UNIVERSIDAD POLITÉCNICA DE MADRID
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Fluid Dynamic Problems of High-Speed Trains in Tunnels

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Madrid, 2018

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Agradecimientos

Esta tesis es el fruto de trabajo de más de tres años de investigación, esfuerzo, frustraciones y victorias; decir que Manuel Rodríguez, mi tutor y mentor, es el mayor co-autor de ella es quedarse corto. Sin el, nada de esto se habría logrado. No solo me guió a través del intrincado laberinto que puede ser realizar un doctorado, sino que además me enseñó a utilizar herramientas, como los ordenes de magnitud, que serán imprescindibles para el resto de mi vida profesional. Su tenacidad, sabiduría y paciencia estarán siempre en mi corazón.

Quiero agradecer también a Benigno Lázaro por toda la ayuda con la parte numérica de la investigación; a Ezequiel González, que además de haberme apoyado en incontables ocasiones, es co-autor de los artículos que fueron resultado de estos años de investigación; a José Manuel Vega de Prada por sus consejos para el último capítulo de la tesis; y a Rafael Rebolo, por su aportación invaluable con el tema de temperatura en túneles.

A Mukh, por echarme la mano con la gramática del inglés, y el apoyo moral en tantos jams.

A Susie Q, por todo el impulso, motivación y cariño que me dió para volar en la recta final.

A mis padres, mi hermana y mis hermanos, que siempre me han dado su amor incondicional.

A Ramón, por el cariño y pasión a esta vocación.

A todos mis amigos que me dieron ánimo, fuera estando cerca o lejos.

Y por último, pero no por ello menos importante, al CONACYT que creyó en mí, y que me dió los recursos para poder cumplir este sueño.

Nada de esto hubiera sido posible sin ellos.

Resumen

Es bien conocido que España es uno de los países con más kilómetros de vía de alta velocidad, de los cuales la mayoría se encuentran a la mayor altitud en Europa, atravesando zonas montañosas a través de un gran número de túneles. El movimiento de trenes de alta velocidad en túneles genera una multitud de fenómenos fluidodinámicos que generalmente se encuentran asociados a ondas de presión que viajan por su interior; dada la magnitud de las dimensiones de los túneles, es necesario predecir la dinámica de dichas ondas con herramientas de cálculo lo más simples posible, pero que describan adecuadamente el proceso físico.

En el presente trabajo se analizan los detalles del flujo en las distintas regiones del movimiento, reteniendo en cada una de ellas los efectos más relevantes, para así generar un programa de cálculo que permite determinar el complejo flujo en el interior del túnel de forma rápida y eficiente. De este modo se puede determinar, en las etapas preliminares del proyecto de diseño de un túnel, los denominados límites de salud y confort, que se deben cumplir de forma obligatoria, en especial el de salud. Se podría pensar que utilizando programas comerciales de Mecánica de Fluidos Computacional (CFD) se obtendrían resultados de forma sencilla, pero nada más lejos de la realidad ya que para tener una resolución adecuada de las ondas que viajan por todo el túnel, el número de nodos requerido es inmenso y los tiempos de cálculo prohibitivos. Los métodos CFD se pueden utilizar para generar la onda inicial durante el par de décimas de segundo que tarda en entrar la cabeza del tren en el túnel, y para ello es necesario mallar adecuadamente los primeros 100 ó 200 primeros metros del túnel; sin embargo, la propagación a lo largo de kilómetros de túnel, dada la naturaleza casi unidimensional del flujo, se puede modelar utilizando ecuaciones simplificadas que ahorran tiempo de implementación y cálculo computacional, sin perder la información física relevante.

En este trabajo se analiza también la propagación y distorsión de la primera onda generada por la cabeza del tren al entrar en el túnel, que está caracterizada por dos parámetros básicos: el incremento máximo de presiones inicial y la pendiente máxima de la distribución de presiones inicial. Esta onda es susceptible de producir una onda de choque en el interior del túnel, que al reflejarse en la boca opuesta del mismo, genera ondas de micro presión en el exterior, dando lugar a un estallido sónico que puede afectar a las zonas pobladas en las cercanías del túnel. Por medio de un modelo unidimensional, que utiliza el método de las características, se determinan los parámetros de la onda citados anteriormente, que permiten asegurar que el estallido sónico no se produzca en un túnel dado por el que circulen trenes de alta velocidad.

Como última parte, se analiza la evolución de la temperatura en el interior de túneles. Cuando el túnel no es lo suficientemente largo, la temperatura se va renovando como consecuencia del paso de los trenes en ambas direcciones. Sin embargo, cuando el túnel es largo (del orden de los 10 km o más), la circulación de trenes es en un solo sentido (en túneles tan largos hay dos túneles paralelos) y la renovación del aire se hace más complicada, alcanzándose temperaturas elevadas al cabo de los años (la temperatura dentro del túnel puede incrementar alrededor de 20 °C con respecto a la temperatura ambiente). Partiendo de las ecuaciones integrales de conservación de continuidad, cantidad de movimiento y energía se proporciona una herramienta numérica capaz de predecir la temperatura que alcanzará el aire en el túnel al cabo del tiempo, a fin de poder tomar las medidas pertinentes para evitarlo cambiando el diseño preliminar del túnel.

Se estudia además la posibilidad de introducir una chimenea que conecte el túnel con el exterior, como una solución pasiva para reducir el aumento de presión y temperatura del flujo dentro del túnel.

Abstract

It is well known that Spain is one of the countries with more kilometers of high speed railway track, the majority of which are at the highest altitude in Europe, crossing mountain zones through a vast number of tunnels. The movement of high speed trains in tunnels generates a multitud of fluid-dynamic phenomena that are generally associated with the pressure waves that travel inside the tunnel; given the dimensions of the system, it is necessary to predict the aforementioned waves with a calculation as simple as possible, but that describes the physical process in an adequate way.

The present work analyzes the flow details throughout the different regions of flow, retaining in each one the most relevant effects, so that a calculus program can be generated to determine the complex flow on the inside of the tunnel in a fast and efficient way. In this way, the health and comfort limits that must be met, can be determined in the preliminary stages of design, particularly the health limit. It could be thought that commercial programs of Computational Fluid Dynamics (CFD) could be used to obtain this results, but this is far from reality, because in order to have a proper simulation of the traveling waves, the number of nodes required its immense and the computational times of calculation are prohibitive. The CFD methods can be used to generate the initial wave during the first couple of tenths of a second that it takes for the train head to enter the tunnel, and for that it is necessary to mesh adequately the first 100 or 200 meters of tunnel; however, the propagation throughout kilometers along the tunnel, given the one dimensional nature of the flow, can be modeled using simplified equations that save implementation time and computational calculus.

The propagation and distortion of the first compression wave generated by the entrance of the train head in the tunnel is also analyzed in this work, and it is characterized by two basic parameters: the maximum initial pressure increment and the maximum initial pressure gradient. This wave is susceptible to produce a weak shock wave in the interior of the tunnel, which at reaching the exit portal is partly reflected and partly emitted to the exterior generating micro-pressure waves and the so called sonic boom, which can be harmful to the populated areas near the tunnel's exit. By means of a one dimensional model, using the characteristics method, the aforementioned parameters are determined, so that it can be assured that, with those calculated parameters, no sonic boom will be generated for a given high speed railway tunnel.

As a last part, the evolution of the air and wall temperature inside a high speed tunnel is analyzed. When a tunnel is not long enough, the temperature is maintained near the ambient temperature thanks to the constant passing of the trains in both directions.

However when the tunnel is long (of the order of 10 km or more), the circulation of trains is only in one direction (in tunnels with this length there are two separated tunnels, one for each direction) and the air renovation becomes more complicated, reaching high temperatures inside the tunnel throughout the years (the temperature inside can increment around 20 °C above the ambient temperature). Starting from the integral conservation equations of continuity, momentum and energy a numerical tool is developed, capable to predict the air temperature that will be reached inside the tunnel throughout time, so that countermeasures can be taken to avoid it, changing the preliminary design of the tunnel.

The use of a chimney that connects the tunnel with the exterior is studied as a passive measure to reduce the pressure and temperature of the flow in the tunnel.

Chapter 1

Introduction

1.1 Problem description

The present work aims to explain the general aerodynamic effects that occur when a high speed train enters a tunnel, particularly a long one. These can be resumed as the generation, propagation and emission of pressure waves that create problems such as sonic booms at the exit portals, aural discomfort on the train passengers, mechanical fatigue in some of the tunnel and train elements, as well as an air temperature rise caused by the power dissipated by the locomotive.

The aerodynamics occurring in a tunnel as a train moves through it are totally different from those observed in the open air and their amplitude and severity grow as the train speed is increased (Raghunathan et al., 2002; Reinke and Ravn, 2004; Schetz, 2001). When a train enters a tunnel it generates an overpressure in front of it that is much larger than the one generated when it circulates in the open air (Baron et al., 2001; Choi and Kim, 2014). This over pressure is due to the confinement of the air between the tunnel and train walls (Cross et al., 2015). This induces a wave along the tunnel upstream the train (in front of the head), leaving the air moving behind the wave, and forces part of the air to leave through the region between the train and tunnel. The over pressure in front of the train grows as the train enters the tunnel, since the volume occupied by the train grows, and there is a larger amount of air that needs to flow from the front of the train to the entry portal which is at atmospheric pressure; that requires to overcome a larger friction force, and hence, a larger pressure at the front of the train (Ko et al., 2012).

Once the train has completely entered the tunnel, the above mentioned changes. An observer fixed with the train sees air coming to the front of the train with a speed lower than the train speed, and downstream where the train wake has vanished, he sees that the velocity with which the air escapes is quite similar to the one with which it came in the front, due to the continuity and incompressible nature caused by the low Mach number (William-Louis and Tournier, 2005). There is no more momentum variation and the train resistance is due to the pressure difference in the front and the back of the train, and the friction in the train and tunnel walls (Raghunathan et al., 2002; Schetz, 2001). Once the tail enters the tunnel, a change in the aforementioned conditions is noted as a decrease

in pressure which generates an expansion wave that travels in the train direction.

The compression and expansion waves get reflected when reaching the tunnel portals, as well as being partially reflected when reaching the train inside the tunnel (Vardy, 2008; Yoon et al., 2001); this creates a vast and complex pattern that depends on the train speed, the tunnel length, the ratio between the train and tunnel diameter, the friction in the tunnel and train walls, which dampens the waves, and the existence of interior shafts and junctions that makes the problem even more complex (Fukuda et al., 2006; Yoon et al., 2001). Nevertheless, the first compression and expansion waves are the most critical (Maeda et al., 2000; Vardy, 2008; Yoon and Lee, 2001), since the rest get damped with time, particularly if the tunnel length is of the order of kilometers. As the distance traveled by the train inside the tunnel grows, the pressure loss between the entry portal and the train tail also grows, so that it can maintain the flow behind the train; as such the pressure downstream the train goes below the ambient pressure.

The compressibility and friction effects in the tunnel become important for the description of the flow upstream as well as downstream the train, and if the tunnel is long enough, the waves can be damped and disappear (Raghunathan et al., 2002).

The flow in the whole tunnel (far-field) needs to be considered as well as the flow near the vehicle. Both domains are strongly dependent on each other. One major flow feature inside the tunnel are pressure waves travelling along it (Anthoine, 2009; Ko et al., 2012) upstream and downstream the train. The downstream and upstream evolution of the pressure waves have to be coupled with the flow over the train. In chapter 6 the different flow regimes that appear inside the tunnel and around the train are analyzed, to have an approximation of flow velocities and temperatures in the problem.

With the train completely inside the tunnel three characteristic regions can be distinguished by the value of the parameter D/L , where D is the equivalent diameter of the considered region, and L the characteristic length of the region. The region between tunnel and train has a characteristic value of $D/L \approx 0.03$ for a train of 200 m and a tunnel of 50 m². On the upstream and downstream regions of the train the value of D/L is even smaller, since L is a larger length.

When $D/L \ll 1$ (Liñan et al., 2016; Shapiro, 1953, 1964), a one dimensional approach can be taken, as it has been widely used (Barrow and Pope, 1987; Fukuda et al., 2006; Hieke et al., 2011; Mashimo et al., 1997; Miyachi et al., 2013; Woods and Pope, 1981; Yoon et al., 2001). This also applies in the gap between train and tunnel. In this approach the fluid magnitudes are uniform in each section, but change from one section to the other; besides the compressibility and unsteady effects can be retained. One dimensional flow equations are solved usually making use of control volume techniques (Baron et al., 2001; Fukuda et al., 2006; Mashimo et al., 1997; Ricco et al., 2007) which additionally need complex geometry routines in order to simulate the train evolution inside the tunnel (Baron et al., 2001; Maeda et al., 2000; Ogawa and Fujii, 1997; Yoon et al., 2001); an even more complex scheme is needed if a full three dimensional resolution is desired (Ogawa and Fujii, 1997). Meanwhile, the discretization of the linearized version of Riemann invariants can provide comparable results while being at the same time a robust numerical scheme, as it has been done and presented in chapter 6.

The flow around the nose and tail is 3-D, but the continuity, momentum and energy equa-

tions, applied in integral form to the appropriate control volume, allows to connect easily the full tunnel with the gap between tunnel and train by using pressure loss coefficients for the nose and the tail of the train (Baron et al., 2001).

An analysis of each of the terms in the motion equations is performed in order to unveil their relative importance and to simplify the problem to be solved. This kind of formulation is useful for the design of high speed lines with dozens of tunnels where the internal pressure on the train, external pressure outside tunnel portals, etc., should be evaluated. The aiming of this theoretical development is to obtain a model that can be solved numerically in the order of minutes for problems that require the general information of the flow inside the tunnel, such as the overall power dissipated by the train along the tunnel, or the general temperature rise on the air. Particularly, the problem of temperature rise on the tunnel wall along the years requires the calculation of hundreds of thousands of train runs; doing a complex computation for each passing would make the problem unsolvable, and that is where a simplified analysis such as the one proposed here can provide a general and robust tool for the computation of the flow inside the tunnel.

Among the complex wave pattern, the initial compression wave is the most critical one, since its magnitude and maximum pressure gradient are the largest when compared with the rest of the generated, propagated and reflected waves. If the track has ballast, the material will act as a porous medious, damping the wave and reducing its intensity (Vardy, 2008; Yoon et al., 2001; Yoon and Lee, 2001), but if the tunnel wall and floor are smooth by the use of slab track (which, as mentoned by Fukuda et al. (2006), is becoming quite common, since for security reasons emergency vehicles must be able to circulate between the rails) then its intensity will grow, given the non linear nature of the flow, in which at larger pressure, larger speed of sound, forcing the pressure wave to become steeper.

When the wave reaches the end of the tunnel a part is reflected and travels back in the form of an expansion wave, but another part is radiated to the ambient in the form of a so called micro-pressure wave. The term micro means that the amplitude is small compared with the atmospheric pressure. Nevertheless, if this first compression wave traveling inside the tunnel becomes steep enough to acquire the nature of a weak shock wave, then it will cause a sonic boom once it reaches the exit portal; its intensity can sometimes be compared to the sonic boom created by a supersonic airfract (Bellenoue et al., 2001; Kashimura et al., 2000). As such, countermeasures must be taken to avoid this issue in order to protect the environment, buildings, and people that are near the tunnel's portals. The whole phenomenon involves three parts: the generation of the initial wave in the inlet portal caused by an entering train, the propagation of the wave along the tunnel, and the emission to the ambience of the wave as a micro-pressure wave. The generation has been predicted analitically with one-dimensional models (Howe, 1998a,b, 1999; Howe et al., 2000, 2003; Howe, 2005) and numerically with one-dimensional (Hagenah et al., 2006; Ricco et al., 2007; Kikuchi et al., 2011), two-dimensional (Ku et al., 2010) and three-dimensional models (Bellenoue et al., 2002; Deeg et al., nd; Mok and Yoo, 2001; Ogawa and Fujii, 1997; Schlammer and Hieke, 2008). To decrease the initial steepness in the compression wave countermeasures can be applied on the tunnel, on the train, or on both (which at the end will reduce the amplitude of the radiated wave at the tunnel exit). The most common measures involve smoothing the initial compression wave at the

beginning (tunnel entrance). For the tunnel these include flared portals (Howe, 1999; Howe et al., 2000), airshafts (Anthoine, 2009; Deeg et al., nd; Hagenah et al., 2006; Ricco et al., 2007; Yoon et al., 2001) and portal hoods (Bellenoue et al., 2001; Howe et al., 2003; Howe, 2005; Schlammer and Hieke, 2008); for the train it usually involves the optimization of the design of the locomotive nose (Bellenoue et al., 2002; Kikuchi et al., 2011; Ku et al., 2010).

To study and predict the changing behaviour of the initial wave, the simulation of the pressure wave propagating inside the tunnel can be made with three dimensional models (Deeg et al., nd; Yoon et al., 2001; Yoon and Lee, 2001), but computationally speaking, this tends to be highly demanding. Luckily, as mentioned before, the behaviour of the flow inside the tunnel can be well approximated by one-dimensional equations along the axial axis (Baron et al., 2001; Fukuda et al., 2006; Hieke et al., 2011; Martínez et al., 2008; Miyachi et al., 2013; Rodríguez, 2013).

To make a first approximate approach to the wave-propagating phenomenon inside the tunnel a piston analogy was applied by Rodríguez (2013) to create the pressure wave in a one-dimensional scheme, solving analytically the Euler equations without friction in characteristic form. This is a good first approach, since it gives a security factor: if a shock wave is not generated without friction, then it certainly will not generate in a real life situation where the friction will attenuate the wave and reduce the possibility of a sonic boom at the tunnel's exit. In chapter 7 the present work proposes a numerical method based on the one presented by Rodríguez (2013) adding friction terms (steady and unsteady) in order to find the regression point. This can work as a fast tool to determine if a tunnel being designed will end up having the formation of a sonic boom at the exit, with the initial pressure wave generated by the train known a priori experimentally or numerically; it can also be used to establish the limit parameters that the initial compression wave must have in order to avoid the sonic boom, so that the designers create portals and/or locomotive noses with an optimal shape.

The emission can be calculated by using the acoustical Kirchhoff integral formulation (shown in section 5.4.5), which has a solid theoretical fundament and has been used in many occasions with good results when compared with real measurements (Baron et al., 2006; Hieke et al., 2011; Yoon and Lee, 2001).

Last but not least, in chapter 8 the rise of the tunnel wall temperature over the years for a long tunnel is analyzed with a simplified model. In long tunnels a significant amount of thermal energy may be transferred to the tunnel environment. The quantity of heat released per unit time in the tunnel is the power consumed by the train which includes the temperature rise caused by the aerodynamic drag, the heat losses of the electric engines, the AC inside the train, the mechanical friction of the moving parts, etc. (Barrow and Pope, 1987; Sadokierski and Thiffeault, 2008; Yanfeng et al., 2008; Yanfeng and Yaping, 2009). This thermal energy increases the temperature of the tunnel wall. The piston effect of the train (Ko et al., 2012) only cools a small proportion of the tunnel length. The accumulative effects of trains circulating daily during years can rise the tunnel wall temperature to undesirable values (Baron et al., 2001; Barrow and Pope, 1987; Sadokierski and Thiffeault, 2008; Thompson et al., 2011; Yanfeng et al., 2008). To approach this

problem, it is necessary to describe the flow when the train is inside the tunnel and also the remaining flow in the tunnel until the next train arrives.

A similar problem to this has been studied for underground railway systems, like the subway, but in systems like these where trains pass in the order of every 5 minutes, the movement of trains create a passive cooling system in the tunnels and stations by mixing the cold and hot air from the different regions (Abi-Zadeh et al., 2003; Ampofo et al., 2004; Ninikas et al., 2016; Sadokierski and Thiffeault, 2008). However, in a high speed line with long tunnels where the trains pass in the order of a half an hour the situation is different. The cooling that can be achieved by the entering ambient air impulsed by the stream caused by the train, penetrates just a small portion of the tunnel before the air temperature rises to the wall temperature. During the passing of the train, the penetration distance inside the rock is small when compared with the tunnel diameter (Sadokierski and Thiffeault, 2008) which can be seen with an order analysis of the heat equation on the rock

$$\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t_c}, \quad \alpha_R \frac{\partial^2 T}{\partial y^2} \sim \frac{\Delta T}{\ell_R} \rightarrow \ell_R \sim \sqrt{\alpha_R t_c}$$

where α_R is the thermal diffusion coefficient of the rock, so that for typical values ℓ_R is of the order of centimeters. With this in mind the wall can be considered isothermal, but once the train has passed there is a certain time between the leaving of a train and the entering of another; during this time there is forced convection caused by the flow left by the train movement until it gets damped by the friction on the tunnel walls.

The model has to be coupled with the heat equation for the rock temperature, where the heat transfer to the wall provides one of the boundary conditions, and the far temperature on the rock provides another.

The model has to be solved for each interval between trains, with the initial conditions changing according to the temperature rise created by the train. This can be estimated by using the integral energy equation along the whole tunnel during the time the train is inside, which will yield to a simple one dimensional model in the radial direction for the wall temperature in the interior of the tunnel.

The description, understanding and prediction of these subjects allow for the optimal design of tunnels in high speed train lines, which can be plenty in mountain zones given that high speed lines require the least amount of curves as possible (the high speed western corridor of Taiwan has 48 tunnels, with plenty of them being of the order of kilometers in length (Ko et al., 2012)). As such, fast and robust tools for the prediction of the general aerodynamic effects becomes fundamental for the development of high speed train lines.

1.2 Bibliographical research

For the study of the aerodynamic phenomena caused by a high speed train inside a tunnel Schetz (2001) reviewed the main differences between this and other types of transportation vehicles. Some of the key differences arise from the fact that the trains operate near the ground, and have a much larger length-to-diameter ratios than other vehicles; they

pass close to each other and to trackside structures, are more subject to crosswinds and operate in tunnels. He presented a compendium of experimental and numerical data about the different subjects of high-speed train aerodynamic, mentioning that the aerodynamic consequences of high-speed train operation in tunnels is centered on two interdependent phenomena: the generation of pressure waves and an increase in drag. He also stated that in long tunnels the drag increase is the most important effect. Raghunathan et al. (2002) presented the state of the art in the knowledge of aerodynamics of high-speed railway train. Is a concise and well documented paper with empirical, semi empirical and analytical equations, as well as numerical and experimental data about subjects such as the aerodynamic noise generated when trains pass next to nearby structures, the drag caused by the pressure difference between head and tail of the train, its skin and external elements like pantographs, the aerodynamics of train inside tunnels and the pressure wave generated because of their entering, the resulting sonic-boom, among others. Reinke and Ravn (2004) recounted the aerodynamic problems aroused by high-speed twin-tube tunnels which might cause more extreme aerodynamic conditions than single-tube double-track tunnels, which were more common in the past for short and long tunnels. Twin-tube tunnels were mainly used for very long distances, but currently, twin-tube is preferred for increasingly shorter tunnel length because of several safety features, such as a better utilization of the mechanical ventilation.

