A THEORETICAL APPROACH TO IMPULSIVE MOTION OF VISCOUS LIQUID BRIDGES

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Abstract—This paper deals with the dynamics of isothermal, axisymmetric, viscous liquid columns held by capillary forces between two circular, concentric, solid disks. The transient response of the bridge to an excitation consisting of a small change in the value of the acceleration acting along its axis has been solved by using a linearized one-dimensional Cosserat model which includes viscosity effects. The main hypothesis of this model is that the axial velocity is considered constant in each section of the liquid bridge. The analysis has been performed by using the Laplace transform.

INTRODUCTION

The liquid bridge configuration considered here consists, as sketched in Fig. 1, of an isothermal, axisymmetric mass of liquid held by surface tension forces between two parallel, coaxial, circular solid disks. Such a fluid configuration can be uniquely identified by the following set of dimensionless parameters: the ratio of the radius of the smaller disk, \( R_1 \), to the radius of the larger one \( R_2 \); the slenderness \( \alpha = L/2R_0 \), where \( R_0 = (R_1 + R_2)/2 \) and \( L \) is the distance between both disks; the dimensionless volume of liquid, \( V = V/\pi R_0^2 \); the Bond number, \( B = \rho \pi R_0^4 \sigma \), and the capillary number, \( C = \sqrt{\rho \pi R_0^4 \sigma} \) (the square root of the liquid density, \( \rho \), and the surface tension, \( \sigma \)); the axial velocity \( A \) (assumed to be uniform in each plane parallel to the disks) whereas \( W(z, t) \) represents the cross-sectional area at \( z \), \( t \) and \( S \) the interface (surface tension) are assumed uniform and constant, and \( \rho \) the liquid density, \( \nu \) the kinematic viscosity.

In this paper, a linear theoretical model of the dynamic response of viscous axisymmetric liquid bridges due to small changes in the value of the axial acceleration acting on the liquid is presented. Most of the published studies related to the liquid bridge response to forcing perturbations are concerned with harmonic perturbations [1-5]. Non-harmonic perturbations have been considered in [6], where a one-dimensional inviscid slice model was used to analyze the liquid injection or removal in the liquid bridge and in [7], where the analysis is restricted to cylindrical liquid bridges \( (V = 2\pi \alpha A) \) between equal disks \( (K = 1) \) and in gravitational conditions \( (\alpha = 0) \) (in this case the interface at rest is a cylinder).

To solve the problem Cosserat's one-dimensional model for continuum media has been used. This model has been used for capillary jet problems and for liquid bridge problems, and has proved to give satisfactory results, when compared with 3-D models results, provided that the slenderness of the liquid bridge is large enough \( (\alpha > 1) \). In the following, unless otherwise stated, all physical quantities are made dimensionless using the characteristic length \( R_0 \) and the characteristic time \( (\rho R_0/\sigma)^{1/2} \).

MATHEMATICAL MODEL

In carrying out the analysis the following assumptions are introduced: first the properties of both the liquid (density and viscosity) and the interface (surface tension) are assumed uniform and constant, and second the effects of the gas surrounding the liquid bridge are negligible. In addition, since only axisymmetric configurations are considered, the solution of the problem is assumed to be independent of the azimuthal coordinate. Under such assumptions the set of non-dimensional differential equations and boundary conditions for the axisymmetric, non-rotating viscous flow, according to the Cosserat model [7, 8], are

\[ \frac{\partial S}{\partial t} + Q_1 = 0 \]

\[ \frac{\partial \rho S}{\partial t} - \frac{1}{8} \left\{ \frac{\partial^2}{\partial z^2} \left[ \rho \frac{3}{2} \left( Q \frac{S}{S_0} \right)^2 \right] \right\} = -SP_1 + \frac{1}{8} \left\{ S \left( \frac{Q}{S_0} \right)_n \right\}_z + 3C \left( S \frac{Q}{S_0} \right)_n, \]

where

\[ \frac{\partial \rho}{\partial t} = \frac{Q_1}{S} \left( S \frac{Q}{S_0} \right)_n, \]

\[ P = 4(2S + S^2_1 - SS_1)(4S + S) + B(t)z. \]

In these expressions \( S = F^2 \) and \( Q = F^3 W \) represent the cross-sectional area at \( z, t \) and the axial momentum of a slice, respectively. \( F(z, t) \) is the dimensionless position of the liquid-gas interface and \( W(z, t) \) the axial velocity (assumed to be uniform in each plane parallel to the disks) whereas \( P(z, t) \) accounts for both hydrostatic pressure and capillary pressure jump. The subscripts \( z \) and \( t \) indicate derivatives with respect to time and axial coordinate, respectively.
Fig. 1. Geometry and coordinate system for the liquid bridge problem and sketch of the spatial grid used in computations.

