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Doctoral Thesis

**Stability Analysis of Complex
Aerodynamic Flows**

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Stability Analysis of Complex Aerodynamic Flows

Abstract

In this work, several approaches for the stability analysis of complex aerodynamic flows are investigated in order to estimate their applicability to industrial flow problems and their computational limitations. The results obtained in this work significantly contribute to understand of how to choose a numerical method for a stability problem, which was unsufficiently investigated until now.

The numerical solution of large eigenvalue problems is becoming increasingly demanding by the scientific community since it allows a better understanding of the intrinsic nature of a phenomenon: vibrations in structural mechanics, oscillations in plasma physics, acoustic resonances, magneto-hydrodynamics are different examples of physically relevant eigenvalue problems.

In particular, in the field of Fluid Mechanics, the asymptotic behavior of stationary solutions of the Navier Stokes equations is defined by the spectrum of a non-self-adjoint eigenvalue problem. In a temporal framework, the linearization of the aforementioned Navier Stokes equations around a equilibrium state, or base flow, permits the linear stability analysis of aerodynamic flows. Depending on the eigenvalues in the spectrum of the discretized linear differential operator, small disturbances, superimposed on the steady solution, may decay or grow in time, thus characterizing the base flow as linearly stable or unstable. The corresponding eigenvectors provide relevant information about the transition mechanism, which allows to characterize and control the studied aerodynamic flow. Nowadays, the advent of high performance computing facilities, with increasing resources in terms of memory and computational capability, is permitting the study of more demanding problems on complex geometries aligned with industrial real needs. However, as a consequence of this expanded scenario for the stability analysis applications, the leading dimension of the involved matrices has risen from hundreds of thousands to millions. Several two-dimensional analyses evolved into three-dimensional cases, even if not affordable yet but for the simplest configurations. In the last decades, the number of available algorithms has increased significantly and different available variants can give insights into certain properties of the eigenvalue problem. However, the effective applicability of these numerical methods in the stability analysis framework, has been still not clarified. In this context, the aim of this thesis is to evaluate new numerical approaches to overcome the computational cost limitations, opening the stability analysis scenario to complex aerodynamic configurations for industrial applications. Furthermore, a new approach to reduce the computational costs, based on a geometrical concept instead of numerical proprieties, is introduced. This novel method, here presented for the first time, represents in an innovative approach with exceptional results.

Keywords: Stability analysis, incompressible Navier Stokes, compressible Navier Stokes, numerical methods for large and sparse EVP, buffet phenomenon, high lift configuration, domain reduction strategy

Análisi de Estabilidad de Flujos Aerodinámicos Complejos

Resumen

En este trabajo se investigan varias estrategias para el análisis de estabilidad de flujos aerodinámicos complejos. Se han analizado diferentes problemas industriales para estimar la aplicabilidad del análisis de estabilidad lineal y sus limitaciones computacionales. Los resultados obtenidos en este trabajo pueden contribuir significativamente en la elección de un método numérico eficiente para problemas de estabilidad en casos de interés industrial, teniendo en cuenta la falta, en el actual estado del arte, de una completa investigación sobre este tema. La solución numérica de grandes problemas de autovalores representa un punto esencial y desafiante, para la, siempre exigente, comunidad científica, ya que permite comprender mejor la naturaleza intrínseca de fenómenos complejos. En particular, en el campo de la mecánica de fluidos, el comportamiento asintótico de las soluciones estacionarias de las ecuaciones de Navier Stokes se relaciona con el espectro de un problema de autovalores no-autoadjunto. En un marco temporal, la linealización de dichas ecuaciones de Navier Stokes alrededor de un estado de equilibrio, o flujo base, permite el análisis de estabilidad lineal de flujos aerodinámicos. Dependiendo de los autovalores del espectro del operador diferencial lineal discretizado, pequeñas perturbaciones superpuestas sobre la solución estable pueden decaer o crecer en el tiempo, caracterizando así el flujo base como linealmente estable o inestable. Los correspondientes autovectores aportan información relevante sobre el mecanismo de transición, que permite caracterizar y controlar el flujo aerodinámico estudiado. Actualmente, el desarrollo y popularización de instalaciones de computación de alto rendimiento, con recursos siempre más eficientes en términos de memoria y capacidad computacional, está permitiendo el estudio de problemas más exigentes, con geometrías complejas alineadas con las necesidades reales industriales. Sin embargo, como consecuencia de este nuevo escenario en el análisis de estabilidad, la dimensión principal de las matrices involucradas ha aumentado de cientos de miles a millones. El análisis bidimensional ha dado paso a casos tridimensionales, aunque sea todavía resoluble solamente para las configuraciones más simples. En las últimas décadas, el número de algoritmos disponibles también ha aumentado, y se ha profundizado en desarrollar variaciones de los ya existentes para aprovechar mejor las propiedades intrínsecas de los problemas de autovalores derivados del problema de estabilidad fluidodinámica. Sin embargo, todavía no existe consenso sobre la manera óptima de abordar dichos problemas. En este sentido, el objetivo de esta tesis es evaluar nuevas estrategias numéricas para superar las limitaciones computacionales, abriendo el escenario del análisis de estabilidad a configuraciones aerodinámicas complejas de aplicaciones industriales. Además, se presenta una nueva herramienta de análisis para reducir el tamaño del problema, basada en consideraciones geométricas en lugar de propiedades numéricas. Este nuevo método, presentado aquí por la primera vez, representa una innovadora estrategia, capaz de obtener resultados prometedores.

Keywords: Analisis de Estabilidad, Navier Stokes compresible , Navier Stokes incompresibles, métodos numéricos por EVP, fenómeno de buffeting, configuración de high lift, estrategia de reducción de dominio

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Introduction

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1.1 Content

The prediction of the onset of flow unsteadiness is one of the main problems in aeronautical engineering industrial applications. This unsteadiness is associated to undesirable load variations that can damage the structures, affecting the flight efficiency. This problem is of crucial importance in several industrial applications, from airfoil design to internal flows in turbines. Interested readers are referred to [Abbas 2013] for a detailed review on aircraft performance, where it is highlighted that the limits of the flight envelope are mainly caused by the undesirable growth of unsteady features.

The global (linear) stability analysis represents a powerful method to predict the onset of flow bifurcations leading to these instabilities [Theofilis 2003, Theofilis 2011].

The Linear stability analysis, in the modal approach [Juniper 2014], relies on the decomposition of the flow variables as the superposition of a time-dependent small amplitude perturbation on a steady base flow. This decomposition is introduced into the non-linear NS equations, then linearized around the steady state flow and discretized into a computational domain, leading to a generalized eigenvalue problem (EVP). The solution of this EVP contains the geometrical extension and the (stable/unstable) time-evolution of this (initially small) disturbance, revealing the stability properties of the base flow.

1.1.1 State of the art

Multiple applications of stability analysis for the prediction of flow unsteadiness can be found in the literature, since simple parallel flows were investigated in [Lin 1955, Drazin 1981]. Further parallel and weakly not parallel flows were studied in [Huerre 1990, Chomaz 1988, Huerre 1985]. This methodology has been then improved and applied on increasing complex geometries. A brief overview of the state of the art of the stability

analysis applications is presented in this section.

Pioneery studies on stability analysis of trailing and co-rotating vortices, using the vortex-filament approach, were carried out by S. Crow and J. Jimenez in [Crow 1970, Jimenez 1975]. More recently, the results obtained by S. Crow were used for the validation of a viscous modal stability analysis on vortex filament flow in [Tendero 2015]. Furthermore, applications on trailing-line vortex detection are presented in [Duck 2011, Pierrehumbert 1982], where the concept of global instabilities analysis appeared for the first time on the international scenario. Stability analysis applications with two-dimensional basic flow in a rectangular channel were performed in [Luijkx 1981], while the first three-dimensional (TriGlobal) stability analysis applications on a laminar flow in a rectangular duct is shown in [Tatsumi 1990]. The two-dimensional vortex shedding detection for flows around a cylinder and around variously shaped bodies were investigated for the first time in [Zebib 1987, Jackson 1987], and later a three-dimensional study for the periodic wave of the flow around a circular cylinder were presented in [Barkley 1996]. Relevant results for the axisymmetric separating/reattaching turbulent flow around a cylinder can be found in [Weiss 2009].

More recently, the transonic channel flow configuration over a bump at high Reynolds number were investigated in [Sartor 2015], with the purpose to detect the instability related to the shock-wave/boundary layer interaction. The results of this analysis were then compared with experimental data released by ONERA.

Separated flows on flat plates were investigated in [Ehrenstein 2011, Joe 2011], and theory extensions were provided in [Alizard 2007, Losse 2011], while wing and bluff bodies were studied in [Rodríguez 2011, Stalnov 2011].

Further relevant two-dimensional applications, including the onset of shear layer instabilities, were investigated in [Ehrenstein 2008a, Ehrenstein 2008b, Morzynski 1991, Crouch 1997], while three-dimensional analysis for the Rayleigh-Bénard convective problem were performed in [Gelfgat 1999]. Finally, roughness elements in hypersonic regime were presented in [Chang 2011].

In the early 2000s, V. Theofilis performed a BiGlobal stability analysis of the incompressible flow around a NACA0012 airfoil, detecting the wake instability as a consequence of the leading edge separation mode [Barkley 2002b]. After this successful application, many other industrial flow configurations were investigated as shown in the works [Theofilis 2003, Collis 2004, Crouch 2005, Schlatter 2011, Meliga 2011, Sharma 2011]. Further remarkable results regarding the BiGlobal and TriGlobal stability analysis for the detection of the instability related to the buffet phenomenon for a NACA0012 airfoil were shown in [Iorio 2014, Iorio 2015], where also the structural receptivity and sensitivity region for the buffet phenomenon were obtained. Similar investigation, focused on Triglobal stability analysis of the buffet phenomenon on a NACA0012 was performed by S. Timme in [Timme 2016]. Further BiGlobal and TriGlobal stability analysis for a standard transonic airfoil and for a high-aspect-ratio-wing dynamic test case, using rigid pitch modes, were carried by R. Thormann in [Thormann 2013], which concluded that TriGlobal analysis shows larger differences than the BiGlobal analysis with respect to the experimental data. Moreover, in [Sartor 2015], F. Sartor et al. exploited the information resulting from the sta-

bility and sensitivity analysis in order to stabilize the buffet phenomenon over a OAT15A supercritical profile, introducing steady streamwise forces in specific regions around the airfoil. In its work [Kitsios 2011], V. Kitsios performed the stability analysis of a flow around a NACA0015 airfoil with laminar leading-edge separation and turbulent recirculation, detecting eigenmodes temporal frequencies in agreement with the forcing frequency used for the experiments in [Tuck 2008].

Relevants results of Biglobal stability analysis, including the sensitivity region detection, of high-lift flow configuration at high angle of attack was presented in [Iorio 2016]. Finally, further extensive review of the stability analysis progress of the last decades can be found in [Theofilis 2011].

1.1.2 Dominant limiting factors for the stability analysis applications

Despite all these successful results obtained applying stability analysis, the state of the art is still far from the possibility to solve real complex aerodynamic problems, such as those relevant for industrial applications.

The limitations, which affects the stability analysis applications to industrial case, are due to different factors. In this section, an overall overview regarding these factors is provided, in order to clarify why the stability analysis of complex aerodynamic flows is such challenging issue.

One of the most relevant factors, related to the geometrical properties of the problems, is the grid points number which is directly proportional to the leading dimension of the EVP. Simple two-dimensional problems are obviously related to small size EVP, that could be solved with affordable computational effort. More complex geometries, even two-dimensional, requires specific refinement in critical regions of the computational domain, leading to large size EVP. Complex flow configurations, usually three-dimensional and complex geometries and turbulence, such as industrial flows, can be extremely large, with an order of magnitude of millions of grid points, and the computational costs needed to recover the solution can exceed the available resources.

The second factor that affects the applicability of the stability analysis applicability is related to the nature of the flow problem which determines the characteristic time scales (duration) of the physical phenomenon involved. These time scales, in the modal approach for the stability analysis used in this thesis, are directly correlated to the order of magnitude of the eigenvalues which belongs to spectrum of the EVP. Turbulence configurations are characterized by high ratios between the larger (2-norm modulus) and smallest eigenvalue. This ratio, also known as conditional number, represents the index for one of the most relevant characteristic for matrix of the EVP: the stiffness. Stiffness EVPs, also called ill-conditioned or bad-conditioned EVPs, are the most challenging problems to solve, and they requires specific numerical approaches, which usually include relaxation techniques and iterative processes. The numerical procedure used to solve the stiffness problems can easily fail, resulting in diverged solution or getting stuck in stagnation points for the iterative process, due to the impossibility to find a time stepping for the time integration which results suitable for all the time scales. Additionally to the numerical complexity related to the stiffness, turbulent flows usually require high-refined meshes in order

to solve the small length scale phenomenon. For these reasons, EVP related to turbulent flow configurations are very large and difficult to solve, even for two dimensional problems.

A last relevant factor for the affordability of the solution of EVPs, in stability analysis of aerodynamic flows, is related to the discretization of the equations which describe the flow motion into the computational domain. The discretization of these equations leads to EVP which matrix is highly sparse. The sparsity is an important characteristic that can significantly affect the performances of the numerical methods used for the solution of the EVP. Sparsity also depends on the physical characterization of the flow field, as for the incompressible flows where all the derivatives of the pressure are equal to zero, increasing the sparsity of the matrix itself. The most common procedure to solve the EVP in stability analysis, as more widely explained in chapter [3], uses a direct LU decomposition technique, which decomposes the EVP matrix into the two lower L and upper U matrices. When the LU decomposition is applied to sparse matrices, the resulting upper U and lower L matrices can be dense. From a computational point of view, the storage of these two matrices is not affordable, depending of the size of the EVP, due to the huge amount of memory required. This memory issue is nowadays considered as the main bottleneck for the applications of the stability analysis for complex flow configurations. Furthermore, the numerical scheme used to perform the discretization, such as the choice of mono-domain or multi-domain approach, determines the pattern of the matrix of the EVP and can be crucial for the performances of the numerical methods used to solve it. Some discretization schemes give favorable properties to the EVP, such as diagonal predominance, that can facilitate the recovering of the solution. Unfortunately, discretization schemes for industrial computational fluid dynamics have to be suitable for the specific mesh features. For example, for unstructured meshes the discretization of the flow equation is usually performed using Finite Volume Methods.

The collection of all these factors determines a wide range of possibility for the EVP properties, giving a specific mark to each possible flow configuration. As consequence, a generalization in terms of suitable numerical approaches to perform the stability analysis, represents a challenging issue.

Moreover, the relevant number of possible numerical methods for solving large EVP, available nowadays, substantially increases the number of variables for the detection of the most appropriate numerical approach.

Summarizing, cutting-edge stability analyses applications refer to three-dimensional problems with low degrees of freedom, related to simple flow configurations at relatives low Reynolds numbers [Gómez 2012, Feldman 2010]. However, the amount of memory demanded for solving EVPs resulting from large three-dimensional problems, or full aircraft configurations even in a two dimensional space, leads to inefficient or even not applicable approaches. Over the last decades, many efforts have been focused on the detection of lower-memory demanding approaches, but the problem still remains open.

1.1.3 Motivations and project development

Nowadays, several industrial applications are still not affordable for stability analysis, and dedicated investigation is required. Moreover, to the authors opinions, a lack of information regarding the applicability and performances of the wide-number of available

numerical methods for solving EVP, in the framework of stability analysis for aeronautical applications, is evident.

Motivated by these reasons, back in 2013, the joint action between the Universidad Politécnica de Madrid and the AIRBUS group led to the definition of the guidelines for this P.h.D. project, in the framework of the AIRUP Marie Curie industrial doctorate program.

AIRUP (Airbus-UPM European Industrial Doctorate in mathematical methods applied to aircraft design) is a common PhD initiative between AIRBUS and Polytechnic University of Madrid-UPM within the frame work FP7 THE PEOPLE PROGRAMME. AIRUP is targeting to foster industry-academia cooperations and strengthen the human potential in research in Europe, allowing for enhanced design processes by mitigating existing efficiency, reliability and accuracy problems in the field of numerical simulation and aerodynamic. AIRUP is focus on three lines of research with strong impact on: first the accuracy of the computational fluid dynamic (CFD) solutions (mesh quality and meshing techniques), second detailed understanding of the flow physics involved (features detection and stability analysis) and third the development of new methods for improved efficiency (surrogate models for design of experiments and aerodynamic shape optimization).

Since its first draft, this work was intended to be developed within both the academic and the industrial environments. In fact, the knowledge deriving from this multi-environmental experience represents the best chance to exploit the academic approach and industrial know-how, leading to a common progress for both sides.

Based on that, the first part of this P.h.D. was conducted in the academic environment, where an intensive literature survey to collect information on the available numerical methods for stability analysis, classic approach apart, was conducted. All the collected information regarding these numerical approaches are summarized in the first part of the chapter [3]. This chapter includes detailed reviews of the most relevant numerical methods, recalling and applying some of the concepts introduced in the chapter [2]. The effects of some of the relevant factors introduced in the previous sections, such as grid points, sparsity level for the EVP matrix and differences between mono or multi domain approach for the discretization of the flow equation into the computational domain, are also investigated. Further characteristics, specific of the numerical methods, are also considered as parameters and their influences on the performance of the numerical methods are estimated. Once that a general overview was depicted, a chosen number of numerical methods were investigated, and their performances compared, using several mono or multi-domains configurations of two simple academic benchmarks: a lid driven cavity flow and the flow over a flat plate. All the information obtained from these comparisons are shown in the results section of the chapter [3].

At the end of this academic experience, a list of alternative numerical methods as possible candidates for industrial applications was obtained.

The powerful resources available in the AIRBUS industrial environment were then used to investigate the applicability of these alternative numerical methods, and to contribute to state of the art of the classical approach, for industrial applications.

Several challenging test cases were investigated, including three-dimensional flows configuration over M6 ONERA wing, two-dimensional transonic flow over a NACA0012 airfoil and two-dimensional multicomponent airfoil configuration in turbulent regime.

The increased level of difficulty related to these industrial test cases, with respect to the academic benchmark investigated until that moment, due to the occurrence of many of the limiting factors commented in section [1.1.2], resulted in poor successful applications. The classic, and most memory demanding, approach for the stability analysis was identified as the best numerical methods for industrial applications. However, only the two-dimensional industrial configurations were successfully analyzed. At the end of these analyses, the obtained results clarified the limitations of the stability analysis, in its classic and alternative approach, for industrial applications.

As consequence, the stability analysis of a challenging two-dimensional turbulent flow of a highlift configuration was performed for a wide range of angle of attack, using the classic numerical approach. The results of this analysis, which represents a remarkable contribution to the state of the art for the industrial applications of the stability analysis, are shown in chapter [4].

Finally, all the collected information and results were merged in order to define a strategy to overcome the limitations which affect the industrial application of the stability analysis. The solid academic background, built during the university experience, and the industrial know-how, acquired in the AIRBUS environment, led to the definition of a new line for tackling the computational issues associated to the stability analysis. This new approach is based on the concept of geometrical projection, and for this reason is hereafter referred as "domain reduction strategy". It permits to reduce the leading dimension of the matrix involves in the stability analysis EVP exploiting the geometrical features of the flow field. As a consequence the leading dimension of the matrix of the EVP can be reduced and computationally affordable. In few words, the key idea behind this technique is to consider only specific regions of the computational domain where the physical phenomenon of study is confined. Starting from a reference known fluid configuration, a smaller region of the domain is studied in order to understand if the stability analysis on the reduced domain is able to recover the stability features of the flow in the region of interest. To precisely determine the region of interest for this analysis, the structural sensitivity region is calculated, as suggested in [Giannetti 2007]. Different domains, comprising or not the structural sensitivity region, are then investigated in order to determine the minimum region that permits to recover the correct spectrum.

The domain reduction strategy, more extensively explained in chapter [5], was validated using both academic (an open cavity flow) and industrial (the buffet phenomenon on NACA0012 airfoil) benchmarks. In particular, the transonic buffet phenomenon on the NACA0012 represents the perfect test case for this analysis, since its leading mode (a shock wave) remains localized on the top of the airfoil, making easier the detection of the region of interest for the domain redaction.

The successfully results, obtained applying the domain reduction strategy, represents the most significant contribution of this thesis to the development of alternative approaches for stability analysis in industrial applications.

1.2 Significance

This work contributes to provide a detailed investigation regarding the numerical methods that are computationally cheaper for the stability analysis, with respect to the classic approach, which involves a full LU-decomposition and projection methods. Before this work the scientific community was missing an extensive review of numerical methods for stability analysis. This gap has been closed with this thesis.

The choice of the right numerical method can be very difficult considering all the parameters which influence the performance, leading to a consequent general sense of uncertainty regarding the choice of the most efficient numerical approach. The results presented in this thesis aim to provide the essential instruments to identify the best approach for stability analysis applications, taking into account the characteristic features of the flow configuration and the state of the art of this powerful methodology. This work represents a relevant contribution to clarify the effects of some of the several factors that influence the performances of the numerical approach. Based on the results in this thesis, the efficiency and rate of success of the numerical approach can be significantly improved.

A further contribution on the investigation of the state of the art for stability analysis is also provided in this thesis, where a relevant industrial application on high lift configuration is discussed. The complexity of the physic phenomena involved in high lift devices still represent a challenge for the stability analysis and an exhaustive investigation of this flow configuration is currently object of several studies. In this context, the results contained in this thesis contribute to understand the underlying mechanism which characterize the onset of instability for multi component airfoils and the potentiality of stability analysis for the industrial applications.

The improvements achieved in this thesis not only concern about the evaluation of cutting-edge numerical approach, but rather about the development of an independent new line, able to alleviate the massive computational costs related to complex flow configurations. This new approach defines a domain size reduction strategy, based on the concept of geometric projection. Specific and representative features of the considered flow configuration are used to identify the most relevant regions of the domain for the stability analysis. In this way, the stability analysis can be restricted to the only relevant regions, reducing the overall computational cost of the procedure. The approach is fully supported by a mathematical formulation, that clearly explain the concepts which define the strategy. A methodology to estimate the errors committed in performing the stability analysis in the reduced domain, with respect to a full domain analysis, is also provided. The domain reduction strategy, successfully validated on academic and industrial benchmarks, represents, to the authors opinion, a very promising and powerful tool for the always more challenging applications of the stability analysis.

1.3 Publication and outreach

During this PhD thesis various publications and presentations were made, in international journal and conferences.

Publications:

- S Sanvido, J Garicano-Mena, J de Vicente, E Valero, "Critical assessment of Domain Reduction Strategies for Global Stability Analysis", submitted for publication to AIAA Journal.
- J Garicano-Mena, S Sanvido, E Ferrer, E Valero, "A Stability Analysis of the Compressible Boundary Layer Flow Over Indented Surfaces", published on Computer and Fluids Journal, vol 160 (2018)10.1016/j.compfluid.2017.10.011
- S Sanvido, J de Vicente, E Valero, "Eigenvalue Problems in Flow Stability Analysis", submitted for publication to SIAM Journal.

Presentations:

- S Sanvido, J de Vicente, E Valero, "Domain Reduction Strategy for Stability Analysis of a NACA0012 airfoil". 5th ENG OPT Conference - Iguassu Falls, Brazil, June 2016.
- S Sanvido, J de Vicente, E Valero, Eigenvalue Problems in flow Stability Analysis, ILAS2016 Conference, Ku Leuven, Belgium, July 2016.
- S Sanvido, J de Vicente, E Valero, Survey of Numerical Methods for the Stability Analysis Eigenvalue Problems, USNCCM13 Conference San Diego, California, 27-30 July 2015.

