|\nabla u\|_{L^{\infty}(\Omega)}, \|\nabla v\|_{L^{\infty}(\Omega)})}{|\beta|} \int_{\Omega} \eta^2 w_{\tau}^{\beta+1} dx \\
& \quad + C(L, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}) \int_{\Omega} \eta^2 w_{\tau}^{\beta+1} dx + \frac{C(p)}{|\beta|} \int_{\Omega} \rho |\nabla \eta|^2 w_{\tau}^{\beta+1} dx.
\end{aligned}$$

Therefore, up to redefining constants, we obtain

$$(3.11) \quad \int_{\Omega} \rho |\nabla w_{\tau}|^2 \eta^2 w_{\tau}^{\beta-1} dx \leq \frac{\dot{C}}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) \int_{\Omega} w_{\tau}^{\beta+1} (\eta^2 + \rho |\nabla \eta|^2) dx,$$

where $\dot{C} = \dot{C}(p, q, L, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}, \|\nabla v\|_{L^{\infty}(\Omega)})$ is a positive constant.

Let us now set

$$(3.12) \quad \tilde{w}_\tau = \begin{cases} w_\tau^{\frac{\beta+1}{2}} & \text{if } \beta \neq -1, \\ \log(w_\tau) & \text{if } \beta = -1 \end{cases}$$

and

$$(3.13) \quad r := \beta + 1.$$

Then, if $\beta \neq -1$, by (3.12) it follows

$$(3.14) \quad \int_{\Omega} \rho \eta^2 |\nabla \tilde{w}_\tau|^2 dx = \int_{\Omega} \rho \eta^2 \left| \nabla (w_\tau)^{\frac{\beta+1}{2}} \right|^2 = \left(\frac{\beta+1}{2} \right)^2 \int_{\Omega} \rho \eta^2 w_\tau^{\beta-1} |\nabla w_\tau|^2 dx$$

and taking into account (3.11), it follows

$$(3.15) \quad \int_{\Omega} \rho \eta^2 |\nabla \tilde{w}_\tau|^2 dx \leq \dot{C} \left(\frac{\beta+1}{2} \right)^2 \frac{1}{|\beta|} \left(1 + \frac{1}{|\beta|} \right) \int_{\Omega} \tilde{w}_\tau^2 (\eta^2 + \rho |\nabla \eta|)^2 dx.$$

Note that, since

$$\frac{1}{(|\nabla u| + |\nabla v|)^{p-2}} \leq \frac{1}{|\nabla u|^{p-2}} \quad \text{and} \quad \frac{1}{(|\nabla u| + |\nabla v|)^{p-2}} \leq \frac{1}{|\nabla v|^{p-2}},$$

then the weight ρ satisfies the same properties of $|\nabla u|^{p-2}$ and $|\nabla v|^{p-2}$. Therefore since either u or v is a solution to (1.1), a weighted Sobolev inequality is available in this case (see Theorem 2.2). Hence, since we can assume that $2_p > 2$, let $2 < \nu < \bar{2}_p$. Using Theorem 2.2 (since $\eta \tilde{w}_\tau \in H_0^{1,2}(\Omega, \rho)$), it follows

$$\begin{aligned} \|\eta \tilde{w}_\tau\|_{L^\nu(\Omega)}^2 &\leq C_p \int_{\Omega} \rho |\nabla \eta \tilde{w}_\tau + \eta \nabla \tilde{w}_\tau|^2 dx \\ &\leq 2C_p \int_{\Omega} \rho \tilde{w}_\tau^2 |\nabla \eta|^2 + \rho \eta^2 |\nabla \tilde{w}_\tau|^2 dx. \end{aligned}$$

Using now (3.15) (together with (3.12) and (3.13)), it follows

$$(3.16) \quad \begin{aligned} \|\eta \tilde{w}_\tau\|_{L^\nu(\Omega)}^2 &\leq \dot{C} \frac{r^2}{|\beta|} \left(1 + \frac{1}{|\beta|} \right) \int_{\Omega} \tilde{w}_\tau^2 (\eta^2 + \rho |\nabla \eta|^2) dx \\ &\leq \dot{C} \frac{r^2}{|\beta|} \left(1 + \frac{1}{|\beta|} \right) \|\tilde{w}_\tau (\eta + |\nabla \eta|)\|_{L^2(\Omega)}^2, \end{aligned}$$

up to redefine the constant $\dot{C}$. Moreover we note that the quantity

$$\frac{1}{|\beta|} \left(1 + \frac{1}{|\beta|} \right)$$

is bounded if $|\beta| \geq C > 0$ and then, from now on, it will be included in the constant $\dot{C}$.

Consider now $\delta \leq h' \leq h'' \leq 5\delta$ and let us suppose $\eta \equiv 1$ in $B(x, h')$ and $\eta \equiv 0$ outside $B(x, h'')$ with $|\nabla \eta| \leq \frac{2}{h'' - h'}$. Taking into account these assumptions (3.16) becomes

$$(3.17) \quad \begin{aligned} \|\tilde{w}_\tau\|_{L^\nu(B(x, h'))} &\leq \dot{C} |r| \|\tilde{w}_\tau (\eta + |\nabla \eta|)\|_{L^2(B(x, h''))} \\ &\leq \dot{C} \frac{|r|}{h'' - h'} \|\tilde{w}_\tau\|_{L^2(B(x, h''))}. \end{aligned}$$