The formulation must be completed with suitable boundary conditions, which state that the interface must remain anchored to the disk edges and that the axial velocity at each one of the disks must be zero:

\[ \pi \int_{-A}^{A} S_0(z) \, dz = V \]  

The initial conditions are assumed to be:

\[ S(z, 0) = S_0(z), \quad Q(z, 0) = 0 \]  

Note that the zero-order problem corresponds to the static problem which defines the interface equilibrium shape at \( t = 0 \); therefore, since \( S_0(z) \) also appears in the \( \epsilon \)-order problem, the zero-order problem must be solved before solving the liquid bridge dynamics.

The introduction of these expressions in (1)-(5), neglecting second order terms, gives the following set of problems:

**Zero-order problem**

\[ 4(2S_0 + (S_0)^2 - S_0 S_a) \times [(4S_0 + (S_0)^2)^{1/2} + B_v z - P_0 = 0 \]  

\[ S_0(\pm A) = [1 \pm (1 - K)/(1 + K)]^2 \]  

\[ \pi \int_{-A}^{A} S_0(z) \, dz = V \]  

\[ s(\pm A, t) = q(\pm A, t) = 0 \]

\[ s(z, 0) = q(z, 0) = 0 \]

\[ \int_{-A}^{A} s(z, t) \, dz = 0. \]

**\( \epsilon \)-order problem**

\[ s_\epsilon + q_\epsilon = 0 \]  

\[ q_\epsilon - \frac{1}{4}(S_0 q_{zz} + S_a q_z) = -S_0 p_\epsilon \]  

\[ \rho = 12[S_0 + (S_0)^2]^{1/2}(2S_0 + S_a) \]  

\[ \times (2S_0 + S_a z + 4S_0 + (S_0)^2)^{1/2} \]  

\[ \times [(4 + S_0)z + S_0 z_z + S_0 z_z + S_0 z_z] + B(t) \sigma \]  

\[ s(\pm A, t) = q(\pm A, t) = 0 \]  

\[ s(z, 0) = q(z, 0) = 0 \]  

\[ \int_{-A}^{A} s(z, t) \, dz = 0. \]

Note that the zero-order problem corresponds to the static problem which defines the interface equilibrium shape at \( t = 0 \); therefore, since \( S_0(z) \) also appears in the \( \epsilon \)-order problem, the zero-order problem must be solved before solving the liquid bridge dynamics.

The introduction of (13) into (12) allows the elimination of \( p(z, t) \). To eliminate \( s(z, t) \) from the formulation, equations (11) and (12) should be used simultaneously. Differentiation of (12) with respect to time and of continuity equation (11) with respect to \( z \) as many times as necessary allows to eliminate the variable \( s(z, t) \) and its derivatives appearing in the momentum equation, the final expression being of the form

\[ D_1 q_{zzz} + D_2 q_{zz} + D_3 q_{z} + D_4 q = 0 \]

which must be solved with the boundary and initial conditions

\[ q(\pm A, t) = q_\epsilon(\pm A, t) = 0 \]

\[ q(z, 0) = q_\epsilon(z, 0) = 0 \]

where the boundary conditions \( q_\epsilon(\pm A, t) = 0 \) comes from (11) and \( s(\pm A, t) = 0 \) whereas initial conditions \( q_\epsilon(z, 0) = 0 \) results from (12) and (15). Note that in equation (17) all functions \( D_0 \) depend only on \( z \) but not on the time \( t \) [the expressions of the different \( D_0 \) (2) can be obtained upon request to the authors].
Let \( Q(z,h) \), \( S(z,h) \) and \( B(h) \) be the Laplace transform of \( q(z,t) \), \( s(z,t) \) and \( \mathcal{A}(t) \) respectively. Then, in view of initial conditions, the subsidiary equation of (17) is

\[
D_{j}Q_{m+3} + D_{j}Q_{m+1} + 2D_{j}Q_{m} + 3D_{m}Q_{j} = S_{j}hB(h)
\]