1.4 Structure

This work is organized as follows:

Chapter 2 Chapter [2] aims to introduce the basic concept of the stability analysis, in order to provide the essential knowledge required to understand the more detailed and technical chapters included in this thesis. In the first part, the Navier-Stokes equations which describe the spatial and time evolution of the flow field are discussed, and details regarding their numerical solution, such as spatial and temporal discretization techniques, are summarized. In the second part, the linear stability theory is explained, and the specific approximations used for the BiGlobal stability analysis are discussed. The last part of the chapter is focused on the definition of the structural sensitivity analysis, that will be later applied in chapters [5] and [4].

Chapter 3 Chapter [3] summarizes the results obtained during the investigation of possible numerical approaches, as alternatives to the classic memory demanding approach currently used in stability analysis. The chapter starts with a comprehensive review of the numerical methods available for the stability analysis, recalling some of the concepts introduced in the chapter [2]. The aim of this chapter is to compare a considerable number of numerical methods for the stability analysis of aerodynamic problems, in order to identify the most suitable approach in terms of computational costs and precision. Two benchmark test cases were used to perform this comparison: a lid driven cavity flow and an open flow over a flat plate.

Chapter 4 Chapter [4] presents the first industrial test case investigated in this thesis: a two-dimensional high lift configuration. More precisely, the stability analysis for this flow configuration was performed for a wide range of angles of attack. The evolution of the spectra with respect to this parameter is analyzed and the most unstable modes are shown. This complex flow configuration, characterized by vortex structures, represents a cutting-edge example of industrial application for the stability analysis, in its classic numerical approach.

Chapter 5 In chapter [5] the domain reduction strategy is presented. This new approach is based on the idea to reduce the size problem performing a geometrical projection instead of the classical projection based on mathematical properties of the matrix (Krylov projection method). Two benchmarks were chosen to test this strategy: the open cavity flow configuration and the buffet phenomenon on the NACA0012 airfoil.

Chapter 6 Conclusions and an outlook are given in chapter [6].

Stability Analysis

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This chapter contains an introduction to global stability analysis, starting from the numerical solution of the Linearized Navier-Stokes Equations (LNSE), until the formulation of the related EigenValue Problem (EVP), passing through the definition of the BiGlobal linear stability. Furthermore, adjoint formulation and sensitivity analysis, for a complete overview of the stability properties of the flow configuration, are also discussed. More extensive reviews of the concepts here introduced can be found in [Theofilis 2011, Theofilis 2003, de Vicente Buendía 2010, Iorio 2014, Iorio 2015, Alizard 2007, Ramanan 1994, Noack 1994, Giannetti 2007, Browne 2014, Browne 2016, Garrido 2014].

This chapter is organized as follows:

In the first part, section [2.1], the Navier-Stokes equations, which describe the motion of a flow field, are introduced and it is shown how to solve them numerically. The second part [2.2], introduces the basic concepts of the linear stability theory. The two main approaches (continuous and discrete) are presented. Furthermore, the definition of the EigenValue Problem (EVP) which contains important information about the stability features of the flow, is given. In the following, the BiGlobal stability analysis, and its assumptions for the flow features, is discussed. The section [2.3] introduces the structural sensitivity analysis, a powerful instruments used to investigate the underlines features of the flow configuration. Furthermore, it contains an introduction to the BiGlobal adjoint formulation for the stability analysis, intrinsically connected to the concept of the geometrical receptivity.

2.1 Governing equations

2.1.1 The Navier-Stokes Equations

Navier-Stokes Equations (NSEs) are a set of Partial Differential Equations (PDE) that depend both on time and space, which describe the non-linear dynamic behavior of a fluid. For a compressible viscus flow, the NSEs can be expressed as follow:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}(\rho u_i) &= 0 \\ \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) &= -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \\ \frac{\partial}{\partial t}[\rho(e + \frac{1}{2}u_i u_i)] + \frac{\partial}{\partial x_j}[\rho u_j(h + \frac{1}{2}u_i u_i)] &= \frac{\partial}{\partial x_j}(u_j \tau_{ij}) - \frac{\partial q_j}{\partial x_j}\end{aligned}\tag{2.1}$$

where t stands for the time, p for the pressure, ρ for the density and u_i is the vector comprising the velocity components $(u_1, u_2, u_3) \equiv (u, v, w)$.

Furthermore, the classical ideal gas law is used to close the system:

$$p = \rho R T = (\gamma - 1)\rho e\tag{2.2}$$

where e is the specific internal energy and $h = e + p/\rho$ is the specific enthalpy, also expressed by:

$$e = c_v T, \quad h = c_p T\tag{2.3}$$

where c_v and c_p are the specific-heat coefficients ($\gamma = c_p/c_v$ and $R = c_p - c_v$). The convective heat flux q_j is defined as follow:

$$q_j = -k_{eff} \frac{\partial T}{\partial x_j}\tag{2.4}$$

where T represents the temperature.

Finally, the effective stress tensor is expressed as:

$$\tau_{ij} = 2\mu_{eff} \frac{\partial u_i}{\partial x_j} - \frac{2}{3}\mu_{eff} \frac{\partial u_k}{\partial x_k}.\tag{2.5}$$

with effective dynamic viscosity μ_{eff} and effective heat conductivity k_{eff} defined as

$$\mu_{eff} = \mu + \rho \nu_t; \quad k_{eff} = \frac{\mu}{Pr} + \frac{\rho \nu_t}{Pr_t}\tag{2.6}$$

Being μ the dynamic viscosity. Pr and Pr_t are the Prandtl number and the turbulent Prandtl number.

The set of equations (2.1) do not have an explicit solution, except for geometrically simple problems and restrictive hypothesis, for which the analytic solution may be obtained. For other more complex geometries, the only possibility is to solve the equation numerically. In this case, the common procedure consists of discretizing the flow field into a mesh. This converts the problem into a system of algebraic equations, for both flow variables and their derivatives. Since NS equations depend on time and space, both spatial and temporal discretizations have to be performed to numerically solve the systems of PDEs. However, these two discretizations can be performed separately and several numerical

methods are available for this purpose. The spatial discretization is performed first, then a Cauchy problem is defined for the temporal discretization.

A generic introduction to the theory regarding the methodologies used to perform these discretizations is provided in the next two sections, [2.1.2] and [2.1.3]. A more widely and extensively discussion about discretization techniques is available in the literature [Drazin 1981, Hildebrand 1987, Isaacson 1994, Canuto 1988, Canuto 2006, Canuto 2007, Boyd 2001, Zienkiewicz 1983, Wesseling 2001, Hirsch 1989, Hesthaven 2011].

2.1.2 Spatial discretization

In this section, the *Methods of Weighted Residuals* (WRM) is introduced. Any system of partial differential equations, can be expressed in the following form:

$$\frac{\partial \mathbf{u}}{\partial t} - L(\mathbf{u}, \lambda) = 0 \quad \text{in } D \quad (2.7)$$

where $\mathbf{u}$ is the (scalar or vector) unknown function. L is the spatial operator which depends on the $\mathbf{u}$ function and its spatial derivatives, and on a set of parameters λ , defined on the domain D . Note that if the operator L is substituted with the terms $-\nabla \mathbf{F}(\mathbf{u})$, then the system of equations (2.7) becomes equivalent to equations (2.1). The theory of the *Methods of Weighted Residuals* implies that the solution of equations (2.7) can be locally approximate with the following series:

$$\mathbf{u} \approx \tilde{\mathbf{u}} = \sum_{k=1}^N \alpha_k \varphi_k \quad (2.8)$$

Where α_k are coefficients and φ_k are base, or trial, (independent) functions. According to the definition of approximation, if the series in equation (2.8) is substituted into the equation (2.7), the Right-Hand-Term (RHT) of the same equations might not be equal to zero, leading to the definition of the Residual R :

$$R = \frac{\partial \mathbf{u}}{\partial t} - \sum_{k=1}^N \alpha_k \varphi_k \neq 0 \quad (2.9)$$

The class of *Methods of Weighted Residuals* aims to minimize R to produce a converged solution. Different WRM can be defined, depending on the way to calculate the coefficients and the base functions for the expansion (2.8), and on the procedure to force the residual to zero in (2.9). The WRM include several methods, and different type of classifications are available for them. The first classification distinguishes between *weak* and *strong* formulations: the weak formulation uses integral form to define the residual R , while the strong formulation uses a differential approach for the residual. According to these definitions, in weak formulation the residual is forced to zero in an integral way, as follows:

$$\int_{\Omega} R W_k d\Omega = 0 \quad k = 1, \dots, N \quad (2.10)$$

Where W_k are weight (or test) functions. Different definitions of these test functions distinguish the type of method: Sub-Domain Method, Least-Squares Method and Galerkin Method are some of those. The strong formulations (Collocation Method) impose the residual to be zero in a set of points of the domain D , and it is then characterized by the

following conditions:

$$R(x_k) = 0 \quad k = 1, \dots, N \quad (2.11)$$

The equation (2.11) is equivalent to the fact that the approximating function assumes the exact value of the unknown function on the collocation points. It is also possible to define an *integral formulation* of equation (2.11), using the Dirac delta function as weight functions:

$$W_k = \delta(x - x_k) \quad k = 1, \dots, N \quad (2.12)$$

Equation (2.12) represents the equivalent *weak* formulation of the collocation method.

The classification into weak or strong formulation is done depending on the definition of the residual, but also the choice of the trial function W_K is determinant, leading to another classification: local and global WRM. Local WRM use local polynomials as base functions to approximate the unknown function $\mathbf{u}$: Finite Difference Methods (FDM), Finite Element Methods (FEM) [Zienkiewicz 1983] and Finite Volume Methods (FVM) [LeVeque 2002, Toro 1999] are examples of local WRM methods. Global WRM are also known as Spectral methods [Boyd 2001, Canuto 1988, Canuto 2006, Hesthaven 2011].

In FDM, the domain is sub-divided into a grid and local overlapping low-order polynomials are used to define the approximating solution. In FDM the derivatives of the unknown functions are approximated using values of the approximating functions themselves, calculated on a set of discrete points. For this reason, these methods are conceptually enclosed in the category of collocation methods and are usually used to solve PDEs in strong formulations.

FEM first divide the entire domain into a set of simple sub-domains (i.e. finite elements), then a set of local non-overlapping polynomials, of fixed order, is chosen to approximate the solution on each sub-domain. FEM derives from a weak formulation of the WRM and so a variational equation is imposed on each element to define the residual R .

A similar approach is used in the Finite Volume Methods (FVM), which subdivide the mesh into small control volumes around each grid points. FVM defines the PDEs using volume integrals. The divergence theorem is applied to convert the volume integral containing divergence terms into surface integral, that are then evaluated as entering/exiting fluxes through the finite volume surfaces. FVM is a conservative approach, and it is usually applied for unstructured meshes.

The Spectral Methods are a class of WRM that use global functions (i.e. defined on the entire domain) mutually orthogonal to approximate the solution. Examples of possible base functions can be: Fourier series to approximate functions with periodic behavior or Chebyshev polynomials [Boyd 2001] to approximate non-periodic functions.

This last class of WRM gives the possibility to obtain better accuracy than the previous two FDM and FEM, but its implementation for complex geometries is complicated. In order to avoid this drawback, Domain Decomposition Methods were specifically implemented for Spectral Methods [Canuto 2007].

2.1.3 Temporal discretization

Assuming that the spatial discretization of the system has already been performed, the system of PDEs (2.1) can be rewritten in a discrete form, as follow:

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}) \quad (2.13)$$

where now $\mathbf{u}$ is a time-dependent N-dimensional vector, originates from the spatial discretization of the computational domain D , and $\mathbf{F}$ represents the numerical discretization of the fluxes on the same domain D . The temporal discretization is then employed to calculate the evolution in time of the system (2.13), when a solution at the initial time t_0 (initial value problem) is given. Performing this discretization, a set of snapshots of the solution is obtained. These snapshots represent the solution of the unknown function at consecutive instants.

$$\{\mathbf{u}(t_0), \mathbf{u}(t_1), \mathbf{u}(t_2), \dots\} \quad (2.14)$$

Numerical methods used for the temporal discretization of the equations (2.1) are basically divided into two groups: implicit and explicit schemes. Explicit schemes refer to all the schemes that require to know only the solution at previous steps in order to obtain the solution at the current step. In other words, using the solutions at times $t_{n-1}, t_{n-2}, \dots$, it is possible to obtain the solution at the time t_n .

The implicit schemes are all the schemes where the function at the current step t_n depends on the current step t_n itself. Explicit schemes are usually easy to implement, but they are limited in obtaining a stable solution. On the other hand, implicit schemes are more stable, but lead to high computational costs since they always require to invert the function used to describe the time evolution, which is defined by the specific schemes. Within all the possible numerical methods for the numerical integration, the explicit and implicit Euler methods are the simplest choices, but many other methods are available, such as Runge Kutta family methods and Backward Differential Formula (BDF) methods. Interested reader can find more details in [Wesseling 2001, Hirsch 1989]

2.2 Linear stability analysis

2.2.1 The linear stability theory

Flow unsteadiness are related to undesirable load variations that can represent a crucial problem for engineering applications. Stability Analysis is a powerful method to obtain a physical understanding of the underlying mechanisms which lead to the unsteady configuration.

The linear stability analysis is based on the linearization of the NS equation around a steady flow, also called base flow. The solution of this base flow is then perturbed and the evolution of the flow under the effects of these perturbations is analyzed in order to detect the stable/unstable response of the flow itself. The final goal is to detect and prevent the onset of flow unsteadiness. The base flow is defined *steady* according to the Lyapunov sense: the introduction of a negligible perturbation into the flow will produce an infinitely small variation in the basic flow variables, at some future time.

$$\|\mathbf{u}(\mathbf{x}, 0) - \mathbf{u}^*(\mathbf{x}, 0)\| < \delta \rightarrow \|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}^*(\mathbf{x}, t)\| < \varepsilon \quad (2.15)$$

Moreover, the solution is said asymptotically stable if the following criteria is satisfied:

$$\|\mathbf{u}(\mathbf{x}, t) - \mathbf{u}^*(\mathbf{x}, t)\| \rightarrow 0 \quad as \quad t \rightarrow \infty \quad (2.16)$$

Once that the stable base flow is provided, stability analysis approach proceeds with the flow decomposition in equation (2.17), that superimposes a small-amplitude time-dependent

perturbation $\tilde{\mathbf{u}}$, which satisfied the NS equations, to the base flow $\bar{\mathbf{u}}$:

$$\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}(\mathbf{x}) + \varepsilon \tilde{\mathbf{u}}(\mathbf{x}, t) \quad (2.17)$$

with $\varepsilon \ll 1$

Then the linearization of the NS is performed. This linearization can follow two different approaches: continuous and discrete.

In the continuous approach the stability equations are derived directly from the NS equations, which are then linearized and later discretized. In the discrete approach the (non-linear) NS equations are preliminary discretized into a computational mesh, defined over the problem geometry, and then linearized. Both approaches lead to a generalized eigenvalue problem (EVP) which contains the stability properties of the base flow, but the related matrices can be significantly different.

2.2.2 Continuous approach

Considering, for example, the incompressible NS equations:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \nabla) \mathbf{u} &= \frac{1}{Re} \nabla^2 \mathbf{u} - \nabla p \\ \nabla \mathbf{u} &= 0 \end{aligned} \quad (2.18)$$

Introducing the decomposition (2.17) into the system of equation (2.18), the following system of equations is obtained:

$$\begin{aligned} \frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon \frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \nabla) \bar{\mathbf{u}} + \epsilon (\bar{\mathbf{u}} \nabla) \tilde{\mathbf{u}} + \epsilon (\tilde{\mathbf{u}} \nabla) \bar{\mathbf{u}} + \epsilon^2 (\tilde{\mathbf{u}} \nabla) \tilde{\mathbf{u}} &= \\ = \frac{1}{Re} \nabla^2 \bar{\mathbf{u}} + \epsilon \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} - \nabla \bar{p} - \epsilon \nabla \tilde{p} & \\ \nabla \bar{\mathbf{u}} + \epsilon \nabla \tilde{\mathbf{u}} &= 0 \end{aligned} \quad (2.19)$$

The basic state ($O(0)$ terms) is a steady solution of the NS equations, so it is possible to subtract it in equation (2.19), obtaining the NS equation for the perturbing terms only:

$$\begin{aligned} \epsilon \left[\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \nabla) \bar{\mathbf{u}} + \epsilon (\tilde{\mathbf{u}} \nabla) \tilde{\mathbf{u}} \right] &= \left[\frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} - \nabla \tilde{p} \right] \epsilon \\ \nabla \tilde{\mathbf{u}} &= 0 \end{aligned} \quad (2.20)$$

Furthermore, in equation (2.20), the $O(\varepsilon^2)$ non-linear term " $(\tilde{\mathbf{u}} \nabla) \tilde{\mathbf{u}}$ " is neglected, leading to a perturbed definition of the problem composed of only $O(\varepsilon)$ terms. According to this, the linearized NS equations in continuous formulation are:

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \nabla) \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \nabla) \bar{\mathbf{u}} &= \frac{1}{Re} \nabla^2 \tilde{\mathbf{u}} - \nabla \tilde{p} \\ \nabla \tilde{\mathbf{u}} &= 0 \end{aligned} \quad (2.21)$$

From this point on, two approaches are possible: modal and non-modal. The non-modal approach rewrites the equation (2.21) in a compact form as follow:

$$\frac{d\tilde{\mathbf{u}}}{dt} = L_{NSE}(Re, \bar{\mathbf{u}}) \tilde{\mathbf{u}} \quad (2.22)$$

Where the linear operator L_{NSE} describes the growth/decay of the perturbation, depending on Re and base flow $\bar{\mathbf{u}}$, and it completely defines the stability of the flow. Modal approach is based on the linear theory and asserts that, if the base flow is a steady solution, the coefficients of the linearized problem are independent of time. This also means that, applying the separation of variables concept, it is possible to express the solution for $\tilde{\mathbf{u}}$ as linear superposition of normal modes, each of them described by the following expression:

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x}) \exp^{-i(\omega t)} \quad (2.23)$$

Introducing the equation (2.23) into the linearized equations (2.21), and performing the discretizations steps as explained in sections [2.1.2] and [2.1.3], leads to the definition of an eigenvalue problem. This EVP contains all the information about stability properties of the flow. The problem of stability analysis assumes then the following form:

$$A\hat{\mathbf{u}} = \omega M\hat{\mathbf{u}} \quad (2.24)$$

where $\omega = \omega_r + i\omega_i$ is a complex number (eigenvalue), A represents the discretized operator for the stability linearized equations and the M matrix is the mass matrix. The expressions for the boundary conditions depend on the formulation used for the discretization. According to the modal approach, defined in equation (2.23), the necessary condition to declare a base flow as stable is to have all the eigenvalues of the EVP in equation (2.24) with negative imaginary part. In fact, the imaginary part ω_i of these eigenvalues is associated to the growth rate of the exponential function and, depending on its sign, the perturbation can decay or increase in time. The real part of the eigenvalues (ω_r), is associated to the frequency of the mode. In case even only one of these eigenvalues presents a positive imaginary part, the solution diverges due to the contribution of this unstable mode, leading to an unstable flow configuration.

2.2.3 Discrete approach

Independently of the numerical method used for the discretization, which does not affect the formulation, this approach leads to the following discrete definition for the PDEs:

$$M \frac{d\mathbf{u}^h}{dt} = \mathbf{F}(\mathbf{u}^h) \quad (2.25)$$

where $\mathbf{u}^h$ is the unknown N -dimensional vector, M is the mass matrix and $\mathbf{F}(\mathbf{u}^h)$ is a N -dimensional vector which represents the numerical fluxes, obtained after the discretization

The flux decomposition expressed in equation (2.17) is introduced into equation (2.25), and the RHT is linerized using the Taylor series around the base flow (i.e. neglecting the second order terms $(\tilde{\mathbf{u}})^2$ in the Taylor series). Since the steady solution $\bar{\mathbf{u}}$ satisfies $F(\bar{\mathbf{u}}) = 0$, the following expression holds:

$$M \frac{d\tilde{\mathbf{u}}}{dt} = \left. \frac{\partial \mathbf{F}(\mathbf{u})}{\partial \mathbf{u}} \right|_{\bar{\mathbf{u}}} \tilde{\mathbf{u}} \quad (2.26)$$

At this point, equivalently to what happen for the continuous approach, is possible to express the perturbation flow according to modal or non-modal theory. Opting for a modal approach, as defined in equation (2.23), it is possible to recover the EVP definition

for the discrete approach:

$$A(\bar{\mathbf{u}}) \hat{\mathbf{u}} = \omega M(\bar{\mathbf{u}}) \hat{\mathbf{u}} \quad (2.27)$$

Where the A matrix (now) is the Jacobian of the system:

$$A(\bar{\mathbf{u}}) = \left. \frac{\partial \mathbf{F}(\mathbf{u})}{\partial \mathbf{u}} \right|_{\bar{\mathbf{u}}} \quad (2.28)$$

The equations (2.24) and (2.27) are characterized by different matrices, but their conclusion regarding the stability analysis for the base flow is equivalent.

2.2.4 BiGlobal stability analysis

The modal decomposition defined in equation (2.23), in its most generic interpretation, defines an inhomogeneous flow field, for both base flow and perturbations, along the three spatial directions and time (TriGlobal linear instability analysis). A complete TriGlobal analysis is really demanding in terms of computational costs, and it is currently possible only for simple flow configurations. However, a set of possible assumptions to simplify the decomposition itself is available (depending on the case examined), in order to make the system of PDEs in (2.1) solvable with a reasonable computational cost. Between all the possible flow characterizations, this thesis is focus on the applications of the BiGlobal linear stability analysis.

BiGlobal linear theory considers a two dimensional definition of the base flow (the dependency by the third spatial dimension is negligible) and a three-dimensional flow field for the perturbations, for the which one of the directions (usually z -direction) is supposed to be periodic and expressed by means of an harmonic expansion, with wavelength l_z [Theofilis 2011].

$$\mathbf{u}(x, y, z, t) = \bar{\mathbf{u}}(x, y) + \varepsilon \tilde{\mathbf{u}}(x, y, z, t) \quad (2.29)$$

with

$$\tilde{\mathbf{u}}(x, y, z, t) = \hat{\mathbf{u}}(x, y) \exp \{i(\beta z - \omega t)\} \quad (2.30)$$

$$l_z = \frac{2\pi}{\beta} \quad (2.31)$$

Where, β , the wavenumber of the disturbance, is a real number.

When the flow decomposition described in equation (2.29) is introduced into the Linearized Navier-Stokes equations (LNSE), the following complex BiGlobal EVP is defined, [Theofilis 2003]:

$$\begin{aligned} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + i\beta \hat{w} &= 0 \\ (\mathcal{L} - \frac{\partial \bar{u}}{\partial x} + i\omega) \hat{u} + \frac{\partial \bar{u}}{\partial y} \hat{v} + \frac{\partial \hat{p}}{\partial x} &= 0 \\ -\frac{\partial \bar{v}}{\partial x} \hat{u} + (\mathcal{L} - \frac{\partial \bar{v}}{\partial y} + i\omega) \hat{v} + \frac{\partial \hat{p}}{\partial x} &= 0 \\ -\frac{\partial \bar{w}}{\partial x} \hat{u} - \frac{\partial \bar{w}}{\partial y} \hat{v} + \mathcal{L} \hat{w} - i\beta \hat{p} &= 0 \end{aligned} \quad (2.32)$$

with

$$\mathcal{L} = \frac{1}{Re} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^2 \right) - \bar{u} \frac{\partial}{\partial x} - \bar{v} \frac{\partial}{\partial y} - i\beta \bar{w} \quad (2.33)$$

The equation (2.32) can be also expressed in a compact EVP form, becoming the system of equations (2.24) or (2.27), depending if the continuous or discrete approach is used.