Set $\chi = \nu/2$ and notice that $\chi > 1$. Considering $0 < r < 1$ (i.e. $-1 < \beta \leq C < 0$), it follows

$$(3.18) \quad \left(\int_{B(x, h')} (w_\tau)^{\frac{\nu(\beta+1)}{2}} dx \right)^{\frac{1}{\chi \cdot r}} = \left(\int_{B(x, h')} w_\tau^{\chi \cdot r} dx \right)^{\frac{1}{\chi \cdot r}} \\ \leq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \left(\int_{B(x, h'')} w_\tau^r dx \right)^{\frac{1}{r}}.$$

Defining now the functional

$$(3.19) \quad \phi(p, r, v) = \left(\int_{B(x, r)} |v|^p dx \right)^{\frac{1}{p}}.$$

Then for $0 < r < 1$ (and in general for $r > 0$ for future use), it follows from (3.18) that

$$(3.20) \quad \phi(\chi r, h', w_\tau) \leq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \phi(r, h'', w_\tau).$$

If instead $r < 0$ (i.e. $\beta < -1$), using (3.17) and arguing as above, we have

$$(3.21) \quad \phi(\chi r, h', w_\tau) \geq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \phi(r, h'', w_\tau).$$

We exploit now the Moser's iterative technique, see [18]. For $r_0 > 0$ given, we define

$$(3.22) \quad r_k = (-r_0)\chi^k \quad \text{and} \quad h_k = \delta \left(1 + \frac{3}{2} \left(\frac{1}{2} \right)^k \right).$$

It follows that $r_k \rightarrow -\infty$ and $\beta_k := r_k - 1 \rightarrow -\infty$. Moreover

$$h_0 = \frac{5\delta}{2} \quad \text{and} \quad h_k \rightarrow \delta \quad \text{as } k \rightarrow +\infty$$

and

$$h_k - h_{k+1} = \frac{3}{2} \frac{\delta}{2^{k+1}}.$$

Using these definitions, we iterate the expression of $\phi(\cdot, \cdot, \cdot)$ in (3.21) obtaining

$$(3.23) \quad \phi(r_{k+1}, h_{k+1}, w_\tau) = \phi(\chi r_k, h_{k+1}, w_\tau) \geq \frac{(\dot{C}|r_k|)^{\frac{2}{r_k}}}{(h_k - h_{k+1})^{\frac{2}{r_k}}} \phi(r_k, h_k, w_\tau) \\ = \frac{(\dot{C}|r_0|\chi^k)^{\frac{2}{(-r_0)\chi^k}}}{\left(\frac{3}{2} \frac{\delta}{2^{k+1}} \right)^{\frac{2}{(-r_0)\chi^k}}} \phi(r_k, h_k, w_\tau) \\ = \left(\dot{C}^{\frac{2}{-r_0}} \right)^{\frac{1}{\chi^k}} \left[(|r_0|\chi^k)^{\frac{2}{-r_0}} \right]^{\frac{1}{\chi^k}} \left[\left(\frac{3}{2} \frac{\delta}{2^{k+1}} \right)^{\frac{2}{r_0}} \right]^{\frac{1}{\chi^k}} \phi(r_k, h_k, w_\tau).$$

Reiterating (3.23) and collecting terms we get

$$(3.24) \quad \phi(r_{k+1}, h_{k+1}, w_\tau) \geq \dot{C}^{\sum_{k \geq 0} \frac{1}{\chi^k}} (2\chi)^{\frac{2}{-r_0} \cdot \sum_{k \geq 0} \frac{k}{\chi^k}} \delta^{\frac{2}{r_0} \cdot \sum_{0 \leq j \leq k} \frac{1}{\chi^j}} \phi(-r_0, \frac{5\delta}{2}, w_\tau).$$

We remark that in (3.24) we have redefined the constant $\dot{C}$. Since, by definition $\chi > 1$, the series converge and it is possible to find a positive constant $C = C(p, q, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ such that

$$(3.25) \quad \phi(-\infty, \delta, w_\tau) \geq C\phi(-r_0, \frac{5\delta}{2}, w_\tau).$$

We are going to suppose now (we will prove it later) that there exists $r_0 > 0$ and a constant $C > 0$ such that

$$(3.26) \quad \phi(r_0, \frac{5\delta}{2}, w_\tau) \leq C\phi(-r_0, \frac{5\delta}{2}, w_\tau).$$

For $0 < s \leq r_0$, using Hölder inequality, we have that

$$(3.27) \quad \begin{aligned} \phi(s, \frac{5\delta}{2}, w_\tau) &= \left(\int_{B(x, \frac{5\delta}{2})} (w_\tau)^s \right)^{\frac{1}{s}} \leq \left(\int_{B(x, \frac{5\delta}{2})} (w_\tau)^{r_0} \right)^{\frac{1}{r_0}} \left| B(x, \frac{5\delta}{2}) \right|^{\frac{r_0-s}{r_0}} \\ &\leq C(r_0, s, \delta) \phi(r_0, \frac{5\delta}{2}, w_\tau). \end{aligned}$$

Using (3.25), (3.26), (3.27) and the fact that $\phi(s, \frac{5\delta}{2}, w_\tau) \geq \phi(s, 2\delta, w_\tau)$, we get

$$(3.28) \quad \phi(-\infty, \delta, w_\tau) \geq C\phi(s, 2\delta, w_\tau),$$

with $C = C(p, q, L, \Omega, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ a positive constant. Therefore, taking the limit for τ that goes to zero in (3.28), since

$$(3.29) \quad \phi(-\infty, \delta, w_\tau) = \inf_{B(x, \delta)} w_\tau,$$

then (3.3) follows for $0 < s \leq r_0$.

