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Studies on Chaotic Intermittency

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Studies on Chaotic Intermittency

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POLITÉCNICA

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Calificación

EL PRESIDENTE

LOS VOCALES

EL SECRETARIO

To my wife María Inés,

my children Silvina and Daniel

and my mother Delia

La duda es uno de los nombres de la inteligencia.

Jorge Luis Borges (1899 - 1986)

La ciencia, la técnica y la investigación son la base de la salud, bienestar, riqueza, poder e independencia de los pueblos modernos. Hay quienes creen que la investigación científica es un lujo o un entretenimiento interesante pero dispensable. Grave error; es una necesidad urgente, inmediata e ineludible para adelantar. La disyuntiva es clara, o bien se cultiva la ciencia, la técnica y la investigación y el país es próspero, poderoso y adelanta; o bien no se la práctica debidamente y el país se estanca y retrocede, vive en la pobreza y la mediocridad. Los países ricos lo son porque dedican dinero al desarrollo científico tecnológico. Y los países pobres lo siguen siendo si no lo hacen. La ciencia no es cara, cara es la ignorancia.

Bernardo Houssay (1887 - 1971)

Proceder con honestidad en aras de la dignidad del hombre es el compromiso más trascendente en nuestro corto paso por este mundo.

René Favaloro (1923 - 2000)

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Abstract

Intermittency has been observed in several physical, chemical, economic and medical systems. The accurate description of intermittency helps to improve the knowledge about these phenomena. On the other hand, the proper characterization of intermittency has great importance for systems whose exact governing equations are totally or partially unknown.

Classically, intermittency has been classified in type I, II and III intermittencies according to the Floquet multipliers of the continuous-system periodic solution or to the fixed point eigenvalues in the local Poincar map. For these types of intermittency, when a control parameter exceeds a threshold value, a fixed point of the local Poincar map (or a periodic solution of the system) becomes unstable or even vanishes.

Type I intermittency appears when a multiplier leaves the unit circle across $+1$, then the periodic solution (orbit) loses its stability by a cyclic-fold bifurcation. Type II intermittency departures from a subcritical Hopf bifurcation or a Neimark-Sacker bifurcation, then two complex-conjugate multipliers of the system move away from the unit circle. Type III intermittency is produced when a multiplier leaves the unit circle by -1 . In this case a subcritical period-doubling bifurcation occurs.

Chaotic intermittency may be studied using Poincar maps. Then, two main features characterize chaotic intermittency: 1) a specific local map and 2) a reinjection mechanism. The local map determines the intermittency type, and the reinjection mechanism maps back the system into the local regular or laminar zone from the chaotic one. This mechanism is described by the so-called reinjection probability density function (RPD), which is determined by the chaotic dynamics of the system itself. The RPD gives the probability that trajectories are reinjected into the laminar zone –close to the unstable fixed point–, and together with the local map allow to describe the intermittency process.

The accurate evaluation of the RPD function is extremely important to correctly describe the chaotic intermittency phenomenon. Still, the experimental or numerical

evaluation of the RPD function is not a simple task due to the huge amount of data needed. Diverse approaches were used to obtain the intermittent RPD function. The main results on classical chaotic intermittency was based on uniform reinjection on the laminar region. Other implemented approaches build the RPD using peculiar features of the nonlinear processes.

A more general methodology to obtain the RPD, called here M function methodology, which includes the uniform reinjection as a particular case, was introduced in the last years. This methodology establishes a generalized power law reinjection and it has shown to be very accurate for a wide class of one-dimensional maps exhibiting type I, II and III intermittencies.

In the thesis, we extend the M function methodology and develop new schemes to obtain the intermittency characteristic functions and to describe the following topics: 1 - The theoretical background of the M function methodology. 2 - Type III intermittency with noise and with lower reinjection boundary. 3 - Type V intermittency. 4 - Intermittency with nonlinear M functions (application of the Perron-Frobenius operator).

Theoretical background of the M function methodology: we have developed analytical methods to estimate the RPD function. We have considered the null or infinite derivatives of the map at the pre-reinjection points. Once the theoretical RPD function was established, we have estimated an analytic expression for the length of the laminar phase of intermittent behaviors. We have carried out a generalization of the classical intermittency theory.

Intermittency with noise and with lower reinjection boundary: We have extended the M function methodology to consider different values of the noise intensity and the lower boundary of reinjection (LBR) acting at the same time for type III intermittency. We have obtained accurate analytic expressions for the reinjection probability density which depends on the non-linear behavior of the system or map, the LBR value and the noise intensity.

Type V intermittency: we have used the M function methodology to obtain the analytical expressions for the reinjection probability density and the probability density of laminar lengths. We have found that these functions can be nondifferentiable functions when the local map has a nondifferentiable point inside the laminar interval.

Intermittency with nonlinear M functions. We have developed a new scheme to calculate the RPD, which uses the Perron-Frobenius operator. The new scheme has shown very good accuracy with the numerical results when M is a nonlinear function. We have applied the new scheme to type II and V intermittencies.

Resumen

La intermitencia caótica ha sido observada en fenómenos físicos, químicos, económicos, en neurociencia, medicina, etc. Una descripción precisa de fenómeno de intermitencia ayudará a mejorar el conocimiento de estos sistemas. Por otro lado, la caracterización adecuada de la intermitencia posee gran relevancia en sistemas cuyas ecuaciones son total o parcialmente desconocidas.

Tradicionalmente, el fenómeno de intermitencia caótica ha sido clasificado en tipo I, II y III según los multiplicadores de Floquet de la solución periódica (órbita) del sistema dinámico o los valores propios del punto fijo del mapa local de Poincaré. Para este tipo de intermitencias, cuando un determinado parámetro de control excede el valor crítico, una solución periódica del sistema dinámico o un punto fijo del mapa de Poincaré pierden su estabilidad o incluso pueden desaparecer.

La intermitencia tipo I aparece cuando un multiplicador escapa del círculo unitario a través de $+1$, luego la solución periódica pierde su estabilidad mediante una bifurcación tipo cyclic-fold. La intermitencia tipo II aparece mediante una bifurcación subcrítica de Hopf o una bifurcación de Neimark-Sacker, cuando dos multiplicadores complejos conjugados de la órbita del sistema salen del círculo unitario. Intermitencia de tipo III se produce cuando un multiplicador deja el círculo unitario por -1 . En este caso, se produce una bifurcación subcrítica o de doblamiento del período.

La intermitencia caótica se puede estudiar usando mapas de Poincaré. Por lo tanto, quedará determinada por: 1) un mapa local específico y 2) un mecanismo de reinyección. El mapa local determina el tipo de intermitencia, y el mecanismo de reinyección permite que las trayectorias regresen desde el comportamiento caótico hacia la zona local o laminar. Este mecanismo se describe mediante la llamada función densidad de probabilidad de reinyección (RPD por sus siglas en inglés), que está determinada fundamentalmente por la dinámica caótica del sistema. La RPD establece la probabilidad de que las trayectorias se reinyecten en la zona laminar (cerca del punto fijo inestable).

La función RPD junto con el mapa local permiten describir el proceso de la intermitencia.

La evaluación precisa de la función RPD es muy importante para describir correctamente el fenómeno de intermitencia caótica. Aún así, la determinación experimental o numérica de la función RPD no es una tarea simple debido a la gran cantidad de datos necesarios. Diversos enfoques han sido utilizados para obtener la función RPD. Los principales resultados en la teoría clásica de intermitencia caótica se obtuvieron considerando reinyección uniforme en la zona laminar. Otros enfoques implementados construyen la RPD usando características particulares de los procesos no lineales en estudio, pero que no pueden ser generalizados a otros mapas.

Una metodología más general para obtener la RPD, llamada aquí la metodología de la función M , que incluye la reinyección uniforme como un caso particular, se ha introducido desde el 2010. Esta metodología establece una ley de potencia generalizada para describir la RPD y ha demostrado ser muy precisa en una amplia clase de mapas unidimensionales que exhiben intermitencias tipo I, II y III.

En la tesis, ampliamos la metodología de la función M y desarrollamos nuevos esquemas con la finalidad de obtener las funciones y variables características de la intermitencia y describir los siguientes temas: 1 - Fundamento teórico de la metodología de la función M . 2 - Intermitencia tipo III con ruido y con límite inferior de reinyección. 3 - Intermitencia tipo V. 4 - Intermitencia con funciones M no lineales (aplicación del operador Perron-Frobenius).

Fundamentos teóricos de la metodología de la función M : hemos desarrollado métodos analíticos para estimar la función RPD. Hemos considerado las derivadas nulas o infinitas del mapa en los puntos previos a la reinyección. Una vez que se estableció la función RPD teórica, hemos estimado una expresión analítica para la duración de la fase laminar. El estudio incluye una generalización de la teoría clásica de la intermitencia.

Intermitencia con ruido y con límite inferior de reinyección: hemos ampliado la metodología de la función M para considerar diferentes valores de la intensidad del ruido y del límite inferior de reinyección (LBR por sus siglas en inglés) actuando simultáneamente en intermitencia tipo III. Hemos obtenido expresiones analíticas para la RPD que depende del comportamiento no lineal del mapa, el valor del LBR y la intensidad del ruido.

Tipo intermitencia V: hemos utilizado la metodología de función M para obtener expresiones analíticas para la densidad de probabilidad de reinyección y la densidad de probabilidad de las longitudes laminares. Hemos encontrado que éstas pueden ser funciones no diferenciables cuando el mapa local tiene un punto no diferenciable dentro del intervalo laminar.

Intermitencia con funciones M no lineales: hemos desarrollado un nuevo esquema para calcular la RPD, el mismo utiliza el operador de Perron-Frobenius. El nuevo esquema ha demostrado muy buena precisión con los resultados numéricos cuando M es una función no lineal. Hemos aplicado el nuevo esquema a las intermitencias tipo II y V.

Nomenclature

- a real number
- b normalization constant and real number
- c upper limit of the laminar interval
- D compact manifold
- $F(x), G(x)$ one-dimensional maps
- $G(x, \sigma)$ probability density of the noise term
- $M(x)$ M function
- h integer number
- i integer number
- j integer number
- k real number
- K real number
- l laminar length
- $\bar{l}$ average laminar length
- m slope of the function $M(x)$
- n integer number
- n integer number or number of reinjected points
- $P_\circ$ the Perron-Frobenius operator
- q real number

$SF(x)$ the Schwartzian derivative

t time

$\hat{x}$ lower boundary of reinjection

$\tilde{x}$ lower boundary of return

$\mathbf{f}$ vector field for continuous systems

$\mathbf{F}$ vector field for discrete systems

$\mathbf{H}$ vector containing all control parameters of the system

$\mathbf{n}$ unit normal vector to the hypersurface

$\mathbf{R}$ real numbers set

$\mathbf{x}$ state vector

α exponent of the power law modeling RPD

β characteristic exponent

Δ real number

ε control parameter

ξ_n uniform distributed noise

γ real number and closed orbit

σ noise intensity

$\rho(x)$ density

$\mu(x)$ probability measure

$\phi(x)$ noiseless reinjection probability density function

$\Phi(x)$ noisy reinjection probability density function

$\psi(l)$ probability density of the laminar lengths or probability density of the length of the laminar phase

$\Theta(x)$ Heaviside step function

Σ hypersurface

Acronyms

DNLS	Derivative NonLinear Schrodinger equation
FPE	backward FokkerPlank Equation
HR	Hindmarsh-Rose model
LBR	Lower Boundary of ReInjection: Limit value for the reInjection form the chaotic region into the laminar one.
NDP	Non Differentiable Points
NRPD	Noisy ReInjection Probability Density function: The RPD modified by the noise effect
RHS	Right Hand Side
RGT	Renormalization Group Theory
RPD	ReInjection Probability Density function (form the chaotic region into the laminar one)

Chapter 1

INTRODUCTION AND OBJECTIVES

Abstract In this chapter a short introduction about chaotic intermittency is presented. A description of the four main objectives of this thesis is also carried out.

1.1 Basic concepts

The word intermittency was employed by Batchelor and Townsend in 1949 to describe some observational data of the fluctuating velocity in turbulent flows [1]. Since then, intermittency has been observed in several physical phenomena.

There are, at least, three routes in which dynamical systems or maps can evolve from regular behaviors to chaos: period-doubling scenario, quasiperiodic route and intermittency. Chaotic intermittency is a route to chaos where the dynamical system solutions have transitions between regular or laminar phases and chaotic bursts or non regular phases. The laminar phases are regions of pseudo-equilibrium and/or pseudo-periodic solutions, while the bursts ones are regions where the evolution is chaotic. Traditionally, intermittency was classified into three different types called I, II and III [2, 3] according to the Floquet multipliers of the continuous-system periodic solution or to the fixed point eigenvalues in the local Poincaré map [4, 5, 6, 7]. More recent studies have extended the classification to other types of intermittencies such as type V, X, on-off, eyelet, ring and in-out [8, 9, 10, 11, 12, 13, 14].

In type I, II and III intermittency, when a control parameter exceeds a threshold value, a fixed point of the local Poincaré map becomes unstable or even vanishes. Intermittency has been observed in several physical studies such as Lorenz system, Rayleigh-Bénard convection, forced nonlinear oscillators, plasma physics, electronic circuits, turbulent flows [15, 16, 17, 18, 19, 20, 21, 22, 23]. Also, intermittency was used to describe the behavior of economics and medical systems [24, 25, 26]. The accurate description of intermittency helps to improve the knowledge about these phe-

nomena. On the other hand, the proper characterization of intermittency has great importance for systems whose exact governing equations are totally or partially unknown.

The occurrence of one specific intermittency type depends on the monodromy operator multiplier [27]. We can summarize type I, II and III intermittencies depending on these multipliers. Type I intermittency appears when a multiplier leaves the unit circle across +1, then the periodic solution (orbit) loses its stability by a cyclic-fold bifurcation [28]. Type II intermittency departs from a subcritical Hopf bifurcation or a Neimark-Sacker bifurcation [5, 29], then two complex-conjugate Floquet multipliers of the system or two complex-conjugate eigenvalues of the local Poincaré map must move away from the unit circle. It was found in a driven double scroll circuit as a consequence of a global bifurcation scenario for T^2 torus breakdown [30, 31]. Also, characteristic relations for type II intermittency were studied in [32]. Type III intermittency occurs in a subcritical period-doubling bifurcation in which an unstable period-2 orbit meets and destabilizes a stable period-1 orbit; this type of intermittency is characterized by the gradual increasing during the laminar phase of a period-2 component in the motion [33]. A one-dimensional map $F(x)$ that displays a subcritical period-doubling bifurcation has a positive Schwartzian derivative: $SF(x) = F'''(x)/F'(x) - 1.5 (F''(x)/F'(x))^2 > 0$.

Chaotic intermittency may be studied using Poincaré maps [4, 5, 7]. Then, two main features characterize chaotic intermittency: 1) a specific local map and 2) a reinjection mechanism. The local map determines the intermittency type, and the reinjection mechanism maps back the system into the local regular or laminar zone from the chaotic one. This mechanism is described by the so-called reinjection probability density function (RPD), which is determined by the chaotic dynamics of the system itself. The RPD gives the probability that trajectories are reinjected into the laminar zone—close to the unstable fixed point—, and together with the local map determine all the dynamical properties of the system.

The accurate evaluation of the RPD function is extremely important to correctly describe the chaotic intermittency phenomenon. Still, the experimental or numerical evaluation of the RPD function is not a simple task due to the huge amount of data needed. Diverse approaches were used to obtain the intermittent RPD function. The main results on classical chaotic intermittency was based on uniform reinjection on the laminar region. Other implemented approaches build the RPD using peculiar features of the nonlinear processes [34, 35]. Nevertheless, these RPD functions cannot be successfully applied to other nonlinear systems.

A new methodology to obtain the RPD, called here M function methodology, which includes the uniform reinjection as a particular case, was introduced in the last years [36, 37, 38, 39, 40, 41, 42, 43]. This methodology establishes a generalized power law reinjection and it has shown to be very accurate for a wide class of one-dimensional

maps exhibiting type I, II and III intermittencies. In the thesis, we will continue working with this methodology.

1.2 Main objectives

In this research we want to extend the M function methodology or to develop new schemes to obtain or describe the following topics:

- 1 - The theoretical background of the new theory (M function methodology).
- 2 - Type III intermittency with noise and with lower boundary of reinjection.
- 3 - Type V intermittency.
- 4 - Intermittency with nonlinear $M(x)$ functions.

We will briefly describe each objective.

1.2.1 Theoretical background of the new theory

We will try to obtain an analytical method to estimate the RPD function (called in this thesis as $\phi(x)$). We will take in consideration the null or infinite derivatives of the map at the pre-reinjection points. Once the theoretical RPD function is established, we will be able to estimated an analytic expression for the length of the laminar phase of intermittent behaviors. Then, we will try to develop a new characteristic exponent which could be a function depending on the whole map, not on the only the local region. In conclusion, we wish to obtain a generalization of the classical intermittency theory. Finally, the theoretical formulation will be compared with numerical results of some celebrated classical maps used in intermittency theory.

1.2.2 Intermittency with noise and with lower boundary of reinjection

As noise is always present in nature [44], it is important to describe how it affects the intermittent RPD. The noise effects on the intermittency phenomenon was studied

using a renormalization group analysis and the Fokker-Plank equation [35, 45, 46]. Note that all the mentioned references analyze the effects of noise on the laminar zone; however, it has been recently shown that noise strongly influences the RPD function [38]. Moreover, as there is experimental evidence of the lower boundary of reinjection (LBR) for type III intermittency [15] we have decided to focus in the analysis of phenomena with type III intermittency where LBR and noise are present.

We want to extend the M function methodology to consider different values of the noise intensity and the lower boundary of reinjection (LBR) acting at the same time for type III intermittency. We want to obtain accurate analytic expressions for the reinjection probability density which could depend on the non-linear behavior of the system or map, the LBR value and the noise intensity. We will compare the new RPD functions with those given by the M function methodology without noise and LBR. We will carry out numerical simulations to verify the theoretical results.

1.2.3 Type V intermittency

Type V intermittency will be studied using the M function methodology. This methodology will be applied on two different maps to evaluate the reinjection probability density function, the probability density of laminar lengths (called in this thesis $\psi(l)$) and the characteristic relation. We want to find analytical expressions for the reinjection probability density and the probability density of laminar lengths. We want to verify if these functions can be nondifferentiable functions when the local map has a nondifferentiable point inside the laminar interval. On the other hand, we want to obtain the characteristic relation and analyze if it has an unique form or if it depends on the local map.

1.2.4 Intermittency with nonlinear $M(x)$ functions

We want to apply the Perron-Frobenius operator to study intermittency [47]. We think that we will be able to obtain theoretical expressions for the reinjection probability density function, $\phi(x)$, and the probability density of the laminar lengths, $\psi(l)$, for several maps with linear and nonlinear $M(x)$ functions. To validate the new expressions we will compare the analytical $\phi(x)$ and $\psi(l)$ functions with numerical data. Also, we will carry out comparisons between the new $\phi(x)$ and $\psi(l)$ functions with those obtained using the M function methodology. Finally, we will describe the advantages and difficulties of both methodologies.

For an application $y = F(x, t)$, i.e. an evolution operator $F^t(x) : \mathbf{R} \rightarrow \mathbf{R}$ ($\mathbf{R}$ represents the real numbers), the Perron-Frobenius operator allows to evaluate the density of trajectories $\rho(y, t)$ at time t starting from the density at a previous time, for example for $t = 0$, i.e. $\rho(x, 0)$. Similarly, for maps, the Perron-Frobenius operator allows to evaluate the density evolution after one iteration.

Let us to consider a general map $y = F(x)$, that transforms some interval $\Delta_0 \subset \mathbf{R}$ in other interval $\Delta \subset \mathbf{R}$. Hence, $x \in \Delta_0$ and $y \in \Delta$. In Δ_0 the density of the trajectories can be defined as $\rho_0(x)$, also in Δ there is other trajectories density: $\rho(y)$. The Perron-Frobenius operator, $P_\circ$, transforms the density $\rho_0(x)$ in $\rho(y)$, and we can write:

$$\rho(y) = P_\circ \rho_0(x) \quad (1.1)$$

1.3 Thesis organization

The thesis contains 8 chapters. After this introduction chapter, the second chapter describes the classical theory on intermittency. Chapter 3 introduces the M function methodology. In Chapter 4 the theoretical bases of the new theory are presented. Chapter 5 describes type III intermittency with noise and lower boundary of reinjection. In Chapter 6 we apply the M function methodology to type V intermittency. In Chapter 7 the Perron-Frobenius operator to obtain the RPD function is introduced, mainly when the $M(x)$ function is nonlinear. The last chapter is dedicated to describe the conclusions. Also, one appendix is included to enumerate the scientific publications produced in this thesis.

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Chapter 2

CLASSICAL THEORY OF CHAOTIC INTERMITTENCY

Abstract Intermittency phenomenon is characterized by a signal that alternates randomly regular or laminar phases and irregular bursts. It has been experimentally verified that the number of chaotic bursts intensifies with an external or control parameter. Intermittency has been observed in a large number of experiments and it offers a continuous route from regular to chaotic motion. In this thesis, we focus on temporal intermittency, which can occur in low-order dynamical systems. Traditionally intermittency was classified into three different types called I, II and III. Nevertheless, later studies have introduced other types of intermittencies: type V, X, eyelet, off-on, ring, etc. In this chapter a brief description of types I, II, III and V is carried out. On the other hand, the classical formulation to study the noise effects on intermittency is also described. Two classical approaches addressing this problem are the most important ones: Fokker-Plank and renormalization group analysis. In any case, they only consider the noise effect acting on the local map.

2.1 Basic concepts

All natural and man-made systems evolve in time. As examples of these systems, we can mention: turbulent and convective flows, electronic circuits, neurons, planetary systems, lasers, heart behavior, economical systems, plasma flows, radioactive decay, chemical processes, earthquakes, galaxies, pendulums, financial markets, stars, insects population, weather, among many others systems.

In many occasions it is necessary to describe the large time evolution of these systems, i.e. time tends to infinity ($t \rightarrow \infty$). Then, for large time evolution, the systems may elaborate several different dynamic behaviors, such as, steady states – time independent mechanisms –, regular oscillations, nonperiodic oscillations, etc.

Some systems evolving in time can be represented by nonlinear dynamical systems:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{H}, t) \quad (2.1)$$

where $\mathbf{x} \in \mathbf{R}^n$ is the vector of dependent variables, $\mathbf{H} \in \mathbf{R}^m$ is the vector containing all control parameters of the system, and $t \in \mathbf{R}$ is the time. Frequently, the vector $\mathbf{f}$ is called vector field, and the vector $\mathbf{x}$ is referred as state vector.

Eq. (2.1) is a differential equations system, and it represents a continuous time evolution process. We can mention two types of continuous systems: nonautonomous and autonomous. Eq. (2.1) corresponds to a nonautonomous system. For the autonomous systems time does not appear explicitly as independent variable, and they can be written as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{H}) \quad (2.2)$$

For Eq. (2.2), the system solutions, $\mathbf{x} = \mathbf{x}(t)$, will be defined in a n -dimensional space, $\mathbf{R}^n$, which is called state space.

Also, for nonautonomous systems, the solutions can be defined in the extended state space, which is a $n + 1$ -dimensional space, $\mathbf{R}^n \times \mathbf{R}$, where the supplementary dimension corresponds to time. An integral curve is the drawing of a nonautonomous system solution in the extended state space. The projection onto the state space of an integral curve is called trajectory or orbit.

It is possible to demonstrate that for a given initial condition $\mathbf{x}_i$, an autonomous system has just a single solution if the scalar components of the vector field, $\mathbf{f}$, have continuous and bounded first partial derivatives with respect to the scalar components of state vector, $\mathbf{x}$. Consequently two orbits or trajectories can not intersect in the system state space [1, 2, 3].

On the other hand, there are non-continuous time evolution mechanisms. They are called discrete time evolution systems or maps, and they are represented by difference equations systems:

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \mathbf{H}) \quad (2.3)$$

If the map $\mathbf{F}(\mathbf{x}, \mathbf{H})$ has inverse, it is invertible, and the inverse results:

$$\mathbf{x}_n = \mathbf{F}^{-1}(\mathbf{x}_{n+1}, \mathbf{H}) \quad (2.4)$$

If a map $\mathbf{F}(\mathbf{x}, \mathbf{H})$ does not have inverse, it is called a noninvertible map.

The vectors $\mathbf{F}(\mathbf{x}, \mathbf{H})$ and $\mathbf{x}$ have N scalar components:

$$\begin{aligned} \mathbf{F}(\mathbf{x}_n, \mathbf{H}) &= [F_1(\mathbf{x}, \mathbf{H}), \dots, F_n(\mathbf{x}, \mathbf{H}), \dots, F_N(\mathbf{x}, \mathbf{H})]^T \\ \mathbf{x} &= [x_1, \dots, x_n, \dots, x_N] \end{aligned} \quad (2.5)$$

$\mathbf{F}(\mathbf{x}, \mathbf{H})$ is called a C^q function when the scalar components of the vector $\mathbf{F}(\mathbf{x}, \mathbf{H})$ are q -times continuously derivable with respect to each component of the vector $\mathbf{x}$. If $\mathbf{F}$ and

its inverse $\mathbf{F}^{-1}$ are C^q functions with $q \geq 1$, the map $\mathbf{F}(\mathbf{x}, \mathbf{H})$ is called diffeomorphism [2]. If the map $\mathbf{F}(\mathbf{x}, \mathbf{H})$ and its inverse $\mathbf{F}^{-1}(\mathbf{x}, \mathbf{H})$ are only continuous functions for each component, then $\mathbf{F}(\mathbf{x}, \mathbf{H})$ is called an homeomorphism [3].

A trajectory $[\mathbf{x}_n]$ is developed by applying the map $\mathbf{F}(\mathbf{x}, \mathbf{H})$, given by Eq. (2.3), repeatedly:

$$\mathbf{F}^0(\mathbf{x}, \mathbf{H}) = \mathbf{x}; \quad \mathbf{F}^1(\mathbf{x}, \mathbf{H}) = \mathbf{F}(\mathbf{x}, \mathbf{H}); \quad \mathbf{F}^2(\mathbf{x}, \mathbf{H}) = \mathbf{F}(\mathbf{F}(\mathbf{x}, \mathbf{H}), \mathbf{H}); \dots \quad (2.6)$$

Also, the trajectory must verify:

$$\mathbf{F}^{r+s}(\mathbf{x}, \mathbf{H}) = \mathbf{F}^r(\mathbf{F}^s(\mathbf{x}, \mathbf{H}), \mathbf{H}); \quad (2.7)$$

where r and s are integers.

In this thesis we are concerning with maps, more specifically with one-dimensional maps:

$$x_{n+1} = F(x_n, \varepsilon) \quad (2.8)$$

where ε represents the control parameter.

The one-dimensional maps may show a very complex dynamical behavior. Notwithstanding, they are simple enough to analytically handle and to obtain numerical results without need large CPU-time [4].

As examples, we show two well known maps: the logistic and the Hénon maps. The first is a one-dimensional map and the second one is a two-dimensional map.

Example 1. The logistic map

One of the most studied one-dimensional map is the logistic map [5, 6], which can be written as:

$$x_{n+1} = \varepsilon x_n(1 - x_n) \quad (2.9)$$

The variable x is defined in the interval $[0, 1]$. Figure 2.1 shows this map together with the bisector line.

The logistic map was used to evaluate the evolution of the organism populations in limited space, with supply of food and without predators [6].

Example 2. The Hénon map

The Hénon map is a two-dimensional map [6, 7]:

$$\begin{cases} x_{n+1} = 1 + y_n - \delta x_n^2 \\ y_{n+1} = \varepsilon x_n \end{cases} \quad (2.10)$$

Fig. 2.1 Logistic map

where $(x, y) \in \mathbf{R}^2$, δ and ε are two control parameters.

There are several routes in which dynamical systems (maps) can evolve from regular behaviors to chaos. As we explain in Chapter 1, three of them are period-doubling scenario, quasiperiodic and intermittency. This research deals only with one of these routes: chaotic intermittency or intermittency (both terms are used indistinctly in this thesis).

The importance of intermittency has been observed in several physical phenomena. We mention only some of them: forced nonlinear oscillators, turbulent flows, plasma physics, Rayleigh-Bénard convection, Lorenz system, electronic circuits, derivative nonlinear Schrodinger equation, etc [8, 9, 10, 11, 12, 13, 14, 15, 16]. Moreover, chaotic intermittency has been taken into account to explain the behavior of economy and medicine phenomena [17, 18, 19].

Batchelor and Townsend in 1949 employed the word intermittency to describe some observational data of the fluctuating velocity in turbulent flows [20]. The term was

introduced to represent signals that alternated between approximately flat periods and burst ones. Since then, intermittency has been observed in several other phenomena in physics, engineering and chemistry [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21].

For over thirty years, the term intermittency has been extensively used to describe systems that alternate almost regular and non-regular behaviors. Chaotic intermittency mechanism is characterized by transitions between regular phases – also called laminar phases – and chaotic bursts or non regular phases. The regular or laminar phases are regions of pseudo-equilibrium, pseudo-periodic solutions or quasi-invariant objects near to which the system may consume long times; and the bursts ones are regions where the system evolution is chaotic.

To describe chaotic intermittency, there are some statistical parameters, such as: the probability density of the laminar lengths, the average laminar length and the characteristic relation. Nevertheless, to evaluate these statistical properties, we previously should know the reinjection probability density function, or RPD.

The RPD function determines the distribution of the reinjected trajectories inside the laminar region. Therefore, this function becomes the more important statistical property to describe the intermittency process, and its accurate evaluation is essential to correctly characterize the chaotic intermittency phenomenon.

The reinjection probability density function provides the probability that trajectories are reinjected into the laminar zone, close to the unstable or vanished fixed point of the map. This function depends on the nonlinear dynamics of the system itself and it can lead to a broad range of different behaviors. The classic theory of intermittency, and most popular approach, considers an uniform reinjection, therefore $RPD = \text{constant}$ [3, 8, 9, 22, 23, 24, 25, 26, 27, 28]. However, we must note that this assumption is not always applicable, there are many processes with more complex RPD.

On the other hand, different approaches have been implemented using a characteristic of the particular nonlinear processes into consideration. But, these RPD functions cannot be applied for other systems. Some approaches were used to obtain the intermittent RPD function, which built the RPD using peculiar features of the nonlinear processes under study. Two cases were given to investigate the effect of noise in type I and III intermittencies. For type I intermittency was assumed that the reinjection was localized in a fixed point [29]; and for type III intermittency the RPD in an electronic circuit was considered proportional to $1/\sqrt{x-a}$, where a is a constant [30]. Nevertheless, we highlight that these RPD functions cannot be successfully applied to other nonlinear systems [31, 32].

We highlight that it has not been an easy activity to obtain an general analytical expression for the RPD function. A direct experimental or numerical evaluation of the RPD function has not also been a simple task due to the huge amount of data needed to

deal with. Moreover, the statistical fluctuations induced in numerical and experimental measurements are generally difficult to estimate.

A more general and robust estimation of the RPD has been introduced in the last years, which includes the uniform reinjection as a particular case. This new methodology is called here the *M function methodology*. The $M(x)$ function was used for the first time in the intermittency context in [33] and it has recently been broadly extended [31, 32, 34, 35, 36, 37, 38, 39, 40, 41]. This methodology has shown to be very accurate for a wide class of maps exhibiting type I, II and III intermittencies, with and without noise, and it will be explained in follows chapters.

The accurate description of intermittency will help to improve the knowledge about several phenomena. On the other hand, the suitable description of chaotic intermittency has a large importance for systems whose exact governing equations are totally or partially unknown.

The latest advances will be described in the follows chapters. However, in this chapter, let us introduce the main results on the classical theory. This will be useful to generalize the intermittency theory.

2.2 Types of chaotic intermittency

The classic classification of chaotic intermittency was first introduced, more than thirty years ago, by Manneville and Pomeau [26, 27]. They classified the chaotic intermittency into three different types called I, II and III according to the eigenvalues of the fixed point in the local Poincaré map or the Floquet multipliers of the periodic solution in the continuous-system [2, 3, 23, 42]. Nevertheless, more recent studies have extended the classification including other types of intermittencies such as type V, X, on-off, eyelet, ring, etc. [43, 44, 45, 46, 47, 48]. A more general case of on-off intermittency is the so called “in-out” intermittency [49].

In this chapter we only consider classical intermittency – type I, II, III and V intermittencies because only these types of intermittency are studied in this thesis.

Let us consider a stable periodic solution of an autonomous continuous-time system given by Eq. (2.2) for specific values of the control parameters. Then, we modify a control parameter until the periodic solution loses its stability. The evolution of the system solution will depend on the way that the Floquet multipliers leave out the unit circle in the complex plane [2, 3, 23]. We highlight that the metamorphoses of a family of solution curves in the neighborhood of a closed orbit is a complex problem of bifurcation theory [50].

There are different alternatives to study the stability of a closed orbit (called here γ). One of them is the Floquet theory by means the Floquet multipliers [2, 51]. An

Fig. 2.2 Poincaré map. There are two orbits: 1 - The orbit γ is a limit cycle. 2 - The second orbit first crosses the surface in q and after in $F(q)$; where $F(x)$ describes the Poincaré map.

alternative assessment is selecting a point x_0 on the closed orbit. For this point we pass a transverse hypersurface to the flow (Σ). Therefore, $\mathbf{f}(\mathbf{x}, \mathbf{H}) \cdot \mathbf{n}(\mathbf{x}) \neq 0$, where $\mathbf{n}(\mathbf{x})$ is the unit normal vector to the hypersurface and $\mathbf{f}(\mathbf{x}, \mathbf{H})$ is the vector field given by Eq. (2.2). On this hypersurface, we obtain the corresponding Poincaré map, and we carry out the linearization of this Poincaré map at the point x_0 . The resulting linear operator is called monodromy operator, and its eigenvalues are called multipliers. This monodromy operator has a real characteristic equation; then every nonreal multiplier must have a complex conjugate eigenvalue [50]. Figure 2.2 shows the hypersurface Σ and the orbit γ for $\mathbf{f}(\mathbf{x}, \mathbf{H}) \in \mathbf{R}^3$. A more detailed description about the relationship between the Floquet multipliers and the eigenvalues of the Poincaré map is given in [52].

The multipliers (or Floquet multipliers) can leave the unit circle in three different scenarios: one of them occurs when one multiplier leaves the unit circle along the positive real axis (by +1). An alternative scenario results when the multiplier leaves the unit circle along the negative real axis (by -1). Finally, the last scenario appears

Fig. 2.3 Floquet multipliers

when two multipliers are complex conjugate numbers. Then they leave the unit circle away from the real axis. Figure 2.3 shows the three scenarios.

The occurrence of one specific intermittency type depends on the monodromy operator multiplier. We can summarize type I, II and III intermittencies depending on these multipliers:

- Type I Type I intermittency appears when a multiplier leaves the unit circle across $+1$, then the periodic solution (orbit) loses its stability by a cyclic-fold bifurcation.
- Type II Type II intermittency departs from a subcritical Hopf bifurcation or a Neimark-Sacker bifurcation [2, 53]. Then, two complex-conjugate multipliers of the system move away from the unit circle.
- Type III Type III intermittency is produced when a multiplier leaves the unit circle by -1 . In this case a subcritical period-doubling bifurcation occurs.

2.3 Classical theory of chaotic intermittency

As we described previously, classic chaotic intermittency is distributed in three categories or types: I, II and III. They can be studied using Poincaré maps [2, 23].

In type I, II and III intermittency phenomena, for values of a control parameter ε less than a threshold value, the map has an attracting fixed point and the system is stable to small perturbations. However, when the control parameter exceeds this critical value, a bifurcation leads to a change of the system behavior towards a chaotic motion that converges into a larger attractor [2]. For type II and III intermittencies, the old attractor remains as a subset of the new attractor. In all cases, a fixed point of the local Poincaré map becomes unstable or even vanishes for some values of the control parameter ε . As the control parameter increases, the chaotic bursts occur more frequently and the duration of the regular or laminar behaviors decreases.

We must note a very important characteristic of the chaotic intermittency phenomenon: the laminar or regular periods do not have the same number of iterations (same length). Accordingly, to quantify the intermittency, it is necessary to introduce statistical variables as the average laminar length.

An example of a one-dimensional map with intermittency is given by [32]:

$$x_{n+1} = F(x_n) = -(1 + \varepsilon)x_n - ax_n^3 + bx_n^6 \sin(x_n) \quad (2.11)$$

where ε is the control parameter that drives the chaotic intermittency. The origin, $x_0 = 0$, is a fixed point of the map and this point is stable for $-2 < \varepsilon < 0$. When $\varepsilon > 0$, the fixed point becomes unstable and type III intermittency appears. Figure 2.4 shows the system evolution when the control parameter is increasing. For the subfigures (a) and (b) the system is stable and it evolves to the fixed point. However, if $\varepsilon \gtrsim 0$ there is intermittency (see subfigure (c)).

In order to chaotic intermittency can appear, two different features must exist: the first is the specific local map, and the second one corresponds to the reinjection mechanism also called relaminarization mechanism.

The local map describes the type of intermittency and it drives the dynamics in the laminar zone. The reinjection mechanism maps back the system into the local regular or laminar zone from the chaotic one. This mechanism is described by means of a probability function called reinjection probability density function which, as we have indicated previously, defines the probability that trajectories are reinjected into the laminar zone, close to the unstable or vanished fixed point. This function depends on the nonlinear dynamics of the system itself and can lead to a broad range of different behaviors.

For Eq. (2.11), the local map is represented by the first and second terms of the RHS:

Fig. 2.4 Evolution of the map (2.11) for three different ε . When $\varepsilon >$ critical value, intermittency rises (see subfigure (c)).

$$-(1 + \varepsilon)x_n - ax_n^3 \quad (2.12)$$

The iteration procedure, which is governed by the parameters ε , a and the cubic exponent, leads to increasing values of x_n , generated from an initial one, close to the origin. For large enough values of n , the influence of the RHS third term in Eq. (2.11)

becomes important as x_n approaches to an extreme point, rendering the reinjection into the laminar zone. Therefore, the term:

$$bx_n^6 \sin(x_n) \quad (2.13)$$

produces the nonlinear reinjection mechanism.

In the next subsections we briefly describe the classical theory of type I, II, III and V intermittencies.

2.3.1 Type I intermittency

Type I intermittency has been extensively studied. It possesses a different behavior than type II and III intermittencies, because for type I intermittency the fixed point of the Poincaré map disappears when the control parameter is higher than the critical value. By contrast, for type II and III intermittencies the fixed point only loses its stability, but it does not disappear. The cyclic-fold bifurcation, associated with type I intermittency, produces this specific behavior.

We present the classic theoretical model for type I intermittency. Let us start considering a nonlinear dynamical system such as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \varepsilon) \quad (2.14)$$

This equation is similar to those given by Eq. (2.2); however there is difference between them: Eq. (2.14) has only one real number as control parameter. When this control parameter, ε , reaches the threshold value, one multiplier (or Floquet multiplier) associated with the periodic solution of Eq. (2.14) leaves the unit circle by +1. Note that the multiplier is a real number. A codimension-one bifurcation occurs, in this case is a cyclic-fold bifurcation which is a discontinuous or catastrophic bifurcation. The manifold corresponding to this Floquet multiplier has a very important information associated with intermittency – in other directions there are dissipation – [2]. Then, to study intermittency, we can consider only a one-dimensional map related to this manifold; and this map can be written as Eq. (2.8): $x_{n+1} = F(x_n, \varepsilon)$.

The local map for type I intermittency can be written as:

$$x_{n+1} = \varepsilon + x_n + ax_n^2 \quad (2.15)$$

For this map the number and stability of fixed points depend on the control parameter value. When $\varepsilon < 0$ the map has two fixed points: one is stable, and the other one is unstable. For $\varepsilon = 0$ both fixed points meet in one fixed point. However, for $\varepsilon > 0$ there

are not fixed points. Figure 2.5 shows the three different behaviors. Figure 2.5.a shows the behavior for $\varepsilon < 0$, where there are two fixed points, x_s and x_u . The fixed point x_s is stable and x_u is unstable. In Figure 2.5.b both fixed points have met in one fixed point called x_0 when $\varepsilon = 0$. The eigenvalue for the x_0 fixed point is $+1$. Figure 2.5.c shows the behavior for $\varepsilon > 0$, where the fixed point has vanished, hence a narrow channel appears. The width of this channel depends on ε . The trajectories spend a long time passing through the tunnel. When the trajectories are passing through the tunnel they are close to the vanished fixed point. The laminar zone is defined as a zone around to the vanished fixed point. Therefore, inside of the laminar zone the trajectories are very close to the vanished fixed point. However, they can never reach it.

Fig. 2.5 Type I intermittency. Tangent bifurcation. (a) Two fixed points. (b) Both fixed points coalesce in one fixed point. (c) Intermittency.

For maps, the equivalent of a cyclic-fold bifurcation is called tangent bifurcation. Figure 2.5 shows a tangent bifurcation, where two fixed points meet in one fixed point and later it disappears.

To obtain type I intermittency, two conditions must exist: a local map with a tangent bifurcation, and a reinjection or relaminarization mechanism. The local map can be written as Eq. (2.15); however the reinjection mechanism depends on the complete nonlinear map.

Outside the channel the trajectories oscillate chaotically, but the relaminarization mechanism returns the trajectories close to the vanished fixed point, inside the narrow channel. The point where each trajectory is reinjected generally changes, i.e it is not the same point for each reinjection process. Therefore, the reinjection probability density function, $\phi(x)$, is the kernel of the statistical behavior of the intermittency. Consequently, $\phi(x)$ describes the probability of the trajectories being reinjected in the point x . Higher values of $\phi(x)$ mean that there are more trajectories reinjecting in the point x .

In some investigations it has been considered that the reinjection mechanisms only reinject points for values of x that exceed a threshold value, let's say $\hat{x}$. Then, the reinjection process verifies $\phi(x) = 0$ for $x < \hat{x}$. This threshold value is referred as lower boundary reinjection point (LBR) and its effect on the RPD will be described in Chapter 3. Consequently, in the present chapter the value of the LBR is just the lower limit point delimiting the laminar region, so we consider that the reinjection mechanism takes place in the whole laminar region.

We have previously established that the RPD is a very important tool to describe the intermittency behavior, and the classical theory of intermittency considers a constant RPD; hence there is the same reinjection probability for all points inside the laminar interval.

Summarizing, the classical theory of intermittency establishes:

$$\phi(x) = k \quad (2.16)$$

where k is a constant value. Therefore, there is uniform reinjection.

In order to obtain a better description of intermittency phenomena, other statistical parameters are employed: the probability density of the laminar lengths, the average laminar length, and the characteristic relation.

Previously, to define the probability density of the laminar lengths, we introduce the laminar length concept, also called the length of the laminar region. When a trajectory is reinjected inside the laminar zone, it evolves through the narrow channel until it

leaves the laminar interval. The number of iterations used for the trajectory to pass through the laminar interval is called the laminar length. Therefore, it is an integer number. We can understand the laminar length as the “time” needed by each trajectory to cross the laminar zone. It is clear that the laminar length depends on the reinjection point x .

To calculate the number of iterations used by a trajectory to pass through the laminar zone, we consider that $0 < \varepsilon \ll 1$ – the control parameter ε is positive and small. Then, from Eq. (2.15) the difference between two successive iterations is very small. Therefore, we can approximate $x_{n+1} - x_n \approx dx/dl$. The difference equation (2.15) can be replaced by a differential equation [23]:

$$\frac{dx}{dl} = \varepsilon + ax^2 \quad (2.17)$$

To obtain $l = l(x)$ we should solve Eq. (2.17) for a reinjected point x :

$$\int_x^c \frac{dx}{\varepsilon + ax^2} = \int_0^l dl \quad (2.18)$$

where l is the laminar length; and c is the upper limit of the laminar interval, which is a positive and small value. Figure 2.6 shows type I intermittency development by a tangent bifurcation with the laminar interval and its limits. The laminar interval is defined as $[x_0 - c, x_0 + c]$, where x_0 is the fixed point.

From Eq. (2.18), the laminar length, l , results:

$$l(x, c) = \frac{1}{\sqrt{a\varepsilon}} \left[\arctan \left(\frac{c}{\sqrt{\varepsilon/a}} \right) - \arctan \left(\frac{x}{\sqrt{\varepsilon/a}} \right) \right] \quad (2.19)$$

Note that a very small control parameter, $\varepsilon \ll 1$, was used to obtain Eq. (2.18). On the other hand, Eq. (2.19) only depends on the local map (2.15), and the reinjection mechanism does not influence the laminar length, which depends on the reinjected point x .

The probability density of the laminar lengths gives the probability of finding laminar lengths between l and $l + dl$. In this thesis, the probability density of the laminar lengths is called $\psi(l, c)$. This probability is given by [23]:

$$\psi(l, c) = \phi[X(l, c)] \left| \frac{dX(l, c)}{dl} \right|, \quad (2.20)$$

where $X(l, c)$ is the inverse of $l(x, c)$ given by Eq. (2.19):

Fig. 2.6 Type I intermittency

$$X(l, c) = \sqrt{\frac{\varepsilon}{a}} \tan \left[\tan^{-1} \left(\sqrt{\frac{a}{\varepsilon}} c \right) - \sqrt{a\varepsilon} l \right], \quad (2.21)$$

Now, we can obtain the average laminar length $\bar{l}$:

$$\bar{l} = \int_0^{l_m} \psi(l, c) l(x, c) dl \quad (2.22)$$

where l_m is the highest laminar length, which is calculated by Eqs. (2.18) or (2.19) with $x = -c$. Therefore, $l_m = l(-c, c)$, on the other hand, $l(c, c) = 0$.

The global map produces the reinjection of the trajectories in some point verifying $-c \leq x \leq c$. As the RPD is constant – uniform reinjection – and considering Eqs. (2.19) and (2.20), the Eq. (2.22) results as:

$$\bar{l} = \int_{-c}^c \phi(x) l(x, c) dx = \frac{1}{\sqrt{a\varepsilon}} \arctan \left(c \sqrt{\frac{a}{\varepsilon}} \right) \quad (2.23)$$

The characteristic relation is other statistical property used to describe the chaotic intermittency phenomena. It establishes the relation between the average laminar length, $\bar{l}$, and the control parameter, ε . For type I intermittency this relation was presented by the first time in [28]. If $c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \gg 1$, from Eq. (2.23) we may obtain:

$$\bar{l} \propto \frac{1}{\sqrt{\varepsilon}} \quad (2.24)$$

Therefore, as ε decreases, $\bar{l}$ increases, and the trajectories need more iterations to pass through the narrow channel close to the vanished fixed point.

The characteristic relation can be written as:

$$\bar{l} \propto \varepsilon^{-\beta} \quad (2.25)$$

where the exponent β indicates how fast the average laminar length $\bar{l}$ increases as the parameter ε decreases. The classical theory of intermittency establishes that β is a constant [2, 23]. If we compare Eqs. (2.24) and (2.25) we find, for type I intermittency: $\beta = 1/2$.

We note that the average laminar length and the characteristic relation permit estimating the “duration” of the laminar phases.

2.3.2 Type III intermittency

For dynamical systems, type III intermittency occurs in a subcritical period-doubling bifurcation in which an unstable period-2 orbit meets and destabilizes a stable period-1 orbit. This type of intermittency is characterized by the gradual increasing during the laminar phase of a period-2 component in the motion [54].

For type III intermittency a Floquet multiplier leaves out the unit circle by -1 , and at the same time the absolute value of the other Floquet multipliers are less than 1. Therefore, the needed information to describe intermittency resides in the manifold associated to the unstable Floquet multiplier, because along the other directions the absolute values of the monodromic multipliers are less than one. Hence, there is a map associated with this manifold, $F(x, \varepsilon)$, and type III intermittency can be studied using maps [2, 23], which must have two main chaotic intermittency characteristics: 1) a specific local map and 2) a reinjection mechanism, which maps the system back into the local laminar zone from the chaotic one –relaminarization mechanism.

A local map for type III intermittency can be written as:

$$x_{n+1} = F(x_n) = -(1 + \varepsilon)x_n + a_2 x_n^2 + a_3 x_n^3 + \dots \quad (2.26)$$

where the control parameter ε is small but larger than zero [32, 54]. The coefficients a_2 and a_3 depend on the dynamical process in study [2]. For this map, $x_0 = 0$ is a fixed point for all ε . This fixed point is stable for $\varepsilon < 0$ and it becomes unstable for $\varepsilon > 0$.

To study type III intermittency we consider a second iteration of the map (2.26):

$$\begin{aligned} x_{n+2} = F^2(x_n) = & -(1 + \varepsilon) [-(1 + \varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3] + \\ & a_2 [-(1 + \varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3]^2 + \\ & a_3 [-(1 + \varepsilon)x_n + a_1 x_n^2 + a_2 x_n^3]^3 + \dots \end{aligned} \quad (2.27)$$

As the local map is defined around the fixed point $x_0 = 0$, we only consider small values of $|x|$ and ε ($0 < \varepsilon \ll 1$). Consequently, we can neglect the following terms in Eq. (2.27): terms with ε^s , where s is an integer and it verifies $s \geq 2$; terms with x^p , where p is an integer and $p \geq 4$; and terms with εx^h , where h is an integer and it verifies $h \geq 2$. Finally, the second iteration of map (2.27) can be written in a simple form:

$$x_{n+2} = F^2(x_n) = (1 + 2\varepsilon)x_n + a x_n^3 + \dots \quad (2.28)$$

where $a = -2(a_3 + a_2^2)$.

In Eq. (2.28) the occurrence of intermittency is driven by the coefficient of the linear term, which determines the slope of the map for $x_0 = 0$. This behavior is different from that of the type I intermittency where there is a channel between the map and the bisector.

We analyze the behavior of map (2.28) when $\varepsilon > 0$. There are two cases depending on the sign of the cubic term coefficient, a :

- Case 1: $a < 0$. Besides the $x_0 = 0$, there are two stable fixed points $x_{1,2} = \pm (2\varepsilon/|a|)^{0.5}$. When x_0 loses its stability, these fixed points attract the iteration process – for $F^2(x)$ a supercritical pitchfork bifurcation appears. This behavior does not lead to intermittency.