2.3 Structural Sensitivity Analysis

In [Giannetti 2007], authors comment the evident differences in shape between the direct and adjoint eigenvectors for the stability analysis of the flow around a cylinder. This suggests that a reliable investigation of the flow features can not be performed considering the two eigenfunctions separately. Consequently, the concept of sensitivity region S is introduced and both direct and adjoint formulation of the stability analysis have to be solved to calculate it. The adjoint formulation represents the complementary problem of the direct formulation, defined in equation (2.24)/(2.27), and it is used to recover the structural receptivity of each modes which compose the flow field [Lucchini 2014, Browne 2014, Iorio 2014].

2.3.1 BiGlobal adjoint formulation

Adjoint formulation is related to concepts of flow control and geometrical receptivity, and it was introduced by [Zhigulev 1987] and more recently discussed in [Hill 1992, Giannetti 2007, Browne 2014].

The receptivity of the eigenmode is used to evaluate the effects of initial conditions and forcing terms on the time evolution of the LNEs, such those in equation (2.32). Interested reader can find a more detailed investigation in [Bewley 2001, Dobrinsky 2000, Morse 1953, Pralits 2003].

Given two generic vectors $\mathbf{p}$ and $\mathbf{q}$, the following inner product is defined:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \mathbf{p}^H M \mathbf{q} \quad (2.34)$$

where the H superscript indicates the complex conjugates transpose and M is (again) the volume matrix.

Once that an inner product is selected, the following relation defines the adjoint operator associated to the A matrix of the direct problem, defined in equations (2.24)/(2.27):

$$\langle \mathbf{p}, A\mathbf{q} \rangle = \langle A_{adj}\mathbf{p}, \mathbf{q} \rangle \quad (2.35)$$

Combining the definition of the inner product in equation (2.34) and the previous equation (2.35), the following relations is obtained:

$$(A\mathbf{q})^H M \mathbf{p} = (\mathbf{q})^H M A_{adj} \mathbf{p} \quad (2.36)$$

$$\mathbf{q}^H A^H M \mathbf{p} = \mathbf{q}^H M A_{adj} \mathbf{p} \quad (2.37)$$

Manipulating the previous equation (2.37), the definition for the adjoint matrix A_{adj} can be expressed as:

$$A_{adj} = M^{-1} A^H M \quad (2.38)$$

That correlates the adjoint matrix A_{adj} in equation (2.38) with the complex conjugate transpose of the A matrix in equation (2.24)/ (2.27).

Following the definition of the adjoint matrix A_{adj} , the adjoint EVP problem which lead to obtain the adjoint eigenmodes is defined by:

$$A_{adj}\mathbf{u}_{Adj} = \omega_{Adj}M\mathbf{u}_{Adj} \quad (2.39)$$

Where ω_{Adj} and $\mathbf{u}_{Adj}$ represent the adjoint eigenvalue and the adjoint eigenvector respectively. Finally, defining $\tilde{\mathbf{u}}_{Adj} = M\mathbf{u}_{Adj}$ and left pre-multiplying the equation (2.39) with the M matrix, and considering the relation in equation (2.38), the following relation holds:

$$A^H\tilde{\mathbf{u}}_{Adj} = \omega_{Adj}M\tilde{\mathbf{u}}_{Adj} \quad (2.40)$$

If the Hermitian operator is applied to equation (2.40), then:

$$\tilde{\mathbf{u}}_{Adj}^H A = \omega_{Adj}^H \tilde{\mathbf{u}}_{Adj}^H M^H \quad (2.41)$$

The equation (2.41), if compared with equation (2.24), clarifies that the eigenvalue of the adjoint EVP are the complex conjugates of the correspondent direct EVP, while the adjoint eigenvectors $\tilde{\mathbf{u}}$ represents the left eigenvectors of the A matrix from the direct EVP. For the particular case when both the matrices A and M of the direct problem are real, as in all the applications investigated in this thesis, the complex conjugate transpose coincides with the transpose itself and the equation (2.40) become:

$$\tilde{\mathbf{u}}_{Adj}^T A = \omega_{Adj}^H \tilde{\mathbf{u}}_{Adj}^T M \quad (2.42)$$

From a computational point of view, equation (2.42) this implies that, in real arithmetic, the direct and adjoint EVP can be solved at the same time, calculating both the left and the right eigenvectors of the A matrix of the direct formulation, defined in equations (2.24)/(2.27). However, a further step involving the matrix-vector product between the inverse of the M matrix and the left eigenvector $\tilde{\mathbf{u}}_{Adj}$ is needed in order to recover the correct scaled adjoint mode $\mathbf{u}_{Adj}$, since $\mathbf{u}_{Adj} = M^{-1}\tilde{\mathbf{u}}_{Adj}$.

2.3.2 Structural sensitivity

The sensitivity region of an eigenmode is defined as the region where a small perturbation can cause the largest drift in the related eigenvalue [Sartor 2015, Giannetti 2007, Meneghello 2015, Browne 2014]. Specifically, the sensitivity region the i -th mode is obtained performing the scalar product between the i -th direct mode $\mathbf{u}$ and the i -th adjoint mode $\mathbf{u}_{Adj}$.

$$S = \|\mathbf{u}\| \cdot \|\mathbf{u}_{Adj}\| \quad (2.43)$$

with

$$\langle \mathbf{u}, \mathbf{u}_{Adj} \rangle = 1 \quad (2.44)$$

where $\langle \bullet, \bullet \rangle$ is the same inner product defined in equation (2.34), with the correspondent norm defined as $\|\bullet\| = \langle \bullet, \bullet \rangle^{1/2}$. In [Giannetti 2007], the derivation of concept related to the stability region are modeled introducing forcing terms into the momentum equation, so only the velocity components of the eigenfuncions are actually relevant and used in the equation (2.43).

Another definition of the Sensitivity region, defined as maximum sensitivity region, refers to the following equation:

$$S_{max} = \| \mathbf{u} \cdot \mathbf{u}_{Adj} \| \quad (2.45)$$

In equation (2.45), the maximum region where the introduction of a perturbation or external force can affect the flow feature is defined. For this case, all the components of the eigenfunction are involve in the inner product.

Eigenvalue Problems for Stability Analysis

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The aim of this chapter is to revisit and present a critical assessment of the different methods to obtain the most significant eigenvalues and eigenvectors in the flow stability problem. In fact, once that the stability analysis is applied and the associated EVP is defined, as described in chapter [2], the variety of the numerical technique available make it difficult to determine the most suitable algorithm. This usually translates in poor implementations and not satisfactory results.

A direct approach, recovering the full-spectrum, is the best practice to avoid missing significant physical phenomena, since direct computation, such as the QZ algorithm, returns all the information contained in the eigenspectrum. The direct approach is still used in the stability analysis, limited by its applicability to simple flows configurations, as a consequence of the cost associated: $O(10N^3)$ operations, with N representing the dimension of the EVP. In industrial applications, its ineffectiveness exploiting sparsity patterns and the necessity to obtain specific part of the spectrum, preclude full spectrum computation to be applied for most of the multidimensional problems of practical relevance. Alternatively, iterative eigensolvers reduce the computational requirements and are especially suitable for linear stability analysis, whenever only the leading eigenvalues are sought or when an

explicit branch of modes is of interest. This approach to the stability problem is by far the most widely used in the literature.

Nowdays, the number of algorithms available has increased significantly and several variants exploit specific properties of the eigenvalue problem. A wide-range comprehensive overview on performance and applicability of these numerical approaches specifically for fluid dynamic applications is currently, at author's knowledge, not available in literature, and it is intended to be provided in this chapter. Figure [3.1] reassumes the steps of direct and iterative numerical approaches used for solving the EVP that will be here reviewed.

Figure 3.1: Numerical schemes to solve EVPs

Moreover, in order to compare the performances of the different numerical methods, two benchmark problems, the three-dimensional lid-driven cavity flow and the two-dimensional flat plate parallel flow, were re-visited and the algorithms previously discussed, evaluated. The linear stability analysis was accomplished using a modal Biglobal approach. A high order spectral method based on Chebyshev polynomials was employed to discretize the spatial operator using strong formulation and the computational domain was decomposed, when necessary, with multidomain techniques. Leading eigenvalues for matrices corresponding to both benchmark test cases were retrieved with different algorithms provided by PETSc and SLEPc libraries, specifically dedicated to solve large eigenvalue problems.

This chapter is organized as follows: The first part, section [3.1], contains a comprehensive survey of the numerical methods used to solve eigenvalue problem. The second part [3.2], introduces the selected test cases for a comparison of the performances of the numerical methods: the Lid-Driven cavity and the flat plate. Part [3.3] contains the details regarding the practical implementation of the numerical methods chosen for the comparison. The results of the comparison of the numerical methods, for both test cases, are presented in [3.4].

3.1 Review of Numerical Methods in Flow Stability Analysis

In this section, an overview of the most used numerical methods to solve the large, real or complex, nonsymmetric generalised eigenvalue problem, associated to the global

instability analysis, is presented. Different examples of stability analysis can be found in the literature where the choice of the eigenspectrum computation algorithm differs significantly. Usually the choice is dictated mainly by computational constraints. However there are other properties of the eigenvalue problem that affect the choice of the method, the most will be explained in the next sections. In its generalized formulation, as already introduced in chapter [2], the EVP aims to determinate λ and $x \in \mathbb{C}^n$ such that:

$$Ax = \lambda Bx \quad (3.1)$$

has no trivial solutions, with A and $B \in \mathbb{C}^{n \times n}$.

Different numerical approaches are available to solve the EVP in (3.1). The methods are mainly subdivided into direct and iterative methods. Although this nomenclature can be somehow debatable, it is widely accepted to differentiate full-spectrum computation -direct- from subspace projection -iterative- techniques, and it is used throughout this work to be consistent to other publications

3.1.1 Direct Methods

The first (direct) category includes the well-known LR and QR algorithms for computing the eigenvalues of matrix A . Both are iterative procedures to factorize the original matrix in a advantageous form. A pioneering study by Pierrehumbert and Widnall [Pierrehumbert 1982] employed the LR method to study the stability properties of shear-layer vortices. Two main instabilities were found: subharmonic and translative. The authors used a steady solution of the two-dimensional Euler equations as a base flow to model the streamwise disturbances of the shear layer. The discretization of the linear operator using spectral collocation techniques led to a eigenvalue problem of the form:

$$Ax = \lambda x \quad (3.2)$$

where A is a complex general matrix of $N \times N$ entries ($N \sim 10^2$).

The QR algorithm, based on QR decomposition, has found to be more stable than its precursor LR. Many works in the last decades have employed this method to obtain all the eigenmodes related to a flow stability problem in one, two or three dimensional analysis. One of the most studied configuration is the stability analysis of the plane Poiseuille flow by Orszag [Orszag 1971] where the author employed a Chebyshev expansion to discretize the 1D Orr-Sommerfeld eigenvalue problem. Later studies by the same author, also using the QR algorithm, analyze the three-dimensional transition to turbulence in plane channel flows [Orszag 1980], the stability of a pipe flow to axisymmetric disturbances [Patera 1981] or the secondary instability of wall-bounded parallel shear flows [Orszag 1983]. In all these cases the rank of the eigenvalue problem is relatively low ($N \sim 10^2$).

More recently Tatsumi and Yoshimura [Tatsumi 1990] studied the stability of two-dimensional rectangular channel ducts of different aspect ratios and obtained the neutral curves of stability in terms of Reynolds number and the wavenumber of the perturbation. To do so, the authors used Legendre polynomials for the expansion of the flow variables and meshes with resolution varying from 20×20 to 20×40 leading to matrices with $\sim (10^3 \times 10^3)$ entries.

The QZ algorithm is based on Generalized Schur Decomposition and represents a variant of QR algorithm for generalised eigenvalue problems. [Zebib 1987] employed the QZ

algorithm to study the stability of incompressible laminar stationary flow past a circular cylinder. The author divided the analysis in two parts, analyzing the growth of symmetric and asymmetric perturbations separately. Disturbances were considered in two dimension, while a Fourier expansions is used to describe the behavior along the azimuthal direction, for the symmetric and asymmetric components. The resulting linear stability generalized eigenvalue matrices, with leading dimension $\simeq 30 \times 30$, were solved using standard library subroutines.

A step beyond in the analysis of the flow past a cylinder is the work by Noack and Eckelmann [Noack 1994] where the periodic cylinder wake was also analyzed using a low-dimensional Galerkin method. The discrepancies between numerical and previous experimental works were the motivation for this three dimensional stability analysis. A modal decomposition, using 63 two-dimensional Galerkin modes, was used to calculate a stability matrix whose eigenvalues were computed using Numerical Recipes [Press 1988] subroutines for the QR algorithm.

During the nineties Gelfgat [Gelfgat 1994, Gelfgat 1996] applied a Spectral Galerkin method and linear stability analysis to study the onset of an oscillatory state from a stationary regime. The two investigated problems, both characterized by oscillatory instability, were the convective flows in confined regions [Gelfgat 1994] and the swirling flow in a cylinder with a rotating lid [Gelfgat 1996]. In both cases, the flow variables of the problem were discretized using a Galerkin expansion with 20-30 basis functions in each spatial direction. The resulting ODE system was solved using the Newton method. Finally, the eigenvalues of the Jacobian matrix ($\simeq 30 \times 30$), calculated at the stationary solution, were computed using QR decomposition.

A common feature of all the previous works cited in this review is the limited order of magnetude of the matrices involved in the eigenvalue problem. Many of them even anticipated further research in reduced methods to deal with the limitations in term of applicability caused by the high memory costs required for solving large EVP.

Considering that Linear Stability Analysis focuses its attention on the detection of the least unstable modes only, a less expensive iterative method, able to recover only a selected part of the spectrum, can be a justified and convenient choice [Edwards 1994].

However, discarding full-spectrum computations can be a risky decision. Full-spectrum computation have shown crucial relevance in detecting several hydrodynamic and aeroacoustic instabilities. There are physical phenomena associated to branches of eigenvalues that may not be located in the leading part of the spectrum and could be missed, if iterative techniques are used. Dongarra et al. [Dongarra 1996], focused on the one dimensional Orr-Sommerfeld operator to implement a variant of the QZ-method in an efficient way, with the aim of extending its applicability to large systems. Authors' motivation for this work came from the previous research, [Butler 1992, Reddy 1993], where non-linear interaction of stable modes, far from the dominant one, were found to be responsible of instabilities at Reynolds numbers below linear analysis predictions.

In the late years of the 20th century, full-spectrum methodologies were still employed where the computational demand was not exceeding the available resources. Kuhlmann et al. [Kuhlmann 1997], used the QZ-algorithm subroutines from IMSL library for the generalized eigenvalue problem arising from the stability analysis of two-sided lid-driven cavities. Stability matrices were built employing mixed (finite-difference and Chebyshev collocation) discretization of flow variables in the two dimensional space (x, y) while periodic behaviour was assumed in the z -direction. Conversely, Lin and Malik [Lin 1996, Lin 1997], employed Chebyshev collocation for both normal (y) and chordwise (x) directions and,

again, periodic expansion in spanwise (z) direction to define disturbances arising in the stability analysis of attachment-line boundary layers. The eigenvalues of the complex generalized eigenvalue problem were obtained by QR decomposition of the Hessenberg form of the $A^{-1}B$ matrix.

In both cases, low grid resolution ($\simeq 100 \times 50$ in [Kuhlmann 1997], while $\simeq 25 \times 45$ in [Lin 1996, Lin 1997]) and improvement in hardware facilities made the full-spectrum computation affordable.

3.1.2 Iterative Methods

In the late nineties, Ding and Kawahara [Ding 1998] compared the performance of the *direct* QR method and an iterative (Arnoldi) method. The benchmark problem employed was the aforementioned lid-driven cavity for a set of Reynolds numbers varying from $Re = 50$ to $Re = 1500$ and grids also varying from (11×11) points to (81×81) points. In the particular case of $Re = 50$ and (31×31) grid, the CPU time was 235 min for the QR computations while 200 secs for the 60 steps of the iterative process. This example is quite illustrative of the limitations associated to the direct approach. Time consumption and memory requirements make full-spectrum computations only feasible for small-dimensional problems.

Iterative methods constitute an affordable alternative for obtaining some of the eigenvalues and eigenvectors of the EVP. These methods are based on the determination of approximations of the solution of the problem, by repeatedly applying a recurrence relation. In the following, the most commonly used iterative methods are introduced.

3.1.2.1 Power Methods

The simplest iterative method is the power iteration method. The mathematical procedure of this iterative method is simple, and it consists to repeatedly multiply the matrix A by a -random- initial vector x . As a results, this procedure amplifies the component of the initial x vector in the direction of the eigenvector associated to the largest eigenvalue.

For sake of simplicity and without loss of generality some assumptions are done to easily describe the algorithm. First premise states that the eigenvalues of the $n \times n$ matrix A , not necessarily different, can be sorted according to the relationship:

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

The corresponding eigenvectors $\{\vec{v}_i\}$ are solutions of the original eigenvalue equation, $Ax = \lambda x$, so sucesively evaluations of the matrix-vector product lead to $A^m \vec{v}_i = A^{m-1}(A \vec{v}_i) = A^{m-1}(\lambda_i \vec{v}_i) = \lambda_i (A^{m-1} \vec{v}_i) = \dots = \lambda_i^m \vec{v}_i$. The second assumption is that the set of eigenvectors $\{\vec{v}_i\}$ $i = 1, \dots, n$ forms a basis of $\mathbb{R}^n$ where any vector x can be linearly represented as:

$$x = \sum_{i=1}^n k_i \vec{v}_i = k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_n \vec{v}_n$$

Under those conditions, for a starting vector x , the iterative multiplication $A^m x$ can be

expressed as an expansion in the basis of eigenvectors in the following way:

$$A^m x = A^m \left(\sum_{i=1}^n k_i \vec{v}_i \right) = \sum_{i=1}^n k_i A^m(\vec{v}_i) = \sum_{i=1}^n k_i \lambda_i^m \vec{v}_i$$

So, the component in the direction of the eigenvector associated to the largest-magnitude eigenvalue, λ_1 , (hereafter dominant eigenvector) is prevailing if compared with the others

$$A^m x = \sum_{i=1}^n k_i \lambda_i^m \vec{v}_i = \lambda_1^m \sum_{i=1}^n k_i \frac{\lambda_i^m}{\lambda_1^m} \vec{v}_i \simeq \lambda_1^m \cdot K \vec{v}_1$$

Consequently:

$$\vec{v}_1 \simeq \frac{A^m x}{\|A^m x\|} \quad (3.3)$$

The corresponding eigenvalue λ_1 is obtained using the Rayleigh quotient

$$\lambda_1 = \frac{\vec{v}_1^T A \vec{v}_1}{\vec{v}_1^T \vec{v}_1} \quad (3.4)$$

The convergence of the power iteration method strongly depends on the choice of the initial vector and on the absolute value of the ratio between the second largest-magnitude eigenvalue and the dominant eigenvalue. This method enables the determination of only one eigenvector and one eigenvalue, the largest in magnitude of the matrix A . Usually, in stability analysis, the part of the spectrum of interest is the region which contains the smallest component of the eigenvalue (i.e. the real or imaginary part, depending on the convention adopted) associated to the damping rate. This region is usually located near the origin or in proximity of one of the two axes.

Inverse iteration method, in its most general description, transforms the original eigenvalue problem in a new system which still preserves the same eigenvectors, but associates the leading/dominant eigenvalue with the targeted one. If matrix A is replaced by its inverse B , with $B = A^{-1}$, the method will converge to $\frac{1}{\lambda_1}$, i.e. the smallest eigenvalue in magnitude. The advantage of inverse iteration technique is the flexibility to find any eigenvector associated to a specific eigenvalue μ of the matrix A . The transformation $B = (A - \mu I)^{-1}$ leads to the eigenvalue problem $Bx = \beta x$ whose eigenvalues are:

$$\beta_1 = \frac{1}{\lambda_1 - \mu}, \beta_2 = \frac{1}{\lambda_2 - \mu}, \dots, \beta_n = \frac{1}{\lambda_n - \mu} \quad (3.5)$$

If λ_i is the closest eigenvalue of A to μ , then the dominant eigenvalue of $Bx = \beta x$ will be β_i . This methodology, commonly known as *shift and invert*, yields in the broad category of rational preconditioners and is widely used and combined with different eigenvalue algorithms for the detection of eigenvalues in definite regions of the complex plane. The forthcoming section [3.1.3] will give more details about spectral transformations commonly used in flow stability analysis.

Several works in stability analysis have used the inverse power iteration method to determine the onset of stability. [Jackson 1987] studied the periodic vortex shedding in the wake of several two-dimensional bodies. Paying special attention to a circular cylinder and ellipses oriented at various angles to the flow. Two grids were used with 8320 and 32832 degrees of freedom respectively. The leading dimension of the matrices in-

involved, prevented the author from using QR or LR techniques. However, for this case, the inverse iteration technique for a generalized eigenvalue problem combined with single LU-decomposition, permitted to determine the critical Reynolds numbers and critical Strouhal numbers. Previous results by other authors, in time-dependent calculations and experiments, demonstrated that the used of an adequate initial guess for the iterative process can reduce the number of iterations to reach convergence. A similar solution procedure was employed by [Morzynski 1991] for the same study of the onset of periodic instabilities around an elliptical cylinder varying the axis ratio. Finite differences were used to discretize the incompressible Navier-Stokes equations in streamfunction formulation. The results were in good agreement with those provided by [Jackson 1987]. The inverse iteration was also applied to study the stability of thermocapillary convection. [Neitzel 1993] used a modification of the algorithm to determine the critical Marangoni number for given Prandtl, Grashof and Biot numbers in the linearized Boussinesq equations.

The fundamental drawback of power and inverse iteration is that these iterative methods are able to recover only one eigenvalue per time. In the stability analysis framework, they were soon replaced by variants able to compute several eigenpairs at once. However they are still used when only one eigenvalue is of interest. For instance, [Ramanan 1994] employed inverse iteration with an exponential transformation to validate the leading eigenvalue calculation in coarse grids obtained with different techniques. The authors focused in the classic problem of the flow in a square cavity driven by a moving lid at different Reynolds numbers. More recently [Giannetti 2007] revisited the first instability of the cylinder wake to analyse the structural sensitivity of the problem. To do that, the authors recovered the dominant eigenmode from the direct and adjoint problems. The investigation of the spatial structure of the product between the direct and adjoint eigenfunctions revealed that the region of the flow most sensitive to momentum forcing was located in two symmetric lobes at the end of the separation bubble.