If instead, $r_0 < s < \chi$, we take a finite number of iterations in (3.20). In this case we define

$$(3.30) \quad r_1 = \frac{s}{\chi^{k_0+1}} \leq r_0,$$

for a natural number k_0 large enough.

We want to point out that, in order to apply (3.20), we need to choose $r < 1$ in such formula. For example in the first iteration of (3.20) (see also equation (3.31) below) we set $r_{k_0+1} = (r_1 \chi^{k_0}) \chi$. Then we need to impose $r_1 \chi^{k_0} < 1$. This holds (as it can be proved using the definition (3.30)) for $r_0 < s < \chi$. Such condition is also sufficient for the other steps.

Hence we consider, for $k = 0, \dots, k_0 + 1$, the values

$$r_k = r_1 \chi^k$$

and

$$h_0 = \frac{5\delta}{2} > h_1 > \dots > h_{k_0+1} = 2\delta.$$

Using these assumptions, we can iterate (3.20) k_0 times, obtaining

$$(3.31) \quad \phi(s, 2\delta, w_\tau) \leq C\phi(r_1, \frac{5\delta}{2}, w_\tau).$$

Since $0 < r_1 \leq r_0$, we claim that (3.26) holds for r_0 replaced by r_1 . In fact we readily have

$$(3.32) \quad \phi(r_1, \frac{5\delta}{2}, w_\tau) \leq \phi(r_0, \frac{5\delta}{2}, w_\tau) \leq C\phi(-r_0, \frac{5\delta}{2}, w_\tau).$$

Moreover, using Hölder inequality, we also obtain

$$\left(\int_{B(x, \frac{5\delta}{2})} \left(\frac{1}{w_\tau} \right)^{r_1} \right)^{\frac{1}{r_1}} \leq C(r_0, r_1, \delta) \left(\int_{B(x, \frac{5\delta}{2})} \left(\frac{1}{w_\tau} \right)^{r_0} \right)^{\frac{1}{r_0}},$$

namely

$$\phi(-r_1, \frac{5\delta}{2}, w_\tau) = \left(\int_{B(x, \frac{5\delta}{2})} \left(\frac{1}{w_\tau} \right)^{r_1} \right)^{-\frac{1}{r_1}} \geq C(r_0, r_1, \delta) \left(\int_{B(x, \frac{5\delta}{2})} \left(\frac{1}{w_\tau} \right)^{r_0} \right)^{-\frac{1}{r_0}}.$$

Including the last expression in (3.32), it follows

$$\phi(r_1, \frac{5\delta}{2}, w_\tau) \leq \phi(r_0, \frac{5\delta}{2}, w_\tau) \leq C\phi(-r_0, \frac{5\delta}{2}, w_\tau) \leq C\phi(-r_1, \frac{5\delta}{2}, w_\tau),$$

that is our claim. Therefore, taking into account also (3.31), we obtain

$$(3.33) \quad \phi(s, 2\delta, w_\tau) \leq C\phi(-r_1, \frac{5\delta}{2}, w_\tau).$$

Arguing exactly as above, also (3.25) can be obtained with r_0 replaced by r_1 getting

$$(3.34) \quad \phi(-\infty, \delta, w_\tau) \geq C\phi(-r_1, \frac{5\delta}{2}, w_\tau).$$

Finally, from (3.33) and (3.34), using (3.29), we have (3.3) for $r_0 < s < \chi$.

To conclude the proof of the theorem, we must show that there exists $r_0 > 0$ and a positive constant C , for which (3.26) holds. We follow closely the technique introduced in [25].

We define

$$\tilde{w}_\tau = \log(w_\tau),$$

that is $\tilde{w}_\tau$ in (3.12) for $\beta = -1$. In this case using (3.11) we have

$$(3.35) \quad \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 dx = \int_{\Omega} \rho \frac{|\nabla w_\tau|^2}{w_\tau^2} \eta^2 dx \leq \dot{C} \int_{\Omega} (\eta^2 + \rho |\nabla \eta|^2) dx,$$

for some positive constant $\dot{C} = \dot{C}(p, q, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$. From (3.35), with $\eta = 1$ in $B(x, 5\delta)$, we obtain

$$(3.36) \quad \int_{B(x, 5\delta)} \rho |\nabla \tilde{w}_\tau|^2 dx \leq C,$$

with C not depending on $\tilde{w}_\tau$. Replacing (in (3.35)) w_τ by w_τ/k , with

$$k := e^{\frac{1}{|B(x, 5\delta)|} \int_{B(x, 5\delta)} \log \tilde{w}_\tau dx},$$

we can suppose that $\tilde{w}_\tau$ has zero mean in $B(x, 5\delta)$. Then we exploit Theorem 2.2 and by (3.36) it follows that

$$(3.37) \quad \|\tilde{w}_\tau\|_{L^\nu(B(x, 5\delta))} \leq C,$$

where C is a constant not depending on $\tilde{w}_\tau$. This will be crucial when we will pass to the limit for $\tau \rightarrow 0$. The constant k that we introduced does not modify the following calculations and can be cancelled in the conclusive inequality, i.e. (3.62) below.