The sonic boom problem can be decomposed in three main parts: generation of the pressure wave, propagation of the wave, and emission of the micro-pressure wave. Vardy (2008) described the origins of sonic booms emitted from railway tunnel portals, providing simple design expressions to enable the estimation of their amplitude, particularly for the case of relatively short tunnels. This paper was targeted primarily at designers that wish to make initial statements without the need of specialized software or specialists in the field. The potential effectiveness of various remedial measures was described, especially at tunnel entrances and exits. For the generation of the pressure wave an analytical approach was developed by Howe (1998a), where the train was modeled as a continuous distribution of monopole sources whose strengths are determined by the train nose profile. The initial wavelength greatly exceeds the tunnel diameter at typical train Mach numbers of about 0.2 and so the analytical problem can be solved by using a compact Greens function. The form of this function depends on the tunnel entrance geometry and on the proximity of other inhomogeneities, such as embankments, buildings, and bridge structures. Detailed predictions were given for axisymmetric trains entering a long circular cylindrical tunnel. The results were found to be in excellent agreement with experimental data for this configuration. Later on Howe (1998b) extended the approach for a flanged cylinder using the Rayleighs method for the calculation of potential flow. Howe (1999) studied the same approach for the case of a flared portal, showing that it was the optimal shape to generate a pressure wave with a constant gradient (leading to a minimal micro-pressure emission at the end of the tunnel). Howe et al. (2000) applied the analytical approach with monopole and second-order dipole sources to account for vortex, and developed experiments with axisymmetric model scale trains to validate the results. Howe et al. (2003), made an analytical approach for a portal hood that is unvented, using the

same approach as Howe et al. (2000) with dipoles to account for the vortex generated by the air that exits with the train entrance; they found that the compression wave is generated by the two successive interactions of the train nose with the hood portal and with the junction between the hood and tunnel. These interactions produce a system of compression and expansion waves, each having characteristic wavelengths that are much smaller than the hood length. Two years later Howe (2005) applied the Greens function approach for a portal hood with a window, where the window was modeled as a pressure node for the solving of the potential flow; the results were in excellent agreement with measurements of compression wave profiles obtained in model scale experiments.

Experimental work was conducted by Bellenoue et al. (2001), who studied the effects of a blind hood (without hole) with a constant section on the compression wave generation. It showed that for a fixed train/tunnel blockage ratio, there exists a unique optimum hood for which the pressure gradient can be minimized. A year later Bellenoue et al. (2002) developed a reduced-scale test method using low-sound-speed gas mixtures. They found that as far as the characteristics of the first compression wave are concerned, axially symmetrical models can advantageously replace three-dimensional models. Ricco et al. (2007) studied the pressure wave generation experimentally and numerically with the aim of gaining a solid understanding of the flow in the standard tunnel geometry and in the configuration with airshafts along the tunnel surface. Laboratory experiments were conducted in a scaled facility where train models travelled at a maximum velocity of about 150 km/h through a 6-meter-long tunnel. The flow was simulated by a one-dimensional numerical code modified to include the effect of the separation bubble forming near the train head. The numerical simulations showed good agreement with the experimental results. Anthoine (2009) presented a review of the current state of understanding of tunnel entrance aerodynamics for high-speed trains and an experimental assessment of the performance of countermeasures to reduce the slope of the initial pressure rise, which is proportional to the strength of the micropressure wave. It was shown that replacing an abrupt entrance with a progressive one has a beneficial effect on the gradient of the compression wave, as was found by Howe (1999). Miyachi et al. (2016) investigated the compression waves generated by axisymmetric trains running at the offset position in double-track tunnels using a train launcher facility. Paraboloids of revolution, ellipsoids of revolution, and cones were used as the simplest nose shapes. It was shown that the cone-shaped train generated the highest pressure gradient, while the paraboloid one generated the smallest.

Plenty of numerical studies has been developed on this subject. Ogawa and Fujii (1997) studied the three dimensional flow induced by a practical high-speed train moving into a tunnel using the computation of the compressible Navier-Stokes equations with the zonal method. The results reveal a pressure increase inside the tunnel before tunnel entry, the one-dimensionality of the compression wave and the histories of the aerodynamic forces; these histories agree with field measurement data. Mok and Yoo (2001) reported numerical computations of the train-tunnel interactions at a tunnel entrance with real dimensions. They showed the possibility of a partial change in the compression wave front by means of the optimal combination of the degree of the tunnel entrance slopes and holes in the tunnel entrance ceiling. Schlammer and Hieke (2008) studied the effects

of different portal modifications on the generation of the entry pressure wave using 3D CFD simulations. The results were compared to TRAIR model experiments at DeltaRail (UK) and to the results from BUHOOD, a portal hood dimensioning tool. They found that a hood increases the entering time, but additionally the shape of the entry pressure wave can be changed by openings in the hood. By means of a 2D simulation Ku et al. (2010) founded that the optimized cross-sectional area distributions of a train nose (in order to reduce the emitted micro-pressure wave) has an extremely blunt front end and a negative gradient around a middle section. The steep change of the cross-sectional area from the positive to negative gradient causes a strong expansion effect. This phenomenon divides one large compression wave into two small waves. Using a 1D model, Kikuchi et al. (2011) determined the optimal longitudinal distribution of the cross-sectional area of the train nose shape by using the rapid computational scheme and a genetic algorithm. The effect of the nose shape optimization was confirmed through experiments using scale models. Uystepruyst et al. (2011) presented a new methodology for the 3D numerical simulation of the entrance of high-speed trains in a tunnel. The governing equations are the Euler equations. The movement of the train is made thanks to a technique of sliding meshes and a conservative treatment of the faces between two domains. The method was validated on model tests as well as on measurements *in situ*, with good agreements.

For studies that involve the propagation of the waves along the tunnel most of the work has been numerical with experiments or purely experimental. Woods and Pope (1981) presented a generalised one-dimensional flow prediction method for calculating the flow generated by a train in a single-track tunnel. The underlying theory was based on the method of characteristics and was built into a computer program. An experiment, performed in a full-size rail tunnel, for providing validation data was described, showing very good agreement between simulation and real data. Baron et al. (2001) analyzed the aerodynamic phenomena generated by a train travelling at high speed through a long tunnel of small cross-section by means of quasi one-dimensional numerical simulation. Several tunnel configurations at high blockage ratio were discussed, together with the positive and negative effects of pressure relief ducts and of partial air vacuum. They found that the best configuration for a long-range, high-blockage ratio tunnel network seems to consist in two coupled tunnels connected by a number of pressure relief ducts. William-Louis and Tournier (2005) presented a new method for predicting the evolution of the pressure in the tunnel. The train wave signature (TWS) generated with the entrance of the train in the tunnel propagates at the speed of sound. The train nearfield signature (TNS) which is linked to the train, moves with the speed of the train. The principle of the TWS method is then to propagate these profiles and at each point of the tunnel the sum of the effects due to the passage of the profiles is calculated to have the resulting pressure at this location. This was compared with the method of characteristics (MOC) and with experiments, showing that this new procedure is reliable, faster than the MOC method and can handle complex scenarios as the circulation of multiple trains in a tunnel. However, as described, it predicts only the pressure changes but not the air temperature or velocity and is valid for simple tunnels only. Hieke et al. (2011) presented a prediction method and an acoustical assessment procedure currently in use at Deutsche Bahn AG.

They calculated not only the wave amplitude, but the full pressure signal in the time domain using a one-dimensional model for the wave propagation and the Kirchhoff integral formula for the micro-pressure wave emission.

Yanfeng et al. (2011) presented a 3D numerical simulation of air flow around a train body with different nose angles and train velocities inside a very long tunnel. Results showed that the pressure drag increases with the rising of the nose angles and train speed. Choi and Kim (2014) performed an analysis to estimate the aerodynamic drag of the Great Train eXpress (GTX), a subterranean high-speed train in South Korea, using 3D CFD. They found that when the train speed increases by a factor of two, the aerodynamic drag is increased approximately four times. The aerodynamic drag is reduced up to approximately 50% by changing of the nose from a blunt to a streamlined shape and it decreases up to approximately 50% again when the cross-sectional area of the tunnel increases. Using a 3D simulation Cross et al. (2015) investigated the effect of altering the blockage ratio of an underground train upon the ventilating air flows driven by the train. The results of this study showed that ventilating air flows can be increased significantly during periods of constant train motion and acceleration, by factors of 1.4 and 2 respectively, while increasing the train drag at the same rate. They also found that the total train drag was strongly influenced by tunnel length through increasing the pressure drag while the train length had a less significant impact, only increasing the viscous drag.

As well, measurements on real tunnels have been done. Mashimo et al. (1997) measured the attenuation and distortion of the compression wave in three Shinkansen tunnels and their results were compared with the numerical values calculated using a second-order TVD scheme. They found that the strength of a compression wave is exponentially attenuated with distance as it propagates along the tunnel in both slab and ballast track tunnels, and that the attenuation in the ballast track tunnel is considerably larger than in the slab track tunnel; in slab track tunnels the compression wave is steepened as it propagates, while it spreads in the ballast track tunnel. Mancini and Malfatti (2002) carried out tests on an Italian high speed line using two Etr 500 trains and a freight train as part of the TRANSAERO project, to produce an extensive and well-documented database about pressures generated by trains passing at high speed in open air and in tunnels. The survey of test data presented leaves in evidence that full-scale tests are not to be considered only as a device to validate numerical simulation or scale model test-rigs, but as essential to a full reliable comprehension of transient aerodynamics. Fukuda et al. (2006) performed field measurement and numerical simulation on the distortion of the compression wave generated by the train entry and its propagation through a slab track Shinkansen tunnel. This was calculated by a one-dimensional compressible flow analysis, which takes account of steady and unsteady friction, combined with acoustic analysis on the effect of side branches in the tunnel. It was found that there exists a tunnel length at which the maximum value of the pressure gradient of the compression wavefront reaches a peak, and it becomes shorter as the initial compression wavefront generated by the train entering the tunnel is steeper. Martínez et al. (2008) performed from November, 2006 to March, 2008, a series of tests onboard a wide variety of trains in order to check their response to pressure waves generated while passing through tunnels. Part of the ex-

perimental results were presented focusing on the differences caused by some parameters involved, such as train length and shape, or tunnel lengths. Ko et al. (2012) performed a series of measurements near the portal and the shaft of a tunnel of a high-speed line in Taiwan during normal operation. They found that when the pressure wave encounters a sudden expansion in its cross sectional area (such as when the wave travels from inside the tunnel to the portal), the reflected wave is out-of-phase with the incident one (*e.g.* the reflection of a compression is an expansion), while a sudden contraction of the cross section causes an in-phase reflection. The field measurements showed that the entry/exit of the train nose generates a compression wave travelling along the tunnel, and the entry/exit of the train tail generates an expansion wave with a pressure drop.

With a more analytical approach, Miyachi et al. (2013) presented a simple equation governing distortion of the tunnel compression wave propagating through a Shinkansen tunnel with concrete slab-tracks. They proposed a simple scheme for numerical calculations based on the characteristics scheme. A space evolution type equation with one variable was derived from the three conservation equations of the 1D CFD by assuming small disturbances excited by the compression wave. The numerical calculation scheme based on the simple equation reduces the computing time remarkably because its CFL condition is relaxed. The calculation results obtained agreed well with those by a conventional scheme based on the 1D CFD.

For the stage of the emission, as a passive solution, Aoki et al. (1999) determined the optimum dimensions of an expansion chamber at the tunnel exit portal for the purpose of reducing the micro-pressure wave by means of numerical simulations and experiments. It was found that simple expansion chambers close to a tunnel exit portal can cause significant reductions in the amplitudes of micro-pressure waves propagating outside the tunnel. The optimum shape of such chambers depends upon the steepness of the reflecting wavefront. Greatest percentage reductions are achieved for the steepest wavefronts. Typically, a well-designed chamber with a volume of only $\frac{1}{4}\pi D^3$, with D being the tunnel diameter, may cause a reduction of about 30% in the magnitudes of micro-pressure waves. Kashimura et al. (2000) carried out numerical (TVD scheme) and experimental investigations of the pressure emitted by a weak shock wave out of a tube's end with different flange diameters, distances from the tube's exit, and strengths of the shock wave. They found that the average strength of an impulsive wave varies by the Mach number of the shock wave, and the diameter of the flange. In this experimental condition the value of the impulse wave turned out more affected by the diameter of a flange rather than the Mach number. Based on their experimental data, for the case of an infinite flange, they found a simple correlation of the impulse wave and the shock wave as a function of the distance between the tubes exit and an observer aligned with its axis. Maeda et al. (2000) presented a brief analysis of the micro-pressure wave emitted at the exit of a tunnel caused by a compression wave generated when a high-speed train enters the tunnel. They presented the basic characteristics of the compression wave at its generation, together with its propagation and its emission. They also mentioned some countermeasures that can be effective to reduce the magnitude of the micro-pressure wave. For the prediction of the sonic boom caused by the micro-pressure wave Yoon et al. (2001) presented a

new method considering the effect of the nose shape in the resultant noise. The Euler equations were first solved, after which the linear Kirchhoff formulation was used for the prediction of farfield acoustics from the flow-field data. An experimental investigation was also carried out on the pressure fluctuations in the tunnel and the micro-pressure wave with parameters such as train speed, blockage ratio, nose shape of train and air-shafts. The computational prediction and experimentally measured data showed a good agreement with each other. That same year Yoon and Lee (2001) proposed a new method for the prediction of sonic-boom noise. It combined an acoustic monopole analysis and the method of characteristics with the Kirchhoff method. The compression wave from a train entering a tunnel was calculated by an approximate compact Greens function (using the work of Howe (1998a) as a base), and the resultant noise at the tunnel exit was predicted by a linear Kirchhoff formulation. This was compared with the method of Yoon et al. (2001) and measured data. The numerical results exhibited a reasonable agreement with the experimental data. The method can take into account the effect of nose shape of the train and the tunnel geometry and is very efficient in that less computation time is involved; it also provides the means of treating a long tunnel with concrete slab or ballasted tracks. Baron et al. (2006) also used a method capable of predicting the pressure disturbances radiated from tunnel portal based on the classical linear acoustic formulation by Kirchhoff, in which the source data are obtained by the solution of the unsteady quasi-1D equations of gas dynamics. The comparison with full-scale and reduced-scale experiments about the micro-pressure wave radiated, showed good agreement with the numerical method, which supports the efficiency of the Kirchhoff formulation for predicting the micro-pressure waves emitted from the exit portal.

For the subject of the temperature rise at the tunnel wall Barrow and Pope (1987) presented a theoretical analysis of turbulent flow and heat transfer for the situation when a train is in transit through a very long tunnel. They devised a new model with the flows adjacent to the rough and smooth surfaces being simulated by parallel-wall ducted flows, and developed an iterative computational scheme for the prediction of the flow and heat transfer parameters. They obtained a good agreement between the flow results of the new analysis and those from a previous theoretical and experimental investigation. Interesting results were found by Krarti and Kreider (1996), who developed a simplified analytical model to determine the energy performance of an underground air tunnel. They assumed that air tunnel-ground system reaches periodic and quasi-steady state behavior after some days of operation. A parametric analysis was conducted to determine the effect of tunnel hydraulic diameter and air flow rate on the heat transfer between ground and air inside the tunnel. It was shown that an increase in the pipe diameter is thermally preferable to an increase in the air velocity when a given cooling rate is needed. The model was validated against measured data. It was found that the earth temperature for light dry soil at a depth of about 3m varies by approx ± 3 °C from the mean soil temperature, which is approximately equal to the mean annual air temperature, and has a phase lag of about 75 days behind the ambient air temperature. The work done by Abi-Zadeh et al. (2003) discusses key issues relating to the comfort of passengers within the Kings Cross St Pancras Underground station and the use of computer modelling to predict changes to the

station environment resulting from redevelopment work and changes to the operation of the station. They found that the main factors that influence the environmental conditions in the station are the trains moving through an underground system; inefficiencies within the traction power system generates heat. They mention that in a typical underground station, trains directly account for approximately 85% of the total heat load in the stations, that the aerodynamic effects are the result of the train piston effect, and that in a passively ventilated system, the piston effect is the main mechanism that moves air in and out of the station. Their simulation was made using a 1D model (Subway Environmental Simulation (SES) model). They found the simulated data agreeing with the measured data for the station without redevelopment, so that the model could be used to predict the behaviour once the modifications have been made to the station; they also found that the increased heat load of the redeveloped station would be counteracted by increased passive ventilation. However, they mention that the cooling associated with passive ventilation is dependent on ambient temperature. Ampofo et al. (2004) investigated the heat load in a generic underground railway network using a purposely-developed mathematical model written using EES, which is an engineering equation solving language that solves problems iteratively. In their model the tunnel was assumed to be cylindrical with a hollow concrete cylinder enclosing the tunnel. Thermal penetration into the surrounding earth was assumed to be negligible beyond a far field radius. Their analysis showed that the major contributor of heat to the tunnel is from the breaking mechanism, as was also mentioned by Abi-Zadeh et al. (2003), and that for the train carriage is from the passengers. The model also showed that additional cooling to the existing rolling stock may be provided by cooling the tunnels within which they operate. They mention that according to the New York City Transit Authority the operation of underground railway systems can generate enough heat to raise tunnel and station temperatures as much as 8-11 °K above ambient temperature. Li et al. (2016) performed a measurement of heat transfer characteristics in a tunnel model. They found that the surface roughness and air velocity both have influence on the heat transfer of the underground tunnels: with the relative roughness increasing, the temperature drop and cooling efficiency increase gradually, and the temperature drop and cooling efficiency increase sharply with the air velocity decreasing; meanwhile, the effect of air velocity on the temperature drop and cooling efficiency is more significant than that of the relative roughness. In addition, they found that the air temperature decreased rapidly with the increase of length; after a certain length, the air temperature and cooling efficiency almost changed no more, and cooling efficiency reached a stable value of 90%-95%. According to their results of field test there is a sudden drop of the wall temperature at the entrance of an underground tunnel, then the wall temperature changes slightly and close to a constant value quickly. They also found that the outlet air temperature can almost achieve the rock temperature near the outlet when the length of the tunnel is long enough. Sadokierski and Thiffeault (2008) investigated the transfer of heat between the air and surrounding soil in an underground tunnel. They used standard turbulent modelling assumptions to obtain the flow profiles in both open tunnels and in the annulus between a tunnel wall and a moving train, from which the heat transfer coefficient between the air and tunnel wall was computed. They gave a model for the coupled evolution of the air and surrounding soil temperature along a tunnel of finite

length, which can be applied to a simple rail tunnel as described, or to other engineering applications where the effect of periodic temperature variations influence an effectively infinite solid. They coupled the heat equation for the rock, and the newton cooling law for the air temperature, first with an established and known function for the air temperature, and then using the energy equation for the flow with a known velocity. Using parameters for the Piccadilly line tunnels, they found that diurnal temperature fluctuations die out to 0.1 of the peak value at the tunnel wall in approximately 0.1 meters and that yearly fluctuations die out within approximately 1.8 meters of the tunnel wall. Using a numerical 3D model with finite volumes Yanfeng et al. (2008) calculated the heating caused by the aerodynamic drag of the train inside the tunnel in terms of pressure drag and friction resistance as well. They found that the higher blockage ratio leads to a more pronounced piston effect, and that for the typical high speed tunnel-train system, when the blockage ratio is around 0.23 and six trains travel through the tunnel per hour, in the conditions of adiabatic walls with no forced ventilation, the air temperature increases in the tunnel by 17.6 °C. They also state that in case of a lack of ventilation, major thermal energy would be absorbed by the tunnel wall, which could affect the safe use of long tunnels. A year later Yanfeng and Yaping (2009) presented a simulation of a high-speed train traveling through a deep buried long tunnel. The Navier-Stokes equations of a three dimensional, unsteady, compressible, and turbulent flow was solved by a finite volume method. They evaluated various speed conditions, pressure drag between train nose and train tail, friction drag on train body and the quantity of heat produced by the train. They showed that the friction drag is increased with train speed, tunnel wall shear stress is little compared with the train surface shear stress, (because the tunnel wall is relatively smoother than the train surface) and that the increased temperature may not reach normal temperature decaying for an hour. Without a proper ventilation this leads to a temperature increase in the rock, as mentioned before by Yanfeng et al. (2008).

1.3 Publications

During the course of this PhD work some contents herein presented have been published or submitted for publication.

Most of the content from chapter 6 has been published as Rivero, J. M., González-Martínez, E., and Rodríguez-Fernández, M. (2018). 'Description of the flow equations around a high speed train inside a tunnel'. *Journal of Wind Engineering and Industrial Aerodynamics*, 172:212229.

As well, the content from chapter 7 has been submitted for publication at the *Journal of Wind Engineering and Industrial Aerodynamics* under the title 'A Methodology for the Prediction of the Sonic Boom in Tunnels of High-Speed Trains', and with authors Rivero, J. M., González-Martínez, E., and Rodríguez-Fernández, M.

It is intended that a paper based on the contents from chapter 8 will be developed in the future.

Chapter 2

Conservation equations

2.1 Mass conservation

The mass conservation, also known as continuity equation, states that the derivative in time of the integral of the density in a fluid volume (that can change in time) is equal to zero, *i.e.* the mean density in a fluid volume doesn't change in time. Mathematically this means

$$\frac{d}{dt} \int_{V_f(t)} \rho(\vec{x}, t) d\Omega = 0, \quad (2.1)$$

where V_f is a fluid volume. Using the Reynolds transport Theorem

$$\frac{d}{dt} \int_{V_c(t)} \rho(\vec{x}, t) d\Omega + \int_{\Sigma_c(t)} \rho(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = 0, \quad (2.2)$$

where V_c is a control volume and Σ_c the surface enclosing the control volume, and $\vec{n}$ is the normal to the control surface, positive when points outside the control volume. For a fixed control volume and fixed surface, (2.2) reduces to

$$\int_{V_c} \frac{\partial \rho}{\partial t} d\Omega + \int_{\Sigma_c} \rho \vec{v} \cdot \vec{n} d\sigma = 0. \quad (2.3)$$

Using the Gauss Theorem

$$\int_{V_c} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] d\Omega = 0. \quad (2.4)$$

Equation (2.4) is true for all fixed control volumes; that is: the integral does not depend on the recint of integration. Then, the integrand must be zero. This yields to the mass conservation or continuity equation in differential form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \quad (2.5)$$

Two alternative ways of writing it are

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0 \quad (2.6)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0 \quad (2.7)$$

Where $D/Dt = \partial/\partial t + \vec{v} \cdot \nabla$ is the material derivative.