3.1.2.2 Simultaneous Iteration Method

Power or Inverse iterative procedure may be reapplied with different starting vectors to obtain several eigenpairs. The first attempt, proposed by Wielandt in 1944, is a method called deflation. Deflation consists in transforming the original matrix A shifting the dominant eigenvalue to zero $B = A - \lambda_1 \vec{v}_1 \vec{v}_1^T$. So, when the iterative method is re-applied, the second eigenvalue in magnitude of matrix A becomes dominant in the modified matrix B . However, nowadays more reliable algorithms have been implemented and this technique is no longer competitive even for computing few eigenvectors. Another possibility is to work with a set of orthogonal vectors as initial values and to apply the power (or inverse) iteration method to each of them. This natural extension of the power method to a limited number of eigenvalues is the Subspace Iteration Method also named Simultaneous Iteration Method. It was firstly described by [Bauer 1957], modified by [Rutishauser 1969], and [Jennings 1975] proposed the most adopted version of the algorithm. The simplest approach to this method is to consider a basis of k linearly independent vectors $\{\vec{q}_1, \dots, \vec{q}_k\}$ as the columns of the initial matrix V_k^0 . The fundamental part of this iterative procedure is to obtain an orthogonal set of approximated eigenvectors in each step. This is done by QR factorization of V_k^i ,

$$V_k^i = Q_k^i R_k^i \quad (3.6)$$

followed by a multiplication of the original matrix A by the obtained Q_k^i factor:

$$V_k^{i+1} = A Q_k^i. \quad (3.7)$$

This algorithm guarantees that the eigenvector approximations remain orthogonal and the power method applied to every column of matrix V^i converges to eigenvectors corresponding to the eigenvalues of largest magnitude. Previously cited work by [Jackson 1987] already described a subspace iteration algorithm comprising the steps detailed above. However no further information are given regarding the number of eigenpairs calculated and only results of the critical parameters are provided. Currently, [Jennings 1975] represents one of the most relevant reference for Simultaneous Iteration Method in the stability analysis problem. The authors describe the modified algorithm for the generalized eigenvalue problem consisting of four steps: premultiplication, reorientation, normalization and convergence check. [Dijkstra 1992] studied the selection of flow patterns in rectangular containers due to buoyancy driven convection. The value of the critical Marangoni number for the first two primary bifurcations were found in an array of numerical setups, changing the aspect ratio (up to 4), the Biot number and the number of grid points (from 128 to 4096 in different configurations). Later, [Dijkstra 1995] extended the former investigation to bigger containers and finer grids by adding an iterative linear solver (Bi-CGSTAB) in the reorientation step of the algorithm. The aforementioned work by [Ramanan 1994] used simultaneous iteration to capture 10-15 eigenvalues in the lid driven cavity flow problem discretized on grids from 9×9 to 65×65 . Simultaneous iteration method showed a better performance than inverse iteration, all cases converged in less than 30 iterations and the reconstructed flow showed good agreement to experiments.

The work of Morzynski deserves a special mention in the adoption of Simultaneous Iteration methods to the field of flow stability analysis. In [Morzynski 1999] the authors presented the eigensolution method for global non-parallel flow stability analysis. The linearized Navier-Stokes equations are discretized using Galerkin Finite Element Method resulting a generalized eigenvalue problem solved following the algorithm proposed by [Jennings 1975]. To achieve an efficient inverse premultiplication in the procedure, the authors performs an ILU precondition of the equations. In addition, an inverse Cayley transformation preconditioned the matrix to drive the iteration to the required eigenvalues. The flow around a circular cylinder was chosen as test case to demonstrate the features of the methods. Matrices employed vary from 2994 to 198598 degrees of freedom (DoF). At its publication this number was among the highest used for computing non-parallel flow.

In [Morzynski 2008] the authors studied three-dimensional flows, extending the previous 2D work with domain decomposition and parallel computation of the eigensolver. The eigenproblem, with 2×10^6 DoF, was solved on 8-16 CPU clusters.

3.1.2.3 Krylov Subspace projection methods

Subspace projection methods are mainly based on the concept to project the matrix into a low-dimensional subspace, appropriately built, for extracting an approximation of the spectrum. In many cases this procedure is based on a Rayleigh-Ritz projection [JIA 2000], described below:

Let V be a $n \times m$ matrix with $m \ll n$, with the columns of V representing an orthonormal basis of a given subspace V , such that $V^T V = I_m$. The Rayleigh-Ritz procedure computes

$$H = V^T A V \quad (3.8)$$

where H is a square matrix $m \times m$, and solve the associated EVP

$$Hy_i = \theta_i y_i \quad (3.9)$$

The eigenpairs (θ_i, y_i) in (3.9) are then used to obtain an approximation $(\tilde{\lambda}_i, \tilde{x}_i)$ of the original eigenpairs (λ_i, x_i) in (3.2), according with the relations:

$$\tilde{\lambda}_i = \theta_i \quad (3.10)$$

$$\tilde{x}_i = Vy_i \quad (3.11)$$

$\tilde{\lambda}_i$ and $\tilde{x}_i$ are called, respectively, *Ritz value* and *Ritz vector* and the matrix H is the projection of A into the subspace V . The Rayleigh-Ritz procedure improves its accuracy if the subspace V is an invariant of the A matrix.

In 1931, Krylov proposed the set of vectors generated by the power method $x, Ax, A^2x, \dots$ to conform the basis of the projection subspace and consequently the columns of V . However, this technique fails due to the ill-conditioning of the generated basis. To overcome this, Hessenberg first suggested a modification of the Krylov basis, making it orthogonal to a given-set. So, the matrix H in (3.8) adopted an upper Hessenberg form.

Later, Lanczos developed an algorithm for symmetric matrices using orthogonal projection on the Krylov basis as it is constructed to obtained a tridiagonal matrix. A non-symmetric version of Lanczos's method requires the use of both the matrix A and its transpose. This unsymmetric approach employs oblique projection methods to ensure orthogonality of the generated basis to the original Krylov span $x, Ax, A^2x, \dots$ and to the transpose span $y, A^T y, (A^T)^2 y, \dots$ with $\langle x, y \rangle = 1$. So the reduced matrix H in (3.8) kept the tridiagonal form.

In 1951 Arnoldi proposed a version of the original symmetric Lanczos algorithm for unsymmetric matrices. As a member of the family of Krylov subspace methods the starting point is an initial (usually random) vector x_1 , for which a m dimensional Krylov space is defined as:

$$k_m(A, x_1) = \text{span} \{x_1, Ax_1, A^2x_1, \dots, A^{m-1}x_1\} \quad (3.12)$$

At each iteration in the projection procedure, the new vector in the m dimensional space is orthonormalized and added as a new column into matrix V . Simultaneously the corresponding reduced matrix H is updated. After m iterations the method reduces the $A \in \mathbb{C}^{n \times n}$ matrix into a Hessenberg matrix $\in \mathbb{C}^{m \times m}$, whose eigenvalues and eigenvectors may be obtained by full-spectrum (direct) techniques. The Arnoldi algorithm converges to the first m dominant eigenvectors associated to the m largest eigenvalues of A , in terms of norm. The simplicity of the Arnoldi algorithm, presented in what follow, make its implementation into a code particularly easy.

Algorithm 1 - Arnoldi Algorithm

```

Start with an arbitrary vector  $x_1$ 
 $v_1 = x_1 / \|x_1\|_2$ 
for  $j = 1, \dots, m - 1$  do
     $x_{j+1} = Av_j$ 
    for  $i = 1, \dots, j$  do
         $h_{i,j} = \langle x_{j+1}, v_i \rangle$ 
         $x_{j+1} = x_{j+1} - h_{i,j}v_i$ 
     $h_{j+1,j} = \|x_{j+1}\|_2$ 
    if  $h_{j+1,j} = 0$  then
        stop
    else
         $v_{j+1} = x_{j+1} / h_{j+1,j}$ 

```

A revealing comparison between Lanczos and Arnoldi's algorithm can be found in [Nayar 1993], where the authors examined both techniques combined with shift and invert strategy. The selected test case is the linear stability of compressible boundary layers. Navier-Stokes equations were discretized using finite differences obtaining complex, non-symmetric EVPs of dimension $5N \times 5N$ with N varying from 16 to 60. Results showed a slight advantage of Arnoldi procedure. Even when Lanczos method required less number of iteration for reaching the desired accuracy, the time employed was higher. Each Lanczos iteration comprises two forward and backward linear solves for only one in Arnoldi method.

The use of Arnoldi algorithm was extended in the literature for the computation of eigenvalues of the linearized steady state. [Eriksson 1985] presented an Arnoldi based eigensolver, with a spectrum transformation, to approximate eigenvalues with small negative damping rate. The authors chose a NACA 0012 airfoil and analyzed the linearized Euler equations at Mach 0.8 using different meshes. Another important contribution of this work is the use of Frechet derivative to develop a *matrix-free* algorithm for extracting the spectrum without storing the full matrix. The *matrix-free* approach was successfully exploited by Tezuka and Suzuki [Tezuka 2006] to analyse the three-dimensional stability of incompressible flow around a spheroid. More details about this technique may be found in [Bagheri 2009] or [de Pando 2012]. Another early example of the application of the basic Arnoldi algorithm is the work by Goldhirsch et al. [Goldhirsch 1987], where the authors re-visited the classic Orr-Sommerfeld equation to test their approach. To obtain the eigenvalues with the largest damping rate, the matrix is previously preconditioned with an exponential filtering, then the orthogonalization of the resulting Krylov space is achieved using Householder transformations. The exponential transformation used to recover the largest damping rate eigenvalues was numerically accomplished with a time-stepping approach in several hydrodynamic stability works. The procedure is well detailed in [Mamun 1995]. The time derivative equation describing the evolution of a perturbation is integrated to dismiss the decaying eigenmodes. Then, the temporal solutions at (K) consecutive time steps are taken, creating a set of vectors which are orthonormalized to build the Krylov subspace base of the Arnoldi procedure. The authors employed this technique to determine the bifurcation mechanisms in spherical Couette flow, where $K = 4$ time intervals were enough to obtain two converged eigenvectors, and [Barkley 1999] in plane three-dimensional Couette flow. In this later work $K = 8$ vectors in the Krylov space were converged. This time-stepping approach for the construction of the Krylov space was

successfully applied in numerous works in stability analysis in the last decades. In order of publication, Schatz et al. [M.F.Schatz 1995] studied the transition to turbulence in plane Poiseuille flow, characterizing the onset of primary and secondary instabilities. The three dimensional instability analysis of the wake of a circular cylinder [Barkley 1996], or over a backward-facing step [Barkley 2002a] are two more excellent examples of the use of the basic Arnoldi algorithm in combination with time-stepping techniques.

The random initial vector, x_1 , strongly affects the convergence of the method: if this is rich in the direction of the wanted eigenvector, the convergence will be fast, but this does not happen normally. Instead, the method usually requires a relevant number of iterations to converge, which lead to a several increase of the memory demand. Moreover, the calculation of the columns of matrix V is usually affected by finite precision errors. Previously mentioned work, by Nayar and Ortega [Nayar 1993], mentioned the loss of accuracy due to rounding errors that lead in the appearance of spurious eigenvalues. Consequently, the practical implementation of this method is usually coupled with an explicit/implicit restarting and/or deflation techniques, for increasing the stability and improving the time and memory requirements. Saad [Saad 1980] proposed a way to recovered accuracy by a process that correct the loss of orthogonality when V_{j+1} is small compared to Ax_j . The idea is based on the already mentioned deflation procedure, which consists on eliminate the already converged eigenvalues. Later in 1991, Sorensen [Sorensen 1992] presented a variant of this technique that became a key improvement in the Arnoldi method, the *Implicit Restart Technique*. This technique, implemented in the open-source library ARPACK [Lehoucq 1998], was successfully employed in many works to perform stability analysis. A simple and stable Krylov-Schur method, *equivalent* to the Arnoldi Algorithm, was proposed by Stewart in [Stewart 2001]. This method reduced the A matrix into a H square matrix $m \times m$ Which is a Real Schur Form, i.e. a quasi-triangular form displaying eigenvalues in 1×1 or 2×2 diagonal blocks.

This last method for subspace projection was implemented in SLEPc (*EPSKRYLOV-SCHUR* solver), and it is currently available for generalized and non-Hermitian EVP also [Hernández 2007].

3.1.2.4 Jacobi Davidson (JD) iteration

Jacobi-Davidson method is akin to Arnoldi iteration, because generates an orthonormal basis of vectors where a projection is performed, so it belongs to the class of subspace methods, but it does not employ the Krylov base to build the subspace of projection.

Each iteration of this method has two parts: a *subspace extraction* for selecting the best eigenpair from the subspace V , and a *subspace expansion* which adds a correction for the searched eigenvector and expand the V . In this projection method the orthonormal base is increased at each step, performing an orthogonalization of the new vector over all the vectors previously calculated, the amount of computational effort increases at each iteration. The main difference between the Jacobi-Davidson method and Krylov methods is the criteria used for expanding the orthonormal based, which is not linked to the powers of the A matrix but it is proportional to the residual (error) committed in the approximation of the eigenpair of A at each iteration.

Let (λ_i, x_i) be an eigenpair of A and let (μ_i, u_i) the corresponding approximation, with x

unit vector. The residual is defined as:

$$r_i = (Au_i - \mu Bu_i) \quad (3.13)$$

The standard extraction technique associated to the Jacobi-Davidson method is based on Rayleigh-Ritz algorithm and it is characterized by a good convergence ratio, if the external eigenvalues (i.e. the largest-modulus eigenvalues) are the target. Otherwise, for calculating the internal eigenvalues (i.e. the smallest-modulus eigenvalues), a good possibility is represented by the harmonic Rayleigh-Ritz method [Sleijpen 2000].

Assuming that the wanted region of the spectrum is closed to a given target τ , the harmonic Rayleigh-Ritz technique imposes the Petrov-Galerkin condition to the wanted eigenpair, such that:

$$(A - \tau I)u - \xi u \perp \mathcal{W} \quad (3.14)$$

with

$$\xi = \theta - \tau \quad (3.15)$$

and

$$\mathcal{W} \equiv (A - \tau I)\mathcal{V} \quad (3.16)$$

V and W (respectively a basis of $\mathcal{V}$ and $\mathcal{W}$), are built orthogonal such that $(A - \tau I)V = WS$ with S upper triangular. It is now possible to define the following projected eigenproblem:

$$W^H(A - \tau I)Vv = \xi W^H Vv \quad (3.17)$$

The eigenpair (ξ, v) , solution of the EVP (3.17), represents the pair $(\xi + \tau, Vv)$ close to the target τ of the subspace $\mathcal{V}$.

The expansion step is performed calculating a correction vector t , imposed to be orthogonal to the last obtained eigenvector u , by mean of the *Jacobi-Davidson correction equation*:

$$(I - z_i z_i^H)(A - \mu B)(I - u_i u_i^H)t_i = -r_i \quad (3.18)$$

Where

$$z_i = \theta Au_i + \nu Bu_i \quad (3.19)$$

for a suitable pair $(\theta, \nu) \in C$. The research space is then expanded using t , while the test space is expanded with $\theta At + \nu Bt$. The approximated eigenpair (μ_i, u_i) is considered acceptable if it satisfies the criteria:

$$\|r_i\|_2 = \|(Au_i - \mu Bu_i)\|_2 < tolerance \quad (3.20)$$

The correction equation (3.18) is generally solved iteratively for t , so a Krylov iterative solver, like Generalized Minimal Residual (GMRES) or Bi-Conjugate Gradient (BICG), is usually involved Jacobi-Davidson method. Jacobi-Davidson method promises to be a valid alternative to the Krylov subspace methods because it is able to recover the internal eigenvalues of the A matrix without involving spectral transformation, so with a sensible reduction of the computational costs, avoiding the inversion of the projected operator. However, the method converges fast only if few eigenvalues are searched and it is characterized by a greater relative error on the eigenvalue with respect to the Krylov methods. Jacobi Davidson method is a relatively young algorithm, suggested for the first time by Sleijpen and van der Vorst in 1996 for solving linear problems and later extended to

higher degree Polynomial Eigenvalue Problems (PEPs). Its popularity is easily increased in the last decades, and nowadays its applications cover several fields. In Sleijpen, Booten, Fokkema, and van der Vorst (1996) [Sleijpen 1996], JD algorithm with GMRES is used to confirm the reliability of the method, solving several eigenvalues problems associated to a finite element discretization of the Maxwell equation for mode-propagation and magnetic field profiles of a rectangular waveguide filled with dielectric and PEC structures. J. Van Dorsselaer(1997) [Dorsselaer 1997] applied the JD algorithm for solving the two-dimensional Rayleigh–Bernard problem, using the Rayleigh number as a physical parameter. In his work, Dorsselaer compares the results obtained using GMRES or BiCGSTAB(1) for solving the correction equation, and the influence of different initial guess vectors for the Krylov linear solver on the overall performances. Another relevant application is presented in R. Heeg and B. Geurts(1998) [Heeg 1998], where linear stability of incompressible attachment–line swept Hiemenz flow is solved using the Jacobi–Davidson algorithm for polynomial eigenvalue problems (JDPOL). According with these results, obtained performing an incomplete LU, a slight improvement is associated to the JDPOL algorithm if compared to the JDQZ algorithm, and relevant improvement are observed if the JDPOL is compared with QZ and QR algorithms. Comparison between the JDQZ algorithm, with GMRES linear solver for the correction equation, and implicit restarted Arnoldi for a flow over a backward facing step and in a driven cavity flow are presented in [Rommes 2008], where the author concludes that the JDQZ is a valid alternative approach to the most used Arnoldi method. A parallel implementation of the variant Davidson for Several Eigenvalues (DSE), preconditioned with ADI or multigrid, is presented in L. Borges and S. Oliveira(1998) [Borges 1998]. The results, obtained solving a simple mathematical test case, show the promising applicability of the JDs algorithms in parallel implementations. More applications of the JD methods on electromagnetics problems were investigated in P. Arbenz and M. Hochstenbach(2004) [Arbenz 2004], where the fundamental role of a good precondition is highlighted. Finally, a remarkable application could be found in F. Hwang(2010) [Hwang 2010], where the JD method, preconditioned with ASM and GMRES, is used for solving a large sparse polynomial eigenvalues problem (PEPs), arise from the discretization of the Schroedinger equations which describe a 3D Quantum Dots (QD) problem.

3.1.3 Spectral transformation: Shift & Inverse transformation

As previously stated, the Krylov-base methods converge to the portion of space linked to the dominant direction, associated to the largest-modulus eigenvalues, of the A matrix. In order to merge this characteristic with the stability analysis purpose, focus on the detection of the internal eigenvalues, specific type of preconditions, also called spectral transformations, were implemented to force the projection to converge in a selected region of the spectrum.

The main objective of a spectral transformation is to map the eigenvalues of a selected region of the A matrix spectrum into the external eigenvalues of another matrix operator. The Arnoldi method (or an alternative projection method) is then applied to the operator, instead to the A matrix, in order to recover these external eigenvalues. A specific relation, which depends on how the operator is defined, allows to recover the wanted eigenvalues of A , once that the external eigenvalues of the operator are known. From a practical point of view, the operator acts on the initial vector, or on the initial subspace, reducing its com-

ponents in the unwanted part of the spectrum, before the projection starts. [Sorensen 1992] [Saad 2003]

The most common spectral transformations used to obtain the internal eigenvalues, especially in fluid dynamic applications, is the Shift&Inverse transformations (or Inverse transformation if $\sigma = 0$), which replaces the A matrix with its σ -shifted inverse, with σ complex number, in equation (3.12), so that:

$$k_m \left((A - \sigma I)^{-1}, x \right) = \text{span} \left\{ x, (A - \sigma I)^{-1}x, \dots, (A - \sigma I)^{-(m-1)}x \right\} \quad (3.21)$$

After the projection, the corresponding reduced H matrix includes the part of the spectrum close to the σ parameter, and then the eigenvalues of H are an approximation of the eigenvalues of A which lie in the part of the complex plane close to σ . Defining μ as eigenvalues of the matrix $(A - \sigma I)^{-1}$, the relation with the eigenvalue λ of the original matrix A is:

$$\mu = \frac{1}{\lambda - \sigma} \quad (3.22)$$

Where

σ is the shift

λ is eigenvalue of A

μ is eigenvalue of $(A - \sigma I)^{-1}$

According with its definition, this spectral transformation maps the internal eigenvalues of A into the external eigenvalues of the operator $(A - \sigma I)^{-1}$ (or in those of the operator A^{-1} for the Inverse transformation).

3.1.4 Precondition techniques

The Inverse and the Shift&Inverse transformations are powerful instruments to recover the part of the spectrum close to the origin or near a target parameter σ , with some examples in [Meerbergen 1994]. However, in both of these transformations, the inversion of the A matrix is required to define the operator to project. This operation is usually computationally demanding for fluid dynamic applications.

Considering the Inverse transformation, without loss of generality:

$$k_m (A^{-1}, x) = \text{span} \left\{ x, A^{-1}x, \dots, A^{-(m-1)}x \right\} = \text{span} \{x_0, x_1, \dots, x_{m-1}\} \quad (3.23)$$

So that:

$$x_{i+1} = A^{-1}x_i \rightarrow Ax_{i+1} = x_i \quad (3.24)$$

That turns the problem of inverting the A matrix into a linear system problem, such that the subspace k_m in equation (3.23) is then obtained solving $m-1$ linear systems as equation (3.24). These linear systems could be solved by means of a direct precondition, such as LU decomposition [Bunch 1974], or using some other type of approximate precondition technique which needs to be coupled with an iterative solver. In any case the precondition of the A matrix is performed just once and then stored for the successive iteration steps. Several alternatives to solve the system (3.24) are listed below:

LU Decomposition

In this method, the A matrix is decomposed into the product of a lower diagonal matrix L and an upper diagonal matrix U , so that

$$A = LU \quad (3.25)$$

Then the linear system is decompose in two steps:

$$Ax = b \rightarrow LUx = b \rightarrow \begin{cases} Ly = b \\ Ux = y \end{cases} \quad (3.26)$$

When the systems (3.26) is solved, the exact solution for the linear system (3.25) is obtained. The LU decomposition solves the linear system with an approximatively cost of $(2n^3)/3$ floating points operations, but the LU matrices obtained are usually more dense of the A matrix.

In fact, LU decomposition for solving an EVP associated to a simple geometry 3D incompressible turbulent flow could already requires an LU element storage of 9×10^9 , that could be also expressed as 1.2 TBytes in term of memory [Iorio 2014]. The LU precondition could represent a good solution for 2D fluid dynamic problems of relevant interest, but the demand of memory costs associated to the decomposition makes this option not suitable for 3D complex applications. Interested readers can find more detailed information regarding the LU decomposition in [Bunch 1974, Chinchole 2014].

ILU Decomposition

Incomplete LU decomposition (ILU) is considered a valid alternative for the precondition of the linear system (3.24), to reduce the massive computational cost associated to the complete LU decomposition for industrial applications. The Incomplete LU factorization of a sparse matrix is an approximation of the LU decomposition, such as:

$$A \approx LU \quad (3.27)$$

Where the LU matrices are still respectively a lower and upper matrix, but defined such that the residual matrix R :

$$R = LU - A \quad (3.28)$$

satisfies some kind of constrains, such having zero entries in some positions where A has nonzero elements.

In this way it is possible to choose the sparsity level of the L and U matrices and, consequently, to regulate the computational cost associated to the precondition. If the sparsity pattern of the matrices L and U is chosen for being similar to that one of the A matrix, the precondition is called ILU factorization with no fill-in, or ILU(0), (*zero-level*) [Saad 2003]. More in general, the precision in the approximation of the exact LU decomposition could be increased using a higher level for the ILU decomposition, taking into account more element of the A matrix. Commonly, the rule indicates the ILU(k) as having similar sparsity pattern of the matrix A^{k+1} . Obviously, if the ILU level increases, the computational costs associated to the precondition is expected to increase as well.

This precondition technique does not yield to the exact solution of the linear systems (3.26), so it is coupled to iterative linear solvers, like for example Krylov Iterative methods.

A relevant application of this precondition in fluid dynamics field was presented in [Zeng 1992], where a variant of the ILU technique is applied for flows in skewed driven cavity and in a L-shaped driven cavity. In [Morzynski 2011], ILU method is employed to perform stability analysis of a flow over a circular cylinder, in both stable and unstable configurations, and on a flow around a NACA0012 airfoil. A further application of this precondition scheme could be found in [Konshin 2016], where the blood flow in a right coronary artery is studied and an exhaustive ILU parameter threshold is investigated. Alternative criteria for the fill-in of the L and U matrices in equation (3.28), based on nodes connectivity instead of the classical non-zero pattern approach, is presented in [Dahl 1992]. Finally, in [Segal 2010] [Rehman 2007] a comprehensive comparison between a modified version of ILU precondition and other efficient block precondition techniques is presented for several fluid dynamic problems, such as a 2D flow in a square cavity and a 2D and 3D backward-facing step flow.