(20)

\[
Q(\pm A, h) = Q_{j}(\pm A, h) = 0
\]

(21)

where the functions \( D_{j} \) are

\[
D_{j} = D_{2j} + hD_{4j}, \quad D_{j} = D_{2j}
\]

(23)

The subsidiary continuity equation is

\[
hS(z,h) + Q_{j}(z,h) = 0.
\]

(22)

Equation (18) has to be solved numerically [the functions \( D_{j} \) depend on the equilibrium interface shape \( S_{j}(z) \) and its derivatives and even \( S_{j}(z) \), except in a few cases in which there is known analytical expressions of \( S_{j}(z) \), has to be calculated numerically]. To solve the problem an implicit finite-difference method, similar to the one described in [1] and [4], has been used, with a five-point scheme for the evaluation of the spatial derivatives.

\[
(Q_{m}) = \begin{pmatrix}
(Q_{-2}, -8Q_{-1} + 8Q_{1}) \\
(Q_{1}, 12A) \\
(Q_{m}) = \begin{pmatrix}
(-Q_{-2} + 16Q_{-1} - 30Q_{0} + 16Q_{1} - 4Q_{2})/12A^{2}
\end{pmatrix} \\
(Q_{-1}, 2Q_{0} - 2Q_{1})/2A^{2}
\end{pmatrix}
\]

(23)

\[
(Q_{m+1}) = \begin{pmatrix}
(Q_{-2}, 6Q_{-1} - 4Q_{0} + 6Q_{1} - 4Q_{2})/A^{2}
\end{pmatrix}
\]

with 0 < \( j < m \) and

\[
D_{2j}^{+} = \begin{pmatrix}
(D_{2j}, \pm 6D_{3j} - D_{1j})/3 \pm D_{1j}A^{2}
\end{pmatrix}
\]

(24)

\[
D_{2j}^{-} = \begin{pmatrix}
(-12D_{3j} + 6D_{2j} - D_{1j}A^{2} + 4D_{1j}A^{3})
\end{pmatrix}
\]

(25)

Boundary conditions are

\[
Q_{-A}(h) = 0 \rightarrow Q_{0} = 0
\]

\[
Q_{m}(h) = 0 \rightarrow Q_{m} = 0
\]

(26)

\[
Q_{-A}(h) - 8Q_{-1} + 8Q_{1} - Q_{0} = 0
\]

\[
Q_{m}(h) - 8Q_{m-1} + 8Q_{m+1} = 0
\]

In conclusion, there are \( m + 1 \) unknowns: \( m - 1 \) unknowns come from the velocity at nodes inside the liquid bridge (note that according to (26) is \( Q_{j} = Q_{m-j} = 0 \)) plus the velocities at the four nodes outside (two at each side, Fig. 1) needed to fulfill boundary conditions, and there are \( m + 3 \) equations. \( m + 1 \) results from (24) plus the two last from (26). This system of \( m + 3 \) equations with \( m + 3 \) unknowns can be written as

\[
[D]Q = [T]
\]

(27)

so that the problem solution would be

\[
Q = [D]^{-1}[T].
\]

(28)

Once \( Q \) is known at \( j = -2, \ldots, m + 2 \), equation (22) allows to calculate \( S_{j} \) at \( j = 0, \ldots, m \) and then, by using the inversion theorem [9] \( q(z,t) \) and \( s(z,t) \) can be obtained.

Taking into account the kind of perturbation here considered (a sudden displacement of the whole liquid bridge along its axis) it is clear that equation (28) has no poles at \( h = 0 \), so that the inverse transform of \( Q(z,h) \) would be formally

\[
q(z,t) = \sum_{n, \nu} c_{n, \nu}(z,h) \exp(h_{n,t})
\]

(29)

\( h_{n} \) being the poles of \( Q(z,h) \) or, in other words, the roots of \( \det(D) = 0 \).

Thus, it will be enough to calculate these poles \( h_{n} \) if we are only interested in the damping process experienced by the liquid bridge after it has been perturbed. According to analytical results presented in [7], if the liquid bridge configuration is stable these roots are either of the form \( h_{n} = \gamma_{n} + \omega_{0} \), if the viscosity parameter \( C \) is smaller than a critical value \( C_{*} \) (where \( C_{*} \) depends on \( n \) and on the parameters that define the liquid bridge configuration) or of the form \( h_{n} = \gamma_{n} + \omega_{2n} \), where both \( \gamma_{n} \) and \( \omega_{2n} \) are negative real numbers if \( C > C_{*} \).