- Case 2: $a > 0$. The only real fixed point is $x_0 = 0$, which is unstable for $\varepsilon > 0$. If there is a relaminarization mechanism, intermittency occurs. Figure 2.7 shows the bifurcation diagram for this case. Type III intermittency is associated with a subcritical pitchfork bifurcation of $F^2(x)$, or a subcritical period-doubling bifurcation of $F(x)$. We highlight that $a > 0$ implies $a_3 < -a_2^2$. Consequently, several books and papers just use Eq. (2.28) as a local map for type III intermittency –instead of Eqs. (2.26) or (2.27)–.

When one-dimensional map, $F(x)$, displays a subcritical period-doubling bifurcation, the Schwartzian derivative must be positive:

Fig. 2.7 Bifurcation diagram for type III intermittency.

$$SF(x) = \frac{d^3F(x)/dx^3}{dF(x)/dx} - \frac{3}{2} \left[\frac{d^2F(x)/dx^2}{dF(x)/dx} \right]^2 > 0 \quad (2.29)$$

This equation, for $\varepsilon = 0$, also implies $a_3 < -a_2^2$. Therefore, for one-dimensional maps, type III intermittency appears if Eq. (2.29) is verified.

As well as type I intermittency, we will describe the following statistical properties: the reinjection probability density function, the probability density of the laminar lengths, the average laminar length, and the characteristic relation.

For the classic theory of intermittency the RPD is constant, $\phi(x) = k$, and it must verify the normalization condition:

$$\int_{-c}^c k dx = 1 \quad (2.30)$$

then, we can obtain: $\phi(x) = k = 0.5/c$.

Inside of the laminar region and for very small ε , we can approximate the difference equation, Eq. (2.28), by a differential equation [32]:

$$\frac{dx}{dl} = 2\varepsilon x + a x^3; \quad (2.31)$$

By solving this equation, we can calculate the laminar length:

$$l(x, c) = \int_0^l dl = \int_x^c \frac{d\tau}{\tau(2\varepsilon + a\tau^2)} \quad (2.32)$$

The laminar length for a reinjected point x results:

$$l(x, c) = \frac{1}{2\varepsilon} \ln \left(\frac{a + \varepsilon/x^2}{a + \varepsilon/c^2} \right) \quad (2.33)$$

where $l(x, c)$ represents the laminar length for $F^2(x)$ map. For simplicity we have rewritten $2\varepsilon \rightarrow \varepsilon$ and c is the upper limit of the laminar interval. Hence, the laminar length for the $F(x)$ map – given by Eq. (2.26) – is $2l(x, c)$.

The probability of finding a laminar phase of length laying between l and $l + dl$ is given by the probability of laminar lengths, $\psi(l)$ (see Eq. (2.20)). Where $X(l, c)$ is the inverse function of $l(x, c)$ with respect to x , which is calculated using Eq. (2.33):

$$X(l, c) = \left[\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^2}\right) e^{2\varepsilon l} - a} \right]^{0.5} \quad (2.34)$$

Then $\psi(l)$ can be written as:

$$\psi(l) = \left[\frac{\varepsilon}{\left(\frac{\varepsilon}{c^2} + a\right) e^{2\varepsilon l} - a} \right]^{3/2} \left(\frac{\varepsilon}{c^2} + a \right) \frac{e^{2\varepsilon l}}{2c} \quad (2.35)$$

Figure 2.8 shows an example of Eq. (2.35) for $a = 1$, $c = 0, 1$ and $\varepsilon = 0.001$.

Now, we can calculate the average laminar length:

$$\bar{l} = \int_0^c \phi(x) 2l(x, c) dx = \frac{1}{c} \int_0^c l(x, c) dx \quad (2.36)$$

From the previous expression and using Eq. (2.33) we can obtain:

$$\bar{l} = \frac{1}{2c\varepsilon} \int_0^c \left[\ln \left(a + \frac{\varepsilon}{x^2} \right) - \ln \left(a + \frac{\varepsilon}{c^2} \right) \right] dx \quad (2.37)$$

After some calculations, we obtain:

$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan \left(c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \right) \quad (2.38)$$

When $c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \gg 1$ the last equation can be written as $\bar{l} \propto \varepsilon^{-1/2}$. Therefore, the characteristic relation for type III intermittency has the same form that those obtained for

Fig. 2.8 $\psi(l)$ for type III intermittency

type I. Both relations show that the average laminar length rises for small values of ε . Accordingly, the duration of the laminar phases increases when the control parameter, ε , decreases.

We notice that in Ref [23] the characteristic relation $\bar{l} \propto \varepsilon^{-1}$ is provided, hence two characteristic relations have been proposed for type III intermittency. As a result of this, some interesting debates took place. Nowadays there is not paradox at all as was explained in Ref. [41].

2.3.3 Type II intermittency

Type II intermittency appears in two or higher dimensional maps because two complex-conjugate eigenvalues of the fixed point move away from the unit circle. There is a two dimensional manifold associated to this pair of complex-conjugate eigenvalues (two dimensional space spanned by the eigenvectors associated with the complex conjugate eigenvalues). We analyze the dynamics on this manifold, because the other eigenvalues are less than 1 in absolute value.

A Hopf bifurcation for the fixed point of the map occurs. And the complex monodromic eigenvalues can be written as: $\nu = (1 + \varepsilon)e^{i\theta}$. Where θ is an angle.

Locally, the dynamic is driven by a two-dimensional map [2]:

$$\begin{aligned} r_{n+1} &= (1 + \varepsilon)r_n + a r_n^3; \\ \theta_{n+1} &= \theta_n + b + q r_n^2 \end{aligned} \quad (2.39)$$

where a , b and q are constants, which depend on the dynamical system; and ε is the control parameter. When $\varepsilon > 0$ intermittency can appear.

To describe type II intermittency we will evaluate the same statistical parameters used to type I and III intermittencies: the reinjection probability density function, the probability density of the laminar lengths, the average laminar length, and the characteristic relation.

The classical theory considers an uniform and constant RPD. However, the other statistical parameters are not constant, and they are calculated using the concepts previously employed for type I and III intermittencies.

For very small values of ε we can approximate Eq. (2.39):

$$\frac{dr}{dl} = \varepsilon r + a r^3; \quad (2.40)$$

Hence, we have:

$$l(x, c) = \int_x^c \frac{dr}{\varepsilon r + a r^3} \quad (2.41)$$

where x is the reinjected point and c the upper limit of the laminar interval, i.e. the laminar interval is: $[0, c]$. Then, we can write:

$$l(x, c) = \frac{1}{2\varepsilon} \ln \left(\frac{a + \varepsilon/x^2}{a + \varepsilon/c^2} \right) \quad (2.42)$$

The probability density of the laminar lengths, $\psi(l)$ is given by Eq. (2.20); where $X(l, c)$ is the inverse of $l(x, c)$ given by Eq. (2.42):

$$\psi(l) = \left[\frac{\varepsilon}{\left(\frac{\varepsilon}{c^2} + a \right) e^{2\varepsilon l} - a} \right]^{3/2} \left(\frac{\varepsilon}{c^2} + a \right) \frac{e^{2\varepsilon l}}{c} \quad (2.43)$$

The average laminar length is given by

$$\bar{l} = \int_0^c \phi(x) l(x) dx = k \int_0^c l(x) dx = (1/c) \int_0^c l(x) dx \quad (2.44)$$

Finally we obtain:

$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan \left(c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \right) \quad (2.45)$$

where the RPD is constant and verifies the normalization condition given by Eq. (2.30): $\phi(x) = k = 1/c$.

The characteristic relation represents the variation of the average laminar length with the control parameter. From Eq. (2.45) this relation is nonlinear, and for $c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \gg 1$ it can be written as $\bar{l} \propto \varepsilon^{-1/2}$. Therefore, the characteristic relation is similar than those obtained for type I and type III intermittencies.

Example 3. Type II intermittency

We study the characteristic relation for the following map:

$$y = G(x) = \begin{cases} G_1(x) = (1 + \varepsilon)x + ax^3 & \text{if } 0 \leq x < x_r \\ G_2(x) = (x - x_r)/(1 - x_r) & \text{if } x_r \leq x \leq 1 \end{cases} \quad (2.46)$$

where $G_1(x_r) = 1$. The map has a fixed point for $x_0 = 0$ which is unstable for $\varepsilon > 0$. The map $G_2(x)$ allows the reinjection mechanism. Then for $\varepsilon > 0$ appears type II intermittency.

We carried out some numerical tests to evaluate the average laminar length. For all tests the reinjected points are 10000, and the RPD verifies $\phi(x) \approx 1/c$. Table 2.1 shows the used parameters and the results. The numerical results show a good accuracy with Eq. (2.45).

Table 2.1 Average laminar length for type II intermittency.

ε	c	Numerical	Eq. (2.45)	Relative error
0.00005	0.05	4296.5	4441.9	3.2%
0.0001	0.05	2895.7	3140.6	7.9%
0.0001	0.10	1610.3	1569.8	2.5%
0.0001	0.15	1018.9	1046	2.6%

2.4 Type V intermittency

Type V intermittency was introduced in Refs. [55, 56, 57]. It appears when a stable fixed point in a nondifferentiable, even discontinuous, map collides with a nondifferentiable or discontinuous point.

Type I and III intermittencies can occur for everywhere differentiable one-dimensional map. Both intermittencies happen when one eigenvalue corresponding to the fixed point reaches +1 or -1 respectively. But, if the map has discontinuous or non differentiable points (NPD) a different process can come about. When a control parameter is adjusted so that one NPD moves toward and collides with a stable fixed point, a channel between the bisector line and the map is obtained. Figures 2.9.a-c describe the process.

For a tangent bifurcation, the map becomes tangent to the diagonal at critically. Therefore, the eigenvalue associated to the critical point is equal to +1. However, for a non differentiable point there is not a tangent because at this point the map is nondifferentiable or discontinuous. Then, for a nondifferentiable point a tangent bifurcation can not appear. For some maps, we can obtain two slopes around the NPD, one for the left part of the map and other for the right one. From Figure 2.9 we can see that the right slope is less than +1, but the left one is high than +1. Then, the intermittency process and its main characteristics should depend on the “V” part in Figures 2.9.a-c. On the other hand, Figures 2.10.a-c show a limit case with infinite left derivative.

Refs. [55] and [56] study different maps showing type V intermittency. One of them is [55]:

$$F(x) = \begin{cases} F_1(x) = (x-a)^2 + b & \text{for } x \geq x_d \\ F_2(x) = A \text{ rnd}(x) + B & \text{for } x < x_d \end{cases} \quad (2.47)$$

where $\text{rnd}(x)$ is a random number from the interval $[0, 1]$, and x_d is the discontinuous point. The parameters A and B are selected to verify that the reinjection occurs below the intersection between the diagonal $x_{n+1} = x_n$ with the function $F_1(x)$. On the other hand, the parameters a and b regulate the variation of the function $F_1(x)$ with the control parameter $\varepsilon = F_1(x_d) - x_d$. As ε decreases, the value $F_1(x_d)$ moves toward the bisector line $x_{n+1} = x_n$. For $\varepsilon = 0$, x_d is a fixed point: $F_1(x_d) = x_d$. The slope of $F_1(x)$ for $x \rightarrow x_d$ is $dF_1(x)/dx|_{x \rightarrow x_d} = s$, which is considered independent of the control parameter ε . The dependence of a and b on the slope s and ε is given by:

$$\begin{aligned} a &= \frac{4\varepsilon - s}{2} \\ b &= \varepsilon - \frac{s^2}{4} \end{aligned} \quad (2.48)$$

Fig. 2.9 Type V intermittency.

For $s \neq 0$ and $s \neq 1$, with $\varepsilon \rightarrow 0$, and with random reinjection in an interval around a point, the average laminar length $\bar{l}$ can be written as:

$$\bar{l} = \frac{\log(\varepsilon)}{\log(s)} + \beta(s) \quad (2.49)$$

where $\beta(s)$ does not depend on ε .

For $s = 1$:

$$\bar{l} = \frac{a}{\varepsilon} + \beta(s) \quad (2.50)$$

Finally, for $s = 0$:

$$\bar{l} = a \log[-\log(\varepsilon)] + \beta(s) \quad (2.51)$$

Fig. 2.10 Type V intermittency: limit case.

In [56] the scaling law for two more general maps is analyzed. To define the first map, Figures 2.10.a-c can be considered. From these figures the right part of the map can be written as:

$$x_{n+1} = F_2(x_n) = x_n - a \sin(x_n) - \varepsilon \quad (2.52)$$

where we have considered the zero point of the axes in position o . The control parameter is $\varepsilon \geq 0$, and the parameter a verifies: $0 < a < 1$. When $\varepsilon = 0$, the point o becomes the critical point.

The laminar length results:

$$l = \left| \int_x^{x_{out}} \frac{dx}{a \sin(x) + \varepsilon} \right| = \frac{|\ln(\varepsilon)|}{a} + \Delta(x) \quad (2.53)$$

where x is the reinjected point, $x_{out} = \frac{\varepsilon}{1-a}$, and $\Delta(x)$ is:

$$\Delta(x) = \frac{1}{a} \ln \left(\frac{2a(1-a) \sin(x)}{1 + \cos(x)} \right) \quad (2.54)$$

Let us consider a random reinjection process with probability density $P(x)$ between the entrance at point c , and the exit point x_{out} . Accordingly, the average laminar length is:

$$\bar{l} = \int_{x_{out}}^c \Delta(x) P(x) dx + \frac{1}{a} |\ln(\varepsilon)| \quad (2.55)$$

The right part of the second map analyzed in [56] is:

$$x_{n+1} = F_2(x_n) = \zeta x_n - \delta x_n^2 - \varepsilon \quad (2.56)$$

This map is a generalization of Eq. (2.47). Following a similar calculation, the average laminar length results:

$$\bar{l} = \frac{|\ln(\varepsilon)|}{1-\zeta} + \int_0^c \frac{1}{1-\zeta} \ln \left(\frac{(1-\zeta)^2 \zeta x}{|x\delta + \zeta - 1|} \right) P(x) dx \quad (2.57)$$

Eqs. (2.49), (2.55) and (2.57) indicate that a logarithmic scaling law $\bar{l} = \bar{l}(\ln(\varepsilon))$ adjusts for type V intermittency.

Type V intermittency has been observed in real systems. Intermittent chaotic bursting in Hindmarsh-Rose (HR) model had been identified to display type V intermittency characteristics in a nonsmooth map. Near of the period-3 bursting, the intermittent chaotic behavior showed the alternation between phases close to the period-3 bursting and irregular bursting. The scale law of bursting was similar to those of type V intermittent chaos generated in a nonsmooth map [58]. In [59, 60] a exhaustive research of the intermittent chaotic bursting and spiking from theoretical models and biological experiments was carried out, and the difference between chaotic spiking and intermittent chaotic bursting was studied. The authors investigated and compared the intermittent chaotic spiking with smooth characteristics and the intermittent chaotic bursting with nonsmooth-like characteristics. The scale law of the averaged length of periodic phase (characteristic relation) for intermittent chaotic bursting was similar to those of type V intermittency generated in nonsmooth systems and type I intermittency simulated in smooth system, while the intermittent chaotic spiking was only classified as type I.

2.5 Classical theory about noise effects in chaotic intermittency

Since the noise is always present in nature, it is a fundamental importance to know the effect of noise. From the beginning there was a great effort to understand the noise role in the intermittency phenomenon.

There are several techniques to study the noise in the intermittent maps. The most popular are the renormalization group analysis and by using the Fokker-Plank equation. The mentioned techniques use very different methods but both techniques have in common that they are focused on the noise effect in the laminar region. It is clear the the noise affects the whole system, but the chaotic part of the system is considered by means of the RPD. Hence in most of the papers devoted to the noise effect in intermittency is considered only the laminar region.

Now, we describe the main result concerning with the noise in classical intermittency theory. In Section 2.6 the noise effects using the Fokker-Plank approach is considered. Section 2.7 introduces the Renormalization Group Theory applied to study the noise effects in chaotic intermittency.

2.6 Noise effects. Fokker-Plank approach

In this section the Fokker-Plank equation is used to analyze the noise effects in classical types of intermittency. Type I intermittency is analyzed in the next subsection. Type II and III intermitencies with noise are studied in Subsection 2.6.2.

2.6.1 Type I intermittency

In the noiseless case, Eq. (2.15) describes the local map for type I intermittency in the laminar region. For $|\varepsilon| \ll 1$ we can describe the local map by the continuous equation (2.17).

Now we introduce an additive noise to the local map as following:

$$x_{n+1} = F(x_n) = \varepsilon + x_n + a x_n^2 + \sqrt{2D}\xi_n \quad (2.58)$$

where D is the spatial diffusion and the noise ξ_n is taken white Gaussian of zero mean and delta-correlated.

The difference equation (2.58) can be approximated by the stochastic differential equation as follows [23]

$$\frac{dx}{dt} = -\frac{dV(x)}{dx} + \sqrt{2D}\xi(t) \quad (2.59)$$

where $\xi(t)$ is taken white Gaussian of zero mean and delta-correlated in time, that is $\langle \xi(t) \rangle = 0$ and $\langle \xi(t')\xi(t) \rangle = \delta(t' - t)$. The potential $V(x)$ is given by

$$V(x) = -\frac{1}{3}ax^3 - \epsilon x + c \quad (2.60)$$

Figure 2.11 represents the local map and the corresponding potential for values of $\epsilon < 0$. For $\epsilon > 0$ there is not fixed points in the map and it presents a narrow corridor of the laminar region which is typical in type I intermittency. However, for $\epsilon < 0$ the map presents two fixed points, that meet in a single one for $\epsilon = 0$. The stable and unstable fixed points of the map correspond to the extreme points of the potential, minimum for the stable point and maximum for the unstable one. The extreme points are given by

$$x_s = -\sqrt{-\epsilon/a} \quad \text{and} \quad x_u = \sqrt{-\epsilon/a} \quad (2.61)$$

Note that in the pure deterministic map, this scenario does not present intermittency because there is a stable fixed point. However, the noise perturbation can move the

Fig. 2.11 Local map of Eq. (2.58) for $\epsilon < 0$ and its associated potential Eq. (2.60). The potential barrier is also indicated.

system outside on the potential well and then the trajectory goes through the laminar region as in the usual intermittency case.

The stochastic differential equation (2.59) can be interpreted as the equation of motion of a zero mass particle in the potential $V(x)$ affected by an external random perturbation, hence by following the standard methodology used in the stochastic theory [29], we can obtain the corresponding backward Fokker-Plank equation (FPE).

$$\frac{\partial G(x,t)}{\partial t} = -\frac{dV(x)}{dx} \frac{\partial G(x,t)}{\partial x} + D \frac{\partial^2 G(x,t)}{\partial x^2} \quad (2.62)$$

In Eq. (2.62) $G(x,t)$ is the probability density of finding a particle at the position x in the time t . In this context, to investigate the average laminar length we must consider the mean first-passage time (MFPT) over the potential barrier ($V(x_s) - V(x_u)$), that is the mean time escaping time $T(x)$ defined by

$$T(x) = - \int_0^\infty \frac{\partial G(x,t)}{\partial t} t dt, \quad (2.63)$$

where the following boundary conditions, $G(x,0) = 1$ and $\lim_{t \rightarrow \infty} G(x,t) = 0$, that is, all particles are placed in x at $t = 0$ and no particle can be found at x for long time enough. By integrating Eq. (2.62) we get a differential equation for $T(x)$ as follow:

$$0 = 1 - \frac{dV(x)}{dx} T(x) + D \frac{d^2 T(x)}{dx^2} \quad (2.64)$$

The general solution of Eq. (2.64) can be written by means of integrals as follows:

$$T(x) = k \int_{\hat{x}}^x \exp\left(\frac{1}{D} V(x')\right) dx' - \frac{1}{D} \int_{\hat{x}}^x \int_{\hat{x}}^{x'} \exp\left(\frac{1}{D} (V(x') - V(x''))\right) dx'' dx' \quad (2.65)$$

Notice that in the previous equation, the second term is dominant due to the factor $\frac{1}{D}$. In general, the second term is not integrable analytically, hence we need to approach the potential by Taylor expansion at the extreme point x_s as follows:

$$V(x_s) \approx V(x_s) + \frac{1}{2} V''(x_s) (x - x_s)^2 + O((x - x_s)^3). \quad (2.66)$$

where V'' indicates the second derivative of the function $V(x)$. We can derive a similar expansion at the point x_u . With this procedure, the solution of Eq. (2.64) can be approximated by

$$T(x) \approx -\frac{1}{D} \exp\left(\frac{1}{D}(V(x_u) - V(x_s))\right) \times \int_{\hat{x}}^x \int_{\hat{x}}^{x'} \exp\left(\frac{1}{2D}(V''(x_u)(x' - x_u)^2 - V''(x_s)(x'' - x_s)^2)\right) dx'' dx' \quad (2.67)$$

The previous integral still can not be evaluated analytically, so it is necessary to introduce further approximation. In this context, we assume that $\hat{x} \ll x_s$ that is $\hat{x} \ll \sqrt{-\varepsilon}$, hence the reinjection takes place very close to $x = 0$, and finally we can get an estimation for MFPT in the following form:

$$T = \frac{2\pi}{\sqrt{V''(x_s)V''(x_u)}} \exp\left(\frac{-B}{D}\right) \quad (2.68)$$

where B is the potential barrier to spring over by the particle in the potential well. For the potential (2.60) we have (see Figure 2.11)

$$B = V(x_u) - V(x_s) = \frac{2a}{3} \left(\frac{|\varepsilon|}{a}\right)^{\frac{3}{2}} \quad (2.69)$$

and taking into account the Eq. (2.61), we get the approximate estimation for the MFPT, that is for $\bar{l}$ in the case of $\varepsilon < 0$ as

$$\bar{l} \propto T = \frac{\pi}{\sqrt{a|\varepsilon|}} \exp\left(\frac{4}{3D\sqrt{a}}|\varepsilon|^{\frac{3}{2}}\right) \quad (2.70)$$

Some comments can be pointed out about the previous equation. Note that as we approximate the potential by using the extreme points, the Eq. (2.70) can not be applied to the case $\varepsilon > 0$, where there are not extreme points. Therefore, the particle will cross the laminar region in a similar way than in the pure determinism case, because the map iterations will be dominant over the small noise perturbation. On the other hand, in the limit $D \rightarrow 0$ that corresponds to absence of random perturbation, from Eq. (2.70) we have $T \rightarrow \infty$, hence there is no escape at all for a particle trapped in the potential well.

The solution given by Eq. (2.70) is consistent with the result from the renormalization group analysis as we will see in Section 2.7.

To check the Eq. (2.70) numerically or experimentally, we take logarithm to get $\ln T \approx -1/2 \ln |\varepsilon| + 1/D |\varepsilon|^{3/2}$, hence for large $|\varepsilon| \gg 0$ the second term is dominant and we have $\ln T \approx 1/D |\varepsilon|^{3/2}$. This approximation has been observed numerically in the two coupled Rössler oscillators [29].

2.6.2 Type II and III intermittency

For type II and III intermittencies, the perturbed map by a random noise can be obtained from the noiseless map described for these types of intermittency in Chapter 2 as following:

$$x_{n+1} = \mp(1 + \varepsilon)x_n \mp a x_n^3 + \sqrt{2D}\xi_n \quad (2.71)$$

where for type II we must select the plus sign and the minus one for type III. As in the previous case, for very small values of $(x_{n+1} - x_n)$, the difference equation (2.71) can be approximated by Eq. (2.59) where the potential $V(x)$ is a symmetric function given by:

$$V(x) = -\frac{1}{4}ax^4 - \frac{\varepsilon}{2}x^2 + c \quad (2.72)$$

having two maxima and a single potential well at $(x = 0)$ as it is represented in Figure 2.12.

Hence we can apply the previous procedure. After applying the Eq. (2.68) and Eq. (2.69) to the potential Eq. (2.72) and taking into account that

Fig. 2.12 Local map of Eq. (2.71) for $\varepsilon < 0$ and its associated potential Eq. (2.72). The potential barrier is also indicated.

$$B = V\left(\sqrt{-\frac{\varepsilon}{a}}\right) - V(0) = \frac{\varepsilon^2}{4a} \quad (2.73)$$

the expression (2.67) transforms into the new one

$$\bar{l} \propto T = \frac{\sqrt{2}\pi}{|\varepsilon|} \exp\left(\frac{1}{3Da}|\varepsilon|^2\right) \quad (2.74)$$

The previous approximation for $\bar{l}$ has been observed in an electronic circuit [30].

Notice that the Eq. (2.68) is very similar to the Arrhenius formula for chemical reaction theory. As the approximation given by Eq. (2.68) depends only on the potential expression, it can be extended to other nonlinear terms in the local maps.

2.7 Renormalization Group and scaling theory

The Renormalization Group Theory (RGT) for deterministic chaos was firstly applied to the Feigenbaum cascade. Notice that the period-doubling context provides a natural structure to be renormalized.

A parallel method has been applied to the map having intermittency, or better speaking, it has been applied only to the laminar map. This is an important difference with respect to the period-doubling case, because in the intermittency case the general expression for the RPD is not considered enough in the RGT, in particular the roll playing by the LBR [61].

Let us consider a more general local map than the map (2.15) as following

$$x_{n+1} = F(x_n) = \varepsilon + x_n + a |x_n|^p + \sigma \xi_n \quad (2.75)$$

where we rewrite the diffusion term as $\sigma = \sqrt{2D}$.

Suppose that we have the usual case $p = 2$ and we map the point x' twice getting x'' , that is, $x'' = F^2(x') = F(F(x'))$. Let be $L(\varepsilon, \sigma)$ the number of iteration of the map F needed by x' to go through the whole laminar region. It is clear that if we used the map F^2 instead F , the number of iteration should be only a half. Suppose also that there is a new set of parameters ε' and σ' mapping a single iteration x' into x'' , hence we have

$$L(\varepsilon', \sigma') = \frac{1}{2}L(\varepsilon, \sigma) \quad (2.76)$$

The same argument can be applied to l iterations getting in this case

$$L(\varepsilon', \sigma') = \frac{1}{l}L(\varepsilon, \sigma) \quad (2.77)$$

where the relation between the parameter set (ε', σ') and (ε, σ) is unknown. It is convenient to write such relation by means of two new scaling exponents ρ and ν as following

$$\varepsilon' = l^\rho \varepsilon \quad \text{and} \quad \sigma' = l^\nu \sigma \quad (2.78)$$

Now the Eq. (2.77) can be rewritten in a new form

$$L(\varepsilon, \sigma) = lL(l^\rho \varepsilon, l^\nu \sigma) \quad (2.79)$$

If we multiply the previous equation by a common factor, we just re-scale the "length" of the laminar corridor, hence we can choose a new scale as following:

$$l^\rho \varepsilon = 1 \quad (2.80)$$

so the Eq. (2.79) becomes

$$L(\varepsilon, \sigma) = \varepsilon^{\frac{-1}{\rho}} Z\left(\frac{\sigma}{\varepsilon^{\frac{\nu}{\rho}}}\right) \quad (2.81)$$

where the new function Z is defined as $Z(\cdot) = L(1, \cdot)$.

Now the main task is to evaluate the exponents ρ and ν . This can be done by solving and linearizing the renormalization group equation. This is explained in the following subsection.

2.7.1 Exact solution for renormalization group equation

Before considering the noise perturbation we need to know the fixed point in the functional space. Following a similar argument used to develop the RG in the period doubling context, we postulated the existence of a function $g(x)$ satisfying the following equation

$$\frac{1}{\alpha} g(\alpha x) = g^l(x) \quad (2.82)$$

where α is the scaling factor affecting the function $g(x)$ after taking l steps in the laminar region.

The boundary conditions used in this case can be derived from the function $F(x)$ defined by Eq. (2.75). That is, for $\varepsilon = 0$, both functions must be similar at the origin, hence

$$g(0) = 0 \quad \text{and} \quad g'(0) = 1 \quad (2.83)$$

A solution of Eq. (2.82) with the restrictions (2.83) can be written in the form

$$g(x) = \frac{x}{\left(1 - dx^{\frac{1}{q}}\right)^q} \quad (2.84)$$

where d is an arbitrary constant and we have made the following identification

$$\alpha = l^q \quad (2.85)$$

The boundary conditions (2.83) determine a single solution, hence we must expand the general solution (2.84) at the origin and compare it with the map $F(x)$ of Eq. (2.58),

$$g(x) = x + qdx^{\frac{1}{q}+1} + \frac{q(q+1)}{2}d^2x^{\frac{2}{q}+1} + \dots \quad (2.86)$$

and by comparison with Eq. (2.58) we get

$$q = \frac{1}{p-1}. \quad (2.87)$$

Taking into account the relation (2.85), we obtain the scaling factor as a function of p :

$$\alpha = l^{\frac{1}{p-1}} \quad (2.88)$$

Now, the fixed point function for the renormalization equation (2.82) has been determined. The following step is to obtain the exponent ρ and ν . For this end, we can use the perturbation theory. In the case of the exponent ρ it is used a small deterministic perturbation of the fixed point, that is

$$g_n(x) = g(x) + h_n(x) \quad (2.89)$$

It is necessary to postulate a ratio of convergence, that is

$$h_n(x) = \lambda^n h(x) \quad (2.90)$$

Now by using Eq (2.82) it is defined an eigenvalue problem which is generated by imposing $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ and by linearization of Eq. (2.82). Finally, the eigenvalues obtained are [62]

$$\lambda = l^\rho, \quad \text{with} \quad \rho = \frac{p}{p-1} \quad (2.91)$$

Notice that the characteristic exponent β is defined for $\varepsilon \rightarrow 0$ as:

$$\bar{l} \propto \varepsilon^{-\beta}, \quad (2.92)$$

hence we have from Eq. (2.91)

$$\beta = -\frac{p-1}{p} \quad (2.93)$$

In a similar way, it is possible to introduce a noise perturbation to the fixed point $g(x)$

$$g_n(x) = g(x) + \xi(t)D(x) \quad (2.94)$$

where $D(x)$ is a small function. By following a similar method to the deterministic case the convergent ratio is given as

$$\lambda = l^v, \quad \text{with} \quad v = \frac{p+1}{p-1} \quad (2.95)$$

hence the Eq. (2.81) changes into the new one

$$L(\varepsilon, \sigma) = \varepsilon^{\frac{1-p}{p}} Z\left(\frac{\sigma}{\varepsilon^{\frac{p+1}{p}}}\right) \quad (2.96)$$

Notice that whereas the renormalization group has a clear meaning in the context of the Feigenbaum cascade, here the real meaning is a little different. As much as we know from Eq. (2.81) is that in the nonperturbed case ($\sigma = 0$), the function $L(\varepsilon, 0)$ determines how fast L increases as ε decreases and then, we must have

$$\beta = -\frac{p-1}{p} \quad (2.97)$$

But, for $\sigma \neq 0$ it is difficult to determine the characteristic exponent because the function Z in Eq. (2.96) remains unknown.

Note also that for the particular value $p = 2$, Eq. (2.96) takes the specific form

$$L(\varepsilon, \sigma) = \frac{1}{\sqrt{\varepsilon}} Z\left(\frac{\sigma}{\varepsilon^{\frac{3}{2}}}\right) \quad (2.98)$$

so it is consistent with Eq. (2.70). It is interesting to emphasize that Eq. (2.70) was derived for $\varepsilon < 0$ whereas this is not the case for Eq. (2.98). In fact for $\varepsilon \gg 0$ and $\sigma > 0$ the scaling properties of the intermittency do not follow the Eq. (2.98) and the deterministic component of the map is dominant over the small noise [29].

The cases of type II and type III present an additional difficulty because the Eq. (2.81) for $\sigma = 0$ has the following form [63]:

$$L(\varepsilon, x_0, c) = 2^{-n} L(\lambda^n \varepsilon, \alpha^n x_0, \alpha^n c) \quad (2.99)$$

where x_0 is the starting point in the laminar region and c determines the end of the laminar region. The important feature of the Eq. (2.99) is that the laminar region

present three characteristic exponents, that is $L(\lambda^n \varepsilon, \alpha^n x_o, \alpha^n c)$ remains constant under change of both $\alpha^n x_o$ and $\alpha^n c$. The Renormalization Group Theory can not determine whether this is true or not.

Finally, we remember that in this chapter devoted to the classical noise theory applied to intermittencies, it has only been considered the noise on the local map. In Chapter 8 the noise on the RPD will be considered. Even in the noiseless case, the Eq. (2.97) provides the correct value of β only for specific cases of the RPD. The general case will be explained in Chapter 3.

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Chapter 3

NEW FORMULATION OF THE CHAOTIC INTERMITTENCY. THE M FUNCTION METHODOLOGY

Abstract As we have seen in previous chapters, in the classical theory of intermittency the uniform density of points reinjected from the chaotic to laminar region is an usual hypothesis. In this chapter we reported on how the reinjection probability density (RPD) can be generalized. Estimation of the universal RPD is based on fitting a linear function to experimental data and it does not require a priori knowledge on the dynamical model behind. We provide special fitting procedure that enables robust estimation of the RPD from relatively short data sets. Thus, the method is applicable for a wide variety of data sets including numerical simulations and real-life experiments. Estimated RPD enables analytic evaluation of the length of the laminar phase of intermittent behaviors. The new characteristic exponent is developed, that now is not a single number but is a function depending on the whole map, not on the only the local region. In conclusion, a generalization of the classical intermittency theory is present. Also, this chapter is devoted to introduce the noise effects in the new theory of intermittency. A method to obtain theoretical expressions for the noisy reinjection probability density function is developed, which basically consists in extending the $M(x)$ function technique to derive the noiseless reinjection probability density. Moreover, this methodology allows to study the noise effect on the intermittency statistical properties for relatively large noise strengths. We also derive analytical approximations for other statistical variables, such as the probability density of the laminar lengths and the average laminar length. This methodology can describe the statistical properties of the noiseless system using the noisy data. Also, we show that occasionally the presence of noise could be not detected due to the results that behave as they would be corresponding to a noiseless system.

3.1 Introduction

As we have seen in previous chapters, different approximations have been used in literature to describe the RPD function. An interesting and more used approach was to consider PRD uniform and thus independent of the reinjection point (see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9]). For uniform reinjection, analytical expressions for the probability density of the length of laminar phase, $\psi(l)$, and also the characteristic exponent, β , were obtained as we have shown in Chapter 2. Other different investigations assumed a limit approximation and it is considered that the reinjection occurs in a fixed point [6, 10].

From early times evidences was found on dynamical system having intermittency which statistical properties departing from the classical result obtained for uniform reinjection. Some of this cases have been analyzed in [11]. Also was reported experimental Poincaré map displaying a hold around the fixed point that can be related with a no uniform reinjection [2, 12, 13].

This demands a generalization of the intermittency theory looking for a new RPD in order to include in a single theory all of the particular cases that are no well explained by the classical intermittency theory.

This chapter is organized as following: In Section 3.2 an introduction on the reinjection probability density function is presented. Section 3.3 describes the M function methodology to evaluate the new RPD. Sections 3.4, 3.5 and 3.6 are devoted to type II, III and I intermittencies respectively, we extend the classical results on intermittencies by applying the M function methodology. Sections 3.7-3.11 analyze the noise effects on intermittency using the new methodology.

3.2 The reinjection probability density function (RPD)

Let us consider the scenario described in Chapter 2, where

$$x_{n+1} = F(x_n, \varepsilon), \quad F : \mathbf{R} \rightarrow \mathbf{R} \quad (3.1)$$

is a general one-dimensional map which exhibits intermittency. Where $\mathbf{R}$ is the 1-dimensional space, and ε is a controlling parameter ($\varepsilon \ll 1$).

First of all, let us generalize the local dynamics corresponding to the three types of intermittencies around the unstable fixed point. The local laminar dynamics of type I intermittency is determined by the Poincaré map in the form:

$$x_{n+1} = \varepsilon + x_n + a x_n^p \quad \text{type I} \quad (3.2)$$

where $a > 0$ accounts for the weight of the nonlinear component. The laminar behaviors of type II and type III intermittencies develop around the fixed point of generalized Poincaré maps are:

$$x_{n+1} = (1 + \varepsilon)x_n + a x_n^p \quad \text{type II} \quad (3.3)$$

$$x_{n+1} = -(1 + \varepsilon)x_n - a x_n^p \quad \text{type III} \quad (3.4)$$

For $\varepsilon \gtrsim 0$, the fixed point $x_0 = 0$ becomes unstable, and hence trajectories slowly escape from the origin preserving and reversing orientation for type II and type III intermittencies, respectively.

Note that the nonlinear component in Eq. (3.2) is not necessary quadratic, (i.e. the parameter p can be $p \neq 2$) as it happens in Eq. (2.15) of Chapter 2. In the same way, for type II and type III, it can be $p \neq 3$ in Eq. (3.3) and Eq. (3.4), whereas for the corresponding classical local maps for this types of intermittencies (see Eq. (2.39) and Eq. (2.28) in Chapter 2) it is assumed $p = 3$. In any case, for $\varepsilon > 0$, there is an unstable fixed point at $x = 0$ for type II and type III and there is not a fixed point at $x = 0$ for type I. Consequently, the trajectories move slowly along the narrow channel formed with the bisecting line as illustrates Figure 2.5.

To fix ideas, let us introduce an illustrating model having type II intermittency

$$x_{n+1} = F(x_n) \equiv \begin{cases} F_1(x_n), & x_n < x_r \\ F_2(x_n), & x_n \geq x_r \end{cases} \quad (3.5)$$

where

$$F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p \quad (3.6)$$

and x_r is the root of the equation $F_1(x_r) = 1$ (see Figure 3.1). Note that F_1 drives the laminar dynamics whereas F_2 drives the reinjection mechanism from the chaotic region into the laminar region as it is represented in Figure 3.1 by a green arrow. The reinjection probability density, that is the RPD function, determines the statistical distribution of the reinjected trajectories and it depends on the particular shape of $F(x)$ in the no linear region, that is $F_2(x)$.

Concerning with the laminar region, the origin $x = 0$ is always a fixed point of F_1 , and for small values of ε it is stable for $\varepsilon < 0$. On the contrary, it is unstable for $\varepsilon > 0$, and the iterated points x_n of a starting point x_1 closed to the origin, increases in a process driven by parameters ε and p . When x_n becomes larger than x_r , a chaotic burst occurs that will be interrupted when x_n is again mapped into the laminar region by means of $F_2(x)$ which determines the RPD.

In the seminar paper [3] uniform reinjection was reported for the map (3.5)-(3.6) with $F_2(x_n) = F_1(x_n) - 1$ and $p = 2$, whereas in [14] was proposed the following generalization of the reinjection mechanism

$$F_2(x_n) = (F_1(x_n) - 1)^\gamma, \quad (3.7)$$

where for $\gamma = 1$ the original map is recovered. However, for $\gamma \neq 1$ it is interesting to estimate the behavior of $\phi(x)$ in a neighborhood of $x = 0$. Note that when γ increases, the number of points that will be mapped around the unstable fixed point $x = 0$ also increase. Consequently we expect that the classical hypothesis of uniform RPD used to develop the classical intermittency theory of the Chapter 2, can not be applied here.

Note that all point reinjected in the laminar region which is defined by the interval $[0, c]$, are coming from the points closed to x_r , as the green arrow shows in Figure 3.1.

Fig. 3.1 Top subplot: Maps defined by Eqs. (3.5) and (3.6) having type II intermittency. Three reinjected mechanisms (Eq. (3.7)) are also indicated for different values of γ . Bottom subplot: RPDs of Eq. (3.19) defined in the laminar interval $[0, c]$. The RPDs have the same color that the region where they are generated. See the colored region of the map in the top subplot.

In this thesis, x_r is called pre-reinjection point. The pre-reinjection points govern –have strong influence in– the reinjection process.

The RPD function is the most important one to determine the intermittent behavior of the system, but before embarking on a discussion concerning RPD, we describe the relationship between the RPD and the probability measure

The probability measure of an interval $S \subset [0, 1]$ is given by (see Chapter 7):

$$\mu(S) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N I_S(x_n) \quad (3.8)$$

where $I_S(x)$ denotes the characteristic function of the interval defined as

$$I_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases} \quad (3.9)$$

Thus the probability measure indicates the frequency of the signal in the corresponding interval of the attractor, and it is related with the density $\rho(x)$ by

$$\mu(S) = \int_S \rho(x) dx \quad (3.10)$$

It is clear that in our context $\mu([0, 1]) = 1$. Let L be the laminar region and s an interval in the laminar region $s \subset L$. Before to consider the probability $\mu(s)$ given by definition (3.8), let us to split the whole data series into three subset

$$\{x_n\} = \{x_{n'}\} \cup \{x_{n''}\} \cup \{x_{n'''}\} \quad (3.11)$$

having empty intersection between them

$$\{x_{n'}\} \cap \{x_{n''}\} = \{x_{n''}\} \cap \{x_{n'''}\} = \{x_{n'''}\} \cap \{x_{n'}\} = \emptyset \quad (3.12)$$

with the following definitions. Firstly, we chose $x_{n'} \in s$ and in the preceding period it has already been there, that is $I_s(x_{n'}) = 1$ and $I_s(x_{n'-1}) = 1$. For the next one we have $x_{n''} \in s$ but in the preceding period it has not been there, that is $I_s(x_{n''}) = 1$ and $I_s(x_{n''-1}) = 0$. Finally, $x_{n'''} \notin s$.

Now it is clear that $\mu(s)$ is given by

$$\mu(s) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N I_s(x_{n'}) + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N I_s(x_{n''}) + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N I_s(x_{n'''}) \quad (3.13)$$

where, in fact, the last term is not necessary because it is zero. The first term in Eq. (3.13) denotes the probability for the signal to be in s when in the preceding period it has already been there. Let us consider $s = L$, then, the second term in the RHS of

Eq. (3.13) defines the RPD, denoted here by $\phi(x)$, by means of the following relation

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^N I_L(x_{n''}) = w \int_L \phi(x) dx \quad (3.14)$$

where the weight w is introduced because it is usual to normalize the function $\phi(x)$ over the whole laminar interval L as

$$\int_L \phi(x) dx = 1 \quad (3.15)$$

We note that $\phi(x)$ determines the fundamental characteristics of intermittency, as for instance, the probability density of the length of the laminar phase, denoted here by $\psi(l)$, and the characteristic exponent β , used to identify the intermittency type as it will be defined below.

Going back on the reinjection mechanism (3.7), for $\gamma = 1$ the original map is recovered [3]. However, for $\gamma \neq 1$ it is interesting to estimate some characteristic of $\phi(x)$ in a neighbor of $x = 0$. Note that all points reinjected in the laminar region which is defined by the interval $[0, c]$, are coming from the points close to x_r , as the green arrow shows in Figure 3.1. That is, for $x' \gtrsim x_r$, all points in the interval $(x', x' + dx')$ are directly mapped into the interval $(F_2(x'), F_2(x' + dx')) \approx (x, x + F_2'(x') dx')$, where $x = F_2(x')$, hence the probability to find a point in $(x', x' + dx')$ is the same that to be reinjected into the interval $(x, x + F_2'(x') dx')$, where $F_2'(x)$ indicates the derivative of the function $F_2(x)$, hence we have:

$$k \rho(x') dx' = \phi(x) F_2'(x') dx' \quad (3.16)$$

The weight k is introduced because the $\rho(x)$ density is normalized on the whole interval $[0, 1]$, whereas $\phi(x)$ is normalized only on the laminar interval, that is, $\int_0^c \phi(\tau) d\tau = 1$. Finally, we can approximate $\phi(x)$ as follow

$$\phi(x) = \rho(x') \frac{k}{\left. \frac{dF_2(\tau)}{d\tau} \right|_{\tau=x'}}, \quad (3.17)$$

where k is a normalization constant such that $\int_0^c \phi(\tau) d\tau = 1$. For the expression (3.17) evaluated for the function given by Eq. (3.7) we have

$$\phi(x) = \frac{k \rho(x')}{\gamma F_1'(x')} x^{\frac{1}{\gamma}-1}, \quad (3.18)$$

where $F_1'(x)$ indicates the derivative of the function $F_1(x)$. In the linear approximation of $F_1(x)$ in the interval $(x_r, F_2^{-1}(c))$, we can consider $F_1'(x)$ as a constant. Now, if the density $\rho(x')$ is uniform, we get the follow reinjection probability density

$$\phi(x) = b x^\alpha \quad \text{where} \quad \alpha = \frac{1}{\gamma} - 1. \quad (3.19)$$

where b is a normalization constant given by Eq. (3.15). Note that the function PRD will strongly depend on parameter γ , that determines the curvature of the map in region marked by colored segments in Figure 3.1. Only the points in that region will be mapped inside of the laminar region. For a numerical verification of Eq. (3.19) see [14]. It is interesting to note that a power law like (3.19) is already verified in a wide class of one-dimensional maps even in some classical “pathological” cases that deviate significantly from the classical predictions as will be explained in [11].

Regarding with the classical hypothesis of uniform RPD, it holds for the map of Figure 3.1 only in the case of $\gamma = 1$, x_r^+ has non zero and finite derivative. On the other hand, for $\gamma \neq 1$, x_r^+ has zero or infinite derivative and the RPD is not constant – it is given by Eq. (3.19). We notice that the hypothesis of uniform reinjection does not work in general, but it usually works for $\rho(x')$ when it is generated in no extreme points as it is indicated by the green arrow in Figure 3.1.

To consider the effect of a $\text{LBR} \neq 0$ we propose the following modification of the map (3.5) having type I intermittency

Fig. 3.2 Map (3.20) having type I intermittency. There are indicated two LBR corresponding with two reinjected mechanisms in Eq. (3.20). The red arrow indicates the corridor followed by the trajectories in the laminar zone.

$$x_{n+1} = G(x_n) = \begin{cases} \varepsilon + x_n + ax_n^p & \text{if } x_n < x_r \\ (1 - \hat{x}) \left(\frac{x_n - x_r}{1 - x_r} \right)^\gamma + \hat{x} & \text{otherwise} \end{cases} \quad (3.20)$$

where x_r is the root of the equation $\varepsilon + x_n + x_n^p = 1$. The parameter γ , as in the previous example, drives the nonlinear term of the reinjection mechanism. The parameter $\hat{x}$ corresponds with the so called lower boundary reinjection point (LBR) and it indicates the limit value for the reinjection from the chaotic region into the laminar one as it is illustrated in Figure 3.2. Recently it has been demonstrated that the statistical behavior of the intermittency can be strongly affected by the value of the LBR [5, 15, 16, 17].

Note that in the map shown in Figure 3.1 the LBR is $\hat{x} = 0$. In the case of Eq. (3.20) where $\hat{x} \neq 0$, a similar argument proposed by type II intermittency provides the same power law, but now including $\hat{x}$ as following:

$$\phi(x) = b (x - \hat{x})^\alpha \quad (3.21)$$

The RPD (3.21) has been observed in a number of maps having intermittencies of the type I, II and III [14, 18, 19]. Note that the RPD (3.21) includes the classical approach as the particular case $\alpha = 0$. The free parameters $\hat{x}$ and α are determined by the dynamics in the chaotic region. The parameter $\hat{x}$ corresponds with the lower boundary of reinjection (LBR). The exponent α is generated by the trajectories within chaotic regime in a vicinity of a point in the map with infinite or zero tangent. Finally, b is a normalization constant.

3.3 The function $M(x)$

The RPD function determines the statistical distribution of trajectories leaving chaotic region going back into the laminar region. The key point to solve the problem of model-fitting is to introduce the following integral characteristic:

$$M(x) = \begin{cases} \frac{\int_{x_t}^x \tau \phi(\tau) d\tau}{\int_{x_t}^x \phi(\tau) d\tau} & \text{if } \int_{x_t}^x \phi(\tau) d\tau \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.22)$$

where x_t is some “starting” point. The function $M(x)$ was used for the first time in the intermittency context in [12] and it has recently been broadly extended as we are going to explain.

As $M(x)$ is an integral characteristic, its numerical estimation is more robust than direct evaluation of $\phi(x)$. This allows reducing statistical fluctuations even for a relatively small data set or data with high level of noise [14, 18, 20].

3.3.1 Fitting linear model to experimental or numerical data

To approximate numerically $M(x)$, we notice that it is an average over reinjection points in the interval $[x_i, x]$, hence we can write

$$M(x) = M_j \equiv \frac{1}{j} \sum_{k=1}^j x_k, \quad x_{j-1} < x \leq x_j \quad (3.23)$$

where the data set (N reinjection points) $\{x_j\}_{j=1}^N$ has been previously ordered, i.e. $x_j \leq x_{j+1}$. For the RPD given by Eq. (3.21), the definition (3.22) provides the function $M(x)$ follows the linear law

$$M(x) = \begin{cases} m(x - \hat{x}) + \hat{x} & \text{if } x \geq \hat{x} \\ 0 & \text{otherwise} \end{cases} \quad (3.24)$$

where $m \in (0, 1)$ is a free parameter and $\hat{x}$ can be approximated by $\hat{x} \approx \inf\{x_j\}$. Then using Eqs. (3.22) and (3.24) we obtain the corresponding RPD:

$$\phi(x) = b(\alpha)(x - \hat{x})^\alpha, \quad \text{with} \quad \alpha = \frac{2m - 1}{1 - m} \quad (3.25)$$

where $b(\alpha)$ is a normalization constant chosen to satisfy $\int_L \phi(x) dx = 1$, where L denotes the laminar interval. At this point, we note that by a simple least squares fitting we can estimate the parameter of the linear approximation (3.24) that determines the RPD given by Eq. (3.25).

The lower part of Figure (3.1) shows different RPD for different exponent α . The red lines correspond to $\gamma < 1$, the black and blue lines correspond to $\gamma = 1$ and $\gamma > 1$ respectively.

Fig. 3.3 RPD for different values of α showing decreasing and non-decreasing functions. It is sketched the corresponding slope for the function $M(x)$. Dashed line represents the limit value $m = 1$.