2.2 Momentum conservation

The integral form of the momentum conservation equation states that the change in time of the momentum inside a fluid volume is equal to the resultant of surface and body forces acting over it. Mathematically this is written as

$$\frac{d}{dt} \int_{V_f(t)} \rho \vec{v} d\Omega = \int_{\Sigma_f(t)} \tau \cdot \vec{n} d\sigma + \int_{V_f(t)} \rho \vec{f}_m d\Omega. \quad (2.8)$$

Using the Reynolds Transport Theorem, it can be written for any control volume

$$\frac{d}{dt} \int_{V_c(t)} \rho \vec{v} d\Omega + \int_{\Sigma_c(t)} \rho \vec{v} (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \int_{\Sigma_c(t)} \tau \cdot \vec{n} d\sigma + \int_{V_c(t)} \rho \vec{f}_m d\Omega, \quad (2.9)$$

where $\vec{v}_c \cdot \vec{n}$ is the normal velocity of the surface that limits the control volume. If $V_c(t) = V_0$ and $\Sigma_c(t) = \Sigma_0$ then the momentum equation takes the form

$$\int_{V_0} \frac{\partial(\rho \vec{v})}{\partial t} d\Omega + \int_{\Sigma_0} \rho \vec{v} \vec{v} \cdot \vec{n} d\sigma = \int_{\Sigma_0} \tau \cdot \vec{n} d\sigma + \int_{V_0} \rho \vec{f}_m d\Omega, \quad (2.10)$$

and applying the Gauss theorem as before

$$\int_{V_0} \frac{\partial(\rho \vec{v})}{\partial t} d\Omega + \int_{V_0} \nabla \cdot (\rho \vec{v} \vec{v}) d\Omega = \int_{V_0} \nabla \cdot \tau d\Omega + \int_{V_0} \rho \vec{f}_m d\Omega. \quad (2.11)$$

Grouping in a volume integral

$$\int_{V_0} \left[\frac{\partial(\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) - \nabla \cdot \tau - \rho \vec{f}_m \right] d\Omega = 0, \quad (2.12)$$

results the momentum equation in conservative form

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla \cdot (\rho\vec{v}\vec{v}) = \nabla \cdot \tau + \rho\vec{f}_m. \quad (2.13)$$

Using the continuity, (2.13) can be written as

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = \nabla \cdot \tau + \rho \vec{f}_m, \quad (2.14)$$

and using the material derivative

$$\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \tau + \rho \vec{f}_m \quad (2.15)$$

2.3 Energy conservation

The external forces acting over a fluid volume are the body forces and the superficial ones. The work of the body forces is $\rho \vec{f}_m \cdot \vec{v}$ by unit volume and time. The work by unit time over all the fluid volume is $\int_{V_f} \rho \vec{f}_m \cdot \vec{v} d\Omega$.

The surface forces are of molecular nature and, per unit area, can be expressed as $\vec{n} \cdot \tau$, being τ the stress tensor. The work of the surface forces is $\vec{n} \cdot \tau \cdot \vec{v}$ by area unity and time, where $\vec{n}$ is the normal exterior of the fluid volume. The work of this forces by time unity over the full fluid surface is $\int_{\Sigma_f} \vec{n} \cdot \tau \cdot \vec{v} d\sigma$. Through out an area element $d\sigma$ of any fictitious surface, with a local normal $\vec{n}$, there is a heat flux by conduction $q_n d\sigma$. Where $q_n = \vec{q} \cdot \vec{n}$. The heat flux through a finite surface is $\int_{\Sigma_f} \vec{q} \cdot \vec{n} d\sigma$

2.3.1 Energy conservation principle

The change of total energy in a V_f is equal to the job done by unit time by the exterior forces plus the heat recibed from the exterior in the time unity.

The total energy of a fluid volume is the internal and kinetic energy, that for unit mass read as $e + \frac{1}{2}v^2$. For a fluid volume is $\int_{V_f} \rho(e + \frac{1}{2}v^2) d\Omega$, so the change of total energy with

respect to time will be equal to the work by unit time that surface and body forces do over the fluid surface and fluid volume respectively, plus the heat flux through the fluid surface and the external heat by unit time added on the fluid volume. This is expressed as

$$\frac{d}{dt} \int_{V_f} \rho(e + \frac{1}{2}v^2) d\Omega = \int_{\Sigma_f} \vec{n} \cdot \tau \cdot \vec{v} d\sigma + \int_{V_f} \rho \vec{f}_m \cdot \vec{v} d\Omega - \int_{\Sigma_f} \vec{q} \cdot \vec{n} d\sigma + \int_{V_f} Q d\Omega. \quad (2.16)$$

Using the Reynolds Transport Theorem for every control volume $V_c(t)$

$$\begin{aligned}
& \frac{d}{dt} \int_{V_c} \rho(e + \frac{1}{2}v^2) d\Omega + \int_{\Sigma_c} \rho(e + \frac{1}{2}v^2)(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \\
& \int_{\Sigma_c} \vec{n} \cdot \tau \cdot \vec{v} d\sigma + \int_{V_c} \rho \vec{f}_m \cdot \vec{v} d\Omega - \int_{\Sigma_c} \vec{q} \cdot \vec{n} d\sigma + \int_{V_c} Q d\Omega.
\end{aligned} \tag{2.17}$$

For a fix V_c and Σ_c and using the Gauss Theorem (2.17) can be written as

$$\begin{aligned}
& \int_{V_c} \left\{ \left(\frac{\partial}{\partial t} \left[\rho(e + \frac{1}{2}v^2) \right] + \nabla \cdot \left[\rho(e + \frac{1}{2}v^2) \vec{v} \right] \right. \right. \\
& \left. \left. - \nabla \cdot (\tau \cdot \vec{v}) - \rho \vec{f}_m \cdot \vec{v} + \nabla \cdot \vec{q} - Q \right) \right\} d\Omega = 0,
\end{aligned} \tag{2.18}$$

which yields to the differential form in conservative form

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\rho(e + \frac{1}{2}v^2) \right] + \nabla \cdot \left[\rho(e + \frac{1}{2}v^2) \vec{v} \right] = \\
& \nabla \cdot (\tau \cdot \vec{v}) + \rho \vec{f}_m \cdot \vec{v} - \nabla \cdot \vec{q} + Q
\end{aligned} \tag{2.19}$$

making use of the continuity equation (2.5) and the material derivative, (2.19) yields

$$\rho \frac{D}{Dt} (e + \frac{1}{2}v^2) = \nabla \cdot (\tau \cdot \vec{v}) + \rho \vec{f}_m \cdot \vec{v} - \nabla \cdot \vec{q} + Q \tag{2.20}$$

2.3.2 State equations

In thermodynamic equilibrium the specific entropy S is determined by the local values of internal energy and specific volume

$$S = S(e, v) \text{ or } e = e(S, v). \tag{2.21}$$

The rest of the state equations and thermodynamic variables can be calculated from these; so that

$$T = \left(\frac{\partial e}{\partial S} \right)_v, \quad p = - \left(\frac{\partial e}{\partial v} \right)_S, \quad h = e + pv. \tag{2.22}$$

Differentiating $e = e(S, v)$

$$de = \left(\frac{\partial e}{\partial S} \right)_v ds + \left(\frac{\partial e}{\partial v} \right)_S dv = T ds - p dv, \tag{2.23}$$

which is equal to

$$de = TdS - pd(1/\rho), \quad (2.24)$$

This means that

$$\nabla e = T\nabla S - p\nabla(1/\rho), \quad (2.25)$$

$$\frac{De}{Dt} = T\frac{DS}{Dt} - p\frac{D(1/\rho)}{Dt}, \quad (2.26)$$

$$\frac{\partial e}{\partial t} = T\frac{\partial S}{\partial t} - P\frac{\partial(1/\rho)}{\partial t}. \quad (2.27)$$

For a perfect gas, the known state equation that relates p , ρ and T is

$$\frac{p}{\rho} = R_g T, \quad (2.28)$$

where $R_g = R/M$ with R being the universal gas constant and M the molecular mass of the gas. Here the internal energy e and entalpy h are just temperature functions

$$c_v = \frac{de}{dT}, \quad c_p = \frac{dh}{dT}, \quad (2.29)$$

that fullfils the relation $c_p - c_v = R_g$

If c_v and c_p are constant, the gas is called to be calorically perfect, in which case

$$e = e_0 + c_v T \quad (2.30)$$

and

$$h = e + \frac{p}{\rho} = e_0 + c_p T \quad (2.31)$$

because

$$h = e_0 + c_v T + \frac{p}{\rho} = e_0 + (c_v + c_p - c_v)T = e_0 + c_p T \quad (2.32)$$

and

$$S = S_0 + c_v \ln \left(\frac{p}{\rho^\gamma} \right) \quad (2.33)$$

where $\gamma = c_p/c_v$

$$dS = \frac{1}{T} de + \frac{p}{T} dv \quad (2.34)$$

$$\int dS = \int \frac{1}{T} c_v dT + \int \frac{p}{T} dv \quad (2.35)$$

$$\begin{aligned}
S &= c_v \ln T + R_g \ln v + c_v \ln K = \\
c_v \ln T + c_v \frac{R_g}{c_v} \ln v + K &= c_v \ln(T v^{R_g/c_v}) + c_v \ln K
\end{aligned} \tag{2.36}$$

where K is a constant and $R_g/c_v = (c_p - c_v)/c_v = \gamma - 1$

$$\begin{aligned}
S &= c_v \ln(T v^{\gamma-1}) + c_v \ln K = c_v \ln \left(\frac{T}{\rho^{\gamma-1}} \right) + c_v \ln K = \\
c_v \ln \left(\frac{p}{R_g \rho^\gamma} \right) + c_v \ln K &= c_v \ln \left(\frac{p}{\rho^\gamma} \right) + c_v \ln(K/R_g)
\end{aligned} \tag{2.37}$$

$$S = c_v \ln \left(\frac{p}{\rho^\gamma} \right) + S_0 \tag{2.38}$$

with $S_0 = c_v \ln(K/R_g)$. These relations valid in thermodynamic equilibrium, can be used in a gas in motion because local thermodynamic equilibrium (or quasi-equilibrium) exist. This is justified because the mean free path and the time between collision of two atoms (or molecules) are very small compared with the characteristic length and characteristic of motion respectively.

2.4 Navier-Poisson and Fourier Laws

In thermodynamic the spherical stress tensor is $\tau_{i,j} = -p\delta_{i,j}$ where p is the pressure; and the heat conduction vector q_i is equal to zero. For a fluid in motion, because of the existence of quasi-equilibrium, the stress tensor and the heat flux deviate from the equilibrium value and can be written as

$$\tau_{i,j} = -p\delta_{i,j} + \tau'_{i,j}, \tag{2.39}$$

where $\tau'_{i,j}$ is the viscous stress tensor and $q_i \neq 0$.

The kinetic theory shows that

$$\tau'_{i,j} = A_{i,j,k,l} \frac{\partial v_l}{\partial x_k}, \tag{2.40}$$

and

$$q_i = k_{i,j} \frac{\partial T}{\partial x_j}, \tag{2.41}$$

where $A_{i,j,k,l}$ and $k_{i,j}$ are constants that depend on the thermodynamic local conditions. The cartesian components of the tensor $\nabla \vec{v}$ are

$$(\nabla \vec{v})_{i,j} = \frac{\partial v_j}{\partial x_i} = \gamma_{i,j} + \frac{1}{2} \epsilon_{i,j,k} \omega_k \tag{2.42}$$

where the last term represents the rotation of the fluid as a rigid solid with angular rotation $\frac{1}{2}\vec{\omega}$, being $\vec{\omega} = \nabla \times \vec{v}$. This term might be canceled if a reference system with $\frac{1}{2}\vec{\omega}$ angular velocity is choosen, while $\tau'_{i,j}$ is independent to the changes. So, the viscous stress cannot depend on the antisymetric part of $\nabla \vec{v}$ and $A_{i,j,k,l}\epsilon_{klm}\omega_m = 0$ for any $\vec{\omega}$ and any couple of i and j . This means that $A_{i,j,k,l} = A_{i,j,l,k}$, so that

$$\tau'_{i,j} = A_{i,j,k,l}\gamma_{k,l}. \quad (2.43)$$

If the flow is isotropic, which means that the stress generated in a fluid element by a speed of deformation is independent from the orientation of such element. This also reduces the number of elements of A to

$$\tau'_i = A_{i,j}\gamma_j, \quad (2.44)$$

where τ'_i and γ_j are the principal components of the viscous stress and deformation speed tensors respectively. Because of the isotropy

$$A_{1,1} = A_{2,2} = A_{3,3} = a, \quad (2.45)$$

and

$$A_{1,2} = A_{1,3} = A_{2,1} = A_{2,3} = A_{3,1} = A_{3,2} = b, \quad (2.46)$$

using this

$$\tau'_i = (a - b)\gamma_i + b(\gamma_1 + \gamma_2 + \gamma_3). \quad (2.47)$$

It's usually written $a - b = 2\mu$ and $b = \mu_v - \frac{2}{3}\mu$ where μ and μ_v are the viscosity and the bulk viscosity coefficients respectively, chich depend on the local thermodynamyc state of the fluid. In an arbitrary coordinate system

$$\tau'_{i,j} = 2\mu\gamma_{i,j} + (\mu_v - \frac{2}{3}\mu)(\nabla \cdot \vec{v})\delta_{i,j}, \quad (2.48)$$

which is the Navier-Poisson law. The normal stress due to viscosity is

$$\frac{1}{3} \sum_{i=1}^3 \tau'_{i,i} = \mu_v \nabla \cdot \vec{v}, \quad (2.49)$$

which is zero for incompressible fluids $\nabla \cdot \vec{v} = 0$ and fluids with $\mu_v = 0$. The kinetic theory shows that for monoatomic gases $\mu_v = 0$, which constitutes the Stokes Law.

In an statistically isotropic fluid

$$k_{i,j} = -k\delta_{i,j}, \quad (2.50)$$

where k is the coefficient of thermal conduction of a fluid, and is a function of its local thermodynamic state. This yields the Fourier Law

$$\vec{q} = -k\nabla T. \quad (2.51)$$

The minus sign implies that heat travels from higher to lower temperatures.

The kinetic theory of gases shows that μ , μ_v , and k depend only on the temperature and grow with it. On liquids, on the contrary μ usually decreases when temperature rises. There are not simple relations that give these transport coefficients as a function of pressure and temperature, so it's mostly necessary to measure them. The bulk viscosity μ_v plays an important role in the damping of acoustic waves in fluids. Measuring this damping μ_v can be determined. For monoatomic gases $\mu_v = 0$, although that for other gases μ and μ_v are usually of the same order.

2.4.1 Kinematic coefficients

The coefficient $\nu = \mu/\rho$ is called the coefficient of kinematic viscosity or viscous diffusivity. The coefficient $\alpha = k/\rho c_p$ is called coefficient of thermal diffusivity. The relation $Pr = \nu/\alpha$ is called the Prandtl number. In gases is usually of order unity, it doesn't vary with pressure and it barely does it with temperature. Adding the Fourier-Poisson law to the momentum equation

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \nabla \cdot \tau' + \rho \vec{f}_m. \quad (2.52)$$

If $\frac{D\vec{v}}{Dt}$ is expressed as $\frac{\partial \vec{v}}{\partial t} + \nabla \left(\frac{v^2}{2} \right) - \vec{v} \times (\nabla \times \vec{v})$ and the momentum equation is multiplied with the dot product by the speed $\vec{v}$, then

$$\rho \frac{D}{Dt} \left(\frac{1}{2} v^2 \right) = -\vec{v} \cdot \nabla p + \vec{v} \cdot (\nabla \cdot \tau') + \rho \vec{f}_m \cdot \vec{v} \quad (2.53)$$

which is the equation of kinetic energy. $-\vec{v} \cdot \nabla p$ and $\vec{v} \cdot \nabla \cdot \tau'$ represent the parts of work by volume and time units generated by pressure and viscosity that changes the kinetic energy.

Once that τ and $\vec{q}$ are substituted in the total energy equation, then the equation becomes

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} v^2 \right) = -\nabla \cdot (p\vec{v}) + \nabla \cdot (\tau' \cdot \vec{v}) + \rho \vec{f}_m \cdot \vec{v} + \nabla \cdot (k\nabla T) + Q. \quad (2.54)$$

Taking in account that

$$-\nabla \cdot (p\vec{v}) = -p\nabla \cdot \vec{v} - \vec{v} \cdot \nabla p, \quad (2.55)$$

and

$$\nabla \cdot (\tau' \cdot \vec{v}) = \tau' : \nabla \vec{v} + \vec{v} \cdot (\nabla \cdot \tau'), \quad (2.56)$$

subtracting the kinetic equation from the total energy

$$\rho \frac{De}{Dt} = -p\nabla \cdot \vec{v} + \phi_v + \nabla \cdot (k\nabla T) + Q, \quad (2.57)$$

where $\phi_v = \tau' : \nabla \vec{v} = \nabla \cdot (\tau' \cdot \vec{v}) - \vec{v} \cdot (\nabla \cdot \tau')$ and is known as the Rayleigh function.

Using $\frac{De}{Dt} = T \frac{DS}{Dt} + \left(\frac{P}{\rho^2}\right) \frac{D\rho}{Dt}$ and $\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \vec{v}$ the entropy equation is obtained

$$\rho T \frac{Ds}{Dt} = \phi_v + \nabla \cdot (k \nabla T) + Q. \quad (2.58)$$

In resume the conservation equations in differential form are:

Mass conservation

In conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (2.59)$$

and non conservative form

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho + \rho \nabla \cdot \vec{v} = 0. \quad (2.60)$$

Momentum conservation

In conservative form

$$\frac{\partial(\rho \vec{v})}{\partial t} + \nabla \cdot (\rho \vec{v} \vec{v}) = \nabla \cdot (-pI + \tau') + \rho \vec{f}_m, \quad (2.61)$$

where I is the unity tensor with components $\delta_{i,j}$ and remembering

$$\tau'_{i,j} = 2\mu\gamma_{i,j} + \left(\mu_v - \frac{2}{3}\mu\right)(\nabla \cdot \vec{v})\delta_{i,j},$$

with

$$\gamma_{i,j} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right), \quad (2.62)$$

then the non conservative form is

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \nabla \cdot \tau' + \rho \vec{f}_m. \quad (2.63)$$

Energy conservation

The conservative form is

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho \left(e + \frac{1}{2} v^2 \right) \right] + \nabla \cdot \left[\rho \left(e + \frac{1}{2} v^2 \right) \vec{v} \right] = \\ \nabla \cdot (-p \vec{v} + \tau' \cdot \vec{v} + k \nabla T) + \rho \vec{f}_m \cdot \vec{v} + Q, \end{aligned} \quad (2.64)$$

and in non conservative form

$$\rho \frac{D}{Dt} \left(e + \frac{1}{2} v^2 \right) = -\nabla \cdot (p\vec{v}) + \nabla \cdot (\tau' \cdot \vec{v}) + \nabla \cdot (k\nabla T) + \rho \vec{f}_m \cdot \vec{v} + Q, \quad (2.65)$$

and the entropy equation is

$$\rho T \frac{DS}{Dt} = \nabla \cdot (k\nabla T) + \phi_v + Q, \quad (2.66)$$

where ϕ_v is the Rayleigh function of dissipation, as it was mentioned before.

Chapter 3

Flow in conducts

Given certain initial and boundary conditions for a flow inside a conduct of variable section, it is tried to be determined the distribution of speed and state variables for each instant, taking in account that the conduct geometry is known, retaining the variation of the area in time caused by the over pressures. This is vital when the compressibility in liquids wants to be maintained.

The equations needed are continuity, momentum conservation, and energy conservation. Certain supositions must be taken in account:

1. There are not sudden changes in the section or direction in the conduct.
2. The hidraulic radius of the cross section, $r_h = A/\ell$, (where A is the cross section area, and ℓ is the perimeter), is small compared to the curvature radius of the conduct, and its length L . If $r_h \ll L$, the continuity equation show that the transversal velocity is quite small compared to the longitudinal velocity, so that the movement is basically along the mean line of the conduct. $r_h \ll L$ also allows to show that the transversal pressure variations can be neglected when compared with the longitudinal ones; this means that the pressure is uniform across the section.
3. It is supposed that the movement is turbulent which in the case of tubes is a Reynolds > 2000 , so that the speed and temperature are uniform in the section, except for a very thin layer near the walls. Given that the pressure and temperature are uniform in the section, all the thermodynamic variables will only change with respect to time and along the length of the conduct. Since the velocity is uniform in the cross section the movement equations can be transversaly integrated and maintained a differential form along the conduct.

To justify the use of a one dimensional model, lets write the full continuity and Navier Stokes equations in cylindrical coordinates (this analysis also works with cartesian coordinates), making an order analysis to see what terms can be neglected.

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r} \frac{\partial(r\rho u_\phi)}{\partial \phi} + \frac{\partial(\rho u_x)}{\partial x} = 0, \quad (3.1)$$

the momentum in the x direction is

$$\rho \left(\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_x}{\partial \phi} + u_x \frac{\partial u_x}{\partial x} \right) = - \frac{\partial p}{\partial x} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \phi^2} + \frac{\partial^2 u_x}{\partial x^2} \right] + \rho g_x, \quad (3.2)$$

in the r direction

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_r}{\partial \phi} + u_x \frac{\partial u_r}{\partial x} - \frac{u_\phi^2}{r} \right) = - \frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial x^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} \right] + \rho g_r, \quad (3.3)$$

and in the ϕ direction

$$\rho \left(\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} + u_x \frac{\partial u_\phi}{\partial x} + \frac{u_r u_\phi}{r} \right) = - \frac{1}{r} \frac{\partial p}{\partial \phi} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial x^2} - \frac{u_\phi}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} \right] + \rho g_\phi. \quad (3.4)$$

The convective terms from the continuity equation (3.1) must be of the same order, so that by acknowledging that $r \sim r_h$ and $x \sim L$ with $r_h \ll L$

$$\begin{aligned} \frac{1}{r} \frac{\partial(r \rho u_r)}{\partial r} &\sim \frac{\rho_a u_r}{r_h}, & \frac{1}{r} \frac{\partial(\rho u_\phi)}{\partial \phi} &\sim \frac{\rho_a}{r_h}, & \frac{\partial(\rho u_x)}{\partial x} &\sim \frac{\rho_a u_x}{L}, \\ \frac{\rho_a u_r}{r_h} &\sim \frac{\rho_a}{r_h} \sim \frac{\rho_a u_x}{L}, \end{aligned}$$

leading to

$$u_r \sim u_\phi \sim \frac{r_h}{L} u_x \ll u_x,$$

so that the radial and angular velocities can be neglected when compared with the axial velocity.