We are going to use (in the weak expression of (3.1)) the test function

$$\phi = \eta^2 \frac{1}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) \quad \beta \geq 1,$$

with $\eta \geq 0$ and $\eta \in C_0^1(B(x, 5\delta))$. Therefore

$$(3.38) \quad \nabla \phi = \frac{2\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) \nabla \eta + \eta^2 \frac{1}{w_\tau^2} (\beta \operatorname{sign}(\tilde{w}_\tau) |\tilde{w}_\tau|^{\beta-1} - |\tilde{w}_\tau|^\beta - (2\beta)^\beta) \nabla w_\tau$$

and

$$\begin{aligned} & \frac{1}{k} \int_\Omega (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla \phi) dx + \frac{1}{k} \int_\Omega (a(x, v) |\nabla v|^q - a(x, u) |\nabla u|^q) \phi dx \\ & \geq \frac{1}{k} \int_\Omega (f(x, v) - f(x, u)) \phi dx. \end{aligned}$$

Hence, using (3.38) and the definition of ϕ , we get

$$\begin{aligned} (3.39) \quad & \frac{1}{k} \int_\Omega (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla \eta) \frac{2\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \\ & + \frac{1}{k} \int_\Omega (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla w_\tau) \eta^2 \frac{1}{w_\tau^2} (\beta \operatorname{sign}(\tilde{w}_\tau) |\tilde{w}_\tau|^{\beta-1} - |\tilde{w}_\tau|^\beta - (2\beta)^\beta) dx \\ & + \frac{1}{k} \int_\Omega (a(x, v) |\nabla v|^q - a(x, u) |\nabla u|^q) \eta^2 \frac{1}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \\ & \geq \frac{1}{k} \int_\Omega (f(x, v) - f(x, u)) \eta^2 \frac{1}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx. \end{aligned}$$

We estimate and rearrange the terms in the inequality (3.39). We start from

$$\begin{aligned} (3.40) \quad & \frac{1}{k} \int_\Omega (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla \eta) \frac{2\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \\ & \leq \check{C}(p) \int_\Omega \rho \frac{2\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) |\nabla w_\tau| |\nabla \eta| dx, \end{aligned}$$

where we used the second of (2.5).

In the following, if $\beta \geq 1$, we will use the following inequality

$$(3.41) \quad 2|\tilde{w}_\tau|^{\beta-1} \leq \frac{\beta-1}{\beta} |\tilde{w}_\tau|^\beta + \frac{1}{\beta} (2\beta)^\beta \leq |\tilde{w}_\tau|^\beta + (2\beta)^\beta,$$

or

$$(3.42) \quad -\beta \operatorname{sign}(\tilde{w}_\tau) |\tilde{w}_\tau|^{\beta-1} + |\tilde{w}_\tau|^\beta + (2\beta)^\beta \geq \beta |\tilde{w}_\tau|^{\beta-1} > 0.$$

Using the inequalities (3.42) and (2.5), the second term of (3.39) can be estimated as

$$\begin{aligned} (3.43) \quad & \frac{1}{k} \int_\Omega (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla w_\tau) \eta^2 \frac{1}{w_\tau^2} (\beta \operatorname{sign}(\tilde{w}_\tau) |\tilde{w}_\tau|^{\beta-1} - |\tilde{w}_\tau|^\beta - (2\beta)^\beta) dx \\ & \leq \int_\Omega (|\nabla v| + |\nabla u|)^{p-2} |\nabla w_\tau|^2 \eta^2 \frac{1}{w_\tau^2} (\beta \operatorname{sign}(\tilde{w}_\tau) |\tilde{w}_\tau|^{\beta-1} - |\tilde{w}_\tau|^\beta - (2\beta)^\beta) dx \\ & \leq \int_\Omega (|\nabla v| + |\nabla u|)^{p-2} |\nabla w_\tau|^2 \eta^2 \frac{1}{w_\tau^2} (-\beta |\tilde{w}_\tau|^{\beta-1}) dx = \int_\Omega \rho |\nabla \tilde{w}_\tau|^2 \eta^2 (-\beta |\tilde{w}_\tau|^{\beta-1}) dx. \end{aligned}$$

For the third term of (3.39) we get

$$\begin{aligned}
 (3.44) \quad & \frac{1}{k} \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, u)|\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &= \frac{1}{k} \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, v)|\nabla u|^q + a(x, v)|\nabla u|^q - a(x, u)|\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &= \frac{1}{k} \int_{\Omega} a(x, v) (|\nabla v|^q - |\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &+ \frac{1}{k} \int_{\Omega} (a(x, v) - a(x, u)) |\nabla u|^q \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx.
 \end{aligned}$$

Taking into account the hypotheses (hp^*), that $\nabla u, \nabla v$ are bounded in Ω and that τ is a positive constant, for the two terms on the right-hand side of (3.44) we have

$$\begin{aligned}
 (3.45) \quad & \frac{1}{k} \left| \int_{\Omega} a(x, v) (|\nabla v|^q - |\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \right| \\
 &\leq \int_{\Omega} |a(x, v)| q (|\nabla v| + |\nabla u|)^{q-1} |\nabla w_{\tau}| \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &\leq C(q, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}) \int_{\Omega} (|\nabla v| + |\nabla u|)^{q-1} |\nabla w_{\tau}| \eta^2 \frac{1}{w_{\tau}} |\tilde{w}_{\tau}|^{\beta} dx \\
 &+ C(q, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}) \int_{\Omega} (|\nabla v| + |\nabla u|)^{q-1} |\nabla w_{\tau}| \eta^2 \frac{1}{w_{\tau}} (2\beta)^{\beta} dx,
 \end{aligned}$$

where we used the mean value theorem. Furthermore

$$\begin{aligned}
 (3.46) \quad & \frac{1}{k} \int_{\Omega} |a(x, v) - a(x, u)| |\nabla u|^q \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &\leq C(q, L, \|\nabla u\|_{L^{\infty}(\Omega)}) \int_{\Omega} \eta^2 (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx.
 \end{aligned}$$