Restricted Additive Schwarz precondition

Restricted Additive Schwarz (RAS) method [Cai 2001] is a cheaper and faster version of the Additive Schwarz(AS) preconditioner [Smith 1996], that was accidentally discovered removing part of the communication routine from the original AS algorithm. The RAS technique defines the graph $G(W, E)$ associated to the A matrix, under the assumption of no-zero and symmetric pattern for the A matrix itself, where W is the set of vertices $W = \{1, N\}$ whose represent the n -unknown and $E = \{(i, j), a_{i,j} \neq 0\}$ represents the pairs of vertices that are coupled by no-zero elements in A .

Applying the graph partitioning and defining each of the N non-overlapping subset as $W_i^{(0)}$, the union on these subset(W), is expressed by:

$$W = \bigcup_{i=1}^N W_i^{(0)} \quad (3.29)$$

Starting from this definition, the one-overlapping partition $W_i^{(1)}$ is obtained by $W_i^{(0)}$ including all the immediate neighboring vertices and then $W_i^{(1)} \supset W_i^{(0)}$. Using this idea recursively, it is possible to define the δ -overlapping partition :

$$W = \bigcup_{i=1}^N W_i^{\delta} \quad (3.30)$$

with $W_i^{(\delta)} \supset W_i^{(0)}$.

For each $W_i^{(0)}$ ($W_i^{(\delta)}$) is possible to define an associated restriction $R_i^{(0)}$ ($R_i^{(\delta)}$): a diagonal matrix with diagonal entries equal to one for the nodes which belong to $W_i^{(0)}$ ($W_i^{(\delta)}$) and entries equal to zero otherwise.

Leading from this definition, the RAS precondition techniques for a δ -overlapping partition, is performed solving the equation:

$$M_{RAS}^{-1}Ax = M_{RAS}^{-1}b \quad (3.31)$$

Where:

$$M_{RAS}^{-1} = R_1^0 A_1^{-1} R_1^{\delta} + + R_N^0 A_N^{-1} R_N^{\delta} \quad (3.32)$$

and

$$A_i = R_i^\delta A R_i^\delta \quad (3.33)$$

Where the inverse of the A matrix in equation (3.32) is obtained inverting the restriction of A to the subspace of vectors spanned by the set $W_i^{(\delta)}$ in $\mathbb{R}^n$.

Again, this precondition technique, expressed in equation (3.31), does not yield the exact solution of the linear systems in (3.31), so needs to be coupled with an iterative solver.

In its first formulation, AS precondition was targeting symmetric positive definite elliptic finite element problems [Smith 1996], but it was soon extended for solving non-linear local problems [Dryja 1997]. Other variants of the AS method are compared in [Lottes 2005], where the preconditioned problems are the 2D Poisson and Helmholtz equations. Some CFD applications of AS preconditioning could be found in [Cai 1994], where an academic non-linear convection-diffusion Dirichlet Problem, the full potential equation for the velocity and a 2D transonic airflow are solved. In [Cai 1997], the method is applied again to solve unsteady compressible Navier-Stokes equations. Some applications of the Restricted AS(RAS) variant are presented in [Cai 2001, Cai 2006], where the method is used to precondition several problems such as Poisson equation on unit square with zero Dirichlet boundary conditions, 2D Helmholtz equation and a 3D compressible Euler flow. Other relevant applications of the RAS precondition could be found in [Cai 2011] where the method is used on a 2D incompressible flow on driven cavity at high Reynold number and in [Cai 1998b] where an unsteady 3D Euler flow around a flexible wing and a supersonic flow passing a complete aircraft are investigated.

3.1.5 Krylov Iterative methods for solving linear systems

Krylov subspace are also used for solving linear system as (3.24), such as described in [Gutknecht 2005]. Considering the generic linear system with $A \in \mathbb{C}^{n \times n}$ and b and $x \in \mathbb{C}^n$:

$$Ax = b \quad (3.34)$$

It is possible to extract an approximated solution x_m of the exact solution x using an affine subspace $x_0 + K_m$ of dimension $m < n$, imposing the Petrov-Galerkin condition for the residual:

$$r_m = b - Ax_m \perp L_m \quad (3.35)$$

Where L_m is another subspace of dimension m . Krylov iterative methods use a Krylov base for defining K_m :

$$k_m(A, r_0) = \text{span} \left\{ r_0, A r_0, A^2 r_0, \dots, A^{(m-1)} r_0 \right\} \quad (3.36)$$

where

$$r_0 = Ax_0 - b \quad (3.37)$$

Even if all the iterative Krylov methods use the same polynomial approximation for K_m , several variants were implemented, each with a different definition for the complementary subspace L_m , which choices significantly affects the iterative technique proprieties. Two broad choice for L_m define the best-known Krylov iterative techniques: The first one is simply $L_m = AK_m$, which leads to the definition of the Minimal Residual (MR) category of methods, which also includes the best-known GMRES (Generalized Minimal RESid-

ual).

GMRES method was suggested in 1986 by Saad and Schulz [Saad 1986], and it aims to minimize the Euclidean norm of the residual vector r , solving a *Least Square problem* [Saad 1982] at each iteration. Its convergence behaviour is related to the eigenvalues position, and it is fast if all of them are far from the origin, for this reason it works well if coupled with spectral transformation. A restarted version of the GMRES algorithm is always to prefer due to the memory demand associated to this method, but the convergence behaviour in this case heavily depends on the Krylov subspace dimension. GMRES method, preconditioned with RAS, is employed in [Cai 2006] to solve several problems, such as Poisson equation, 2D Helmholtz equation and a 3D compressible Euler flow and again in [Cai 2011], to solve a 2D driven cavity incompressible flow at high Reynold number. Further examples of GMRES applications to the full potential equation for some 2D aerodynamics cases could be found in [Cai 1998a, Cai 1994, Cai 1997], while 3D applications for transonic flow around a flexible wing and supersonic flow passing a complete aircraft are shown in [Cai 1998b]. Unfortunately, using a restarting techniques, the information regarding the smallest eigenvalues could be lost after restarting. For this reason a variant of the method, the Deflated GMRES (DGMRES) [Erhel 1996], was implemented for keeping these information, using the invariant subspace associated to the smallest eigenvalues.

The Flexible GMRES (FGMRES) [Saad 1993] is another variant of the preconditioned GMRES algorithm, where the precondition matrix changes from step to step. In [Saad 1996], the DGMRES, FGMRES and GMRES, such as also BiCG and TFQMR, linear solvers are applied to solve linear systems related to fully implicit black oil simulator, thermal simulation of steam injection and Navier-Stokes equations.

In the second case:

$$L_m = k_m (A^T, r_0) \quad (3.38)$$

Where A^T is the transpose of the A matrix, and it brings to the definition of the class of Conjugate Methods (CG)[Hestenes 1952], which aims to minimized the energetic norm of the residual associated to the linear system (3.24). In its basic definition, the CG methods could be used for solving only linear system with A real, symmetric and positive defined or A Hermitian positive defined. An application of CG method could be found in [Ebeida 2009], where the method is used for solving the poisson equation, or in [Cai 2001] where the CG method, preconditioned with AS and RAS, is applied to solve the Poisson equation on a unit square. A generalization of the CG family methods for non-symmetric matrices is represented by the family of Bi-CG methods. In Bi-CG methods there is not a research for the minimum of the residual norm vector, instead the expansion of the Krylov subspace is performed such as the residual accomplishes the orthogonality to a dual krylov subspace, built using the A^T matrix. Unlike the CG methods, whose require one Matrix-Vector Multiplication(MVM) per step, the Bi-CG family methods requires 2* MVM at each step: one which involves the matrix A and one which involves its transpose A^T . Further information regarding performances of the Bi-CG family methods could be found in [Gutknecht 1997]. In [Sonneveld 1989], Sonneveld presented the CG Square (CGS) method, where the second MVM per step associated to the A^T of the Bi-CG method is replaced by a MVM which involved again the A matrix. The CGS method, which formally derives from the Bi-CG methods and not from the CG ones, converges with double speed compared with the Bi-CG methods, but it also less stable and could

more easily arise to errors. The stabilized Bi-CG (BiCGSTAB), proposed by Van der Vorst in [der Vorst 1992], represents an improved variant of the Bi-CG methods, where local optimization techniques are employed for smoothing the convergence of the residual norm. The BiCGSTAB method is a hybrid method which also adapts the Bi-CG for avoiding the use of A^T , such as happened for the CGS method. An application of the BiCGSTAB method, preconditioned with ILU, could be found in [Konshin 2016], where the blood flow in a right coronary artery is examined. Others relevant applications of the BiCGSTAB, as also of the GMRES method, for solving a 2D square lid-driven cavity and a 2D and 3D backward facing steps, are presented in [Segal 2010]. In its original version, the BiCGSTAB method requires the spectrum of the A matrix to be completely real, but further variants were implemented to deal with complex arithmetic and to improve the convergence behaviour. One of this variant lead to the definition of the Bi-CGSTAB(l) family methods [Sleijpen 1994, Gutknecht 1993, Yang 2002], with " l " being the degree of the minimal residual polynomial used for the minimization at each step. CGS, BiCGSTAB and BiCGSTAB(l) are all characterized by double convergence speed if compared with the Bi-CG, cause they avoid to use the A^T , and so they belong to the transport-free family of methods for linear systems. In [Sonneveld 1989], CGS and BiCG are compared for solving a convection diffusion equation on the unit square and the Van der Vorst problem, while several applications of BiCGSTAB(l) are available in [Sleijpen 1993].

A further Bi-CG-like approach, was presented by R. Freund and N. Nachtigal in 1991 with the name of Quasi Minimal-Residual (QMR) [Freund 1991b, Freund 1991a, Saad 1985]. This method is born to overcome the irregular convergence behaviour of the Bi-CG method, and basically use the same algorithm and bi-orthogonal krylov base to reduce the system in (3.1) into a tridiagonal linear system, but employing a least square approach, similarly to those of the GMRES method, instead of an implicit LU decomposition. The least square approach in this case, leads to a quasi minimal-residual solution, giving the name to the algorithm. Finally, the QMR, and its Transport-Free variant (TFQMR), is an improved algorithms considered more stable than the Bi-CG algorithm. A comparison between the TFQMR method and the GMRES method, to solve unsteady incompressible flow, is shown in [Ebeida 2009], while comparison between CGS and TFQMR methods applied to test matrices from the Harwell-Boing set and to linear system derived to thermal stressed material problem are presented in [Bucker].

For interested readers, a more exhaustive explanation on all this iterative linear solvers, and the respective complete algorithms, could be found in [Saad 1982, Paige 1975, Greenbaum 1997, Meurant 1999, Simoncini pear].

3.2 Benchmark problems

In order to effectively compare the different algorithms, it is necessary to perform, in the first place, comparisons on simple tests cases. Two different test cases: a confined flow into a square cavity and an open flow over a flat plate, were studied. For both problems, the steady flow, needed for the stability analysis, is obtained solving the two-dimensional Navier-Stokes equations for incompressible flow in primitive formulation. A semi-implicit Euler scheme (implicit for pressure and viscous terms while explicit for convective terms) is used for advancing in time, adding artificial compressibility to overcome the absence of time evolution for pressure. High-order Chebyshev spectral collocation technique with Gauss-Lobatto points is used for spatial discretization. More details

on base flow calculations can be found in [de Vicente Buendía 2010].

The eigenvalue problem, resulting from applying the BiGlobal analysis, is then solved using a selected set of strategies between those described in the previous section. Besides, when possible, a direct QZ algorithm, or earlier results available on others relevant publications [de Vicente Buendía 2010], [Alizard 2007], were used to compared the studied methodologies.

3.2.1 Lid-Driven-Cavity (LDC)

The square lid-driven cavity problem provides a perfect environment for the purpose of this work. Confined cavity flows are probably the most studied problems in computational fluid dynamics. The relatively simple geometry and, even more important, the absence of open boundary conditions for the Navier-Stokes equations, make this kind of configurations an excellent benchmark for testing numerical methods concerning accuracy, convergence, consistency and reliability. Among cavity flows, lid-driven cavity (LDC) flows were a mainstay of fluid dynamic research for decades [de Vicente Buendía 2010] [Shankar 2000]. By far, the most studied configuration is the two-dimensional flow, set up by constant speed motion of a flat surface over a square or rectangular container.

In the present work a square cavity (aspect ratio, length over depth, $L/D = 1$) was considered. Reynolds number was fixed to 1000, while the wavenumber in the spanwise direction for the BiGlobal analysis, β , was set to 2.

Figure [3.2] shows the streamfunction and vorticity of the base flow solution for the LDC case.

Figure 3.2: LDC base flow

According with the stability analysis, the chosen configuration for this LDC problem was proven to be stable. The associated eigenspectrum completely lies in the negative part of the space for the damping rate (damped eigenmodes). Least stable mode is associated with a pair of complex conjugated eigenvalues with frequency $\omega_i = + - 0.5115$ and damping rate $\omega_r = 0.08$. Second mode is a non-oscillatory ($\omega_i = 0$), disturbance with a negative damping rate around $\omega_r = 0.095$. In previous works [de Vicente Buendía 2010] the full eigenspectrum was recovered using two different algorithms. A classic QZ algorithm, im-

plemented using LAPACK, and a Krylov projection method with LU decomposition using ARPACK and SUPERLU libraries. LAPACK implementation of QZ algorithm presents several drawbacks, LAPACK does not manage sparse format so storage capabilities are due to important limitations. Moreover, The demanding computational resource associated to the QZ algorithm constrains the analysis to coarse meshes, resulting in an inefficient methodology. The Krylov projection (ARPACK) with LU decomposition (SuperLU) approach was tested in the past and validated in a wide number of stability problems concerning confined and open cavity flows [several references] with excellent results. Therefore the solution obtained with this method was used as a reference value in this work. Figure [3.3] shows the mentioned eigenspectrum for the 72×72 mesh. The eigenvalue associated to the aforementioned second non-oscillatory mode will be used as reference to compare the different methodologies examined.

LDC: Reference value [de Vicente Buendía 2010]	
Damping rate	Frequency
$-9.517667596(-2)$	0.00

Table 3.1: LDC reference eigenvalue

Figure 3.3: LDC eigenspectrum - (reference)

3.2.2 Flat Plate

The second test case considered is a two-dimensional boundary layer flow over a flat plate at Reynolds number equal to 610, based on boundary layer displacement thickness

at the inflow of the computational domain. This flow configuration is characterized by the onset of the so called Tollmien-Schlichting (TS hereafter), convective instabilities which develop during the first phase of the transition to turbulent flow into the boundary layer of the flat plate. This phenomenon was widely observed in experiments since the beginning of the last century and successively many efforts were focused on reproducing it in numerical simulations. However some discrepancies between theory and experiments still remain not explained [Alvarez 2010].

The computational domain is dimensionalized using the displacement thickness at the inflow as dimensionless length. The length of the domain in the streamwise direction is $L_x = 400$, while in the wall normal direction, the boundary was placed at $L_y = 20$ in order to minimize numerical effects due to the imposition of artificial boundary conditions for open flows. The different lengths of the computational domain in streamwise and wall normal direction, in addition to the clustering of Chebyshev-Gauss-Lobatto points, when spectral collocation is applied for the spatial discretization, are good reasons for the use of domain decomposition techniques. Domain decomposition methods are based on dividing the computational region in several domains in which the solution is calculated independently but taking into account information from the neighboring domains. In the flat plate case, the configuration considered for the base flow calculations and for the EVP resulting from the linear stability analysis is a multidomain approach with four conforming domains along the streamwise direction, each of them with length $L_{i_x} = 100$. In the base flow calculations of the flat plate problem, four types of boundary conditions appear: non-slip wall, inflow, farfield and outflow. For viscous wall bounded flows, the adequate boundary conditions are the no-slip condition for velocity and homogeneous Neumann condition for pressure derivative in the normal direction. Velocity components of the laminar boundary layer at the inflow are calculated integrating the Blasius equation. Homogeneous Neumann conditions for pressure in the normal direction were set. The farfield boundary denotes the upper artificial limit, far enough from the boundary layer, where the streamwise component of the velocity is equal to a nominal value. Normal velocity is considered constant. Finally, in the downstream limit of the computational domain, and according to [Gresho 1987] [Sani 1994] the second derivatives of the flux velocity components are set to zero and the pressure is kept constant. Figure [3.4] shows the streamwise velocity component and the vorticity of the base flow solution for the flat plate test case.

Figure 3.4: Flat plate base flow - streamwise velocity and vorticity -

This particular flow configuration is chosen to be compared with the work of Alizard and Robinet in [Alizard 2007], where the onset of the TS convective global modes was examined. The authors study the linear stability of the flat plate boundary layer, and particular attention is focus on the features of one mode belongin to the TS branch. Table [3.2] contains the real and imaginary parts of one of the eigenvalues reported in Alizard and Robinet paper that will be considered as the reference value for the flat plate case studied in this paper. Figure [3.5] shown the eigenspectra associated to the flat plate configuration, with a black square highlighting the reference eigenvalue of table [3.2].

Figure 3.5: Flat plate eigenspectra

Flat plate - reference TS value [Alizard 2007]	
DAMPING RATE	FREQUENCY
$\simeq 5.56(-3)$	$\simeq 0.113$

Table 3.2: Flat plate eigenvalue associated to reference TS mode

3.3 Implementation description

3.3.1 Softwares and cluster

The different techniques for solving large eigenvalue problems described in section [3.1] were implemented using SLEPc library. SLEPc (*Scalable Library for Eigenvalue Problem Computations*) is a scalable library for solving EVP [Hernández 2005], built on top of

PETSc (*Portable, Extensible Toolkit for Scientific Computation*, [Satish 2014]), able to manage Hermitian and non-Hermitian problems, for both Standard and generalized EVP, in real or complex arithmetics, on parallel computer.

PETSc is an object-oriented library for solving partial differential equations on parallel framework, which uses linear, non-linear and time-stepping solver objects. In the framework of sparse and large EVP solvers, the combination of PETSc and SLEPc represents a cutting edge approach able to deal with a wide number of possible numerical methods, in terms of spectral transformations, precondition techniques, iterative linear solvers and projection methods. For the purpose of this chapter, focus on the comparison of numerical approaches, SLEPc represents the most natural choice, which adds consistency and coherency to the test methodology and results comparison.

The simulations were performed on Magerit cluster from CESVIMA [Centro de Supercomputacin y Visualizacin de Madrid 2016], 41 computer nodes, connected by Infiniband QDR net, with 2 Intel Xeon processors each featuring 8 cores, 2.6 GHz and 64 GB DDR memory. The following versions of software were employed: PETSc 3.5.2 (installed with MUMPS 5.0.0 and HYPRE 2.10.0) and SLEPc 3.5.2.

3.3.2 Survey on matrix characteristics

The generalized eigenvalue problem arising from the stability linear analysis of the Navier-Stokes equations has some crucial features that influence the applicability of some methods and their implementation, so the A matrix properties (A leading dimension *in primis*) have to be taken into account. In fact, even for the very simple geometry whose define the lid-driven cavity and flat plate problems, the computational resources needed to store the matrix are significantly influenced by the matrix size itself. To illustrate this assessment, a collection of 12 different monodomain meshes, for the LDC test case, were examined, see table [3.3]. The effects of the leading matrix dimension on the numerical approach used to obtain the EVP solution will be investigated in the next paragraph [3.4], where all the matrices of this set of benchmarks will be solved.

Lid Driven Cavity				
MESH	A Dimension (n)	Memory consumption for storing A [Mb]	CPU time for storing A [s]	
16x16	1156	19.23	0	
24x24	2500	21.52	8.00(-3)	
32x32	4356	25.85	1.20(-2)	
40x40	6724	33.02	2.00(-2)	
48x48	9604	43.74	3.60(-2)	
56x56	12996	58.71	5.60(-2)	
64x64	16900	78.60	7.60(-2)	
72x72	21316	104.13	0.11	
80x80	26244	136.01	0.15	
88x88	31684	174.91	0.20	
96x96	37636	221.23	0.27	
104x104	44100	276.26	0.35	

Table 3.3: Size-problem for the LDC cases

A similar benchtest of simulations was performed for the flat plate case, leading to similar results. For this second test case, three different size-problem, all 4-domains, were studied, as shown in table [3.4]

Flat plate				
MESH	A Dimension (n)	Memory consume for storing A [Mb]	CPU time for storing A [s]	
24x16	6800	27.56640625	1.6(-2)	
56x32	30096	108.7304687	0.124008	
64x48	50960	220.5117187	0.280017	

Table 3.4: Size-problem for the flat plate cases

In the present work the methodology employed to study the stability of the flow over the flat plate includes the use of the domain decomposition. A first consequence linked to this technique is the number of non zero elements in the matrices of the associated EVP. To illustrate this, a first comparison is done between the matrices corresponding to four different configurations: a single domain calculation with a resolution of $1 \times 256 \times 48$ ($N_{domain} \times N_x \times N_y$) points, a four aligned computational domains of $4 \times 64 \times 48$ nodes, an eight computational domains of $8 \times 30 \times 48$ nodes and a sixteen computational domains with $16 \times 15 \times 48$ nodes. In all cases the leading dimension of the left-hand side matrix of the EVP is around $n \simeq 5 \times 10^4$. However the number of nonzero elements differs significantly, from $\simeq 6 \times 10^7$ in the 16-multidomain approach to $\simeq 18 \times 10^7$ nonzero coefficients in the monodomain case, as shown in table [3.5].

Flat plate: sparsity		
Mesh (Ndom $\times$ Nx $\times$ Ny)	A Dimension	No-zero elements
$1 \times 256 \times 48$	50372	18669820
$4 \times 64 \times 48$	50960	8756830
$8 \times 30 \times 48$	48608	6573066
$16 \times 15 \times 48$	50176	5911080

Table 3.5: Flat plate sparsity for monodomain and multidomain approaches

The previous table highlights how the sparsity of the A matrix increases with the number of subdomains for the spatial discretization. More in details, multidomain approach leads to a dominant block diagonal structure associated to the BiGlobal discretized EVP in each subdomain plus a low rate imposition of nonzero entries due to the coupling conditions among those domains. Figures [3.6] and [3.7] show, respectively, the matrix patterns of the A matrix for the BiGlobal analysis on a mono-domain and on a four-domain Cavity flow, discretized with high order methods based on Chebyshev polynomials.

These figures clearly show how the structure of the matrix is affected by the subdivision on the computational domain. Particularly relevant, in this term, is the diagonal dominance of the matrix, which can strongly favorites some specific class of precondition techniques. Multi-domain approach increments the diagonal dominance of the matrix, making the matrix structure particularly suitable for preconditioning which base their performance on this propriety.

Figure 3.6: Matrix pattern (left) and grid (right) for a monodomain cavity flow

Figure 3.7: Matrix pattern (left) and grid (right) for a four-domain cavity flow

Since most of the methods to compute internal eigenvalues required the inverse of the A matrix, a first test to compare the influence on the numerical performance with respect to the domain decomposition technique was done, for the flat plate test case, using two solvers: direct LU decomposition or flexible GMRES (FGMRES) linear solver preconditioned with ILU, both with Shift–Inverse and krylov projection techniques, in a subspace of dimension 2000. Results are shown in table [3.6]. The overall number of grid points is similar for all the cases, but it is clearly shown that the increment of number of domains results in lower computational costs, for both memory consumption and computational times, for both direct and iterative approach.

Flat plate: Comparison computational performance				
	Direct LU		Iterative ILU+FGMRES	
	Memory consume [Mb]	CPU time [s]	Memory consume [Mb]	CPU time [s]
$1 \times 256 \times 48$	27851.8	8509.5	16756.5	91620.3
$4 \times 64 \times 48$	13518.0	1364	6897.6	6859.8
$8 \times 32 \times 48$	6457.3	803.8	3117.2	4499.8
$16 \times 16 \times 48$	3879.1	662.5	2209.7	2214.6

Table 3.6: Memory consumption and computational time for the flat plate using single and multi computational domain(s)

The following figures [3.8] and [3.9] show the stability analysis results, eigenspectra and u-velocity component of the reference eigenvalue respectively, for the test cases in table [3.6], obtained with direct LU decomposition and Shift–Inverse ($\sigma = 0$). The eigenspectra are qualitatively similar, but continuous spectrum appears shifted for the multidomain case, probably depending on the spatial discretization. Anyway, TS branch is well resolved in all cases and reference eigenvalue and eigenvector are adequately calculated.