The order of magnitude of the unsteady and convective terms in the momentum equation in the axial direction (3.2) are of order

$$\rho \frac{\partial u_x}{\partial t} \sim \frac{\rho_a u_x}{t_c}, \quad \rho u_x \frac{\partial u_x}{\partial x} \sim \frac{\rho_a u_x^2}{L},$$

where t_c is the characteristic time for temporal changes on the flow. The pressure gradient $\partial p / \partial x \sim (\Delta p)_x / L$ must be either of the order of the unsteady term

$$\frac{(\Delta p)_x}{L} \sim \frac{\rho_a u_x}{t_c}, \quad (\Delta p)_x \sim \frac{\rho_a u_x L}{t_c}$$

or of the convective term

$$\frac{(\Delta p)_x}{L} \sim \frac{\rho_a u_x^2}{L}, \quad (\Delta p)_x \sim \rho_a u_x^2.$$

The order of magnitude of the unsteady and convective terms in the momentum equation in the radial direction (3.3) are of order

$$\rho \frac{\partial u_r}{\partial t} \sim \frac{\rho_a u_r}{t_c} \sim \frac{\rho_a r_h u_x}{L t_c}, \quad \rho u_r \frac{\partial u_r}{\partial r} \sim \frac{\rho_a u_r^2}{r_h} \sim \frac{\rho_a r_h u_x^2}{L^2},$$

The pressure gradient $\partial p / \partial r \sim (\Delta p)_r / r_h$ must be either of the order of the unsteady term

$$\frac{(\Delta p)_r}{r_h} \sim \frac{\rho_a r_h u_x}{L t_c}, \quad (\Delta p)_r \sim \frac{\rho_a r_h^2 u_x}{L t_c}$$

or of the convective term

$$\frac{(\Delta p)_r}{r_h} \sim \frac{\rho_a r_h u_x^2}{L^2}, \quad (\Delta p)_r \sim \frac{r_h^2 \rho_a u_x^2}{L^2}.$$

The order of magnitude of the unsteady and convective terms in the momentum equation in the angular direction (3.4) are of order

$$\rho \frac{\partial u_\phi}{\partial t} \sim \frac{\rho_a u_\phi}{t_c} \sim \frac{\rho_a r_h u_x}{L t_c}, \quad \rho \frac{u_\phi}{r} \frac{\partial u_\phi}{\partial \phi} \sim \frac{\rho_a u_\phi^2}{r_h} \sim \frac{\rho_a r_h u_x^2}{L^2},$$

The pressure gradient $(\partial p) / (r \partial \phi) \sim (\Delta p)_\phi / r_h$ must be either of the order of the unsteady term

$$\frac{(\Delta p)_\phi}{r_h} \sim \frac{\rho_a r_h u_x}{L t_c}, \quad (\Delta p)_\phi \sim \frac{\rho_a r_h^2 u_x}{L t_c}$$

or of the convective term

$$\frac{(\Delta p)_\phi}{r_h} \sim \frac{\rho_a r_h u_x^2}{L^2}, \quad (\Delta p)_\phi \sim \frac{r_h^2 \rho_a u_x^2}{L^2}.$$

The pressure difference in the radial direction, $(\Delta p)_r$, is of the same order as the one in the angular direction, $(\Delta p)_\phi$. When the pressure gradients are of the order of the unsteady terms, the ratio between the pressure difference in the radial (or angular) direction over the one in the axial direction is of order

$$\frac{(\Delta p)_r}{(\Delta p)_x} \sim \frac{\rho_a r_h^2 u_x / (L t_c)}{\rho_a u_x L / t_c} \sim \frac{r_h^2}{L^2} \ll 1.$$

When the pressure gradients are of the order of the convective terms, the ratio between the pressure difference in the radial (or angular) direction over the one in the axial direction is of order

$$\frac{(\Delta p)_r}{(\Delta p)_x} \sim \frac{r_h^2 \rho_a u_x^2 / L^2}{\rho_a u_x^2} \sim \frac{r_h^2}{L^2} \ll 1,$$

yielding the same order either if the unsteady terms are more important than the convective ones or vice versa. It is reasonable to consider that the pressure only changes in the axial direction, and that the only component needed from the momentum equation is in the x direction, as it will be shown further on when the integral equations for a small control volume inside a tube are written.

3.1 Governing equations for a tube

The conservation of mass, momentum and energy will be applied to a control volume limited by two cross section of the tube, which are separated by a $\Delta s \ll 1$ distance, and the lateral wall between them (Fig. 3.1)

Figure 3.1: Scheme of the flow inside a tube of variable cross section along the s variable.

3.1.1 Continuity equation

In integral form

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega + \int_{\Sigma} \rho(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = 0,$$

and since $d\Omega = A\Delta s$

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega = \Delta s \frac{\partial(\rho A)}{\partial t}.$$

The surface integral in the wall of the conduct is null given that $\vec{v} = \vec{v}_c$. In the inlet and outlet surfaces $\vec{v}_c = 0$, so that

$$\int_{\Sigma} \rho \vec{v} \cdot \vec{n} d\sigma = -(\rho v A)_s + (\rho v A)_{s+\Delta s},$$

but also

$$-(\rho v A)_s + (\rho v A)_{s+\Delta s} = \Delta s \frac{\partial(\rho v A)}{\partial s},$$

so that

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho v A)}{\partial s} = 0. \quad (3.5)$$

3.1.2 Momentum conservation

The projection of the momentum conservation equations along the mean line of the conduct, with unitary vector $\vec{i}$, gives

$$\begin{aligned} & \vec{i} \cdot \left\{ \frac{d}{dt} \int_{\Omega} \rho \vec{v} d\Omega + \int_{\Sigma} \rho \vec{v}(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma \right\} \\ &= \vec{i} \cdot \left\{ - \int_{\Sigma} p \vec{n} d\sigma + \int_{\Sigma} \tau' \cdot \vec{n} d\sigma + \int_{\Omega} \rho \vec{f}_m d\Omega \right\}, \end{aligned}$$

so each of the terms take the form

$$\vec{i} \cdot \frac{d}{dt} \int_{\Omega} \rho \vec{v} d\Omega = \Delta s \frac{\partial(\rho v A)}{\partial t},$$

$$\vec{i} \cdot \int_{\Sigma} \rho \vec{v} \vec{v} \cdot \vec{n} d\sigma = -(\rho v^2 A)_s + (\rho v^2 A)_{s+\Delta s} = \Delta s \frac{\partial(\rho v^2 A)}{\partial s},$$

$$\begin{aligned}
-\vec{i} \cdot \int_{\Sigma} p \vec{n} d\sigma &= (pA)_s - (pA)_{s+\Delta s} - \vec{i} \cdot \int_{\Sigma_{lateral}} p \vec{n} d\sigma \\
&= -\Delta s \frac{\partial(pA)}{\partial s} - \vec{i} \cdot \int_{\Sigma_{lateral}} p \vec{n} d\sigma.
\end{aligned}$$

If the section was constant, the last integral would be null; it represent the component in the direction of the mean line of the conduct, of the pressure force applied at the lateral wall of the tube. When the cross section varies

$$-\vec{i} \cdot \int_{\Sigma_{lateral}} p \vec{n} d\sigma = p \frac{\partial A}{\partial s} \Delta s,$$

so the definite integral of the pressures is

$$-\vec{i} \cdot \int_{\Sigma} p \vec{n} d\sigma = \Delta s \left\{ -\frac{\partial(pA)}{\partial s} + p \frac{\partial A}{\partial s} \right\} = -A \frac{\partial p}{\partial s} \Delta s.$$

The viscous term is

$$\vec{i} \cdot \int_{\Sigma} \tau' \cdot \vec{n} d\sigma = \vec{i} \cdot \int_{A(s)} \tau' \cdot \vec{n} d\sigma + \vec{i} \cdot \int_{A(s+\Delta s)} \tau' \cdot \vec{n} d\sigma + \vec{i} \cdot \int_{\Sigma_{lateral}} \tau' \cdot \vec{n} d\sigma.$$

The two first integrals of the second member are, at most¹, of the order of the characteristic stress τ_c multiplied by the cross section area A , while the last integral is of the order of τ_c multiplied by the lateral area of tube, of order $L\sqrt{A}$, so that this last term is the dominant. This means that

$$\vec{i} \cdot \int_{\Sigma} \tau' \cdot \vec{n} d\sigma = \vec{i} \cdot \int_{\Sigma_{lateral}} \tau' \cdot \vec{n} d\sigma = -\tau_w \ell \Delta s = -\tau_w \frac{A}{r_h} \Delta s,$$

where τ_w is the stress in the wall, ℓ is the perimeter of the section and $r_h = A/\ell$ is the hydraulic radius.

By last, but not least, the term of the body forces may be written as

¹Looking with more detail, using the axisymmetric cylindrical coordinates for illustrative purposes, in the case of the entrance and exit faces

$$\vec{i} \cdot \vec{n} \cdot \tau' = \vec{i} \cdot (1, 0) \cdot \begin{pmatrix} \tau'_{xx} & \tau'_{xr} \\ \tau'_{rx} & \tau'_{rr} \end{pmatrix} = \vec{i} \cdot (\tau'_{xx}, \tau'_{xr}) = \tau'_{xx} = 2\mu \frac{\partial u_x}{\partial x} \sim \mu \frac{u_c}{L_{Tu}}$$

and in the case of the tunnel wall

$$\vec{i} \cdot \vec{n} \cdot \tau' = \vec{i} \cdot (0, 1) \cdot \begin{pmatrix} \tau'_{xx} & \tau'_{xr} \\ \tau'_{rx} & \tau'_{rr} \end{pmatrix} = \vec{i} \cdot (\tau'_{rx}, \tau'_{rr}) = \tau'_{rx} = \mu \left(\frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} \right) \sim \mu \frac{u_c}{D_{Tu}},$$

so that the latter over the former is of order $D_{Tu}/L_{Tu} \ll 1$; adding also the areas where the stresses are being applied, makes the stress term on entrance and exit surfaces even more negligible when compared with the one in the tunnel wall.

$$\vec{i} \cdot \int_{\Omega} \rho \vec{f}_m d\sigma = \rho f_{ms} A \Delta s,$$

where f_{ms} is the projection of the body forces in s direction. If the body forces comes from a potential U , then

$$\vec{i} \cdot \int_{\Omega} \rho \vec{f}_m d\sigma = -\rho A \frac{\partial U}{\partial s} \Delta s.$$

Grouping all terms of the momentum equation

$$\frac{\partial(\rho v A)}{\partial t} + \frac{\partial(\rho v^2 A)}{\partial s} = -A \frac{\partial p}{\partial s} - \tau_w \frac{A}{r_h} + \rho f_{ms} A. \quad (3.6)$$

Taking in account the continuity equation, the first member of (3.6) may be written as

$$\begin{aligned} \frac{\partial(\rho v A)}{\partial t} + \frac{\partial(\rho v^2 A)}{\partial s} &= v \left\{ \frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho v A)}{\partial s} \right\} + \rho A \left\{ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right\} \\ &= \rho A \left\{ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right\}, \end{aligned}$$

so that the momentum equation reads

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial s} = -\frac{\partial p}{\partial s} - \frac{\tau_w}{r_h} + \rho f_{ms},$$

and if the body forces comes from a potential, then

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial s} = -\frac{\partial p}{\partial s} - \frac{\tau_w}{r_h} - \rho \frac{\partial U}{\partial s}.$$

For the case of turbulent movement, the wall stress can be written as

$$\tau_w = \frac{1}{2} c_f \rho v |v| = \frac{\lambda}{8} \rho v |v|, \quad (3.7)$$

where c_f is the friction coefficient and λ the Darcy friction coefficient, which is a function of the Reynolds number and the relative rugosity of the conduct. If the wall has a velocity v_w then $\tau_w = \frac{\lambda}{8} \rho (v - v_w) |v - v_w|$. The stress has been written in terms of $v|v|$ instead of v^2 , because otherwise, if v changed its sign, the stress wouldn't oppose the movement. Substituting τ_w in the momentum equation, then

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial s} = -\frac{\partial p}{\partial s} - \frac{\lambda}{8 r_h} \rho v |v| - \rho \frac{\partial U}{\partial s}. \quad (3.8)$$

3.1.3 Energy equation

In its integral form, reads

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho(e + \frac{1}{2}v^2) d\Omega + \int_{\Sigma} \rho(e + \frac{1}{2}v^2)(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma \\ &= - \int_{\Sigma} p\vec{v} \cdot \vec{n} d\sigma + \int_{\Sigma} \vec{v} \cdot \tau' \cdot \vec{n} d\sigma - \int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma + \int_{\Omega} \rho \vec{f}_m \cdot \vec{v} d\Omega + \int_{\Omega} Q d\Omega, \end{aligned}$$

and applied to the control volume in the Fig. 3.1 it yields

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho(e + \frac{1}{2}v^2) d\Omega = \Delta s \frac{\partial}{\partial t} \left\{ \rho A(e + \frac{1}{2}v^2) \right\}, \\ & \int_{\Sigma} \rho(e + \frac{1}{2}v^2) \vec{v} \cdot \vec{n} d\sigma = - \left\{ \rho v A(e + \frac{1}{2}v^2) \right\}_s + \left\{ \rho v A(e + \frac{1}{2}v^2) \right\}_{s+\Delta s} \\ & = \Delta s \frac{\partial}{\partial s} \left\{ \rho v A(e + \frac{1}{2}v^2) \right\}. \end{aligned}$$

The sum of these two terms, that constitute the first member of the energy equation, taking in account the continuity equation is

$$\begin{aligned} & \Delta s \frac{\partial}{\partial t} \left\{ \rho A(e + \frac{1}{2}v^2) \right\} + \Delta s \frac{\partial}{\partial s} \left\{ \rho v A(e + \frac{1}{2}v^2) \right\} \\ &= \rho A \left\{ \frac{\partial}{\partial t} (e + \frac{1}{2}v^2) + v \frac{\partial}{\partial s} (e + \frac{1}{2}v^2) \right\} \Delta s. \end{aligned}$$

These last terms are written for the total energy (internal plus kinetic), but they can also be written as a function of the stagnation enthalpy, taking in account the state equation $h = e + P/\rho$, so that

$$\begin{aligned} & \frac{\partial}{\partial t} \left\{ \rho A(e + \frac{1}{2}v^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho v A(e + \frac{1}{2}v^2) \right\} \\ &= \frac{\partial}{\partial t} \left\{ \rho A(h + \frac{1}{2}v^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho v A(h + \frac{1}{2}v^2) \right\} - \frac{\partial(pA)}{\partial t} - \frac{\partial(pvA)}{\partial s}, \end{aligned}$$

and taking in account the continuity equation

$$\begin{aligned} & \Delta s \left[\frac{\partial}{\partial t} \left\{ \rho A(e + \frac{1}{2}v^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho v A(e + \frac{1}{2}v^2) \right\} \right] \\ &= \Delta s \left[\rho A \left\{ \frac{\partial}{\partial t} (h + \frac{1}{2}v^2) + v \frac{\partial}{\partial s} (h + \frac{1}{2}v^2) \right\} - \frac{\partial(pA)}{\partial t} - \frac{\partial(pvA)}{\partial s} \right]. \end{aligned}$$

The work of the pressure forces can be written as

$$\begin{aligned} - \int_{\Sigma} p \vec{v} \cdot \vec{n} d\sigma &= (pvA)_s - (pvA)_{s+\Delta s} - \int_{\Sigma_{lateral}} p \vec{v} \cdot \vec{n} d\sigma \\ &= -\Delta s \left[\frac{\partial}{\partial s}(pvA) - p \frac{\partial A}{\partial t} \right], \end{aligned}$$

where the last term of this last equation is given to the variation of the area in time, which yields a normal velocity.

The work of the viscosity forces is of order $\frac{\lambda}{8}\rho u^3 A$ and the order of the pressure forces is $\rho u^3 A$, so that the ratio between them is $\lambda/8 \ll 1$, which means that the latter can be neglected when compared with the former. The heat recibed by conduction is only important if it's recibed through the conduct wall, as happens with the stresses in the momentum equation, so that

$$- \int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma = - \int_{\Sigma_{lateral}} \vec{q} \cdot \vec{n} d\sigma = q_s \ell \Delta s = q_s \frac{A}{r_h} \Delta s,$$

where q_s is the heat recibed by time and area througout the conduct wall.

The work by time unit of the body forces is

$$\int_{\Omega} \rho \vec{f}_m \cdot \vec{v} d\Omega = \rho v A f_{ms} \Delta s,$$

and the external heat is

$$\int_{\Omega} Q d\Omega = Q A \Delta s,$$

being Q the external heat by volume and time unit. Grouping all of these terms the energy equation reads as

$$\begin{aligned} &\rho A \left\{ \frac{\partial}{\partial t} \left(e + \frac{1}{2} v^2 \right) + v \frac{\partial}{\partial s} \left(e + \frac{1}{2} v^2 \right) \right\} \\ &= - \frac{\partial}{\partial s}(pvA) - p \frac{\partial A}{\partial t} + q_s \frac{A}{r_h} + \rho v A f_{ms} + Q A, \end{aligned}$$

which is the equation of the total energy by mass unit.

In the same way, the energy equation for the stagnation entalpy is (once it has been divided by A)

$$\rho \left\{ \frac{\partial}{\partial t} \left(h + \frac{1}{2} v^2 \right) + v \frac{\partial}{\partial s} \left(h + \frac{1}{2} v^2 \right) \right\} = \frac{\partial p}{\partial t} + \frac{q_s}{r_h} - \rho v \frac{\partial U}{\partial s} + Q, \quad (3.9)$$

where it has also been supposed that the body forces comes from a potential, so that $f_{ms} = -\partial U/\partial s$.

The static entalpy equation can be obtained deducting (3.9) from (3.8), multiplied by v , so that

$$\rho \frac{\partial h}{\partial t} + \rho v \frac{\partial h}{\partial s} = \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial s} + \frac{\lambda}{8r_h} \rho v^2 |v| + \frac{q_s}{r_h} + Q.$$

Knowing that the state equation for the entropy is $TdS = de + pd(1/\rho) = dh - dp/\rho$, so that the entropy equation reads

$$\rho T \left(\frac{\partial S}{\partial t} + v \frac{\partial S}{\partial s} \right) = \frac{\lambda}{8r_h} \rho v^2 |v| + \frac{q_s}{r_h} + Q. \quad (3.10)$$

Equation (3.10) may be written for liquids taking in account that $TdS = cdT$, where c is the specific heat. So it yields

$$\rho c \frac{\partial T}{\partial t} + \rho c v \frac{\partial T}{\partial s} = \frac{\lambda}{8r_h} \rho v^2 |v| + \frac{q_s}{r_h} + Q.$$

The given problem requires the solution of a system of partial differential equations of first order for three dependent variables, the speed and two thermodynamic variables, as a function of the independent variables t and s .

It's necessary to impose initial conditions that are associated to the temporal derivatives.

$$t = 0 : T(s, 0), p(s, 0), v(s, 0).$$

Given that in the continuity equation (3.5) there are derivatives of the cross section area with respect to time, and since these temporal variations of the area are associated with the overpressures in the interior and the elasticity of the material, it's necessary to add an equation that relates the variation of the area with the overpressure, elasticity and thickness of the wall. Although for the case that concerns this thesis, this won't be necessary, since the area is constant along t and s , so that the only derivatives that are maintained in the continuity equation are the ones that involve the product ρv .

The boundary conditions associated to the derivatives with respect to s , might be of the form

$$s = 0 : T(0, t), p(0, t), v(0, t),$$

or something equivalent. For example, the thermodynamic variables can be specified at the entrance of the conduct, and the pressure at the exit, in which case the velocity would be an unknown to be determined. The boundary conditions might be also specified as algebraic combinations of the dependent variables. For example, if the fluid access to the tube from a deposit (with p_d pressure and T_d temperature), and the movement in the inlet region is such that the stagnation magnitudes are maintained, the boundary conditions in $s = 0$ are

$$p(0, t) = p_d(t) \left\{ 1 + \frac{\gamma - 1}{2} M^2(0, t) \right\}^{-\frac{\gamma}{\gamma-1}}; \quad T(0, t) = T_d(t) \left\{ 1 + \frac{\gamma - 1}{2} M^2(0, t) \right\}^{-1},$$

which needs to be closed with another condition. For example

$$p(L, t) = p_a,$$

if the conduct ends open in a subsonic regime to a zone where the pressure is p_a . If the discharge pressure is low enough, then the $M = 1$ is reached in some section, and this condition substitutes the former.

For the case of an incompressible fluid with $A = A(s)$ (3.5) is reduced to $v(s, t) = G(t)/\rho A(s)$, *i.e.*, given a velocity in a certain point of the conduct and the geometry of the conduct, the velocity in any other point (in the same instant) can be determined. This also means that an arbitrary initial condition $v(s, 0)$ can't be imposed, since $v(s, 0) = G(0)/\rho A(s)$. On the other hand, there are not time derivatives of pressure, and it is only required for the initial distribution of temperature $T(s, 0)$ to be given, if it is of interest to determine $T(s, t)$.

3.2 Friction coefficient and heat flux

In the movement equations for fluids in conducts there are two additional unknowns that require relations for each. One of this is the stress in the wall, which can be expressed using the Darcy friction coefficient λ , the density and the speed, as it's shown in (3.7). The friction coefficient λ is a function of the Reynolds number, based in the diameter of the tube D (when the conduct is not a tube, then the equivalent diameter, $4r_h$, is used), $Re = U_0 D/\nu$, and the relative rugosity $\varepsilon = h/D$ of the tube wall. The numeric values of λ are given in Fig. 3.2, which is called the Moody Diagram.

An explicit approximation to this diagram, that works well in the interval $3000 < Re < 10^8$, is

$$\lambda = 1.325 \left\{ \ln \left(\frac{\varepsilon}{3.7} + \frac{5.74}{Re^{0.9}} \right) \right\}^{-2},$$

and is called the Swamee-Jain equation.

For the calculation of q_s an analogy of Reynolds for the Stanton number is given, which says that

$$Sta = \frac{q_s}{\rho v (h_p - h - v^2/2)},$$

and the friction coefficient in the wall

$$C_f = \frac{\tau_w}{\rho v^2/2} = \frac{\lambda}{4},$$

Figure 3.2: Moody diagram. The Darcy friction coefficient λ is in function of the Reynolds number $Re = U_0 D / \nu$, for different values of the relative rugosity $\varepsilon = h/D$.

fulfils the condition

$$Sta = \frac{C_f}{2} = \frac{\lambda}{8},$$

if the termic and viscous diffusions are equal (Prandtl and turbulent Prandtl equal to unity). From this result it is obtained

$$q_s = \frac{\lambda}{8} \rho |v| \left(h_p - h - \frac{1}{2} v^2 \right), \quad (3.11)$$

where h_p is the enthalpy of the fluid at wall temperature. For perfect gases the former equation becomes

$$q_s = \frac{\lambda}{8} \rho |v| \left\{ c_p T_p - \left(c_p T + \frac{1}{2} v^2 \right) \right\}.$$

If the tube was isolated then $q_s = 0$, so that the wall temperature would be $T_p = T + v^2/2c_p$, which is higher than the gas temperature in the tube because of the viscous

dissipation produced by the braking effect in the wall. This temperature is called recovery temperature. If $T_p > T + v^2/2c_p$ then there's a heat flux to the gas in the interior of the conduct, and viceversa if it's $T_p < T + v^2/2c_p$.

For the liquids, the viscous dissipation effect is negligible and it can be assumed that

$$q_s = \frac{\lambda}{8} \rho |v| c (T_p - T).$$

Another approximation is using experimental correlations that relate the Nussel number

$$\text{Nu} = \frac{q_s D}{k(T_p - T)},$$

with the Reynolds, Prandtl and Mach numbers. An explicit correlation among the Nusselt, Reynolds and Prandtl numbers, valide for conducts in wich the ratio of the longitudinal length and the diameter is $L/D > 10$, at low Mach number, with Reynolds number equal or higher to 10^4 and Prandtl number between 0.7 and 160, is the Dittus-Boelter relation

$$\text{Nu} = 0.023 \text{Re}^{4/5} \text{Pr}^n,$$

where $n = 0.4$ for the case of heating, and $n = 0.3$ for the case of cooling.

3.3 Governing equations for a cylinder in a tube

The general movement equations for the gap between a train inside a tunnel can be obtained by simplifying the problem to a moving cylinder inside a tube with no change of area, setting a system of reference fixed to the cylinder (the train), so that the tube wall (which would be the tunnel wall) moves with the train speed, as shown in figure 3.3. As before, the transversal variations have been neglected when compared with the axial variations, since either the train or the tunnel diameter is very small when compared with the tunnel length. The flow is assumed turbulent, so that the variables are uniform along the transversal section, except for a very thin layer in the tunnel and train surfaces.

The volume of the control volume is $(A_{Tu} - A_{Tr})\Delta s$ where A_{Tu} is the tunnel transversal area, A_{Tr} the train transversal area and Δs an infinitesimal length that will tend to zero. The surface of the control volume is composed by the train surface ($\ell_{Tr}\Delta s$, with ℓ_{Tr} being the train perimeter), the tunnel surface ($\ell_{Tu}\Delta s$, with ℓ_{Tu} being the tunnel perimeter) and the transversal areas at s and $s + \Delta s$ ($A_s = A_{s+\Delta s} = A_{Tu} - A_{Tr}$). The velocity at the surfaces, $\vec{v}_c$, is $\vec{v}_c = 0$ for the transversal faces and the train surface, and $\vec{v}_c = U\vec{i}$ for the tunnel surface, being $\vec{i}$ the unitary vector of s , which on this occasion goes from right to left.