Considering (3.45), since $q \geq p - 1$, we obtain

$$\begin{aligned}
 & \frac{1}{k} \int_{\Omega} a(x, v) (|\nabla v|^q - |\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &\leq C(p, q, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}, \|\nabla v\|_{L^{\infty}(\Omega)}) \int_{\Omega} \rho |\nabla \tilde{w}_{\tau}| \eta^2 |\tilde{w}_{\tau}|^{\beta} dx \\
 &+ C(p, q, \|v\|_{L^{\infty}(\Omega)}, \|\nabla u\|_{L^{\infty}(\Omega)}, \|\nabla v\|_{L^{\infty}(\Omega)}) \int_{\Omega} \rho |\nabla \tilde{w}_{\tau}| \eta^2 (2\beta)^{\beta} dx
 \end{aligned}$$

and then applying (weighted) Young's inequality

$$\begin{aligned}
 (3.47) \quad & \frac{1}{k} \int_{\Omega} a(x, v) (|\nabla v|^q - |\nabla u|^q) \eta^2 \frac{1}{w_{\tau}} (|\tilde{w}_{\tau}|^{\beta} + (2\beta)^{\beta}) dx \\
 &\leq \varepsilon C \int_{\Omega} \rho |\nabla \tilde{w}_{\tau}|^2 \eta^2 |\tilde{w}_{\tau}|^{\beta-1} dx + \frac{C}{\varepsilon} \int_{\Omega} \rho \eta^2 |\tilde{w}_{\tau}|^{\beta+1} dx \\
 &+ C \int_{\Omega} \rho |\nabla \tilde{w}_{\tau}|^2 \eta^2 (2\beta)^{\beta} dx + C \int_{\Omega} \rho \eta^2 (2\beta)^{\beta} dx,
 \end{aligned}$$

where $\varepsilon > 0$ and $C = C(p, q, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ is a positive constant. Using (3.46) and (3.47), from (3.44) we get

$$\begin{aligned}
 (3.48) \quad & \frac{1}{k} \left| \int_{\Omega} (a(x, v)|\nabla v|^q - a(x, u)|\nabla u|^q) \eta^2 \frac{1}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \right| \\
 & \leq C(q, L, \|\nabla u\|_{L^\infty(\Omega)}) \int_{\Omega} \eta^2 (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) \\
 & + C \left(\varepsilon \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx + \frac{1}{\varepsilon} \int_{\Omega} \rho \eta^2 |\tilde{w}_\tau|^{\beta+1} dx \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 (2\beta)^\beta dx + \int_{\Omega} \rho \eta^2 (2\beta)^\beta dx \right),
 \end{aligned}$$

where $C = C(p, q, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ is a positive constant.

Finally, using hypothesis (hf^*) and the fact that τ is a positive constant, we have

$$(3.49) \quad \frac{1}{k} \int_{\Omega} |f(x, v) - f(x, u)| \eta^2 \frac{1}{w_\tau} (|\tilde{w}|^\beta + (2\beta)^\beta) dx \leq L \int_{\Omega} \eta^2 (|\tilde{w}|^\beta + (2\beta)^\beta) dx.$$

Collecting estimations (3.40), (3.43), (3.48) and (3.49), from (3.39) we obtain

$$\begin{aligned}
 (3.50) \quad & \beta \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx \\
 & \leq C \int_{\Omega} \rho \frac{\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) |\nabla w_\tau| |\nabla \eta| dx \\
 & + C \left(\varepsilon \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx + \frac{1}{\varepsilon} \int_{\Omega} \rho \eta^2 |\tilde{w}_\tau|^{\beta+1} dx + \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 (2\beta)^\beta dx + \int_{\Omega} \rho \eta^2 (2\beta)^\beta dx \right) \\
 & + C \int_{\Omega} \eta^2 (|\tilde{w}|^\beta + (2\beta)^\beta) dx
 \end{aligned}$$

where $C = C(p, q, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ is a positive constant. Now we use Young's inequality in the second term of (3.50) and we get

$$\begin{aligned}
 (3.51) \quad & \int_{\Omega} \rho \frac{\eta}{w_\tau} (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) |\nabla w_\tau| |\nabla \eta| dx = \int_{\Omega} \rho \eta |\tilde{w}_\tau|^\beta |\nabla \tilde{w}_\tau| |\nabla \eta| dx + \int_{\Omega} \rho \eta (2\beta)^\beta |\nabla \tilde{w}_\tau| |\nabla \eta| dx \\
 & \leq \frac{\varepsilon}{2} \int_{\Omega} \rho \eta^2 |\tilde{w}_\tau|^{\beta-1} |\nabla \tilde{w}_\tau|^2 dx + \frac{1}{2\varepsilon} \int_{\Omega} \rho |\tilde{w}_\tau|^{\beta+1} |\nabla \eta|^2 dx \\
 & + \frac{1}{2} \int_{\Omega} \rho (2\beta)^\beta |\nabla \eta|^2 dx + \frac{1}{2} \int_{\Omega} \rho \eta^2 (2\beta)^\beta |\nabla \tilde{w}_\tau|^2 dx.
 \end{aligned}$$