Before pursuing further it would be convenient to point out some characteristics of the liquid bridge response, already stated in [7] for cylindrical liquid bridges between equal disks (obviously the behaviour is qualitatively the same no matter the values of \( V, K \) and \( B \) are, provided the liquid bridge configuration is stable).

For a given liquid bridge configuration, that is, once \( A, k, B \) and \( V \) are fixed, all the roots are imaginary (\( \gamma_{n} = 0 \)) if \( C = 0 \), and consequently the liquid bridge response will be oscillatory, without damping, as one would expect from an inviscid motion, and all the oscillation modes will be present in the liquid bridge response. As \( C \) increases, the value of \( \gamma_{n} \) becomes more and more negative, whereas the absolute value of \( \omega_{2n} \) decreases. That means that all oscillation modes are damped, their oscillation frequencies being smaller as \( C \) grows. For each oscillation mode there is a critical value \( C = C^{*} \) beyond which the associated movement is only damped, without oscillation (\( \omega_{0} = 0 \)).

Concerning the importance of the different oscillation modes in the liquid bridge dynamics it must be pointed out that the only significative oscillation mode is the first one by two reasons. The first one concerns the amplitude of the different oscillation modes, much larger in the first oscillation mode than...
in the following ones and the second is that, in the
case of damped oscillations, the absolute value of $\gamma_i$ increases as the index of the oscillation mode, $n$, grows; therefore, all oscillation modes are damped much quicker than the first one (oscillation modes different from the first one are only important for short times when all of them appear to fulfill the initial conditions [7]).

The dependence on $K$, $B$, $V$ and $C$ of the roots corresponding to the first oscillation mode, which is the only significant for $C \neq 0$, of a liquid bridge with slenderness $A = 2$ is shown in Figs 2–4.

The influence of the liquid volume is shown in Fig. 2 ($K = 1$, $B = 0$) whereas the influence of either the ratio of the diameters of the disks or the Bond number (for liquid bridges having cylindrical volume) are shown in Figs 3 and 4, respectively. For each of these plots the same behavior can be observed; for a given value of $C$ the resonance frequency, $\omega_1$, decreases and the absolute value of damping factor, $\gamma_1$, increases as the liquid bridge configuration approaches the corresponding stability limit; close enough of the stability limit there is a region in which $\omega_1 = 0$ and the absolute value of $\gamma_1$ decreases. The damping factor becomes zero at the stability limit and it is greater than zero for unstable liquid bridge configurations [7, 10].

The dependence on the dimensionless parameter of viscosity, $C$, of the real and imaginary parts of the root corresponding the first oscillation mode, $\gamma_i$ and $\omega_i$, respectively, of a liquid bridge with $A = 2$, $K = 1$, $B$ and different volumes is shown in Fig. 5. The main characteristic to be pointed out from this plot is that the damping factor $\gamma_i$ varies almost linearly with $C$ no matter the values of the remaining liquid bridge parameters are. This linear dependence of $\gamma_i$ on $C$ is of great help when theoretical results are compared with experimental ones because it allows the comparison of both types of results independently of the value of surface tension, which is difficult to measure.

In order to check theoretical predictions, the transient response of liquid bridges subjected to a small perturbation in the axial acceleration has been measured. Experiments have been carried out by working with very small liquid bridges...
Fig. 5. Variation with the dimensionless viscosity parameter, $C$, of the real and imaginary part of the root of $D(A,h) = 0$ corresponding to the first oscillation mode, $h = \gamma_1 \pm i \omega_1$. Numbers on lines indicate the value of $K/K_c$. These results correspond to liquid bridges with $A = 2, \varepsilon = 1$ and $B = 0$.

$(R_0 = 5 \times 10^{-4}$ m) and using distilled water as working fluid. The liquid bridge, initially at rest, is suddenly displaced vertically a distance of the order of $R_0$ and the subsequent evolution recorded, so that the damping time constant of the evolution until the final equilibrium state is measured. A number of experiments have been performed, varying either the volume of liquid or the distance between disks, experimental results have been published elsewhere [11] and they show the same trends as theoretical predictions.

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