3.3.1 Continuity equation

The integral of mass conservation is

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega + \int_{\Sigma} \rho (\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma = 0, \quad (3.12)$$

Figure 3.3: Scheme of the flow between train and tunnel, with system of reference fixed to the train.

where $\vec{w}$ is the velocity in the gap. The unsteady term is

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega = \Delta s (A_{Tu} - A_{Tr}) \frac{\partial \rho}{\partial t},$$

and

$$\int_{\Sigma} \rho (\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma = (A_{Tu} - A_{Tr}) [(\rho w)_{s\Delta s} - (\rho w)_s] = (A_{Tu} - A_{Tr}) \frac{\partial(\rho w)}{\partial s} \Delta s,$$

remembering that $\Delta s \rightarrow 0$. Then equation (3.12) can be written in differential form as

$$\frac{\partial p}{\partial t} + \frac{\partial(\rho w)}{\partial s} = 0. \quad (3.13)$$

3.3.2 Momentum equation

The integral momentum equation projected along the axial direction, $\vec{i}$ (notice that $\vec{i}$ goes from right to left), is

$$\vec{i} \cdot \left\{ \frac{d}{dt} \int_{\Omega} \rho \vec{w} d\Omega + \int_{\Sigma} \vec{w}(\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma \right\} = \vec{i} \cdot \left\{ - \int_{\Sigma} p \vec{n} d\sigma + \int_{\Sigma} \tau' \cdot \vec{n} d\sigma + \int_{\Omega} \rho \vec{f}_m d\Omega \right\}. \quad (3.14)$$

Each term takes the form

$$\vec{i} \cdot \frac{d}{dt} \int_{\Omega} \rho \vec{w} d\Omega = \Delta s (A_{Tu} - A_{Tr}) \frac{\partial(\rho w)}{\partial t},$$

$$\begin{aligned} \vec{i} \cdot \int_{\Sigma} \vec{w}(\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma &= (A_{Tu} - A_{Tr}) [(\rho w^2)_{s+\Delta s} - (\rho w^2)_s] \\ &= (A_{Tu} - A_{Tr}) \Delta s \frac{\partial(\rho w^2)}{\partial s}, \end{aligned}$$

$$- \vec{i} \cdot \int_{\Sigma} p \vec{n} d\sigma = -(A_{Tu} - A_{Tr}) [p_{s+\Delta s} - p_s] = -(A_{Tu} - A_{Tr}) \Delta s \frac{\partial p}{\partial s},$$

$$\begin{aligned} \vec{i} \cdot \int_{\Sigma} \tau' \cdot \vec{n} d\sigma &= \vec{i} \cdot \int_{\Sigma_{Tunnel}} \tau' \cdot \vec{n} d\sigma + \vec{i} \cdot \int_{\Sigma_{Train}} \tau' \cdot \vec{n} d\sigma \\ &= -\tau_{Tu} \ell_{Tu} \Delta s - \tau_{Tr} \ell_{Tr} \Delta s = -\tau_{Tu} \frac{A_{Tu}}{r_{hTu}} \Delta s - \tau_{Tr} \frac{A_{Tr}}{r_{hTr}} \Delta s, \end{aligned}$$

where r_{hTu} and r_{hTr} are the hydraulic radius of the tunnel and train respectively. The last term on the right of (3.14) is

$$\vec{i} \cdot \int_{\Omega} \rho \vec{f}_m d\Omega = \rho (A_{Tu} - A_{Tr}) \frac{dU}{dt} \Delta s.$$

In this way, the momentum equation in differential form is

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho w^2)}{\partial s} = -\frac{\partial p}{\partial s} - \frac{\tau_{Tu}}{r_{hTu}} \left(\frac{1}{1-\beta} \right) - \frac{\tau_{Tr}}{r_{hTr}} \left(\frac{\beta}{1-\beta} \right) + \rho \frac{dU}{dt}$$

which, by the use of the continuity equation (3.13) can be written as

$$\rho \frac{\partial w}{\partial t} + \rho w \frac{\partial w}{\partial s} = -\frac{\partial p}{\partial s} - \frac{\tau_{Tu}}{r_{hTu}} \left(\frac{1}{1-\beta} \right) - \frac{\tau_{Tr}}{r_{hTr}} \left(\frac{\beta}{1-\beta} \right) + \rho \frac{dU}{dt}, \quad (3.15)$$

where $\beta = A_{Tr}/A_{Tu}$, $\tau_{Tu} = \frac{\lambda_{Tu}}{8} \rho (w - U) |w - U|$ and $\tau_{Tr} = \frac{\lambda_{Tr}}{8} \rho w |w|$.

3.3.3 Energy equation

The integral energy equation in the gap is

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho(e + \frac{1}{2}w^2) d\Omega + \int_{\Sigma} \rho(e + \frac{1}{2}w^2)(\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma = \\ - \int_{\Sigma} p\vec{w} \cdot \vec{n} d\sigma + \int_{\Sigma} \vec{w} \cdot \tau' \cdot \vec{n} d\sigma - \int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma + \int_{\Omega} \rho \vec{f}_m \cdot \vec{w} d\sigma + \int_{\Omega} Q_r d\Omega, \end{aligned} \quad (3.16)$$

where Q_r is the external heat released by the train (AC units, heat losses from the electric engines, etc.). Applying the control volume

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho(e + \frac{1}{2}w^2) d\Omega &= (A_{Tu} - A_{Tr}) \Delta s \frac{\partial}{\partial s} \left\{ \rho(e + \frac{1}{2}w^2) \right\}, \\ \int_{\Sigma} \rho(e + \frac{1}{2}w^2)(\vec{w} - \vec{v}_c) \cdot \vec{n} d\sigma &= (A_{Tu} - A_{Tr}) \Delta s \frac{\partial}{\partial s} \left\{ \rho w(e + \frac{1}{2}w^2) \right\}, \end{aligned}$$

and by use of the continuity equation (3.13)

$$\frac{\partial}{\partial t} \left\{ \rho(e + \frac{1}{2}w^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho w(e + \frac{1}{2}w^2) \right\} = \rho \frac{\partial}{\partial t} (e + \frac{1}{2}w^2) + \rho w \frac{\partial}{\partial s} (e + \frac{1}{2}w^2).$$

Knowing that $h = e + p/\rho$ where h is the enthalpy

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \rho(e + \frac{1}{2}w^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho w(e + \frac{1}{2}w^2) \right\} = \\ \frac{\partial}{\partial t} \left\{ \rho(h + \frac{1}{2}w^2) \right\} + \frac{\partial}{\partial s} \left\{ \rho w(h + \frac{1}{2}w^2) \right\} - \frac{\partial p}{\partial t} - \frac{\partial(pw)}{\partial s} = \\ \rho \frac{\partial}{\partial t} (h + \frac{1}{2}w^2) + \rho w \frac{\partial}{\partial s} (h + \frac{1}{2}w^2) - \frac{\partial p}{\partial t} - \frac{\partial(pw)}{\partial s}, \end{aligned}$$

where the continuity equation was used.

The work by the pressure forces is

$$- \int_{\Sigma} p\vec{w} \cdot \vec{n} d\sigma = (A_{Tu} - A_{Tr}) [(pw)_{s+\Delta s} - (pw)_s] = (A_{Tu} - A_{Tr}) \frac{\partial(pw)}{\partial s} \Delta s.$$

The work by the viscous forces is

$$\int_{\Sigma} \vec{w} \cdot \tau' \cdot \vec{n} d\sigma = -U \tau_{Tu} \frac{A_{Tu}}{r_{hTu}} \Delta s,$$

since the tunnel wall moves with $U\vec{i}$ speed (for an observer fixed to the train).

The heat recieved through conduction is only important in the train and tunnel surfaces, so that

$$\begin{aligned}
-\int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma &= -\int_{\Sigma_{lateral}} \vec{q} \cdot \vec{n} d\sigma = q_{Tu} \ell_{Tu} \Delta s + q_{Tr} \ell_{Tr} \Delta s \\
&= q_{Tu} \frac{A_{Tu}}{r_{hTu}} \Delta s + q_{Tr} \frac{A_{Tr}}{r_{hTr}} \Delta s,
\end{aligned}$$

where

$$q_{Tu} = \frac{\lambda_{Tu}}{8} \rho |w - U| \left[c_p (T_{Tu} - T) - \frac{1}{2} (w - U)^2 \right]$$

and

$$q_{Tr} = \frac{\lambda_{Tr}}{8} \rho |w| \left[c_p (T_{Tr} - T) - \frac{1}{2} w^2 \right].$$

where T_{Tu} and T_{Tr} is the temperature of the tunnel and the train walls respectively. The work of the mass forces is

$$\int_{\Omega} \rho \vec{f}_m \cdot \vec{w} d\Omega = \rho w \frac{dU}{dt} (A_{Tu} - A_{Tr}) \Delta s,$$

and the external heat term is

$$\int_{\Omega} Q_r d\Omega = Q_r (A_{Tu} - A_{Tr}) \Delta s.$$

The total energy equation in differential form becomes

$$\begin{aligned}
\rho \frac{\partial}{\partial t} \left(e + \frac{1}{2} w^2 \right) + \rho w \frac{\partial}{\partial s} \left(e + \frac{1}{2} w^2 \right) = \\
- \frac{\partial(pw)}{\partial s} - U \frac{\tau_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + \frac{q_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + \frac{q_{Tr}}{r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) + Q_r + \rho w \frac{dU}{dt}, \quad (3.17)
\end{aligned}$$

and the enthalpy

$$\begin{aligned}
\rho \frac{\partial}{\partial t} \left(h + \frac{1}{2} w^2 \right) + \rho w \frac{\partial}{\partial s} \left(h + \frac{1}{2} w^2 \right) = \\
\frac{\partial p}{\partial t} - U \frac{\tau_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + \frac{q_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + \frac{q_{Tr}}{r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) + Q_r + \rho w \frac{dU}{dt}. \quad (3.18)
\end{aligned}$$

Multiplying the momentum equation (3.15) by w and resting it to (3.18) yields

$$\begin{aligned}
\rho c_p \frac{\partial T}{\partial t} + \rho c_p w \frac{\partial T}{\partial s} - \left(\frac{\partial p}{\partial t} + w \frac{\partial p}{\partial s} \right) = \\
(w - U) \frac{\tau_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + w \frac{\tau_{Tr}}{r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) + \frac{q_{Tu}}{r_{hTu}} \left(\frac{1}{1 - \beta} \right) + \frac{q_{Tr}}{r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) + Q_r \quad (3.19)
\end{aligned}$$

which was obtained by using the relation $h = c_p T$.

3.3.4 Change to a system of reference fixed to the tunnel

Figure 3.4 shows the scheme of the control volume in the gap with a system of reference fixed to the tunnel. The relations used for the change are

$$w = U - u, \quad dx = U dt - ds, \quad dt' = dt,$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + U \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial s} = -\frac{\partial}{\partial x},$$

Figure 3.4: Scheme of the flow between train and tunnel, with system of reference fixed to the tunnel.

so that the continuity equation (3.19) can be written as

$$\frac{\partial \rho}{\partial t'} + U \frac{\partial \rho}{\partial x} - \frac{\partial(\rho[U - u])}{\partial x} = 0,$$

which takes to

$$\frac{\partial \rho}{\partial t'} + \frac{\partial(\rho u)}{\partial x} = 0. \quad (3.20)$$

Applying the same to the momentum equation (3.15) leaves to

$$\rho \frac{\partial(U-u)}{\partial t'} + \rho U \frac{\partial(U-u)}{\partial x} - \rho(U-u) \frac{\partial(U-u)}{\partial x} =$$

$$\frac{\partial p}{\partial x} - \frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho(-u)|-u| - \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho(U-u)|U-u| + \rho \frac{dU}{dt'},$$

which becomes

$$\rho \frac{\partial u}{\partial t'} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} - \frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho(u)|u| - \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho(u-U)|u-U|. \quad (3.21)$$

And in the energy equation (3.19)

$$\rho c_p \left[\frac{\partial T}{\partial t'} + U \frac{\partial T}{\partial x} - (U-u) \frac{\partial T}{\partial x} \right] - \left[\frac{\partial p}{\partial t'} + U \frac{\partial p}{\partial x} - (U-u) \frac{\partial p}{\partial x} \right] =$$

$$(-u) \frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho(-u)|-u| + (U-u) \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho(U-u)|U-u|$$

$$+ \frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho|-u| \left[c_p(T_{Tu} - T) - \frac{1}{2}(-u)^2 \right]$$

$$+ \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho|U-u| \left[c_p(T_{Tr} - T) - \frac{1}{2}(U-u)^2 \right] + Q_r,$$

this yields

$$\rho c_p \left[\frac{\partial T}{\partial t'} + u \frac{\partial T}{\partial x} \right] - \left[\frac{\partial p}{\partial t'} + u \frac{\partial p}{\partial x} \right] =$$

$$\frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho u^2|u| + \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho(u-U)^2|u-U|$$

$$+ \frac{\lambda_{Tu}}{8r_{hTu}} \left(\frac{1}{1-\beta} \right) \rho|u| \left[c_p(T_{Tu} - T) - \frac{1}{2}u^2 \right]$$

$$+ \frac{\lambda_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1-\beta} \right) \rho|u-U| \left[c_p(T_{Tr} - T) - \frac{1}{2}(u-U)^2 \right] + Q_r, \quad (3.22)$$

Chapter 4

Ideal Fluids: Euler equations

4.1 Movements at high Reynolds

Doing a scale analysis to the momentum conservation equation with $u \sim U$, $t \sim t_c$, $x_i \sim \ell$, $\rho \sim \rho_c$ and $f_m \sim f_{m_c}$, and refering all terms to the convective one, the relative orders of magnitude are

$$\underbrace{\rho \frac{\partial \vec{v}}{\partial t}}_{\sim \text{St}} + \underbrace{\rho \vec{v} \cdot \nabla \vec{v}}_{O(1)} = \underbrace{-\nabla p}_{\sim \Delta_s p / \rho_c U^2} + \underbrace{\nabla \cdot \tau'}_{\sim 1/\text{Re}} + \underbrace{\rho \vec{f}_m}_{\sim 1/\text{Fr}} \quad (4.1)$$

where $\text{St} = \ell / Ut_c$ is the Strouhal number, $\text{Re} = \rho_c U \ell / \mu_c$ is the Reynolds number with the characteristic viscosity μ_c , $\text{Fr} = U^2 / \ell f_{m_c}$ is the Froude number, and $\Delta_s p$ is the characteristic spatial increment of pressure. If $\text{Re} \gg 1$ with Strouhal of order unity or small, the viscous term is negligible. The same happens if the Strouhal number is large compared with $1/\text{Re}$, but in this case the convective terms are also negligibles. A similar discussion can be done with the Froude number, but it is not of practical interest. Then in general the viscous terms are negligible in the momentum equation if $\text{Re} \gg 1$ (or $\text{ReSt} \gg 1$), so that

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \rho \vec{f}_m. \quad (4.2)$$

The pressure gradient term is as important as any other, except for trivial cases. When $\text{Re} \gg 1$ and $\text{St} \sim 1$ or $\text{St} \ll 1$, the pressure increment is $\Delta_s p \sim \rho_c U^2$. If $\text{ReSt} \gg 1$ and $\text{Re} \ll \text{St}$, the pressure increment is $\Delta_s p \sim \text{St} \rho_c U^2 \sim \rho_c U \ell / t_c$.

For the entropy equation T_m is the mean temperature and $\Delta_s T$ is the characteristic spatial increment of temperature, so that

$$\underbrace{\frac{DS}{Dt}}_{\sim \frac{\Delta_t S}{cv}} = \underbrace{\frac{\nabla \cdot (k \nabla T)}{\rho T}}_{\sim \frac{\gamma}{\text{RePr}} \frac{\Delta_s T}{T_m} \frac{t_0}{t_r}} + \underbrace{\frac{\phi_v}{\rho T}}_{\sim \frac{1}{\text{Re}} \frac{U^2 t_0}{cv T_m}} + \underbrace{\frac{Q}{\rho T}}_{\sim \frac{Q_c t_0}{\rho_c cv T_m}}. \quad (4.3)$$

where $\Delta_t S$ is the characteristic increment of entropy during time t_0 , which is the smallest time between the local characteristic time, t_c and t_r , which is the residence time ($t_r = \ell/U$). Given that $S = c_v \ln T + S_0$, for liquids, and $S = c_v \ln \left(\frac{p}{\rho^\gamma} \right) + S_0$ for gases. The increment of entropy is, at least, of the order of c_v because the logarithm is only large for very, very large values of its argument. Then $\Delta_t S / c_v$ is, at least, of order unity (but it also can be small). The same happens with the ratio $\Delta_s T / T_m$ that it is at least of order unity. According to this, the dissipation and conductive terms are negligibles if $\text{Re} \gg 1$ for gases, because $\text{Pr} \sim 1$. In the case of liquids the conduction term is negligible if the product $\text{RePr} \gg 1$ (the product $\text{RePr} = \text{Pe}$ is the Peclet number). The term $U^2 / c_v T_m$ is always very small for liquids, and for gases is of the order of the squared Mach number ($M^2 \sim U^2 / c_v T_m$) that in practical cases is not too large. Then, the entropy equation reads

$$\frac{DS}{Dt} = \frac{Q}{\rho T}. \quad (4.4)$$

The continuity and state equations stay the same if $\text{Re} \gg 1$. So the Euler equations become

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{v}, \quad (4.5)$$

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{f}_m, \quad (4.6)$$

$$\frac{DS}{Dt} = \frac{Q}{\rho T}, \quad (4.7)$$

which can also be written in terms of the internal energy

$$\rho \frac{De}{Dt} = -p \nabla \cdot \vec{v} + Q. \quad (4.8)$$

4.2 Initial and boundary conditions

Removing viscous terms and heat conduction from Navier-Stokes brings out two consequences:

1. The equations lose their parabolic nature.
2. It cannot be applied all of the boundary conditions.

So, for this, first, the conditions in the infinite must be known and applied. Second, if the fluid is delimited by an impermeable solid, the relative normal velocity must be zero. And third, when the fluid is limited by another immiscible fluid, there are two conditions:

1. The normal velocity to the surface matches with the advancing normal speed of the surface.

2. The pressure jump through the surface compensates the normal component to the superficial tension so that $p_1 - p_2 = \sigma(1/R_1 + 1/R_2)$, being $(1/R_1 + 1/R_2)$ the mean curvature of the surface.

If $f(\vec{x}, t) = 0$ is the surface limiting the fluid, being a solid or in contact with another fluid, the null normal velocity condition in the surface is written as $\frac{Df}{Dt} = 0$. Temperature conditions are not applied in $f(\vec{x}, t) = 0$.

4.3 Continuity and existence of the solution

In general, there is no continuous solution for the Euler equations, so solutions with continuities in the fluid magnitudes and its derivatives must be found (shock waves and tangential discontinuities). Because of this, Euler equations can't represent uniformly the Navier-Stokes equations. What happens in reality is that there exist regions of very small thickness compared to the characteristic length ℓ , around the discontinuous surface, where the fluid magnitudes have variations of the order of the characteristic values. In these regions viscosity effects and heat conduction can't be ignored.

With the increment of the Reynolds number, the thickness of the region decreases.

4.4 The speed of sound

The speed of sound is a thermodynamic variable determined by the relation

$$a = \sqrt{-\frac{(\partial S / \partial \rho)_p}{(\partial S / \partial p)_\rho}} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \quad (4.9)$$

that for a calorically perfect gas is

$$a = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma R_g T} = \sqrt{(\gamma - 1)h}. \quad (4.10)$$

The speed of sound is the speed of propagation of a small perturbation generated in a static fluid. According to this, a perturbation generated in a certain instant from a point in the fluid after a time t is in a sphere of radius at , with center in the emitting point. If the flow moves with constant velocity u , the perturbation will be in a sphere of radius at , with center in ut . A flow is subsonic if $u < a$ (Fig. 4.1) and supersonic if $u > a$ (Fig. 4.2), in which the zone affected for the perturbations is inside a cone with semiangle $\alpha = \arcsin(1/M)$ known as the Mach Cone. The normal component of the speed normal to the cone surface is a .

4.5 Isentropic and homentropic movements

When the addition of external heat is neglected, the entropy equation reduces to

Figure 4.1: Subsonic regime of a current moving with speed $u < a$ and a perturbation travelling at a with respect to the current.

$$\frac{DS}{Dt} = 0 \quad (4.11)$$

which gives $S = S_p$, where S_p is the particle entropy and maintains as a constant following the fluid particles. This means that a particle that moves will maintain its initial entropy, but initially each particle may have a different entropy, which means the entropy won't be spatially constant in the full domain; then this is an isentropic movement.

In the case where all the particles have the same initial entropy $S_p = S_0$, then $S = S_0$ for all particles and $\nabla S = 0$; this is a homentropic movement. For a movement to be homentropic, it just has to be isentropic and with the same initial entropy for all the particles.

In isentropic and almost stationaries (when $t_c \gg t_r = t_0$) we get

$$\vec{v} \cdot \nabla S = 0; \quad \frac{\partial S}{\partial \ell} = 0, \quad (4.12)$$

Figure 4.2: Supersonic regime of a current moving with speed $u > a$ and a perturbation travelling at a with respect to the current. Notice the Mach Cone.

where $\partial S / \partial \ell$ is the derivative in the direction of the stream line. This means that $S = S_\ell$ is a constant for a given instant over a certain stream line. If all the streamlines come from a region where the entropy is uniform, $S_\ell = S_0$, then the movement is homentropic.

4.6 Stagnation magnitudes

If the movement is at large Reynolds numbers, quasi stationary, without work of the body forces, and without external heat added, then $h + \frac{1}{2}v^2$ and S will be constant along the streamlines. This is deduced from the equations

$$\rho \frac{D(h + \frac{1}{2}v^2)}{Dt} = \frac{\partial p}{\partial t} + \rho \vec{f}_m \cdot \vec{v} + \nabla \cdot (\tau' \cdot \vec{v}) + \nabla \cdot (k \nabla T) + Q, \quad (4.13)$$

and

$$\rho T \frac{DS}{Dt} = \phi_v + \nabla \cdot (k \nabla T) + Q, \quad (4.14)$$

where all the right terms are neglected. These, among the state equations $\rho = \rho(S, h)$,

$p = p(S, h)$, $T = T(S, h)$, and $e = e(S, h)$ allows to calculate all thermodynamic variables in function of the local velocity and the constants S_0 and h_0 . substituting S_0 and h_0 in the state equations ρ_0 , p_0 , T_0 , and e_0 are obtained. These are called the stagnation magnitudes, which, as S_0 and h_0 , may vary from one streamline to the other. For a calorically perfect gas

$$\frac{p}{\rho} = R_g T ; e = c_v T ; a = \sqrt{\gamma R_g T} ; S = c_v \ln \left(\frac{p}{\rho^\gamma} \right) ; h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho},$$

then the homentropic relations are

$$S_0 = S$$

$$\frac{h_0}{h} = \frac{T_0}{T} = \frac{e_0}{e} = \left(\frac{a_0}{a} \right)^2 = \left(\frac{\rho_0}{\rho} \right)^{\gamma-1} = \left(\frac{p_0}{p} \right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma-1}{2} M^2 \quad (4.15)$$

where $M = v/a$ is the local Mach number that relates the local velocity of the flow, and the local speed of sound. It is worth to notice that for liquids Q (the external heat) is irrelevant to calculate the stagnation pressure p_0 ; this is given to the fact that if the density doesn't change, the continuity and momentum conservation equations are not coupled with the energy equation.