Using (3.51) (recall (3.36)) and then grouping the term in (3.50), for ε small, we obtain

$$\begin{aligned}
 (3.52) \quad & \beta \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx \\
 & \leq C \int_{\Omega} \rho (|\tilde{w}_\tau|^{\beta+1} + (2\beta)^\beta) |\nabla \eta|^2 dx + C \int_{\Omega} \eta^2 (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \\
 & + C \int_{\Omega} \rho \eta^2 |\tilde{w}_\tau|^{\beta+1} dx + C(2\beta)^\beta,
 \end{aligned}$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ some positive constant. For future use we collect that, to group the term

$$(3.53) \quad \int_{\Omega} \rho \eta^2 (2\beta)^\beta dx$$

as $C(2\beta)^\beta$ in (3.50), we used that $p \geq 2$. In the Case (b) here below, we will also consider the quantity in (3.53) added to the term $C(2\beta)^\beta$ exploiting Theorem 2.1 that gives the right L^1 -integrability of the weight ρ .

Since the support of η depends on δ , we can write

$$(2\beta)^\beta = C(\delta) \int_{B(x_0, 5\delta)} (2\beta)^\beta \eta^2 dx.$$

Thus, recalling that we are supposing $\beta \geq 1$, (3.52) (up to redefining the constant C there) becomes

$$(3.54) \quad \begin{aligned} & \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx \\ & \leq C \left(\int_{\Omega} \rho (|\tilde{w}_\tau|^{\beta+1} + (2\beta)^\beta) |\nabla \eta|^2 dx + \int_{\Omega} \eta^2 (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx + \int_{\Omega} \rho \eta^2 |\tilde{w}_\tau|^{\beta+1} dx \right). \end{aligned}$$

By (3.41), we also have

$$\int_{\Omega} \eta^2 (|\tilde{w}_\tau|^\beta + (2\beta)^\beta) dx \leq C \int_{\Omega} \eta^2 (|\tilde{w}_\tau|^{\beta+1} + (2\beta)^\beta) dx.$$

Therefore (3.54) can be written as

$$(3.55) \quad \int_{\Omega} \rho |\nabla \tilde{w}_\tau|^2 \eta^2 |\tilde{w}_\tau|^{\beta-1} dx \leq C \int_{\Omega} (|\tilde{w}_\tau|^{\beta+1} + (2\beta)^\beta) (\eta^2 + \rho |\nabla \eta|^2) dx,$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ some positive constant.

We note that (3.55) is similar to (3.11), except for the extra term $(2\beta)^\beta$. Then we are going to apply again the iterative method as we did above.

Following (taking in account the extra term $(2\beta)^\beta$) verbatim the technique from equation (3.11) to equation (3.18), recalling that $r = \beta + 1$ (here we are in the hypothesis $\beta \geq 1$), using (3.55) we obtain for $r > 0$

$$(3.56) \quad \begin{aligned} \left(\int_{B(x, h')} (\tilde{w}_\tau)^{\frac{\nu(\beta+1)}{2}} dx \right)^{\frac{1}{\chi r}} &= \left(\int_{B(x, h')} \tilde{w}_\tau^{\chi r} dx \right)^{\frac{1}{\chi r}} \\ &\leq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \left[\left(\int_{B(x, h'')} \tilde{w}_\tau^r dx \right)^{\frac{1}{r}} + \ddot{C}r \right], \end{aligned}$$

where $\dot{C} = \dot{C}(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ is a positive constant and $\ddot{C} = \ddot{C}(p, \delta)$ is a positive constant too that takes into account the extra term $(2\beta)^\beta$. Recalling (3.19), from (3.56) we infer that

$$(3.57) \quad \phi(\chi r, h', \tilde{w}_\tau) \leq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} [\phi(r, h'', \tilde{w}_\tau) + \ddot{C}r].$$

We claim that there exists a constant C such that

$$(3.58) \quad \phi(m, \frac{5\delta}{2}, \tilde{w}_\tau) \leq C(\phi(\nu, 5\delta, \tilde{w}_\tau) + m) \quad \forall m \geq \nu.$$

To prove the last inequality, we are going to choose

$$h_k \equiv \frac{5\delta}{2} [1 + \frac{1}{2^k}] \quad k = 0, 1, \dots$$

and

$$(3.59) \quad \chi^k \nu = \chi r = \chi(\beta + 1)$$

in (3.57) (noting that there exists $\beta > 1$ such that (3.59) holds), getting

$$\phi(\chi^k \nu, h_k, \tilde{w}_\tau) \leq \left(\frac{2\dot{C}\chi^k}{h_{k-1} - h_k} \right)^{\frac{1}{\chi^k}} [\phi(\chi^{k-1} \nu, h_{k-1}, \tilde{w}_\tau) + \ddot{C}\chi^{k-1} \nu].$$

Iterating we obtain the following

$$(3.60) \quad \begin{aligned} \phi(\chi^k \nu, h_k, \tilde{w}_\tau) &\leq (2\dot{C})^{\sum_{k=1}^{+\infty} \frac{1}{\chi^k}} \frac{\prod_{k=1} (\chi^k)^{\frac{1}{\chi^k}}}{\prod_{k=1} (h_{k-1} - h_k)^{\frac{1}{\chi^k}}} \phi(\nu, h_0, \tilde{w}_\tau) \\ &\quad + \sum_{h=0}^{k-1} \ddot{C} \chi^{h-k} \left[\prod_{\tau=h}^{k-1} \left(\frac{2C\chi^{\tau+1}}{h_\tau - h_{\tau+1}} \right)^{\frac{1}{\chi^{\tau+1}}} \right] \chi^k \nu. \end{aligned}$$

Therefore, estimating the products (using e.g. the logarithm function), we obtain a positive constant C such that

$$(3.61) \quad \phi(\chi^k \nu, h_k, \tilde{w}_\tau) \leq C(\phi(\nu, 5\delta, \tilde{w}_\tau) + \chi^k \nu),$$

where $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ (not depending on k) and where we have used also the fact that $h_0 \equiv 5\delta$.