On the other hand Figure (3.3) displays different RPD depending on the exponent α for $\hat{x} = 0$ and $c = 0.5$. It is also shown how the parameter α depends on the slope m according with Eq. (3.25). For $m = 1/2$ we recover the most common approach with uniform RPD, i.e. $\phi(x) = \text{cst}$, widely considered in the literature. For $m < 1/2$ we have $\alpha < 0$ and the RPD increases without bound for $x \rightarrow \hat{x}$ as it is shown in Figure (3.3). In the opposite case $m > 1/2$ we have $\phi(\hat{x}) = \phi(0) = 0$. In this last case, the two possibilities for the RPD, concave or convex are separated by the slope $m = 2/3$ (see Figure 3.3).

We highlight that the RPD function (3.25) possess two limit cases:

$$\phi_0(x) = \lim_{m \rightarrow 0} \phi(x) \rightarrow \delta(x - \hat{x}) \quad (3.26)$$

$$\phi_1(x) = \lim_{m \rightarrow 1} \phi(x) \rightarrow \delta(x - c) \quad (3.27)$$

Note that $b(\alpha) \rightarrow 0$ in these cases.

3.3.2 How to deal with short data sets

For relatively big data sets (thousands of points), Eq. (3.23) provides faithful description of the RPD. However, for small data sets (usually available in experiments) it may lead to a bias in estimation of the parameters because the standard least squares fitting using Eqs. (3.23) and (3.24) tends to underestimate the value of m .

To avoid this problem was published a modification of the standard fitting as follow [20]. Let $\{x_j\}_{j=1}^N$ be an available properly ordered (say, $x_j \leq x_{j+1}$) small data set of experimental points. Assuming that m_{exc} and $\hat{x}_{\text{exc}}$ are the exact values of m and $\hat{x}$, in general unknown, we can evaluate the error provided by Eqs. (3.23) and (3.24). For the first data we have:

$$E_1 = (1 - m_{\text{exc}})(\hat{x}_{\text{exc}} - x_1) \quad (3.28)$$

and in general, for $j = 2, 3, \dots, N$ the error will be

$$E_j \equiv M(x_j) - M_j = E_1 + \sum_{k=1}^{j-1} \left(\frac{k}{j} - m_{\text{exc}} \right) (x_k - x_{k+1}) \quad (3.29)$$

where M_j are provided by Eq. (3.23) applied over the data set $\{x_j\}_{j=1}^N$. We note that the straight line (3.24) intersects the bisector line in the point $(\hat{x}_{\text{exc}}, M(\hat{x}_{\text{exc}}))$. Thus E_1 quantifies the deviation of the first data point from this value. Since $x_1 \leq \hat{x}_{\text{exc}}$ for any data set, the error at the first data point is always nonnegative, $E_1 \geq 0$, which leads to a

systematic error. Moreover, E_1 propagates to the other errors, which causes significant bias in the least squares fitting of small data sets.

Since $E_1 \propto (\hat{x}_{\text{exc}} - x_1)$, to reduce the effect of E_1 , the data set must have a point close to $\hat{x}_{\text{exc}}$. This is usually the case for intermittencies with $0 < m < 1/2$, because then $\lim_{x \rightarrow \hat{x}_{\text{exc}}} \phi(x) = \infty$, i.e. the probability to find $x_1 \approx x_{\text{exc}}$ is high enough.

In view of the above mentioned we modify the fitting procedure. The main idea on how to reduce the bias is to introduce an new point, z , into the data set. This new point satisfies $z > x_1$. Then we determine z in such a way that the newly obtained values of M_j would not have significant bias.

Before proceed, we introduce the following notation. Given two vectors $u, v \in \mathbb{R}^N$ we define their mean and covariance:

$$S_{uv} = \frac{1}{N} \sum_{j=1}^N (u_j - \bar{u})(v_j - \bar{v}) \quad (3.30)$$

where

$$\bar{u} = \frac{1}{N} \sum_{j=1}^N u_j, \quad \bar{v} = \frac{1}{N} \sum_{j=1}^N v_j. \quad (3.31)$$

By using Eq. (3.30) we define the variances of u and v as S_{uu} and S_{vv} are the respectively.

Let us now introduce three vectors $w, h, y \in \mathbb{R}^N$:

$$w_j = \frac{1}{j+1}, \quad h_j = jM_j w_j, \quad y_j = h_j + z w_j \quad (3.32)$$

The vectors w, h , and y define weights, weighted values of M_j , and ordinates of new data points, respectively. At this point we can apply the standard least squares fitting to $\{(x_j, y_j(z))\}_{j=1}^N$, which gives ($y = mx + p$):

$$m(z) = \frac{S_{xy}}{S_{xx}}, \quad p(z) = \bar{y} - \bar{x} \frac{S_{xy}}{S_{xx}} \quad (3.33)$$

After tedious algebraic manipulations we reach to the variance of residuals:

$$S_{rr}(z) = S_{ww}z^2 + 2S_{wh}z + S_{hh} - \frac{(S_{xh} + zS_{xw})^2}{S_{xx}} \quad (3.34)$$

we then select z by minimizing $S_{rr}(z)$:

$$z = \frac{S_{xh}S_{xw} - S_{wh}S_{xx}}{S_{xx}S_{ww} - S_{xw}^2} \quad (3.35)$$

Finally, the optimal value of m is given by

$$m_{\text{opt}} = \frac{S_{xh} + zS_{xw}}{S_{xx}} \quad (3.36)$$

where z is given by (3.35). The interested reader can find in Ref. [20] an application of this method to study a classical map with $m > 1/2$. In the following sections we shall show how the new RPD (3.25) produces quite different results from which we know from the classical theory. A numerical comparison between this method and the standard one for small number of data can be found in [20].

The duration of the trajectories in the laminar region depends on the RPD. Such duration is described by $\psi(l)$, that is, the duration probability density of the laminar phase, hence, the probability of finding a laminar phase of length between l and $l + dl$ is $dl\psi(l)$. The function $\psi(l)$ is useful to characterize the type of intermittency by comparing its analytical prediction with its numerical or experimental evaluation. In the following sections we explain how the RPD of Eq.(3.25) modifies the classical result explained in Chapter 2 about $\psi(l)$. The method used is similar for the three types of intermittencies studied here. However, whereas for type II and type III it is possible to find the analytical solution. On the contrary, for type I it is not possible in the general case.

3.4 Type II intermittency

3.4.1 Length of laminar phase

Firstly we study type II intermittency following [14, 18]. To do this, we introduce a continuous differential equation to approximate the dynamics of the local map (2.40) in the laminar region

$$\frac{dx}{dl} = \epsilon x + ax^p \quad (3.37)$$

where l approximates the number of iterations in the laminar region, i.e. the length of the laminar phase. Note that l depends on the reinjection point x , and from Eq. (3.37) it is given by

$$l(x, c) = \int_x^c \frac{d\tau}{a\tau^p + \epsilon\tau} \quad (3.38)$$

After integration it yields

$$l(x, c) = \frac{1}{\epsilon} \left[\ln \left(\frac{c}{x} \right) - \frac{1}{p-1} \ln \left(\frac{ac^{(p-1)} + \epsilon}{ax^{(p-1)} + \epsilon} \right) \right]. \quad (3.39)$$

We notice that Eqs. (3.37), (3.38) and (3.39) which describe the local behavior of the map in the laminar region are similar to Eqs. (2.40), (2.41) and (2.42), but now the local parameter p is included. Moreover, the probability density of the length of laminar phase $\psi(l)$, is also affected by the density $\phi(x)$, which is a global property as

$$\psi(l) = \phi(X(l)) \left| \frac{dX(l)}{dl} \right| = \phi(X(l)) |\varepsilon X(l) + aX(l)^p| \quad (3.40)$$

where $X(l)$ is the inverse function of $l(x, c)$ and we have used the Eq. (3.37). We notice that $\psi(l)$ depends on the RPD, which is an important difference with the classical theory, see for instance Eq. (2.43) for type II intermittency. Finally, by using the previous equations and after some algebraic manipulation we get

$$\begin{aligned} \psi(l) = & b \left(\left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}}\right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{1}{p-1}} - \hat{x} \right)^{\alpha} \times \\ & \left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}}\right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{p}{p-1}} \left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l} \end{aligned} \quad (3.41)$$

that for $\hat{x} = 0$ takes a more compact form

$$\psi(l) = b \left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}}\right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{p+\alpha}{p-1}} \left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l} \quad (3.42)$$

which depends on the global parameter α determined by the slope m of the linear function $M(x)$. The analytical expression for $\psi(l)$ given by Eq. (3.42) was firstly verified by numerical simulations in Ref. [14].

3.4.2 Characteristic relations

Now we consider the average laminar length $\bar{l}$ given by:

$$\bar{l} = \int_0^\infty l(x, c) \psi(l) dl = \int_{\hat{x}}^c l(x, c) \phi(x) dx. \quad (3.43)$$

where the integral in Eq. (3.43) depends on the LBR, hence we need to go through the different cases. Let us start our analysis with $\hat{x} = 0$. Then, from Eq. (3.25) and Eq.

(3.39) and taking into account Eq. (3.38) we get

$$\bar{l} = \lim_{x \rightarrow 0} \frac{b}{\alpha + 1} l(x, c) x^{\alpha+1} \Big|_x^c + \frac{b}{\alpha + 1} \int_{\hat{x}}^c \frac{x^\alpha}{\varepsilon + ax^{(p-1)}} dx \quad (3.44)$$

If $-1 < \alpha$, the limit in Eq. (3.44) is zero and the second term can be written as

$$\bar{l} = \frac{1}{ac^{\alpha+1}} \left(\frac{a}{\varepsilon} \right)^{\frac{p-\alpha-2}{p-1}} \left(\int_0^\infty \frac{y^\alpha}{1+y^{(p-1)}} dy - \int_{c_y}^\infty \frac{y^\alpha}{1+y^{(p-1)}} dy \right) \quad (3.45)$$

where $y = (a/\varepsilon)^{1/(p-1)}x$ and $c_y = (a/\varepsilon)^{1/(p-1)}c$. The second integral goes to zero as ε goes to zero, whereas the first one converges in the parameter region $-1 < \alpha < p-2$ or equivalent $0 < m < \frac{p-1}{p}$, so we get for small values of ε

$$\bar{l} \approx \frac{1}{ac^{\alpha+1}} \left(\frac{a}{\varepsilon} \right)^{\frac{p-\alpha-2}{p-1}} \frac{\pi}{p-1} \sin^{-1} \left(\frac{\pi(1+\alpha)}{p-1} \right) \quad (3.46)$$

Assuming that α remains constant as ε changes, the characteristic relation yields

$$\bar{l} \propto \varepsilon^{-\beta} \quad (3.47)$$

where the critical exponent β is given by:

$$\beta = \frac{p-\alpha-2}{p-1} = \frac{p(1-m)-1}{(p-1)(1-m)} \quad (3.48)$$

Note that now β does not take a single value as it happens in classical theory explained in Chapter 2. Now it depends on the behavior of the local map around the origin (parameter p), and also on the global dynamics of reinjection described by the parameter m .

As $-1 < \alpha < p-2$ and according with Eq. (3.48), the critical exponent β can take any value in the interval $(-1, 0)$, for any value of p . We expect that m (and then α) weakly depend on parameters ε and p , hence for several values of γ we evaluate $\bar{l}$ as ε changes. The results are shown in Figure (3.4) for different values of γ and p in equations (3.5), (3.6) and (3.7).

The particular value $\gamma = 1$ produces uniform reinjection in both cases, $p = 3$ and $p = 2$ with $m \approx 0.5$ (see dashed lines in Figures 3.4 and 3.5). Hence for $p = 3$ the map exhibits the classical characteristic relation reported for type II intermittency $\bar{l} \propto \varepsilon^{0.5}$ which is a particular case of Eq. (3.48) with $p = 3$.

For $\gamma = 1$ and $p = 2$, due to $m \approx 0.5$, we are out of the range of the application of Eq. (3.46) and also Eq. (3.48). Note, however that, as $\alpha = 0$ we can evaluate the integral in Eq. (3.44), getting

Fig. 3.4 Characteristic relation for the map (3.5)-(3.6) different values of the parameter γ in (3.7). Dots show numerical data and lines show the best fit straight. For lines B,C and D we used $P = 3$ and the values of γ are 1, 2 and 3 for lines B, C and D respectively. The value of parameters for line A are $P = 2$ and $\gamma = 1.5$. Reprinted from: *International Journal of Bifurcation and Chaos* **20**, Ezequiel del Rio and Sergio Elaskar, "New Characteristic relations in type II intermittency", 1185-1191 (2010), Copyright (2010) with permission from World Scientific Publishing Company.

Fig. 3.5 Characteristic relation for the map (3.5)-(3.6)-(3.7) with $\gamma = 1$ and $p = 2$. Dashed line fits points and its slope is in good agreement with Eq. (3.50). Reprinted from: *International Journal of Bifurcation and Chaos* **20**, Ezequiel del Rio and Sergio Elaskar, "New Characteristic relations in type II intermittency", 1185-1191 (2010), Copyright (2010) with permission from World Scientific Publishing Company.

$$\bar{l} = \frac{1}{ac} \ln \left(\frac{\varepsilon + ac}{\varepsilon} \right). \quad (3.49)$$

As a result we have $\beta = 0$ for small values of ε , so the asymptotic behavior of $\bar{l}$ is given by

$$\bar{l} \approx \frac{1}{c} (\ln c - \ln \varepsilon) \quad (3.50)$$

Figure 3.5 shows a numerical verification of Eq. (3.50).

3.5 Type III intermittency

A one dimensional map $F(x)$ that displays type III intermittency has a positive Schwartzian derivative (see Chapter 2):

$$SF(x) = \frac{F'''(x)}{F'(x)} - 1.5 \left(\frac{F''(x)}{F'(x)} \right)^2 > 0. \quad (3.51)$$

In Figure 3.6 a typical map exhibiting type III intermittency is represented,

$$x_{n+1} = F_{III}(x_n) = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n) \quad (3.52)$$

This map has two main differences with respect the previous one. Firstly, points in the neighbor of the extreme point –pre-reinjection point– need two map iterations to reach the laminar reinjection and secondly it has a symmetric reinjection mechanism into the laminar region.

To understand the reinjection mechanism, we highlight that a trajectory needs two iterations on the map to go from points close to $dF_{III}(x)/dx = 0$ until the laminar interval. Therefore, we can write

$$\rho(x') \propto \rho(x'') \frac{1}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x''}} \quad (3.53)$$

where x' and x'' need one and two iterations to reach the laminar region respectively. $\rho(x')$ and $\rho(x'')$ are the respective densities.

In a second step we connect $\rho(x')$ and $\phi(x)$ by

$$\phi(x) = \rho(x') \frac{C}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x'}} \quad (3.54)$$

where C is a suitable constant. As $\frac{dF(x')}{dx'}$ is non-zero and bounded, by means of the approximation

$$K \approx \frac{C}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x'}}. \quad (3.55)$$

From Eqs. (3.53) and (3.54) we can obtain the RPD, but now referring to $\rho(x'')$, that is, two steps back from the laminar region:

$$\phi(x) = K \frac{\rho(x'')}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x''}}. \quad (3.56)$$

Notice that, to be reinjected into the laminar region in two iterations, x'' must be close to x_m (see Figure 3.6). Now from Eq. (3.56) we can recover for the RPD the general power law (3.25).

Another important characteristic of the map of Figure 3.6 is that it has two symmetric reinjection mechanisms. Note that if $\hat{x} > 0$ we recover the RPD given by Eq. (3.25),

Fig. 3.6 Maps illustrating type III intermittencies. Also the reinjection mechanism is indicated into the laminar region by empty arrows. Dashed arrow illustrates the trajectory inside of laminar region, and x_m indicates the maximum of the map.

but in the case of $\hat{x} < 0$ the RPD must take into account the overlapping of the two symmetry reinjections and should be described by the following function:

$$\phi(x) = \begin{cases} b [(|\hat{x}| + x)^\alpha + (|\hat{x}| - x)^\alpha] & \text{if } |x| \leq |\hat{x}| \\ b (|\hat{x}| + x)^\alpha & \text{if } |\hat{x}| < x \leq c \\ b (|\hat{x}| - x)^\alpha & \text{if } -c < x \leq -|\hat{x}| \end{cases} \quad (3.57)$$

where $b > 0$ is again obtained by the standard normalization condition. For $\hat{x} < 0$ it is important to size that RPD given by Eq. (3.57) is specified by the two parameters α and $\hat{x}$, but now the function $M(x)$ is not linear. However, it is still useful to determine the RPD. In this way, we notice that $\phi(x)$ given by Eq. (3.57) is non-continuous for $x = |\hat{x}|$, hence $M(x)$ is non-differentiable at this point. Due to this fact, $\hat{x}$ must appear as a vertex point for $M(x)$ and then $\hat{x}$ can be directly determined. The next step will be to find α . To do this we evaluate $M(x)$ by means of the definition by Eq.(3.22). For $\phi(x)$ given by Eq. (3.57) we have for $0 < x < |\hat{x}|$

$$M(x) = \frac{1}{(2 + \alpha)} \left[(1 + \alpha)x - |\hat{x}| + 2 \frac{|\hat{x}| (|\hat{x}| - x)^{1+\alpha} - |\hat{x}|^{2+\alpha}}{(|\hat{x}| - x)^{1+\alpha} - (|\hat{x}| + x)^{1+\alpha}} \right] \quad (3.58)$$

which for the particular value $x = |\hat{x}|$ we have

$$M(|\hat{x}|) = \frac{\alpha + 2^{-\alpha}}{\alpha + 2} |\hat{x}| \quad (3.59)$$

from which α can be finally obtained.

3.5.1 Length of laminar phase

In Chapter 2 it has been shown that the length of laminar phase of type III is similar to the type II intermittency. It is clear because from Eq. (3.4) we have:

$$|x_{n+1}| = (1 + \varepsilon)|x_n| + a|x_n|^p \quad (3.60)$$

that gives the continuous differential equation

$$\frac{d|x|}{dl} = \varepsilon|x| + a|x|^3 \quad (3.61)$$

where l indicates the number of iterations in the laminar region. As the Eq. (3.61) is quite similar to Eq. (3.37), we recover all the results on type II intermittency, in particular the functions determining the exponent β .

3.6 Type I intermittency

In this section we describe the main results for type I intermittency.

3.6.1 Length of laminar phase

For type I intermittency, from Eq. (3.2) we get the continuous differential equation equivalent to Eq. (3.37) as following

$$\frac{dx}{dl} = \varepsilon + ax^p, \quad (3.62)$$

from which we obtain $l = l(x, c)$ as a function of x

$$l(x, c) = \frac{c}{\varepsilon} {}_2F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ac}{\varepsilon}\right) - \frac{x}{\varepsilon} {}_2F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ax}{\varepsilon}\right) \quad (3.63)$$

in terms of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ [21]. In the case of $p = 2$, $l(x, c)$ can be given by

$$l(x, c) = \frac{1}{\sqrt{a\varepsilon}} \left[\tan^{-1} \left(\sqrt{\frac{a}{\varepsilon}} c \right) - \tan^{-1} \left(\sqrt{\frac{a}{\varepsilon}} x \right) \right]. \quad (3.64)$$

In the case of type I intermittency, the Eq. (3.40) transforms into the follow

$$\psi(l) = \phi(X(l, c)) \left| \frac{dX(l, c)}{dl} \right| = \phi(X(l, c)) |aX(l, c)^p + \varepsilon| \quad (3.65)$$

where the explicit expression for $X(l, c)$ can be obtained only for a few cases. However, ψ can be plotted in all cases by using the parametrization suggested by Eq. (3.65)

$$(l(x, c), \psi^x(x)) = (l(x, c), \phi(x) |\varepsilon + ax^p|). \quad (3.66)$$

where we have taken the coordinate of the reinjected points x as the free parameter. ψ^x is the probability density of the laminar lengths as function of x . Let us describe the different shapes of $\psi(l)$ according with the values of α and $\hat{x}$. Taking into account that the maximum length of $l = l_{max}$ is given for $x = \hat{x}$, we can determine the value of the function $\psi(l_{max})$ by means of Eq.(3.66) as

$$\lim_{l \rightarrow l_{max}} \psi(l) = \lim_{x \rightarrow \hat{x}} \psi^x(x) = \begin{cases} 0 & \text{if } \alpha > 0 \\ b(\varepsilon + a\hat{x}^p) & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha < 0 \end{cases} \quad (3.67)$$

which depends on α . Concerning with the extreme points of the function $\psi(l)$, and considering Eqs. (3.62) and (3.65), such points should be the root of the follow equation:

$$\frac{d\psi(l)}{dl} = \left[(\varepsilon + ax^p) \frac{d\phi(x)}{dx} + ap\phi(x)x^{p-1} \right] \left| \frac{dX(l)}{dl} \right| = 0 \quad (3.68)$$

Since Eq. (3.62) imposes $dX(l)/dl \neq 0$ for $\varepsilon \neq 0$, the expression between the square brackets in (3.68) must be zero for $x \in (\hat{x}, c)$ and $\varepsilon \approx 0$, hence the roots can be approximated as

$$x_{r1} \approx 0 \quad \text{and} \quad x_{r2} \approx \frac{p\hat{x}}{\alpha + p}. \quad (3.69)$$

This way, the extreme points for the density $\psi(l)$ can occur at two points, $l(x_{r1}, c)$ and $l(x_{r2}, c)$, provided that x_{r1} and x_{r2} lie in $(\hat{x}, c)$.

Let us go through the three cases splitting in Eq. (3.67). The first case with $\alpha > 0$ can be split in two other ones. If $\hat{x} > 0$ we have $x_{r1} \notin (\hat{x}, c)$ and $x_{r2} \notin (\hat{x}, c)$ hence in this case the shape of $\psi(l)$ is like Figure 3.7(a) shows.

Suppose that $\alpha > 0$ but now $\hat{x} < 0$, hence $x_{r1} \in (\hat{x}, c)$ and $x_{r2} \in (\hat{x}, c)$ and the function $\psi(l)$ has two local extreme points, a minimum at x_{r1} and a maximum at x_{r2} as it is shown in Figure 3.7(b).

In the third case of Eq. (3.67) we have $\alpha < 0$. Note that now $\psi(l) \rightarrow \infty$ as $l \rightarrow l_{max}$ we have only an extreme point that corresponds with x_{r2} for $\hat{x} > 0$ and x_{r1} for $\hat{x} < 0$, as it is shown in Figure 3.7(c) and Figure 3.7(d) respectively.

Finally we consider the case $\alpha = 0$, corresponding with the uniform reinjection. In this case the exact roots $x_{r1} = 0$ and $x_{r2} = \hat{x}$ hold for any value of ε . Observe that x_{r2} lies on the lower endpoints of the laminar interval, whereas x_{r1} yields to a minimum only if $\hat{x} < 0$.

The Table 3.6.1 shows all possibilities matching this restriction for both roots. Notice that in second line of the table 3.6.1 we have $\psi(l_{max}) = 0$, but now $\psi(l)$ has a local maximum, a remarkable characteristic that is not given by the classical theory on type I intermittency.

It is interesting to observe that if $\alpha > 0$ we have $\psi(l_{max}) = 0$ and the graph for this function is very different from the obtained for the classical $\psi(l)$ that can be seen in [1, 22], for instance. The reader can find all possible shapes for the $\psi(l)$ in Ref. [19].

The slope of $\psi(l)$ at the final interval point is determined by the factor $(x - \hat{x})^{(\alpha-1)}$, thus, we have

$$\lim_{l \rightarrow l_{max}} \frac{d\psi(l)}{dl} = \begin{cases} \infty & \text{if } \alpha < 1 \\ (\varepsilon + a\hat{x}^p)^2 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases} \quad (3.70)$$

Table 3.1 Classification of the $\psi(l)$ local extreme types, minimum (min) or maximum (MAX), at $l(x_{r1}, c)$ and $l(x_{r2}, c)$, according to α and $\hat{x}$ values in the RPD. The limits $\lim_{l \rightarrow l_{max}} \psi(l)$, depending on α , are also given. Typical shapes for each case are plotted in subfigures in Figure 3.7 as it is specified in the last column. Reprinted from: *Communications in Nonlinear Science and Numerical Simulation* **19**, Ezequiel del Rio, Sergio Elaskar and J. M. Donoso, “Laminar length and characteristic relation in Type I intermittency”, 967-976 (2014), Copyright (2013) with permission from Elsevier.

	$l(x_{r1}, c)$	$l(x_{r2}, c)$	$\lim_{l \rightarrow l_{max}} \psi(l)$	Subfigure
$\alpha > 0 \ \hat{x} > 0$	$\nexists$	$\nexists$	0	a
$\alpha > 0 \ \hat{x} < 0$	min	MAX	0	b
$\alpha < 0 \ \hat{x} > 0$	$\nexists$	min	∞	c
$\alpha < 0 \ \hat{x} < 0$	min	$\nexists$	∞	d
$\alpha = 0 \ \hat{x} < 0$	min	$\nexists$	$\frac{\varepsilon + a\hat{x}^p}{ \hat{x} - c }$	e
$\alpha = 0 \ \hat{x} > 0$	$\nexists$	$\nexists$	$\frac{\varepsilon + a\hat{x}^p}{ \hat{x} - c }$	f

3.6.2 Characteristic relations

Let us describe how the characteristic exponent is affected by the RPD of Eq. (3.25). As the exponent β is defined by the characteristic relation Eq. (3.47), we need to evaluate the mean value of $\bar{l}$ defined by Eq. (3.43).

Taking into account the function ψ , depending on the parameter $\hat{x}$ and α , as it has been described before, a similar analysis that had been carried out for type II intermittency provides the characteristic exponent β for type I intermittency, that again is not a single value, and it is given according with the following cases:

- Case A: $\hat{x} \approx x_0$

- A1: $m \in (0, 1 - \frac{1}{p})$ or equivalent $\alpha \in (-1, p - 2)$. Equations (3.25), (3.65) and (3.43) give:

$$\beta = \frac{p - \alpha - 2}{p} = 1 - \frac{1}{(1 - m)p}. \quad (3.71)$$

Particularly $\lim_{m \rightarrow 0} \beta = 1 - \frac{1}{p}$ and $\lim_{m \rightarrow 1 - \frac{1}{p}} \beta = 0$.

- A2: $m \in [1 - \frac{1}{p}, 1)$ or equivalent $p - 2 < \alpha$. Now we have:

$$\beta = 0 \quad (3.72)$$

- Case B: $\hat{x} > x_0$.

$$\beta = 0 \quad (3.73)$$

Fig. 3.7 Different $\psi(l)$ profiles from Eq. (3.66) as a function of $\hat{x}$ and α . The figures correspond to the parameters of table 3.6.1. The numerical values selected to display the figures are: a) $\hat{x} = 0.1$, $\alpha = 0.2$, b) $\hat{x} = -0.4$, $\alpha = 0.3$, c) $\hat{x} = 0.1$, $\alpha = -0.6$, d) $\hat{x} = -0.4$, $\alpha = -0.3$, e) $\hat{x} = -0.3$, $\alpha = 0$ and f) $\hat{x} = 0.1$, $\alpha = 0$. In all cases $p = 2$, $a = 1$, $\varepsilon = 0.001$ and $c = 0.5$. Reprinted from: *Communications in Nonlinear Science and Numerical Simulation* **19**, Ezequiel del Rio, Sergio Elaskar and J. M. Donoso, “Laminar length and characteristic relation in Type I intermittency”, 967-976 (2014), Copyright (2013) with permission from Elsevier.

- Case C: $\hat{x} < x_0$.

- C1: $\alpha > 0$

$$\beta = \frac{p-2}{p}. \quad (3.74)$$

- C2: $\alpha < 0$

$$\beta = \frac{p-1}{p}. \quad (3.75)$$

3.7 New formulation of the noise effects in chaotic intermittency

In the following sections we study the noise influence on the statistical properties of intermittency using the new theory. It is clear that noise affects the whole dynamical system, therefore the RPD function will be perturbed too. We note that in the previous studies about noise, only the local effect was assumed and the noise strength σ is much smaller than the control parameter ε , as we have seen in Chapter 2.

Since the noise is always present in nature, it is of a fundamental importance to know the effect of noise on the intermittency phenomenon. There are many papers devoted to study such effect, by means of the renormalisation group analysis or by using the Fokker-Plank equation as we have seen in Chapter 2. In those approaches it is assumed that the noise strength σ is much smaller than ε , ($\sigma \ll \varepsilon$). In this chapter we consider a situation where this assumption should not be necessary.

Many researches devoted to the noise on the local Poincaré map have been published so far, but in spite of the importance of the RPD in the classical intermittency theory, there are no studies focused on the effect of noise on the RPD as far as the authors know.

This fact may occur because the RPD was considered as uniform, hence it remains uniform under the effect of a wide class of noise. As a consequence it was not necessary to include the noise effect on the RPD. On one hand, it is clear that noise affects the whole region where the system dynamics takes place. On the other hand, as we have explained in [4], the RPD can not be considered as uniform in general. In view of this, a natural demand is an investigation of the effect of noise on the chaotic region and the changes that it produces on the RPD. In this chapter we address this demand and we develop an analytical approach to the noise reinjection probability density (NRPD). We organize the following sections starting with the Section 3.8 where the noisy reinjection probability density (NRPD) function is developed. The effect of this NRPD on type II, type III and type I intermittencies are studied in Sections 3.9, 3.10 and 3.11 respectively.

3.8 Noisy reinjection probability density function (NRPD)

In this section, instead of only studying the perturbed local map, we consider the whole perturbed map. Let us introduce an illustrating example having type II intermittency as

$$x'_{n+1} = \begin{cases} F(x_n) + \sigma \xi_n & x_n \leq x_r \\ (F(x_n) - 1)^\gamma + \sigma \xi_n & x_n > x_r, \end{cases} \quad (3.76)$$

To get an analytical expression for the NRPD, denoted here by capital case $\Phi(x)$ whereas we reserve the lower case $\phi(x)$ for the noiseless RPD, we analyze the effect of noise on the reinjection trajectories, as it is sketched in Figure 3.8. In such a figure, a noiseless trajectory represented by a dashed line, is perturbed by noise. As a consequence of this, the reinjection point must end inside of an interval represented in Figure 3.8 by l_0 . That is, the noiseless density $\phi'(x)$ should be transformed into a new density $\Phi(x)$ according to the convolution

where $G(x, \sigma)$ is the probability density of the noise term $\sigma \xi_n$ in Eq. (3.76).

Fig. 3.8 Map shown in top subplot of Figure 3.1. The dashed arrow indicates the noiseless trajectory going into the laminar region. The noisy trajectory should be shifted to end inside the interval l_0 .

Note that whereas the function $\phi'(x)$ is in general unknown, we have information on $\Phi(x)$ by means of the function $M(x)$. It is clear that the noise is present in the whole map and then the density $\rho(x')$ can change. We remember that to develop the power law reinjection in [4], $\rho(x')$ was considered constant, hence it will remain constant under the effects of a large class of noises. The constant approximation is motivated because, as it is shown in Figure 3.8, the density $\rho(x')$ is generated in region without extreme points, hence in Eq. (3.77) we can assume $\phi'(x) \approx \phi(x)$ given by Eq. (3.21).

Moreover, expanding $\phi(x)$ at to the first term, the Eq. (3.77) becomes as the following one:

$$\Phi(x) \approx \int \phi(x) G(x-y, \sigma) dy + \int \frac{d\phi(x)}{dx} (x-y) G(x-y, \sigma) dy. \quad (3.78)$$

Taking into account that $\int G(x, \sigma) dx = 1$, we have

$$\Phi(x) \approx \phi(x) + \frac{d\phi(x)}{dx} \int (x-y) G(x-y, \sigma) dy \quad (3.79)$$

so we expect $\Phi(x) \approx \phi(x)$ in regions where the slope of $\phi(x)$ is small enough.

A consequence of the previous argument is that, for regions with small $\Phi(x)$, that is far from the LBR $\hat{x}$, the slope m of the noisy function $M(x)$ approaches the corresponding noiseless slope. This is an important tool to determine the density $\phi(x)$ corresponding to noiseless behavior, that we need to evaluate the integral (3.77). In the following sections we deal with some relevant cases of noisy intermittencies.

Concerning with length of laminar period in the noisy intermittency, we can still use Eq. (3.40) to determine duration probability density of the laminar phase, but now we replace the function $\phi(x)$ by the noisy case, that is $\Phi(x)$. We notice that this method takes into account only the noise effect on the RPD and ignores the noise effect on the laminar phase. However, it provides a good approximation as it has been confirmed by numerical analysis and also by experimental evidences.

3.9 Noise effect on type II intermittency

In this section we describe how the reinjection probability density function can be perturbed by the noise. Let us consider the map (3.76) that maps the interval $[0, 1]$ into itself. However, because of the noise, the value of x'_{n+1} may be mapped out of the unit interval, so to keep x'_{n+1} in the unit interval we define the following map

$$x_{n+1} = \begin{cases} |x'_{n+1}| & x'_{n+1} \leq 1 \\ |x'_{n+1}| - 2 \bmod(|x'_{n+1}|, 1) & x'_{n+1} > 1. \end{cases} \quad (3.80)$$

Note that $x_{n+1} = x'_{n+1}$ for $x'_{n+1} \in [0, 1]$.

In the case of the map (3.76) we have $\hat{x} = 0$, hence $\phi'(x) \approx \phi(x) = b|x|^\alpha$ in Eq. (3.77). To get an analytical expression for the integral (3.77) it is necessary to know the noise distribution present in the system. To get some general qualitative conclusions, let us assume that the noise source is a random variable ξ in the interval $[-1, 1]$, hence its probability density G in Eq. (3.77) is given by

$$G(x, \sigma) = \frac{\Theta(x + \sigma) - \Theta(x - \sigma)}{2\sigma}, \quad (3.81)$$

where $\Theta(x)$ is the Heaviside step function. Finally, after integrating Eq. (3.77), we get the NRPD as

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + \sigma)^{1+\alpha} - Sg(|x| - \sigma)|x| - \sigma|^{1+\alpha}}{2\sigma}, \quad (3.82)$$

where we denote by $Sg(x)$ the sign function that extracts the sign from its argument. The expression (3.82) has been verified numerically with a one-dimensional map [23]. Figure 3.9 shows the general noisy scenario described by Eq. (3.82). The scheme shown in Figure 3.9 is general for type I, II and III intermittencies but the Eq. (3.82) can change depending on the boundary conditions of the maps. For instance, to derive the Eq. (3.82) we have taken into account the Eq. (3.80). A similar equation to Eq. (3.82) is obtained for symmetric reinjection. Figure 3.9 displays the cases where the $\hat{x}$ is not affected by the noise. Different scenarios will be described in the context of type I intermittency. We also notice that different expressions should be obtained by considering different noise distributions in Eq. (3.77). However, from Eq. (3.82) we can get some general results concerning with the transformation of the RPD into NRPD due to the noise, as it is shown in Figure 3.9.

The middle sub-figure shows the typical noisy function $M(x)$ split in two cases separated by the line with $m = 0.5$ that corresponds with the uniform reinjection. Let us describe the case with $m < 0.5$ where we can distinguish two regions separated by $x = \chi$. For $\chi < x$, that is far from $\hat{x}$, the value of m for the noisy function $M(x)$ approaches to the corresponding value for the noiseless case, hence the functions $M(x)$ corresponding to the noiseless and noisy cases should be almost parallel. Mathematically it is clear, that is, if $\sigma \ll x$, Eq. (3.82) approaches the following one

$$\Phi(x) \approx \frac{1}{c^{1+\alpha}} \frac{d(|x| + \sigma)^{1+\alpha}}{dx} = \phi(x) \quad (3.83)$$

This region is very important to identify the parameter α of $\phi(x)$ by means of Eq. (3.25). In Figure 3.9, the down sub-figure shows the reinjection probability densities for the noiseless and noisy cases according with equations (3.21) and (3.82)

respectively. Note that for $\chi > x$ both functions approach each other. We notice that the uniform reinjection looks robust against the noise. This is because the noise effect on the RPD is not enough considered in the classical intermittency theory.

A numerical simulation for the map Eqs. (3.76) and (3.80) was presented in [23] where the scenario that was previously described can be identified, as the Figure 3.10 shows.

A different scenario appears for $x < \chi$ because now we have $\sigma \gg x$, hence, in the limit $x \rightarrow 0$, from Eq. (3.82) we obtain:

$$\frac{d\Phi(x)}{dx} \approx 0, \quad (3.84)$$

Fig. 3.9 Upper and lower sub-figures show the RPD in dashed line and the NRPD in solid line. The middle sub-figure sketches the different cases for the noisy $M(x)$ (solid lines) compared with the corresponding noiseless case. Arrows link the regions with different slopes of $M(x)$ with the corresponding region of the NRPD.

that is, the noisy slope m approaching to $1/2$ like in the uniform reinjection and the NRPD departs from the RPD as it is shown in the down sub-figure 3.9 and also can be identified in the numerical simulation displayed in Figure 3.10. Regarding to the case with $m > 0.5$, the scheme is quite similar to the $m < 0.5$ case as it is indicated in the upper sub-figure 3.9. In particular, both equations (3.83) and (3.84) are hold.

Numerical evaluation of the NRPD for the map Eqs. (3.76) and (3.80) are presented in Figure 3.11. In that figure the analytical expressions given by Eq. (3.82) are also plotted. The parameters α and $\hat{x}$ used in Eq. (3.82) are coming from the fit square analysis of the $M(x)$ plotted in Figure 3.10. We call singular points to those points where the behavior of $M(x)$ changes, e.g., x_s . More complete information can be found in Ref. [23].

3.10 Noise effect on type III Intermittency

For typical one-dimensional maps with type III intermittency the noise effect on the RPD can be stronger than for the map with type II intermittency. To understand this, we consider the following map formed by composition of the following noiseless map:

Fig. 3.10 Function $M(x)$ for the map Eqs. (3.76) and (3.80). The dashed line with slope $1/2$ shows the uniform reinjection case. The lines above the dashed one correspond to $\gamma = 0.65$ for two values of the noise strength as indicated. The same values of noise strength are used for the two lines below the dashed one, that correspond to $\gamma = 2$. For all the cases $\varepsilon = 0.001$. The solid lines show the corresponding least mean square fit, very close to the numerical simulations indicated by dots. Reprinted from: *Communications in Nonlinear Science and Numerical Simulation* **17**, Ezequiel del Rio, Miguel A.F. Sanjuán and Sergio Elaskar, "Effect of noise on the reinjection probability density in intermittency", 3587-3596 (2012), Copyright (2012) with permission from Elsevier.

$$x'_n = F(x_n) = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n) \quad (3.85)$$

and the random perturbed map defined as:

$$x_{n+1} = x'_n + \sigma \xi_n, \quad (3.86)$$

where we just add a noise to the variable.

In Figure 3.12, is sketched by a dashed line the noiseless trajectory of a point starting near the maximum of the map. Because of the noise, this trajectory may spread over a region of some width, say l_0 . Note that l_0 will be rescaled by a suitable factor K increasing the length up to $l_1 = Kl_0$ on the graph of the map.

Let us consider the noise effect on a point x_m lying close to the maximum of the map Eq. (3.85) as it is indicated in Figure 3.12. As in the type II intermittency, if $\rho'(x)$ is the invariant density of the map Eq. (3.85), the noise effect on this density is modeled by the map Eq. (3.86) and can be obtained according to the convolution

$$\rho(x) = \int \rho'(\tau) G(\tau - x, \sigma) d\tau, \quad (3.87)$$

where $\rho(x)$ is the invariant density observed in the interval l_0 . On the contrary of type II case, now points on l_0 are not directly mapped on the laminar region, so to get the NRPD we must propagate the density $\rho(x)$ by the map Eq. (3.85) as following

$$\rho_I(x) = \frac{dF^{-1}(x)}{dx} \rho(F^{-1}(x)). \quad (3.88)$$

Fig. 3.11 NRPD for the map Eq. (3.76-3.80) using two sets of parameters: a) $c = 0.1$, $\gamma = 0.65$, $\sigma = 0.03$ and b) $c = 0.1$, $\gamma = 2$ and $\sigma = 0.01$. Dots correspond to numerical data whereas the analytical approximation given by (3.82) is plotted as a solid line. Reprinted from: *Communications in Nonlinear Science and Numerical Simulation* **17**, Ezequiel del Rio, Miguel A.F. Sanjuán and Sergio Elaskar, "Effect of noise on the reinjection probability density in intermittency", 3587-3596 (2012), Copyright (2012) with permission from Elsevier.

Finally, to include the noise effect into the density $\rho_I(x)$, we must propagate the density $\rho_I(x)$ as we did in (3.87), that is, the NRPD is given by

$$\Phi(x) = \int \rho_I(y) G(y-x, \sigma) dy. \quad (3.89)$$

By using Eq. (3.87) and Eq. (3.88), we get

$$\Phi(x) = \int \int \frac{dF^{-1}(y)}{dy} \rho'(\tau') G(\tau' - F^{-1}(y), \sigma) G(y-x, \sigma) dy d\tau' \quad (3.90)$$

and by rescaling the dummy variable $\tau' = F^{-1}(\tau)$ we can rewrite the last equation in a more convenient form to be compared with the noiseless case as

$$\Phi(x) = \iint \phi(\tau) G_F(\tau, y) G(y-x, \sigma) dy d\tau, \quad (3.91)$$

where we use the following definitions

$$\phi(x) = \frac{dF^{-1}(x)}{dx} \rho'(F^{-1}(x)) \quad (3.92)$$

Fig. 3.12 Map of Eq. (3.85). Dashed line between the two solid lines indicate the effect of the map on a point near the maximum. These solid lines indicate the effect of the noisy map on the same point, that will be mapped on the interval I on the graph of the map.

and

$$G_F(\tau, y) = \frac{dF^{-1}(y)}{dy} G(F^{-1}(\tau) - F^{-1}(y), \sigma). \quad (3.93)$$

Equation (3.92) shows the effect of the noiseless map on the density ρ' . By using the aforementioned hypotheses for type II intermittency, we can approximate ρ' by the same power law that in the noiseless map. Using the linear approximation $F(x) \approx F(x_{-1}) + K(x_{-1} - x)$, where $K = dF/dx$ corresponds to the mean slope of the curve I , the density ϕ in Eq. (3.92) can be approximated by a similar power law than ρ' [18], that is $\phi(x) \approx b|x|^\alpha$ even in the noisy case. By considering the previous linear approximation we get for the function $G_F(\tau, y)$

$$G_F(\tau, y) \approx G(\tau - y, K\sigma). \quad (3.94)$$

For large values of the slope K , a case shown in Figure 3.12, the density $G(y - x, \sigma)$ in the integral (3.91) is very narrow as compared with G_F provided by Eq. (3.94). This way, in this context we can introduce the approximation $G(y - x, \sigma) \approx \delta(y - x)$ in Eq. (3.91) to get the following expression for NRPD

$$\Phi(x) = \int \phi(\tau) G(\tau - x, K\sigma) d\tau, \quad (3.95)$$

that is similar to Eq.(3.77) obtained to describe the noise in the type II intermittency with direct reinjection from the extreme point. Hence we can proceed in the same way that in the previous case,

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + K\sigma)^{1+\alpha} - Sg(|x| - K\sigma) |x| - K\sigma|^{1+\alpha}}{2K\sigma}. \quad (3.96)$$

It is interesting to note that the factor K in Eq. (3.95) produces an amplification of the noise effect as it appears also in Eq.(3.96) that is similar to Eq.(3.82). As a result of this, the whole analysis made previously in the context of type II intermittency and displayed in Figure 3.9 works also in this case. In particular, in spite of the factor K in Eq. (3.96), it is possible to predict the RPD from the noisy data, because the slope of $M(x)$ is robust against the noise, at least for x far from $\hat{x}$. This technique was used in [23] for the map (3.85). In Figure 3.13 it is shown the result of several predictions of the RPD for different values of the noise strength σ .

We notice that due to the factor K in Eq. (3.96) the noise effect can be strong enough to make significant differences between the slopes of $M(x)$ with and without noise. However, even in this case it is possible to estimate the function $M(x)$ for points $x > x_p$ far enough from $\hat{x}$, that is $x_p > \chi$. To do this, according with Eq. (3.77) we expect $\Phi(x) \approx \phi'(x)$ for $x_p \gg x_s$, so by using the definition of $M(x)$, (Eq. (3.22)) we have

$$M_n(x) \approx M(x) + \frac{\int_0^{x_p} \tau(\Phi(\tau) - \Phi(\tau)) d\tau}{\int_0^x \phi(\tau) d\tau} \quad \text{if } x > x_s, \quad (3.97)$$

where $M_n(x)$ and $M(x)$ are the noisy and noiseless functions respectively and we assume that the integral in the denominator does not depend on noise. With the approximation given by Eq. (3.23), we get for $x > x_p$

$$M(x_l) \approx M_n(x_q) - \frac{C}{q} \approx \frac{\sum_{j=1}^q x_j}{q} - \frac{C}{q}. \quad (3.98)$$

As $M(\hat{x}) = \hat{x}$ we can choose the value C to approximate $M(\hat{x}) \approx \hat{x}$ to recover the noiseless function $M(x)$. This means that even in this case with a very high value of noise strength, the value of α corresponding to the density $\phi(x)$ of the noiseless map is recovered. This technique has been used in a map with type III intermittency providing a good agreement with numerical simulations [23].

Fig. 3.13 RPD for the map Eq. (3.85). Dots correspond to numerical data. For clarity we have omitted the points on the right hand side. Solid lines correspond to Eq. (3.25) evaluated using the three values of m found using $\sigma = 0$, $\sigma = 0.02$ and $\sigma = 0.03$ where the three of them are almost coincident. The parameters are: $c = 0.6$, $a = 1.1$, $d = 1.35 \varepsilon = 10^4$. Reprinted from: *Communications in Nonlinear Science and Numerical Simulation* **17**, Ezequiel del Rio, Miguel A.F. Sanjuán and Sergio Elaskar, “Effect of noise on the reinjection probability density in intermittency”, 3587-3596 (2012), Copyright (2012) with permission from Elsevier.

3.11 Noise effect in type I intermittency

To illustrate the effect of noise in type I intermittency we have chosen a model with direct reinjection, as happened in the type II case,

$$F(x) = \begin{cases} F_1(x) = \varepsilon + x + ax^2 + \sigma \xi_n, & \text{if } x < x_r, \\ F_2(x) = \hat{x} + \frac{1 - \hat{x}}{(1 - x_r)^\gamma} (x - x_r)^\gamma + \sigma \xi_n & \text{if } x > x_r \end{cases} \quad (3.99)$$

where x_r verifies: $\varepsilon + x_r + ax_r^2 = 1$. The map (3.99) is represented in Figure 3.2 for $\sigma = 0$. Note that this figure is similar to Figure 3.8. However, now the boundary conditions are different and a noisy reinjected point can be mapped out of the unity range.

In this case, the integral (3.77) gives us the NRPD as

$$\Phi(x) = \frac{b}{2\sigma(\alpha + 1)} \left\{ [x - (\hat{x} - \sigma)]^{\alpha+1} - \Theta[x - (\hat{x} + \sigma)] [x - (\hat{x} + \sigma)]^{\alpha+1} \right\}. \quad (3.100)$$

where Θ is the Heaviside step function.

Let us compare the Eq. (3.100) with the corresponding equation for type II intermittency, that is, Eq. (3.82). Note that in Eq. (3.100) the position of the LBR is shifted to a new position given by $(\hat{x} - \sigma)$. On one hand, Eq. (3.83) works in both cases, hence far from $\hat{x}$ the noisy $M(x)$ should be parallel to the noiseless case, as it is represented in Figure 3.9. On the other, Eq. (3.84) works for Eq. (3.100) and now, for $x \rightarrow \hat{x}$ we have

Fig. 3.14 NRPD according with Eq. (3.100) with $\hat{x} = 0$. Dashed and solid lines correspond with RPD and NRPD respectively. Note that the $\Phi(x) \neq 0$ for $x < \hat{x}$ where $\phi(x) = 0$.

$$\Phi(x) \approx \frac{b}{2\sigma(\alpha+1)} [x - (\hat{x} - \sigma)]^{\alpha+1} \quad (3.101)$$

as in the noiseless RPD with exponent $\alpha + 1$ and the LBR shifted to $\hat{x} + \sigma$. As a consequence, the function $M(x)$ can still be approximated by a piecewise linear function but now the noisy slope m_n for $x \rightarrow \hat{x}$ will be not zero, as was the case shown in Figure 3.9. Now the new slope is given by Eq. (3.25) with $\alpha + 1$ hence we have

$$m_n = \frac{\alpha + 2}{\alpha + 3} \quad (3.102)$$

so the two slopes of the noisy function $M(x)$ are related by

$$m_n = \frac{1}{2 - m} \quad (3.103)$$

where m is the slope of the noiseless case. Note that, as $m > 0$ we have $m_n > 1/2$ and consequently for $x \rightarrow \hat{x}$ we have $\Phi(x) \rightarrow 0$ as it is displayed in Figure 3.14.

The relations (3.100) and (3.103) are confirmed in one-dimensional maps perturbed by noise [24].

It is interesting to mention a consequence of Eq. (3.100). As it is shown in Figure 3.14 the value of $\hat{x}$ and α are shifted by the noise, hence it is possible to select a set of noisy parameter values that correspond with set of noiseless values. This case is shown in Figure 3.15 where two values γ are selected in the map (3.99). As the noisy and noiseless values are related by $\alpha_n = \alpha + 1$, taking into account the Eq. (3.19), we get a relation for noisy γ_n and noiseless γ as following

$$\frac{1}{\gamma_n} = \frac{1}{\gamma} + 1 \quad (3.104)$$

By selecting the values of γ according with Eq. (3.104) we can get similar statistic behaviors in both cases as Figure 3.7(b) displays. Here, the parameter values obtained numerically are very similar for both cases, that is, the slope of the noisy $M(x)$ is $m_1 = 0.623$. Hence, from Eq. (3.25) we have $\alpha_1 = 0.652$ and for the noiseless case $m = 0.624$, so we have $\alpha = 0.660$.