4.7 Unidimensional flow

For a case where it is considered that there are not mass forces and no heat addition by radiation or chemical reactions, the continuity, momentum, and energy equations for an onedimensional flow are

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (4.16)$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \quad (4.17)$$

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0. \quad (4.18)$$

Using the state equation $S = S(p, \rho)$ and given that

$$dS = \left(\frac{\partial S}{\partial p} \right)_\rho dp + \left(\frac{\partial S}{\partial \rho} \right)_p d\rho,$$

and together with the continuity equation, the entropy equation becomes

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} - a^2 \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) = 0, \quad (4.19)$$

with a being the speed of sound, $a^2 = -(\partial S / \partial \rho) / (\partial S / \partial p)$.

4.8 Lineal waves

In the case of small perturbations with respect to the initial conditions p_0 and ρ_0 , $a(p_0, \rho_0) = a_0$, and $u = 0$

$$u = u' , a = a_0 + a' , p = p_0 + p' , \rho = \rho_0 + \rho' ,$$

where we consider that

$$u \ll a_0 , a' \ll a_0 , p' \ll p_0 , \rho' \ll \rho_0 ,$$

substituting these in the continuity, momentum, and entropy equations and neglecting square terms of perturbations

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \quad (4.20)$$

$$\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \quad (4.21)$$

$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial \rho'}{\partial t} = 0. \quad (4.22)$$

Deriving equation (4.20) with respect to time, (4.21) respect to x , and (4.22) with respect to time, it can be obtained the wave equations for each one of the variables: $\partial^2 \phi / \partial t^2 - a_0^2 \partial^2 \phi / \partial x^2 = 0$ being $\phi = u', \rho'$, or p' . On the other hand, Eqs. (4.20), (4.21), and (4.22) can be written along the characteristic lines. If (4.20) is multiplied by a_0^2 and substitute $a_0^2 \partial \rho' / \partial t$ by $\partial p' / \partial t$ (using (4.22)) and if (4.21) is added or subtracted and multiplied by a_0 , the next equations are obtained

$$\frac{\partial}{\partial t}(p' + \rho_0 a_0 u') + a_0 \frac{\partial}{\partial x}(p' + \rho_0 a_0 u') = 0,$$

$$\frac{\partial}{\partial t}(p' - \rho_0 a_0 u') - a_0 \frac{\partial}{\partial x}(p' - \rho_0 a_0 u') = 0.$$

Introducing the independent variables $\xi = x - a_0 t$ and $\eta = x + a_0 t$ these yields

$$\frac{\partial}{\partial \eta}(p' + \rho_0 a_0 u') = 0,$$

and

$$\frac{\partial}{\partial \xi}(p' - \rho_0 a_0 u') = 0,$$

this means that

$$p' + \rho_0 a_0 u' = F(\xi),$$

and

$$p' - \rho_0 a_0 u' = G(\eta),$$

which can also be expressed as

$$p' = \frac{1}{2}[F(x - a_0 t) + G(x + a_0 t)],$$

$$u' = \frac{1}{2\rho_0 a_0}[F(x - a_0 t) - G(x + a_0 t)],$$

it is shown that $p' + \rho_0 a_0 u'$ is constant over the lines $x - a_0 t = \xi$, constant, and $p' - \rho_0 a_0 u'$ is constant over the lines $x + a_0 t = \eta$, constant.

If the initial perturbation is such that $G = 0$, $p' = \rho_0 a_0 u' = \frac{1}{2}F(x - a_0 t)$ which represents a wave that travels to the right, without modification, with speed a_0 . If the initial perturbation is such that $F = 0$, $p' = -\rho_0 a_0 u' = \frac{1}{2}G(x + a_0 t)$, then the wave travels to the left with speed a_0 .

The lines $x - a_0 t = \xi$, constant, and $x + a_0 t = \eta$, constant, are the characteristic lines of the system of equations. The functions F and G are determined by the initial conditions and boundary conditions. If they or their derivatives are discontinuous, the fluid magnitudes or their derivatives will present jumps in the characteristics. Also

$$\frac{\partial p'}{\partial t} - a_0^2 \frac{\partial \rho'}{\partial t} = 0,$$

this could be integrated to generate

$$p' - a_0^2 \rho' = H(x),$$

and from $S(p, \rho)$

$$\begin{aligned} dS &= \left(\frac{\partial S}{\partial p} \right)_\rho dp + \left(\frac{\partial S}{\partial \rho} \right)_p d\rho, \\ dS &= \left(\frac{\partial S}{\partial p} \right)_\rho \left(dp + \frac{(\partial S / \partial \rho)_p}{(\partial S / \partial p)_\rho} d\rho \right) = \left(\frac{\partial S}{\partial p} \right)_\rho (dp - a^2 d\rho), \end{aligned}$$

so that

$$S' = \left(\frac{\partial S}{\partial p} \right)_{\rho_0, p_0} (p' - a_0^2 \rho') = \left(\frac{\partial S}{\partial p} \right)_{\rho_0, p_0} H(x),$$

also

$$a_0^2 \rho' = p' - H(x) = \frac{1}{2}[F(x - a_0 t) + G(x + a_0 t)] - H(x).$$

If the initial perturbation of entropy is equal to zero, $H(x) = 0$, then

$$\rho' = p' / a_0^2.$$

Supposing that the fluid is initially moving with a speed u_0 , it's only needed to change the referential system to

$$x_1 = x - u_0 t.$$

The perturbations of velocity and pressure are propagated with respect of the fluid at the speed of sound on the ambient, $\pm a_0$, while the entropy perturbation does not propagate with respect to the fluid. This means that the speed of propagation with respect to the ground are $u_0 \pm a_0$, and u_0 , for the velocity and pressure, and entropy, respectively. Thus the solution becomes

$$p' + \rho_0 a_0 u' = F[x - (u_0 + a_0)t],$$

$$p' - \rho_0 a_0 u' = G[x - (u_0 - a_0)t],$$

$$p' - a_0^2 \rho' = H(x - u_0 t).$$

The system of equations, linearizing around $u = u_0$, $p = p_0$, $\rho = \rho_0$, so that $u = u_0 + u'$, with $u' \ll u_0$, becomes

$$\begin{aligned} \frac{\partial}{\partial t}(p' + \rho_0 a_0 u') + (u_0 + a_0) \frac{\partial}{\partial x}(p' + \rho_0 a_0 u') &= 0, \\ \frac{\partial}{\partial t}(p' - \rho_0 a_0 u') + (u_0 - a_0) \frac{\partial}{\partial x}(p' - \rho_0 a_0 u') &= 0, \\ \frac{\partial}{\partial t}(p' - a_0^2 \rho') + u_0 \frac{\partial}{\partial x}(p' - a_0^2 \rho') &= 0. \end{aligned} \tag{4.23}$$

4.9 Non lineal waves

In the case of lineal waves $p' + \rho_0 a_0 u'$ and $p' - \rho_0 a_0 u'$ don't change along the lines $x - (u_0 + a_0)t = \xi$, constant, and $x - (u_0 - a_0)t = \eta$, constant, respectively, while the entropy doesn't change along the lines $x - u_0 t = \xi$, constant. This way of writing the equations is called characteristic form, and is convenient because each equation in characteristic form is a condition about how the variables change in a direction defined in the $x - t$ plane. A system of equations is called hyperbolic if it can be written in characteristic form. Parting again from Eqs. (4.16)-(4.19), it can be seen that (4.18) is already in a characteristic form, and show that the entropy doesn't vary along the trajectory of every fluid particle (lines in the $x - t$ plane that verifies $dx/dt = u$). To write Eqs. (4.16) and (4.17) in the characteristic form, (4.19) is used to eliminate the density derivative in (4.16), so that

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho a^2 \frac{\partial u}{\partial x} = 0.$$

If (4.17) multiplied by an undetermined λ is added,

$$\frac{\partial p}{\partial t} + (u + \lambda) \frac{\partial p}{\partial x} + \rho \lambda \left[\frac{\partial u}{\partial t} + \left(u + \frac{a^2}{\lambda} \right) \frac{\partial u}{\partial x} \right] = 0.$$

To get the characteristic form

$$u + \lambda = u + \frac{a^2}{\lambda} ; \lambda = \pm a,$$

so that

$$\frac{\partial p}{\partial t} + (u + a) \frac{\partial p}{\partial x} + \rho a \left[\frac{\partial u}{\partial t} + (u + a) \frac{\partial u}{\partial x} \right] = 0,$$

$$\frac{\partial p}{\partial t} + (u - a) \frac{\partial p}{\partial x} - \rho a \left[\frac{\partial u}{\partial t} + (u - a) \frac{\partial u}{\partial x} \right] = 0.$$

The characteristics of the system are the family of the equations of curves $dx/dt = u \pm a$ and $dx/dt = u$. In general, these are no longer straight lines because the approximations $u \simeq u_0$ and $a \simeq a_0$ cannot be applied. So if

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (u + a) \frac{\partial}{\partial x},$$

$$\frac{dp}{dt} + \rho a \frac{du}{dt} = 0 \quad \text{along} \quad C^+ : \frac{dx}{dt} = u + a, \quad (4.24)$$

$$\frac{dp}{dt} - \rho a \frac{du}{dt} = 0 \quad \text{along} \quad C^- : \frac{dx}{dt} = u - a, \quad (4.25)$$

$$\frac{dp}{dt} - a^2 \frac{d\rho}{dt} = 0 \quad \text{along} \quad C^0 : \frac{dx}{dt} = u. \quad (4.26)$$

4.9.1 Riemann variables

If $\partial S/\partial t + u(\partial S/\partial x) = 0$ is integrated along the line $u = dx/dt$, then $dS/dt = 0$ so that $S = S_p$. If every fluid particle had the same initial entropy $S_p = S_0$ without shockwaves, then the movement would be homentropic. In this case the thermodynamic variables can be expressed in the terms of the constant S_0 , and another thermodynamic variable, so $\rho = \rho(S_0, p)$, $a = a(S_0, p)$, etc.

Being dependables of only one variable the system of equations becomes

$$\frac{dR_+}{dt} = 0 \quad \text{along} \quad C^+ : \frac{dx}{dt} = u + a,$$

$$\frac{dR_-}{dt} = 0 \quad \text{along} \quad C^- : \frac{dx}{dt} = u - a,$$

with

$$R_+ = \int \frac{dp}{\rho a} + u,$$

and

$$R_- = \int \frac{dp}{\rho a} - u,$$

which are called the Riemann variables.

For calorically perfect gases, the relations $\rho = \rho(p)$ and $a(p)$ are

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \left(\frac{a}{a_0}\right)^2,$$

with p_0 , ρ_0 , and a_0 as constants, yields the Riemann variables as

$$R_+ = \frac{2}{\gamma-1}a + u$$

and

$$R_- = \frac{2}{\gamma-1}a - u$$

4.9.2 Simple waves

In the particular case that initially one of the Riemann variables is uniform, this will stay uniform for the rest of the time in the zone where characteristic has reached the fluid. These movements are called simple waves. For example if $R_- = A$ with A constant, then

$$\frac{2}{\gamma-1}a - u = A,$$

which relates every thermodynamic variable with the speed (thanks to the homentropic relations). Using

$$R_+ = \frac{2}{\gamma-1}a + u,$$

and

$$R_- = \frac{2}{\gamma-1}a - u,$$

so that

$$u = \frac{R_+ - A}{2}; a = \frac{\gamma-1}{4}(R_+ + A),$$

given that R_+ is constant along lines C^+ , so u and a are also constant along C^+ . This means that $dx/dt = u + a$ are a family of straight lines. For example with initial values $u = F(x)$ is a known function for all x in the time $t = 0$. In order for this to be a simple wave movement, it's just needed (from (4.9.2)) that

$$a = (\gamma - 1)[A + F(x)]/2,$$

for a given value of A in the initial instant. So the family of lines C^+ has the form

$$x = \xi + (u + a)t,$$

which means $x = \xi$ for $t = 0$. Over the corresponding line to each value of ξ , $u = F(\xi)$ and $a = (\gamma - 1)[A + F(\xi)]/2$; so the solution is $u = F[x - (u + a)t]$ with a given by

$$\frac{2}{\gamma - 1}a - u = A.$$

Figure 4.3: Characteristic lines in the $x - t$ plane. Notice the expansion in the left side and the compression in the right side as the wave advances in time.

The Fig. 4.3 is an scheme of the characteristics in the $x - t$ plane, with the solution for two different times. In the region where $F(x)$ grows, the slope of the straight line

$u(\xi) + a(\xi)$ growing with ξ , which will yield an expansion wave; on the contrary, where $F(x)$ decreases with x there will be a compression wave. The moment at which a line crosses another, because a multiform solution would not make sense, a shock wave is generated, the movement have cease to be homentropic. The position of the shock wave must be calculated as part of the solution, using the conservation equations accros the shock together the differential equations (4.24)-(4.26).

4.9.3 Examples

As an application, the case of an unidirectional movement of a gas inside an infinitely long tube is presented. Initially the gas is at pressure p_0 and with a speed of sound a_0 , and the movement is generated because of the displacement of the piston, as shown in the Fig. 4.4

Figure 4.4: Scheme of a piston moving inside an infinitely long tube.

Self-similar expansion:

Supossing that the piston is initially static and begins to move at left ($x < 0$) with constant speed u_p at $t = 0$, this movement will generate an expansion wave that will propagate to the right with respect to the gas, generating a gas movement to the left. A dimensional analisis gives

$$u = f_1(x, t, p_0, a_0, u_p, \gamma),$$

and

$$a = f_2(x, t, p_0, a_0, u_p, \gamma).$$

Aplying the π theorem

$$\frac{u}{a_0} = \phi_1 \left(\frac{x}{a_0 t}, \frac{u_p}{a_0}, \gamma \right),$$

and

$$\frac{a}{a_0} = \phi_2 \left(\frac{x}{a_0 t}, \frac{u_p}{a_0}, \gamma \right),$$

so that the solution depends of the combination of $x/(a_0 t)$ and not of x and t separately. To find a solution it can be seen that the movement is homentropic and the Riemann variable R_- is uniform in the fluid field

$$R_- = \frac{2}{\gamma-1}a - u = \frac{2}{\gamma-1}a_0,$$

given that the zone of interest is the space at the right of the piston, covered by the family of characteristics C^- coming from $x > 0$, where $u = 0$, and $a = a_0$. The characteristic lines C^+ are straight lines.

In a point like A in the Fig. 4.5, C^+ comes from $x > 0$, where $u = 0$ and $a = a_0$. So

$$R_+ = \frac{2}{\gamma-1}a_A + u_A = \frac{2}{\gamma-1}a_0,$$

together with

$$R_- = \frac{2}{\gamma-1}a_A - u_A = \frac{2}{\gamma-1}a_0,$$

which yields $a_A = a_0$ and $u_A = 0$. This happens for all the points under the line $x = a_0 t$, which is the region not affected by the movement of the piston.

Figure 4.5: Characteristic lines in the $x - t$ plane for a piston travelling to the left with a speed u_p inside an infinitely long tube.

In the point B

$$R_+ = \frac{2}{\gamma - 1}a_B + u_B = \frac{2}{\gamma - 1}a_C + u_C,$$

$$R_- = \frac{2}{\gamma - 1}a_B - u_B = \frac{2}{\gamma - 1}a_0,$$

and a_C and u_C are found using

$$R_- = \frac{2}{\gamma - 1}a_C - u_C = \frac{2}{\gamma - 1}a_0, \quad (4.27)$$

with $u_C = -u_p$, which is the speed of the piston. and from (4.27)

$$a_C = a_0 - \frac{\gamma - 1}{2}u_p,$$

so that

$$u_B = -u_p,$$

and

$$a_B = a_0 - \frac{\gamma - 1}{2}u_p = a_C.$$

The line that unites C and B has a slope of $u + a = a_0 - (\gamma + 1)u_p/2$. This is true for every point in the piston trajectory; this means the region over $x = [a_0 - (\gamma + 1)u_p/2]t$, which is the place where the expansion wave has already showed its effect.

Lastly, the point D represents the region that covers the expansion wave, the transition C^+ that gets at D has a slope $u_D + a_D$ and comes from the origin, so that the line C^+ is $x = (u_D + a_D)t$. This together with

$$R_- = \frac{2}{\gamma - 1}a_D - u_D = \frac{2}{\gamma - 1}a_0,$$

yields

$$u_D = \frac{2a_0}{\gamma + 1} \left(\frac{x}{a_0 t} - 1 \right),$$

and

$$a_D = \frac{2a_0}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} \frac{x}{a_0 t} \right).$$

Inside the expansion area u and a varies linealy with x in this region. u goes to zero and a goes to a_0 when D is aproximated to the line $x = a_0 t$, which is the first characteristic of the expansion, while u goes to $-u_p$ and a goes to $a_0 - (\gamma - 1)u_p/2$ when D gets near the characteristic of the expansion $x = [a_0 - (\gamma + 1)u_p/2]t$; the derivatives of u and a with respect to x are discontinuous over these two characteristics. The rest of the thermodynamic variables may be obtained with

$$\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \left(\frac{a}{a_0}\right)^2,$$

so that

$$p_p = p_0 \left(1 - \frac{\gamma-1}{2} \frac{u_p}{a_0}\right)^{\frac{2\gamma}{\gamma-1}}.$$

The first characteristic always go to the right with respect to the ground (because $a_0 > 0$). The last one only goes to the right if $u_p < 2a_0/(\gamma+1)$; if on the contrary $u_p > 2a_0/(\gamma+1)$, the slope of the last characteristic is negative. In this case a_C is less than the absolute value of u_C , so that the flow pass the expansion is supersonic and the characteristics are propagated to the left with respect to the ground ($u+a < 0$), and propagated to the right with respect to the fluid ($a > 0$). The transition from subsonic to supersonic happens inside the expansion, on the characteristic of slope $u+a=0$, that coincides with the t axis. The maximum possible value is obtained by doing $a_C = 0$, which gives $|u_{max}| = 2a_0/(\gamma-1)$. If $u_p > 2a_0/(\gamma-1)$ then a vacuum region appears between $x = -u_p t$ and $x = -(2a_0 t)/(\gamma-1)$.

Expansion followed by compression:

Lets suppose that the speed of the piston, moving to the left, grows from zero to a maximum value, and then decreases again. The equation of the trajectory of the piston is $x = x_p(t) < 0$, and its speed is $u_p = dx_p/dt < 0$. In a given instant $t_1 > 0$, the fluid speed in contact with the piston is $u = u_p(t_1)$ and the speed of sound is $a = a_p = a_0 + (\gamma-1)u_p/2$ which comes from

$$R_- = \frac{2}{\gamma-1}a_p - u_p = \frac{2}{\gamma-1}a_0,$$

so that the characteristic equation that comes out of the piston at instant t_1 is

$$\begin{aligned} x &= x_p(t_1) + [u_p(t_1) + a_p(t_1)](t - t_1) \\ &= \left[a_0 + \frac{\gamma+1}{2}u_p(t_1)\right](t - t_1) + x_p(t_1). \end{aligned} \tag{4.28}$$

The speed of the fluid is always $u_p(t_1)$ on the points of a given line. These lines for different values of t_1 diverge while $du_p/dt_1 < 0$ and converge when $du_p/dt_1 > 0$. To find the envelope curve of the characteristics (4.28) is derivated with respect to t_1 at t fix, and equall to zero, which will yield the values of t where the characteristic lines cross each other (hence $(\partial x/\partial t_1)_t = 0$) which conforms the envelope curve. So

$$\left(\frac{\partial x}{\partial t_1}\right)_t = u_p(t_1) + \frac{\gamma+1}{2} \frac{du_p}{dt_1}(t - t_1) - \left[a_0 + \frac{\gamma+1}{2}u_p(t_1)\right] = 0,$$

which gives

$$t = t_1 + \frac{2}{\gamma + 1} \frac{a_0 + \frac{\gamma-1}{2} u_p(t_1)}{du_p/dt_1}, \quad (4.29)$$

and substituting (4.29) in (4.28) yields to the parametric form of the envelope curve with t_1 as parameter

$$x = x_p(t_1) + \frac{2}{\gamma + 1} \left[a_0 + \frac{\gamma + 1}{2} u_p(t_1) \right] \left[a_0 + \frac{\gamma - 1}{2} u_p(t_1) \right] \frac{1}{du_p/dt}. \quad (4.30)$$

Deriving (4.29) with respect to t_1 and making it equal to zero, gives the t_1 that corresponds for the minimum t , which is the t where the regression point appears.

$$\frac{dt}{dt_1} = 1 - \frac{2}{\gamma + 1} \left(a_0 + \frac{\gamma - 1}{2} u_p(t_1) \right) \left(\frac{du_p}{dt_1} \right)^{-2} \frac{d^2 u_p}{dt_1^2} + \frac{\gamma - 1}{\gamma + 1} \frac{du_p}{dt_1} \left(\frac{du_p}{dt_1} \right)^{-1} = 0,$$

which gives the t_1 value corresponding to the regression point

$$\gamma = \left(a_0 + \frac{\gamma - 1}{2} u_p(t_1) \right) \frac{d^2 u_p / dt_1^2}{(du_p / dt_1)^2}. \quad (4.31)$$

Figure 4.6: Characteristic lines in the $x - t$ plane for a piston travelling to the right with a speed $u_p(t)$ inside an infinitely long tube, which eventually generates a shockwave.

Compression and shock wave:

The figure 4.6 shows the trajectory of the piston together with some characteristic lines of the C^+ family and its envelope curve for the case $u_p = \alpha t$ for $t \geq 0$ (with $u_p = 0$ for

$t < 0$). Since the beginning there is a compression wave. Equations (4.28)-(4.31) can be used for $t_1 > 0$; the point where the shockwave appears corresponds to $t_1 = 0$, where d^2u_p/dt_1^2 does not exist, so (4.31) cannot be applied (but there's no need for it to be used, because it's already known to which t_1 the shockwave corresponds). The coordinates for the point where the shock wave begins are obtained using (4.29) and (4.30) with $t_1 = 0$

$$(x, t) = \left(\frac{2a_0^2/\alpha}{\gamma + 1}, \frac{2a_0/\alpha}{\gamma + 1} \right).$$

It can be seen that the shockwave appears as soon as the acceleration of the piston α is bigger. For the case that the piston speed changes suddenly from $u_p = 0$ for $t < 0$ to $u_p = U > 0$, constant, for $t > 0$, the shockwave appears immediately in $t = 0$ and $x = 0$, and the speed of the shockwave, D , is constant.

Chapter 5

Kirchhoff's integral formula

The following chapter presents the deduction of the Kirchhoff's integral formula, which is recurrently used for the prediction of the micro-pressure wave generated when a compression pressure wave caused by a moving high-speed train inside a tunnel reaches the exit portal.

This formula parts from the concept of sound waves, with the surrounding air outside the tunnel behaving as a potential flow reacting to monopole and dipole pressure distributions, generated by the pressure wave reaching from the inside of the tunnel.

The strength of the micro-pressure wave is directly proportional to the time and spatial derivatives of the compression wave, and inversely proportional to the distance between the observer perceiving the micro-pressure wave and the tunnel's exit.