Setting now

$$k_m \equiv \inf_{h \in \mathbb{N}} \{h \mid \chi^h \nu \geq m\},$$

we obtain

$$\begin{aligned} \phi(m, \frac{5\delta}{2}, \tilde{w}_\tau) &\leq C_1 \phi(\chi^{k_m} \nu, h_{k_m}, \tilde{w}_\tau) \\ &\stackrel{\text{by (3.61)}}{\leq} C(\phi(\nu, 5\delta, \tilde{w}_\tau) + \chi^{k_m} \nu) \\ &\leq C(\phi(\nu, 5\delta, \tilde{w}_\tau) + \chi m) \leq C(\phi(\nu, 5\delta, \tilde{w}_\tau) + m), \end{aligned}$$

up to redefine the constant C and with $C = C(p, q, \delta, \Omega, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$. This proves (3.58).

We are going to apply now (3.58) in order to prove that there exists $r_0 > 0$ for which (3.26) holds. For $r_0 > 0$ given, taking into account the power series expansion of $e^{r_0|\tilde{w}_\tau|}$, we obtain

$$\begin{aligned} \int_{B(x, \frac{5\delta}{2})} e^{r_0|\tilde{w}_\tau|} dx &\leq \sum_{k=0}^{+\infty} \int_{B(x, \frac{5\delta}{2})} \frac{(r_0|\tilde{w}_\tau|)^k}{k!} dx \leq \sum_{k=0}^{+\infty} \frac{(r_0\phi(k, \frac{5\delta}{2}, \tilde{w}_\tau))^k}{k!} \\ &\leq \sum_{k=0}^{+\infty} \frac{(Cr_0)^k (\phi(\nu, 5\delta, \tilde{w}_\tau)^k + k^k)}{k!}. \end{aligned}$$

We can prove (using the ratio test) that, if $r_0 > 0$ is small enough, the last series are convergent. Therefore, it follows that

$$\int_{B(x, \frac{5\delta}{2})} e^{r_0|\tilde{w}_\tau|} dx \leq C$$

and then by monotonicity

$$(3.62) \quad \int_{B(x, \frac{5\delta}{2})} e^{r_0\tilde{w}_\tau} dx \int_{B(x, \frac{5\delta}{2})} e^{-r_0\tilde{w}_\tau} dx \leq \left(\int_{B(x, \frac{5\delta}{2})} e^{r_0|\tilde{w}_\tau|} dx \right)^2 \leq C^2.$$

Taking now the power $1/r_0$ in the inequality (3.62), using (3.19) and recalling that $\tilde{w}_\tau = \log(w_\tau)$, we prove that (3.26) holds for this choice of r_0 . Moreover, the constant C in (3.62) does not depend on τ .

Case (b): $(2N+2)/(N+2) < p < 2$. Arguing exactly as in the Case (a) we are able to get (3.11). Using (3.12), we still get (3.15) if $\beta \neq 1$ or (3.35) if $\beta = 1$.

Since $u \in C^1(\bar{\Omega})$, if $p < 2$, the weight

$$\rho = (|\nabla u| + |\nabla v|)^{p-2} \geq \lambda > 0.$$

Then as in the Case (a), using the classic Sobolev's inequality (instead of Theorem 2.2), we get

$$\|\eta\tilde{w}_\tau\|_{L^{2^*}(\Omega)}^2 \leq C_S \int_{\Omega} |\nabla(\eta\tilde{w}_\tau)|^2 dx \leq C(\lambda, S) \int_{\Omega} \rho |\nabla(\eta\tilde{w}_\tau)|^2 dx,$$

where 2^* is the classical Sobolev's exponent and therefore (see (3.16))

$$\|\eta\tilde{w}_\tau\|_{L^{2^*}(\Omega)}^2 \leq \dot{C} \frac{1}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) r^2 \int_{\Omega} \tilde{w}_\tau^2 (\eta^2 + \rho |\nabla \eta|^2) dx.$$

Now if we suppose $\rho \in L^t(\Omega)$, applying Hölder inequality with exponents t and $t' = t/(t-1)$, we have

$$(3.63) \quad \|\eta\tilde{w}_\tau\|_{L^{2^*}(\Omega)}^2 \leq Cr^2 \|\rho\|_{L^t(\Omega)} \|\tilde{w}_\tau(\eta + |\nabla \eta|)\|_{L^{t^\sharp}(\Omega)}^2,$$

where

$$t^\sharp = 2t'.$$

We set now $\chi' = 2^*/t^\sharp$ and, in order to run over again the arguments in the Case (a), we only need $\chi' > 1$. This condition is obviously satisfied if

$$(3.64) \quad t > \frac{N}{2}.$$

By Theorem 2.1 (see in particular (2.2)), if $p < 2$, it follows that $\rho \in L^{\frac{p-1}{2-p}\theta}(\Omega)$, for every $0 < \theta < 1$. Then (3.64) holds if $(2N+2)/(N+2) < p < 2$. $\square$