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Fig. 3.15 $M(x)$, NRPD and probability density of laminar lengths for map (3.99) show a shape corresponding to the case of Figure 3.7(b). The values used are $\varepsilon = 10^{-4}$, and (a) Noisy case with $\gamma_n = 0.60$, $\hat{x} = 0$ and $\sigma = 0.05$ (b) Noiseless case with $\gamma = 1.50$, $\hat{x} = -0.05$ and $\sigma = 0$. Reprinted from: *Physica A* **402**, Gustavo Krause, Sergio Elaskar and Ezequiel del Rio, “Noise effect on statistical properties of type I intermittency”, 318–329 (2014), Copyright (2014) with permission from Elsevier.

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Chapter 4

SOME THEORETICAL CONSIDERATIONS ABOUT THE NEW THEORY

Abstract The classical theory of intermittency developed for return maps assumes uniform density of points reinjected from the chaotic to laminar region. In previous chapters, we have reported how the reinjection probability density (RPD) can be generalized. Estimation of the universal RPD is based on fitting a linear function to experimental or numerical data. Here we present analytical approaches to estimate the RPD. After this, we can get an analytic evaluation of the characteristic exponent traditionally used to characterize the intermittency type. The proposed theoretical methods are general and very simple to use. It is compared with numerical computation, showing a good agreement between both. Our analytical results are compared with some celebrated classical numerical results on intermittency theory.

4.1 Introduction

We have explained that intermittency is a particular route to the deterministic chaos characterized by spontaneous transitions between laminar and chaotic dynamics.

Proper qualitative and quantitative characterizations of intermittency based on experimental data are especially useful for studying problems with partial or complete lack of knowledge on exact governing equations, as it frequently happens e.g. in Economics, Biology, Medicine, etc. (see e.g. Refs. [1, 2]).

It is interesting to note that some of the cited references are devoted to systems having more than one dimension. In spite of this, they can be described by one dimensional map. This phenomenon is typical of systems that contract volume in phase space [3].

The local laminar dynamics of type I intermittency evolves in a narrow channel, whereas the laminar behaviors of type II and type III intermittencies develop around a fixed point of its generalized Poincaré map [4]. Another representative attribute of intermittency is the global reinjection mechanism that maps trajectories of the system

from the chaotic region back into the local laminar phase. This mechanism can be described by the corresponding reinjection probability density function (RPD). The RPD is determined by the chaotic dynamics of the system and drives the statistical properties of the system. Note that in the linear region, the invariant density of the map is formed by the RPD and by the density of trajectories on reinjected point going out of the linear region. The RPD contributes to the invariant density of the map in the linear region only with the reinjected points.

In previous chapters, we have explained that analytical expressions for the RPD are available for a few maps only, hence to describe main statistical properties of intermittency different approximations of the RPD have been employed. The most common classical approach uses the uniform RPD, which, however, works fine in a few model cases only [5, 6, 7]. Another approach for RPD deals with a δ -function limit as it considers reinjection into a given point in the presence of noise [8, 9, 10]. A more recent theory was introduced in Chapter 3; the M function methodology has shown to be robust to evaluate the RPD function with and without noise. In this chapter we propose a simple method to get an analytical approximation of the RPD, in good agreement with the numerical result given in the literature [11].

After this introduction, in Section 2 we briefly describe the state of the art and how the RPD is generated by the map. Section 3 is devoted to present the main result of this chapter, and we propose an analytical method to estimate the RPD and the characteristic exponent β . A comparison between numerical and analytical results are presented in Section 4. Finally, in Section 5 we summarize the main results.

4.2 Theoretical framework

First, let us briefly describe the theoretical framework that accounts for a wide class of dynamical systems exhibiting intermittency. We consider a general one-dimensional map

$$x_{n+1} = F(x_n), \quad F : \mathbf{R} \rightarrow \mathbf{R} \quad (4.1)$$

which exhibits intermittency, hence it presents two different behaviors according where the trajectory takes place: laminar and chaotic. Remember that $\mathbf{R}$ represents the real numbers.

The local laminar dynamics of type I intermittency is determined by the one-dimensional map in the form:

$$x_{n+1} = \varepsilon + x_n + a x_n^p \quad \text{type I} \quad (4.2)$$

where $a > 0$ accounts for the weight of the nonlinear component and ε is a control parameter ($\varepsilon \ll 1$).

On the other hand, for type II and type III intermittencies, the laminar behaviors around the fixed point are determined for the following local maps:

$$x_{n+1} = (1 + \varepsilon)x_n + ax_n^p \quad \text{type II} \quad (4.3)$$

$$x_{n+1} = -(1 + \varepsilon)x_n - ax_n^p \quad \text{type III} \quad (4.4)$$

Traditionally, the laminar interval L is delimited by a small constant c as following: $L = (0, c)$ for type I and type II and by the symmetric interval $L = (-c, c)$ in the case of type III.

In some pioneer papers devoted to type I and II intermittency, the nonlinear component is quadratic, (*i.e.* $p = 2$). In the same way, in the classical theory on type III was fixed $p = 3$ (see Chapter 2).

In all of the cases $\varepsilon \gtrsim 0$, the point $x_0 = 0$ becomes unstable or disappears, and hence trajectories slowly escape from the origin preserving orientation for type I and II and reversing orientation for type III intermittency.

Regarding to the nonlinear dynamics, the RPD describes the reinjection mechanism that maps back the trajectories from chaotic region of the map into the laminar region where the dynamics is determined by the local maps above referred. Notice that this mechanics is necessary to sustain the alternation between laminar and chaotic bursts. As we have explained previously, the RPD function is the most important one to determine the intermittent behavior of the system. Remember that the relationship between the RPD and the probability measure of an interval S was introduced in Chapter 3.

From the mathematical RPD shape for each case it is possible to analytically estimate the fundamental characteristic of the intermittency, the probability density of the length of laminar phase $\psi(l)$, depending on l , that approximates the number of iterations in the laminar region, *i.e.* the length of the laminar phase. Note that the function $\psi(l)$ can be estimated from time series, as it is usual to characterize the intermittency type. The characteristic exponent β , depending on $\psi(l)$ through the relation $\bar{l} \propto \varepsilon^{-\beta}$, is also a good indicator of the intermittency type.

To illustrate how the RPD determines $\psi(l)$ and β , let us consider the type II intermittency. For the other case the methodology is similar. To estimate $\psi(l)$, we take, as usual, the approximation of the particular map for the local dynamics by the differential equation [4]. In the case of the map (4.3) we have:

$$\frac{dx}{dl} = \varepsilon x + ax^p \quad (4.5)$$

where l approximates the number of iterations in the laminar region, i.e. the length of the laminar phase. Note that the Eq. (4.5) provides a good approximation when $x_{n+1} - x_n$ is small enough. This fact determines the limit of the laminar interval defined by the constant c .

Solving (4.5) for l we get

$$l = \frac{1}{\varepsilon} \ln \left[\ln \left(\frac{x}{a} \right) - \frac{1}{p-1} \ln \left(\frac{ac^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon} \right) \right] \quad (4.6)$$

Since x in (4.6) is a random variable described by $\phi(x)$, the statistics of l is also governed by the global properties of the RPD.

Let $\psi(l)$ be the probability density function of l , then it can be obtained by

$$\psi(l) = \phi(X(l)) \left| \frac{dX(l)}{dl} \right| \quad (4.7)$$

where $X(l)$ is the inverse function of (4.6), that is,

$$X(l) = \left[\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{p-1}}\right)e^{(p-1)\varepsilon l} - a} \right]^{\frac{1}{p-1}} \quad (4.8)$$

Therefore, by means of Eqs. (4.5), (4.7) and (4.8), the probability density function of the length of the laminar phase is given by

$$\psi(l) = \phi(X(l)) [\varepsilon X(l) + aX^p(l)] \quad (4.9)$$

Also, using (4.9), we can determine the mean value of l which is an important property to characterize the intermittency

$$\bar{l} = \int_0^\infty \tau \psi(\tau) d\tau \quad (4.10)$$

Notice that $\bar{l}$ is directly related to the characteristic exponent β , commonly used to characterize any type of intermittency, by means of the characteristic relation $\bar{l} \propto \varepsilon^{-\beta}$, which describes, for small values of ε , how the length of the laminar phase increases as ε decreases.

At this point, we mention that $\psi(l)$ and β depend on the RPD, hence the main task here is to determine the RPD.

To describe the reinjection mechanisms present in a wide class of dynamical systems exhibiting intermittency, the RPD is given by the following power law (see Chapter 3)

$$\phi(x) = b(x - \hat{x})^\alpha \quad (4.11)$$

where $\hat{x}$ is the lower boundary of reinjection (LBR).

The RPD (4.11) has been observed in a number of maps having intermittencies of the type I, II and III [12, 13, 14]. Note that the RPD (4.11) includes the classical approach as the particular case $\alpha = 0$. The free parameters $\hat{x}$ and α are determined by the dynamics in the chaotic region. The exponent α is generated by the trajectories within chaotic regime in a vicinity of a point in the map with infinite or zero tangent (pre-reinjection point), and b is a normalization constant.

We highlight that $\hat{x}$ can be obtained from the definition of the map, however, there is not a general method to get the parameter α from the expression of the map.

Regarding to the characteristic exponent, for $\hat{x} \approx 0$ in Eq. (4.11), β has been established as a function of α , as follow (see Chapter 3):

- In the type I case we have [14]

$$\beta = \frac{p - \alpha - 2}{p} \quad (4.12)$$

- For type II the expression for β is given by [12]

$$\beta = \frac{p - \alpha - 2}{p - 1} \quad (4.13)$$

- Finally, for type III, the characteristic exponent is also given by Eq. (4.13) [13].

4.2.1 Considerations about the RPD function

Here, we introduce some analytical features on the RPD, that can help us in the following section. To carry out this task we follow the explanation given in Chapter 3.

As we have described, the RPD function depends on the particular shape of $F(x)$. To fix ideas, let us introduce an illustrating model:

$$x_{n+1} = F(x_n) \equiv \begin{cases} F_1(x_n), & x_n < x_m \\ F_2(x_n), & x_n \geq x_m \end{cases} \quad (4.14)$$

where $F_1(x) = (1 + \varepsilon)x_n + (1 - \varepsilon)x_n^p$, and x_m is the root of the equation $F_1(x_m) = 1$ (see Figure 4.1). Note that F_1 drives the laminar dynamics whereas F_2 govern the reinjection mechanism from the chaotic region into the laminar zone as it is represented in Figure 4.1 by a green arrow.

In Ref. [5] was reported uniform reinjection for the map (4.14) with $F_2(x_n) = F_1(x_n) - 1$ and $p = 2$, whereas in Ref. [12] was proposed the generalization to the map $F_2(x_n) = (F_1(x_n) - 1)^\gamma$ giving the RPD of Eq. (4.11).

For $\gamma = 1$ the original map is recovered, however, for $\gamma \neq 1$ it is interesting to estimate some characteristic of $\phi(x)$ in a neighborhood of the lower boundary of reinjection, $\hat{x}$.

Without loss of generality for our purpose, we set $\hat{x} = 0$ in the map (4.14). The value of $\hat{x}$ is determined by $x_m = F^{-1}(x)$ where x_m is a pre-reinjection point. Note that all points reinjected in the laminar region which is defined by the interval $(0, c)$, are coming from the points close to x_m as the green arrow shows in Figure 4.1. That is, for $x' \gtrsim x_m$, all points in the interval $(x', x' + dx')$ are directly mapped into the interval $(F_2(x'), F_2(x' + dx')) \approx (x, x + F_2'(x')dx')$, where $x = F_2(x')$. Then, following Chapter 3, we can obtain the Eq. (3.17):

$$\phi(x) = \rho(x') \frac{k}{\left. \frac{dF_2(\tau)}{d\tau} \right|_{\tau=x'}} \quad (4.15)$$

Remember that the constant k account of $\rho(x)$ is normalized on the whole interval $[0, 1]$ whereas $\phi(x)$ is normalized only on the laminar interval, that is, $\int_0^c \phi(\tau) d\tau = 1$ (see Chapter 3).

In Ref. [12] the case $F_2(x) = (F_1(x) - 1)^\gamma$ was investigated. The expression (4.15) in this case gives:

$$\phi(x) = \frac{k \rho(x')}{\gamma F_1'(x')} x^{\frac{1}{\gamma} - 1}, \quad (4.16)$$

where F_1' indicates the derivative of the function F_1 . In the linear approximation of F_1 in the interval $(x_m, F_2^{-1}(c))$, we can consider F_1' as a constant. Now, if the density $\rho(x')$ is uniform, we get for the reinjection probability density the aforementioned power law $\phi(x) = bx^\alpha$ where

$$\alpha = \frac{1}{\gamma} - 1. \quad (4.17)$$

was verified in [12].

It is interesting to note that the power law $\phi(x) = bx^\alpha$ is already verified in a wide class of one-dimensional maps even in some classical 'pathological' cases that deviate significantly from the classical predictions [15]. Regarding with the classical hypothesis of uniform RPD, it holds for the map (4.14), only in the case of $\gamma = 1$, where x_m is not an extreme point, however it is false for $\gamma \neq 1$ where the RPD is given by Eq. (4.11). This means that whereas the hypothesis of uniform reinjection does not work in general, it usually works for $\phi(x)$ when it is generated in points with non-zero and finite derivative as is indicated by the black arrow in Figure 4.1.

In this scenario, where $\phi(x)$ is generated around the point x_m as Eq. (4.16) describes, the parameter γ determines $\phi(x)$ following Eq. (4.17). Note that whereas in this case

Fig. 4.1 Map of Eq. (4.14) for $p = 2$ with the bisecting line. The vertical dashed line indicates x_m . Cases of $F_2(x_n) = F_1(x_n) - 1$ and $F_2(x) = 1 - \cos\left(\pi \frac{F_1(x)-1}{2}\right)$ correspond to dashed and solid lines respectively. The green arrow indicate the corresponding reinjection mechanism into the laminar zone. The reinjection around x_m is indicated by the black arrow.

the parameter γ appears explicitly in the definition of the map, this is not the general case. In this way, a further generalization of the reinjection mechanism can be proposed as follow: $F_2(x_n) = f(F_1(x_n) - 1)$ with $f(0) = 0$ and $f(1) = 1$. In the next sections we develop a method to associate a value of γ to a general reinjection mechanism, hence we extend the application of Eq. (4.17) to practically any map.

Concerning with type III intermittency the scenario is more complicated as we explained in Chapter 3. To illustrate this point, let us to consider the map (3.52):

$$x_{n+1} = F_{III}(x_n) = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n) \quad \text{with } a > 0. \quad (4.18)$$

Figure 4.2 shows the map of Eq. (4.18) with the reinjection mechanism depending on the value of $F_{III}(x_m)$ at the extreme points x_m satisfying $dF_{III}(x)/dx = 0$. In spite of our odd map has two extreme points, for simplicity in Figure 4.2 is indicated only the maximum. As the number of iterations increases, any point x_n close to the origin goes away in a process driven by the parameters ε and a in the cubic term of the map. For large enough values of x_n , the influence of the third term in the right hand side of Eq. (4.18) increases and x_n approaches to the maximum x_m rendering the reinjection into the laminar zone indicated in Figure 4.2 by a dashed circle. Note that in this case, points in the neighborhood of x_m need two map iterations to be reinjected into the laminar region, however, this class of maps still has a RPD given by Eq. (4.11), hence

should be possible to obtain from the Eq. (4.18) a value of γ providing the exponent α by means of Eq. (4.17).

Fig. 4.2 Map (4.18) exhibiting type III intermittency. Thick arrow illustrates mapping of points from the chaotic region (around the maximum of $F_{III}(x)$) into the laminar region. The parameters used are $a = 1$, $\varepsilon = 0.01$ and $d = 1.1$.

In the next sections we propose an analytical method to approximate the parameter γ . We also apply the result to maps having type I, II and III intermittency. Ref. [16] provides more detailed information on this topic.

4.3 Estimation of the RPD

In this section, we propose two methods to estimate analytically the RPD by means of parameter γ . We call them: Alternative 1 and Alternative 2. Besides, we describe the RPD estimation when there is a composition of single maps.

4.3.1 Alternative 1

We start with a direct reinjection from the extreme point as the map (4.14) shows. Based on the argument of the preceding section, we approximate the map (4.14) around the extreme point x_m by:

$$F(x) \approx F(x_m) + w(x - x_m)^\gamma, \quad (4.19)$$

where w is a suitable constant. Hence the value of γ is given by next limit

$$\gamma = \lim_{x \rightarrow x_m} \frac{\ln(F(x) - F(x_m)) - \ln w}{\ln(x - x_m)} \quad (4.20)$$

To convert the limit (4.20) from indeterminate forms to an expression that can be evaluated, we can apply the L'Hôpital rule, but usually we get an indeterminate limit again. In the general case, for $x \rightarrow x_m$ we have $F^{(i)}(x) \rightarrow 0$ for $i \leq q$ where $F^{(i)}(x)$ denote the i -derivative of the function $F(x)$, so by using the L'Hôpital theorem $q + 1$ times we have

$$\gamma = q + \lim_{x \rightarrow x_m} F^{(q+1)}(x) \frac{x - x_m}{F^{(q)}(x)}. \quad (4.21)$$

Now, if the derivative $F^{(q+1)}$ exists, the limit (4.21) gives $\gamma = q + 1$ and according with Eq. (4.17) we have:

$$\alpha = -\frac{q}{q+1} = \frac{1}{\gamma} - 1 \quad (4.22)$$

It is interesting to note that the values of q in Eq. (4.22) are confined to odd natural number, a condition imposed because x_m is an extreme point. In this context, if $F'(x)$ is a polynomial having x_m as a root of multiplicity q , then the exponent α is given by Eq. (4.22).

The approximation (4.22) gives two natural limits. For $q = 0$ we recover the uniform reinjection and on the other hand, for $q \rightarrow \infty$ (all derivatives are zero), $\alpha \rightarrow -1$ and the RPD collapses into a δ -function.

4.3.2 Alternative 2

Other way to estimate the RPD function can be obtained using Taylor Series. To carry out this task, we approximate the map $F(x)$ around x_m as:

$$\begin{aligned} F(x) \approx & F(x_m) + (x - x_m)F'(x_m) + \frac{1}{2!}(x - x_m)^2 F''(x_m) + \frac{1}{3!}(x - x_m)^3 F'''(x_m) \\ & + \dots + \frac{1}{s!}(x - x_m)^s \frac{d^s F(x_m)}{dx^s} \end{aligned} \quad (4.23)$$

Remember that the derivatives must be continuous inside the interval $x - x_m$. On the other hand, as the distance $x - x_m \ll 1$, the lower order non-zero derivative drives the

series. Therefore, if $F^1(x_m) = \dots = F^{n-1}(x_m) = 0$ and $F^n(x_m) \neq 0$, this last derivative has a strong influence on the series. Then, we can write:

$$F(x) \approx F(x_m) + \frac{1}{n!}(x - x_m)^n \frac{d^n F(x_m)}{dx^n} \quad (4.24)$$

If we compare this last equation with Eq. (4.19), we obtain $\gamma = n$, and the constant $w = \frac{d^n F(x_m)}{dx^n} / (n!)$.

In both methods, Alternatives 1 and 2, the lower order non-zero derivative determines the exponent γ . Afterward, the RPD exponent, α , can be directly evaluated from it (see Eq. (4.17)).

4.3.3 Composition of single maps

Let us consider a map $F(x)$ defined by a composition of single maps $\{F_i\}$. See for instance [6, 17]. Even in the case of a not composed map as in Eq. (4.18), to reinject into laminar region a point lying in the vicinity of the maximum or minimum, there are necessary two iterations of the map, hence the point $x = F_{III}(F_{III}(x_m))$ lies in the laminar region. In this case, whereas the intermittency is referred to the map $x_{n+1} = F_{III}(x_n)$, the limit (4.20) must be referred to the functional composition of maps $x_{n+1} = F(x_n) \equiv F_{III} \circ F_{III}(x_n)$.

For a general view, we consider the map

$$x_{n+1} = F(x_n) \equiv F_i \circ F_{i-1} \circ \dots \circ F_1(x_n), \quad (4.25)$$

where the function $F_1(x)$ has an extreme point at x_m , that will be mapped into the laminar region by successive application of the single functions F_i forming the composed map (4.25).

We will demonstrate that even in this case, the RPD will be approximated by applying Eq. (4.21) just to the function $F_1(x_n)$, instead of applying it to the complete function $F(x)$

Theorem 4.1. *Let $F(x)$ be the function defined by Eq. (4.25). If for a positive integer q , $F_1^{(i)}(x_m) = 0$ with $i \leq q$ and $F_1^{(q+1)}(x_m) \neq 0$, where $F_1^{(i)}$ indicates the i -derivative of the function F_1 , the following equality holds:*

$$\lim_{x \rightarrow x_m} F^{(q+1)}(x) \frac{x - x_m}{F^{(q)}(x)} = \lim_{x \rightarrow x_m} F_1^{(q+1)}(x) \frac{x - x_m}{F_1^{(q)}(x)} \quad (4.26)$$

Proof. Let us prove the equality (4.26) for $q = 1$, that is

$$\lim_{x \rightarrow x_m} F''(x) \frac{x - x_m}{F'(x)} = \lim_{x \rightarrow x_m} F_1''(x) \frac{x - x_m}{F_1'(x)} \quad (4.27)$$

To prove Eq. (4.27) let us evaluate $F'(x)$ from its definition in Eq. (4.25),

$$F'(x) = \prod_{i=1}^q F_i' \circ G_{i-1} \quad (4.28)$$

where we define $G_0(x) \equiv x$ and $G_i(x) \equiv F_i \circ \dots \circ F_1(x)$. Note that $F'(x_m) = 0$ because $F_1'(x_m) = 0$, hence x_m is an extreme point also for $F(x)$. The second derivative is given by

$$F''(x) = \sum_{i=1}^q \left(F_j'' \circ G_{j-1} \prod_{i \neq j} F_i' \circ G_{i-1} \right) \quad (4.29)$$

To evaluate the limit (4.26) at $x = x_m$ we point out that all brackets in (4.29) with $i \neq 1$ are zero because $F_i'(x_m) = 0$ so we have

$$\lim_{x \rightarrow x_m} F''(x) \frac{x - x_m}{F'(x)} = \lim_{x \rightarrow x_m} F_1'' \prod_{i \neq 1} (F_i' \circ G_{i-1}) \frac{x - x_m}{\prod_{i=1}^q F_i' \circ G_{i-1}} \quad (4.30)$$

so after eliminating the common factor in the fraction we have the equation (4.27). In the case of $q > 1$ the Eq. (4.26) is provided by means of L'Hôpital rule applied to Eq. (4.27).

A direct consequence of Theorem 4.1 is that the estimation of γ in the case of a map defined like Eq. (4.25) is given by

$$\gamma = q + \lim_{x \rightarrow x_m} F_1^{(q+1)}(x) \frac{x - x_m}{F_1^{(q)}(x)}. \quad (4.31)$$

Note that the function F_1 in Eq. (4.31) is just F_{III} in the map (4.18), but in the general case the function F_1 can be quite different from the function used to define the map. Examples of this point will be shown in the next section.

Once γ is estimated, by means of Eq. (4.17) we can calculate the exponent α for the power law (4.11) which describes the RPD. In Section 4.5 we apply this result to one-dimensional map having intermittency.

Regarding to $\bar{l}$, by using Eq. (4.22) we get an analytical expression for β depending on q as follow:

In the type I case, Eq. (4.12) becomes

$$\beta = 1 - \frac{q+2}{p(q+1)} \quad (4.32)$$

and type II the expression (4.13) transforms into

$$\beta = 1 - \frac{1}{(p-1)(q+1)} \quad (4.33)$$

Finally, for type III, the characteristic exponent is given by Eq. (4.33) with $p = 3$ [13].

4.4 Maps with infinite derivative

The previous methods allow to calculate the exponent γ when the derivatives of the map at the pre-reinjection points are not infinite. Remember that, we call pre-reinjection points to those points that govern the reinjection process. In previous sections, the pre-reinjection points were extreme points.

In this section we extend these methods to evaluate the exponent γ when, at least, the first derivative at the pre-reinjection point tends to infinite.

Let us consider the map (4.14) with:

$$\left. \frac{dF_2(x)}{dx} \right|_{x=x_m} \longrightarrow \infty \quad (4.34)$$

For this map, the two described methods can not be directly used to determine the exponents γ and α . However, we can work with the inverse function of $F_2(x)$. This inverse function must verify:

$$\left. \frac{dF_2^{-1}(x)}{dx} \right|_{x=x_m} = 0 \quad (4.35)$$

and the origin of $F_2^{-1}(x)$ must be the pre-reinjection point.

We should note that by working with $F_2^{-1}(x)$ instead of $F_2(x)$, we have transformed a pre-reinjection point with infinite derivative into an extreme point. Therefore, once obtained the inverse function, $F_2^{-1}(x)$, we can apply the previously described methods to obtain the exponent γ , and later α .

For Alternative 1 method, we consider:

$$F_2^{-1}(x) \approx F_2^{-1}(x_m) + w(x - x_m)^\gamma, \quad (4.36)$$

From this equation, the exponent γ , for $F_2^{-1}(x)$, is calculated from Eq. (4.21). To obtain the exponent γ for $F_2(x)$, we must determine the inverse function of Eq. (4.36):

$$F_2(x) \approx x_m + \left(\frac{x}{w} \right)^{1/\gamma} \quad (4.37)$$

where γ is calculated for $F_2^{-1}(x)$. Note that to apply the last equation we must shift the origin of $F_2(x)$ to x_m .

Summarizing, if we can obtain the exponent γ for $F_2^{-1}(x)$, the exponent γ for $F_2(x)$ is its inverse.

For Alternative 2 method, we consider:

$$F_2^{-1}(x) \approx F_2^{-1}(x_m) + (x - x_m)(F_2^{-1}(x_m))' + \frac{1}{2!}(x - x_m)^2(F_2^{-1}(x_m))'' + \frac{1}{3!}(x - x_m)^3(F_2^{-1}(x_m))''' + \dots + \frac{1}{s!}(x - x_m)^s \frac{d^s F_2^{-1}(x_m)}{dx^s} \quad (4.38)$$

If all derivatives of $F_2^{-1}(x)$ with order lower than n are zero, and the lower order finite and non-zero derivative is n ; the term $\frac{1}{n!}(x - x_m)^n(F_2^{-1}(x_m))^{(n)}$ is the more important in the Taylor Series (4.38), and $\gamma = n$. The function $F_2^{-1}(x)$ can be approximate by:

$$F_2^{-1}(x) \approx F_2^{-1}(x_m) + \frac{1}{\gamma!}(x - x_m)^\gamma \frac{d^\gamma F_2^{-1}(x_m)}{dx^\gamma} \quad (4.39)$$

The $F_2(x)$ approximation can be obtained as the inverse of Eq. (4.39). In this way $F_2(x)$ results: $F_{2a}(x) \propto (x - x_m)^{1/\gamma}$

Note that, both methods (Alternative 1 and 2) produce the same result: the exponent γ for $F_2(x)$ is equal to $1/\gamma$, where γ is calculated for $F_2^{-1}(x)$.

4.5 Numerical results

We apply the proposed methods to one-dimensional maps having type I, II and III intermittency to approach their RPDs and the characteristic exponents β . To apply the new methods, we consider four maps.

4.5.1 Map1. Type II intermittency

Firstly we study the map (4.14) having type II intermittency. In the case of $F_2(x_n) = (F_1(x_n) - 1)^\gamma$ it is clear that value of the limit (4.21) is equal to the parameter γ and the expression (4.17) was already checked in reference [12]. Here we are focus in a more general function as, for instance, the next one

$$F_2(x) = 1 - \cos\left(\pi \frac{F_1(x) - 1}{2}\right) \quad (4.40)$$

The reinjection mechanism proposed by Eq. (4.40) is represented by a solid line in Figure 4.1. Following the last section, from Eq. (4.40) we have $F_2'(x_m) = 0$ and the second derivative exits with $F_2''(x_m) \neq 0$, hence we can use Eq. (4.22) with $q = 1$ giving $\alpha = -\frac{1}{2}$ and according with Eq. (4.33) the predicted value of β is $1/2$.

It is interesting to note that argument is quite general and the finding values of α and β do not depend on the particular parameter values of the map. To evaluated this analytical prediction, we set $p = 2$ and $\varepsilon = 0.01$ in the map (4.14). Note that for this map $\hat{x} = 0$, hence α is the only free parameter to determine $\phi(x)$ and $\psi(l)$ of Eqs. (4.11) and (4.7) respectively. In Figure 4.3.a and 4.3.b we compare, this functions using $\alpha = -\frac{1}{2}$ with the numerical estimation.

Note that, on the contrary of the previous published results on this kind of intermitencies, dots and curves are plotted in Figure 4.3 without any fitting analysis. We just superimpose the numerical estimations with our analytical predictions.

Fig. 4.3 Dots are referring to the numerical estimation obtained from 10^6 iterations of the map (4.14)-(4.40) having 9074 reinjections into the laminar region $(0,0.1)$. Curves superimpose the analytical prediction using $\alpha = -1/2$ for: a) $\phi(x)$ and b) $\psi(l)$. The other parameters are $p = 2$ and $\varepsilon = 0.01$.

Concerning to the characteristic relation, in Figure 4.4 are represented the numerical estimation for $\bar{l}$ depending on ε . The slope of the line of the log-log plot should be approach to $-\beta$, hence Figure 4.4 displays a good agreement with our analytical prediction fog β .

4.5.2 Map2. Type III intermittency

To analyze type III intermittency we consider the map (4.18). Note that in this case the power law (4.11) is generated in the neighborhood of the maximum x_m (also around the minimum) but this region is not mapped into the laminar region by the map $x_{n+1} = F_{III}(x_n)$, however, it is done by the composed map $F(x) \equiv F_{III} \circ F_{III}(x)$. According

Fig. 4.4 Characteristic relations. Dots show numerical data for the map (4.14)-(4.40) with $p = 2$, while the solid line represents the least squares straight fitting with slope $-\beta = -0.48$, in good agreement with the analytical prediction giving $\beta = 0.5$ indicated by a dashes line.

with the Theorem 1, we can still applied Eq. (4.21) to the function F_{III} . With this considerations, we get a similar case that the previous one, hence $q = 1$ so we have $\alpha = -1/2$ and according with (4.33) evaluated for $p = 3$ we get $\beta = 3/4$, closed to the reported value $\beta \approx 0.8$ in ref. [13]. The error around 6% between both data is due to the large value $c = 0.6$ used to delimit the laminar interval. This affect our approximation around the extreme point. Note that to deal with Eq. (4.33), following [4], we approximate in the laminar region a finite difference equation by a continuous differential equation, and this approximation also introduces error in the case of large length of the laminar region.

4.5.3 Map3. Type I intermittency

Regarding to type I intermittency we consider the classical cases reported in [18], where the authors study the three maps shown in Figure 4.5. The mentioned paper is one of the pioneer works reporting a *non standard* characteristic exponents β , where an uniform RPD fails to explain these values.

The authors evaluated the RPD by solving numerically the Shaw relation [19]. In all of the cases the reinjection from the extreme point into the laminar region is doing by the composed maps $x_{n+1} = F \circ F(x_n)$ as it is sketched in Figure 4.5. For the three cases, by Taylor expansion the map in the laminar region can be approximated by F_1 of Eq. (4.14) with $p = 2$ (see [18]).

Let us apply the method proposed in previous section to the maps investigated in ref. [18]. The first map is the next composition of logistic maps $F(x) = f^{(A)}[f^{(B)}(x)]$ (we follow the same notation than in the cited reference) where $f^{(A)}(x) = 4Ax(1-x)$ and $f^{(B)}(x) = 4Bx(1-x)$. By means of Theorem 1, we can study only the map $f^{(B)}(x)$ at its extreme point $x_m = 0.5$. As $f^{(B)}(x)$ is a second order polynomial, the second derivative of $f^{(B)}$ must be different from zero hence $q = 1$, the first odd natural number, and according with Eq. (4.21) we have $\alpha = -1/2$ that by means of Eq. (4.32) we get $\beta = 1/4$. In this simple way we get the same result reported by ref. [18].

The second map is defined again by a composed function in a similar form that the first one except that $f^{(A)}(x) = A(1 - 16(x - 1/2)^4)$ and $f^{(B)}(x) = B(1 - 16(x - 1/2)^4)$. By applying the same argument that in the first case to the forth order polynomial of $f^{(B)}$ we conclude that now $q = 3$, the second odd natural number, hence $\alpha = -3/4$ and $\beta = 3/8$, getting the same values reported by the mentioned reference.

Finally, the third map has a similar shape as the first one except in the range $a < x < b$ where the map is a flat constant at value x^* (see Figure 4.5). In this case the derivative $F^{(q)}$ is zero for all value of q , hence in the limit $q \rightarrow \infty$ we get $\alpha \rightarrow -1$ and $\beta = 1/2$, that coincides again with the value reported in [18].

In other cases reported in the literature (see for instance [20]) the map is constructed by a number of composed identical logistic maps, but the reinjection into the laminar region is coming from several regions of the map without local extreme points hence the RPD is like a constant reinjection.

4.5.4 Map4. Map with infinite first derivative

The first derivatives, at pre-reinjection points for the previously studied maps, do not tend to infinite. For the three maps, the first derivatives are zero. Therefore, the pre-reinjection points are extreme points. In this section, we analyze a map with infinite first derivative at the pre-reinjection point.

Let us to introduce the following map with logarithmic nonlinearity (see Eq. (7.49)):

$$F(x) = \begin{cases} F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & 0 \leq x < x_m, \\ F_2(x) = \sqrt{[1 + \ln(x^{-1/\ln(x_m)})]}, & x_m \leq x \leq 1, \end{cases} \quad (4.41)$$

where x_m is once more time defined by $(1 + \varepsilon)x_m + (1 - \varepsilon)x_m^p = 1$. The origin, $x_0 = 0$ is a fixed point of the map, which is stable for $-2 < \varepsilon < 0$. For $\varepsilon > 0$ type II intermittency occurs.

The derivative $dF_2(x)/dx$ results:

Fig. 4.5 Maps of [18] exhibiting type I intermittency. Arrow illustrates mapping of points from the chaotic region into the laminar region. The parameter values are the same that used in [18] as follow: a) $A = 0.9416195$, b) $A = 0.98115325$ and c) $A = 0.9416$, $B = 0.83023023$, $a = 0.743$, $b = 0.874$ and $x^* = 0.9414793$.

$$\frac{dF_2(x)}{dx} = \left[-2x \ln(x_m) \sqrt{1 + \ln(x^{-1/\ln(x_m)})} \right]^{-1} \quad (4.42)$$

Note that x_m is a pre-reinjection point, for which: $dF_2(x)/dx|_{x_m} \rightarrow \infty$. Therefore, we need to calculate the inverse function $F_2^{-1}(x)$, which have to verify:

$$F_2^{-1}(x_m) = 0; \quad F_2^{-1}(1) = 1; \quad dF_2^{-1}(x)/dx|_{x_m} = 0 \quad (4.43)$$

It results:

$$F_2^{-1}(x) = \frac{x_m^{1-(x-x_m)^2} - x_m}{x_m^{1-(1-x_m)^2} - x_m} \quad (4.44)$$

Note that $F_2(x)$ and $F_2^{-1}(x)$ are symmetrical regarding the straight line joining the points $(x_m, 0)$ and $(1, 1)$.

The first and second derivatives of Eq. (4.44) are:

$$\frac{dF_2^{-1}(x)}{dx} = \frac{2(x_m - x) \ln(x_m) x_m^{1-(x-x_m)^2}}{x_m^{1-(1-x_m)^2} - x_m} \quad (4.45)$$

$$\frac{d^2 F_2^{-1}(x)}{dx^2} = \frac{2x_m^{(1-x)(1+x-2x_m)} \ln(x_m) (1 - 2(x-x_m)^2 \ln(x_m))}{x_m^{(x_m-1)^2} - 1} \quad (4.46)$$

To calculate numerical data, we consider $p = 2$ and $\varepsilon = 0.001$. From $(1 + \varepsilon)x_m + (1 - \varepsilon)x_m^2 = 1$, we obtain $x_m = 0.6179284$. With this value in Eqs. (4.44), (4.45) and (4.46), we have:

$$F_2^{-1}(x_m) = 0; \quad \left. \frac{d^2 F_2^{-1}(x)}{dx^2} \right|_{x=x_m} = 0; \quad \left. \frac{d^2 F_2^{-1}(x)}{dx^2} \right|_{x=x_m} \cong 13.225 \quad (4.47)$$

The second derivative is not zero at x_m , then $\gamma = 2$. Therefore, if we use Eq. (4.39), $F_2^{-1}(x)$ can be approximated as:

$$F_2^{-1}(x) \approx F_2^{-1}(x_m) + \frac{(x-x_m)^\gamma}{\gamma!} \frac{d^\gamma F_2^{-1}(x_m)}{dx^\gamma} = w(x-x_m)^2 \quad (4.48)$$

where $w = 2 d^2 F_2^{-1}(x_m)/dx^2$ is a constant.

To obtain the approximation of the function $F_2(x)$, the inverse of Eq. (4.48) must be calculated. This approached function must verify $F_{2a}(x_m) = 0$ and $F_{2a}(1) = 1$

$$F_{2a}(x) \approx \frac{(x-x_m)^{1/2}}{(1-x_m)^{1/2}} \propto (x-x_m)^{1/2} \quad (4.49)$$

Note that the exponent γ for $F_2(x)$ is $1/2$. Now, we can evaluate α , which results: $\alpha = 1/\gamma - 1 = 1$.

To validate this result we compare the $F_2(x)$ function, Eq. (4.41), and its approximation given by Eq. (4.49). Figure 4.6 shows both functions, which are very similar. The difference between Eqs. (4.41) and (4.49) occurs because high order even derivatives of $F_2^{-1}(x)$ are not zero.

Fig. 4.6 Comparison between $F_2(x)$ function and its approximation for $x_m = 0.6179284$. Red: $F_2(x)$ function, Eq. (4.41). Blue: $F_2(x)$ approximation, Eq. (4.49).

Also, to validate the new method, we carry out the comparison between the RPDs calculated using the M function methodology for $F_2(x)$ (given by Eq. (4.41)), and its approximation function (given by Eq. (4.49)). Beside, we compare these RPDs with the numerical reinjection points calculated using the $F_2(x)$ function. For the two test 50000 reinjected points were considered. The results are shown in Figure 4.7. We can observe a very good accuracy between the theoretical RPDs and the numerical data. Finally, we highlight that the exponents α evaluated using the M function methodology are very close to the theoretical prediction: $\alpha = 1$. For $F_2(x)$, the exponent is: $\alpha \cong 1.0169$. For the $F_2(x)$ approximation function, we obtain $\alpha \cong 0.9782$.

4.6 Some characteristic of the $M(x)$ function

Let us to suppose that the following two conditions on the behavior of the RPD in a neighborhood of the fixed point $x_0 = 0$ are true:

- i) $\phi(0) \neq 0$
- ii) $\left. \frac{d\phi(x)}{dx} \right|_0$ is bounded.

Fig. 4.7 Comparison between the RPDs calculated using the M function methodology for $F_2(x)$ (Eq. (4.41)), its approach (Eq. (4.49)) and the numerical reinjection points for $F_2(x)$ function. Red: RPD for $F_2(x)$ function, Eq. (4.41). Blue: RPD for $F_{2a}(x)$ approximation, Eq. (4.49). Green: numerical data calculated using $F_2(x)$ function.

Then the function $M(x)$ can be approximated as $M(x) = x/2$ for points closed enough to $x = 0$. We notice that this approximation produces the same value of β that should be found in the case of uniform reinjection, because of $m = 1/2$ [18].

To prove this, note that for $\phi(0) \neq 0$, we have

$$\lim_{x \rightarrow 0} M(x) = \lim_{x \rightarrow 0} \frac{x \phi(x)}{\phi(x)} = 0 \quad (4.50)$$

and the slope of the function $M(x)$ for x close to zero can be derived from

$$\lim_{x \rightarrow 0} \frac{dM(x)}{dx} = \lim_{x \rightarrow 0} \left[\frac{x \phi(x) \int_0^x \phi(\tau) d\tau - \phi(x) \int_0^x \tau \phi(\tau) d\tau}{\left(\int_0^x \phi(\tau) d\tau \right)^2} \right] \quad (4.51)$$

which by the L'Hôpital theorem, gives

$$\lim_{x \rightarrow 0} \frac{dM(x)}{dx} = 1 - \frac{1}{2} \lim_{x \rightarrow 0} \left[\frac{\phi'(x) \int_0^x \tau \phi(\tau) d\tau}{\phi(x) \int_0^x \phi(\tau) d\tau} \right] - \frac{1}{2} \lim_{x \rightarrow 0} \left[\frac{x \phi(x)}{\int_0^x \phi(\tau) d\tau} \right] \quad (4.52)$$

Now, because of $\left. \frac{d\phi(x)}{dx} \right|_{x=0}$ is bounded, we obtain $1/2$ for the limit (4.52). As a consequence $M(x) \approx 0.5x$, which means that the classical values of the characteristic exponent can be obtained not only by uniform reinjection, but also by any RPD holding the previous two conditions.

This results is important, because if the conditions (i) and (ii) are verified, then for a very small laminar interval ($c \rightarrow 0$), the RPD is uniform and the classical theory is valid.

4.7 Conclusions

Based on the well confirmed result for many one-dimensional maps, the reinjection probability density RPD can be approached by the power law given by Eq. (4.11). In this chapter we investigate how the RPD is generated in return maps. We show that in several cases the RPD is generated around the extreme (pre-reinjection) points of the map. Based on this fact, we propose two analytical methods providing the RPD, hence by simple calculus it is possible to estimate the RPD given by Eq. (4.11). In particular we propose a formula given the parameter α as a function of the number of null derivatives of the map at the extreme point. The extreme condition of the point imposes that this number must be odd, hence there is a restriction on the possible values of α .

Once the value of α is determined, the value of the characteristic exponent is also determined, depending on the intermittency type. This analytical prediction have been compared with numerical estimation showing good agreement between both. For the maps investigated in the celebrated paper [18], our method provide the exact value of the characteristic exponent β in agreement with the numerically reported values in the mentioned reference.

We emphasize that the values of α and β found by our method usually do not depend on the parameter values of the map, in particular on ε .

Furthermore, we have developed a new method to evaluate the RPD function when the reinjection process is govern by a pre-reinjection point with infinite derivative. For these cases, the reinjection process is different to those generated by extreme points, and we can not directly apply the Alternative 1 and Alternative 2 methods. To find the exponent γ for these cases, we need to obtain an inverse function and over it we can apply the Alternative 1 or 2 methods. After that, we must evaluate a new inverse function. The numerical results verify with high accuracy the theoretical predictions of this new scheme.

Finally, we have shown that under analytical specific conditions the RPD function can be uniform.

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Chapter 5

RPD FOR TYPE III INTERMITTENCY WITH NOISE AND LOWER BOUNDARY OF REINJECTION

Abstract In this chapter we extend M function methodology to study type III intermittency considering different values of the noise intensity and the lower boundary of reinjection (LBR). We obtain accurate analytic expressions for the reinjection probability density (RPD). The proposed RPD has a piecewise definition depending on the non-linear behavior, the LBR value and the noise intensity. The new RPD is a sum of exponential functions with exponent $\alpha + 2$, where α is the exponent of the noiseless RPD. The theoretical results are verified with numerical simulations.

5.1 Introduction

As we have described in Chapter 2, type III intermittency occurs in a subcritical period-doubling bifurcation in which an unstable period-2 orbit meets and destabilizes a stable period-1 orbit; this type of intermittency is characterized by the gradual increasing during the laminar phase of a period-2 component in the motion [1]. A one-dimensional map $F(x)$ that displays a subcritical period-doubling bifurcation has a positive Schwartzian derivative.

We use Poincaré maps (one-dimensional maps) to study type III intermittency [2, 3, 4], which have two main chaotic intermittency characteristics: 1) a specific local map and 2) a reinjection mechanism. The local Poincaré map for type III intermittency is given by:

$$x_{n+1} = -(1 + \varepsilon)x_n - a x_n^3 \quad (5.1)$$

where the control parameter ε and the coefficient a are larger than zero [1, 5]. The reinjection mechanism maps the system back into the local regular or laminar zone from the chaotic one. This mechanism is described by means the RPD function which gives the probability that trajectories are reinjected into the laminar zone, close to the unstable fixed point (see Chapters 1, 2, 3 and 4). As we have explained previously,

this function depends on the nonlinear dynamics of the system itself and can lead to a broad range of different behaviors.

On the other hand, as noise is always present in nature [6], it is important to describe how it affects the intermittent RPD. In Chapter 2 we have study the noise effects on the intermittency phenomenon using the Fokker-Plank equation and the renormalization group analysis [7, 8, 9]. Note that the mentioned references analyze the effects of noise only on the laminar zone; however, it has been recently shown that noise strongly influences the RPD function [10, 11] (see Chapter 3). Moreover, as there is experimental evidence of the lower boundary of reinjection (LBR) for type III intermittency [12] we decided to focus in the analysis of phenomena with type III intermittency where LBR and noise are present.

The RPD function allows to determine the statistical properties of the intermittency process like the average laminar length -the average time that trajectories spend close to the vanished or unstable limit cycle-, and the characteristic equation -the scaling relation of the average laminar length. The characteristic equation gives the relation between the average laminar length, $\bar{l}$ and the control parameter ε : $\bar{l} \sim \varepsilon^{-\beta}$, where β is the characteristic exponent of the scaling relation and it depends on of reinjection probability distribution (see Eq. (3.48)). This characteristic relation allows to estimate the duration of the laminar phases -the time interval between chaotic phases. These functions depend on the RPD, which depends on the global maps behavior.

There was not an efficient method to obtain the RPD function until a more general technique, called *M* methodology, which includes the uniform reinjection as a particular case, was introduced in the last years [4, 5, 10, 13, 14, 15, 16, 17, 18, 19]. In this chapter this general methodology is used to describe the reinjection processes for type III intermittency considering noise effects, and to extend the effects of noise to cases not previously considered as lower boundaries of reinjection different from zero (LBR $\neq 0$). Several new theoretical noisy RPD functions defined as the addition of exponential functions are obtained. The new noisy RPD functions are compared with numerical results showing a very good agreement for two maps with different nonlinear behaviors.

5.2 Type III Intermittency

We analyze two one-dimensional maps with type III intermittency. We select these maps because they were studied using the *M* function methodology in preceding papers [5, 15]. However, these papers do not analyze the effects of noise and LBR on the RPD function. Both maps have a fixed point in $x = 0$, and they are symmetric around this point. The first map was presented in [5], and can be written as:

$$x_{n+1} = F(x_n) = -(1 + \varepsilon)x_n - ax_n^3 + dx_n^6 \sin(x_n) \quad (5.2)$$

where ε is the control parameter. The map has a fix point $x_0 = 0$ at the origin, which is stable for $-2 < \varepsilon < 0$. When $\varepsilon > 0$, the fix point becomes unstable and type III intermittency arises. The reinjection mechanism depends on the map extreme values, x_m , satisfying $dF(x_m)/dx = 0$ (see Figure 5.1). In particular, this odd map has two extreme points. The iteration procedure, which is governed by the parameters ε , a and the cubic exponent, leads to increasing values of x_n generated from an initial one, close to the origin. For large enough values of n , the influence of the RHS third terms in Eq. (5.2) increases as x_n approaches to an extreme point x_m , rendering the reinjection into the laminar zone.

The second map was given in [1]:

$$x_{n+1} = F(x_n) = -((1 + \varepsilon)x_n + x_n^3)e^{-bx_n^2} \quad (5.3)$$

This map has a single fix point at $x_0 = 0$ with a slope $dF(x_0)/dx = -(1 + \varepsilon)$, being stable for $-2 < \varepsilon < 0$. There is a subcritical period-doubling bifurcation at $\varepsilon = 0$ for $b < 1$. The reinjection mechanism is governed by the factor $e^{-bx_n^2}$: when the amplitude x_n is large enough, this factor becomes small and the iteration $n + 1$ falls close the unstable fix point (see Figure 5.1).

Both maps have x_m points with $dF(x_m)/dx = 0$ (local maximum or local minimum). These points are pre-reinjection points, they have a strong influence on the reinjection processes, but the reinjection mechanism is different for each map because the derivative $dF(x_n)/dx$, where x_{n+1} is a reinjected point inside the laminar interval, are different. Therefore, the implementation of these two maps permits to analyze different nonlinear behaviors, i.e., different reinjection processes. We highlight that the points close to x_m need two iterations to be reinjected.

5.3 Evaluation of the RPD function. The **M** function methodology

Let us consider a general one-dimensional map: $x_{n+1} = F(x_n)$. The noiseless RPD function, $\phi(x)$ specifies the statistical behavior of the reinjection trajectories, which depends on the specific form of $F(x)$ [2, 3].

As the maps (5.2) and (5.3) are symmetrical around the origin of coordinates, the laminar interval is split in two sub-intervals $-c \leq x \leq -\hat{x}$ and $\hat{x} \leq x \leq c$. Where c is a constant verifying $c > 0$ which specifies the limits of the laminar region around the unstable fixed point x_0 , and $\hat{x}$ is the LBR. Therefore, the laminar zone of type III intermittency can be defined by $[x_0 - c, -\hat{x}]$ and $[\hat{x}, x_0 + c]$. In previous works, it was

Fig. 5.1 Top figure: Map (5.2). The parameters are: $\varepsilon = 0.01$, $a = 1, 0$ and $b = 1.1$. Bottom figure: Map (5.3). The parameters are: $\varepsilon = 0.1$ and $b = 0.1$.

assumed $\hat{x} \equiv x_0$ (LBR = 0) [15, 16, 17]. Here, we consider the more general case, $\hat{x} \neq x_0$. Note that the LBR corresponds to the second iteration of x_m , i.e., $\hat{x} = F^2(x_m)$.

To obtain the noiseless RPD, the M function methodology is implemented. Therefore, a briefly description about the M function methodology is here presented.

The calculation of $M(x)$ from the data series is very simple: $M(x) \cong \frac{1}{N} \sum_{j=1}^N x_j$, where the data set (reinjection points) $\{x_j\}_{j=1}^N$ must be sorted from the lowest to the highest, i.e. $x_j \leq x_{j+1}$ (see Chapter 3).