5.1 Governing equations

An adequate set of equations to describe the sound waves are the linear approximation of the Euler equations, *i.e.* the propagation of the small perturbations of the fluid magnitudes on a compressible flow. The effects of the viscosity forces, heat conduction and viscous dissipation can be neglected when compared with the terms of local variation, that is

$$\rho \left| \frac{\partial \mathbf{u}}{\partial t} \right| \gg \left| \nabla \cdot \bar{\bar{\tau}} \right|, \rho \frac{\partial e}{\partial t} \gg \nabla \cdot (K \nabla T), \rho \frac{\partial e}{\partial t} \gg \bar{\bar{\tau}}^j : \nabla \mathbf{u}. \quad (5.1)$$

The last inequality in (5.1) its checked for being quadratically small in the perturbations. The other two conditions in (5.1) are verified if

$$\frac{\nu}{\omega_0 \lambda_0^2} \sim \frac{\nu \omega_0}{a_0^2} \ll 1, \quad \frac{\nu}{\omega_0 \lambda_0^2} \frac{1}{Pr} \sim \frac{\nu \omega}{a_0^2} \frac{1}{Pr} \ll 1, \quad (5.2)$$

where $Pr = \mu c_p / K$ is the Prandtl number. For gases $Pr = O(1)$ and the two conditions in (5.2) are equivalent; so, for atmospheric air at 20 °C, $\nu \simeq 1.5 \times 10^{-5}$ m/s, $a_0 \simeq 340$ m/s, and taking the most unfavorable case with a typical frequency of 20,000 Hz, it is obtained $\nu \omega_0 / a_0^2 \sim 2.5 \times 10^{-6}$, so it can be seen that is plainly justified to considerate an isentropic propagation of the sound in the air, and for extension, to any gas in normal

conditions.

Under the conditions from (5.2), the governing equations of the fluid field are the Euler equations, which in the absence of body forces and volumetric contributions of heat, are written as

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0, \quad (5.3)$$

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = 0, \quad (5.4)$$

$$\frac{DS}{Dt} = 0. \quad (5.5)$$

It is considered that the fluid is initially uniform and in rest, so that the initial entropy is the same for all the fluid particles, and, according with (5.5) these maintain their value, S_0 , along the movement. The state equation $S(p, \rho) = S_0$ implies that the movement is barotropic, $p = p(\rho)$, and the pressure and density variations satisfy

$$\frac{\partial p}{\partial \rho} = \left. \frac{\partial p}{\partial \rho} \right|_S = \gamma \left. \frac{\partial p}{\partial \rho} \right|_T, \quad (5.6)$$

where $\gamma = c_p/c_v$ is the relation of specific heats; it will be seen shortly that $a = [(\partial p/\partial \rho)_S]^{1/2}$ is the propagation velocity for sound waves. For a perfect gas $a = \sqrt{\gamma R_g T}$. To obtain the governing equations for the sonic field the fluid variables are expressed in (5.3)-(5.5) as

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad \mathbf{u} = \mathbf{u}', \quad (5.7)$$

where p_0 and ρ_0 are the pressure and the density in the undisturbed medium, and p' , ρ' and $\mathbf{u}'$ represent the acoustic perturbations of that state, and are linearized taking into account that $p' \ll p_0$ and $\rho' \ll \rho_0$. For an homogenous undisturbed and resting medium, the following system of linear partial differential equations is obtained

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0, \quad (5.8)$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = 0, \quad (5.9)$$

$$p' = a_0^2 \rho', \quad (5.10)$$

where (5.10) has been obtained by developing in a Taylor series the isentropic state equation $p = p(\rho, S_0)$ until first order terms in the perturbations,

$$p - p_0 = \left. \frac{\partial p}{\partial \rho} \right|_S (\rho - \rho_0). \quad (5.11)$$

The equations (5.8)-(5.10) show the essential role of the compressibility of the fluid in the propagation of the sound perturbations. Consider a fixed volume element in the fluid domain and supposed that in a given instant $\nabla \cdot \mathbf{u}' < 0$: (5.8) shows that fluid is being accumulated in that given element at the same time that, as shown by (5.10), the pressure rises in that same element. In this way a formed pressure gradient, as stated by (5.9), slows down the coming fluid in the element and, later on, pushes it to the adjacent elements, propagating in that way the perturbation in the medium.

If $\mathbf{u}'$ is eliminated from (5.8) and (5.9), and (5.10) is used, then the following equations are obtained

$$\frac{\partial^2 \rho'}{\partial t^2} - a_0^2 \nabla^2 \rho' = 0, \quad (5.12)$$

and

$$\frac{\partial^2 p'}{\partial t^2} - a_0^2 \nabla^2 p' = 0; \quad (5.13)$$

in a similar fashion, eliminating ρ' and p' , it is obtained

$$\frac{\partial^2 \mathbf{u}'}{\partial t^2} - a_0^2 \nabla^2 \mathbf{u}' = 0. \quad (5.14)$$

Equations (5.12)-(5.14) show that the acoustic perturbations in an uniform and steady medium satisfy the *wave equation*. This equation admits solutions that represent waves propagating with a speed a_0 which, as such, is the speed of propagation of the small perturbations in the medium. Instead of calculate ρ' , p' and $\mathbf{u}'$ by the use of (5.12)-(5.14), it is more useful to operate with the velocity potential, which reduces the problem to only one incognita from which the other incognitas may be found. This is valid since the flow is considered isentropic, without body forces and initially irrotational; then it will be irrotational for every instant, and as such, there exists a velocity potential ϕ that accomplishes

$$\mathbf{u}' = \nabla \phi. \quad (5.15)$$

Substituting (5.15) in (5.9) the next equation is obtained

$$\rho_0 \nabla \frac{\partial \phi}{\partial t} + \nabla p' = 0, \quad (5.16)$$

which when integrated provides

$$p' = -\rho_0 \frac{\partial \phi}{\partial t}, \quad (5.17)$$

where the integrating constant has been taken as zero, which can be time dependent, given that in acoustic it is not of interest the situations where there are at the same time pressure (and density) perturbations that vary on time and stationary velocity perturbations ($\partial(\nabla \phi)/\partial t = 0$). By use of (5.10) the next relation is obtained

$$\rho' = -\frac{\rho_0}{a_0^2} \frac{\partial \phi}{\partial t}. \quad (5.18)$$

Finally, substituting (5.15) in (5.14) the wave equation is obtained for ϕ :

$$\frac{\partial^2 \phi}{\partial t^2} - a_0^2 \nabla^2 \phi = 0. \quad (5.19)$$

5.2 Energy equation for the sonic field

If (5.9) is multiplied scalarly by $\mathbf{u}'$, then it is obtained the equation

$$\frac{\partial(\rho_0 \mathbf{u}'^2/2)}{\partial t} = -\mathbf{u}' \cdot \nabla p', \quad (5.20)$$

which shows that the acoustic kinetic energy variation per time unit contained in a volume unit is equal to the mechanic potency applied on it by the resulting forces of the pressure perturbation. On the other hand, the internal energy equation in absence of dissipative and conduction effects, and without heat addition, is written as

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{u}') = -p \nabla \cdot \mathbf{u}' = -(p_0 + p') \nabla \cdot \mathbf{u}'. \quad (5.21)$$

The state equation of internal energy can be expressed as $e = e(\rho, S_0)$, where S_0 is the entropy in the sonic field, and the function ρe , which only depends on ρ' , can be developed in a Taylor series; until second order terms in the perturbations, it is obtained

$$\rho e = \rho_0 e_0 + \rho' \left. \frac{\partial(\rho e)}{\partial \rho} \right|_0 + \frac{1}{2} \rho'^2 \left. \frac{\partial^2(\rho e)}{\partial \rho^2} \right|_0, \quad (5.22)$$

where the derivatives are taken at constant entropy and evaluated in an undisturbed state; those derivatives can be calculated easily from the thermodynamic relation $de = TdS + p/\rho^2 d\rho$; this is, $\rho de = (p\rho)d\rho$. If ρde is added to the two terms of the last relation, then

$$d(\rho e) = \left(e + \frac{p}{\rho} \right) d\rho \quad \text{and} \quad \left. \frac{\partial(\rho e)}{\partial \rho} \right|_S = e + \frac{p}{\rho} = h. \quad (5.23)$$

Derivating once more, and taking into account the relation $dh = dp/\rho = a^2 d\rho/\rho$, which is valid for an isentropic process, it is obtained

$$\left. \frac{\partial^2(\rho e)}{\partial \rho^2} \right|_S = \left. \frac{\partial h}{\partial \rho} \right|_S = \left. \frac{\partial h}{\partial p} \right|_S \left. \frac{\partial p}{\partial \rho} \right|_S = \frac{1}{\rho} a^2. \quad (5.24)$$

Using (5.22) and (5.23) in (5.21) the internal energy per volume unit of fluid until quadratic terms in the perturbations, is obtained

$$\rho e = \rho_0 e_0 + h_0 \rho' + \frac{a_0^2}{2\rho_0} \rho'^2, \quad (5.25)$$

that once substituted in (5.21) gives

$$\frac{\partial}{\partial t} (a_0^2 \rho'^2 / 2\rho_0) = -p' \nabla \cdot \mathbf{u}' \quad (5.26)$$

where the cubic terms of the perturbations have been neglected (specifically $\rho'^2 \mathbf{u}'$) and the relation $h_0[\partial \rho' / \partial t + \nabla \cdot (\rho' + \rho_0) \mathbf{u}'] = 0$ has been taken into account. The term $a_0^2 \rho'^2 / 2\rho_0 = p'^2 / (2\rho_0 a_0^2)$ in (5.26) represents the internal energy stored in a volume unit of flow given by the compression work of the pressure fluctuations. Adding (5.20) and (5.26) the conservation of acoustic energy equation is obtained

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \mathbf{u}'^2 + \frac{1}{2} \frac{p'^2}{\rho_0 a_0^2} \right) + \nabla \cdot (p' \mathbf{u}') = 0, \quad (5.27)$$

which shows that the variation of acoustic energy per time unit, kinetic plus intern, contained in the volume unit is equal to the total potency communicated by the fluctuations of pressure to the volume unit. The vector

$$\mathbf{I} = p' \mathbf{u}' \quad (5.28)$$

is the acoustic energy flux, or *acoustic intensity vector*. Its projection according to the unitary vector $\mathbf{n}$,

$$I_n = p' \mathbf{u}' \cdot \mathbf{n} = p' u'_n, \quad (5.29)$$

is named *acoustic intensity*, and is the quantity of acoustic energy that in a time unit passes through a surface unit of $\mathbf{n}$ orientation.

For typical frequencys of sound waves the acoustic intensity is a function highly variable in time, so that the sound receptors, given their limited response in time to sound signals, do not react instantaneously to the acoustic intensity, but rather to its mean value in a period of time which is long compared with the characteristic time of the sound vibrations, As such, it is convenient to define the mean acoustic intensity as

$$\bar{I}_n(\mathbf{x}, t) = \frac{1}{2T_0} \int_{t-T_0}^{t+T_0} I_n(\mathbf{x}, \hat{t}) d\hat{t}, \quad (5.30)$$

where the averaging time, $2T_0$, has to be long compared with the characteristic time of the sound waves, $2\pi/\omega_0$, but at the same time short compared with the characteristic time of the variation of the integral (5.30); if $\bar{I}_n$ is independent of time in the sonic field it is called to be *stationary in the mean* or, *statistically stationary*. Besides, in the case that the acoustic perturbations vary with time in a harmonic way with an only frequency ω (monocromatic oscillations), the averaging time in (5.30) can be taken equal to the oscillation period $2\pi/\omega$. The characteristic values of the mean acoustic intensity vary among a very wide rank; for example, for the human ear, and for a typical frequency of 5000 Hz the sound threshold corresponds to $\bar{I}_n \sim 10^{-12}$ W/m² while for the pain threshold corresponds to $\bar{I}_n \sim 10^3$ W/m². Since the values of acoustic intensity are between a rank of fifteen magnitud orders it is convenient to use a logarithmic scale and express the value of $\bar{I}_n$ in decibels so that

$$I_{DB} = 10 \log_{10} \frac{\bar{I}_n (\text{W/m}^2)}{10^{-12} \text{W/m}^2}. \quad (5.31)$$

As an example, at a distance of 1 m a whisper corresponds to a typical value of acoustic intensity $I_{DB} \sim 20$, a scream $I_{DB} \sim 70$, and the pain threshold corresponds to $I_{DB} \sim 150$.

5.3 Plane waves

The simplest case of a sonic field is one in which its distribution depends of only one coordinate, for example the coordinate in x . In this case, the acoustic perturbations will be uniform in each plane $x = \text{const.}$: $p' = p'(x, t)$, $\mathbf{u}' = u'(x, t)\mathbf{e}_x$ and $\phi = \phi(x, t)$, so that (5.19) is reduced to

$$\frac{\partial^2 \phi}{\partial t^2} - a_0^2 \frac{\partial^2 \phi}{\partial x^2} = 0. \quad (5.32)$$

To find the solution of (5.32) it is convenient to introduce as new variables $\xi = x - a_0 t$ and $\eta = x + a_0 t$ with which the equation is reduced to $\partial^2 \phi / \partial \xi \partial \eta = 0$. Integrating with respect to ξ yields $\partial \phi / \partial \eta = G'(\eta)$ being G' an arbitrary function of η , and if this is integrated with respect to η then this yields $\phi = F(\xi) + G(\eta)$ where F and $G = \int^\eta G' d\eta$ are arbitrary functions of their arguments. As such, the general solution of (5.32) is

$$\phi = F(x - a_0 t) + G(x + a_0 t), \quad (5.33)$$

and the pressure and density perturbations are determined from (5.15)-(5.16) as

$$\mathbf{u}' = u'\mathbf{e}_x = \frac{\partial \phi}{\partial x} \mathbf{e}_x = [F_1(x - a_0 t) + G_1(x + a_0 t)]\mathbf{e}_x \quad (5.34)$$

$$\frac{p'}{\rho_0 a_0} = -\frac{1}{a_0} \frac{\partial \phi}{\partial t} = F_1(x - a_0 t) - G_1(x + a_0 t) = a_0 \frac{\rho'}{\rho_0} \quad (5.35)$$

where $F_1 \equiv \partial F / \partial x = -\partial F / \partial(a_0 t)$ and $G_1 \equiv \partial G / \partial x = \partial G / \partial(a_0 t)$. The physical meaning of this solutions is simple; for example that $G \equiv 0 \equiv G_1$, so that $u' = F(x - a_0 t) = p' / \rho_0 a_0 = a_0 \rho' / \rho_0$. The perturbations vary in time and space in such a way that they belong constant for an observer that moves following any trajectory $x - a_0 t = \text{const.}$ As such, if in some instant t_0 the perturbations posses a certain value in the plane $x = x_0$, those values will be found in an instant $t > t_0$ at the position $x = x_0 + a_0(t - t_0)$. In this sense, the fluid magnitudes are propagated through the medium in the x direction with the speed of sound a_0 . The functions $F(x - a_0 t)$ and $F_1(x - a_0 t)$ represent a travelling plane wave that propagates in the positive direction of the x axis. As such, the functions $G(x + a_0 t)$ and $G_1(x + a_0 t)$ represent a wave that propagates in the opposite direction. According with (5.34)-(5.35), for wave that propagates to the right ($G = G_1 = 0$) it is given

$$\frac{u'}{a_0} = \frac{\rho'}{\rho_0} = \frac{p'}{\rho_0 a_0^2}, \quad (5.36)$$

and for a wave that propagates to the left ($F = F_1 = 0$) it is verified

$$\frac{u'}{a_0} = -\frac{\rho'}{\rho_0} = -\frac{p'}{\rho_0 a_0^2}. \quad (5.37)$$

The functions F_1 and G_1 are determined from the initial and boundary conditions; for example, if a fluid that is initially uniform and steady ($u' = 0$) is disturbed using a small local increment in the pressure given by $p' = \rho_0 a_0 f(x)$ being $f(x)$ a known function, F_1 and G_1 are determined by imposing the equations (5.34) and (5.35) in $t = 0$,

$$0 = F_1(x) + G_1(x), \quad f(x) = F_1(x) - G_1(x) \quad (5.38)$$

which yields

$$F_1(x) = -G_1(x) = f(x)/2. \quad (5.39)$$

As such, the problem's solution is

$$\frac{p'}{\rho_0 a_0} = \frac{1}{2}f(x - a_0 t) + \frac{1}{2}f(x + a_0 t) = \frac{\rho' a_0}{\rho_0}, \quad (5.40)$$

$$u' = \frac{1}{2}f(x - a_0 t) - \frac{1}{2}f(x + a_0 t). \quad (5.41)$$

Equation (5.40) indicates that the perturbation of the initial pressure is divided in two parts, each one with a magnitude equal to the half of the initial perturbation, that travel with a speed a_0 in opposite directions.

Another example of unidimensional sound propagation is the movement of a gas inside a very long (infinite) cylinder generated by a piston that is moving inside of it with a speed very small compared with the one of sound, so that the perturbations that are introduced in the gas are quite small; if u_{p0} is the characteristic speed of the piston then the pressure variations are of the order $p' = p - p_0 \sim \rho_0 u_{p0}^2 \sim p_0 u_{p0}^2 / a_0^2 \ll p_0$. If $x_p(t)$ defines the position of the piston with respect to the coordinate's origin then the speed of the gas in contact with the piston must be $u = \dot{x}_p(t)$. It is supposed that there are no perturbations propagating from the infinite to the piston, so that in the zone $x > x_p(t)$ there is only one wave propagating to the right (the one generated by the piston),

$$u' = F_1(x - a_0 t) = \frac{p'}{\rho_0 a_0}, \quad (5.42)$$

while for $x < x_p(t)$ there is a travelling wave to the left,

$$u' = G_1(x + a_0 t) = -\frac{p'}{\rho_0 a_0}. \quad (5.43)$$

The boundary condition $u' = \dot{x}_p(t)$ in $x = x_p(t)$ determines F_1 and G_1 ; as such, it is given as

$$F_1[x_p(t) - a_0 t] = \dot{x}_p(t), \quad (5.44)$$

and if it is defined $s \equiv -a_0 t$, and it is supposed that $|\dot{x}_p| \ll a_0$, $[x_p(t) \ll a_0 t]$, then

$$F_1(s) \simeq \dot{x}_p(-s/a_0). \quad (5.45)$$

This yields the solution for the sonic field

$$u' = p'/\rho_0 a_0 = \dot{x}_p(t - x/a_0), \quad (5.46)$$

which constitutes a wave that propagates to $x > 0$. Analogously for $x < 0$ it is obtained a wave that propagates to the left

$$u' = \dot{x}_p(t + x/a_0) = -p'/\rho_0 a_0 = -\dot{x}_p(t + x/a_0). \quad (5.47)$$

5.4 Sound emission

5.4.1 Spherical waves. Acoustic monopole

In a problem with spherical symmetry the wave equation (5.19) is written as

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{a_0^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 0, \quad (5.48)$$

which can also be expressed as

$$\frac{\partial^2(r\phi)}{\partial t^2} - a_0^2 \frac{\partial^2(r\phi)}{\partial r^2} = 0. \quad (5.49)$$

Comparing (5.49) and (5.32) it is deduced that the general solution of (5.49) is

$$\phi = \frac{F(t - r/a_0)}{r} + \frac{G(t + r/a_0)}{r}, \quad (5.50)$$

where F and G are arbitrary functions. In what follows, it will be considered that there are only waves that travel in the crecent direction of r , so that $G \equiv 0$, since it is supposed that there are no perturbations in the infinite that get propagated to the origin. Then, the velocity, pressure and density perturbations are

$$u'_r = \frac{\partial \phi}{\partial r} = -\frac{1}{a_0} \frac{F'(t - r/a_0)}{r} - \frac{F(t - r/a_0)}{r^2}, \quad (5.51)$$

$$\frac{p'}{\rho_0 a_0} = a_0 \frac{\rho'}{\rho_0} = -\frac{1}{a_0} \frac{\partial \phi}{\partial t} = -\frac{1}{a_0} \frac{F'(t - r/a_0)}{r}. \quad (5.52)$$

Notice that, in contrast with a plane wave, $u'_r \neq p'/\rho_0 a_0$ except for a sufficiently large distance from the origin so that the second term of the second member of (5.51) can be neglected in comparison with the first, so that the spherical wave behaves locally as a plane one. Notice also that for great values of r the intensity of the spherical waves, $p'u'_r$, falls with the square of the distance to the origin because of the attenuation of the spherical geometry; given that the total energy flux has to be distributed over a surface

which area grows with the distance as r^2 , the acoustic intensity has to diminish as r^{-2} in the distance to the origin. As an example, consider a sphere with a radius R and center in the origin that makes pulsations that generates a known flow $Q(t)$. If it is supposed a solution of the form $F[t - (r - R/a_0)]/r$ and it is considered that in $r=R$ the speed is $Q(t)/4\pi R^2$, equation (5.51) yields

$$-a_0 \frac{Q(t)}{4\pi R} = F'(t) + a_0 \frac{F(t)}{R}. \quad (5.53)$$

Solving this linear differential equation of first order and replacing t for $t - (r - R)/a_0$ in the solution for F gives

$$\phi(r, t) = -\frac{a_0}{4\pi Rr} e^{-a_0[t-(r-R)/a_0]/R} \int_{-\infty}^{t-(r-R)/a_0} e^{a_0\hat{t}/R} Q(\hat{t}) d\hat{t}. \quad (5.54)$$

Notice that if the pulsations of the sphere stop in a given instant t_f [$Q(\hat{t}) = 0$ for $\hat{t} > t_f$] the potential at a distance r from the center will diminish exponentially with time according to the law $\phi = \text{const} \cdot e^{-a_0 t/R}$ for $t > t_f + (r - R)/a_0$.

A punctual acoustic source that generates a flow $Q(t)$ its called an *acoustic monopole* of intensity $Q(t)$. The potential created by an acoustic monopole that is placed in the origin can be obtained from (5.54) taking the limit when $R \rightarrow 0$. Using integration by parts in (5.54) it can be checked that, excluding terms of order R^2 , the slowly variable factor $Q(\hat{t})$ can be taken out of the integral and replaced by $Q(t - r/a_0)$ ($Q(\hat{t})$ don't vary in a time scale of the order of R/a_0), and if the integration is made with respect to the exponential factor then

$$\phi(r, t) = -\frac{1}{4\pi r} Q(t - r/a_0). \quad (5.55)$$

If (5.55) is used in (5.51)-(5.52) then the pressure and velocity field for the acoustic monopole is obtained

$$u'_r = \frac{Q(t - r/a_0)}{4\pi r^2} + \frac{\dot{Q}(t - r/a_0)}{4\pi a_0 r}, \quad (5.56)$$

$$\frac{p'}{\rho_0 a_0} = \frac{\dot{Q}(t - r/a_0)}{4\pi a_0 r}. \quad (5.57)$$

Notice that expression (5.55) for ϕ is formally analogue to the velocity potential created by a punctual and volumetric source in an incompressible fluid, $-Q(t)/4\pi r$, except that t is replaced in this case by $t - r/a_0$. This difference is caused by the compressibility that introduces a delayed time r/a_0 between the value of the signal received in r at an instant t , $Q(t - r/a_0)$, and the flow, $Q(t)$, emitted by a source in that given instant; the interval of the delayed time is the time that a sound wave takes in propagating through a distance r from the origin. Given that the delay depends of r there are two terms in $u'_r = \partial\phi/\partial r$, and as such, the velocity field is more complicated than the one of an incompressible fluid $a_0 \rightarrow \infty$. Nonetheless, if ω_0 is a characteristic frequency of the signal emitted by the

source, at distances r of the source that are very small compared with the characteristic length of the emitted waves, $\omega_0 r/a_0 \ll 1$, the delayed time r/a_0 can be neglected as a first approximation, as also the second term of u'_r , that is of order $\omega_0 r/a_0$ compared with the first one. As such, in this limit the following expressions are obtained

$$\phi = -\frac{Q(t)}{4\pi r}, \quad u'_r \simeq \frac{\dot{Q}(t)}{4\pi r^2}, \quad \omega r/a_0 \ll 1, \quad (5.58)$$

in which this limit is called the *near field* approximation for the sonic field of a monopole. It can be seen in (5.58) how in the near field the fluid behaves in an incompressible wave, and how its velocity potential satisfies, as a first approximation, the Laplace equation $\nabla^2 \phi = 0$. In the opposite limit, $\omega_0 r/a_0 \gg 1$, the second term on the right side of (5.56) is dominating compared with the first one, and as such the *far field* approximation is obtained, in which the velocity field (5.56) is simplified to

$$u'_r \simeq \frac{\dot{Q}(t - r/a_0)}{4\pi a_0 r}, \quad \omega r/a_0 \gg 1, \quad (5.59)$$

while the expressions (5.55) and (5.57) for the potential and the pressure field remain unchanged.