Figure 3.8: Flat plate: eigenspectra - Domain decomposition comparison

Figure 3.9: Flat plate: u-velocity component of the reference eigenvector - Domain decomposition comparison

According to these results, direct and iterative approach perform more efficiently in terms of memory consumption for the multidomain approach, with a reduction of one order of magnitude between the monodomain and the 16-domain cases. Even total time for obtaining the eigenspectrum with the requested tolerance is significantly lower for the multidomain approach, scaling similarly for both direct and iterative approach. Since that, from now on comparison on the different numerical methodologies, for the stability EVP for the flat plate test case, will be done using the multidomain (four-domain) discretization matrix.

3.3.3 Survey on numerical methods characteristics

Beside the characteristics presented in the previous paragraph, the generalized EVP here examined are also characterized of non-Hermitian A matrix and a possibly singular B matrix. As consequence, the numerical methods suitable for these characteristics and available in PETSC/SLEPc is limited. Moreover, some numerical methods like Richardson iteration, and its variant Chebyshev acceleration, are not suitable because does not accomplish the stability analysis necessity. Specifically, those methods require that all the eigenspectrum lie in the same region of the complex plane for the amplification/damping rate. Nevertheless, for the stability analysis purpose it could be considered a possible choice only if the flow configuration is stable. For more generic stability analysis applications, these methods are not usually employed because they are not able to properly handle the condition at the which the flow becomes unstable, which involves at least one eigenvalue with positive damping rate, while the rest of the eigenspectrum still remain in the negative region for the damping rate.

According with the scheme in figure [3.1], table [3.7] reassumes the possible methodologies available in PETSc and SLEPc and, for some of them, the reason why they are not employed.

Overview of numerical methods available in SLEPc/PETSc [Hernández 2005] [Satish 2014]		
Method	suitability ¹	comments
Spectral transformation - (Krylov projection only)		
Shift		Employed for external eigenvalues
Shift-inverse	x	
Cayley	x	
Direct Solver		
QR/QZ	x	
LU	x	
Cholesky		Symmetric operator
Precondition		
Jacobi	x	
Block Jacobi	x	
Successive OverRelaxation (SOR)	x	
Additive Schwarz	x	
Multigrid	x	
ILU	x	
ICC		Symmetric operator
Krylov Subspace Methods		
Minimum Residuals Method		Symmetric operator
GMRES (family)	x	
CG		Symmetric positive(negative)(semi)-definite operator
CGsquared	x	
Conjugate Residuals		Symmetric operator
BiCG (family)	x	
BiCGstab/BiCGstab(l)	x	
TFQMR	x	
Least Square Method	x	
Conjugate Gradient on the normal equation	x	
SYMMLQ		Symmetric operator
Richardson		Good rate of convergency only for positive (negative)-definite operator
Chebyshev		Good rate of convergency only for positive (negative)-definite operator
Linear eigensolver		
Power		Employed for external eigenvalues
Subspace iteration		Employed for external eigenvalues
Lanczos		Hermitian definite problem
Arnoldi	x	
Krylov-Schur	x	
Rayleigh quotient CG		Hermitian definite problem
Generalized - Davidson	x	
Jacobi - Davidson	x	
LOBPCG		Hermitian definite problem
Contour Integral SS	x	

Table 3.7: Main methods available in PETSc/SLEPc

Matrices arising from the Navier-Stock (NS) equations discretization are usually ill-conditioned, independently of the discretization technique employed, due to the different physical scales that characterize the phenomenon or boundary layers. Another source of ill-conditioning comes from the use of spectral methods to discretize the differential operators [Chen 2005] [Peyret 2002]. Ill-conditioned systems are challenging to solve accurately, and consequently, on the set of numerical approaches tested for both the LDC and the flat plate fluid dynamic problems, only a reduced number of methodologies were able to convergence.

Ill-conditioning of the A matrix can be alleviated by preconditioning, which basically acts to reduce the condition number of the matrix, a crucial role for the fluid dynamic applications. For this reason, particular attention is focused on the performances of the precondition technique. Direct methods such as LU decomposition, currently widely used in fluid dynamic problems, requires relevant memory resources, which increases with the size of the problem. A focal point of this work aims to understand the applicability of efficient alternative preconditions with respect to the direct LU decomposition.

Jacobi, Block-Jacobi and SOR were preliminary tested with a symmetric and real test matrix, showing a lower convergency than the more accurate variant ILU, such as also stated in [Saad 1982]. Consecutively, these first three options were no longer considered as convenient choices. Similarly, Generalized Davidson and Arnoldi projection methods, showed lower performances than their respective variants Jacobi-Davidson and Krylov-Schur projection methods.

Before to choose the settings for the ILU precondition techniques, an analysis to investigate the influence of the ILU level on memory scaling was performed. Table [3.8] shows, for the lid-driven cavity stability analysis on the finest mesh (104x104), the memory consumption in Mb and the corresponding relative error for the reference eigenvalue. The relative error is defined as:

$$e_r = \frac{\|Ax - \lambda x\|_2}{\|x\|_2} \quad (3.39)$$

Lid Driven Cavity - ILU level analysis - Mesh 104x104			
	Memory consumption [Mb]	Computational time [sec]	e_{abs}
ILU(2)	16129.61	30842.84	3.62(-3)
ILU(3)	16142.70	33841.74	1.09(-4)
ILU(4)	16143.89	35940.82	1.12(-8)
ILU(5)	16150.93	37740.27	2.87(-8)
LU	27315.22	-	-

Table 3.8: LDC Memory demand for different ILU levels.

All the values reported in the referred table were obtained via ILU precondition, GMRES linear solver and Krylov projection method with Shift-Inverse technique for the iterative approach. For the sake of comparison, the memory requirements associated to direct LU decomposition with Krylov projection method are also shown in Table [3.8]. Krylov subspace dimension was set to 200. For ILU levels lower than 2 or 3, depending on the method, convergence is not reached. On the other hand, ILU levels greater than 5 do not imply a significant improvement in terms of precision on the reference eigenvalue

while an increase in the memory consumption occurs. According with results in table [3.8] the level of ILU was set equal to 4 for the LDC test case. For this value, all the methods converge, the memory cost is sensible lower than that one associated to the LU decomposition and the improvement in the precision is significative compared with the lower level 3.

Same analysis was carried out for the flat plate problem. Computational resources depending on the ILU level are presented in Table [3.9]. To properly describe the eigenspectrum, this time, the Krylov dimension is raised to 2000. This value is enough to recover both the most unstable mode and the reference eigenvalue describe in table [3.2]. Regarding the ILU level: below level 6, no solution is recovered while value higher than 8 does not provide any considerable improvement of the relative error, so the optimum ILU level for this problem was set to 8.

Flat plate - ILU level analysis - MESH 64x48					
	Memory consumption [Mb]	Computational time [sec]	e_{abs} most unstable eig.	e_{abs} reference eig.	
ILU(6)	6854.0156	36041.100	1.17	0.98474912	
ILU(7)	6865.23	7710.382	2.18(-5)	4.39(-3)	
ILU(8)	6865.63	7756.617	2.18(-5)	1.39(-3)	
ILU(9)	6888.39	9173.869	1.36(-5)	4.93(-3)	
LU	13517.29	-	-	-	

Table 3.9: Flat plate Memory demand for different ILU levels.

Another important parameter to take into account is the tolerance. In the framework of the stability analysis EVP and SLEPc library, two different values are defined. The first value is related to the iterative linear solver and, when Krylov subprojection is used, it represents the maximum acceptable error to obtain the approximated solution of the linear system in (3.25). In case of Jacobi-Davidson subprojection, it represents the maximum acceptable error for obtaining the approximated expansion of the Jacobi-Davidson correction equation (3.18). The second tolerance value is imposed to the projection method itself and it corresponds to the minimum requirement for considering the projection successful. It is equivalent to the absolute error for each eigenvalue , defined as:

$$e_{abs} = ||Ax - \lambda x||_2 \quad (3.40)$$

In Jacobi-Davidson subprojection methods, this second tolerance is the tolerance value used in equation (3.20). Finally, regarding the choice of the projection method, Jacobi-Davidson projection is commonly known to be more efficient if just a few inner eigenvalues are searched. For this reason, the dimension of the projection subspace for JD projection is set to be lower than the one used for the Krylov projection. In fact, increasing the number of vectors in the base for the JD severally affects the performance of the method, and it is useless for the purpose of the linear stability analysis itself. To illustrate this, a comparison between the two projection methods, for six different meshes, was performed to the lid-driven cavity problem. Tables [3.10] and [3.11] show, respectively, the computational demands and the relative error for the reference eigenvalue when Krylov and Jacobi-Davidson methods are applied using 200 vectors in the projection subspace. Besides, the comparison is fulfilled considering only 25 vectors for the Jacobi-Davidson

algorithm. According to these results, Jacobi-Davidson projection method requires more computational resources, in terms of memory and time, than Krylov method for a fixed subspace dimension. However a lower number of vectors in the subspace associated to Jacobi-Davidson methodology does not entail a loss of accuracy in the eigenvalue calculation.

LDC: Krylov and JD projection methods comparison						
Mesh	KRYLOV(200)		JD(200)		JD(25)	
	Memory cost[Mb]	CPU time[s]	Memory cost[Mb]	CPU time[s]	Memory cost[Mb]	CPU time[s]
16x16	27.94	0.912	47.25	1388.22	19.86	10.69
24x24	96.77	1.656	136.13	4526.67	87.35	16.04
32x32	274.07	3.956	365.68	11243.10	257.54	33.50
40x40	682.11	12.38	884.39	26990.15	658.53	87.96
48x48	1469.29	56.56	2087.68	92669.81	1441.72	230.71
56x56	2554.83	82.29	3223.59	103585.75	2527.40	392.88

Table 3.10: LDC memory consume for Krylov(200), JD(200) and JD(25)

LDC: Absolute error on the reference eigenvalue						
Mesh	KRYLOV(200)		JD(200)		JD(25)	
	Damping rate	e_{abs}	Damping rate	e_{abs}	Damping rate	e_{abs}
16x16	-9.015637(-2)	5.79(-11)	-9.015637(-2)	7.01(-10)	-9.015637(-2)	1.52(-9)
24x24	-9.521460(-2)	8.85(-11)	-9.521460(-2)	2.50(-9)	-9.521460(-2)	2.13(-13)
32x32	-9.518221(-2)	3.01(-9)	-9.518221(-2)	2.66(-9)	-9.518221(-2)	3.57(-13)
40x40	-9.517789(-2)	7.43(-11)	-9.517789(-2)	3.09(-9)	-9.517789(-2)	1.10(-12)
48x48	-9.517744(-2)	5.35(-10)	-9.517744(-2)	1.79(-9)	-9.517744(-2)	3.46(-10)
56x56	-9.51770215(-2)	1.51(-7)	-9.51770216(-2)	1.60(-9)	9.51770216(-2)	8.11(-13)

Table 3.11: LDC absolute error on the least stable eigenvalue for Krylov(200), JD(200) and JD(25).

This suggests that each projection method should be employed considering its area of applicability, which lead to the best performances.

3.4 Results

This section contains the results obtained applying the linear stability analysis to the lid-driven cavity and flat plate problems, presented in section [3.2].

The performances of the different numerical approaches, in terms of memory demands, computational time and accuracy on the reference eigenvalue, are shown for the both test cases. Since, for the flat plate problem, the reference eigenvalue in table [3.2] does not lie near the origin of the complex plane, two different strategies for recovering the solution were investigated for this test case. In the first, a number of vectors equal to 2000 for the subspace projection methods was used with *no-shift* (shift = 0). The second strategy

uses a number of vectors equal to 200 for the Krylov subspace projection with a complex shift close to the references eigenvalue. As previously stated in section 5.2, the specific properties of the selected test cases curbs to a limited number of numerical approaches. Furthermore, between all the suitable methods shown in table [3.7], only a limited number of them were able to converge for the LDC and flat plate test cases. Tables [3.12] and [3.15] reassume the set of the numerical approaches that successfully converged for the stability analysis of the lid-driven cavity and flat plate test cases respectively.

3.4.1 Lid-driven cavity

The linear stability analysis of a square lid-driven cavity at $Re = 1000$ and $\beta = 2$ is the first test case studied for understanding the applicability of the numerical approaches, which can represent a valid alternative to the widely used LU direct decomposition with krylov projection with Shift–Inverse technique.

The selected spectral transformation (*ST*), preconditioning techniques (*PC*), linear solver (*LS*) and projection method are shown in table [3.12]. Hereafter, the case number in the first column will identify the numerical methodology for the following figures and tables.

CASE	ST	PC	LS	PROJECTION METHOD
1	SHIFT-INVERT	LU	-	Krylov
2	-	LU	-	JD
3	SHIFT-INVERT	ILU	BICGSL(2)	Krylov
4	-	ILU	BICGSL(2)	JD
5	-	RAS	BICGSL(2)	JD
6	SHIFT-INVERT	ILU	GMRES	Krylov
7	SHIFT-INVERT	RAS	GMRES	Krylov
8	SHIFT-INVERT	ILU	TFQMR	Krylov
9	-	ILU	TFQMR	JD
10	SHIFT-INVERT	RAS	TFQMR	Krylov
11	-	RAS	TFQMR	JD
12	SHIFT-INVERT	ILU	CGS	Krylov
13	-	ILU	CGS	JD
14	SHIFT-INVERT	RAS	CGS	Krylov
15	-	RAS	CGS	JD
16	SHIFT-INVERT	ILU	FGMRES	Krylov
17	SHIFT-INVERT	RAS	FGMRES	Krylov
18	SHIFT-INVERT	ILU	DGMRES	Krylov
19	SHIFT-INVERT	RAS	DGMRES	Krylov

Table 3.12: Selected cases for LDC

The previous table does not contain any case which employs GMRES/FGMRES/DGMRES as linear solver with Jacobi-Davidson projection method, because these combinations have shown poor convergency results when applied to BiGlobal stability analysis matrices, for both test cases.

Following, table [3.13] reassumes the parameter setting for LDC test case simulations.

PARAMETER	VALUE
Shift (<i>if needed</i>)	0
ILU level (<i>if needed</i>)	4
RAS subdomain (<i>if needed</i>)	1
RAS overlapping (<i>if needed</i>)	1
Tolerance for the LS (<i>if needed</i>)	10^{-8}
Tolerance for the projection	10^{-8}
Number of vectors for the Krylov subspace	200
Minimum number of eigenvalues required for the Krylov projection	100
Number of vectors for the JD subspace	25
Minimum number of eigenvalues required for the JD projection	15

Table 3.13: Setting for LDC simulations

A further comment regarding the initial guess vector for the linear solver is due: almost all the cases were performed with null vector as initial guess, with exception for the case number 4 in table [3.12], which has required a random no-zero initial vector. As general rules, the convergence of the methods is affected by the initial guess vector for the linear solver and this should be carefully considered before to discharge a methodology as not suitable. Not all the value parameters which characterize the simulation are explicitly reported in table [3.12]. If not explicitly stated, they were set as default value into PETSc/SLEPc libraries.

A first comparison between the different performances is conducted to determine how the computational demands scales with the size of the problem. The graphs [3.10] and [3.11] show, respectively, the memory required and the computational time associated to the numerical methods in table [3.12].

Regarding the memory demand, the first graph shows a common trend for the numerical approaches, varying the mesh dimension. As expected, when the dimension of the problem is increasing, the preconditioning technique reveals itself as the most demanding in term of memory requirements and direct LU decomposition reveals to be less competitive than ILU or RAS, independently by the linear solver. Regarding the computational time, in the second graph, the convergence behaviour shows, again, a common trend for almost all the cases. Jacobi-Davison projection methods exhibit an irregular response as the size of the problem increases. This feature is more appreciable when the linear solver chosen is BICGSL or TFQMR.

Figure 3.10: LDC Memory costs

Figure 3.11: LDC computational time

A first conclusion is directly deduced from these results: direct LU decomposition is the fastest option to obtain the solution, no matter the projection method chosen. However, as previously mentioned, memory requirements are significantly lower when ILU or RAS techniques are employed to precondition the EVP matrix. It is important to remark, at this point, that current implementation of Jacobi-Davidson projection method uses a less number of vectors for the projection-space base, otherwise this gain in term of computational time (against the methods that use iterative precondition), will be lost. For a better understanding of the vast set of results presented here and to compare the performance of the numerical approaches, table [3.14] shows explicitly, for the 72x72 mesh, the computational demands for each method. Besides, the damping rate of the reference mode is shown together with relative error, calculated according with eq.(3.39).

LDC: Least stable eigenvalue - mesh 72x72 -				
Reference value : -9.517667596(-2)				
Case	Damping rate	e_{abs}	Memory consumption[Mb]	CPU time[s]
1	-9.517667598(-2)	1.70(-7)	7257.5664	282.686
2	-9.517667596(-2)	3.53(-12)	7232.8125	1335.367
3	-9.517667596(-2)	4.94(-9)	3733.0391	6043.226
4	-9.517667596(-2)	5.37(-11)	3730.5039	4658.079
5	-9.517667596(-2)	7.29(-13)	3630.6953	5325.197
6	-9.517667596(-2)	2.34(-9)	3728.9336	4053.933
7	-9.517667596(-2)	1.20(-9)	3686.6992	3450.296
8	-9.517667596(-2)	1.65(-9)	3738.9805	4827.654
9	-9.517667821(-2)	2.83(-7)	3730.0469	5607.250
10	-9.517667596(-2)	1.96(-9)	3683.4805	4638.702
11	-9.517667782(-2)	2.38(-7)	3638.1328	8827.304
12	-9.517667596(-2)	6.33(-9)	3729.8633	4328.495
13	-9.517666960(-2)	2.80(-7)	3724.5977	4812.377
14	-9.517667596(-2)	9.83(-10)	3679.4883	4402.203
15	-9.517666960(-2)	2.80(-7)	3630.1328	5165.775
16	-9.517667596(-2)	4.80(-13)	3736.8047	4025.584
17	-9.517667596(-2)	4.66(-13)	3685.9922	4128.974
18	-9.517667596(-2)	1.33(-9)	3730.0430	4041.893
19	-9.517667596(-2)	1.72(-9)	3686.7891	4179.421

Table 3.14: LDC least stable eigenvalues for different numerical approaches mesh 72x72

For all the methods which converged to the solution, the eigenvalue was recovered with sufficient precision, in agreement with the reference value. In general, Jacobi-Davidson method is associated to a lower order of magnitude for the relative error and, consequently, is more accurate than Krylov projection, if the same preconditioning and linear solver are chosen. On the other hand, the dimension of the projection-subspace in Jacobi-Davidson method was chosen lower than in Krylov strategy and this leads to a better performance in number of floating point operations and therefore optimal when few eigenvalues are sought.

Figure [3.12] shows the eigenspectrum for the 72x72 mesh obtained using the numerical

approches in table [3.12]. The reference eigenvalue is highlighted with a black square.

Figure 3.12: LDC : Eigenspectrum comparison - Mesh 72x72

Finally, in figure [3.13], the streamwise velocity component of the eigenmode associated to the eigenvalue in table [3.14] is shown for three different mesh sizes and two different numerical approaches. The quality of the solution is obviously conditioned by the mesh refinement, but not by the numerical approach adopted to recover the solution.

Figure 3.13: LDC u velocity (x-direction) component and meshes

Summarizing the results obtained for the LDC problem, several different strategies

were able to properly recovered the solution of the BiGlobal EVP. As expected, direct LU preconditioning is the fastest approach, however, the approximated ILU and RAS alternatives require considerably less memory. Regarding the linear solvers, all the choices lead to similar performances in terms of memory demand, but FGMRES is particularly precised if compared with the others cases, and it could be suggested as convenient choice.

3.4.2 Flat plate

Similar comparisons among the different strategies for solving the EVP were conducted for the multidomain flat plate problem. As seen before, matrices corresponding to the multi-domains spatial decomposition are less dense than their counterpart mono-domain, for discretization with same number of mesh points. Moreover, the block diagonal structure of the multidomain approach can be exploited by the numerical algorithms involved in the EVP calculations. There are some features of the spectrum associated to this problem that can be explained from the eigenvalue distribution in the complex plane. Figure [3.14] depicts only the half positive imaginary part corresponding to the least stable region of the symmetric eigenspectrum recovered. Least stable disturbances are again close to the origin of the complex coordinate system, however, for the present two-dimensional analysis the reference eigenvalue, marked with a black square, is located in a branch of modes (aforementioned TS branch) with growing frequency while the damping rate is bounded in a specific range of values close to zero.

Figure 3.14: Flat plate: Reference eigenspectrum

From a practical point of view this feature implies that, contrary to what happened in the LDC case, a larger number of vectors for the base of the projection subspace has to be used to properly determine the wanted TS mode.

Another important characteristic of this test case is the presence of the continuous spectrum close to the dominant part of the spectrum. Clustered eigenvalue are related with a loss of accuracy in the eigenvector calculation. As a consequence, the dimension of the projection subspace has to be risen to compute adequately the TS branch, since increasing the projection subspace dimension has a straightforward effect in the choice of the numerical methodology. As previously stated from the LDC results, Jacobi-Davidson algorithm requires a low-dimensional search subspace to be competitive in terms of computational necessities. Moreover, this projection method is specifically affected by the presence of eigenvalue clusters in the search subspace. Because of these drawbacks, Jacobi-Davidson projection method has revealed to be not suitable for the flat plate problem. The table [3.15] show the cases studied for the flat plate, while the table [3.16] contains details on the parameter setting.

CASE	ST	PC	LS	PROJECTION METHOD
1	SHIFT-INVERT	LU	-	Krylov
2	SHIFT-INVERT	RAS	BICGSL(2)	Krylov
3	SHIFT-INVERT	ILU	BICGSL(2)	Krylov
4	SHIFT-INVERT	RAS	GMRES	Krylov
5	SHIFT-INVERT	ILU	GMRES	Krylov
6	SHIFT-INVERT	ILU	TFQMR	Krylov
7	SHIFT-INVERT	RAS	TFQMR	Krylov
8	SHIFT-INVERT	ILU	CGS	Krylov
9	SHIFT-INVERT	RAS	CGS	Krylov
10	SHIFT-INVERT	ILU	FGMRES	Krylov
11	SHIFT-INVERT	RAS	FGMRES	Krylov
12	SHIFT-INVERT	ILU	DGMRES	Krylov
13	SHIFT-INVERT	RAS	DGMRES	Krylov

Table 3.15: Selected cases for flat plate

PARAMETER	VALUE
Shift (<i>if needed</i>)	0
ILU level (<i>if needed</i>)	8
RAS subdomain (<i>if needed</i>)	1
RAS overlapping (<i>if needed</i>)	1
Tolerance for the LS (<i>if needed</i>)	10^{-8}
Tolerance for the projection	10^{-8}
Number of vectors for the Krylov subspace projection	2000
Minimum number of eigenvalues required for the Krylov projection	1000

Table 3.16: Setting for LDC simulations

Preliminary study is dedicated to analyze the mesh discretization effect on the accuracy of the obtained results. The classical direct LU approach was used to compute the spectrum and recover the reference eigenvalue and eigenmode from the TS branch. Three different meshes were tested, all of them with four computational domains, varying the number of discretization points in each of them. A coarser mesh with 24x16 nodes, an intermediate mesh with 56x32 nodes and a finer with 64x48 points, as also already mentioned in section [3.3], table [3.4]. The points in the wall-normal direction (y-axis) are concentrated on the wall using a stretching function to properly describe the boundary layer. Figures [3.16]-[3.17]-[3.18] show the TS-mode associated to the reference eigenvalue. The eigenmode is well resolved for the meshes 56x32 and 64x48. The mode for the smallest mesh 24x16 can not be neither recognized as TS-mode due to the not sufficient number of grid points, which does not allow to properly recover the temporal stability of the disturbance nor its spatial growth and frequency.