For both maps, the $M(x)$ function without noise was numerically evaluated obtaining a linear form $M(x) = m(x - \hat{x}) + \hat{x}$ for different values of b , a and ε [5]. Hence, the RPD function can be expressed by:

$$\phi(x) = b(x - \hat{x})^\alpha, \quad \text{with} \quad \alpha = \frac{2m - 1}{1 - m}, \quad (5.4)$$

where b is the normalization parameter.

As the slope m determines the value of the exponent α in the RPD function (see Eq. (5.4)), it rules the reinjection mechanism and has direct influence in the probability density of the laminar length, in the average laminar length and in the characteristic relation. The probability density of the laminar lengths, $\psi(l, c)$, is a global property and is related to the reinjection probability density function [5, 13]:

$$\psi(l, c) = 2b [x(l, c) - \hat{x}]^\alpha [\varepsilon x(l, c) + ax(l, c)^p] \quad (5.5)$$

where l is the laminar length. For the maps (5.2) and (5.3), the exponent is $p = 3$.

As we have explained in previous chapters, for intermittency without noise, the $M(x)$ function is determined by the nonlinear map; each map produces a different $M(x)$ function, defined by the parameters m and $\hat{x}$. Therefore, this function stores the nonlinear information of the map [4, 19, 4].

5.4 Noise influence for LBR = 0

In this section the influence of noise on the statistical properties of type III intermittency is analyzed [10]. Noise, present in all natural dynamical system, will affect the RPD function. It is usually assumed that the noise strength σ is much smaller than the control parameter ε . Here, we consider a general procedure where this hypothesis is avoided by transforming Eqs. (5.2) and (5.3) into noisy maps:

$$x_{n+1} = F(x_n) = -(1 + \varepsilon)x_n - ax_n^3 + dx_n^3 \sin(x_n) + \sigma \xi_n \quad (5.6)$$

$$x_{n+1} = F(x_n) = -((1 + \varepsilon)x_n + x_n^3) e^{(-bx_n^2)} + \sigma \xi_n \quad (5.7)$$

where ξ_n is a uniform distributed noise verifying that $\langle \xi_m, \xi_n \rangle = \delta(m - n)$ and $\langle \xi_n \rangle = 0$, with noise strength σ . Note that if $\sigma = 0$, the previous maps (5.2) and (5.3) are recovered.

The reinjection processes in both maps (5.2) and (5.3) are different that those for type II intermittency described in [11]. For type III intermittency, a point laying very close to the extreme points (maximum or minimum) will need two iterations to be reinjected into the laminar zone. On the other hand, for type II intermittency only one

iteration was necessary [11]. Through these iterations, type III intermittency trajectories are stretching or contracting by the derivative $K = dF(x_n)/dx$, where $x_{n+1} = F(x_n)$ is a reinjected point into the laminar zone. The map (5.2) strengthens the trajectories, however the map (5.3) can contract them. The slope K varies as x_n changes, however in most maps, K can be approximated by a constant because the laminar interval is small around the unstable fixed point. Accordingly, K corresponds to the mean slope of the map in the interval I , where I is the interval mapped backward of the laminar region.

Figure 5.2 shows the numerical results for the noisy $M(x)$ considering Eq. (5.6) with $\varepsilon = 0.01$, $a = 1.1$, $b = 1.35$, $c = 0.6$ and $\sigma = 0.02$. To compare results we have considered, following the papers [5, 10], that the laminar interval length verifies $c = 0.6$. As for the noiseless case, the functions $M(x)$ smooths down the data. However, for the noisy tests, the function $M(x)$ has different behavior on each side of x_s . Remember that we call singular points to those points where the behavior of $M(x)$ changes, e.g., x_s . $M(x)$ can be approximated by a piecewise linear function with two slopes. The x_s value is approximately equal to the product between the noise intensity and the slope K , $x_s \cong K\sigma$. Note that $\sigma_r = K\sigma$ is the amplified noise intensity in the reinjected point. For the figure $K \cong 11$; therefore, $\sigma_r \cong 0.22$. For $x < x_s$ the slope of $M(x)$, m_l , approaches $1/2$, as expected for the uniform reinjection. For $x > x_s$, the slope of $M(x)$, m_r is similar to the noiseless slope. The total number of reinjected point used was 300000, and 53483 reinjected points were used to evaluate m_r (we calculate m_r using the points indicated by the darker (blue) line in Figure 5.2). This is a remarkable property of $M(x)$ because we can obtain the RPD function for the noiseless case by means of a noisy data analysis. Note that even when noise acts on the complete system, it does not modify the slope of $M(x)$ in the region $x > x_s$. Hence, on the right side of x_s , the RPD function is robust against noise without adjacent reinjection [10, 11]. However, in the region $x < x_s$ the noise modifies the RPD. When noise is considered and $x < x_s$, the RPD approaches the uniform reinjection case, at least locally around the unstable fix point $x_0 = 0$, due to the symmetry of the map around it. Therefore, point x_s is a boundary that separates two regions inside the laminar zone. One region is $[0, x_s]$ in which the noise governs the reinjection process. The other region is $(x_s, c]$ in which the nonlinear map drives the intermittency reinjection.

Following reference [10] we can get an analytical expression for the noisy reinjection probability density function, called NRPD. The noiseless density $\phi(x)$ transforms in a new density $\Phi(x)$ according to the convolution integral:

$$\Phi(x) = \int \phi(y)G(x-y, K\sigma) dy \quad (5.8)$$

where $\Phi(x)$ is the NRPD and $G(x, \sigma)$ is the probability density of the noise term $\sigma\xi_n$. A random variable ξ in the interval $[-1, 1]$ is used as a noise source, then its probability

Fig. 5.2 Map (5.6). Top figure: $M(x)$ from the numerical simulations, the darker (blue) line indicates the data used to calculate the exponent α . Bottom figure: Theoretical and numerical RPD functions, the points corresponds to numerical data, the result of Eq. (5.10) is plotted as a solid line. Parameters: $\varepsilon = 0.0005$, $\sigma = 0.02$, $c = 0.6$, $a = 1.1$ and $b = 1.35$, $K = 10.5833$.

density $G(x, \sigma)$ results:

$$G(x, \sigma) = \frac{\Theta(x + K\sigma) - \Theta(x - K\sigma)}{2K\sigma} \quad (5.9)$$

where $\Theta(x)$ is the Heaviside step function. If the function $\phi(x) = b|x|^\alpha$ is introduced in the convolution integral, the resulting NRPD function is:

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + K\sigma)^{1+\alpha} - \text{Sg}(|x| - K\sigma) ||x| - \sigma|^{1+\alpha}}{2K\sigma} \quad (5.10)$$

where $\text{Sg}(x)$ is the sign function. However, to develop the Eq. (5.10) we considered only the case $K \gg 1$ to use a single convolution integral [10].

The NRPD for the same values of ε, c, a, b is shown in Figure 5.2-Bottom. The slope K is close to 11 ($K = 10.5833$). The points represent the numerical data, and the continuous line is the analytical RPD calculated by Eq. (5.10). There is a good correspondence between the numerical and theoretical data.

Figure 5.3 shows the results for the map (5.7), considering 150000 reinjected points. Eq. (5.8) is used to evaluate the RPD, and considering now the approximation $K \approx 0$ ($K \ll 1$), resulting:

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + \sigma)^{1+\alpha} - \text{Sg}(|x| - \sigma) ||x| - \sigma|^{1+\alpha}}{2\sigma} \quad (5.11)$$

Note that Eq. (5.11) is similar to Eq. (5.10); in Eq. (5.11), K is not explicitly present because $K \approx 0$, therefore σ , the noise intensity, governs the reinjection process.

If the slope $K \neq 0$ and K does not reach a high value, it influences the noisy reinjection process, resulting in a more complex RPD structure. Thus, we need to consider the two iteration process until the reinjection to obtain an analytical expression for the NRPD. We consider the map (5.7) as a composition of the noiseless map (5.3):

$$x'_n = F(x_n) = -((1 + \varepsilon)x_n + x_n^3)e^{(-bx_n^2)} \quad (5.12)$$

and the map defined as

$$x_{n+1} = R(x'_n) = x'_n + \sigma \xi_n \quad (5.13)$$

The density function produced by the map (5.13) can be calculated using the following convolution integral:

$$\rho(x) = \int \rho'(t) G(x - t, \sigma) dt \quad (5.14)$$

where $\rho'(t)$ is the density produced by the noiseless map given by Eq. (5.12). If we consider points x_n placed close to the maximum (or minimum) of the map (5.12), the points x_{n+1} are not directly mapped inside of the laminar interval. It is necessary to consider other iteration to reach the laminar interval:

$$x'_{n+1} = F(x_{n+1}) = -((1 + \varepsilon)x_{n+1} + x_{n+1}^3)e^{(-bx_{n+1}^2)} \quad (5.15)$$

and

$$x_{n+2} = R(x'_{n+1}) = x'_{n+1} + \sigma \xi_{n+1} \quad (5.16)$$

Fig. 5.3 Map (5.7). Top figure: $M(x)$ from the numerical simulations, the darker (blue) line indicates the data used to calculate the exponent α . Bottom figure: Theoretical and numerical RPD function, the points correspond to numerical data and the result of Eq. (5.11) is plotted as a solid line. Parameters: $\varepsilon = 0.005$, $c = 1.0$, $\sigma = 0.3$, $b = 0.05$, $K \cong 0$.

The resulting density function, applying Eq. (5.15), can be written as:

$$\rho_l(x) = \frac{dF^{-1}(x)}{dx} \rho(F^{-1}(x)) = K \rho(F^{-1}(x)) \quad (5.17)$$

Finally, to obtain the noisy reinjection probability density function, $\Phi(x)$, it is necessary to apply a new convolution integral as follow:

$$\Phi(x) = \int \rho_l(y) G(x-y, \sigma) dy \cong \int \rho(y) G(x-y, K\sigma) dy \quad (5.18)$$

From the previous relations (5.14) and (5.18), we note that the amplified noise intensity is given by $(K+1)\sigma$ and the NRPD will have the exponent $\alpha+2$, where α is the exponent for the RPD without noise. Note, that Eqs. (5.10) and (5.11) are two extreme cases. For Eq. (5.10), $K \gg 1$ verifies, we can use the following approximation $(K+1)\sigma \approx K\sigma$. For (5.11), $K \approx 0$ is verified, therefore, $(K+1)\sigma \approx \sigma$.

The presence of noise not only affects the RPD function, due to the deviations produced on the reinjection trajectories, another reinjection mechanism can arise when the noise strength $(K+1)\sigma$ is relatively large: the adjacent reinjection mechanism (AR). The AR mechanism can be described as follows: when a trajectory has a reinjected point close to the unstable fixed point, the following iterative points increase driven by the parameters of the map until a point, x_j , leaves the laminar interval –then x_{j+1} is outside the laminar interval. If the difference between x_{j+1} (evaluated from maps (5.2) and (5.3), without noise) and the laminar boundary c is less or equal than the noise intensity ($x_{j+1} - c \leq (K+1)\sigma$), the trajectory may return into the laminar interval. The AR concept and its influence in type II intermittency was recently described in [11].

5.5 Noisy RPD without AR and LBR $\neq 0$

In this section we study the LBR influence on the noisy reinjection process without considering the AR mechanism. If no restrictions are imposed to the noise strength $((K+1)\sigma)$ and the LBR ($\hat{x}$), two different processes can occur. One of them develops when the amplified noise intensity is lower than the LBR, $(K+1)\sigma < \hat{x}$. The other one occurs when the LBR is lower or equal than the amplified noise intensity, $(K+1)\sigma \geq \hat{x}$. Also, for each one of these cases, different expressions for the NRPD will be found, depending on the slope K .

5.5.1 NRPD without AR and amplified noise intensity $< \text{LBR}$

When $(K+1)\sigma < \hat{x}$, the LBR changes by the noise effect. A “new LBR” arises depending on the original LBR and the amplified noise strength, defined as the difference $x_i = \hat{x} - (K+1)\sigma$. We use the convolution integral (5.18) to evaluate the NRPD (Φ), explicitly given in Eq. (5.19):

$$\Phi(x) = \frac{\int [(y + \sigma - \hat{x})^{(1+\alpha)} - H(y - \sigma - \hat{x})(y - \sigma - \hat{x})^{(1+\alpha)}] G(x - y, K\sigma) dy}{2\sigma(c - \hat{x})^{(1+\alpha)}} \quad (5.19)$$

Where $H(x)$ is the Heaviside function, and the expression inside the square brackets is the density, $\rho(x)$, after the first iteration for points close to the local maximum or minimum, x_m (see Eq. (5.14)). According to the convolution integral the NRPD depends on the value of the slope K , and three cases can appear: $0 < K \leq 1$; $1 \leq K \leq 2$; and $2 \leq K$.

For all the tests that will be considered in this chapter we will apply the M methodology to obtain the NRPD. The first step of the M methodology is to evaluate the M function from the numerical data. Afterwards we obtain information from $M(x)$; finally we solve Eq. (5.19) considering different subintervals inside the laminar zone. These subintervals are determined by the overlap of the functions in the convolution integral (5.19).

5.5.1.1 NRPD for $0 < K \leq 1$

Figure 5.4-Top shows $M(x)$ verifying $(K + 1)\sigma < \hat{x}$ for the map ((5.7)). The used parameters are: $\varepsilon = 0.005$, $b = 0.18$, $\hat{x} = 0.3181$, $K \cong 0.5366$, $c = 1$, $\sigma = 0.2$ and 250000 reinjected points. Then, the new LBR results: $x_i \cong 0.0108$. The function $M(x)$ can be fitted by two straight lines having different slopes that correspond with two regions lying at both side of a small zone around the point $x_s = \hat{x} + (K + 1)\sigma \cong 0.6254$. The first one, m_l , is the slope close to the new LBR point (x_i), and m_r is the slope close to the point c (see the darker (blue) line in the figure). Therefore, m_l approximately corresponds to the interval $[\hat{x} - (K + 1)\sigma, \hat{x} + (K + 1)\sigma]$, and m_r approximately corresponds to $[\hat{x} + (K + 1)\sigma, c]$. Similarly to the case without LBR, m_r is very close to the slope of $M(x)$ without noise. The darker (blue) line indicates the points used to calculate $m_r \cong 0.2922$ ($\alpha_r \cong -0.5872$). In the noiseless case $m \cong 0.2913$ and $\alpha \cong -0.5889$. On the other hand, $m_l \cong 0.685$, on the contrary that was report in [10], here there is not uniform reinjection close to x_i . This occurs because there is not a direct influence of the negative points ($x < 0$) in the evaluation of $M(x)$.

There are two different behaviors between Figures 5.4-Top and 5.3-Top. The first one is associated with the matching between the two straight lines. In Figure 5.4-Top we have small zone, and in Figure 5.3-Top we obtain only a point x_s . The second difference is due to the m_l value; Figure 5.4-Top shows that there is not uniform reinjection close to the LBR.

Using Eq. (5.19), where α is the exponent of the noiseless RPD obtained from $M(x)$ of Figure 5.4-Top, we get the NRPD given by Eqs. (5.20), (5.21), (5.22) and (5.23).

Fig. 5.4 Map (5.7). $(K+1)\sigma < \hat{x}$ and $0 < K \leq 1$. Top figure: $M(x)$ function, the darker (blue) line shows the data used to calculate the exponent α in Eq.(5.4). Bottom figure: NRPD function, numerical data and theoretical result are represented by points and line respectively. Parameters: $c = 1$, $\varepsilon = 0.005$, $\sigma = 0.2$, $\hat{x} \cong 0.3181$, $K \cong 0.5366$.

To obtain Eq. (5.20), the interval boundaries are defined by the new LBR and the point $\hat{x} + (K-1)\sigma$; this is the first point where the total overlap between the probability density functions $\rho(x)$ and $G(x)$ occurs. Therefore, in this first interval, these functions only overlap partially. Eq. (5.21) is obtained solving the convolution integral in the interval $[\hat{x} + (K-1)\sigma; \hat{x} + (1-K)\sigma]$. For points $x \geq \hat{x} + (1-K)\sigma$, $\rho(x)$ changes due to the Heaviside function (see Eq. (5.19)). Thus to calculate the convolution integral we use here only the first term of $\rho(x)$. Eq. (5.22) is evaluated inside the third interval, which considers the convolution integral for the points located between $\hat{x} + (1-K)\sigma$

and $\hat{x} + (1 + K)\sigma$. Thus, inside this interval the convolution integral is calculated as the sum of two integrals. For one of them we need to consider only the first term of $\rho(x)$ and for the second we require the both terms of $\rho(x)$. To evaluate the convolution integral, for points $x \geq \hat{x} + (1 + K)\sigma$, it is necessary to use the both terms in $\rho(x)$. Finally, Eq. (5.23) is calculated inside the fourth interval, which considers the convolution integral for points located between $\hat{x} + (1 + K)\sigma$ and c . Here, the convolution integral is calculated considering the two terms in the expression of $\rho(x)$.

The numerical and theoretical NRPD functions, for the same parameters used in Figure 5.4-Top, are shown in Figure 5.4-Bottom. The points represent the numerical data, and the line represents the analytical NRPD. Note that the analytical results accurately adjust the numerical data.

For $\hat{x} - (K + 1)\sigma \leq x \leq \hat{x} + (K - 1)\sigma$

$$\Phi(x) = \Phi_1(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)} \quad (5.20)$$

For $\hat{x} + (K - 1)\sigma \leq x \leq \hat{x} + (1 - K)\sigma$

$$\Phi(x) = \Phi_2(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)} \quad (5.21)$$

For $\hat{x} + (1 - K)\sigma \leq x \leq \hat{x} + (1 + K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)} \quad (5.22)$$

For $\hat{x} + (1 + K)\sigma \leq x$

$$\begin{aligned} \Phi(x) = & \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)} \\ & + \frac{(x - \sigma(1 + k) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)} \end{aligned} \quad (5.23)$$

Figure 5.4-Bottom shows four different behaviors for the NRPD, each one inside of the intervals $[\hat{x} - (K + 1)\sigma, \hat{x} + (K - 1)\sigma]$, $[\hat{x} + (K - 1)\sigma, \hat{x} + (1 - K)\sigma]$, $[\hat{x} + (1 - K)\sigma, \hat{x} + (1 + K)\sigma]$, and $[\hat{x} + (1 + K)\sigma, c]$, respectively. This NRPD complex behavior is produced by the iterative process: trajectories starting in points close

to local maximum or minimum need, at least, two iterations before the reinjection. Therefore, the noise strongly influences all the reinjection process. Unlike the case with $LBR = 0$, the reinjection is not uniform around the unstable fix point because there is no reinjection from negative x values, thus $x_i > 0$. Three intervals are placed between the new LBR, x_i , and the critical point x_s . Therefore, the combined influence of noise and LBR produces higher effects inside the interval $[x_i, x_s]$. Also, this behavior is shown by the slopes m_l and m_r in Figure 5.4-Top: m_r is very close to the noiseless $M(x)$ slope, but m_l is not equal to the uniform reinjection case, nor to the noiseless reinjection.

Note that the NRPD given by Eq. (5.20) - Eq. (5.23) is represented by a serie of exponential functions with exponent $\alpha + 2$, where α is the noiseless exponent. This results is different from that obtained in [10, 14] where the exponent was $\alpha + 1$ and the NRPD was more simple. This occurs because the NRPD described in [10, 14] are only particular cases of the NRPD developed in this chapter.

5.5.1.2 NRPD for $1 < K \leq 2$

Some differences appear when $K > 1$, we analyze the interval $1 < K \leq 2$. Figure 5.5-Top shows $M(x)$ for $\varepsilon = 0.005$, $b = 0.21$, $\hat{x} \cong 0.8076$, $K \cong 1.1750$, $c = 2$, $\sigma = 0.2$ and 150000 reinjected points. Therefore, $(K + 1)\sigma < \hat{x}$ is verified. The new LBR results $x_i \cong 0.5901$. $M(x)$ has also two straight lines with slopes m_l and m_r , separated by a reduced zone around the point $x_s = \hat{x} + (K + 1)\sigma \cong 1,0251$. The slope m_l corresponds to the interval $[\hat{x} - (K + 1)\sigma, \hat{x} + (K + 1)\sigma]$, and the slope m_r corresponds to $[\hat{x} + (K + 1)\sigma, c]$. m_r is close to the slope of $M(x)$ for the reinjection process without noise. The darker (blue) line indicates the points used to calculate this slope: $m_r \cong 0.32$ ($\alpha_r \cong -0.53$). In the noiseless case $m \cong 0.33$ and $\alpha \cong -0.51$. The behavior of m_l is similar to that described in the previous subsection ($0 < K \leq 1$): $m_l \cong 0.675$.

To obtain the analytical expression for the NRPD function we solve Eq. (5.19). We calculate α using m_r from $M(x)$ of Figure 5.5-Top. The NRPD is given by Eqs. (5.24), (5.25), (5.26) and (5.27). Figure 5.5-Bottom shows the numerical and theoretical NRPD function. To obtain this figure we considered the same parameters used in Figure 5.5-Top. The points and the line represent the numerical data and the analytical NRPD, respectively. The analytical results verify accurately the numerical data.

This test gives similar results as for the $0 < K \leq 1$ case, but m_r for $0 < K \leq 1$ has a better convergence towards the noiseless slope.

Fig. 5.5 Map (5.7). $(K+1)\sigma < \hat{x}$ and $1 < K \leq 2$. Top figure: $M(x)$ function, the darker (blue) line shows the data used to calculate the exponent α . Bottom figure: NRPD function, the points represent the numerical data and the line is the analytical NRPD. Parameters: $c = 2$, $\varepsilon = 0.005$, $\sigma = 0.2$, $\hat{x} \cong 0.8076$, $K \cong 1.1750$.

5.5.1.3 NRPD for $K > 2$

For the case $K > 2$, we calculate the function $\Phi(x)$ using the convolution integral (5.19). It is given by (5.28), (5.29), (5.30), and (5.31). As there is no slope $K > 2$ associated with the map (5.7) we cannot obtain numerical results. Still, the NRPD is similar to those described previously, and is also defined inside four intervals.

For $\hat{x} - (K + 1)\sigma \leq x \leq \hat{x} + (1 - K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha + 2)} \quad (5.24)$$

For $\hat{x} + (1 - K)\sigma \leq x \leq \hat{x} + (K - 1)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha + 2)} \quad (5.25)$$

For $\hat{x} + (K - 1)\sigma \leq x \leq \hat{x} + (1 + K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha + 2)} \quad (5.26)$$

For $\hat{x} + (1 + K)\sigma \leq x$

$$\begin{aligned} \Phi(x) = & \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha + 2)} \\ & + \frac{(x - \sigma(1 + K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha + 2)} \end{aligned} \quad (5.27)$$

5.5.2 NRPD without AR and amplified noise intensity $\geq$ LBR

We analyze the reinjection behavior when $(K + 1)\sigma \geq \hat{x}$ and $\sigma < \hat{x}$. The condition $(K + 1)\sigma \geq \hat{x}$ implies that the “new LBR” is $x_i = 0$ and reinjection trajectories coming from points $x < 0$ will occur. The restriction $\sigma < \hat{x}$ implies that the “new LBR” is the origin only when a two iteration reinjection process is considered for trajectories coming from points close to x_m . Then, σ alone is not enough to produce the reinjection from $x < 0$. Accordingly, the function Φ depends on the value of the slope K . Three cases can appear: $0 < K \leq 1$; $1 \leq K \leq 2$; and $2 \leq K$. To obtain the function Φ we again calculate the convolution integral (5.19).

Again, we consider the Eq. (5.7) to evaluate the reinjection process for different slopes K ,

For $\hat{x} - (K+1)\sigma \leq x \leq \hat{x} + \sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+2)} \quad (5.28)$$

For $\hat{x} + \sigma \leq x \leq \hat{x} + (2K-1)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+2)} \quad (5.29)$$

For $\hat{x} + (2K-1)\sigma \leq x \leq \hat{x} + (3K-1)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+2)} \quad (5.30)$$

For $\hat{x} + (3K-1)\sigma \leq x$

$$\begin{aligned} \Phi(x) = & \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+2)} \\ & + \frac{(x - \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+2)} \end{aligned} \quad (5.31)$$

5.5.2.1 NRPD for $0 < K \leq 1$

We study the reinjection process when the following conditions are verified: $(K+1)\sigma \geq \hat{x}$, $\sigma < \hat{x}$, and $0 < K \leq 1$. Following the proposed methodology, the first step is to evaluate $M(x)$. Function $M(x)$ is shown in Figure 5.6-Top. The parameters are: $b = 0.17$, $\varepsilon = 0.005$, $\hat{x} = 0.194$, $K \cong 0.3459$, $c = 1$, $\sigma = 0.18$ and 150000 reinjected points. Function $M(x)$ shows at least three different behaviors, each one represented by straight lines with different slopes, m_l , m_h and m_r . The first line with slope m_l falls inside the interval $[0, |\hat{x} - (K+1)\sigma|]$, where $\hat{x} - (K+1)\sigma \cong 0.05$. The second line with slope m_h falls inside $[\hat{x} - (K+1)\sigma, x_s]$ and the last one, with slope m_r , corresponds to the interval $[x_s, c]$; where $x_s = \hat{x} + (K+1)\sigma \cong 0.4363$. The slope m_l is close to the slope of the uniform reinjection $m_l \cong 0.5135$. The slope m_r is close to the slope of the reinjection process without noise. The darker (blue) line indicates the points used to calculate this slope, and it is $m_r \cong 0.2560$ ($\alpha_r \cong -0.656$). In the noiseless case, we obtain $m \cong 0.2780$ and $\alpha \cong -0.61$. The difference between α_r and the noiseless α is approximately 7%.

Fig. 5.6 Map (5.7). $(K+1)\sigma \geq \hat{x}$, $\sigma < \hat{x}$ and $0 < K \leq 1$. Top figure: $M(x)$ function, the darker (blue) line indicates the data used to calculate the exponent α . Center figure: NRPD function for $\alpha_r = -0.656$. Bottom figure: Comparison of the NRPD functions for $\alpha_r = -0.656$ and $\alpha = -0.61$. Parameters: $c = 1$, $\varepsilon = 0.005$, $\sigma = 0.18$, $\hat{x} \cong 0.194$, $K \cong 0.3459$.

Using the convolution integral, Eq. (5.19), we obtain the analytical expressions for $\Phi(x)$: Eqs. (5.32), (5.33), (5.34), (5.35) and (5.36). Here, $\Phi(x)$ is defined in five intervals. The first one had not been considered in the previous sections, and it represents the reinjection for $x < 0$. Note that the straight line with slope m_l appears in the interval with reinjection from $x < 0$.

The first interval $[0, (K+1)\sigma - \hat{x}]$ (see Eq. (5.32)) could be understood as if there were a negative LBR ($x_i < 0$). The new LBR, x_i , could be calculated as $x_i = \hat{x} - (K+1)\sigma < 0$. As the map is symmetrical, there is overlapping between the two symmetrical reinjections and $x'_i = (K+1)\sigma - \hat{x}$ is the new LBR for the reinjection process for $x < 0$.

The numerical data and theoretical NRPD function, for the same parameters used in Figure 5.6-Top, are shown in Figure 5.6-Center. To get the theoretical NRPD, we obtain α from $M(x)$ of Figure 5.6-Top. The numerical results are shown by the points and the line represents the analytical NRPD. A good agreement between numerical and theoretical results is obtained. A better correspondence, principally inside of the interval $[0, \hat{x} + (1-K)\sigma]$, between the theoretical NRPD and the numerical results is obtained for the noiseless exponent α . Figure 5.6-Bottom shows the analytical NRPD functions for noisy α_r and noiseless α and the numerical results. The blue (clearer) curve represents the NRPD calculated with $\alpha_r = -0.656$ and the black (darker) one is the NRPD with $\alpha = -0.61$. The last figure shows that the NRPD is sensitive to the exponent α . A good agreement is obtained when the difference between the noisy α_r and the noiseless one is not higher than 10%.

5.5.2.2 NRPD for $1 < K \leq 2$

The reinjection process satisfying the following conditions: $(K+1)\sigma \geq \hat{x}$, $\sigma < \hat{x}$, and $1 < K \leq 2$ is studied. We evaluate the function $M(x)$ shown in Figure 5.7-Top. The parameters are: $b = 0.21$, $\varepsilon = 0.005$, $\hat{x} = 0.7961$, $K \cong 1.1669$, $c = 2$, $\sigma = 0.5$ and 250000 reinjected points. The function $M(x)$ is represented by a continuous line. Unlike the previous cases it is not possible to observe two or three straight lines with different slopes. Following the previous sections, we can calculate the exponent α using the points represented by the darker (blue) line resulting $\alpha_r \cong -0.3195$ ($m_r \cong 0.4049$). In the noiseless case, we obtain $\alpha \cong -0.4937$ ($m = 0.3361$). The difference between α_r and the noiseless α is approximately 35.5%. However, the difference between m_r and the noiseless m only is approximately 17%. Also, we calculate a slope $m_l \cong 0.55$ inside the interval $[0, |\hat{x} - (K+1)\sigma|]$, where $|\hat{x} - (K+1)\sigma| \cong 0.287$. Therefore, very close to the fixed point, the slope m_l is approximately 10% higher than the uniform reinjection slope.

For $0 \leq x \leq (K+1)\sigma - \hat{x}$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (-x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.32)$$

For $(K+1)\sigma - \hat{x} \leq x \leq \hat{x} + (K-1)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.33)$$

For $\hat{x} + (K-1)\sigma \leq x \leq \hat{x} + (1-K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.34)$$

For $\hat{x} + (1-K)\sigma \leq x \leq \hat{x} + (1+K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.35)$$

For $\hat{x} + (1+K)\sigma \leq x$

$$\begin{aligned} \Phi(x) = & \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} + \\ & \frac{(x - \sigma(1+k) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \end{aligned} \quad (5.36)$$

Summarizing, if the noise intensity increases, the exponent α_r obtained, using the noisy $M(x)$, can reduce its accuracy, and, for the last case the analytical results must be evaluated using the noiseless exponent α .

To obtain the NRPD function the convolution integral Eq. (5.19) is solved. In this test α is the noiseless exponent of the RPD, which is obtained from the noiseless evaluation. The analytical expressions for $\Phi(x)$ are: Eqs. (5.37), (5.38), (5.39), (5.40) and (5.41). The numerical and theoretical NRPD functions, for the same parameters used in Figure 5.7-Top, are shown in Figure 5.7-Bottom. The numerical data are represented by points and the line corresponds to the theoretical NRPD. A good correspondence between numerical and theoretical results is observed.

Fig. 5.7 Map (5.7). $(K+1)\sigma \geq \hat{x}$, $\sigma < \hat{x}$, and $0 < K \leq 1$. Top figure: $M(x)$ function. Bottom figure: NRPD function. Parameters: $c = 1$, $\varepsilon = 0.005$, $\sigma = 0.18$, $\hat{x} \cong 0.194$, $K \cong 0.3459$.

Although, the amplified noise is very high $-(K+1)\sigma > 0.5c$ -, the NRPD remains as a sum of exponential functions with exponent $\alpha + 2$; but the noisy $M(x)$ is no longer adequate to correctly calculate the exponent α .

Considering $(K+1)\sigma \geq \hat{x}$ the NRPD is a piecewise function determined by five intervals. Four intervals fall inside $[0, \hat{x} + (K+1)\sigma)$ and the noise strongly affects the zone around to the original LBR, $\hat{x}$.

For $0 \leq x \leq (K+1)\sigma - \hat{x}$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (-x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.37)$$

For $(K+1)\sigma - \hat{x} \leq x \leq \hat{x} + (1-K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.38)$$

For $\hat{x} + (1-K)\sigma \leq x \leq \hat{x} + (K-1)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.39)$$

For $\hat{x} + (K-1)\sigma \leq x \leq \hat{x} + (1+K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha} - (x + \sigma(1+K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \quad (5.40)$$

For $\hat{x} + (1+K)\sigma \leq x$

$$\begin{aligned} \Phi(x) = & \frac{(x + \sigma(1+K) - \hat{x})^{2+\alpha} - (x + \sigma(1-K) - \hat{x})^{2+\alpha} - (x + \sigma(K-1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} + \\ & \frac{(x - \sigma(1+k) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)} \sigma^2 K(\alpha+1)(\alpha+2)} \end{aligned} \quad (5.41)$$

5.6 Effects of adjacent reinjection (AR)

We now analyze the AR effects on maps (5.6) and (5.7). When a trajectory reinjects next to the unstable fixed point, the absolute value of the next iterative points increases driven principally by the parameters ε , a and $(K+1)\sigma$, until a point, x_j , leaves the laminar interval. If the difference between $|x_{j+1}|$ (obtained from (5.2) or (5.3), without noise) and the laminar boundary $|c|$ is less or equal than the amplified noise intensity ($|x_{j+1}| - |c| \leq (K+1)\sigma$) the trajectory can return into the laminar interval. This behavior is produced by the noise and has been recently defined as adjacent reinjection (AR) [11]. If there is no noise ($\sigma = 0$) there is no AR.

The lower difference $d_l = |x_{j+1} = F(x_n)| - |c|$ is given when $|x_j|$ is replaced by c . For the map (5.2), using $\varepsilon = 0.01$, $c = 0.4$, $a = 1$, $b = 1.1$, we obtain: $d_l = 0.4662$, implying that the amplified noise should be higher than the semi-length of the laminar interval ($K\sigma > c$) to obtain AR. In that case the noise effects on the RPD are more important than the intermittency mechanism, and other mathematical techniques are required to analyze the map behavior. A similar behavior is obtained for map (5.3). If we use $\varepsilon = 0.005$, $c = 1$, $b = 1$, we obtain: $d_l = 1.8142$. The amplified noise $(K + 1)\sigma$ should be approximately of the order of c and the noise is stronger than the intermittency phenomenon.

For the type II intermittency studied in [11], the AR mechanism is important, but in the two maps here studied this mechanism is not significant, except for very large amplified noise values where the phenomenon is governed by the noise and the intermittency is hidden.

5.7 Conclusions

In this chapter some recent studies proposed in [5, 10, 14, 11] were extended to consider type III intermittency with nonzero lower boundary of reinjection and high noise intensity. We studied the influence of the noise intensity and the LBR on the reinjection probability density function. Also, the adjacent reinjection mechanism was evaluated.

To carry out this study we have implemented the M function methodology. We found that, for $\text{LBR} = 0$ the laminar interval can be split in subintervals in which the function $M(x)$ preserves the linear form. However, for $\text{LBR} \neq 0$, different behaviors appear. For $(K + 1)\sigma < \text{LBR}$ a zone where the function $M(x)$ is not linear is displayed. This zone is a matching between two linear behaviors of $M(x)$. For $(K + 1)\sigma \geq \text{LBR}$, the function $M(x)$ can be split in three straight lines with different slopes: m_l , m_h and m_r ; where m_l corresponds to a uniform reinjection and m_r is approximately equal to the noiseless slope m . Even, for higher noise intensities the function $M(x)$ is represented by a smooth nonlinear curve.

By means of this methodology, it was possible to calculate the noisy RPD function considering the two iteration processes occurring before the reinjection. Thus, it was necessary to solve two convolution integrals. The NRPD is represented by series of exponential functions with exponent $\alpha + 2$, where α is the noiseless exponent. The new obtained NRPD presents a more complex structure than those previously published [10, 14]. Also, the exponent $\alpha + 2$ is different from that presented before [4, 10, 14, 11].

Through these two iterations occurring before the reinjection, the type III intermittency trajectories are stretched or contracted by the derivative $K = dF(x_n)/dx$, where $x_{n+1} = F(x_n)$ is a reinjected point into the laminar zone. The amplified noise generated

in the reinjection process can be calculated as $(K + 1)\sigma$. The previous studies [10, 14] are only particular cases satisfying $K \approx 0$ or $K \gg 1$.

In all the tests, the numerical data and the theoretical results showed a very good agreement. Note that using the NRPD evaluated from the noisy data, allows to obtain a complete description of the noiseless system, if the noise intensity is not high enough. If the noise strength is high, the noisy $M(x)$ function can introduce errors in the evaluation of the noiseless exponent α . We suggest to use the noiseless α to evaluate the NRPD if the difference between the noisy α and the noiseless α is higher than 10%.

The adjacent reinjection was evaluated and its effects on the RPD were analyzed. For the two maps with type III intermittency studied, the AR mechanism was not important. Only for very high noise intensity values -noise intensities approximately equal to the laminar interval intensity- the AR has influences on the reinjection process. However, for these cases the noise is the principal phenomenon, and the intermittency is hidden by the noise.

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Chapter 6

NON-UNIFORM AND DISCONTINUOUS RPD IN TYPE V INTERMITTENCY

Abstract In this chapter type V intermittency is studied using the M function methodology. This methodology is applied on two different maps to evaluate the reinjection probability density function (RPD), the probability density of laminar lengths and the characteristic relation. We have found two behaviors: 1- The RPD can be nonuniform and it is written as an exponential function, where the uniform reinjection is only a singular case. Also, the probability density of laminar lengths can be a nondifferentiable function when the local map has a nondifferentiable point inside the laminar interval. On the other hand, the characteristic relation is not unique, it depends on the local map. 2- The RPD can be a discontinuous function and it is written as a sum of exponential functions. The RPD shows two behaviors, depending if the reinjection is produced from points higher than the upper limit of the laminar interval or points lower than the lower limit of this interval. Also, the probability density of laminar lengths has two different behaviors in accordance with the RPD function. The dependence of the reinjection probability density function and the probability density of laminar lengths with the reinjection mechanism and the lower boundary of return is considered. For the analyzed map, when the RPD is discontinuous, the characteristic relation verifies $\bar{l} \approx \varepsilon^{-0.5}$. Therefore, the behavior of the reinjection processes and the statistical properties for type V intermittency is wider than the previous studies have described. Finally, it is noted that the M function methodology is a suitable tool to analyze type V intermittency and there are a very high accuracy between the theoretical equations and the numerical data.

6.1 Introduction

In the previous chapters we have described that chaotic intermittency is a route to deterministic chaos, where dynamical systems show transitions between regular phases

and chaotic bursts [1, 2, 3, 4]. Intermittency was classified in three types called I, II and III, in accordance to the Floquet multipliers of the system or to the eigenvalues in the local Poincaré map [5, 6]. Later studies have included other types of chaotic intermittencies: type V, X, on-off (in-out), eyelet and ring [7, 8, 9, 10, 11, 12, 13].

In this chapter we study one of these new types of intermittencies: type V intermittency. The local map and the reinjection mechanism determine the chaotic intermittency for one dimensional maps [1, 2, 4]. The local map specifies the intermittency type (I, II, III or V) and the reinjection mechanism returns the system trajectories from the chaotic zone to the laminar one, which is given by the reinjection probability density (RPD) function. Remember that the RPD establishes the probability that trajectories are reinjected inside the regular zone. Therefore, to calculate accurately the chaotic intermittency phenomenon, the correct determination of the RPD function is significant.

To describe the reinjection mechanism, for type V intermittency, we implement the M function methodology. This methodology has shown to work accurately for several maps exhibiting type I, II and III intermittencies (see previous chapters and references [4, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]).

We will show that this methodology can work accurately for type-V intermittency and the reinjection processes may include nonuniform and discontinuous RPD functions. Also, we shall that the characteristic relations can acquire other forms than those previously published.

The rest of the chapter is organized as follows. In Section 6.2 we present type-V intermittency. Section 6.3 introduces briefly the M function methodology. In Sections 6.4 and 6.5 the first analyzed map is introduced. In Section 6.3 we study numerically several reinjection processes for Map1.

In Sections 5, 6 and 7 we study two maps with type V intermittency. Finally, Section 9 describes the analysis and conclusions.

6.2 Type V intermittency

The concept of type V intermittency was, for the first time, introduced in Refs. [26, 27, 28].

Type I, II and III intermittencies are characterized by continuous local map. However, type V intermittency appears in maps with nondifferentiable, even discontinuous, local maps.

Type I intermittency begins when one eigenvalue of the fixed point reaches +1 and a channel between the bisector line and the local map appears (tangent bifurcation). On the other hand, type V intermittency occurs when a nondifferentiable point (NDP) and

a stable fixed point collide also forming a channel between the bisector line and the map. But here, there is not a tangent bifurcation because the local map is nondifferentiable or discontinuous at this point.

For type V intermittency, the local map is composed of two maps with different slopes: one for the left side of the NDP, and the other one for the right side. These maps describe a “V” [26, 27].

A description about type V intermittency was introduced in Section 2.4 and it will not be repeated here. Only we remember that type V intermittency has been observed in real systems [29, 30, 31].

6.3 Evaluation of the RPD function. The M function methodology

Only for completeness, we present a short description of the M function methodology. Let us take account of a general one dimensional map: $x_{n+1} = F(x_n)$. Then, the $M(x)$ function is [4, 15, 14, 16, 17, 18, 19]:

$$M(x) = \begin{cases} \frac{\int_{\hat{x}}^x \tau \phi(\tau) d\tau}{\int_{\hat{x}}^x \phi(\tau) d\tau}, & \text{if } \int_{\hat{x}}^x \phi(\tau) d\tau \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (6.1)$$

where τ represents the reinjected points around the unstable fixed point, and $\hat{x}$ is the lower boundary of reinjection point (LBR).

$M(x)$ corresponds to the average over the reinjection points in the interval $[\hat{x}, x]$. We can calculate $M(x)$ from a numerical or experimental data series:

$$M(x) \cong \frac{1}{N} \sum_{j=1}^N x_j \quad (6.2)$$

where the data set (reinjection points) $\{x_j\}_{j=1}^N$ must be sorted from the lowest to the highest, i.e. $x_j \leq x_{j+1}$.

In previous chapters, we found that $M(x)$ satisfies a linear approximation for a wide class of maps exhibiting type I, II and III intermittencies [15, 14, 16, 17, 19]:

$$M(x) = \begin{cases} m(x - \hat{x}) + \hat{x}, & \text{if } \hat{x} \leq x \leq c, \\ 0, & \text{otherwise,} \end{cases} \quad (6.3)$$

where the slope $m \in (0, 1)$ governs the reinjection process. Introducing Eq. (6.3) in Eq. (6.1) the corresponding RPD function results [14, 15, 17]:

$$\phi(x) = b(\alpha)(x - \hat{x})^\alpha, \quad \text{with} \quad \alpha = \frac{2m-1}{1-m}, \quad (6.4)$$

where $b(\alpha) = (\alpha + 1)/(c - \hat{x})^{\alpha+1}$ is a normalization parameter.

The $M(x)$ function is determined by the parameters m and $\hat{x}$ (M verifies $M(\hat{x}) = \hat{x}$). Note that the slope m determines the value of the exponent α in the RPD function (see Eq. (6.4)). Thus, the $M(x)$ function stores the nonlinear information of the map. [4, 23, 24].

6.4 Piecewise maps

Maps defined on intervals of the real line have interest in non linear dynamics and chaos [32]. In this chapter we study piecewise maps, $F(x)$:

$$F(x) = \begin{cases} F_1(x), & y_0 \leq x < y_1, \\ F_2(x), & y_1 \leq x < y_2, \\ \cdot \\ \cdot \\ \cdot \\ F_m(x), & y_{m-1} \leq x \leq y_m, \end{cases} \quad (6.5)$$

where $y_1, y_2, \dots, y_m$ are real numbers.

To analyze the behavior of the M function methodology in type V intermittency, two piecewise one dimensional maps with type V intermittency are studied (called here Map1 and Map2). It must be noted that the Map2 is a generalization of the map introduced in [27, 28].

6.5 Map1

The Map1 is built with three functions: linear, quadratic and reinjection. We have selected this map because it is defined by simple functions; but, at the same time, it shows interesting reinjection processes. The map can be written as:

$$F(x) = \begin{cases} F_1(x) = a_1x + \varepsilon, & \tilde{x} \leq x < 0, \\ F_2(x) = \varepsilon + x + a_2x^2, & 0 \leq x < x_m, \\ F_3(x) = \tilde{x} + \frac{(y_m - \tilde{x})(y_m - x)^\gamma}{(y_m - x_m)^\gamma}, & x_m \leq x \leq y_m, \end{cases} \quad (6.6)$$

where $y_m = F(x_m) = 1$, $\tilde{x}$ is the lower boundary of return [4], $0 < a_1 < 1$ is the slope of the straight line, a_2 is the coefficient of the quadratic term in $F_2(x)$, and finally ε is the control parameter. If $\tilde{x} = 0$, the map is defined inside the interval $[0, 1]$. Note that $F_1(x)$ is a straight line and $F_2(x)$ is the local map for type I intermittency. The function $F_3(x)$ produces the reinjection process, and it depends on γ . Therefore, this map allows us to analyze different types of reinjection mechanisms using different values of γ . Figure 6.1 shows this map and one reinjection trajectory.

Fig. 6.1 Map1 for $\gamma = 2$, $\varepsilon = 0.001$, $a_1 = 0.5$, $a_2 = 1$.

The map (6.6) have three fixed points:

$$\begin{aligned} x_1 &= \frac{\varepsilon}{1-a_1}, \\ x_2 &= \left(\frac{-\varepsilon}{a_2} \right)^{0.5}, \\ x_3 &= \tilde{x} + \frac{(y_m - \tilde{x})(y_m - x_3)^\gamma}{(y_m - x_m)^\gamma}, \end{aligned} \tag{6.7}$$

If $\varepsilon < 0$ the fixed point x_1 is less than zero and x_2 is higher than zero. Also, x_1 and x_2 disappear for $\varepsilon > 0$, and they are equal zero if $\varepsilon = 0$. Accordingly, for $\varepsilon = 0$, the fixed points x_1 and x_2 collide in one fixed point $x_0 = 0$.

Also the map (6.6) possesses a discontinuous point at $x = 0$, which collides with the fixed point x_0 for $\varepsilon = 0$. The fixed point vanishes and type V intermittency can appear for $0 < \varepsilon \ll 1$. The iteration procedure —governed by a_1 , a_2 and ε — gives increasing values of x_n generated from an initial one, close to $F(\tilde{x})$. A chaotic burst —governed by $F_2(x)$ and $F_3(x)$ — occurs when $x > 0$, which ends when the trajectory is reinjected into a point in the laminar zone. Then, a new iterative process —governed by ε , a_1 and a_2 — will be developed producing larger values of the new successive iterative points.

This map can generate two reinjection mechanisms. The first one from points $x < x_0 - c$, where $F_1(x)$ returns the iterative process inside the laminar interval. The second mechanism is from $x > x_0 + c$, where $F_3(x)$ maps points inside the laminar interval as it is indicated in Figure 6.1. $x_0 - c$ and $x_0 + c$ are the lower and upper ends of the laminar interval respectively. Note that $2c$ is the length of the laminar interval.

In the first sections, we only consider the second mechanism. It must be noted that the first mechanism produces discontinuous RPD — this phenomenon will be studied in Sections 6 - 11.

When the reinjection mechanism is only given by $F_3(x)$; the exponent γ drives it, whereas a_1 , a_2 and ε determine the laminar phase duration.

To study the intermittency behavior in this map, we shall carry out several numerical experiments or tests considering different reinjection processes.

6.6 Reinjection process for Map1

The first test considers $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$ and $\tilde{x} = x_0 - c$. Where N is the number of reinjected points. Note that the lower boundary of return is the lower limit of the laminar interval; then there is no reinjection from points $x < x_0 - c$. The reinjected points inside the laminar interval, $\Delta = [x_0 - c, x_0 + c]$, are mapped by $F_3(x)$ from points placed inside the interval $\Delta_0(x) = F_3^{-1}[x_0 - c, x_0 + c]$.

The theoretical and numerical $M(x)$ functions are shown in Figure 6.2. Red points are the numerical data and the blue line is the minimum square approximation. The $M(x)$ function is a straight line with slope $m \approx 0.5$ ($\alpha \approx 0.0$). Therefore, by Eq. (6.4), the theoretical RPD can be written as: $\phi(x) = (2c)^{-1}$. Figure 6.3 shows the numerical and theoretical RPDs. From the figure we can observe that the reinjection process produces an uniform RPD, also the numerical and theoretical results have a very good agreement.

Fig. 6.2 $M(x)$ function for map (6.6) with $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Red line is the numerical data and the continuous line represents the theoretical $M(x)$ function. The slope $m \cong 0.5030$, and $\alpha \cong 0.0121$.

Fig. 6.3 RPD for map (6.6) with $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Points are numerical results and the continuous line represents the theoretical RPD.

Other important function to describe the intermittency is the probability density of the laminar lengths, $\psi(l)$, which is a global property of the map [1, 4]. The laminar length, $l(x, c)$, is defined as the number of iterations that a trajectory carries out inside the laminar interval; and the probability density of the laminar lengths determines the probability of finding laminar lengths between l and $l + dl$, and it is given by [1, 4]:

$$\psi(l, c) = \phi[X(l, c)] \left| \frac{dX(l, c)}{dl} \right|, \quad (6.8)$$

where $X(l, c)$ is the inverse of $l(x, c)$. For the map (6.6), $\frac{dX(l, c)}{dl}$ is given by:

$$\begin{aligned} \frac{dX(l, c)}{dl} &= \varepsilon + x(a_1 - 1), & x < 0, \\ \frac{dX(l, c)}{dl} &= \varepsilon + a_2 x^2, & x \geq 0, \end{aligned} \quad (6.9)$$

We highlight that $\frac{dX(l, c)}{dl}$ is a discontinuous function. From these last equations we can evaluate $l(x, c)$:

$$\begin{aligned} l(x, c) &= \frac{\ln(\varepsilon) - \ln((a_1 - 1)x + \varepsilon)}{a_1 - 1} + \frac{\arctan(c \frac{\sqrt{a_2}}{\sqrt{\varepsilon}})}{\sqrt{a_2} \varepsilon}, & x < 0, \\ l(x, c) &= \frac{\arctan(c \frac{\sqrt{a_2}}{\sqrt{\varepsilon}}) - \arctan(x \frac{\sqrt{a_2}}{\sqrt{\varepsilon}})}{\sqrt{a_2} \varepsilon}, & x \geq 0, \end{aligned} \quad (6.10)$$

Figure 6.4 shows the numerical and theoretical probability density of the laminar lengths for the same parameters of Figures 6.2 and 6.3. The theoretical $\psi(l)$ was evaluated using Eq. (6.8) and it is shown as a blue line; whereas the points are the numerical results. There is a very good accuracy between the theoretical evaluation and the numerical data. The probability density of the laminar lengths has two behaviors: one for $0 < l \leq 40$, in which $\psi(l)$ is a decreasing function and corresponds to points reinjected inside $[x_0, x_0 + c]$; the other one appears for $l > 40$ and it is given by points reinjected in $[x_0 - c, x_0)$.