The iteration technique is easier in the next case and it allows us to prove the following

Theorem 3.2. *Let $u, v \in C_{loc}^1(\Omega)$ and assume that either u or v is a weak solution to (1.1), with $q \geq \max\{p-1, 1\}$ and $f(x, u), a(x, u)$ satisfying (hp^*) . Assume that $\overline{B(x, 6\delta)} \subset \Omega' \subset \Omega$ for some $\delta > 0$ and that*

$$(3.65) \quad -\Delta_p v + a(x, v)|\nabla v|^q - f(x, v) \leq -\Delta_p u + a(x, u)|\nabla u|^q - f(x, u), \quad u \leq v \quad \text{in } B(x, 5\delta).$$

We distinguish the two cases:

- *Case (a) : $p \geq 2$. For all $s > 1$, there exists $C > 0$ such that*

$$\sup_{B(x,\delta)} (v - u) \leq C \|v - u\|_{L^s(B(x,2\delta))},$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')})$.

- *Case (b) : $(2N + 2)/(N + 2) < p < 2$. Define*

$$\bar{t}^\sharp = \frac{2(p-1)}{2p-3}.$$

Then, for every $s > \bar{t}^\sharp/2$, there exists $C > 0$ such that

$$\sup_{B(x,\delta)} (v - u) \leq C \|v - u\|_{L^s(B(x,2\delta))},$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega')}, \|\nabla u\|_{L^\infty(\Omega')}, \|\nabla v\|_{L^\infty(\Omega')})$.

Proof. We are going to use the same technique in the proof of Theorem 3.1 and then we will omit some details. As above we relabel the sub-domain Ω' by Ω . From (3.65) it follows that

$$(3.66) \quad -\Delta_p u + a(x, u)|\nabla u|^q - f(x, u) \geq -\Delta_p v + a(x, v)|\nabla v|^q - f(x, v).$$

In this case, given $w := v - u$, let us define the function $\phi = \eta^2 w^\beta$ with $\beta > 0$ and $\eta \in C_0^1(B(x, 5\delta))$. For $p \geq 2$, using ϕ as test function in (3.66) (and repeating the same calculations of the proof to Theorem 3.1), we get (see (3.16))

$$(3.67) \quad \begin{aligned} \|\eta \tilde{w}\|_{L^\nu(\Omega)}^2 &\leq \dot{C} \frac{1}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) r^2 \int_{\Omega} \tilde{w}^2 (\eta^2 + \rho |\nabla \eta|^2) dx \\ &\leq \dot{C} \frac{1}{|\beta|} \left(1 + \frac{1}{|\beta|}\right) r^2 \|\tilde{w}(\eta + |\nabla \eta|)\|_{L^2(\Omega)}^2, \end{aligned}$$

with $r, \beta, \dot{C}$ as in (3.16). Since now $\beta > 0$, it follows that $r > 1$. Then, for $r > 0$ (see (3.20) and (3.57) with $\ddot{C} = 0$) we obtain

$$(3.68) \quad \phi(\chi r, h', w) \leq \frac{(\dot{C}|r|)^{\frac{2}{r}}}{(h'' - h')^{\frac{2}{r}}} \phi(r, h'', w).$$

Hence, taking $s > 1$ and setting $\chi^k s = \chi r$, iterating as in (3.60) and (3.61), we get

$$\phi(\chi^k s, h_k, w) \leq C \phi(s, 2\delta, w)$$

with h_k given by

$$h_k = \delta \left(1 + \left(\frac{1}{2}\right)^k\right).$$

Letting k tending to infinity we have

$$\sup_{B(x,\delta)} (v - u) \leq C \|v - u\|_{L^s(B(x,2\delta))},$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ a positive constant.

For $(2N+2)/(N+2) < p < 2$ arguing as in the proof of Theorem 3.1, using ϕ as test function in (3.66), we get (see (3.63))

$$(3.69) \quad \|\eta \tilde{w}_\tau\|_{L^{2^*}(\Omega)}^2 \leq Cr^2 \|\rho\|_{L^t(\Omega)} \|\tilde{w}_\tau(\eta + |\nabla \eta|)\|_{L^{t^\sharp}(\Omega)}^2,$$

with $t^\sharp = 2t'$. Iterating (3.69) (and repeating the same type of arguments as above) we reach the desired conclusion: given $\tilde{t}^\sharp = 2(p-1)/(2p-3)$, for any $s > \tilde{t}^\sharp/2$ we have

$$\sup_{B(x,\delta)} (v - u) \leq C \|v - u\|_{L^s(B(x,2\delta))},$$

with $C = C(p, q, \delta, L, \|v\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)})$ a positive constant. $\square$

Now it is easy to deduce the

Proof of Theorem 1.1. The proof readily follows applying both Theorem 3.1 and Theorem 3.2. $\square$

As a corollary of the Harnack comparison inequality we have the

Proof of Theorem 1.2. The proof is standard, but we give the details for the reader's convenience. Let us set $w := v - u$. Define the set

$$U_w = \{x \in \Omega \mid w(x) = 0\}.$$

By the continuity of u and v it follows that U_w is a closed set in Ω . On the other hand by Theorem 3.1 we have that U_w is also open. Then the thesis follows. $\square$

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