Figure 3.15: Flat plate : Flat plate: Reference Eigenvector u-velocity

Figure 3.16: Flat plate - u velocity (x-direction) least stable eigenmode and mesh 4x24x16

Figure 3.17: Flat plate - u velocity (x-direction) least stable eigenmode and mesh 4x56x32

The graphs [3.19] and [3.20] show the memory consumption and the computational time associated to the methods in table [3.15]. Also for this second test flow, such as

Figure 3.18: Flat plate - u velocity (x-direction) least stable eigenmode and mesh 4x64x48

for the previous LDC flow, the results show a common trend for the case in table [3.15], varying the mesh dimension. The classic approach with LU and Krylov projection (case 1), is again the fastest and most memory demanding approach. Regarding the others cases, which do not employ the LU decomposition, the performances in terms of memory demand are comparable, while the computational time needed for recovering the solution shows slight differences among the studied methods. Independently of the mesh size, the use of FGMRES solver with incomplete LU (cases 10) is the fastest method to converge.

Figure 3.19: Flat plate - Memory costs

Figure 3.20: Flat plate - computational time

Table [3.17] shows the reference eigenvalue obtained with the different numerical approaches, for the mesh with four computational domains of $4 \times 64 \times 48$ nodes. All these results match the reference value with acceptable relative errors.

Flat plate: Reference eigenvalue - mesh $4 \times 64 \times 48$ -						
Case	Damping rate	Frequency	e_{abs}	Memory Consumption[Mb]	CPU TIME [s]	
1	-5.56215092(-3)	0.11283377	1.67(-3)	13518.6250	1364.349	
2	-5.56186463(-3)	0.11283289	4.37(-3)	6760.7148	33269.479	
3	-5.56171920(-3)	0.11283292	1.39(-3)	6874.9609	14706.235	
4	-5.56179619(-3)	0.11283288	3.90(-3)	6760.3242	14610.769	
5	-5.56173746(-3)	0.11283283	4.39(-3)	6865.6250	7756.617	
6	-5.56170783(-3)	0.11283283	2.64(-3)	6889.6523	15411.283	
7	-5.56168759(-3)	0.11283291	5.68(-3)	6775.6562	37567.880	
8	-5.56188693(-3)	0.11283281	5.98(-3)	6883.1367	11356.262	
9	-5.56176398(-3)	0.11283283	1.97(-3)	6777.5938	24183.195	
10	-5.56174534(-3)	0.11283287	3.42(-5)	6897.6016	6859.785	
11	-5.56174533(-3)	0.11283287	1.04(-4)	6782.9805	11625.831	
12	-5.56173929(-3)	0.11283283	4.93(-3)	6879.7305	7844.794	
13	-5.56169418(-3)	0.11283292	5.45(-3)	6776.6094	14843.848	

Table 3.17: Flat plate - Reference eigenvalues for different numerical approaches - mesh $4 \times 64 \times 48$

Regarding the accuracy obtained for the reference eigenvalue, the FGMRES linear solver, preconditioned with ILU and RAS (cases number 10 and 11) is again associated to the best performances. This suggests that the gain in term of precision could be mainly associated to a capability of the FGMRES to recover the solution using a lower number

of numerical operations than the others approaches, reducing both the errors associated to the floating-point arithmetic operations and the computational time.

The use of Krylov projection with ILU or RAS preconditioning using FGMRES as linear solver is then here suggested as convinient choice for solving the EVP, confirming the results obtained for the LDC.

Next figures [3.21]-[3.22] depict the obtained eigenspectrum with the classic approach (direct LU) and the recommended iterative method, case 10. Both results are qualitatively and quantitatively analogous and the TS branch, which contains the reference eigenvalue highlighted with a black square, are well detected. These results agree with [Alizard 2007].

Figure 3.21: Flat plate : Eigenspectrum - Case 1: LU precondition

Figure 3.22: Flat plate : Eigenspectrum - Case 10: ILU + FGMRES

Finally, the second strategies is adopted. This strategy, based on the shifted technique as expained in section [3.1.3], follows the T-S branch with a lower dimension of the krylov subspace for a faster recovery of the desired region of the eigenspectrum. As the shift is defined complex, a complex version of the libraries was used since introducing a complex shift turn the problem into complex arithmetics formulation. The incomplete LU is not available in HYPRE Library for complex matrices so RAS preconditioning with FGMRES linear solver were chosen to recover the set of shifted spectra. Figure [3.23] depicts the combined spectrum resulting from computing the EVP solver with three different shifts located along the T-S branch. From a computational point of view the memory cost associated to the three runs in complex arithmetic with a projection subspace dimension of 50 is $8500Mb$, almost a reduction of 30% compared to the complex version of the original 2000 Krylov dimension. CPU time is also reduced from 15245 seconds to 6501 when the shifted version is used.

Figure 3.23: Flat plate : Combined spectrum obtained using three shifted runs

Based on these results, this second strategy that involves the use of shifting is associated to better performances. However, it is also associated to more restrictions, since complex arithmetic is involved and a previous knowledge of the spectrum is required in order to identify the shift values close to the targeted region.

3.5 Conclusion

Several numerical approaches were compared in the framework of the fluid dynamic application in order to caught a general overview of the applicability and performances, specifically for avoiding the use of the huge memory-demanding LU decomposition. Between all the available numerical approaches, alternative to the typical direct LU method with Krylov subspace projection, any relevant difference were found from a memory demand point of view and few differences were found regarding the computational time.

In particular, the use of an incomplete LU decomposition associated to FGMRES linear solver and Krylov projection methods lead a higher performance in term of convergency behaviour.

Particularly care has to be focused on the choice of the ILU level, that affects the memory demand of the methods and the quality of the solution obtained.

For the final convergence of the tested cases, also the initial guess vectors for the krylov iterative solver could make the difference to converge to the solution.

Regarding the two different projection methods, JD and Krylov, they are comparable both for memory and time cost only if a lower number of vectors for the JD subspace base is

chosen, otherwise Krylov method has to be considered as the better choice.

If only few and inner eigenvalues are required, as for the LDC, both JD and Krylov (with a proper setting) are a suitable choices, but for more complex and ill-conditioned problems or if a great part of the eigenspectrum is required, Krylov projection could be the only suitable choice.

These results have to be contextualized into the fluid dynamic problems here examined, different results could be obtained for better-conditioned problems, as reported in [Romero 2010]. Again, the results obtained shown that the iterative processes are cheaper than the LU decomposition and also able to recover the solution with a sufficient precision. Increasing the complexity of the fluid dynamic problem, the number of available alternative numerical approaches is reducing, but the capability to recover a good solution with a sensible reduction of the memory costs is still possible.

High Lift Study

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Instabilities generated by separation at high lift condition are still one of the most relevant industrial problem that requires further investigations. In this chapter, the stability analysis of an A310 airfoil for a wide range of angle of attack (AoA) is performed and the results are compared to experimental data. A comprehensive evolution of the spectrum, increasing the AoA, is presented. Moreover, a structural sensitivity analyses to detect the area where a perturbation of the flow have a maximum impact is performed for relevant AoAs, providing the most sensible regions that can be identified to stabilize the modes.

High lift devices are used to reduce the take-off and landing maneuver times, in order to allow to complete these maneuvers shortly. In particular, the three-component airfoil, here used as test case, shows better performances in terms of maximum lift coefficient and a delayed progressive stall with respect to the mono-component airfoil of same chord. Moreover, the recirculating flow region between slat and central element leads to lower adverse pressure gradients on the upper part of the central body itself, delaying the onset of the boundary layer separation conditions. In addition, the boundary layer of each elements results to be thinner, and then more stable, if compared with the boundary layer of a mono-component airfoil of equal chord. For all these reasons, High Lift devices are now part of the daily base projection phases in the industrial scenario.

The phenomenon which characterizes a turbulent flow for a high lift configuration is complex and can exhibit several patterns, such as shock-wave/boundary layer interaction, suction pockets and flow separation. An overall scenario of the possible flow phenomenon that can occur, summarized in figure [4.1], was presented by Ramsey in [Ramsey 2002]. The complexity of the flow features which characterizes the high lift configuration, combined with the increasing industrial interest for the multi-component airfoil, are the reasons which motivate the analysis presented in this chapter. A complete stability analysis in order to understand the underlying mechanism which govern the flow evolution for this industrial application is missing in literature and next sections are intended to provided a preliminary study to fill this lack.

Figure 4.1: Possible flow phenomenon for multi-component devices

4.1 High Lift test case configuration

In [Manie 1989] several multi-component flow configurations for the two dimensional A310 airfoil were tested in the ONERA F1 wind tunnel. Among those, the flow condition at $Re = 6 \cdot 10^6$ and Mach Number $M = 0.22$, for the GARTEUR Take Off (TO) airfoil configuration, is here analyzed for a wide range of angles of attack (AoA). The results are then compared with the experimental data, in order to validate the stability analysis approach for this test case.

Here, the high lift flow is described by means of compressible Reynolds averaged Navier-Stokes equations (RANS). The turbulence model employed is the Spalart-Allmaras [Spalart 2000b]. The computational domain Ω is discretized using a finite volume approach, into an unstructured mesh of radius equal to 100 chords, shown in figure [4.2].

In order to obtain a steady-state solution, the time integration is performed using a pseudo time stepping τ and a backward Euler implicit scheme. The linear solver used to solve the implicit scheme is a Lower-Upper Symmetric Gauss Seidel Methods (LU-SGS). No slip and farfield boundary conditions are imposed on the body surface and the external contour respectively.

This flow configuration has been solved for a wide range of values of AoA, spanning from 0° to 30° . For each of this AoA, the lift coefficient (Cl) was calculated, in order to recover the polar curve Cl -AoA. Figure [4.3] shows the curve obtained with RANS simulations compared with the experimental data available in [Manie 1989]. In that research, the RANS simulations slightly overestimated the maximum value of the lift coefficient, Cl_{max} , in agreement with what observed by Iorio et al. in [Iorio 2016], but the progressive stall prediction, between AoA of 23° and 24° was coherent with the experimental results.

Full illustration of the base flow, obtained with TAU-DLR flow simulator [201 2011], for the configuration at AoA equal to 24° is shown in figures [4.4] and [4.5]. The base flow clearly shows the low pressure area above the slat component, which corresponds to the higher streamline velocity and higher Mach number regions, typical of this multi-component airfoil.

Figure 4.2: A310 airfoil mesh

Figure 4.3: A310 airfoil Cl -AoA curve

Figure 4.4: A310 airfoil base flow: pressure (top) and x-component velocity (bottom)

Figure 4.5: A310 airfoil base flow: Eddy viscosity (top) and Mach number contours (bottom)

4.2 Stability analysis

In order to obtain the eigenspectrum evolution with respect to the AoA variation, the stability analysis of the flow configuration described in the previous section was carried out for the AoA spanning from 0° to 30° . The stability analysis performed here follows the classic approach, which involves the use of *LU* decomposition, spectral inverse transformation and Krylov-base projection, as widely discuss in the chapter [3].

For all the range of AoA examined, the multi-component eigenspectrum, shown in picture [4.6] for the AoA= 0° and Krylov subspace dimension of 1000, shows two main parts: a clustered region of eigenvalues around the origin of the complex plane and a lateral-extended branch of eigenvalues which spans for a large range of frequencies. Many simulations were performed in order to obtain a fully converged eigenspectrum, but the spectral inverse transformation only permits to trustly recover the eigenvalues which belong to a short range area close to the shift. Picture [4.7-top] shows the half-right part of the central converged region for the symmetric eigenspectrum, obtained applying the inverse transformation with no-shift. Picture [4.7-bottom], instead, shows the half-right part of the not converged branch of eigenvalues, for subspace Krylov dimensions among 1000 and 3000. For all the AoA and the subspace Krylov dimensions considered for this work, the, not fully converged, lateral-extended branch shows unstable modes at high frequencies, as shown in picture [4.7-bottom] and picture [4.8]. The frequencies of these unstable eigenvalues increase if the dimension of the Krylov subspace is increased. However, a further analysis using inverse transformation with several shift factors along the real axis, spanning for a large range for the frequency, did not detect any instability. In conclusion, it was not possible to confirm or exclude the existence of unstable eigenvalues at very high frequency.

Two complex pairs, corresponding to the lateral branch of high frequency modes, were investigated, hereafter referred with the letters A and B. Although they are not fully converge in this preliminary studied, (an exhaustive campaign of cases varying the chosen shift is required) the richness of the structures existing deserves a close examination.

The A-eigenpair is a convective mode characterized by very thin boundary layer and no flow separation. Figure [4.9] shows the direct and adjoint mode related to the A-eigenpair. Figure [4.10] shows the structural sensitivity of this A-eigenpair. It reveals that this mode can be modified if an external force is applied to the flow field around the slat, in the flow region between slat and central body and in the flow region between the central body and the flap.

The second unstable eigenpair (B) is more complex in terms of physical explanation and semblance. This mode develops after the onset of the shear layer instability, which in this case appears on the slat trailing edge, where the flows from the lower cavity and upper surface of the slat join with different velocities. Figure [4.11] clearly shows the consecutive sequences of clockwise and counterclockwise vortices for the density and x-velocity components of the direct mode related to the B-eigenpair, while figure [4.12] shows adjoint and sensitivity eigenfunctions. The structural sensitivity, in this case, is mainly located in the region between the trailing edge of the slat and the central body.

Figure 4.6: A310 airfoil spectrum, $\text{AoA} = 0^\circ$

4.2.1 High Lift condition

According with the Cl -AoA curve in figure [4.3], progressive stall starts between AoA equal to 23° and 24° . However, the converged low-frequency region of the eigenspectrum, obtained when performing the stability analysis across this transition, appears to be stable.

In previous turbulent flow stability analyses on a NACA0012 airfoil by Wales et al. [Wales 2011, Wales 2012] and on High Lift configuration in Iorio et al. [Iorio 2016], an eigenmode is found to become unstable after the condition of Cl_{max} is overcome. In particular, the investigation of Iorio et al. is performed on a A310 airfoil, as the one here analyzed, but at different Reynolds number. Since the experimental results in [Manie 1989] clarified that the Cl_{max} is not particularly affected by Reynolds number variation, the Iorio et al. results locate the Cl_{max} condition in the vicinity of AoA equal to 24° , as obtained in this analysis, and identified the onset on an unstable mode when the AoA 27° is overcome. Similarly, between the AoA $26^\circ - 27^\circ$ the flow configuration here studied presents a pair of complex conjugate eigenvalues, shows in figure [4.13], which turn from stable to unstable conditions. This mode, depicted in figures [4.14] and [4.15], is a wake type mode which evolves from the boundary layer until it separates.

Transducers data, applied on the three components of the A310 airfoil during the wind tunnel tests in [Manie 1989], led to the conclusion that, for this specific configuration, the progressive stall start from the wing-body central component. In agreement with this results, the eigenfunction in figures [4.14], shows a flow evolution which particularly affects the central element, even if the sensitivity eigenfunctions in figure [4.16] is mainly located on the slat.

The onset of this instability starts for an AoA of circa 1° lower than the previous results obtained by Iorio et al. A possible explanation can be found in the different Reynolds

Figure 4.7: A310 airfoil spectrum, $AoA = 0^\circ$, converged region (top), not converged branch (bottom)

number used from the two flow configurations. It is plausible to assume that the increment of the Reynolds number, at the other same conditions, can move up the onset of an unstable mode, even if not affecting the Cl^{max} .

Figure 4.9: First A310 airfoil mode, AoA equal to 0° , direct (top) and adjoint (bottom) x-component velocity

Figure 4.10: A310 airfoil mode, AoA equal to 0°, structural sensitivity

Figure 4.11: Second A310 airfoil mode, AoA equal to 0° , density (top) and direct (bottom) x-component velocity

Figure 4.12: Second A310 airfoil mode, AoA equal to 0° , adjoint x-velocity component (top), structural sensitivity (bottom)

Figure 4.13: A310 airfoil eigenspectrum transition between AoA 26° – 27°

Figure 4.14: A310 airfoil unstable mode, post stall condition, direct x-component velocity

Figure 4.15: A310 airfoil unstable mode, post stall condition, adjoint x-component velocity

Figure 4.16: A310 airfoil unstable, post stall condition, structural sensitivity

4.3 Conclusion

In this chapter, the stability analysis, in its classic approach, was applied to investigate one of the most challenging industrial application: the High Lift flow configuration. Experimental data are available for the particular flow configuration used for this stability analysis, in order to validate the results. The base flow calculation, obtained by RANS simulations, shows a good agreement with the experimental results, with a slightly overestimation for the Cl_{max} . The multi-component A310 airfoil shows possible unstable modes even for the smallest AoA considered, even if it is not possible to obtain a fully converged spectrum. Once that the Cl_{max} condition is overcome, a couple of complex eigenvalues were identified to turn into unstable modes, possibly triggering to the flow separation which starts the progressive stall. This unstable mode develops starting from the slat and evolves along the upper part of central element of the multi-component airfoil, as also suggested by the results obtained from the transducers applied to the A310 airfoil during the wind tunnel experiments. These remarkable results, in agreement with previous analysis, confirm the potential of stability analysis to detect the onset of unstable modes and provide valuable information to further understand the phenomenon which lead to the unstable configuration. The structural sensitivity, then, support the results and detect the region of the flow field where an external force can act to possibly stabilize the mode.

Domain reduction strategy

Contents

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The domain reduction methodology is a promising tool to alleviate the relevant, and sometimes prohibitive, amount of memory demanded for solving large EVP. The key of the strategy is to consider only the region of the computational domain where the physical phenomenon of study is located. Consequently, the leading dimension of the EVP is reduced and becomes numerically affordable. The method could also be used to zoom into a region of interest, reducing the domain size but increasing its resolution, balancing the overall computational effort. To the authors knowledge, no detailed analysis of the accuracy of this approach can be found in literature, with the sole exception of Giannetti and Luchini [Giannetti 2007], where the efficiency of the method is suggested to be linked to the structural sensitivity of the mode. In their work, Giannetti and Luchini perform a domain restriction analysis of the flow around a cylinder, resizing the mesh in order to include or exclude parts of the structural sensitivity region and then comparing the eigenvalues.

However, to the authors point of view, the lack of mathematical explanation to clarify this relation motivated further investigations. In order to clarify this, in this chapter the reduction strategy is performed acting directly on the Jacobian matrix of the whole domain, neglecting the part associated to the nodes of the domain whose do not belong to the region of interest. This mathematical formulation is able to explain, quantitatively and qualitatively, the implications associated to the domain reduction only. Moreover, it represents a powerful tool for the a priori-estimation of the error committed on the eigenvalue calculation, depending on which region of the computational domain is considered.

Once the mathematical background is described, the open cavity problem is used as a benchmark to demonstrate that the reduced domain technique is able to recover the stability features of the flow in the region of interest. Besides, the structural sensitivity analysis is used to investigate the limitations of this technique.

Finally, to validate the methodology, a more ambitious test case, the transonic buffet phenomenon on the NACA0012 is investigated as industrial benchmark. Exhaustive stability analysis of this test case is available in [Iorio 2014]. For this phenomenon, the leading mode, i.e. the shock wave, remains localized on the top of the airfoil so the region of interest for the reduction domain is well confined and easily identified.

5.1 Mathematical description of domain reduction strategy

Subspace iteration methods project the matrix into an appropriately built, lower-dimensional subspace to extract an approximation of the spectrum. The projection is performed as:

$$P^T A_n P = A_m, \quad (5.1)$$

where P is the $n \times m$ orthogonal projection matrix and m the dimension of the subspace used for the projection. A_m is the matrix of the reduced system of leading dimension m . The Arnoldi algorithm [Saad 2003] is one of the most efficient and reliable projection techniques that builds the P matrix using a m -dimensional Krylov Subspace. After m iterations the method reduces the A_n matrix into a Hessenberg matrix, whose eigenvalues and eigenvectors may be obtained by full-spectrum (direct) techniques. The domain reduction methodology can also be interpreted as a projection of the original matrix into a subspace of lower dimension. The projection is of geometric nature, instead of algebraic as in the standard Arnoldi algorithm.

The physical domain, Ω , is discretized into a reference computational mesh Ω_n with an associated EVP matrix A_n . The domain Ω_n is cropped and a subset of the computational mesh, $\Omega_m \subset \Omega_n$, containing m degrees of freedom is considered. No assumption on the mesh type or connectivity is made: the derivation remains valid irrespective of the numerical technique used to assemble the EVP. To simplify the exposition, the global vector $\mathbf{q}_n$ is sorted keeping the unknowns in the subdomain Ω_m in the first positions. Namely, $\mathbf{q}_n = \{q_1 \cdots q_n\}^T = \{q_{\{1, \dots, m\}} | q_{\{m+1, \dots, n\}}\}^T$.

Operator P is defined as:

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{(n,m)}, \quad (5.2)$$

where the first m rows of P defines an identity matrix of order m and the remaining $(n - m)$ rows are identically zero. Matrix P is an orthogonal projector whose columns form an orthonormal basis for subdomain Ω_m . The projection of the Jacobian A_n into the m -dimensional subspace is computed as:

$$A_m = P^T A_n P.$$

Note that P^T is the projection operator from Ω_n to Ω_m , $\mathbf{q}_m = P^T \mathbf{q}_n$.

Additionally, the following relations hold:

$$A_n P = P A_m + E_m \quad (5.3)$$

$$P^T A_n = A_m P^T + E_n \quad (5.4)$$

The *error matrix* E_m is a $n \times m$ matrix whose first m rows are zero and the remaining $n - m$ correspond to the $\{m + 1, \dots, n\}$ rows of the Jacobian A_n and columns from 1 to m . Likewise, E_n is a $m \times n$ matrix whose first m columns are zero and the remaining

$n - m$ columns coincide with columns $m + 1, \dots, n$ of the first m rows of A_n :

$$E_m = \left(\frac{0}{A_{i,j}} \right)_{(n,m)} \quad i = m + 1, \dots, n; \quad j = 1, \dots, m$$

$$E_n = \left(0 \mid A_{i,j} \right)_{(m,n)} \quad i = 1, \dots, m; \quad j = m + 1, \dots, n \quad (5.5)$$

The error matrices, E_m and E_n , are -generally- quite sparse, since they only contain those elements of the Jacobian A_n in the subregion $\delta\Omega_m \cap \Omega_n$. From a numerical standpoint, they can be seen as artificial boundary conditions for the reduced domain.

Regarding the EVPs:

$$A_n \mathbf{q}_n = \omega_n \mathbf{q}_n, \quad (\mathbf{q}_n^*)^H A_n = \omega_n (\mathbf{q}_n^*)^H, \quad (5.6)$$

$$A_m \mathbf{q}_m = \omega_m \mathbf{q}_m, \quad (\mathbf{q}_m^*)^H A_m = \omega_m (\mathbf{q}_m^*)^H, \quad (5.7)$$

where ω_n and ω_m are the exact eigenvalues of A_n and A_m respectively, with associated right and left eigenvectors $\mathbf{q}_n, \mathbf{q}_n^*$ and $\mathbf{q}_m, \mathbf{q}_m^*$. Superscript H stands for the conjugate transpose. Note that adjoint and left modes differ in general, since the former requires the definition of a particular dot product implying pre- and post-multiplication by a weight matrix.

domain reduction method are advantageous when the eigenvalues of A_m (computationally affordable provided $m \ll n$) approximate those of A_n well. We quantify errors in both eigenvalues and eigenvectors, and their relation to the structural sensitivity.