We study a second test which considers the same parameters used in the previous test, with the exception of the exponent γ , which now is $\gamma = 2$; i.e. $F_3(x)$ becomes a nonlinear function. The $M(x)$ function is shown in Figure 6.5. The red and blue lines show the numerical data and minimum square approach respectively. From this figure we can note that the $M(x)$ function can be considered as a linear function. From the minimum square approach we have obtained $m \cong 0.3348$ and $\alpha \approx -0.5$. The theoretical RPD results as (see Eq. (6.4)):

$$\phi(x) \cong 1.118 (x + 0.1)^{-0.5} \quad (6.11)$$

Fig. 6.4 $\psi(l)$ for map (6.6) with $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Red points are numerical results and the blue line represents the theoretical probability density of the laminar lengths.

Figure 6.6 shows the numerical and theoretical RPDs. The theoretical RPD is evaluated using the linear approximation for the $M(x)$ function. The red points are the numerical RPD, and the continuous blue line represents the theoretical RPD which is very accurate approximation to the numerical data. The RPD is a decreasing function with a high concentration of reinjection trajectories close to lower limit of the laminar interval, $x_0 - c$. Similar results have been documented for other types of intermittency [14, 15, 18] and there is theoretical studies about it [4, 23].

The third numerical case uses the same parameters of the first test, but in this case $\gamma = 0.5$, i.e. $F_3(x)$ is again a nonlinear function, but different those used in Test 2. Figure 6.7 shows the numerical and theoretical $M(x)$ functions, which result lineal functions with slope $m \cong 0.6639$ ($\alpha \cong 1$). Then the RPD results:

$$\phi(x) \cong 50(x + 0.1) \quad (6.12)$$

Note that the RPD is not uniform, it is an increasing function with high concentration of reinjection points close to the upper end of the laminar interval, $x_0 + c$. The theoretical $\phi(x)$ function increases linearly as x increases.

Figure 6.9 shows the numerical and theoretical probabilities density of the laminar lengths. The continuous line shows the theoretical results and it was calculated by Eq. (6.8). Red points represent the numerical data. The theoretical result shows high accuracy regarding the numerical data.

Fig. 6.5 $M(x)$ function for map (6.6) with $\gamma = 2$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Red line is the numerical data and the continuous line represents the theoretical $M(x)$ function. The slope $m \cong 0.348$, and $\alpha \cong -0.466$.

Fig. 6.6 RPD for map (6.6) with $\gamma = 2$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Points are numerical results and the continuous line represents the theoretical RPD.

Fig. 6.7 $M(x)$ function for map (6.6) with $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Red line is the numerical data and the continuous line represents the theoretical $M(x)$ function. The slope $m \cong 0.6639$, and $\alpha \cong 0.975$.

Fig. 6.8 RPD for map (6.6) with $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Points are numerical results and the continuous line represents the theoretical RPD.

Fig. 6.9 $\psi(l)$ for map (6.6) with $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1$. Points are numerical results and the continuous line represents the theoretical probability density of the laminar lengths.

To evaluate the “duration” of the laminar phases we also consider the average laminar length, which can be evaluated as [1, 2, 4]:

$$\bar{l} = \int_0^{l_m} \psi(l, c) l dl = \int_{-c}^c \phi(x) l(x, c) dx \quad (6.13)$$

where l_m is the highest laminar length.

From the numerical data and the theoretical results shown in Figures 6.4 and 6.9 we can evaluate the first integral in (6.13).

On the other hand, by Eqs. (6.4), (6.10) and (6.13), we can obtain the average laminar length and the characteristic relation:

$$\bar{l} = \int_{-c}^0 \phi(x) l(x, c) |_{x < 0} dx + \int_0^c \phi(x) l(x, c) |_{x > 0} dx \quad (6.14)$$

We have found that for $\gamma = 1$, the exponent $\alpha \cong 0$. For $\alpha = 0$ we can integrate analytically Eq. (6.13), then the average laminar length results:

$$\begin{aligned} \bar{l} = & \left(1 - \frac{\varepsilon}{c(a_1 - 1)}\right) \frac{\ln(\varepsilon)}{2(a_1 - 1)} + \frac{(\varepsilon - (a_1 - 1)c) \ln(\varepsilon - (a_1 - 1)c)}{2c(a_1 - 1)^2} + \\ & + \frac{1}{2(a_1 - 1)} + \frac{\arctan\left(c \frac{\sqrt{a_2}}{\sqrt{\varepsilon}}\right)}{2\sqrt{a_2}\varepsilon} + \frac{\ln\left(1 + \frac{a_2 c^2}{\varepsilon}\right)}{4a_2 c} \end{aligned} \quad (6.15)$$

Figure 6.10 shows the average laminar length as function of the control parameter ε for $\gamma = 1$ –then it shows the characteristic relation. The blue line represents the Eq. (6.15), and the red points are numerical results. From this figure the characteristic relation, for $\varepsilon \rightarrow 0$, can be approximated as:

$$\bar{l} \propto \varepsilon^{-\beta} \quad (6.16)$$

where $\beta \cong 0.5$.

Fig. 6.10 $\ln(\bar{l})$ as function of $\ln(\varepsilon)$ for map (6.6) with $\gamma = 1$. Points are numerical data and the continuous line represents the theoretical results, Eq. (6.15).

For $\gamma = 0.5$, we have calculated $\alpha \cong 1$. Then, the RPD can be written as: $\phi(x) = b(x + c)$, and we can obtain the theoretical expression for the average laminar length, which is given by Eq. (6.17).

$$\begin{aligned} \bar{l} = & \frac{2c^2(a_1-1)^2\sqrt{a_2\varepsilon}\arctan(c\sqrt{a_2/\varepsilon})+a_2\varepsilon((a_1-1)c(3c(a_1-1)-2\varepsilon)+2(\varepsilon-(a_1-1)c)^2(\ln(\varepsilon)-\ln(\varepsilon-(a_1-1)c)))}{8a_2c^2\varepsilon(a_1-1)^3} \\ & + \frac{2c(1-\ln(\varepsilon)+\ln(\varepsilon+a_2c^2))-\frac{2\sqrt{\varepsilon}}{\sqrt{a_2}}\arctan(c\sqrt{a_2/\varepsilon})}{8a_2c^2} \end{aligned} \quad (6.17)$$

Figure 6.11 shows the characteristic relation for $\gamma = 0.5$. The blue line represents the Eq. (6.17), and the red points are numerical evaluations. When $\varepsilon \rightarrow 0$, the characteristic relation can be again expressed by Eq.(6.16) with $\beta \cong 0.5$.

From the previous tests, we have found there are a high accuracy between numerical and theoretical average laminar lengths.

Fig. 6.11 Variation of $\ln(\bar{l})$ as function of $\ln(\varepsilon)$ for map (6.6) with $\gamma = 0.5$. Points are numerical results and the continuous line represents the theoretical approximation, Eq. (6.17).

We also carried out several numerical experiments using $\gamma = 0.75$. We have considered $\varepsilon = 1 \times 10^{-3} - 1 \times 10^{-8}$. For all these tests $\alpha \cong 1/3$ (see Table 6.1).

ε	α
1×10^{-3}	0.3320
1×10^{-4}	0.3322
1×10^{-5}	0.3232
1×10^{-6}	0.3244
1×10^{-7}	0.3376
1×10^{-8}	0.3332

Table 6.1 Values of α , Eq. (6.4), for different values of ε . $\gamma = 0.75$.

Figure 6.12 shows $\ln(\bar{l})$ as function of $\ln(\varepsilon)$ for $\gamma = 0.75$. The relation can be represented by a straight line with slope ≈ -0.5 , then the characteristic relation can be written as $\bar{l} \propto \varepsilon^{-0.5}$

Fig. 6.12 Numerical variation of $\ln(\bar{l})$ as function of $\ln(\varepsilon)$ for map (6.6) with $\gamma = 0.75$.

To study the sensibility of the reinjection process regarding the slope a_1 , we evaluate theoretical and numerically the exponent α for different values of a_1 . The theoretical α for different values of a_1 and γ are given in Table 6.2.

$a_1 - \gamma$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 1$
$a_1 = 0.25$	1.0202	0.3378	0.0145
$a_1 = 0.50$	1.0036	0.3320	0.0113
$a_1 = 0.90$	0.9750	0.3320	0.0121

Table 6.2 Variation of exponent α for different a_1 and γ .

We have found a very good accuracy between them, and we have obtained that a_1 does not influence the reinjection process. The reinjection is mainly ruled by the exponent γ . Therefore, for small c , the local map does not influence the reinjection mechanism.

In previous works, the following relation was analytically found :

$$\alpha = \frac{1}{\gamma} - 1 \quad (6.18)$$

It was applied successfully for maps with continuous local maps [4, 23]. However, Eq. (6.18) has also been valid for all the tests here studied. From Table 6.2, for $\gamma = 1$ we have obtained $\alpha \cong 0$ (see also Figures 6.2 and 6.3); for $\gamma = 0.5$ we have calculated $\alpha \cong 1$ (see Figures 6.7 and 6.8), and for $\gamma = 0.75$ we have found $\alpha \cong 1/3$. On the other hand, for $\gamma = 2$ we have obtained $\alpha \cong -0.5$ (see Figures 6.5 and 6.6).

All previous tests verify $\tilde{x} = x_0 - c$, which does not allow reinjections from points $x < x_0 - c$. However, the RPDs functions show different behaviors depending on the γ value. We have found three different behaviors:

- For $\gamma = 1$ there is uniform reinjection.
- For $\gamma > 1$ the RPD is a decreasing function with a high concentration of reinjection points close to the lower boundary of reinjection.
- For $\gamma < 1$ the RPD is an increasing function with a high concentration of reinjection points close to upper end of the laminar interval.

We highlight that $M(x)$ functions are lineal for all tests, and the M function methodology has captured accurately the numerical results.

On the other hand, we have found that the characteristic relation can be written as: $\bar{l} \propto \varepsilon^{-0.5}$ for different reinjection processes.

6.7 Map2

The Map2 is given by Eqs. (6.19) and (6.20). When $\gamma = 1$ system (6.19) reduces to those presented in a seminal paper about type V intermittency [28]. This map is shown in Figure 6.13.

$$F(x) = \begin{cases} F_1(x) = \left[\frac{f^{1/\gamma}(b_0) - f^{1/\gamma}(b_1)}{b_0 - b_1} (x - b_1) + f^{1/\gamma}(b_1) \right]^\gamma, & -0.8\pi \leq x < -0.5\pi, \\ F_2(x) = \frac{0.8\pi - g + F_3(x_{out})}{x_{out} + 0.5\pi} x + \frac{0.5\pi[F_3(x_{out}) - g] - 0.8\pi x_{out}}{x_{out} + 0.5\pi}, & -0.5\pi \leq x < x_{out}, \\ F_3(x) = x - a \sin(x) - \varepsilon, & x_{out} \leq x < \frac{23\pi}{20}, \\ F_4(x) = \frac{20[0.8\pi + F_3(x_{out})]}{\pi} x - 19.2\pi - 23F_3(x_{out}), & \frac{23\pi}{20} \leq x \leq \frac{6\pi}{5}, \end{cases} \quad (6.19)$$

Where $b_0 = -0.8\pi$, $b_1 = -0.5\pi$, $x_{out} = \varepsilon/(1 - a)$, g is the gap between $F_2(x_{out})$ and $F_3(x_{out})$; and the function $f(x)$ is:

$$f(x) = \frac{10[1.2\pi - F_3(x_{out})]}{3\pi}x + 3.2\pi - \frac{5F_3(x_{out})}{3} \quad (6.20)$$

Fig. 6.13 Map2 for $\gamma = 0.5$, $\varepsilon = 0.001$, $a = 0.25$. The upper end of the laminar interval, $x_0 = c$, is indicated by the vertical dashed-dot line.

Eqs. (6.19) and (6.20) allow us to obtain several reinjection processes, being the case studied in [28] only one process ($\gamma = 1$).

The Map2, for $\varepsilon \neq 0$, has one fixed point: $x_0 = \pi + \arcsin(\varepsilon/a)$. In [28], the laminar interval is $x_{out} \leq x \leq x_0 = c$; where $x_{out} = \varepsilon/(1-a)$. Note that the amplitude of the laminar interval is as large as the turbulent regime amplitude. However, to compare the new results with the previous ones given in [28], we use the same laminar interval (see Figure 6.13). Then, $F_1(x)$ returns the trajectories from the chaotic interval to the laminar one.

6.8 Reinjection process for Map2

For all tests, to obtain the statistical variables of the reinjection processes the M function methodology is applied. Then, the RPD can be written as:

$$\phi(x) = \frac{(x - x_{out})^\alpha}{\gamma(c - x_{out})^{1/\gamma}}, \quad \text{with} \quad \alpha = \frac{1}{\gamma} - 1 \quad (6.21)$$

Three different tests are analyzed: $\gamma = 1$, $\gamma = 0.5$ and $\gamma = 1.5$; each one of them produces a different reinjection mechanism.

6.8.1 Test 1

The first test corresponds to those analyzed in [28], then $\gamma = 1$ and the laminar interval is $x_{out} \leq x \leq x_0$. Figure 6.14 shows the numerical $M(x)$ function (red) and its minimum square approximation (blue) for $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1$, $Nj = 50000$. $M(x)$ can be represented by a linear function with slope $m \cong 0.4845$ ($\alpha \cong -0.06$). Accordingly, the RPD function results:

$$\phi(x) \cong \frac{(x - x_{out})^\alpha}{x_0 - x_{out}} \cong \frac{1}{x_0 - x_{out}} \quad (6.22)$$

Note that Eq. (6.18) is verified: $\alpha \cong 0$.

Fig. 6.14 $M(x)$ functions for Eq. (6.19). Red: numerical. Blue: minimum square. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1$, $Nj = 50000$.

The comparison between numerical and theoretical RPDs is shown in Figure 6.15. The blue line is the theoretical function, which is evaluated using Eq. (6.22).

Fig. 6.15 Theoretical and numerical RPDs functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1$, $Nj = 50000$.

Figure 6.16 shows the comparison between numerical and theoretical probability density of the laminar lengths, $\psi(l, c)$, which is calculated using Eq. (6.8). Where:

$$\frac{dX(l, c)}{dl} = -(a \sin(x) + \varepsilon) \quad (6.23)$$

Integrating this equation, we can obtain the relation between the laminar length and the reinjection point:

$$l(x, x_{out}) = \frac{\ln \left| \frac{\varepsilon \tan(x/2) + a - \sqrt{a^2 - \varepsilon^2}}{\varepsilon \tan(x/2) + a + \sqrt{a^2 - \varepsilon^2}} \right| - \ln \left| \frac{\varepsilon \tan(x_{out}/2) + a - \sqrt{a^2 - \varepsilon^2}}{\varepsilon \tan(x_{out}/2) + a + \sqrt{a^2 - \varepsilon^2}} \right|}{\sqrt{a^2 - \varepsilon^2}} \quad (6.24)$$

The previous results are in agreement with those included in [28]. But, other reinjection processes did not study previously can appear for Map2. Therefore, we will analyze the system (6.19)-(6.20) for different values of the parameter γ .

6.8.2 Test 2

The second test uses the same parameters of the first test, but $\gamma = 0.5$. Then, the main parameter are: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 0.5$, $Nj = 50000$. The laminar interval is

Fig. 6.16 Theoretical and numerical $\psi(l, c)$ functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1$, $Nj = 50000$.

$x_{out} \leq x \leq x_0$. Figure 6.17 shows the numerical M function (red) and its minimum square approximation (blue). $M(x)$ can be approached by a linear function with slope $m \cong 0.66$ ($\alpha \cong 0.94$). Then, by Eq. (6.4), the RPD is:

$$\phi(x) \cong 1.94 \frac{(x - x_{out})^{0.94}}{(x_0 - x_{out})^{1.94}} \quad (6.25)$$

Theoretical and numerical RPDs are displayed in Figure 6.18. The theoretical prediction, Eq. (6.18), establishes $\alpha = 1$. Therefore, the error in the evaluation of the α exponent is $6/100$, which is low considering the large laminar interval (the laminar iterations can reach the same length that the turbulent ones).

Figure 6.19 shows the theoretical and numerical probability density of the laminar lengths, $\psi(l, c)$, which is calculated using Eqs. (6.8) and (6.24). Red points represent numerical data and the theoretical result is the blue line. All the figures show a good accuracy between numerical data and theoretical predictions although the large laminar interval.

It must be noted that M function methodology can capture accurately the RPD. In this test, $\phi(x)$ increases with x . There is a high concentration of reinjection points close to the upper end of the laminar interval x_0 . Again the M function methodology has been a suitable tool to describe the reinjection process.

Fig. 6.17 $M(x)$ functions for Eq. (6.19). Red: numerical. Blue: minimum square. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 0.5$, $Nj = 50000$.

Fig. 6.18 Theoretical and numerical RPD functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 0.5$, $Nj = 50000$.

Fig. 6.19 Theoretical and numerical $\psi(l, c)$ functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 0.5$, $Nj = 50000$.

6.8.3 Test 3

This test has the same parameters that Test 1, but $\gamma = 1.5$. The laminar interval is $x_{out} \leq x \leq x_0$. The numerical $M(x)$ function (red) and its minimum square approach are shown in Figure 6.20. Note that a straight line is a good approach for $M(x)$ function ($m \cong 0.3878$ and $\alpha \cong -0.3772$).

The RPD is calculated using the M function methodology:

$$\phi(x) \cong 0.6228 \frac{(x - x_{out})^{-0.3772}}{(x_0 - x_{out})^{0.6228}} \quad (6.26)$$

The results calculated with Eq. (6.26) are compared with numerical data in Figure 6.21. The RPD is again a nonlinear function, which decreases with increasing x . Note the high concentration of reinjected points close to the lower limit of the laminar interval x_{out} .

Finally, the probability density of the laminar lengths is shown in Figure 6.22. The theoretical $\phi(x)$ and $\psi(l)$ have good accuracy with regard the numerical results.

Fig. 6.20 M functions for Eq. (6.19). Red: numerical. Blue: minimum square. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1.5$, $Nj = 50000$.

Fig. 6.21 Theoretical and numerical RPD functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1.5$, $Nj = 50000$.

Fig. 6.22 Theoretical and numerical $\psi(l)$ functions for Eq. (6.19). Red: numerical. Blue: theoretical. Parameters: $\varepsilon = 0.0001$, $a = 0.25$, $\gamma = 1.5$, $Nj = 50000$.

6.8.4 Characteristic Relation

Eq. (6.24) establishes the relation between the laminar length, l , and the reinjection point, x . For $\varepsilon \rightarrow 0$, this equation can be approximated by [28]:

$$l(x, x_{out}) \cong \frac{|\ln(\varepsilon)| + \ln(2a(1-a)) + \ln[\tan(x/2)]}{a} \quad (6.27)$$

Therefore, using Eqs. (6.4) and (6.27), the average laminar length results:

$$\bar{l} = \int_{x_{out}}^{x_0} \frac{b(x - x_{out})^\alpha [|\ln(\varepsilon)| + \ln(2a(1-a)\tan(\frac{x}{2}))]}{a} dx \quad (6.28)$$

Where $b = (\alpha + 1) [(x_0 - x_{out})^{-\alpha-1}]$. Then, the average laminar length can be approximated by:

$$\bar{l} \approx \frac{|\ln(\varepsilon)|}{a} + k \quad (6.29)$$

Where k is a constant. To obtain this equation we have considered that x_{out} does not depend on ε in the last term of Eq. (6.28).

Figure 6.23 shows $\bar{l}$ as function of $\ln(\varepsilon)$ for uniform reinjection ($\gamma = 1$ and $\alpha = 0$). Numerical data are shown as red points and the blue line is evaluated using minimum

square, which has a slope $|h_n| \cong 3.4$. Remember that Eq. (6.29) is an approximated equation which predicts a slope $|h| = 1/a = 4$.

Fig. 6.23 Characteristic relation for Eq. (6.19) when $a = 0.25$, $\gamma = 1.0$, $Nj = 50000$, $c = 0.1$. Red: numerical data. Blue: minimum square approach.

To analyze nonuniform reinjection process, we evaluate several tests for $\gamma = 1.5$ and different values of ε . Table 6.3 shows the exponent α calculated using the M function methodology for $\varepsilon = 1 \times 10^{-3} - 3 \times 10^{-5}$. The theoretical prediction, Eq. (6.18), is verified ($\alpha \cong -1/3$).

ε	α
1×10^{-3}	-0.3185
5×10^{-4}	-0.3145
3×10^{-4}	-0.323
2×10^{-4}	-0.322
1.5×10^{-4}	-0.32
1×10^{-4}	-0.332
5×10^{-5}	-0.333
3×10^{-5}	-0.343

Table 6.3 Values of α , Eq. (6.4), for different values of ε . $\gamma = 1.5$.

On the other hand, Figure 6.24 shows $\bar{l}$ as function of $\ln(\varepsilon)$ for $\gamma = 1.5$ and $\alpha \cong -1/3$. Red points are the numerical data and the blue line is the minimum square approach. As in the uniform reinjection test, the relation $\bar{l} = \bar{l}(\ln \varepsilon)$ is a straight line,

but the numerical slope is smaller, $|h_n| \cong 3.1$. It must be noted that the assumptions used to obtain Eq. (6.29) introduce errors in the slope evaluation. However, Eq. (6.29) captures the linear relation between $\bar{l}$ and $\ln(\varepsilon)$ for uniform and nonuniform RPDs .

Fig. 6.24 Characteristic relation for Eq. (6.19) when $a = 0.25$, $\gamma = 1.5$, $Nj = 30000$, $c = 0.1$. Red: numerical data. Blue: minimum square approach.

6.9 Discontinuous RPD for piecewise monotonic maps

Let us consider the piecewise one-dimensional map given by Eq. (6.6). We shall analyze tests in which the lower boundary of return does not match the lower boundary of the laminar interval, more specifically $\tilde{x} < x_0 - c$ (where $\tilde{x}$ is the lower boundary of return). Then, trajectories coming from $x < x_0 - c$ could reinject in the laminar interval. Two different cases can appear:

- a- All trayectories passing through the lower boundary of return match the lower boundary of the laminar interval: $F^n(\tilde{x}) = x_0 - c$ for some n .
- b- No iteration of the lower boundary of return matches the lower boundary of the laminar interval: $F^n(\tilde{x}) \neq x_0 - c$ for all n .

To study the intermittency behavior in this map, we shall carry out several numerical experiments or tests considering different reinjection processes for the case (a). Note, that we can always turn a case (b) into a case (a) by moving slightly the laminar interval. Therefore, we only study the case (a).

We consider three tests verifying $F^n(\tilde{x}) = x_0 - c$ for some n . The first test uses the following parameters $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$. Note that $\tilde{x} = F^{-20}(x_0 - c)$. Therefore, for this test there are two reinjection mechanisms. One of them is directly generated by $F_3(x)$; this mechanism is the same one evaluated in the previous sections and in Ref. [33]. The second mechanism corresponds to points reinjected from values $x < x_0 - c$. Then, some points placed in the interval $\Delta_i = [\tilde{x}, x_0 - c)$ will be reinjected in the interval $\Delta_1 = [x_0 - c, x_s)$, being $x_s = F(x_0 - c)$ a singular point where the function $M(x)$ is discontinuous. Note that Δ_1 is a subinterval of the laminar interval $[x_0 - c, x_0 + c]$ [4, 20].

Figure 6.25 shows the numerical and theoretical $M(x)$ functions. The theoretical $M(x)$ function is calculated using Eq. (6.2). In this figure the laminar interval is splitted in two subintervals: $\Delta_1 = [x_0 - c, x_s)$ and $\Delta_2 = [x_s, x_0 + c]$. Δ_1 possesses reinjected points coming from $x < x_0 - c$ and from $x > x_0 + c$. However, the interval Δ_2 only receives points from $x > x_0 + c$.

Figures 6.26 and 6.27 show the $M(x)$ functions inside the intervals $\Delta_1 = [x_0 - c, x_s]$ and $\Delta = [x_0 - c, x_0 + c]$ respectively.

Points reinjected from $x < x_0 - c$ are placed inside the subinterval Δ_1 ; the numerical function $M_i(x)$ is calculated only using these points. $M_i(x)$ is a linear function with slope $m_i \cong 0.5006$ and $\alpha_i \cong -0.00249$ (see Figure 6.26). Points reinjected from $x > x_0 + c$ are distributed along all the laminar interval $[x_0 - c, x_0 + c]$; the numerical $M_s(x)$ function is evaluated using these points, and it is shown in Figure 6.27, which is also a linear function ($m_s \cong 0.5089$ and $\alpha_s \cong 0.03623$).

As the $M(x)$ function has a nondifferentiable point (see Figure 6.25), the RPD will be discontinuous at this point. With $\phi_1(x)$ we restrict the analysis to the interval $[x_0 - c, x_s)$ and with $\phi_2(x)$ we consider only points $x \geq x_s$ (see Eq. (6.30)). Therefore, the RPD can be written as:

$$\phi(x) = \begin{cases} \phi_1(x) = \phi_i(x) + \phi_s(x), & x < x_s, \\ \phi_2(x) = \phi_s(x), & x \geq x_s. \end{cases} \quad (6.30)$$

where:

$$\begin{aligned} \phi_i(x) &= b (x - x_0 + c)^{\alpha_i}, \\ \phi_s(x) &= b k (x - x_0 + c)^{\alpha_s}. \end{aligned} \quad (6.31)$$

The RPD, $\phi(x)$, is obtained by superimposing two reinjection processes represented by $\phi_i(x)$ and $\phi_s(x)$. Where $\phi_i(x)$ and $\phi_s(x)$ are calculated using trajectories coming from $x < x_0 - c$ and $x > x_0 + c$ respectively.

To find the parameters m_i and m_s we firstly sort the numerical data and we apply the M function methodology as described previously. Then, we can calculate the functions $\phi_i(x)$ and $\phi_s(x)$ using Eq. (6.31).

Consequently, the exponents α_i and α_s are obtained from:

Fig. 6.25 $M(x)$ function for map (6.6) with $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\bar{x} = -1.00006$. Red line is the numerical data and the continuous line represents the theoretical $M(x)$ function.

Fig. 6.26 Function $M_i(x)$ inside $\Delta_1 = [x_0 - c, x_s]$ for map (6.6) calculated using only reinjected points coming from $x < x_0 - c$. Parameters: $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\bar{x} = -1.00006$. From this figure we obtain: $m_i \cong 0.5006$ and $\alpha_i \cong -0.00249$.

Fig. 6.27 Function $M_s(x)$ inside $\Delta = [x_0 - c, x_0 + c]$ for map (6.6) obtained for reinjected points coming from $x > x_m$. Parameters: $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\bar{x} = -1.00006$. From this figure we calculate: $m_s \cong 0.5089$ and $\alpha_s \cong 0.03623$.

$$\alpha_{i,s} = \frac{2m_{i,s} - 1}{1 - m_{i,s}}, \quad (6.32)$$

being m_i the slope of the following function

$$M_i(x) = m_i(x - x_0 + c) + x_0 - c \quad (6.33)$$

which is defined in $[x_0 - c, x_s]$, and it considers only reinjected points coming from $x < x_0 - c$; and m_s the slope of

$$M_s(x) = m_s(x - x_0 + c) + x_0 - c \quad (6.34)$$

which is calculated using only reinjected points coming from $x > x_0 + c$ and it is defined in $[x_0 - c, x_0 + c]$.

The factor b is the normalization parameter and it results:

$$b = \left(\frac{(x_s - x_0 + c)^{1+\alpha_i}}{1 + \alpha_i} + k \frac{(2c)^{1+\alpha_s}}{1 + \alpha_s} \right)^{-1} \quad (6.35)$$

To obtain the global $M(x)$ function, we again apply the M function methodology described in Section 2.2. Therefore, the global $M(x)$ function is given by Eqs. (6.36) and (6.37). We highlight that this function has a nondifferentiable point at $x = x_s$.

For $x_0 - c \leq x < x_s$:

$$M(x) = \frac{\int_{x_0-c}^x \tau \phi_i(\tau) \tau + \int_{x_0-c}^x \tau \phi_s(\tau) \tau}{\int_{x_0-c}^x \phi_i(\tau) \tau + \int_{x_0-c}^x \phi_s(\tau) \tau} = \frac{\frac{(c+x)^{1+\alpha_i}(-c+x(1+\alpha_i))}{(1+\alpha_i)(2+\alpha_i)} + \frac{k(c+x)^{1+\alpha_s}(-c+x(1+\alpha_s))}{(1+\alpha_s)(2+\alpha_s)}}{\frac{(c+x)^{1+\alpha_i}}{1+\alpha_i} + \frac{k(c+x)^{1+\alpha_s}}{1+\alpha_s}} \quad (6.36)$$

For $x \geq x_s$:

$$\begin{aligned} M(x) &= \frac{\int_{x_0-c}^{x_s} \tau \phi_i(\tau) \tau + \int_{x_0-c}^{x_s} \tau \phi_s(\tau) \tau + \int_{x_s}^x \tau \phi_s(\tau) \tau}{\int_{x_0-c}^{x_s} \phi_i(\tau) \tau + \int_{x_0-c}^{x_s} \phi_s(\tau) \tau + \int_{x_s}^x \phi_s(\tau) \tau} = \\ &= \frac{\frac{(c+x_s)^{1+\alpha_i}(-c+x_s(1+\alpha_i))}{(1+\alpha_i)(2+\alpha_i)} + \frac{k(c+x_s)^{1+\alpha_s}(-c+x_s(1+\alpha_s))}{(1+\alpha_s)(2+\alpha_s)} + \frac{k((c+x)^{1+\alpha_s}(-c+x(1+\alpha_s))}{(1+\alpha_s)(2+\alpha_s)} - \frac{(c+x_s)^{1+\alpha_s}(-c+x_s(1+\alpha_s))}{(1+\alpha_s)(2+\alpha_s)}}{\frac{(c+x_s)^{1+\alpha_i}}{1+\alpha_i} + \frac{k(c+x_s)^{1+\alpha_s}}{1+\alpha_s} + \frac{k((c+x)^{1+\alpha_s} - (c+x_s)^{1+\alpha_s})}{1+\alpha_s}} \end{aligned} \quad (6.37)$$

The factor k in Eq. (6.31) is used to evaluate the different number of reinjections in the interval Δ_1 with respect those for Δ_2 . This parameter is obtained by means of the definition of $M(x)$. Because the global $M(x)$ function does not depend on the parameter b , we can obtain the factor k from it. To carry out this task the global $M(x)$ function should be evaluated at some point $x_y > x_s$, i.e. $M_y = M(x_y)$. Then, it is possible to explicitly obtain the factor k , which results:

$$k = \frac{\frac{((x_s-x_0+c)^{1+\alpha_i})(x_s(1+\alpha_i)+x_0-c)}{(1+\alpha_i)(2+\alpha_i)} - M_y \frac{(x_s-x_0+c)^{1+\alpha_i}}{1+\alpha_i}}{M_y \frac{(x_y-x_0+c)^{1+\alpha_s}}{1+\alpha_s} - \frac{((x_y-x_0+c)^{1+\alpha_s})(x_y(1+\alpha_s)+x_0-c)}{(1+\alpha_s)(2+\alpha_s)}} \quad (6.38)$$

In few cases, it could be necessary to calculate k using the Eq. (6.38) evaluated at two points called (x_{y_1}, M_{y_1}) and (x_{y_2}, M_{y_2}) ; the factor k will be the mean value: $k = 0.5(k_1 + k_2)$.

Figures 6.25 and 6.28 show the global nondifferentiable $M(x)$ and the discontinuous RPD functions respectively for $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$. From these figures, we can observe a very good accuracy between the numerical data and the theoretical equations (6.30), (6.31), (6.32), (6.36) and (6.37).

Fig. 6.28 RPD for map (6.6) with $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$. Points are numerical results and the continuous line represents the theoretical RPD.

As we have explained, other important function to describe the intermittency is the probability density of the laminar lengths, $\psi(l)$, which is a global property of the map [1, 4]. Remember that the laminar length, $l(x, c)$, is defined as the number of iterations that a trajectory carries out inside the laminar interval; and the probability density of the laminar lengths determines the probability of finding laminar lengths between l and $l + dl$, and it is given by Eq. (6.8). For the map (6.6), $\frac{dX(l, c)}{dl}$ can be approximated by Eq. (6.9) – if $\varepsilon \ll 1$ and $a_1 < 1$ but close to 1. Finally, we can evaluate $l(x, c)$, which is determined by Eq. (6.10).

Figure 6.29 shows $\psi(l)$ for the same parameters of the previous figures. Red points represent the numerical data, and the blue line is the theoretical result calculated using Eqs. (6.30) (6.31), (6.8) and (6.9). Figure 6.29(b) is an enlarged image of Figure 6.29(a). We can observe a very good accuracy between numerical data and theoretical results.

The probability density of the laminar lengths has two behaviors, one for $l \leq 65$ corresponding to the reinjections in the interval Δ_2 , and other one for $l > 65$ for reinjections in the interval Δ_1 .

This test shows that a discontinuous RPD function can appear in type V intermittency. The laminar interval has two subintervals with different RPD inside each one. It is a new behavior, different those previously described [26, 27, 28, 33]. However, the RPD is constant in each subinterval.

Fig. 6.29 Probability density of the laminar length for $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\bar{x} = -1.00006$. Red points represent the numerical data, and the blue line the theoretical results calculated using Eqs. (6.30) (6.31), (6.8) and (6.9). Figure (b) is an enlargement of Figure (a) for reinjected points inside of Δ_2 interval.

To evaluate a non constant RPD in each interval, we consider a reinjection process governed by $\gamma \neq 1$. We use the same parameters of the previous test, but now $\gamma = 0.5$. In Figures 6.30 and 6.31 numerical and theoretical $M(x)$ functions and RPDs are shown. Blue line corresponds to theoretical results and the numerical data are indicated in red color. Again the laminar interval can be divided in two subinterval, $\Delta_1 = [x_0 - c, x_s)$ and $\Delta_2 = [x_s, x_0 + c]$, where $M(x)$ and $\phi(x)$ have different behaviors. Therefore, to evaluate the theoretical RPD we have implemented the M function methodology described by Eqs. (6.30)-(6.38). Figure 6.31(b) is an enlarged image of Figure 6.31(a) inside the Δ_2 interval.

Fig. 6.30 Function $M(x)$ for map (6.6). The parameters are: $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$.

To obtain the exponents α_i and α_s we use the numerical data show in Figures 6.32 and 6.33 respectively: $\alpha_i \cong 0.0389$ ($m_i \cong 0.5095$) and $\alpha_s \cong 0.08847$ ($m_s \cong 0.5212$).

From Figures 6.31(a) and 6.31(b), we can note that the RPD inside each subinterval $\Delta_1 = [x_0 - c, x_s)$ and $\Delta_2 = [x_s, x_0 + c]$ increases as x increases. Also, the theoretical results have a high accuracy with respect the numerical data.

Furthermore, we carried out several numerical tests with $\gamma = 1.5$, and we have used the M function methodology to obtain the RPD function. Similarly to the previous cases, the RPDs are discontinuous.

Tables 6.4 and 6.5 show the values of α_i and α_s for $\gamma = 0.5, 1, 1.5$ and for two lower boundary of return $\tilde{x} = -1.00006$ and $\tilde{x} = -0.15844993141289437$.

(a)

(b)

Fig. 6.31 RPD for map (6.6). The parameters are: $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\bar{x} = -1.00006$. Figure (b) is an enlarged image of the Figure (a) inside the Δ_2 interval.

Fig. 6.32 Function $M(x)$ inside $\Delta_1 = [x_0 - c, x_s]$ for map (6.6). The parameters are: $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$.

Fig. 6.33 Function $M(x)$ inside $\Delta_2 = (x_s, x_0 + c]$ for map (6.6). The parameters are: $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$.

γ	α_i	α_s	m_i	m_s
0.5	0.0389	0.08847	0.5095	0.5212
1.0	0.00249	0.03623	0.5006	0.508897
1.5	-0.082642	-0.03137	0.47845	0.492032

Table 6.4 α_i , α_s , m_i and m_s for different values of γ . The other parameters are $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -1.00006$.

γ	α_i	α_s	m_i	m_s
0.5	0.30308	0.70238	0.5658	0.62995
1.0	-0.00651	0.00597	0.49837	0.50148
1.5	-0.2037	-0.14965	0.4433	0.45956

Table 6.5 α_i , α_s , m_i and m_s for different values of γ . The other parameters are $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and $\tilde{x} = -0.15844993141289437$.

From these tables, we can note that α_i and α_s depend on γ . For $\gamma < 1$, α_i and α_s are higher than 0; for $\gamma = 1$, $\alpha_i \cong \alpha_s \cong 0$; and for $\gamma > 1$, α_i and α_s are lower than 0. Also, we can observe that α_i and α_s depend on $\tilde{x}$.

Therefore, the reinjection process, i.e. α_i and α_s , depends on $\frac{dF_3(x)}{dx}$ valuated at different points (see Refs. [4, 23]):

a - Around $x = y_m$

b - Around $x = F^{-1}(x_0)$, i.e. for $F^{-1}(x_0 - c) \leq x \leq F^{-1}(x_0 + c)$.

6.10 Influence of the lower boundary of return

In this section the influence of the lower boundary of return, $\tilde{x}$, on the reinjection process is analyzed. To carry out this task, the lower boundary of return can get different values, but the other parameters do not change.

The first tests set uses the following parameters: $\gamma = 1.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, $c = 0.1128$ and different $\tilde{x}$. The results are summarized in Table 6.6, which has four columns: the lower boundary of return is in the first column, the exponent α_i and α_s are in the second and third columns respectively, the last one shows the number of iterations to go from $\tilde{x}$ to $x_0 - c$.

If we consider that the lower boundary of return matches the lower limit of the laminar interval, $\tilde{x} = x_0 - c$, there is not reinjection from $x < x_0 - c$; and the only reinjection process is produced by $x > x_m$. In this case the following relation should be satisfied [4]:

$$\alpha = \frac{1}{\gamma} - 1 \quad (6.39)$$

$\tilde{x}$	α_i	α_s	iterations
-0.1264444444444444	-0.33774	-0.20727	1
-0.158449931412894	-0.20367	-0.14965	3
-0.342186145207406	-0.1084	-0.05874	10
-0.6527	-0.0856	-0.00871	16
-1.00006	-0.08264	-0.03137	20

Table 6.6 α_i and α_s for different values of the LBR. The parameters are $\gamma = 1.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, and $c = 0.1128$. The last column indicates the number of needed iterations from $\tilde{x}$ to $x_0 - c$.

For $\gamma = 1.5$, this relation establishes $\alpha \cong -1/3$. From Table 6.6 we can see that for $\tilde{x} = -0.126444444$ – the reinjection process for $x < x_0 - c$ needs only one iteration – the exponent α_i is very close to $1/3$ as we expected. For $\tilde{x}$ located farther away from the lower boundary of the laminar interval, Eq. (6.39) is not verified because points close to $\tilde{x}$ need several iterations to move inside the laminar interval. While this process happens, there are also trajectories moving from $F_3(x)$ to $F_1(x)$. These trajectories are generated at different values of $|dF_3(x)/dx|$. When $|\tilde{x} - (x_0 - c)|$ increases the influence of $\frac{dF_3(x)}{dx}|_{x=y_m}$ on the RPD decreases, and $\alpha_i \rightarrow 0$ when $\tilde{x} \rightarrow -\infty$.

On the other hand, also α_s is influenced by $\tilde{x}$, because α_s depends on the reinjection trajectories coming from $F_3(x)$. When $\tilde{x} \rightarrow -\infty$, the function $F_3(x_n)$ is almost linear with constant and finite $|dF_3(x_n)/dx|$, where $x_{n+1} = F(x_n)$ are the reinjected points. Therefore, when $\tilde{x} \rightarrow -\infty$, $\alpha_s \rightarrow 0$. Conversely, when $\tilde{x}$ approaches to $x_0 - c$, then $\alpha_s \neq 0$.

Hence, it is important to highlight that the distance between the lower boundary of return, $\tilde{x}$, and the lower limit of the laminar interval, $x_0 - c$, has strong influence in the reinjection process.

6.11 The characteristic relation

The characteristic relation sets the relation between the average laminar length, $\bar{l}$, and the control parameter, ε . The average laminar length can be evaluated as (see Chapters 2 - 3 and Ref. [4]):

$$\bar{l} = \int_{x_0-c}^{x_0+c} \phi(x) l(x, c) dx \quad (6.40)$$

where $\phi(x)$ and $l(x, c)$ are given by Eqs. (6.30) and (6.10) respectively. For $x_0 - c < x_s \leq x_0$, the average laminar length results:

$$\bar{l} = \bar{l}_1 + \bar{l}_2 + \bar{l}_3 \quad (6.41)$$

where

$$\begin{aligned}\bar{l}_1 &= \int_{x_0-c}^{x_s} \phi_1(x) l(x, c) |_{x < x_0} dx \\ \bar{l}_2 &= \int_{x_s}^{x_0} \phi_2(x) l(x, c) |_{x < x_0} dx \\ \bar{l}_3 &= \int_{x_0}^{x_0+c} \phi_2(x) l(x, c) |_{x > x_0} dx\end{aligned}\tag{6.42}$$

The last equation only possesses analytical solution in few cases, such as: $\gamma = 1$ ($\alpha_i \approx \alpha_s \approx 0$) and $\gamma = 0.5$ ($\alpha_i \approx \alpha_s \approx 1$). For $\gamma = 1$, the relations between $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$ and ε are settled down in Eqs. (6.43), (6.44), and (6.45) respectively:

$$\begin{aligned}\bar{l}_1 &= b(1+k) \left[\left(\frac{\ln(\varepsilon)}{a_1-1} + \frac{\arctan\left(c\sqrt{\frac{a_2}{\varepsilon}}\right)}{\sqrt{(a_2\varepsilon)}} \right) (c+x_s) + \frac{c+x_s}{a_1-1} \right] - \\ &\quad \frac{b(1+k)}{(a_1-1)^2} [(\varepsilon + (a_1-1)x_s) \ln(\varepsilon + (a_1-1)x_s) - (\varepsilon - (a_1-1)c) \ln(\varepsilon - (a_1-1)c)]\end{aligned}\tag{6.43}$$

$$\bar{l}_2 = -bk \left[x_s + \frac{\ln(\varepsilon)}{a_1-1} \left(\frac{\varepsilon}{a_1-1} + x_s \right) - \frac{(\varepsilon + (a_1-1)x_s) \ln(\varepsilon + (a_1-1)x_s)}{(a_1-1)^2} + \frac{\arctan\left(c\sqrt{\frac{a_2}{\varepsilon}}\right)}{a_2\varepsilon} \right]\tag{6.44}$$

$$\bar{l}_3 = \frac{0.5bk [\ln(\varepsilon + a_2c^2) - \ln(\varepsilon)]}{a_2}\tag{6.45}$$

To analyze the characteristic relation we study several tests, verifying $\gamma = 1$ and $x_0 - c = F^{20}(\tilde{x})$, for $\varepsilon = 0.001 - 0.0000001$. Then a trajectory going by $\tilde{x}$ needs 20 iterations to reach the lower boundary of the laminar interval, $x_0 - c$. Therefore, $\tilde{x}$ must change with ε to satisfy $x_0 - c = F^{20}(\tilde{x})$. Table 6.7 shows $\tilde{x}$ for different ε .

The relation between $\ln(\bar{l})$ with $\ln(\varepsilon)$, the characteristic relation, is shown in Figure 6.34. The blue line represents the theoretical results given by Eqs. (6.41), (6.42), (6.43), (6.44) and (6.45), and the red points are the numerical data. We can note the very good accuracy between the numerical and theoretical results. The relation can be expressed by a linear function with slope $s_n \approx s_t \approx -0.47$ (where s_n corresponds to the numerical

data and s_t to the theoretical approach). Therefore, the characteristic relation can be written as:

$$\bar{l} = \varepsilon^{-\beta} \quad (6.46)$$

with $\beta \approx 0.5$. We highlight although the RPD is discontinuous, the characteristic relation still satisfies Eq. (6.46).

Fig. 6.34 Characteristic relation for $\gamma = 1$, $\varepsilon = 0.001 - 0.0000001$, $a_1 = 0.9$, $a_2 = 1$, number of iterations = 20. Red points: numerical data. Blue line: theoretical approach given by (6.41), (6.42), (6.43), (6.44) and (6.45).

We can find other analytical expression for the characteristic relation, if we assume $\alpha_i = \alpha_s = 1$. This last conditions could be verified when $F_1(\tilde{x}) = x_0 - c$ and $\gamma = 0.5$, but in this case the number of reinjected points from $x < x_0 - c$ is very reduced (less than 1 for 300 global reinjections) and the accurate evaluation of parameters k and b is very difficult. Note that the theoretical characteristic relation has strong dependence on b and k , then the results will be very sensitive with the number of reinjected points from $x < x_0 - c$. As the relation between reinjected points from $x < x_0 - c$ and the total reinjected points is very small, we can introduce errors in the analytical evaluation. Therefore, we will calculate the characteristic relations using directly numerical integration. Besides the numerical integration of both, theoretical and numerical results, will allow us to use different values of γ (not only $\gamma = 0$ or $\gamma = 1$).

We will study several tests with $\gamma = 1.5$ and $\varepsilon = 0.001 - 0.0000001$. To obtain $\bar{l}$ we use Eqs. (6.30), (6.8) and (6.9), from which we carry out the numerical calculation of the following integral:

$$\bar{l} = \int_0^{l_m} \psi(l) l dl \quad (6.47)$$

We consider two set of tests. Both groups have similar parameters, they only differ in the number of iterations that the trajectories need to reinject from $\tilde{x}$ to $x_0 - c$. For the first one the reinjection process, from $\tilde{x}$ to $x < x_0 - c$, needs 20 iterations. The values of $\tilde{x}$ for different ε are given in Table 6.7.

The second tests group uses $F_1(\tilde{x}) = x_0 - c$, therefore the process needs only one iteration to reinject from points $x < x_0 - c$. Table 6.8 shows $\tilde{x}$ for different ε .

The results are shown in Figure 6.35. The upper and lower lines correspond for tests with 20 and 1 iterations between $\tilde{x}$ and $x_0 - c$. In both cases, red points and blue lines represent the numerical and theoretical results respectively. A very high accuracy can be noted. When trajectories need only 1 iteration to be reinjected (lower line) the numerical and theoretical slopes are $s_n \cong -0.46$ and $s_t \cong -0.467$. On the other hand, for trajectories with 20 iterations the slopes are: $s_n \cong s_t \cong -0.47$. Also, for these cases Eq. (6.46) is verified with $\beta \approx 0.5$.

The average laminar length is higher for trajectories needing 20 iterations to go from the $\tilde{x}$ to the laminar interval in comparison with those needing only 1 iteration (see Figure 6.35). It happens because the reinjection processes, governed by α_i and α_s , are different. The values of α_i and α_s are given in Tables 6.7 and 6.8. When the process needs only one iteration $\alpha_i \approx -0.33$ and $\alpha_s \approx -0.21$; however for 20 iterations trajectories $\alpha_i \approx \alpha_s \approx 0$. Therefore, $\bar{l}$ acquires similar values for processes with $\gamma = 1.5$ and trajectories with 20 iterations from $\tilde{x}$ to $x_0 - c$ and processes with $\gamma = 1$. The $\tilde{x}$ influence on the RPD and other functions, such as the characteristic relation, is significant because as the distance $|\tilde{x} - (x_0 - c)|$ increases the influence of γ reduces.

ε	$\tilde{x}$	α_i	α_s
0.001	-1.00006	-0.082642	-0.031372
0.0001	-0.935031	-0.10182	-0.0215356
0.00001	-0.92853	-0.0879117	-0.012836
0.000001	-0.92788	-0.065915	-0.0201744
0.0000001	-0.927816	-0.08006	-0.0236215

Table 6.7 $\tilde{x}$, α_i and α_s for different values of $\tilde{x}$ to maintain 20 iterations from $\tilde{x}$ to $x = x_0 - c$. The parameters are $\gamma = 1.5$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, and $c = 0.1128$.

We emphasize that three groups of tests with different values of γ and $\tilde{x}$ have been analyzed, and the characteristic relation $\bar{l} \approx \varepsilon^{-0.5}$ is verified.

ε	$\tilde{x}$	α_i	α_s
0.001	-0.126444	-0.3306678	-0.201
0.0001	-0.125444	-0.3306678	-0.201
0.00001	-0.125344	-0.34637	-0.2241
0.000001	-0.125334	-0.3292	-0.2112
0.0000001	-0.125333	-0.324895	-0.21541

Table 6.8 $\tilde{x}$, α_i and α_s for different values of $\tilde{x}$ to maintain 1 iteration from $\tilde{x}$ to $x = x_0 - c$. The parameters are $\gamma = 1.5$, $a_1 = 0.9$, $a_2 = 1$, $N = 30000$, and $c = 0.1128$.

Fig. 6.35 Characteristic relation for $\gamma = 1.5$, $\varepsilon = 0.001 - 0.0000001$, $a_1 = 0.9$, $a_2 = 1$. The number of iterations are 1 and 20 for the lower and upper lines respectively. Red points: numerical data. Blue line: theoretical approach.

6.12 Analysis and Conclusions

In this chapter we have applied the M function methodology to describe the reinjection process for type V intermittency, which occurs when the local map has a nondifferentiable or discontinuous point. This methodology had worked accurately for type I, II and III intermittencies, where the local map is continuous (see Chapters 3 and 4). Here, we have shown that this methodology also works accurately for type V intermittency. It can capture very well the RPD function, and other intermittency statistical properties (probability density of the laminar lengths and average laminar length), for different maps and different reinjection mechanisms.

We have studied two maps. The local map for the first one, Eq. (6.6), is composed of two simple functions, one is linear and the other one is quadratic. However, the local part of the second piecewise map, Eqs. (6.19)-(6.20), is given by a sine function. Also,

the used laminar intervals were very dissimilar for both maps. For the map (6.6), it was small ($c \ll 1$) and symmetrical around the vanished fixed point; but for the map (6.19), the laminar interval only considered positive values and it was large ($c \approx \pi$). But, despite of these differences the M function methodology produced accurate results. We have shown that the RPD can be a nonuniform function. It is described by an exponential function with exponent α . To obtain the analytical RPD is only necessary to evaluate the slope m of the $M(x)$ function (α depends on m). The uniform reinjection is only a singular case when $m = 1/2$ ($\alpha = 0$). Also, we have obtained that the local map does not influence the reinjection process when $c \ll 1$. However, if the local map is nondifferentiable, the laminar length, $l(x, c)$, and the probability density of the laminar lengths, $\psi(l)$, are nondifferentiable too. For Map1 the probability density of the laminar lengths, $\psi(l)$, has two behaviors: the first one for reinjected points inside the interval $-c \leq x \leq x_0$, which need more than ≈ 40 iterations to leave the laminar interval; the second one for reinjected points in $x_0 < x \leq c$ which need less than ≈ 40 iterations to leave the laminar interval. Then, $\psi(l)$ is nondifferentiable function for $l \approx 40$ (see Figures 6.4 and 6.9). For Map2, $\psi(l)$ is a differentiable function with a high concentration between $20 < l < 40$ (see Figures 6.16, 6.19 and 6.22). $\psi(l)$ is a differentiable function because the laminar interval only contains points verifying $x > 0$, then it is not symmetrical around the vanished fixed point. The average laminar length, $\bar{l}$, acquires different form for each map. The Map1 verifies $\bar{l} \propto \varepsilon^{-\beta}$; however, the characteristic relation for Map2 is $\bar{l} \propto \ln(\varepsilon)$. Therefore, there is not only one characteristic relation for type V intermittency, it depends on the local map.

We have also shown that, for the map Eq. (6.6), the RPD can be a discontinuous function, because there are two reinjection mechanisms if the lower boundary of return is less than the lower limit of the laminar interval, $\tilde{x} < x_0 - c$. One of them is produced by trajectories coming from points $x < x_0 - c$, and the other one from trajectories coming from $x > x_m > x_0 + c$. The laminar interval is divided in two subintervals, within each of them the RPD function has different behavior. The subintervals are $\Delta_1 = [x_0 - c, x_s]$ and $\Delta_2 = [x_s, x_0 + c]$. Δ_1 receives trajectories coming from $x < x_0 - c$ and $x > x_m$; however Δ_2 only receives trajectories from $x > x_m$. x_s is a singular point where the RPD is discontinuous. The RPD is described by a sum of exponential functions with exponents α_i and α_s , to obtain its analytical expression is only necessary to evaluate the slope m_i and m_s of the $M(x)$ function. Where m_i and m_s are calculated using reinjected points from $x < x_0 - c$ and $x > x_m$ respectively. The influence of $\tilde{x}$ on the RPD, $\phi(x)$, has been evaluated. The lower boundary of return, $\tilde{x}$, has an important influence on the RPD and the characteristic relation: when the distance $|\tilde{x} - (x_0 - c)|$ increases the influence of γ reduces, and the reinjection process resembles the reinjection process for $\gamma = 1$. The influence of γ on the RPD has also been studied. The exponents α_i and α_s decrease when γ increases, and $\alpha_i \approx \alpha_s \approx 0$ for $\gamma = 1$. However,

in all cases the RPD is discontinuous. Similarly to the RPD, the probability density of the laminar lengths also has two distinctive behaviors, one for reinjections inside the subinterval Δ_1 and the other one for reinjections in the subinterval Δ_2 . For all analyzed tests, with different values of γ , ε and $\tilde{x}$, the characteristic relation can be written as: $\bar{l} \approx \varepsilon^{-0.5}$.

Finally, we highlight that the reinjection mechanism for type V intermittency has a wide behavior, and the uniform reinjection is only a particular case. The reinjection depends, mainly, on the global part of the map. Also, the obtained theoretical equations for the reinjection probability density, the probability density of the laminar lengths and the characteristic relation showed a very good accuracy regarding the numerical data for different values of ε , γ and $\tilde{x}$.

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Chapter 7

EVALUATION OF THE INTERMITTENCY PROPERTIES USING THE PERRON-FROBENIUS OPERATOR

Abstract We apply the Perron-Frobenius operator to study type II and V intermittencies. By mean of this operator we can obtain theoretical expressions for the reinjection probability density function, $\phi(x)$, and the probability density of the laminar lengths, $\psi(l)$, for several maps with type II intermittency and for two maps with type V intermittency. To validate these expressions we compare the analytical $\phi(x)$ and $\psi(l)$ functions with numerical data; we find very good accuracy between theoretical equations and numerical simulations. Also, we carry out comparisons between the new $\phi(x)$ and $\psi(l)$ functions with those obtained using the M function methodology. Finally, we describe the advantages and difficulties of both methodologies.

7.1 Introduction

In this chapter a new methodology, using the Perron-Frobenius operator, is implemented to obtain analytical expressions for the reinjection probability density, $\phi(x)$, and the probability density of the laminar lengths, $\psi(l)$, functions. This methodology is called here Perron-Frobenius operator technique, and it is applied in type II intermittency. It allows us to obtain very accurate $\phi(x)$ and $\psi(l)$ functions. However, to use this technique, it is necessary to know explicitly the map and its inverse [1].

Also we carry out several comparisons between the RPD calculated using the Perron-Frobenius operator technique and the M function methodology. Finally, we describe the advantages and drawbacks of both schemes.

We only give a short introduction about concepts such as density, measures and Perron-Frobenius operator. A more complete and deeper description about this subjects can be found in Refs. [2, 3, 4].

The chapter has nine sections: Section 7.2 describes the density and measures concepts. They are necessary to introduce the Perron-Frobenius operator, which is car-

ried out in Section 7.3. Section 7.4 applies the Perron-Frobenius operator to piecewise maps. The evaluation of statistical properties of type II intermittency is given in Section 7.5. Sections 7.5.1 and 7.5.2 show several comparisons between the Perron-Frobenius operator technique and M function methodology. In Section 7.5.3, a map showing nonlinear $M(x)$ function is analyzed. In Section 7.6 type V intermittency is analyzed. Finally, the conclusions are presented in Section 7.7.

7.2 Density and probability measures

Let us to introduce a family of evolution operators $F^t(x) : D \rightarrow D$, such as $F^0 = \text{identity}$ and $F^{t_1+t_2} = F^{t_1} \circ F^{t_2}$. Where $x \in D$ and D is a compact manifold. The parameter t can be seen as time; if t is continuous (t varies over the real numbers), then $F^t(x)$ is a dynamical system. However, if t is discrete, the operator $F^t(x)$ represents a map.

There are, at least, two formulations to describe the evolution of a dynamical system or map $F^t(x) : D \rightarrow D$. One of them considers the evolution of individual trajectories, the other one uses the density concept to explain the system behavior. Therefore, firstly we will define the density concept and subsequently we will consider the probability measure concept.

A trajectory for the map $F^t(x)$ can be defined as:

$$x_0, F(x_0); F^2(x_0); F^3(x_0); F^4(x_0); \dots; F^N(x_0) \quad (7.1)$$

where $F^2(x_0) = F(F(x_0)) = F \circ F(x_0)$, and $F^i(x_0)$ represents i consecutive applications of $F(x)$. In this case, $x_0 \in D$ is an initial point for the iterative process.

To simplify the study here presented, we consider that $F(x)$ maps a closed interval of real numbers onto itself; therefore D is reduced to an interval $[a, b] \in \mathbf{R}$, where $\mathbf{R}$ represents the real numbers. The interval $l = [a, b]$ is divided in $h - 1$ discrete subintervals l_j , which must verify:

$$[a, b] = \cup l_j \quad \text{for } j = 1, \dots, h - 1 \quad (7.2)$$

$$l_i \cap l_j = \emptyset \quad \text{for } i \neq j \quad (7.3)$$

Eq. (7.2) expresses that the subintervals cover completely the interval $l = [a, b]$; and Eq. (7.3) indicates that they are nonintersecting. If all subintervals have the same length, we can write:

$$l_j = \begin{cases} \left[\frac{l(j-1)}{h}, \frac{lj}{h} \right) & \text{for } j = 1, \dots, h-1 \\ \left[\frac{l(j-1)}{h}, l \right] & \text{for } j = h \end{cases} \quad (7.4)$$

Let us consider a system that develops a trajectory like Eq. (7.1) with $N \gg h$; and we calculate the number of points of the trajectory located inside of each interval:

$$N_j = \text{number of } F^i(x_0) \in \left[\frac{l(j-1)}{h}, \frac{lj}{h} \right). \quad (7.5)$$

Therefore, we can determine the fraction $\rho_j = N_j/N$. Where ρ_j represents the trajectory density in the j -subinterval. If we consider the densities in all subintervals, we obtain an histogram which indicates the density distribution inside the interval $l = [a, b]$.

We can assume that $h \rightarrow \infty$, accordingly we can define $\rho_j = \rho(x_j)$. More generally, we define a density function $\rho(x_0, x, t)$, which represents the number of times that the trajectory passes through the state x until a time t if the starting state was x_0 . If the density is independent of the initial state, we can write $\rho(x, t)$.

Probability measure is other important concept to characterize the dynamics of chaotic systems. The differential of probability measure can be defined as:

$$d\mu(x) = \rho(x)dx \quad (7.6)$$

If we implement a discrete number of subintervals, for each subinterval l_j the measure results

$$\Delta\mu_j = \int_{l_j} \rho(x) dx \quad (7.7)$$

The probability measure in the interval $l = [a, b]$ is the sum of all $\Delta\mu_j$, and it is normalized:

$$\mu = \sum_{j=1}^h \Delta\mu_j = 1 \quad (7.8)$$

Again we can assume $h \rightarrow \infty$, hence

$$\mu = \int_l \rho(x) dx = 1 \quad (7.9)$$

From Eqs. (7.6) and (7.9) and analogously with continuum mechanics, we might interpret the density of trajectories and probability measure concepts as the “density” and “mass” of a fluid.

Note that measure is a positive definite quantity, which summed over nonintersecting intervals l_j with $j = 1, \dots, h-1$ equals the quantity evaluated over the sum of all intervals $l = \sum_{j=1}^{h-1} l_j$.

7.3 The Perron-Frobenius operator

Let us consider an application $y = F(x, t)$, i.e. an evolution operator $F^t(x) : \mathbf{R} \rightarrow \mathbf{R}$, the Perron-Frobenius operator allows to evaluate the density of trajectories $\rho(y, t)$ at time t starting from the density at a previous time, for example for $t = 0$, i.e. $\rho(x, 0)$. Similarly, for maps, the Perron-Frobenius operator allows to evaluate the density evolution after one iteration.

Let us to consider a general map $y = F(x)$, that transforms some interval $\Delta_0 \subset \mathbf{R}$ in other interval $\Delta \subset \mathbf{R}$ (remember that $\mathbf{R}$ represents the real numbers). Hence, $x \in \Delta_0$ and $y \in \Delta$. In Δ_0 the density of the trajectories can be defined as $\rho_0(x)$, also in Δ there is other trajectories density: $\rho(y)$. The Perron-Frobenius operator, $P_\circ$, transforms the density $\rho_0(x)$ in $\rho(y)$, and we can write:

$$\rho(y) = P_\circ \rho_0(x) \quad (7.10)$$

If we evaluate $x = F^{-1}(y)$, the inverse of map $F(x)$, we can write: $\Delta_0 = F^{-1}(\Delta)$. The interval Δ possesses all mapped points from the interval Δ_0 . Hence, Δ embraces all the image points of Δ_0 under the map $F(x)$. Then, the following relation must be verified:

$$\int_{\Delta} \rho(y) dy = \int_{F^{-1}(\Delta)} \rho_0(x) dx \quad (7.11)$$

where $\rho_0(x)$ and $\rho(y)$ are the densities before and after the application of the map $F(x)$.

If we consider $\Delta = [a, y]$ – where the upper limit y is variable –, we can calculate the density $\rho(y)$ from Eq. (7.11):

$$\rho(y) = \frac{d}{dy} \int_{F^{-1}[a, y]} \rho_0(x) dx \quad (7.12)$$

We can solve the last equation to obtain the density $\rho(y)$:

$$\rho(y) = \rho_0(F^{-1}(y)) \left| \frac{dF^{-1}(y)}{dy} \right| \quad (7.13)$$

Then, the Perron-Frobenius operator, $P_\circ$, for an application $F(x)$ can be written as:

$$P \circ \rho(x) = \frac{d}{dx} \int_{\Delta(x)} \rho(u) du \quad (7.14)$$

On the other hand, the Perron-Frobenius operator can be defined in an alternative form:

$$\rho(y) = P \circ \rho(x) = \int_{\Delta_0(x)} \rho_0(x) \delta(y - F(x)) dx \quad (7.15)$$

Where $\delta(x)$ is the Dirac delta function: $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$. For one-dimensional $F(x)$, i.e. $x \in \mathbf{R}$, the integral in the second member of Eq. (7.15) results:

$$\rho(y) = \int_{\Delta_0(x)} \rho_0(x) \delta(y - F(x)) dx = \sum_{x=F^{-1}(y)} \frac{\rho_0(x)}{|F'(x)|} \quad (7.16)$$

where $F'(x) = \frac{dF(x)}{dx}$. Therefore, if the map $y = F(x)$ is a one-to-one application, $F(x) : \Delta_0 \rightarrow \Delta$, we can write:

$$\rho(y) = \frac{\rho_0(F^{-1}(y))}{|F'(F^{-1}(y))|} \quad (7.17)$$

Note that this equation coincides with Eq. (7.13).

In the next subsections we will introduce a procedure using the Perron-Frobenius operator to evaluate the statistical properties of intermittency for maps. Also, we will analyze several examples.

7.4 The Perron-Frobenius operator for the evaluation of the RPD function for piecewise monotonic maps

Maps defined on intervals of the real line have received strong attention in nonlinear dynamics and chaos. Here, we consider a map $F(x)$, which maps the closed unit interval onto itself: $F(x) : [0, 1] \rightarrow [0, 1]$. Consequently, the state space of the system is $0 \leq x \leq 1$.

To carry out a broad analysis we deal with piecewise monotonic maps:

$$F(x) = \begin{cases} F_1(x), & a_0 \leq x < a_1, \\ F_2(x), & a_1 \leq x < a_2, \\ \cdot & \\ \cdot & \\ \cdot & \\ F_n(x), & a_{n-1} \leq x \leq a_n, \end{cases} \quad (7.18)$$

where $a_0 = 0$, $a_n = 1$, $F_1(a_0) = 0$ and $F_n(a_n) = 1$; and also the map verifies $F(a_i) = 0$ for $i = 1, \dots, n-1$.

Eq. (7.18) indicates that the unit interval is split in n subintervals; and each function $F_i(x)$ is a C^1 function inside the interval $[a_{i-1}, a_i]$, i.e., a function with first derivative continue.

On account of Eq. (7.18) and for $x \in [0, 1]$, we be able to write:

$$F^{-1}([0, x]) = \bigcup_{i=1}^n [a_{i-1}, F_i^{-1}(x)] \quad (7.19)$$

where $F^{-1}(x)$ is the inverse of the map (7.18).

For maps given by Eq. (7.18), the Perron-Frobenius operator can be calculated using Eq. (7.12) (see Ref. [2]):

$$P \circ \rho(x) = \frac{d}{dx} \int_{F_i^{-1}[0, x]} \rho(u) du = \frac{d}{dx} \sum_{i=1}^n \int_{a_{i-1}}^{F_i^{-1}(x)} \rho(u) du \quad (7.20)$$

or

$$P \circ \rho(x) = \sum_{i=1}^n \left| \frac{dF_i^{-1}(x)}{dx} \right| \rho(F_i^{-1}(x)) \quad (7.21)$$

To calculate the RPD function in intermittency, we should not include the first interval in the sum (7.21), because it involves the laminar interval. In consequence, the RPD function can be written as:

$$\phi(x) = P_{int} \circ \rho(x) = \sum_{i=2}^n \left| \frac{dF_i^{-1}(x)}{dx} \right| \rho(F_i^{-1}(x)) \quad (7.22)$$

where $\rho(x)$ is the density in the intervals previous to reinjection:

$$[a_1, F_1^{-1}(x)), [a_2, F_2^{-1}(x)), \dots, [a_{n-1}, F_n^{-1}(x)] \quad (7.23)$$

If the density, $\rho(x)$, is generated in regions without extreme points or points with infinite derivative, hence we can assume that it is uniform, $\rho(x) = k$. Where k is a constant, which should satisfy:

$$\sum_{i=2}^n \int_{a_{i-1}}^{F_i^{-1}(x)} k dx = 1 \quad (7.24)$$

Remember that the RPD function, $\phi(x)$, determines the probability that trajectories are reinjected into a point x inside the laminar interval. The RPD is the more important function to describe the intermittent reinjection process, because it specifies the statis-

tical behavior of the reinjection trajectories, which depends on the specific form of the map $F(x)$ (see Chapters 3 and 4).

7.5 Application to type II intermittency. Evaluation of statistical properties

In this section an application of the Perron-Frobenius operator technique to study maps with intermittency is developed. To carry out this study we consider a one-dimensional map with type II intermittency. This map was introduced for first time in Ref. [5] to apply the M function methodology to type II intermittency:

$$x_{n+1} = F(x_n) = \begin{cases} F_1(x_n) = (1 + \varepsilon)x_n + (1 - \varepsilon)x_n^p, & 0 \leq x_n < a_1, \\ F_2(x_n) = [F_1(x_n) - 1]^\gamma, & a_1 \leq x_n \leq 1, \end{cases} \quad (7.25)$$

Where a_1 is defined by $(1 + \varepsilon)a_1 + (1 - \varepsilon)a_1^p = 1$. Note that ε is the control parameter. If $\gamma = 1$ and $p = 2$ in Eq. (7.25) the seminal map presented in [6] is recovered. Eq. (7.25) allows us to analyze different types of reinjection mechanism using different values of the exponent γ .

This map has a fixed point at the origin ($x_0 = 0$), which is stable for $-2 < \varepsilon < 0$. When $\varepsilon > 0$ the fixed point becomes unstable and type II intermittency is generated. The iteration procedure – governed by the parameter ε and the exponent p – gives increasing values of x_n generated from an initial one, close to the origin. A chaotic burst occurs if x_n becomes larger than a_1 , and ends when the trajectory is reinjected into a point in the laminar zone. Then, a new iterative process – again governed by ε and p – will be developed producing larger values for the new successive iterative points. Note that γ drives the reinjection mechanism, whereas p and ε determine the laminar phase duration. When $p = 3$, the local Poincaré map for classical type II intermittency is obtained (see Chapters 3 and 4).

The reinjected points in the laminar interval, $\Delta(x) = [0, x]$, are mapped by $F(x)$ from points placed inside the interval $\Delta_0(x) = F^{-1}[0, x]$. Where $0 \leq x \leq c$ and c is the upper extreme of the laminar interval. From Eqs. (7.12) and (7.22) the reinjection probability distribution function $\phi(x)$ results:

$$\phi(x) = \frac{d}{dx} \int_{\Delta_0(x)} k \, du \quad (7.26)$$

where k is the density in the interval $\Delta_0(x)$.

To find an explicit expression of the RPD function for the map (7.25), we firstly consider $p = 2$, and we express the interval Δ_0 as:

$$\Delta_0(x) = [F^{-1}(0), F^{-1}(x)] \quad (7.27)$$

Remember that x verifies: $0 \leq x \leq c$.

It is clear that $F^{-1}(0) = a_1$ and $F^{-1}(x) = F_2^{-1}(x)$, where $F_2^{-1}(x)$ results:

$$F_2^{-1}(x) = \frac{-\lambda + [\lambda^2 + 4\eta(1 + x^{1/\gamma})]^{0.5}}{2\eta} \quad (7.28)$$

where $\eta = 1 - \varepsilon$ and $\lambda = 1 + \varepsilon$.

By means of Eq. (7.26), we have:

$$\phi(x) = \frac{d}{dx} \int_{a_1}^{F_2^{-1}(x)} k \, du, \quad (7.29)$$

Therefore, using Eq. (7.28), the RPD can be written as:

$$\phi(x) = \frac{k}{\gamma} \frac{x^{\frac{1}{\gamma}-1}}{[\lambda^2 + 4\eta(1 + x^{1/\gamma})]^{0.5}} \quad (7.30)$$

To evaluate k , the normalization condition is employed:

$$\int_{a_1}^{F^{-1}(x)} k \, du = 1, \quad (7.31)$$

We solve the last equation to obtain k :

$$k = \frac{2\eta}{-\lambda - 2\eta a_1 + [\lambda^2 + 4\eta(1 + c^{1/\gamma})]^{0.5}} \quad (7.32)$$

Note that Eq. (7.30) can be written as:

$$\phi(x) = x^\alpha \chi(x) \quad (7.33)$$

where $\alpha = \frac{1}{\gamma} - 1$, and:

$$\chi(x) = \frac{k}{\gamma[\lambda^2 + 4\eta(1 + x^{1/\gamma})]^{0.5}} \quad (7.34)$$

To verify the previous methodology, we carry out several numerical test, and we compare the numerical results with these analytical expressions.

The first test corresponds to the following values: $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$, where N is the number of reinjected points and c is the upper limit of the laminar interval. Figure 7.1 shows the numerical and theoretical RPD. The continuous line is the theoretical RPD given by Eqs. (7.30) and (7.32); the points represent the numerical RPD. The accuracy between both results is very good.

Fig. 7.1 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result.

The second test corresponds to $p = 2$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Figure 7.2 shows the numerical and theoretical RPD. The continuous line represents the theoretical RPD, and the points are the numerical data. Also, the accuracy between the numerical results and analytical prediction is very good.

We highlight that the two analyzed tests are very different. The RPD in Figure 7.1 is a decreasing function and it verifies: $\phi(x) \rightarrow \infty$ when $x \rightarrow 0$. On the other hand, the RPD shows in Figure 7.2 is an increasing function and it verifies: $\phi(0) = 0$. However, the Perron-Frobenius operator technique have obtained very good results in both tests.

The previous tests were performed using $p = 2$. Now, the new methodology is applied to describe classical type II intermittency; therefore, we consider $p = 3$ in Eq. (7.25). To calculate the RPD, we use the Eqs. (7.26) and (7.27), but in this case $F_2^{-1}(x)$ is given by:

$$F_2^{-1}(x) = \frac{24^{1/3}(1 - \varepsilon^2) + 2^{1/3}\varphi_2(x)^{2/3}}{6^{2/3}\eta\varphi_2(x)^{1/3}} \quad (7.35)$$

where $\varphi_2(x)$ is:

$$\varphi_2(x) = 9\eta^2(1 + x^{1/\gamma}) + \varphi_1(x) \quad (7.36)$$

and $\varphi_1(x)$ is:

$$\varphi_1(x) = [3\eta^3(27\eta(1 + x^{1/\gamma})^2 + 4\lambda^3)]^{1/2} \quad (7.37)$$

Fig. 7.2 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result.

For $p = 3$ the RPD function results:

$$\phi(x) = x^{\frac{1}{\gamma}-1} \left[\frac{3\eta(1+x^{1/\gamma})(6\eta\lambda + 2^{1/3}\varphi_2(x)^{2/3}) + 2\lambda\varphi_1(x)}{(4/3)^{1/3}\eta^{-2}\gamma\varphi_1(x)\varphi_2(x)^{4/3}} + \varphi_2(x)^{-2/3} \right] \quad (7.38)$$

This last equation can be written as:

$$\phi(x) = x^\alpha \chi(x) \quad (7.39)$$

where $\alpha = \frac{1}{\gamma} - 1$ and $\chi(x)$ is:

$$\chi(x) = \frac{3\eta(1+x^{1/\gamma})(6\eta\lambda + 2^{1/3}\varphi_2(x)^{2/3}) + 2\lambda\varphi_1(x)}{(4/3)^{1/3}\eta^{-2}\gamma\varphi_1(x)\varphi_2(x)^{4/3}} + \varphi_2(x)^{-2/3} \quad (7.40)$$

The RPDs for two cases considering $p = 3$ are studied. The first test uses the following parameters: $p = 3$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$. The second one considers: $p = 3$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. The results are shown in Figures 7.3 and 7.4 respectively. In both figures, the theoretical results are represented by continuous lines and numerical data are given by points. The accuracy is very good for both test.

As in previous chapters, we note that the exponent α has a high influence on the RPD. For $\alpha < 0$ the RPD verifies $\lim_{x \rightarrow 0} \phi(x) \rightarrow \infty$ and $\frac{d\phi(x)}{dx} < 0$ (see Figures 7.1 and

Fig. 7.3 Numerical and theoretical RPD with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Red points: numerical data. Blue line: theoretical result. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Rio and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

7.3). On the other hand, for $\alpha > 0$ the RPD verifies $\lim_{x \rightarrow 0} \phi(x) \rightarrow 0$ and $\frac{d\phi(x)}{dx} \geq 0$ (see Figures 7.2 and 7.4).

Inside the laminar zone, with $\varepsilon \rightarrow 0$, the local map can be approximate by a differential equation (see Chapter 2 and Ref. [7, 8]):

$$\frac{dx}{dl} = \varepsilon x + \eta x^p. \quad (7.41)$$

Integrating this expression the laminar length for each reinjection point, x_{in} , results (see Chapter 3, Eq. (3.39)):

$$\begin{aligned} l(x_{in}, c) &= \int_{x_{in}}^c \frac{dx}{\varepsilon x + \eta x^p} \\ &= \frac{1}{\varepsilon} \left[\ln \left(\frac{c}{x_{in}} \right) - \frac{1}{p-1} \ln \left(\frac{\eta c^{(p-1)} + \varepsilon}{\eta x_{in}^{(p-1)} + \varepsilon} \right) \right]. \end{aligned} \quad (7.42)$$

The probability density of the laminar lengths, $\psi(l)$, is a global property and it is related with the reinjection probability density function, $\phi(l, c)$, by the expression [5, 9]:

$$\psi(l, c) = \phi(x(l, c)) [\varepsilon x(l, c) + \eta x(l, c)^p], \quad (7.43)$$

Fig. 7.4 Numerical and theoretical RPD with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Río and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

where $x(l, c)$ is the inverse of $l(x, c)$ given by Eq. (7.42). Thus, $\psi(l, c)$ depends on the global parameter α and the reinjection probability density function.

Figures 7.5, 7.6, 7.7 and 7.8 show the numerical and theoretical $\psi(l)$ for the four tests analyzed previously. The four figures show a very high accuracy between numerical and theoretical results.

7.5.1 Application of M function methodology: comparisons

We implement a methodology developed in the last years to deal with intermittency phenomenon [5, 9, 10, 11, 12, 13]. Also, this methodology was described in previous chapters. In which the RPD function is not directly evaluated from the numerical or experimental data; an auxiliary function, called $M(x)$, is calculated previously.

For the map (7.25) the $M(x)$ function was numerically evaluated obtaining a linear form $M(x) = m(x - \hat{x}) + \hat{x}$ for different values of γ and ε (see Figure 7.9). Hence, the RPD function can be expressed by:

$$\phi(x) = b(x - \hat{x})^\alpha, \quad \text{with} \quad \alpha = \frac{2m - 1}{1 - m}, \quad (7.44)$$

Fig. 7.5 Numerical and theoretical $\psi(l)$ with $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result.

where b is a normalization parameter:

$$b = (\alpha + 1)/(c - \hat{x})^{\alpha+1}. \quad (7.45)$$

The function $M(x)$ is only determined by the parameter m , and it is easier to obtain among a huge amount of data than the complete RPD function. Note that $M(x)$ satisfies $M(\hat{x}) = \hat{x}$, then, in addition, it allows to evaluate the LBR. In this chapter, only for simplicity, we consider $\hat{x} = 0$, i.e. the LBR = 0. On the other hand we remember that the reinjection probability density function, $\phi(x)$, verifies: $\lim_{x \rightarrow 0} \phi(x)$ is infinity when $0 < m < 1/2$ and zero, when $1/2 < m < 1$.

To carry out comparisons, we analyze the behavior of the RPD and $M(x)$ functions calculated using this methodology for the same four cases studied using the Perron-Frobenius operator technique in the previous section.

The parameters for the first test were: $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$. The $M(x)$ function for these values is shown in Figure 7.9. This figure shows the numerical $M(x)$ (red) and the interpolated $M(x)$ using minimum square technique (blue). From the figure we can obtain $m \cong 0.213$ and $\alpha \cong -0.73$ (see Eq. (7.44)). Figure 7.10 shows the RPDs calculated using the M function methodology and Perron-Frobenius operator technique. We can observe a good accuracy between the two techniques.

The second test considers the following parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Figure 7.11 shows the numerical $M(x)$ function (red) and its minimum

Fig. 7.6 Numerical and theoretical $\psi(l)$ with $p = 2$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result.

square approximation (blue). From this figure we obtain $m \cong 0.625$ and $\alpha \cong 0.665$. Figure 7.12 shows the RPDs calculated using the M function methodology and the Perron-Frobenius operator technique. There is a very high accuracy.

The third and fourth tests use $p = 3$. The remaining parameters for the third test are $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Figure 7.13 shows the $M(x)$ function (red) and the minimum square approximation (blue). From the figure we can calculate $m \cong 0.2$ and $\alpha \cong 0.75$. Fig. 7.14 shows the RPDs evaluated using the M function methodology and the Perron-Frobenius operator technique. Note the good accuracy between them.

For the fourth test, the parameters are $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. In Figure 7.15 the numerical $M(x)$ function and the minimum square approximation are shown. From the figure, we calculate: $m \cong 0.6235$ and $\alpha \cong 0.656$. The RPD function is obtained by Eq. (7.44). Figure 7.16 shows the evaluated RPDs using the M function methodology and the Perron-Frobenius operator technique. Again, there is a very good accuracy.

From the previous results we highlight that the functions $\chi(x)$ given by Eqs. (7.34) and (7.40) have low influence in the RPD expression when the $M(x)$ function is linear in x . Therefore, the power law approximation, Eq. (7.44), is a simple and very good approximation to the RPD function [1].

Fig. 7.7 Numerical and theoretical $\psi(l)$ with $p = 3$, $\varepsilon = 0.01$, $\gamma = 3$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Río and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

7.5.2 Evaluation of intermittency statistical properties for other maps

Until now, we have evaluated the $\phi(x)$ and $\psi(l)$ functions for type II intermittency only using the map given by Eq. (7.25). We have shown, for this map, that the Perron-Frobenius operator technique and the M function methodology obtain accurate theoretical results. However, in this section we apply these schemes to evaluate the statistical properties for maps with other nonlinear behaviors.

We analyze maps with: trigonometry nonlinearity, logarithmic nonlinearity and a piecewise monotonic map with three subintervals.

7.5.2.1 Map with trigonometry nonlinearity

Let us consider the following map:

$$F(x) = \begin{cases} F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & 0 \leq x \leq a_1, \\ F_2(x) = \left[\frac{\sin(\pi(x - a_1))}{\sin(\pi(1 - a_1))} \right]^\gamma, & a_1 < x \leq 1, \end{cases} \quad (7.46)$$

where a_1 is again defined by $(1 + \varepsilon)a_1 + (1 - \varepsilon)a_1^p = 1$.

Fig. 7.8 Numerical and theoretical $\psi(l)$ with $p = 3$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Points: numerical data. Line: theoretical result. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Rio and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

The map has a fixed point at the origin, $x_0 = 0$, which is stable for $-2 < \varepsilon < 0$. For $\varepsilon > 0$ type II intermittency appears.

For the map (7.46), the inverse of $F_2(x)$ can be written as:

$$F_2^{-1}(x) = a_1 + \pi^{-1} \sin^{-1}(x^{1/\gamma} \sin(\pi(1 - a_1))) \quad (7.47)$$

To obtain the RPD, we use Eqs. (7.29) and (7.31). Therefore, the RPD function results:

$$\phi(x) = \frac{\sin(\pi(1 - a_1)) x^\alpha}{\gamma \sin^{-1}(c^{1/\gamma} \sin(\pi(1 - a_1))) \sqrt{1 - x^{(2/\gamma)} \sin^2(\pi(1 - a_1))}} \quad (7.48)$$

where $\alpha = \frac{1}{\gamma} - 1$.

To validate this last relation, we carry out five numerical tests and we compare Eq. (7.48) with numerical results.

The first and second tests use the following parameters: $p = 2$, $\varepsilon = 0.01$, $N = 50000$, $c = 0.1$. However, $\gamma = 3$ and $\gamma = 0.85$ for the first and second test respectively. Figures 7.17 and 7.18 show the numerical and theoretical RPD functions. In both figures the continuous line is the theoretical RPD given by Eq. (7.48), and the red points repre-

Fig. 7.9 $M(x)$ function with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: minimum square approximation.

sent the numerical RPD. We can see that the accuracy between numerical data and theoretical results is very good.

Other two comparisons are made for $p = 3$, $\varepsilon = 0.01$, $N = 50000$, $c = 0.1$, together with $\gamma = 3$ and $\gamma = 0.85$ respectively. Figures 7.19 and 7.20 show the numerical and theoretical RPD. In both figures the continuous line represents the theoretical RPD, and the points are the numerical RPD. Again, the results show a very good accuracy.

Finally we evaluate the following test: $p = 2$, $\varepsilon = 0.01$, $\gamma = 1$, $N = 50000$, $c = 0.1$. Figure 7.21 shows the theoretical results using Perron-Frobenius operator scheme (blue line) and M function methodology (black line) together with the numerical results (red points). Both theoretical results approach very good the numerical data. However, we highlight that the Perron-Frobenius operator technique improves the accuracy with respect to the M function methodology principally for $x \rightarrow x_0$. For this test, the RPD is approximately uniform, i.e. $\alpha \cong 1$, because $\gamma = 1$.

7.5.2.2 Map with logarithmic nonlinearity

Now, we consider a map with logarithmic nonlinearity:

$$F(x) = \begin{cases} F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & 0 \leq x < a_1, \\ F_2(x) = \left[1 + \ln\left(\frac{1}{x^{1/\ln(a_1)}}\right)\right]^\gamma, & a_1 \leq x \leq 1, \end{cases} \quad (7.49)$$

Fig. 7.10 RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red: $M(x)$ methodology. Blue: Perron-Frobenius operator technique.

a_1 is once more defined by $(1 + \varepsilon)a_1 + (1 - \varepsilon)a_1^p = 1$. The origin, $x_0 = 0$ is a fixed point of the map. The fixed point is stable for $-2 < \varepsilon < 0$. For $\varepsilon > 0$ there is type II intermittency.

We apply the Perron-Frobenius methodology given by Eqs. (7.29) and (7.31) to obtain the RPD for the map (7.49). For this map, the expression for $F_2^{-1}(x)$ can be written as:

$$F_2^{-1}(x) = a_1^{1-x^{1/\gamma}} \quad (7.50)$$

Therefore, the RPD function results:

$$\phi(x) = \frac{\ln(a_1) a_1^{1-x^{1/\gamma}} x^\alpha}{\gamma a_1 (a_1^{-c^{1/\gamma}} - 1)} = K a_1^{1-x^{1/\gamma}} x^\alpha \quad (7.51)$$

where K is a constant and $\alpha = \frac{1}{\gamma} - 1$.

Two numerical test are carried out to compare the theoretical results calculated by Eq. (7.51) and the numerical data. Both tests have the following parameters: $p = 3$, $\varepsilon = 0.01$, $N = 50000$, $c = 0.1$. However, $\gamma = 2$ and 0.6 for the first and second test respectively. Figures 7.22 and 7.23 show the numerical and theoretical RPDs, Eq. (7.51). In both figures the continuous line corresponds to the theoretical RPD, and the points are the numerical RPD. The accuracy between numerical data and theoretical results is again very good.

Fig. 7.11 $M(x)$ function with $p = 2$, $\varepsilon = 0.01$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: minimum square approximation.

7.5.2.3 Piecewise monotonic map with three subintervals

To extend the previous studies, in this section the following piecewise map is analyzed:

$$F(x) = \begin{cases} F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & 0 \leq x < a_1, \\ F_2(x) = [(1 + \varepsilon)x + (1 - \varepsilon)x^p - 1]^\gamma, & a_1 \leq x < a_2, \\ F_3(x) = \left[1 + \ln\left(\frac{1}{x^{1/\ln(a_2)}}\right)\right]^\theta, & a_2 \leq x \leq c, \end{cases} \quad (7.52)$$

where a_1 verifies $(1 + \varepsilon)a_1 + (1 - \varepsilon)a_1^p = 1$, and $0 < a_1 < a_2 < c$. Fig. 7.24 shows the map given by Eq. (7.52) for $p = 2$, $\varepsilon = 0.01$, $\gamma = 2$, $\theta = 0.7$, $a_2 = 0.9$. We note that the map possesses three subintervals, and in the second one, the $F_2(x)$ function verifies: $F_2(a_2) < 1$

If we apply Eq. (7.22), we can obtain the RPD:

$$\phi(x) = k \left[\frac{x^{\frac{1}{\gamma}-1}}{\gamma [(1 + \varepsilon)^2 + 4(1 - \varepsilon)(1 + x^{1/\gamma})]^{1/2}} - \frac{\ln(x_z)x_z^{1-x^{1/\theta}}x^{\frac{1}{\theta}-1}}{\theta} \right] \quad (7.53)$$

Where k must verify:

$$\int_{a_1}^{F_2^{-1}(c)} k dx + \int_{a_2}^{F_3^{-1}(c)} k dx = 1 \quad (7.54)$$

Fig. 7.12 RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Red: $M(x)$ methodology. Blue: Perron-Frobenius operator methodology.

$F_2^{-1}(x)$ is the inverse of $[(1 + \varepsilon)x + (1 - \varepsilon)x^p - 1]^\gamma$, which is given by Eqs. (7.28) and (7.35) for $p = 2$ and $p = 3$ respectively. $F_3^{-1}(x)$ is the inverse of $\left[1 + \ln\left(\frac{1}{x^{1/\ln(a_2)}}\right)\right]^\theta$ (see Eq. (7.50)). Thus, k results:

$$k = \frac{1}{F_2^{-1}(c) + F_3^{-1}(c) - a_1 - a_2} \quad (7.55)$$

To verify the accuracy of the previous equations we develop two numerical tests with different γ and θ . The first one has the following parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 2$, $\theta = 0.7$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Figure 7.25 shows the theoretical RPD calculated using Eqs. (7.53) and (7.55) and the RPD obtained by numerical simulation. From this figure we can observe a very good accuracy. On the other hand, Figure 7.26 shows the theoretical $\psi(l)$ calculated by Eq. (7.43) and the numerical $\psi(l)$. Again, there is a high accuracy.

For the second test we consider the following parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $\theta = 2$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Figure 7.27 shows the theoretical RPD calculated by Eqs. (7.53) and (7.55) and the numerical RPD. Figure 7.28 shows the numerical and theoretical $\psi(l)$, this last one was obtained using Eq. (7.43). These figures show a very good accuracy between numerical and theoretical results.

Fig. 7.13 $M(x)$ function with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: minimum square approximation. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Rio and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

7.5.3 Map with nonlinear $M(x)$ function

Until now we have applied the Perron-Frobenius operator technique to calculate the RPD for maps showing type II intermittency and verifying a linear $M(x)$ function. In such cases the RPD has an power law form $\phi(x) = \Gamma x^\alpha$. Where $\Gamma = \text{constant}$ for M function methodology, and $\Gamma = \chi(x)$ for the Perron-Frobenius operator technique.

For the tests developed in the previous sections, the analytical results calculated using both technique, the M function methodology and Perron-Frobenius operator technique, show very good accuracy with respect to numerical data.

In this section we introduce a new map, which also shows type II intermittency. For this map, the $M(x)$ function can be nonlinear for some range of parameters. Therefore, we want to determine if the M function methodology could be applied even when $M(x)$ is nonlinear. Also, we want to obtain the analytical RPD function using the Perron-Frobenius operator technique.

The map can be written as:

$$F(x) = \begin{cases} F_1(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & 0 \leq x \leq a_1, \\ F_2(x) = (e^{x^\nu} - e^{a_1^\nu}) / (e - e^{a_1^\nu}), & a_1 < x \leq 1, \end{cases} \quad (7.56)$$

Fig. 7.14 RPD with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red: $M(x)$ methodology. Blue: Perron-Frobenius operator methodology. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Río and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

The coefficient a_1 satisfies $(1 + \varepsilon)a_1 + (1 - \varepsilon)a_1^p = 1$. Also, $x_0 = 0$ is a fixed point; and for $\varepsilon > 0$ type II intermittency appears. Figure 7.29 shows the map (7.56) for $p = 2$, $\varepsilon = 0.01$, $\gamma = 4$.

Again, to calculate the RPD function, Eq. (7.22) is used:

$$\phi(x) = b \frac{(e - e^{a_1^\gamma}) \left[\ln \left(x(e - e^{a_1^\gamma}) + e^{a_1^\gamma} \right) \right]^{\frac{1}{\gamma} - 1}}{\gamma [x(e - e^{a_1^\gamma}) + e^{a_1^\gamma}]} \quad (7.57)$$

Where b is the normalization parameter:

$$b = \frac{1}{\left[\ln \left(c(e - e^{a_1^\gamma}) + e^{a_1^\gamma} \right) \right]^{1/\gamma} - a_1} \quad (7.58)$$

To obtain Eq. (7.57) we have calculated $F_2^{-1}(x)$:

$$F_2^{-1}(x) = \left[\ln \left(c(e - e^{a_1^\gamma}) + e^{a_1^\gamma} \right) \right]^{1/\gamma} \quad (7.59)$$

Fig. 7.15 $M(x)$ function with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: minimum square approximation. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Rio and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

It is important to highlight that the RPD given by Eq. (7.57) does not have an power law form.

For a generic map $F(x)$, the M function methodology has shown to work correctly if $M(x)$ is linear. For a wide class of maps, $M(x)$ is linear if:

$$\left(\frac{dF(x)}{dx} \right)_{x=a_1} = 0 \quad \text{or} \quad \left(\frac{dF(x)}{dx} \right)_{x=a_1} \rightarrow \infty \quad (7.60)$$

where $a_1 = F^{-1}(x_0)$, and x_0 is the fixed point.

The map, given by Eq. (7.56), shows different reinjection behaviors depending on the control parameter ν . To describe these behaviors, we evaluate the derivative of $F_2(x)$ at $x = a_1$:

$$\left(\frac{dF_2(x)}{dx} \right)_{x=a_1} = \nu \frac{e^{a_1^\nu} a_1^{\nu-1}}{e - e^{a_1^\nu}} \quad (7.61)$$

For a wide range of values of ν , the derivative $dF_2(x)/dx|_{x=a_1}$ is finite and different to zero. Also, the parameter ν can take positive and negative values. We study two cases: for the first test $\nu < 0$, and for the second one $\nu > 0$.

The first test considers the following parameters: $p = 2$, $\varepsilon = 0.01$, $\nu = -1$, $N = 50000$, $c = 0.1$. Figure 7.30 shows the $M(x)$ function, where red points represent the

Fig. 7.16 RPD with $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Red: $M(x)$ methodology. Blue: Perron-Frobenius operator technique. Reprinted from: *Nonlinear Dynamics*, doi:10.1007/s11071-016-2951-6, Sergio Elaskar, Ezequiel del Rio and Eduardo Zapico, “Evaluation of the statistical properties for type II intermittency using the Perron-Frobenius operator”, in press, Copyright (2016) with permission from Springer.

numerical data and the blue line is the least square approximation. There is a very good accuracy between numerical data and least square approximation; and the $M(x)$ function can be represented by a linear equation $M(x) = 0.038657x$. Hence, if we use the M function methodology, the RPD can be written as:

$$\phi(x) = bx^\alpha \cong 11.35x^{0.51} \quad (7.62)$$

Figure 7.31 shows the RPD functions for the same parameters used in Figure 7.30. Red points are the numerical data, the blue line represents the theoretical RPD calculated using the Perron-Frobenius operator technique (Eqs. (7.57) and (7.58)), and the black line is obtained using the M function methodology (Eq. (7.62)). Both analytical curves approach accurately the numerical data, however for x close to x_0 the RPD evaluated by the Perron-Frobenius operator technique presents a more accurate behavior.

We highlight that for this test $(dF_2(x)/dx)_{x=a_1}$ is a nonzero finite value; even with this condition the $M(x)$ function can be linear, i.e. the M function methodology works correctly.

For the second test, we use the following parameters: $p = 2$, $\varepsilon = 0.01$, $v = 6$, $N = 50000$, $c = 0.1$. Again we calculate the RPD using two techniques: Perron-Frobenius operator technique and the M function methodology.

Fig. 7.17 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

Figure 7.32 shows $M(x)$. Red points are the numerical data and the blue continuous line corresponds to the least square approximation. There is not a very good accuracy between numerical data and least square approximation. Therefore, a linear $M(x)$ function is only an approximation to the numerical data because the numerical evaluation of $M(x)$ is nonlinear. If we consider a linear approach for $M(x)$ – using minimum squares –, it results: $M(x) \cong 0.378324x$. Then, the RPD function could be written as:

$$\phi(x) = bx^\alpha \cong 2.47x^{-0.39} \quad (7.63)$$

The RPD functions are shown in Figure 7.33. Once more, red points represent the numerical data, the blue line is the theoretical RPD calculated using Eqs. (7.57) and (7.58) (Perron-Frobenius operator technique), and the black line represents the RPD evaluated by the M function methodology – Eq. (7.63). Note that the RPD calculated using the Perron-Frobenius operator technique possesses a better accuracy. On the other hand, the RPD obtained by the M function methodology gives only approximate results. Inside the interval $0 \leq x < 0.005$ the M function methodology produces a poor accuracy.

The M function methodology is a power tool to calculate the statistical properties of the intermittency, mainly if we do not know the exact map or we only have data series. However, this methodology works with high accuracy when the $M(x)$ function is linear. On the other hand, the Perron-Frobenius operator technique allows us to determine the

Fig. 7.18 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.85$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

RPD even for maps with nonlinear $M(x)$; and the obtained theoretical RPDs are more accurate than those calculated using the M function methodology. However, it presents the disadvantage that it is required to know the map or the equations that rule the system [1].

7.6 Application to type V intermittency

In this section we apply the Perron-Frobenius operator technique to calculate the reinjection probability distribution function for type V intermittency.

Type V intermittency is characterized by a nondifferentiable or discontinuous local map. On the other hand, the Perron-Frobenius operator technique uses integral operators to calculate the evolution of trajectory densities. Therefore, we hope that this technique can work accurately for type V intermittency.

We will analyze the two maps studied in Chapter 6: Map1 given by Eq. (6.6), and Map2 defined by Eqs. (6.19 - 6.20).

Fig. 7.19 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

7.6.1 Map1

The Map1 is (see Eq. (6.6)):

$$F(x) = \begin{cases} F_1(x) = a_1x + \varepsilon, & \tilde{x} \leq x < 0, \\ F_2(x) = \varepsilon + x + a_2x^2, & 0 \leq x < x_m, \\ F_3(x) = \tilde{x} + \frac{(y_m - \tilde{x})(y_m - x)^\gamma}{(y_m - x_m)^\gamma}, & x_m \leq x \leq y_m, \end{cases} \quad (7.64)$$

where $y_m = F(x_m) = 1$, $\tilde{x}$ is the lower boundary of return [4], $0 < a_1 < 1$ is the slope of the straight line, a_2 is the coefficient of the quadratic term in $F_2(x)$, and ε is the control parameter. Figure 7.34 shows this map.

7.6.1.1 Continuous RPD

We consider that the lower boundary of return verifies $\tilde{x} = x_0 - c$, and $y_m = 1$, $x_0 = 0$. Thus, the reinjection process is driven by $F_3(x)$. To calculate the RPD function we can use the Eqs. (7.22) and (7.24) with $n = 2$, which results:

$$\phi(x) = \frac{(x+c)^{1/\gamma-1}}{\gamma(2c)^{1/\gamma}} \quad (7.65)$$

Fig. 7.20 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.85$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

where $|dF_3^{-1}(x)/dx|$ is:

$$\left| \frac{dF_3^{-1}(x)}{dx} \right| = \frac{(1 - x_m)(x + c)^{1/\gamma - 1}}{\gamma(1 + c)^{1/\gamma}} \quad (7.66)$$

and:

$$k = \frac{1}{1 - x_m} \left(\frac{1 + c}{2c} \right)^{1/\gamma} \quad (7.67)$$

Note that Eq. (7.65) is an exponential function with exponent $\alpha = 1/\gamma - 1$. The same results was found in Chapter 6 using the M function methodology.

We carry out the comparison between the RPDs obtained using the Perron-Frobenius operator technique and those calculated with the M function methodology in Chapter 6.

The first test considers $\gamma = 1$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $c = 0.1$ and $\tilde{x} = x_0 - c$. The lower boundary of return is the lower limit of the laminar interval; then there is no reinjection from points $x < x_0 - c$. The reinjected points inside the laminar interval, $\Delta = [x_0 - c, x_0 + c]$, are only mapped by $F_3(x)$ from points placed inside the interval $\Delta_0(x) = F_3^{-1}[x_0 - c, x_0 + c]$. For this test using the M function methodology we have found:

Fig. 7.21 Numerical and theoretical RPD with $p = 2$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 1$, $N = 50000$, $c = 0.1$, $x_0 = 0$. Red points: numerical data. Blue line: Perron-Frobenius operator technique. Black line: M function methodology.

$$\phi(x) = \frac{1}{2c} \quad (7.68)$$

If we evaluate Eq. (7.65) with $\gamma = 1$, the Perron-Frobenius operator technique, by means of Eq. (7.65), obtains the Eq. (7.68). Therefore, both methodologies produce the same result.