Multiplying both sides of equation (5.3) by $\mathbf{q}_m$, using (5.7) and definition (5.2) yields:

$$(A_n - \omega_m I) \tilde{\mathbf{q}}_n = E_m \mathbf{q}_m, \Rightarrow \|(A_n - \omega_m I) \tilde{\mathbf{q}}_n\|_{L_2} = \|E_m \mathbf{q}_m\|_{L_2} \quad (5.8)$$

where $\tilde{\mathbf{q}}_n = P \mathbf{q}_m$ and $\omega_m = \tilde{\omega}_n$.

This means that residual $\|A_n \tilde{\mathbf{q}}_n - \tilde{\omega}_n \tilde{\mathbf{q}}_n\|$, associated to the approximate eigenpair $(\tilde{\omega}_n, \tilde{\mathbf{q}}_n)$, equals the norm of the product of the error matrix E_m times the approximate eigenvector $\mathbf{q}_m$. This error can be measured exactly since prior knowledge of the eigenvalues of A_n is not necessary but only those of A_m , computationally cheaper. The computation of E_m does not require to obtain the full Jacobian A_n , since only boundary nodes are involved. Additionally, the equation implies that the error is proportional to the norm of the reduced eigenvector q_m in subregion $\delta\Omega_m \cap \Omega_n$.

In order to estimate differences between the eigenvalues of the reference and the reduced domain, both sides of equation (5.3) are pre-multiplied by the exact left eigenvector $\mathbf{q}_n^*$ and post-multiplied by $\mathbf{q}_m$, obtaining:

$$(\omega_n - \tilde{\omega}_n) ((\mathbf{q}_n^*)^H \cdot \tilde{\mathbf{q}}_n) = (\mathbf{q}_n^*)^H E_m \mathbf{q}_m \Rightarrow \Delta\omega_n = \frac{(\mathbf{q}_n^*)^H E_m \mathbf{q}_m}{((\mathbf{q}_n^*)^H \cdot \tilde{\mathbf{q}}_n)}, \quad (5.9)$$

Equivalently, pre-multiplying equation (5.4) by the left eigenvector of A_m , $\mathbf{q}_m^*$, and post-multiplying it by $\mathbf{q}_n$ leads to:

$$\Delta\omega_n = \frac{(\mathbf{q}_m^*)^H E_n \mathbf{q}_n}{((\mathbf{q}_m^*)^H \cdot (P^T \mathbf{q}_n))}, \quad (5.10)$$

Equation (5.9), and analogously (5.10), resembles the structural sensitivity, see Luchini et al. ([Giannetti 2007]), Eq.(8.4)). The sensitivity region of an eigenmode is the

region where a small perturbation could cause the largest drift in its associated eigenvalue [Sartor 2015, Meneghello 2015, Browne 2014]. This region is obtained from the product of the right and left eigenmodes at each computational node. Numerator $(\mathbf{q}_n^*)^H E_m \mathbf{q}_m$ in (5.9) deserves further attention: given E_m , only the product of the left mode (for the external/reference domain) and right mode (for the internal/reduced) at the common region $\delta\Omega_m \cap \Omega_n$ contributes to the error in equation (5.9). The aforementioned product is intrinsically related to the sensitivity region and the shift in the eigenvalue is weighted by the magnitude of the error matrix E_m at that boundary. The relevance of the sensitivity region will be assessed in sections 3 and 4.

Regarding the difference between the eigenvectors, from equations (5.3), (5.7), (5.8) and (5.9), the following relation is deduced:

$$((\mathbf{q}_n^*)^H P - (\mathbf{q}_m^*)^H) = (A_m - \omega_n I_m)^{-1} (\mathbf{q}_n^*)^H E_m \left(\frac{\mathbf{q}_m (\mathbf{q}_m^*)^H}{(\mathbf{q}_n^*)^H \tilde{\mathbf{q}}_n} - I_{(m,m)} \right). \quad (5.11)$$

Equivalently,

$$(P^T \mathbf{q}_n - \mathbf{q}_m) = (A_m - \omega_n I_m)^{-1} \left(\frac{\mathbf{q}_m (\mathbf{q}_m^*)^H}{(\mathbf{q}_m^*)^H P^T \mathbf{q}_n} - I_{(m,m)} \right) E_n \mathbf{q}_n. \quad (5.12)$$

As before, errors in left (right) approximated eigenvectors (equations 5.11 or 5.12) are also dominated by the sensitivity region $(\mathbf{q}_m (\mathbf{q}_m^*)^H)$ at $\delta\Omega_m \cap \Omega_n$. Additionally to terms involved in calculating error of eigenvalues, the resolvent of the reduced operator $(A_m - \omega_n I_m)^{-1}$ affects eigenvector error computations. This factor is highly sensitive for non-normal matrices, like those derived from NS equations. As it will be shown in the validation section, this non-normality may lead to higher errors in comparison to those obtained for the corresponding eigenvalues.

Unfortunately, the left eigenvector $\mathbf{q}_n^*$ in equations (5.9) and (5.11) is unknown since the reference problem is computationally unaffordable. An error estimation may be inferred nevertheless -under the assumption that the difference between the “external” and “internal” left modes at the shared boundary is small- by considering a slightly larger reduced domain, $\Omega_m \subset \Omega_{m2}$, encompassing the region where E_m is not zero. The left mode in Ω_{m2} is a good estimation for $\mathbf{q}_n^*$. A similar procedure approximates the right mode $\mathbf{q}_n$ in equations (5.10) and (5.12).

The validation of the discussed error estimation is tested in the last section of this chapter for the buffet transonic phenomena.

5.2 Study and validation: Open cavity problem

The open cavity (OC) problem appears in numerous industrial applications such as landing gears or weapon bays. The richness of the physics involved, showing convective and global instabilities -confined or not inside the cavity- makes this problem an excellent candidate to test the domain reduction methodology.

A schematic representation of the flow configuration is depicted in Figure [5.1]. The flow is defined by the cavity-depth-based Reynolds number (Re_D), and the incoming boundary layer displacement thickness (δ_0/D). The geometrical parameters of the problem are the length-to-depth aspect ratio (L/D), and the wavelength in the spanwise direction (L_z) with wavenumber $\beta = 2\pi/L_z$.

The configuration considered in this section -aspect ratio $L/D = 2$, wavenumber $\beta = 6$, $\text{Re}_D = 1500$ and $\delta_0/D = 0.28$ - has been analyzed in [Bres 2008, de Vicente 2014]. The leading eigenmode is a slightly unstable disturbance associated with a pair of complex conjugated eigenvalues. As seen in figure [5.4], this mode is the first one which gets unstable for the selected parameters. The amplification rate is $\omega_R = 0.009837$ and frequency is $\omega_I = \pm 0.162314$, corresponding to a Strouhal number of $St = 0.0259$.

The growing mode emanates from the cavity. It belongs to a family of disturbances known as centrifugal instabilities, caused by the recirculation bubble, which successively breaks down the flow in the spanwise direction into periodic finger-like structures. Those structures/eigenmodes are mainly confined in the cavity, consequently they are ideal candidates to test the limits of domain reduction methods.

Figure 5.1: Schematic description of the three-dimensional open cavity parameters

The steady two-dimensional basic flow is computed using a multidomain spectral collocation technique. The multidomain methodology decomposes the computational region into several subdomains. Inside each subdomain, the solution is expanded using the tensor product of high order Chebyshev polynomials over Gauss-Lobatto nodes. Collocation points are clustered near $y = 0$ employing a stretching law, which improves the resolution. For illustration the velocity components of the steady base flow are sketched in figure [5.2].

The stability analysis is then performed.

In order to test the validity of the domain reduction method, the following procedure is applied:

Firstly, the multidomain approach is employed to discretize the stability equations in the computational region. Four different configurations are defined. The reference mesh Ω_n comprises eleven rectangular subdomains. The reduced configurations are subsets of the reference mesh: a 6-domains Ω_{m1} case includes the cavity and the whole downstream region of Ω_n . Discarding two downstream domains the 4-domains Ω_{m2} case is obtained, while the 1-domain Ω_{m3} case consists only of the cavity region. Figure [5.3] depicts the computational domain with the discretization employed for the setups mentioned. The discretization inside each subdomain is preserved for every configuration: this eases error analysis and inter-setup comparisons.

The continuous LNS equations for the stability analysis are discretized in Ω_n and the associated EVP is assembled. This EVP is solved using an Arnoldi iterative algorithm with shift and invert methodology. A complete stability analysis of the 3D incompressible cavity flow including a detailed study of the boundary conditions, the parametric dependence of

Figure 5.2: Open cavity base flow

the modes, their morphological structure, characteristic frequencies and neutral stability curves were presented by the authors in [Garrido 2014].

Finally, the domain reduction procedure is applied. From the original EVP matrices (related to 11-domains Ω_n) only rows and columns belonging to region $\Omega_{m,i}$ are kept. This leads to a smaller EVP where no additional boundary conditions are imposed. The solution of the stability analysis of the flow in the reduced $\Omega_{m,i}$ is obtained and compared to that of Ω_n .

Next, the effects and limitations of the domain reduction technique are investigated and identified following the error methodology proposed in section 3.

Figure [5.5] shows the real part of the three velocity components of the selected eigenmode in Ω_n . Despite the main activity being confined inside the cavity for all the amplitude functions, the streamwise velocity component develops downstream reaching the boundary of the computational domain.

Figure [5.6] shows the real part of the three velocity components of the left eigenmode employed in expression (5.9) to evaluate the eigenvalue error. The sensitivity region, $||(\mathbf{q}_n^*)^H \mathbf{q}_n||$ at each node is shown in figure [5.7]. The sensitivity region lies mainly inside the cavity and extends along the shear layer from the upstream corner to the downstream cavity lip. According to this, as long as the reduced domain contains the sensitivity region, the error in the eigenvalue $\delta\omega$ (5.9) will be negligible. To check that, the computed eigenvalues, $\omega_{m,i}$, and their associated errors, according to formula (5.9) are shown in Table [5.1]. The comparison confirms the assumptions in section [5.1]: whenever the sensitivity region is included in the computational domain the shift on the eigenvalue is minimal. Additionally, the first component in the leading mode obtained for the domain reduction setups are shown in figures [5.8],[5.9] and [5.10]. Contrary to intuition, even when

Figure 5.3: Reference and reduced configurations for the OC problem

the spatial structure of the amplitude functions are affected by the domain reduction the eigenvalue $\omega_{m,i}$ remains well resolved.

Open cavity leading eigenvalue				
setup	Leading eigenvalue		Error eq. (5.9)	
	Frequency	Amplification rate	Frequency	Amplification rate
Reference mesh	0.1623139	9.837369(−3)	—	—
6 — domains	0.1623542	9.881542(−3)	4.03(−5)	4.41(−5)
4 — domains	0.1619219	9.516112(−3)	3.92(−4)	3.21(−4)
1 — domain	0.1451093	−1.367971(−2)	1.72(−2)	2.35(−2)

Table 5.1: Leading eigenvalue associated to the OC problem. Error according to equation (5.9)

However, the differences between the eigenmode computed in the reduced domain and the projection of the reference mode, $P^T \mathbf{q}_n - \mathbf{q}_{m,i}$, are not negligible. The differences may even reach the same order of magnitude of the variable, as seen in figure [5.8],[5.9] and [5.10-bottom]. The non-normality of the LNS matrices, linked to the resolvent operator in (5.12), is responsible of the difference observed. Reversely, this non-normality stems from the artificial boundary conditions imposed by the domain reduction technique.

Figure 5.4: Open cavity spectrum: 11-domains case

Figure 5.5: Velocity components of the right leading disturbance from the BiGlobal analysis

Figure 5.6: Velocity components of the left leading disturbance from the BiGlobal analysis

Figure 5.7: Region of maximum sensitivity

Figure 5.8: (Top) Stream-wise velocity components of the leading disturbance for the 6-domain reduction setups. (Bottom) Relative error in the same component with respect to the reference value in the 11-domain configuration

Figure 5.9: (Top) Stream-wise velocity components of the leading disturbance for the 4-domain reduction setups. (Bottom) Relative error in the same component with respect to the reference value in the 11-domain configuration

Figure 5.10: (Top) Stream-wise velocity components of the leading disturbance for the 1-domain reduction setups. (Bottom) Relative error in the same component with respect to the reference value in the 11-domain configuration

5.3 Transonic buffet phenomenon

The buffet phenomenon is defined as a strong shock boundary layer interaction during the transonic condition on an airfoil, see [Crouch 2009, Iorio 2014]. The phenomenon appears for values for the angle of attacks α which overcome a specific critical values α_{cr} that depends on Mach number and Reynolds number, leading to a globally unstable flow configuration. More in details, this cycling and auto-fed phenomenon starts after that the critical condition is reached, when the separation bubble starts to grows reducing the effective camber of the airfoil. As a consequence, the shock is shifted upstream, where the pre-shock Mach number is reduced, losing enough strength to permit the reattachment of the flow. This sudden flow motion is then followed by a shock wave downstream and the wave itself gains strength again. At this point the feedback loop is finished and the cycle restart. The overall phenomenon appears as an oscillating shock wave that could damage the airfoil and is usually also perceived by the pilot as a vibration.

The domain reduction method is applied to the transonic buffet phenomenon on a NACA0012 profile at Reynolds number $Re_c = 10^7$ based on the chord, Mach number $M = 0.76$ and angle of attack of $\alpha = 3.2$ degrees. The base flow for this test case is computed using the industrial solver DLR-TAU [201 2011], which solves the compressible Reynolds averaged Navier-Stokes equations (RANS) using a finite volume discretization. Spalart-Allmaras turbulence model with compressibility correction [Spalart 2000a] is used as turbulence model. Time integration is performed using a pseudo time-stepping and a backward Euler scheme with a LU-Symmetric Gauss-Seidel method for the linear solver. The flow domain Ω_n is bounded by the airfoil and a circumferential boundary of radius $R/c = 100$. The flow region is discretized with a hybrid mesh using 58537 grid points. An adiabatic no-slip condition is enforced at the wall, whereas either a farfield (in Ω_n) or Dirichlet ($\Omega_{m,i}$) condition at the outer boundary. Figure [5.11] shows the mesh, full and close to the profile. Pressure and eddy viscosity of the steady solution, in figure (5.12) in agreement with the results reported in [Iorio 2014].

Figure 5.11: NACA0012 Mesh

The Jacobian matrix is computed following a discrete approach, as described in [Iorio 2014]. The corresponding eigenspectrum is shown in figure [5.13]. The eigenvector associated with the buffet phenomena is the only unstable mode for this configuration. Its right and left eigenvectors are shown in figure [5.13], where the transonic shock wave, typical of the buffet phenomenon, is visible.

Figure 5.12: Base flow NACA0012: pressure (left) and eddy viscosity (right)

The region of maximum sensitivity, defined again as the modulus of $(\mathbf{q}_n^*)^H \mathbf{q}_n$ at each node, is shown on left figure in [5.14]. The values are normalized with the modulus of $\|(\mathbf{q}_n^*)^H \mathbf{q}_n\|$ in the full domain. This region is located above the airfoil, extending from the bottom of the shock. The sensitivity reaches its maximum at $y/c \simeq 0.45$, then decreases progressively, taking negligible values beyond $y/c \simeq 10$.

In order to investigate the domain reduction efficiency, domain Ω_n is reduced into three smaller domains, of radii 40 (Ω_{m40}), 10 (Ω_{m10}) and 2 (Ω_{m2}) chords.

The comparison of the eigenvalue associated to buffet for these meshes is shown in Table [5.2]. As expected, (see Figure [5.14]), domain reduction does not affect the results as long as the reduced domain encompasses completely the sensitivity region. In this particular case, slight deviations appear for Ω_{m10} , and become more important for Ω_{m2} . The explanation can be found in Figure [5.16-left], where $E_m \mathbf{q}_m$ is depicted. Whenever the intersection of this region with the sensitivity zone produces non-negligible values, the domain reduction technique fails to provide accurate results for Ω_n , in accordance with the theoretical estimation presented in Section 3.

Lamentably, the information from Ω_n is -in general- not available. However, the shift in the eigenvalue is strictly related to the intersection between the sensitivity region and $E_m \mathbf{q}_m$. For that reason an estimation ($\mathbf{q}_m^\dagger$) of the unknown reference eigenmode $\mathbf{q}_n$ can be used by computing the reduced EVP in a slightly larger domain $\Omega_m^\dagger$ including that boundary region. Figure [5.15] shows the region of maximum sensitivity for the reduced domains of radius 3 and 2. The differences between these two regions could be used to estimate the error, even if the sensitivity region for the whole domain Ω_n is unknown. The eigenvectors $\mathbf{q}_m^\dagger$ will be used in (5.9) (or 5.10) to estimate $\Delta\omega_n$ (and also in 5.12). The procedure is sketched in figure [5.16-right] for Ω_{m2} , using Ω_{m3} as $\Omega_m^\dagger$. Following this procedure, the error estimation for radii 40, 10 and 2 is shown in Table [5.2].

Furthermore, figure [5.17] confirms that even for Ω_{m2} the mode is detected correctly with a maximum error below 10^{-4} with respect to $\mathbf{q}_n$.

In conclusion, results in table [5.2] confirm the potential of domain reduction method to provide/recover accurate stability results.

Figure 5.13: Spectrum (top), normalized right (bottom-left), and normalized left (bottom-right) $\hat{u}$ component of the perturbation amplitude for the Buffet case

Figure 5.14: Region of maximum sensitivity for the reference domain (right)

Figure 5.15: Region of maximum sensitivity in the reduced domain of radius 3 (left) and in the reduced domain of radius 2 (right)

Figure 5.16: Left: vector $E_m \mathbf{q}_m$ for domain reduction of radius 2. Right: Domain extension for the error estimation approach

Figure 5.17: Normalized $\hat{u}$ component of the perturbation amplitude for the buffet transonic phenomenon, comparison between reference mesh and reduced domain of radius 2

Buffet eigenvalue				
Mesh size (in chords)	Error (computed and equation (5.9))		Estimated	
	Frequency	Damping rate	Frequency	Damping rate
Reference mesh	0.23649	6.15676(−2)	—	—
40	5.15134(−7)	1.19875(−7)	1.62214(−7)	1.84801(−7)
10	1.06141(−3)	4.69014(−4)	6.76330(−4)	3.88361(−4)
2	−1.58915(−2)	1.17123(−2)	−7.03020(−3)	1.09145(−2)

Table 5.2: Eigenvalue associated to the buffet phenomenon, comparison between stability analysis on reduced domains and error estimation

5.4 Conclusions

In this chapter, a detailed mathematical analysis of the domain reduction methodology for stability analysis was conducted. The method is regularly used whenever the region of study Ω_n and/or the mesh density, is too large so the Jacobian of the linear problem and its associate EVP is computational not affordable. domain reduction is also applicable if a refined mesh in Ω_n , able to capture the detailed structures, is of interest. In any case, a reduced domain Ω_m is considered and the associated eigenvalue problem is solved. The accuracy of this approach and the sources of errors are the topic of this work.

The mathematical description shows that the problem can be seen as a geometric projection. The original Jacobian (in Ω_n) comprises the reduced Jacobian (in Ω_m) plus an error matrix whose main non-zero elements correspond to $\partial\Omega_m \cap \Omega_n$. Analytic expressions for the differences between “large” and “reduced” eigenvalues and the associated left and right eigenvectors are presented and validated. The analysis shows that the sensitivity region, i.e. the region where the scalar product between the left and right eigenvectors of the analyzed eigenvalues is non-negligible, is precisely the area inducing the largest differences between the “true” and “estimated” eigenvalues.

An estimation of that difference, without computing the solution of the large Jacobian, is proposed. An extended domain, $\Omega_m^\dagger$, including the area of the intersection between the sensitivity region and the boundary of the reduced domain Ω_m , produces a cheap and accurate estimation of the differences when the reduced domain methodology is used.

The obtained results and formulas were validated in the extensively studied cavity problem using a high-order spectral multidomain solver and a continuous approach for the EVP. As a final test case, the methodology was applied to analyze buffet onset on a NACA profile where several domains of progressively smaller radius were considered. In both cases, the results confirm that, whenever the sensitivity region takes negligible values at the boundary of the reduced domain, the shift of the “reduced” eigenvalue is also negligible. Moreover, using an “extended” domain $\Omega_m^\dagger$ enables to estimate the differences between the large and reduced eigenvalues at an affordable computational cost. This provides valuable information about the limits of domain reduction applicability.

Conclusions and Outlook

6.1 Conclusions

The development of this work shares academic and industrial experiences, under the common objective to progress in the applicability of the stability analysis for industrial applications. The objectives of these work have been twofold: Firstly, to overcome the current limitations of stability analysis in complex industrial problems, a detailed comparison and review of existing methodologies to solve large eigenvalues problems has been performed. Secondly, the most promising and mature methodologies have been checked in a high-lift configuration, finally a new methodology, domain reduction, has been developed to overcome and complement the existing approached.

In chapter [3], the numerical methodologies for stability analysis has been investigated in order to evaluate a computational and more efficient approach, with respect to the classic LU decomposition with Krylov sub-projection and Shift–Inverse technique. A detailed analysis on the Lid Driven Cavity and Flat Plate academic test cases have identified suitable alternative approaches. All of them obtain similar results with comparable performances, thus requiring approximately half of the memory consumption needed to the classic approach, but doubling the computational time. A strong dependency between the numerical methods performances and the properties of the EVP (Eigenvalue Problem) matrices is highlighted. As a final result, under certain circumstances more challenging industrial flow configurations with ill-conditioning associated matrices can be studied with these new methods.

In chapter [4], the methodology was successfully applied to the two-dimensional High-Lift configuration. This challenging industrial test case, which involves turbulence transitions and complex geometries, was tested for a wide range of angle of attack. The results confirm the reliability of the stability analysis approach to identify the onset of unstable configurations and to provided relevant information regarding their flow features. Finally, the strong numerical background developed during the academic experience, joined to the industrial tools knowledge acquired at AIRBUS, brought to the definition of an alternative approach based on geometrical projection: Domain reduction methodology. After a detailed mathematical justification, domain reduction strategy is validated on the open cavity flow and finally applied to detect the onset of the buffet phenomenon on NACA0012 profile in transonic condition. The domain reduction strategy was able to correctly recover the unstable flow features of the transonic condition for a domain reduction up to 90 per cent of the original domain size. Moreover, a mathematical formulation able to perform an a-priori error estimation, associated to the detection of the unstable eigenvalues positions and eigenvectors shapes, was also provided. The results represent the most relevant contribution of this work to the progress on the stability analysis application for industrial flow configurations, and it is currently under review in the AIAA journal.

6.2 Outlook

According with the results obtained in this thesis, the stability analysis applications in industrial framework are still subjected to relevant limitations. Especially regarding possible alternative numerical approaches, the computational resources, available nowadays, make many complex industrial flow configurations still not affordable for the stability analysis. Further investigations on numerical methods, possibly involving more complex precondition techniques which take into account the matrix morphology, can be crucial to overcome these limitations.

Nevertheless, the domain reduction strategies represents, in this scenario, one of the most promising option to overcome the memory consumption, but further studies in this direction are required. In more detail, the applicability of this young methodology to more complex flow configurations, which involve not confined phenomenon (as the transonic buffet) or complex three-dimensional geometries, should be investigated in future works.

Finally, a combined strategy can be defined. The domain reduction strategy can be used as first step of the stability analysis process, to identify the relevant regions. Then, the information arising from the numerical methods comparison can be used to select the best numerical approach, in terms of computational costs, to perform a cheap stability analysis on the reduced domain. The consequent overall gain in terms of memory consumption, with respect to the classic more demanding approach, represents a key point to go beyond the limitations that the stability analysis is currently facing in its industrial applications.

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