The second test uses $\gamma = 2$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $c = 0.1$ and $\tilde{x} = x_0 - c$. The M function methodology obtained (see Eq. (6.11):

$$\phi(x) \cong 1.118(x+0.1)^{-0.5} \quad (7.69)$$

By Eq. (7.65), the Perron-Frobenius operator technique calculates:

$$\phi(x) = \frac{(x+0.1)^{-0.5}}{2(0.2)^{0.5}} = 1.118034(x+0.1)^{-0.5} \quad (7.70)$$

Eqs. (7.69) and (7.70) produce the same RPD.

The third numerical test uses the following parameters: $\gamma = 0.5$, $\varepsilon = 0.001$, $a_1 = 0.9$, $a_2 = 1$, $c = 0.1$ and $\tilde{x} = x_0 - c$. Using the M function methodology, we have obtained (see Eq. (6.12)):

$$\phi(x) \cong 50(x+0.1) \quad (7.71)$$

Fig. 7.22 Numerical and theoretical RPD using Eq. (7.51). Parameters: $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 3$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

The same equation is calculated using the Perron-Frobenius operator technique, Eq. (7.65):

$$\phi(x) = \frac{(x+c)}{2c^2} = 50(x+0.1) \quad (7.72)$$

Again both methodologies work very accurately.

We have verified, in Chapter 6, that the RPDs calculated using the M function methodology have a very good accuracy compared with the numerical data. Therefore, the RPDs obtained with the Perron-Frobenius operator technique have a very good accuracy too (for the three tests).

The high accuracy obtained with both techniques could occur because Map1 has simple functions and $\tilde{x} = x_0 - c$, remember that we introduced it to show that RPD in type V intermittency can be more complex than those presented in previous studies, inclusive in simple maps. Therefore, we will develop similar numerical studies for the Map1 with $\tilde{x} < x_0 - c$ (discontinuous RPD), and a more complex map (Map2).

7.6.1.2 Discontinuous RPD

To obtain discontinuous RPDs, the lower boundary of return should verify $\tilde{x} < x_0 - c$. Therefore, there are two reinjection processes, one is driven by $F_3(x)$ and the other one by $F_1(x)$. To calculate the RPD function we can use the Eq. (7.22) with $n = 3$:

Fig. 7.23 Numerical and theoretical RPD using Eq. (7.51). Parameters: $p = 3$, $\varepsilon = 0.01$, $\hat{x} = 0$, $\gamma = 0.6$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

Fig. 7.24 Map given by Eq. (7.52) for the following parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 2$, $\theta = 0.7$ and $a_2 = 0.9$.

Fig. 7.25 Numerical and theoretical RPD using Eqs. (7.53) and (7.55). Parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 2$, $\theta = 0.7$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Red points: numerical data. Blue line: theoretical result.

$$\phi(x) = \phi_i(x) + \phi_s(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-1}(F_1^{-1}(x)) + \left| \frac{dF_3^{-1}(x)}{dx} \right| \rho_3(F_3^{-1}(x)) \quad (7.73)$$

where x is a point inside the laminar interval. $\rho_1^{-1}(x)$ and $\rho_3(x)$ are the densities in the intervals $I_1^{-1} = [F_1^{-1}(x_0 - c), x_0 - c)$ and $I_3 = [F_3^{-1}(x_0 - c), F_3^{-1}(x_0 + c))$ respectively. Thus, the density:

$$\phi_i(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-1}(F_1^{-1}(x)) \quad (7.74)$$

is defined inside the interval $[x_0 - c, x_s)$, where $x_s = F_1(x_0 - c)$. However, the density $\phi_s(x)$ is defined in the complete laminar interval $[x_0 - c, x_0 + c]$:

$$\phi_s(x) = \left| \frac{dF_3^{-1}(x)}{dx} \right| \rho_3(F_3^{-1}(x)) \quad (7.75)$$

Then, the RPD can be written as (see Eq. (6.30)):

$$\phi(x) = \begin{cases} \phi_1(x) = \phi_i(x) + \phi_s(x), & x < x_s, \\ \phi_2(x) = \phi_s(x), & x \geq x_s. \end{cases} \quad (7.76)$$

Fig. 7.26 Numerical and theoretical $\psi(l)$ using Eq. (7.53) and (7.43). Parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 2$, $\theta = 0.7$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Red points: numerical data. Blue line: theoretical result.

To calculate the Eq. (7.65), a constant $\rho_3(x)$ density was used, and the obtained results showed very good accuracy. Therefore, this assumption is also here considered: $\rho_3(x) = k = \text{constant}$ inside the interval $I_3^s = [F_3^{-1}(\tilde{x}), F_3^{-1}(x_0 + c)]$. As I_3 is included in I_3^s ($I_3 \subset I_3^s$), $\rho_3(F_1^{-1}(x))$ is constant in Eq. (7.73).

On the other hand, $\rho_1^{-1}(x)$ is defined inside the interval $I_1^{-1} = [F_1^{-1}(x_0 - c), x_0 - c]$, and to evaluate this density we need to consider the iteration procedure for the map (7.64) because $\rho_1^{-1}(x)$ depends on the density in the previous iteration, $\rho_3(F_3^{-1}(x))$.

Note that the interval $I_3^a = [F_3^{-1}(F_1^{-1}(x_0 - c)), F_3^{-1}(x_0 - c)]$ maps on the interval I_1^{-1} :

$$I_3^a = [F_3^{-1}(F_1^{-1}(x_0 - c)), F_3^{-1}(x_0 - c)] \longrightarrow I_1^{-1} = [F_1^{-1}(x_0 - c), x_0 - c] \quad (7.77)$$

Besides, points inside the interval $I_1^{-2} = [F_1^{-1}(F_1^{-1}(x_0 - c)), F_1^{-1}(x_0 - c)]$ also map on the interval $I_1^{-1} = [F_1^{-1}(x_0 - c), x_0 - c]$:

$$I_1^{-2} = [F_1^{-1}(F_1^{-1}(x_0 - c)), F_1^{-1}(x_0 - c)] \longrightarrow I_1^{-1} = [F_1^{-1}(x_0 - c), x_0 - c] \quad (7.78)$$

therefore the density $\rho_1^{-1}(x)$ in the interval I_1^{-1} must be calculated.

Fig. 7.27 Numerical and theoretical RPD using Eqs. (7.53) and (7.55). Parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $\theta = 2$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Red points: numerical data. Blue line: theoretical result.

Following Eqs. (7.77) and (7.78), the density $\rho_1^{-1}(x)$ inside I_1^{-1} will have two contributions, one of them is given by $\rho_3(x)$ from the interval I_3^a , the other one is $\rho_1^{-2}(x)$ from the interval I_1^{-2} :

$$\rho_1^{-1}(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-2}(F_1^{-1}(x)) + \left| \frac{dF_3^{-1}(x)}{dx} \right| \rho_3(F_3^{-1}(x)) \quad (7.79)$$

As $\rho_3(x) = k = \text{constant}$ inside the interval $I_3^s = [F_3^{-1}(\tilde{x}), F_3^{-1}(x_0 + c)]$, the previous equation reduces:

$$\rho_1^{-1}(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-2}(F_1^{-1}(x)) + \left| \frac{dF_3^{-1}(x)}{dx} \right| k \quad (7.80)$$

Also, the points inside the I_3^s interval are mapped by $F_3(x)$ to the interval $I_1^s = [\tilde{x}, x_0 + c]$, then the density $\rho_1^s(x)$ results:

$$\rho_1^s(x) = k \left| \frac{dF_3^{-1}(x)}{dx} \right| \quad (7.81)$$

and the Eq. (7.80) can be written as:

Fig. 7.28 Numerical and theoretical $\psi(l)$ using Eq. (7.53) and (7.43). Parameters: $p = 2$, $\varepsilon = 0.01$, $\gamma = 3$, $\theta = 2$, $N = 50000$, $c = 0.1$, $a_2 = 0.9$. Red points: numerical data. Blue line: theoretical result.

Fig. 7.29 Map (7.56) for the following parameters: $p = 2$, $\varepsilon = 0.01$, $\nu = 4$.

$$\rho_1^{-1}(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-2}(F_1^{-1}(x)) + \rho_1^s(x) \quad (7.82)$$

Fig. 7.30 $M(x)$ function for the map (7.56). Parameters: $p = 2$, $\varepsilon = 0.01$, $v = -1$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

where k is calculated using the normalization condition:

$$\int_{F_3^{-1}(x_0+c)}^{F_3^{-1}(\tilde{x})} k dx = 1 \quad (7.83)$$

and it results:

$$k = \frac{(y_m - \tilde{x})^{1/\gamma}}{(x_m - y_m)(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.84)$$

where $y_m = F_3^{-1}(\tilde{x})$.

Now, we have to determine $\rho_1^{-2}(x)$ inside the interval I_1^{-2} .

Let us consider that a trajectory needs n iterations to go from $\tilde{x}$ to $x_0 - c$. Thus, a point placed inside the interval $I_1^{-n} = [\tilde{x}, F_1(\tilde{x})]$ needs n iterations to reinject. The density ρ_1^{-n} inside the interval I_1^{-n} only depends on $\rho_3 = k$:

$$\rho_1^{-n}(x) = k \left| \frac{dF_3^{-1}(x)}{dx} \right| = \rho_1^s(x) \quad (7.85)$$

Then, the density $\rho_1^{-n+1}(x)$ in the interval $I_1^{-n+1} = [F_1(\tilde{x}), F_1(F_1(\tilde{x}))]$ results:

Fig. 7.31 RPD functions for the map (7.56). Parameters: $p = 2$, $\varepsilon = 0.01$, $v = -1$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: Perron-Frobenius methodology. Black line: function $M(x)$ approach.

$$\rho_1^{-n+1}(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-n}(F_1^{-1}(x)) + k \left| \frac{dF_3^{-1}(x)}{dx} \right| \quad (7.86)$$

The density $\rho_1^{-n+2}(x)$ in the interval $I_1^{-n+2} = [F_1(F_1(\tilde{x})), F_1(F_1(F_1(\tilde{x})))]$ results:

$$\rho_1^{-n+2}(x) = \left| \frac{dF_1^{-1}(x)}{dx} \right| \rho_1^{-n+1}(F_1^{-1}(x)) + k \left| \frac{dF_3^{-1}(x)}{dx} \right| \quad (7.87)$$

Where:

$$F_1^{-1}(x) = \frac{x - \varepsilon}{a_1}; \text{ and } \frac{dF_1^{-1}(x)}{dx} = 1/a_1 \quad (7.88)$$

and

$$F_3^{-1}(x) = y_m - \frac{y_m - x_m}{(y_m - \tilde{x})^{1/\gamma}} (x - \tilde{x})^{1/\gamma}; \text{ and } \frac{dF_3^{-1}(x)}{dx} = \frac{x_m - y_m}{\gamma(y_m - \tilde{x})^{1/\gamma}} (x - \tilde{x})^{-1+1/\gamma} \quad (7.89)$$

If we introduce Eqs. (7.84), (7.88) and (7.89) in Eqs. (7.85), (7.86) and (7.87), we can obtain:

Fig. 7.32 Function $M(x)$ for the map (7.56). Parameters: $p = 2$, $\varepsilon = 0.01$, $\nu = 6$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: theoretical result.

$$\rho_1^{-n}(x) = \rho_1^s(x) = \frac{(x - \tilde{x})^{-1+1/\gamma}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.90)$$

$$\rho_1^{-n+1}(x) = \frac{(x - \tilde{x})^{-1+1/\gamma} + \frac{(F_1^{-1}(x) - \tilde{x})^{-1+1/\gamma}}{a_1}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.91)$$

$$\rho_1^{-n+2}(x) = \frac{(x - \tilde{x})^{-1+1/\gamma} + \frac{(F_1^{-1}(x) - \tilde{x})^{-1+1/\gamma}}{a_1} + \frac{(F_1^{-2}(x) - \tilde{x})^{-1+1/\gamma}}{a_1^2}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.92)$$

Therefore, we can generalize:

$$\rho_1^{-n+i}(x) = \frac{\sum_{j=0}^{j=i} \frac{(F_1^{-j}(x) - \tilde{x})^{-1+1/\gamma}}{a_1^j}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.93)$$

Note that for $i = n$, the RPD, $\phi_1(x)$ in Eq. (7.76) is obtained:

$$\phi_1(x) = \frac{\sum_{j=0}^{j=n} \frac{(F_1^{-j}(x) - \tilde{x})^{-1+1/\gamma}}{a_1^j}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.94)$$

The second RPD in Eq. (7.76), $\phi_2(x)$, is:

Fig. 7.33 RPD functions for the map (7.56). Parameters: $p = 2$, $\varepsilon = 0.01$, $v = 6$, $N = 50000$, $c = 0.1$. Red points: numerical data. Blue line: Perron-Frobenius methodology. Black line: function $M(x)$ approach.

$$\phi_2(x) = \rho_1^s(x) = k \left| \frac{dF_3^{-1}(x)}{dx} \right| = \frac{(x - \tilde{x})^{-1+1/\gamma}}{\gamma(x_0 + c - \tilde{x})^{1/\gamma}} \quad (7.95)$$

Note, that $\phi_2(x_n)$ only considers points reinjected for $F_3(x_{n-1})$, where $x_n = F_3(x_{n-1})$.

To validate the previous equations, we carry out four tests. The first one considers the following parameters: $y_m = 1$, $x_0 = 0$, $\varepsilon = 0.001$, $\gamma = 0.5$, $N = 30000$, $c = 0.1128$, $\tilde{x} = -0.158449931412894$. Then, a trajectory needs three (3) iterations to move from $\tilde{x}$ to $x_0 - c$. We have calculated the RPD function using the M function methodology, the Perron-Frobenius operator technique and numerically. The comparison between the three RPDs is shown in Figure 7.35.

The second tests uses: $y_m = 1$, $x_0 = 0$, $\varepsilon = 0.001$, $\gamma = 1.5$, $N = 30000$, $c = 0.1128$, $\tilde{x} = -0.158449931412894$. Again, a trajectory needs three (3) iterations to move from $\tilde{x}$ to $x_0 - c$. The results shown the calculated RPD function using the M function methodology, the Perron-Frobenius operator technique and numerically are shown in Figure 7.36.

The third and fourth use the following parameters: $y_m = 1$, $x_0 = 0$, $\varepsilon = 0.001$, $N = 30000$, $c = 0.1128$, $\tilde{x} = -0.126444444444$. However, for the third test we implement $\gamma = 0.5$, and for the fourth test $\gamma = 1.5$. For both cases a trajectory needs one (1) iteration to travel from $\tilde{x}$ to $x_0 - c$. Figures 7.37 and 7.38 show the theoretical and numerical results.

Fig. 7.34 Map1 for $\gamma = 2$, $\varepsilon = 0.001$, $a_1 = 0.5$, $a_2 = 1$.

Figures 7.35-7.38 show a high accuracy between the numerical data and theoretical evaluations obtained using the Perron-Frobenius operator technique. Also, we can observe a good accuracy between the M function methodology results and the Perron-Frobenius operator technique. Both schemes can capture very well the intermittency reinjection behavior.

Once calculated the RPD we can evaluate the probability density of the laminar lengths, $\psi(l)$. We can use the Eq. (2.20):

$$\psi(l, c) = \phi[X(l, c)] \left| \frac{dX(l, c)}{dl} \right|, \quad (7.96)$$

where $X(l, c)$ is the inverse of $l(x, c)$. For the Map1, $\frac{dX(l, c)}{dl}$ can be approximated by (see Eq. (6.9)):

$$\begin{aligned} \frac{dX(l, c)}{dl} &= \varepsilon + x(a_1 - 1), & x < 0, \\ \frac{dX(l, c)}{dl} &= \varepsilon + a_2 x^2, & x \geq 0, \end{aligned} \quad (7.97)$$

Figures 7.39-7.42 show the theoretical and numerical results for the same four test previously studied. Again, we can observe a very good accuracy between the numerical data with the M function methodology and the Perron-Frobenius operator technique.

Fig. 7.35 Discontinuous RPD function for the Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 0.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.158449931412894$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

7.6.2 Map2

Map2 is given by Eqs. (6.19 - 6.20):

$$F(x) = \begin{cases} F_1(x) = \left[\frac{f^{1/\gamma}(b_0) - f^{1/\gamma}(b_1)}{b_0 - b_1} (x - b_1) + f^{1/\gamma}(b_1) \right]^\gamma, & -0.8\pi \leq x < -0.5\pi, \\ F_2(x) = \frac{0.8\pi - g + F_3(x_{out})}{x_{out} + 0.5\pi} x + \frac{0.5\pi[F_3(x_{out}) - g] - 0.8\pi x_{out}}{x_{out} + 0.5\pi}, & -0.5\pi \leq x < x_{out}, \\ F_3(x) = x - a \sin(x) - \varepsilon, & x_{out} \leq x < \frac{23\pi}{20}, \\ F_4(x) = \frac{20[0.8\pi + F_3(x_{out})]}{\pi} x - 19.2\pi - 23F_3(x_{out}), & \frac{23\pi}{20} \leq x \leq \frac{6\pi}{5}, \end{cases} \quad (7.98)$$

Where $b_0 = -0.8\pi$, $b_1 = -0.5\pi$, $x_{out} = \varepsilon/(1-a)$, g is the gap between $F_2(x_{out})$ and $F_3(x_{out})$; and the function $f(x)$ is:

$$f(x) = \frac{10[1.2\pi - F_3(x_{out})]}{3\pi} x + 3.2\pi - \frac{5F_3(x_{out})}{3} \quad (7.99)$$

This map, for $\varepsilon \neq 0$, has one fixed point: $x_0 = \pi + \arcsin(\varepsilon/a)$. The laminar interval is $x_{out} \leq x \leq x_0 = c$; where $x_{out} = \varepsilon/(1-a)$ (see Chapter 6 and Ref. [28]). Note that the amplitude of the laminar interval is as large as the turbulent regime amplitude. However, to compare the new results with the previous ones given in Chapter 6 and [28],

Fig. 7.36 Discontinuous RPD function for the Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 1.5$, $N = 30000$, $c = 0.1128$, $\tilde{x} = -0.158449931412894$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

we use the same laminar interval (see Figure 7.43). Then, $F_1(x)$ returns the trajectories from the chaotic interval to the laminar one.

Fig. 7.37 Discontinuous RPD function for the Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 0.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.1264444444$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

Fig. 7.43 Map2 for $\gamma = 0.5$, $\varepsilon = 0.001$, $a = 0.25$. The upper end of the laminar interval, $x_0 = c$, is indicated by the vertical dashed-dot line.

Fig. 7.38 Discontinuous RPD function for the Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 1.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.1264444444$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

Eqs. (7.22) and (7.24) with $n = 2$ are utilized to evaluate the RPD. For this map, only trajectories coming out from $F_1(x)$ can reinject inside the laminar interval. Therefore, the RPD can be written as:

$$\phi(x) = \frac{x^{1/\gamma-1}}{\gamma (c^{1/\gamma} - x_{out}^{1/\gamma})} \quad (7.100)$$

where $|dF_1^{-1}(x)/dx|$ is:

$$\left| \frac{dF_1^{-1}(x)}{dx} \right| = \frac{(b_0 - b_1)x^{1/\gamma-1}}{\gamma (f^{1/\gamma}(b_0) - f^{1/\gamma}(b_1))} \quad (7.101)$$

and k is obtained from the normalization condition:

$$\int_{F_1^{-1}(x_{out})}^{F_1^{-1}(c)} k \, dx = 1, \quad (7.102)$$

then:

$$k = \frac{f^{1/\gamma}(b_0) - f^{1/\gamma}(b_1)}{(b_0 - b_1)(c^{1/\gamma} - x_{out}^{1/\gamma})} \quad (7.103)$$

Fig. 7.39 $\psi(l)$ function for Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 0.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.158449931412894$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

7.6.2.1 Map2. Results

Here, we compare the ability of the Perron-Frobenius operator technique and the M function methodology to calculate the RPD for the Map2. We use the same test analyzed in Chapter 6.

The first test is the same studied in [14]: $\gamma = 1$, $\varepsilon = 0.0001$, $a = 0.25$ and the laminar interval is $x_{out} \leq x \leq x_0 = c$. In Chapter 6, we have found that $M(x)$ is a linear function with slope $m \cong 0.4845$ ($\alpha \cong -0.06$). Accordingly, the RPD function results (see Eq. (6.22)):

$$\phi(x) \cong \frac{1}{c - x_{out}} \quad (7.104)$$

We apply the Eq. (7.100) to obtain the RPD function using the Perron-Frobenius operator technique:

$$\phi(x) = \frac{x^{1/\gamma-1}}{\gamma \left(c^{1/\gamma} - x_{out}^{1/\gamma} \right)} = \frac{1}{c - x_{out}} \quad (7.105)$$

Both RPDs, Eqs. (7.104) and (7.105), are very similar.

The second test uses the same parameters of the first test, but $\gamma = 0.5$. The RPD evaluated by the M function methodology was given in Eq. (6.25):

Fig. 7.40 $\psi(l)$ function for Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 1.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.158449931412894$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

$$\phi(x) \cong 1.94 \frac{(x - x_{out})^{0.94}}{(x_0 - x_{out})^{1.94}} \quad (7.106)$$

If we apply the Eq. (7.100), we obtain:

$$\phi(x) = \frac{x^{1/\gamma-1}}{\gamma (c^{1/\gamma} - x_{out}^{1/\gamma})} = \frac{2x}{(c^2 - x_{out}^2)} \quad (7.107)$$

Figure 7.44 shows the comparison between the RPDs calculated using Eqs. (7.106) and (7.107). We can observe that both functions are similar.

Fig. 7.41 $\psi(l)$ function for Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 0.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.126444444444$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

Fig. 7.44 RPD functions for the Map2. Parameters: $\gamma = 0.5$, $\varepsilon = 0.0001$, $a = 0.25$. Red points: M function methodology. Blue line: Perron-Frobenius operator technique.

Finally, we study the third test: $\gamma = 1.5$, $\varepsilon = 0.0001$, $a = 0.25$. The RPD calculated using the M function methodology results (see Eq. (6.26)):

Fig. 7.42 $\psi(l)$ function for Map1. Parameters: $\varepsilon = 0.001$, $\gamma = 1.5$, $N = 30000$, $c = 0.1128$, $\bar{x} = -0.126444444444$. Red points: numerical data. Blue line: M function methodology. Green line: Perron-Frobenius operator technique.

$$\phi(x) \cong 0.6228 \frac{(x - x_{out})^{-0.3772}}{(x_0 - x_{out})^{0.6228}} \quad (7.108)$$

On the other hand, the RPD evaluated by the Perron-Frobenius operator technique is:

$$\phi(x) = \frac{x^{1/\gamma-1}}{\gamma (c^{1/\gamma} - x_{out}^{1/\gamma})} = \frac{2x^{-1/3}}{3 (c^{2/3} - x_{out}^{2/3})} \quad (7.109)$$

Figure 7.45 shows the RPD functions calculated using the M function methodology (red points) and the Perron-Frobenius operator technique (blue line). From this figure we can observe a good concordance between the two functions.

Fig. 7.45 RPD functions for the Map2. Parameters: $\gamma = 1.5$, $\varepsilon = 0.0001$, $a = 0.25$. Red points: M function methodology. Blue line: Perron-Frobenius operator technique.

Finally, we highlight that both methodologies produce similar RPD functions for the three tests.

7.7 Conclusions

In this chapter we have presented a new methodology to evaluate the statistical properties of intermittency. To do this we implemented the Perron-Frobenius operator technique to calculate the reinjection probability distribution function (RPD) for type II and V intermittencies. Once obtained the RPD function, we have calculated other statistical properties such as the probability of laminar lengths [1].

For type II intermittency, we have considered piecewise monotonic maps $F(x)$ defined inside the interval $0 \leq x \leq 1$. We have compared the theoretical results obtained by the Perron-Frobenius operator technique [1] with those calculated by the M function methodology [5, 9, 10, 11, 12, 13, 15, 16]. We have found that for a wide class of maps, the theoretical RPD calculated using the Perron-Frobenius operator technique can be written as: $\phi(x) = \chi(x)x^\alpha$. Nevertheless, if we used the M function methodology the RPD was given by: $\phi(x) = bx^\alpha$. Both theoretical RPDs are very close, but there is a difference given by the factor of the power law function. For the Perron-Frobenius

methodology $\chi(x)$ is a function of x ; however, for the M function methodology b is a constant.

We carried out several computational tests to compare numerical data with theoretical $\phi(x)$ and $\psi(l)$ functions for maps with different nonlinear behaviors: exponential, trigonometric, logarithmic. For tests showing a linear $M(x)$ function, $M(x) = mx$; both schemes, Perron-Frobenius operator technique and M function methodology, obtain very good results. However we note that, in particular cases, the results calculated using the Perron-Frobenius scheme can be somewhat more accurate than those given by the M function methodology.

Also, we have analyzed other map with type II intermittency. The map is given by Eq. (7.56). For some parameters of the map, the numerical $M(x)$ function can become a nonlinear function. If we approximate, even with this condition, the nonlinear $M(x)$ with a linear function the RPD has a power law form like that given by Eq. (7.63). On the other hand, if we employ the Perron-Frobenius operator technique, the RPD reaches a more complex structure, see Eq. (7.57). Hence, the theoretical equation describing the RPD function depends on the employed methodology to obtain it.

We have considered two tests. For the first test, the numerical $M(x)$ function was linear. Hence, both RPD functions obtained by M function methodology and Perron-Frobenius operator technique produce similar results with good accuracy (see Figure 7.31). For the second test, the numerical $M(x)$ was nonlinear (see Figure 7.32). We used a linear $M(x)$ to approximate it; and we calculated a power law RPD. Also, we implemented the Perron-Frobenius operator technique to evaluate the RPD function. This last RPD presents a better accuracy respect the numerical data.

For type V intermittency we have studied the same two maps evaluated in Chapter 6. We have found that both methodologies obtain similar RPD functions for the all studied tests (either with continuous or discontinuous RPD functions).

The Perron-Frobenius operator, in the form here implemented, has shown to be a powerful tool evaluating the RPD and other statistical properties for intermittency. However, this scheme has a disadvantage: it is necessary to know, explicitly, the map and its inverse function.

On the other hand, the M function methodology is a very general tool, and it is an useful scheme to evaluate the intermittency statistical properties from data series. This methodology works accurately even when the map equations are not known. However, it could lose accuracy if the $M(x)$ function, obtained from the data series, is not linear.

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Chapter 8

CONCLUSIONS AND FUTURE WORKS

Abstract In this chapter the main conclusions and the original contributions developed in this thesis are summarized. Also, some suggestions for future works are presented.

8.1 Introduction

In this thesis, we have carried out theoretical and numerical studies on several topics on chaotic intermittency, which have been included in Chapters 4, 5, 6 and 7. At the end of these chapters, specific conclusions were written. Here we compile the main conclusions of the thesis. Besides, the last section describes some suggestions for future works.

8.2 Conclusions

Based on previous results, we have implemented the M function methodology to evaluate the RPD functions [1, 2, 3, 4, 5, 6, 7, 8, 9]. These papers have shown that this methodology is an useful and strong tool to calculate the statistical functions for intermittency processes. For many one-dimensional maps, the reinjection probability density function can be approached by a power law given by Eq. (3.25):

$$\phi(x) = b(\alpha)(x - \hat{x})^\alpha, \quad \text{with} \quad \alpha = \frac{2m-1}{1-m} \quad (8.1)$$

We have shown that for several maps:

$$x_{n+1} = F(x_n) \equiv \begin{cases} F_1(x_n), & x_n < x_m \\ F_2(x_n), & x_n \geq x_m \end{cases} \quad (8.2)$$

where $F_2(x) = (F_1(x) - 1)^\gamma$, the exponent α verifies:

$$\alpha = \frac{1}{\gamma} - 1. \quad (8.3)$$

In Chapter 4 we have investigated the mechanisms by means of which the RPDs are developed in nonlinear one-dimensional maps. We have found that the pre-reinjection points of the map are very important to determine the RPDs, because the structure of the RPDs are generated around them. Two types of pre-reinjection points have a fundamental role to determine the RPDs: those with zero first derivative (extreme points) and those with infinite first derivative.

Based on this fact, we have proposed analytical methods (called Alternative 1 and Alternative 2) to provide the RPD function, hence by simple calculus it is possible to estimate the RPD given by Eq. (8.4). Specifically, for extreme points, we have developed two simple methods determining the parameter γ for other type of maps that those given by Eq. (8.2). From γ , we can obtain the RPD exponent α as a function of the number of null derivatives of the map at the extreme point. The extreme condition of the point imposes that this number must be odd, hence there is a restriction on the possible values of α . Also we have shown that the characteristic exponent, β , is determined, depending on the intermittency type, by the value of α .

We note that the two methods, Alternative 1 and Alternative 2, are directly related by the order of the first non zero derivative of the map.

$$\phi(x) = b(x - x_m)^\alpha \quad (8.4)$$

To verify these results, we have carried out several numerical experiments. We have found that the theoretical predictions and the numerical data have a very good accuracy. The relationships between β and α are given for Eq. (4.32) for type I intermittency and for Eq. (4.33) for type II and III intermittencies. We have shown that for the maps investigated in [10], our new methods provide the exact value of the characteristic exponent β in agreement with the numerically reported values in the mentioned reference. We highlight that the values of the exponents α and β calculated by this new method usually do not depend on the parameter values of the map, in particular on ε .

The previous studies do not include, reinjection processes govern by pre-reinjection points with infinite first derivative. Therefore, a new method to evaluate the RPD function when the reinjection process is govern by a pre-reinjection point with infinite derivative was developed. For these processes, the reinjection mechanism is different to those generated by extreme points, and we can not directly apply the Alternative 1 and Alternative 2 methods. To find the exponent γ , we obtain an inverse function of $F_2(x)$ and over it we apply the Alternative 1 or 2 methods. Then, an approximated

function of the inverse of $F_2(x)$ is evaluated. After that, a new inverse function of this approximated function is obtained. We highlight that the numerical results and the theoretical predictions of this new scheme show a high accuracy.

Furthermore, we have shown that under specific conditions the RPD can be uniform. If the RPD in a neighborhood of the fixed point $x_0 = 0$ verifies the following two conditions:

- i) $\phi(0) \neq 0$
- ii) $\left. \frac{d\phi(x)}{dx} \right|_0$ is bounded.

Then the function $M(x)$ satisfies $M(x) = 0.5x$ for points closed enough to $x = 0$. Note that this approximation produces the same value of β that should be found in the case of uniform reinjection, because of $m = 1/2$.

In Chapter 5 type III intermittency with nonzero lower boundary of reinjection and high noise intensity was studied. To carry out this task, we extended some recent studies proposed in [2, 3, 4, 11]. We have described the influence of the noise intensity, σ , and the LBR, $\hat{x}$, on the reinjection probability density function. The importance of the adjacent reinjection mechanism was also evaluated (this mechanism was introduced for first time in Ref. [11]).

We have implemented the M function methodology and we found that the RPD behavior depends on $\hat{x}$. For $\hat{x} = 0$ the laminar interval can be split in subintervals in which the $M(x)$ function preserves the linear form. However, for $\hat{x} \neq 0$, several behaviors appear:

- i) For $(K+1)\sigma < \hat{x}$ a zone where nonlinear $M(x)$ function is displayed. This zone is a matching between two linear behaviors of $M(x)$.
- ii) For $(K+1)\sigma \geq \hat{x}$, the $M(x)$ function can be split in three straight lines with different slopes: m_l , m_h and m_r ; where m_l corresponds to a uniform reinjection and m_r is approximately equal to the noiseless slope m . Even, for higher noise intensities the $M(x)$ function is represented by a smooth nonlinear curve.

We have calculated the noisy RPD function considering the two iteration processes occurring before the reinjection, for which it was necessary to solve two convolution integrals. The NRPD can be represented by series of exponential functions with exponent $\alpha + 2$, where α is the noiseless exponent. The new obtained NRPD presents a more complex structure than those previously published [3, 4]. Also, the exponent $\alpha + 2$ is different from that presented before [3, 4, 11].

Through the two iterations before the reinjection, from the pre-reinjection points (extreme points) to the laminar interval, the type III intermittency trajectories are stretched

or contracted by the derivative $K = dF(x_n)/dx$, where $x_{n+1} = F(x_n)$ is a reinjected point into the laminar zone. The amplified noise generated in the reinjection process can be calculated as $(K + 1)\sigma$. The previous studies [3, 4] are only particular cases satisfying $K \approx 0$ or $K \gg 1$.

We have made several numerical experiments or tests. In all of them the numerical data and the theoretical results showed a very good agreement.

An important results is that using the NRPD evaluated from the noisy data, we can obtain a complete description of the noiseless system, if the noise intensity is not high enough. If the noise strength is high, the noisy $M(x)$ function can introduce errors in the evaluation of the noiseless exponent α . Based on our studies, we suggest to use the noiseless α to evaluate the NRPD if the difference between the noisy α and the noiseless α is lower than 10%.

On the other hand, the effects of the adjacent reinjection mechanism (AR) on the NRPD were also studied. For the two maps with type III intermittency analyzed in this thesis, the AR mechanism was not important. Only for very high noise intensity values –noise intensities approximately equal to the laminar interval length– the AR has influences on the reinjection process. However, for these cases the noise is the principal phenomenon, and the intermittency is hidden by the noise.

As we explained in Chapters 3 and 4, the M function methodology have worked accurately for type I, II and III intermittencies, which posses continuous and differentiable local maps. In Chapter 6 this methodology was applied to describe the reinjection process for type V intermittency, which happens when the local map has a nondifferentiable or discontinuous point. In type V intermittency the fixed point disappears and the trajectories move inside a narrow channel in a similar way that type I intermittency. Specifically, type V intermittency occurs when a nondifferentiable point, NDP, and a stable fixed point, x_0 , collide forming a channel between the bisector line and the map. We have shown that the M function methodology also works accurately for type V intermittency. This methodology can capture very well the RPD function and with this function getting the probability density of the laminar lengths and average laminar length, for different maps and different reinjection mechanisms.

To describe this phenomenon, we have analyzed two one-dimensional maps. The first one map (called in this thesis Map1) is given by Eq. (6.6); its local map is composed of two very simple functions, one is linear and the other one is quadratic. However, the local part of the second piecewise map, called Map2 and defined by Eqs. (6.19)-(6.20), is given by a sine function. Also, the implemented laminar intervals were very different for both maps. For Eq. (6.6), it was small, $c \approx 0.1 \ll 1$ and symmetrical around the vanished fixed point. However, for the second map, Eq. (6.19), the laminar

interval only considered positive values and it was very large: $c \approx \pi$. But, despite of these differences the M function methodology have produced accurate results.

We have found that the RPD can be a nonuniform or a discontinuous function. The nonuniform RPD is described by an exponential function with exponent α (see Eq. (8.1)). This type of RPD occurs when the lower boundary of return is equal that the lower boundary of the laminar interval: $\tilde{x} = x_0 - c$. To obtain the analytical RPD is only necessary to evaluate the slope m of the $M(x)$ function (α depends on m). The uniform reinjection is only a singular case when $m = 1/2$ ($\alpha = 0$). We have obtained that the local map does not influence the reinjection process if $c \ll 1$. However, if the local map is nondifferentiable, the laminar length, $l(x, c)$, and the probability density of the laminar lengths, $\psi(l)$, are nondifferentiable too. For the Map1 the probability density of the laminar lengths, $\psi(l)$, has two behaviors: the first one for reinjected points inside the interval $-c \leq x \leq x_0$, which need more than ≈ 40 iterations to leave the laminar interval; the second one for reinjected points in $x_0 < x \leq c$ which need less than ≈ 40 iterations to leave the laminar interval. As the Figures 6.4 and 6.9 show, $\psi(l)$ is a nondifferentiable function for $l \approx 40$. The behavior for Map2 is different, it is shown in Figures 6.16, 6.19 and 6.22. we can observe that $\psi(l)$ is a differentiable function with a high concentration between $20 < l < 40$. $\psi(l)$ is a differentiable function because the laminar interval only contains points verifying $x > 0$, then it is not symmetrical around the vanished fixed point. The average laminar length, $\bar{l}$, acquires different form for each map. The Map1 verifies $\bar{l} \propto \varepsilon^{-\beta}$; however, the characteristic relation for Map2 is $\bar{l} \propto \ln(\varepsilon)$. We highlight that there is not only one characteristic relation for type V intermittency, it depends on the local map.

We have also found that, for the Map1, Eq. (6.6), the RPD can be a discontinuous function. This phenomenon happens because there are two reinjection mechanisms if the lower boundary of return is less than the lower limit of the laminar interval: $\tilde{x} < x_0 - c$. One of them is produced by trajectories coming from points $x < x_0 - c$, and the other one from trajectories coming from $x > x_m > x_0 + c$ (see Figure 6.1). The laminar interval is divided in two subintervals, within each of them the RPD function has different behavior. The subintervals are $\Delta_1 = [x_0 - c, x_s)$ and $\Delta_2 = [x_s, x_0 + c]$. Δ_1 receives trajectories coming from $x < x_0 - c$ and $x > x_m$; however Δ_2 only receives trajectories from $x > x_m$. x_s is a singular points where the RPD is discontinuous. The RPD is described by a sum of exponential functions with exponents α_i and α_s , to obtain its analytical expression is only necessary to evaluate the slope m_i and m_s of the $M(x)$ function. Where m_i and m_s are calculated using reinjected points from $x < x_0 - c$ and $x > x_m$ respectively. The influence of $\tilde{x}$ on the RPD, $\phi(x)$, has been evaluated. The lower limit of return, $\tilde{x}$, has an important influence on the RPD and the characteristic relation: when the distance $|\tilde{x} - (x_0 - c)|$ increases the influence of γ reduces, and the reinjection process resembles the reinjection process for $\gamma = 1$. The influence of γ on

the RPD has also been studied. The exponents α_i and α_s decrease when γ increases, and $\alpha_i \approx \alpha_s \approx 0$ for $\gamma = 1$. However, in all cases the RPD is discontinuous. Similarly to the RPD, the probability density of the laminar lengths also has two distinctive behaviors, one for reinjections inside the subinterval Δ_1 and the other one for reinjections in the subinterval Δ_2 . For all analyzed tests, with different values of γ and $\tilde{x}$, the characteristic relation could be written as: $\bar{l} \approx \varepsilon^{-0.5}$.

We highlight that the reinjection mechanism for type V intermittency is very interesting and it has a wide behavior. Until now only uniform reinjection has been considered; however, we have shown that the uniform reinjection is only a particular case. The reinjection depends, mainly, on the global part of the map. Also, the obtained theoretical equations for the reinjection probability density, the probability density of the laminar lengths and the characteristic relation showed a very good accuracy regarding the numerical data for different values of γ and $\tilde{x}$.

In Chapter 7 we have presented the Perron-Frobenius operator technique to evaluate the RPD function for type II and V intermittencies. Once obtained the RPD, we have calculated other statistical properties such as the probability of laminar lengths.

Type II intermittency was studied using piecewise monotonic maps $F(x)$ defined inside the interval $0 \leq x \leq 1$. We have compared the theoretical results obtained by the Perron-Frobenius operator technique [12] with those calculated by the M function methodology [1, 2, 4, 5, 6, 7, 3, 8]. We have found that for a wide class of maps, the theoretical RPD calculated using the Perron-Frobenius operator technique can be written as: $\phi(x) = \chi(x)x^\alpha$. Nevertheless, if we used the $M(x)$ methodology the RPD was given by: $\phi(x) = bx^\alpha$. Both theoretical RPDs are similar, but they differ in the factor of the power law function. For the Perron-Frobenius methodology $\chi(x)$ is a function of x ; however, for the M function methodology b is a constant.

We carried out several computational tests to compare numerical data with theoretical $\phi(x)$ and $\psi(l)$ functions for maps with different nonlinear behaviors: exponential, trigonometric, logarithmic. For tests showing a linear $M(x)$ function, $M(x) = mx$; both schemes, Perron-Frobenius operator technique and M function methodology, obtain very good results. However we note that, in particular cases, the results calculated using the Perron-Frobenius scheme are more accurate than those given by the M function methodology.

Also, we have analyzed other map with type II intermittency (see Eq. (7.56)). For some parameters of the map, the numerical $M(x)$ function can become a nonlinear function. If we approximate, even with this condition, the nonlinear $M(x)$ with a linear function the RPD has a power law form (see Eq. (7.63)):

$$\phi(x) = bx^\alpha \cong 2.47x^{-0.39} \quad (8.5)$$

On the other hand, if we employ the Perron-Frobenius operator technique, the RPD reaches a more complex structure (see Eq. (7.57)):

$$\phi(x) = b \frac{(e - e^{a_1^\gamma}) \left[\ln \left(x(e - e^{a_1^\gamma}) + e^{a_1^\gamma} \right) \right]^{\frac{1}{\gamma} - 1}}{\gamma [x(e - e^{a_1^\gamma}) + e^{a_1^\gamma}]} \quad (8.6)$$

Hence, the theoretical equation describing the RPD function depends on the employed methodology to obtain it. We have considered two tests. For the first test, the numerical $M(x)$ function was linear. Hence, both RPD functions obtained by M function methodology and Perron-Frobenius operator technique produce similar results with good accuracy (see Figure 7.31). For the second test, the numerical $M(x)$ was nonlinear (see Figure 7.32). We used a linear $M(x)$ function to approximate it; and we calculated a power law RPD. Also, we implemented the Perron-Frobenius operator technique to evaluate the RPD function. This last RPD presents a better accuracy regarding the numerical data as Figure 7.33 shows.

For type V intermittency we have studied the same maps evaluated in Chapter 6: Map1 and Map2.

For Map1 we carried out two studies:

- The first one implements the Perron-Frobenius operator technique to obtain continuous RPDs.
- The second study uses the Perron-Frobenius operator technique to obtain discontinuous RPDs. When the RPD is discontinuous, because the reinjection is produced by different part of the map, the implementation of this technique is not direct, and it is necessary to repeatedly apply this operator to find the RPD function.

We have found that both methodologies, M function and Perron-Frobenius operator, obtain similar RPD functions for all studied tests. However, for the second map or Map2, which is a singular map, both techniques produce RPD functions with slight differences. Using the M function methodology, the RPD can be written as (see Eq. (6.21)):

$$\phi(x) = \frac{(x - x_{out})^\alpha}{\gamma (c - x_{out})^{1/\gamma}}, \quad \text{with} \quad \alpha = \frac{1}{\gamma} - 1 \quad (8.7)$$

However, the RPD obtained applying the Perron-Frobenius operator technique can be written as (see Eq. (7.100)):

$$\phi(x) = \frac{x^\alpha}{\gamma (c^{1/\gamma} - x_{out}^{1/\gamma})}, \quad \text{with} \quad \alpha = \frac{1}{\gamma} - 1 \quad (8.8)$$

Finally, we can note that the Perron-Frobenius operator, in the form here implemented, has shown to be a powerful tool evaluating the RPD and other statistical prop-

erties for intermittency. However, this scheme has a disadvantage: it is necessary to know, explicitly, the map and its inverse function. On the other hand, the M function methodology is a very general tool, and it is an useful scheme to evaluate the intermittency statistical properties from data series. This methodology works accurately even when the map equations are not known. However, it could lose accuracy if the $M(x)$ function, obtained from the data series, is not linear.

8.3 Suggestions for future works

In this section some future research topics are briefly explicated.

- The M function methodology works accurately if $M(x)$ is a lineal function; however it can fail when $M(x)$ is nonlinear. To solve this problem we have introduced other technique using the Perron-Frobenius operator. But, it has shown to work accurately if the invariant density previous to the reinjection can be considered constant: $\rho(x') = k$, where $x = F(x')$ is a reinjected point. Therefore, a new task is to extend the theory when invariant density is a non-constant function: $\rho(x') \neq k$. We have found numerical tests, in which the RPD is a discontinuous function, and it can not be explained by the research developed in this thesis. In Chapter 7, we have introduced the first steps to develop this task.

- To extend the M function methodology and the Perron-Frobenius operator technique to work for high dimensional intermittency. An interesting paper [13] studied the two dimensional intermittency. Even though the process is different from one-dimensional maps, to extend both methodologies for high dimensional intermittency could be an attractive future activity.

- To apply the new methodologies, M function and the Perron-Frobenius operator, to physical phenomena such as turbulent flows, plasma discharge, etc.

As first step, we have applied the M function methodology to study type I intermittency with discontinuous RPD in the three wave truncation model of the derivative nonlinear Schrodinger equation (DNLS) [7]. The DNLS equation describes the evolution of circularly polarized Alfvén waves of finite amplitude propagating parallel or near-parallel to an unperturbed uniform magnetic field in a cold, homogeneous plasma, using a two-fluid, quasi-neutral approximation with electron inertia and current displacement neglected. A three wave truncation model allows to obtain an approximated solution which consists of three traveling waves satisfying the resonance condition

$2k_0 = k_1 + k_2$ (the wave that corresponds with k_0 is called the mother wave) [14, 15]. However, it was a simplified study and deeper research could be carry out.

Turbulent flows is other subject where intermittency can be analyzed. Turbulence is one of the most challenging topics in physics, which is characterized by random fluctuations of the thermo-mechanical variables such as pressure, velocity, temperature, enthalpy, etc. Flows far from the walls are traditionally called free shear flows. Examples of free shear flows are wakes, shear layers and jets [16]. Free shear flows show irregular boundary between the turbulent flow and the environment fluid. This concept is reinforce for the results described in many experiments, starting with those developed by Corrsin in 1943 [17]. Also the experiments have shown that exist a very contorted moving surface, called the viscous superlayer. This surface separates zones of turbulent and non-turbulent flow. From the mathematical description, all thermo-mechanical variables change smoothly through the viscous superlayer, there are no discontinuities across it. However, the layer is very thin compared with a characteristic length of the flow. At a fixed position in space, close to the moving boundary, variables as velocity evolves in time intermittently: some times the flow is laminar and in other times is turbulent. Consequently, the time series showing the pressure or velocity evolution can expose intermittency behavior.

- To apply the new methodologies to other type of intermittency like as on-off intermittency, eyelet intermittency, spatiotemporal intermittency, multichannel reinjection with noise, etc.

The on-off intermittency is characterized by sudden changes between nearly constant periods or static states and irregular bursts of large intensity oscillations. Nearly constant periods correspond to the “off” states, while the “on” states are the burst time periods which depart and return abruptly to the “off” states. Clearly the “off” and “on” states correspond to the laminar and chaotic phases [18, 19]. The most elemental version of on-off intermittency needs two elements: 1 - An invariant object, 2 - Orbits entering and leaving each small neighborhood of the invariant object.

Eyelet intermittency, can be detected at the boundary of phase synchronization of coupled chaotic oscillators. The occurrence of eyelet type intermittency can be described by means of the synchronization of unstable periodic saddle orbits embedded in chaotic attractors [20, 21]. Eyelet type intermittency also may appear close to the phase synchronization boundary of periodically forced chaotic system, e.g. a forced oscillator [20].

Spatiotemporal intermittency can be observed in systems with spatial extension – spatially extended dynamical systems. These systems are generally depicted by non-linear partial differential equations (PDEs) and they display intermittency. Therefore, it is a very broad research topic.

We have defined that chaotic intermittency is characterized by transitions between regular phases and chaotic bursts. These regular or laminar phases are regions of pseudo-equilibrium and/or pseudo-periodic solutions and the bursts ones are regions where the system evolution is chaotic. Multichannel intermittency occurs when there are more than one pseudo-equilibrium and/or pseudo-periodic solutions. We note that due to the multiplicity of the laminar zones this phenomenon, sometimes, is also called multintermittency.

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Appendix A

PUBLICATIONS

During the development of this thesis (2016-2018) the following scientific publications have been made:

A.1 Books

One book with scientific peer evaluation was published:

1. *Elaskar, S.; del Río, E.* (2017), New Advances on Chaotic Intermittency and its Applications. **Springer**, New York, 2017. ISBN: 978-3-319-47836-4.

A.2 Papers in Journals

Six papers, with scientific peer evaluation, were written. Four of them have been published and two paper have been submitted.

2. *Elaskar, S.; del Río, E.; Zapico, E.* (2016), Evaluation of the statistical properties for type-II intermittency using the Perron-Frobenius operator. **Nonlinear Dynamics**. Vol. 86, No 2, pp. 1107-1116.

3. *del Río, E.; Elaskar, S.* (2016), On the intermittency theory. **International Journal of Bifurcation and Chaos**. Vol. 26, No. 14, 1650228.

4. *Elaskar, S.; del Río, E.; Costa, A.* (2017), Reinjection probability density for type-III intermittency with noise and lower boundary of reinjection. **Journal of Computational and Nonlinear Dynamics, ASME**. Vol. 12, No 3, pp. 031020-031020-11.

5. *Elaskar, S.; del Río, E.; Gutiérrez Marcantoni, L.* (2018), Non-uniform reinjection probability density function in type V intermittency. **Nonlinear Dynamics**. Vol 92, pp. 683697.

6. *del Río, E.; Elaskar, S.* (2017), Experimental evidence of power law reinjection in chaotic intermittency. **Communications in Nonlinear Science and Numerical Simulation**. Submitted.

7. *Elaskar, S.; del Río, E.* (2018), Discontinuous reinjection probability density function in type V intermittency. **Journal of Computational and Nonlinear Dynamics**. Submitted.

A.3 Book Chapters

One book chapter was published.

8. *del Río, E.; Elaskar, S.* (2016), The Intermittency Route to Chaos. **Handbook of Applications of Chaos Theory**. CRC Press Book. ISBN: 9781466590434. pp. 3-21. www.crcpress.com/Handbook-of-Applications-of-Chaos-Theory/Skiadas-skiadas/9781466590434.

A.4 Conferences

One Invited Lecture.

9. *Elaskar, S.* (2016), New theory of chaotic intermittency with and without noise. **Laboratoire de Mathématiques et de leurs Applications. Université de Pau et de Pays de L'Adour**, France. November 2016. Invited Lecture.