5.4.2 Continuous distribution of monopoles

Now let it be considered a sonic radiation emitted by a continuous spatial distribution of monopoles. That distribution can be volumetric or superficial and is characterized in each point by an intensity per volume unit or surface unit $q(\mathbf{x}, t)$ or $q_s(\mathbf{x}, t)$ respectively. It is worth to indicate that, although in many cases the distribution is known *a priori* there are other cases in which this does not happen, and as such, its determination requires the resolution of the wave equation with the appropriate boundary conditions; for example for the sonic field of a closed room the walls can be considered as sources of sound when they reflect (and transmit) the incident acoustic waves; the same happens with objects immersed in a fluid that disperse the incidental acoustic waves.

Considering a continuous distribution of monopoles in a region Ω . If $q(\mathbf{x}, t)$ denotes the intensity per volume unit of the sources, the resulting acoustic potential is obtained by the use of the elemental solution (5.58) for each of the punctual sources of the distribution and applying the superposition principle, which can be done under the linearity of the problem, yielding

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi} \int_{\Omega} \frac{q(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\Omega, \quad (5.60)$$

where $\mathbf{x}'$ denotes a generic point that shows the position of the monopoles in the interior of Ω , and $\tau = t - |\mathbf{x} - \mathbf{x}'|/a_0$ is the instant in which the signal must be emitted from $\mathbf{x}'$ for it to get to $\mathbf{x}$ in the instant t . If the laplacian is taken from (5.60) taking in account that

$$\nabla \tau = -\nabla |\mathbf{x} - \mathbf{x}'|/a_0 = -\mathbf{e}(\mathbf{x}, \mathbf{x}')/a_0 \quad (5.61)$$

where $\mathbf{e}(\mathbf{x}, \mathbf{x}')$ denotes the unitary vector directing from $\mathbf{x}'$ to $\mathbf{x}$, and that

$$\nabla^2 |\mathbf{x} - \mathbf{x}'|^{-1} = -4\pi \delta(\mathbf{x} - \mathbf{x}'), \quad (5.62)$$

where δ is the Dirac delta function, it can be seen that ϕ satisfies the differential equation

$$\nabla^2 \phi - \frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2} = q(\mathbf{x}, t). \quad (5.63)$$

The pressure field is determined from (5.60) with the form

$$p'(\mathbf{x}, t) = -\rho_0 \frac{\partial \phi}{\partial t} = \frac{\rho_0}{4\pi} \int_{\Omega} \frac{q_t(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\Omega, \quad (5.64)$$

where q_t is the partial derivative with respect to time t of q ; notice that from the expression of τ it is deduced that $q_t = q_\tau$. Analogously, the velocity field is given by

$$\mathbf{u}'(\mathbf{x}', t) = \nabla \phi = \frac{1}{4\pi} \int_{\Omega} \left[\frac{q(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{a_0} q_t(\mathbf{x}', \tau) \right] \frac{\mathbf{e}(\mathbf{x}', \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} d\Omega, \quad (5.65)$$

where the second term in the brackets of the integrand in (5.65) has been obtained by derivating according with the chain rule taking into account that $q_t = q_\tau$ and that $\nabla \tau = -\nabla |\mathbf{x} - \mathbf{x}'|/a_0 = -\mathbf{e}(\mathbf{x}', \mathbf{x})/a_0$.

A magnitude of interest in the analysis of the sonic field created by the source distribution is the mean value of the acoustic intensity in an observation point placed in a far field from all the sources, this is, if $\mathbf{x}'$ is any point of the source and ω_0 is an oscillation frequency of the source, the observation point $\mathbf{x}$ must verify the condition $\omega_0 |\mathbf{x} - \mathbf{x}'|/a_0 \ll 1$. In this case, and supposing that $q_t(\mathbf{x}', t) \sim \omega_0 q(\mathbf{x}', t)$, the second term in the integrand of (5.65) dominates among the first and the velocity field turns out, in first approximation,

$$\mathbf{u}'(\mathbf{x}, t) = \frac{1}{4\pi a_0} \int_{\Omega} \mathbf{e}(\mathbf{x}', \mathbf{x}) \frac{q_t(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\Omega. \quad (5.66)$$

A region of interest in the sonic field produced by the sources distribution is the far field, also called the *radiation zone* of the distribution. If the characteristic dimension of the volume occupied by the sources is L , then the mentioned zone is defined as the one formed by the points located at distances r of the distribution that verify

$$r \gg \lambda_0, \quad \text{and} \quad r \gg L^2/\lambda_0, \quad (5.67)$$

where $\lambda_0 = a_0/\omega_0$ is the characteristic wavelength of the radiation emitted by the sources. Notice that if L is of an order of, or much bigger than, λ_0 , the second condition (5.67) implies also the first, while if $\lambda_0 \gg L$ the first condition implies the second, and as such, the far field condition reduces to the corresponding for a punctual source; notice also that the conditions (5.67) always implice $r \gg L$ despite of the value in the relation L/λ_0 . If a coordinate system with origin in the interior of the sources distribution is taken, the distance from a generic point of the source, to the point of observation in the far field can be develop in a Taylor series around the origin of coordinates as

$$|\mathbf{x} - \mathbf{x}'| \simeq r - \mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x}) + O(|\mathbf{x}'|^2/r), \quad (5.68)$$

where $r \equiv |\mathbf{x}|$ and $\mathbf{e}_r(\mathbf{x})$ is the vector from the origin of coordinates to the point $\mathbf{x}$ (notice that, with the exception of terms $O(|\mathbf{x}'|^2/r)$, $\mathbf{e}_r(\mathbf{x}', \mathbf{x})$ can be substituted in (5.68) by $\mathbf{e}_r(\mathbf{x})$ for every $\mathbf{x}'$). Conditions (5.67) imply that, for every source point, the contribution of the quadratic terms in (5.68) to the time that the signal takes to arrive at $\mathbf{x}$ is much smaller than the typical oscillation period of the sources, $L^2/(ra_0) \ll 1/\omega_0$, so that if those terms are not taken into account, errors will be comitted in the calculation of the function $q(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|/a_0)$ that are of the order

$$\frac{|\mathbf{x}'|^2 q_t}{ra_0 q} \sim \frac{L^2 \omega_0}{ra_0} \ll 1. \quad (5.69)$$

So, the acoustic potential (5.60) might be approximated in the radiation zone by

$$\phi(\mathbf{x}, t) \simeq -\frac{1}{4\pi r} \int_{\Omega} q[\mathbf{x}', t - r/a_0 + \mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x})/a_0] d\Omega, \quad (5.70)$$

and the velocity and pressure fields are given by

$$u'_r(\mathbf{x}, t) \simeq \frac{1}{4\pi a_0 r} \int_{\Omega} q_t[\mathbf{x}', t - r/a_0 + \mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x})/a_0] d\Omega, \quad (5.71)$$

and

$$p'(\mathbf{x}, t) \simeq \frac{\rho_0}{4\pi r} \int_{\Omega} q_t[\mathbf{x}', t - r/a_0 + \mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x})/a_0] d\Omega. \quad (5.72)$$

Equations (5.70)-(5.72) express that, in a first approximation, each point of the distribution send to the observation point an spherical wave (locally plane) that can be considered centered at the origin of coordinates, and which temporal law contains a phase, $\mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x})/a_0$, that depends on the relative orientation between the source point and the observation point. The resulting wave in the observation point is the superposition of the train wave approximately concentric emitted by the source points, and because of the interferences originated by the phase differences of the corresponding waves to the different points of the distribution, their amplitude can present a strong dependency with the direction of the observation point, in contrast with what occurs with a punctual source. Only for the cases where the source dimension is much smaller than the wavelength, $L \gg a_0/\Omega_0$, the difference between phases of the points in Ω can be neglected in (5.70) (which is determined by $\mathbf{x}' \cdot \mathbf{e}_r(\mathbf{x})/a_0$) and, as such, the source distribution radiates as a punctual source of intensity $Q(t) = \int_{\Omega} d\Omega q(\mathbf{x}', t)$; in this case, if $Q(t)$ is not exactly null, (5.70) becomes in a first approximation

$$\phi(\mathbf{x}, t) \simeq -\frac{1}{4\pi r} \int_{\Omega} q[\mathbf{x}', t - r/a_0] d\Omega = -\frac{Q(t - r/a_0)}{4\pi r}. \quad (5.73)$$

5.4.3 Superficial sources distribution

The acoustic potential generated by a monopole distribution over a surface Σ whose intensity per surface unit is given by a function $q_s(\mathbf{x}, t)$ is given by

$$\phi = -\frac{1}{4\pi} \int_{\Sigma} \frac{q_s(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\sigma. \quad (5.74)$$

The surface integral (5.74) can be expressed as a volume integral by the use of the Dirac δ function

$$\phi = -\frac{1}{4\pi} \int \frac{q_s(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \delta[\xi(\mathbf{x}')] d\Omega \quad (5.75)$$

where $\mathbf{x}'$ is the spatial variable of integration over any volume that contains Σ , $q_s(\mathbf{x}')$ is any soft spatial extension of the function q_s defined over the surface Σ , and $\xi(\mathbf{x}')$ denotes the distance from $\mathbf{x}'$ to Σ . If the same reasoning that conducted to (5.63) is applied to (5.75) then the wave equation for a superficial monopole distribution is obtained

$$\nabla^2 \phi - \frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2} = q_s(\mathbf{x}, t) \delta[\xi(\mathbf{x})]. \quad (5.76)$$

From (5.76) a very useful relation can be obtained that relates q_s with the jump in the normal derivative of ϕ through the surface Σ . Consider the volume $\Delta\omega$ shown in Fig. 5.1, that contains the point $\mathbf{x}$ of Σ and is limited by two surfaces of area $\Delta\Sigma$ parallel to Σ that pass through the points $\mathbf{x}_1 = \mathbf{x} + \xi(\mathbf{x}_1)\mathbf{n}$ [$\xi(\mathbf{x}_1) < 0$] and $\mathbf{x}_2 = \mathbf{x} + \xi(\mathbf{x}_2)\mathbf{n}$ [$\xi(\mathbf{x}_2) > 0$] respectively, and by a lateral surface which area can be neglected compared to $\Delta\Sigma$ if $[-\xi(\mathbf{x}_1) + \xi(\mathbf{x}_2)] \ll \sqrt{\Delta\Sigma}$ when $\Delta\Omega \rightarrow 0$. Integrating (5.76) in $\Delta\Omega$ and neglecting terms of order $\Delta\Omega \rightarrow 0$, and applying the Gauss theorem, then

$$[-\mathbf{n} \cdot \nabla \phi(\mathbf{x}_1, t) + \mathbf{n} \cdot \nabla \phi(\mathbf{x}_2, t)] \Delta\Sigma = \int_{\xi(\mathbf{x}_2)}^{\xi(\mathbf{x}_1)} \int_{\Delta\Sigma} q_s \delta(\xi) d\sigma d\xi = q_s(\mathbf{x}, t) \Delta\Sigma. \quad (5.77)$$

If (5.77) is divided by $\Delta\Sigma$, and $\xi(\mathbf{x}_1)$ and $\xi(\mathbf{x}_2)$ tend to zero, then finally

$$\mathbf{n} \cdot \nabla \phi(\mathbf{x}, t) - \mathbf{n} \cdot \nabla \underline{\phi}(\mathbf{x}, t) = q_s(\mathbf{x}, t), \quad \mathbf{x} \in \Sigma, \quad (5.78)$$

where ϕ is the acoustic potential in the zone on the side of $\mathbf{n}$ with respect to Σ , and $\underline{\phi}$ is the acoustic potential in the zone on the side $-\mathbf{n}$. Over Σ the functions ϕ and $\underline{\phi}$ match [$\phi(\mathbf{x}, t) = \underline{\phi}(\mathbf{x}, t)$, $\mathbf{x} \in \Sigma$] in accordance with (5.74), but it so does not happen with the normal derivatives, which are related by the jump condition (5.78).

Figure 5.1: Infinitesimal volume for the integration of (5.76).

5.4.4 Acoustic dipoles

An acoustic dipole of orientation $\mathbf{n}$ and intensity G is the limit configuration formed by two punctual sources in the points $\mathbf{x}'$ and $\mathbf{x}'' = \mathbf{x}' + l\mathbf{n}$ of intensities Q and $-Q$ so that $Q \rightarrow \infty$ and $l \rightarrow 0$ while their product remains finite and equal to G , $Ql = G$. The resulting acoustic potential in $\mathbf{x}$ that comes from the superposition of the two sources is, with $Q(\mathbf{x}', t) = -Q(\mathbf{x}'', t)$,

$$\phi(\mathbf{x}, t) = -\frac{Q(\mathbf{x}', \tau)}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{Q(\mathbf{x}', \tau')}{4\pi|\mathbf{x} - \mathbf{x}''|}, \quad (5.79)$$

where, as before, $\tau = t - |\mathbf{x} - \mathbf{x}'|/a_0$ and $\tau' = t - |\mathbf{x} - \mathbf{x}''|/a_0$. Developing $Q(\mathbf{x}', \tau')$ and $|\mathbf{x}'' - \mathbf{x}|^{-1}$ around $\mathbf{x}'$, and neglecting terms of order l^2 , then

$$Q(\mathbf{x}', \tau') = Q(\mathbf{x}', \tau) + Q_\tau l\mathbf{n} \cdot \nabla' \tau = Q(\mathbf{x}', \tau) - \frac{1}{a_0} Q_\tau l\mathbf{n} \cdot \nabla' |\mathbf{x}' - \mathbf{x}|, \quad (5.80)$$

and

$$|\mathbf{x}'' - \mathbf{x}|^{-1} = |\mathbf{x}' - \mathbf{x}|^{-1} + l\mathbf{n} \cdot \nabla' |\mathbf{x}' - \mathbf{x}|^{-1}, \quad (5.81)$$

where ∇' denotes the gradient operator with respect to the variable $\mathbf{x}'$. If (5.80) and (5.81) are substituted in (5.79) then

$$\begin{aligned} \phi(\mathbf{x}, t) = & -\frac{1}{4\pi a_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|} Q_\tau(\mathbf{x}', \tau) l \mathbf{n} \cdot \nabla' |\mathbf{x} - \mathbf{x}'| \\ & + \frac{1}{4\pi} Q(\mathbf{x}', \tau) l \mathbf{n} \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} + O(l^2). \end{aligned} \quad (5.82)$$

Equation (5.82) is still valid for the limit $l \rightarrow 0$, $Q \rightarrow \infty$, so that if $lQ = G(\mathbf{x}', \tau)$, $\nabla G = G_\tau \nabla \tau = -G_\tau \nabla |\mathbf{x} - \mathbf{x}'|/a_0$, and $\nabla |\mathbf{x} - \mathbf{x}'| = -\nabla' |\mathbf{x} - \mathbf{x}'|$, the equation (5.82) can be written as

$$\phi = -\frac{1}{4\pi} \mathbf{n} \cdot \nabla \left[\frac{G(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right]. \quad (5.83)$$

If ω is a characteristic oscillation frequency of the dipole, then (5.82) can be approximated in the far field, $\omega/a_0 \gg |\mathbf{x} - \mathbf{x}'|$, by

$$\phi \simeq \frac{\mathbf{n} \cdot \mathbf{e}_r}{4\pi a_0 |\mathbf{x} - \mathbf{x}'|} G_\tau(\mathbf{x}', \tau), \quad (5.84)$$

where $\mathbf{e}_r = \nabla |\mathbf{x} - \mathbf{x}'|$ is the unitary vector directed from the position of the dipole $\mathbf{x}'$ to the observation point $\mathbf{x}$.

The formed configuration for a superficial acoustic dipole distribution of intensity $g_s(\mathbf{x}, t)$ by surface unit and oriented in each point according to the normal of the surface is called a *dipolar layer*. Its sonic field may be calculated by the superposition principle from expression (5.83) for the acoustic potential. If Σ is the surface in which the dipoles are distributed, then

$$\phi(\mathbf{x}, t) = -\nabla \cdot \left[\frac{1}{4\pi} \int_{\Sigma} \mathbf{n}' \frac{g_s(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] d\sigma, \quad (5.85)$$

where $\mathbf{n}' = \mathbf{n}(\mathbf{x}')$ is the normal to the surface Σ in the generic point $\mathbf{x}'$. As an analogous form as it was made with (5.74), the surface integral (5.85) can be expressed as a volume integral by the use of the Dirac δ function, so that

$$\phi(\mathbf{x}, t) = -\nabla \cdot \left[\frac{1}{4\pi} \int \delta[\xi(\mathbf{x}')] \mathbf{n}' \frac{g_s(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] d\Omega, \quad (5.86)$$

where, as before, $\mathbf{n}'$ and g_s represent spatial functions that match with the values of the normal and the dipolar intensity of the surface. If the Laplacian and the second temporal derivative are applied in (5.86), and its taken into account that such operators commute with the divergence, analogously to (5.76), then the wave equation for a superficial dipole distribution is

$$\nabla^2 \phi - \frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla \cdot (\mathbf{n}(\mathbf{x}) g_s(\mathbf{x}, t) \delta[\xi(\mathbf{x})]). \quad (5.87)$$

From the last equation is deduced that, with the exception of infinitesimals of superior order, the flux of the vector $\nabla \phi - \mathbf{n}(\mathbf{x}) g_s(\mathbf{x}, t) \delta[\xi(\mathbf{x})]$ is constant through any transversal section of the volume of Figure 5.1; as such, for any ξ

$$\frac{\partial \phi}{\partial \xi} - \left[\frac{\partial \phi}{\partial \xi} \right]_{\xi(\mathbf{x}_1)} = g_s \delta(\xi), \quad (5.88)$$

and a new integration between $\xi = 0^-$ and $\xi = 0^+$ yields

$$\phi(\mathbf{x}, t) - \underline{\phi}(\mathbf{x}, t) = g_s(\mathbf{x}, t). \quad (5.89)$$

Equation (5.89) relates the jump experienced by the acoustic potential for each point of Σ with the local intensity of the superficial acoustic dipole distribution.

5.4.5 Kirchhoff's integral formula

The past results allow to calculate the acoustic field in a fluid that fill an enclosure limited by a set of closed surfaces (one of which could be in the infinite) in which the potential values and their normal derivatives are ϕ and $\partial\phi/\partial n$. If $\mathbf{n}'$ denotes the unitary normal directed to the fluid in each point of the surfaces, a solution to the wave equation (5.76) in every interior point, by using (5.78) in (5.74) and (5.89) in (5.85), and considering that $\underline{\phi} = \mathbf{n} \cdot \nabla \underline{\phi} = 0$ (since it is for the interior), is

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi} \int_{\Sigma} \frac{\mathbf{n}' \cdot \nabla'_{\tau} \phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\sigma - \frac{1}{4\pi} \int_{\Sigma} \mathbf{n}' \cdot \nabla \left[\frac{\phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] d\sigma, \quad (5.90)$$

where Σ is the surface of the enclosure, $\mathbf{n}'$ the normal vector directed to the interior of the enclosure, in the point $\mathbf{x}'$, and ∇'_{τ} denotes the gradient with respect to $\mathbf{x}'$ without taking in account the dependency of τ with $\mathbf{x}'$ (*i.e.* constant τ). Equation (5.90) is called the *Kirchhoff's formula*, and it represents the acoustic potential in the interior of the enclosure as a superposition of the potentials caused by monopole and dipole distributions places over Σ and of intensities $\mathbf{n}' \cdot \nabla'_{\tau} \phi(\mathbf{x}', \tau)$ and $\phi(\mathbf{x}', \tau)$ respectively. As it has been seen in a general form (equations (5.76) and (5.87)), the integrals in (5.90) satisfy the wave equation in the interior of Σ ; also, to check that (5.90) satisfies the specified values of ϕ and $\partial\phi/\partial n$ over Σ is only required to make $\mathbf{x} \rightarrow \mathbf{x}'$,

$$\mathbf{n}' \cdot \nabla'_{\tau} \phi(\mathbf{x}', \tau) \rightarrow \mathbf{n} \cdot \nabla \phi(\mathbf{x}, t) = \partial\phi/\partial n, \quad (5.91)$$

and use the jump conditions (5.78) and (5.89). Notice that this implies that for the exterior surface of Σ (*i.e.* that which is in no contact with the fluid, and whose normal is $-\mathbf{n}'$), as mentioned before, $\mathbf{n}' \cdot \nabla \underline{\phi} = \underline{\phi} = 0$, so that the acoustic field calculated in (5.90) has to be null in the exterior region to the fluid. As such, the values ϕ and $\partial\phi/\partial n$ over Σ can not be arbitrary, and have to satisfy the relation

$$\int_{\Sigma} \frac{\mathbf{n}' \cdot \nabla'_{\tau} \phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\sigma + \int_{\Sigma} \mathbf{n}' \cdot \nabla \left[\frac{\phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] d\sigma = 0, \quad (5.92)$$

if $\mathbf{x}$ belongs to the exterior region to the fluid. The Kirchhoff equation (5.90) is, as such, an integral equation for the determination of ϕ .

Developing the integrand in the second integral in (5.90)

$$\nabla \left[\frac{\phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] = -\frac{1}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{1}{a_0} \frac{\partial \phi}{\partial \tau} + \frac{\phi}{|\mathbf{x} - \mathbf{x}'|} \right] \mathbf{e}(\mathbf{x}', \mathbf{x}), \quad (5.93)$$

remembering that $\mathbf{e}(\mathbf{x}', \mathbf{x}) = \nabla |\mathbf{x} - \mathbf{x}'|$ is the vector directed from $\mathbf{x}'$ to $\mathbf{x}$; so that Kirchhoff equation takes the form

$$\begin{aligned} \phi(\mathbf{x}, t) = & -\frac{1}{4\pi} \int_{\Sigma} \frac{\mathbf{n}' \cdot \nabla'_{\tau} \phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\sigma \\ & + \frac{1}{4\pi} \int_{\Sigma} \left[\frac{1}{a_0} \frac{\partial \phi(\mathbf{x}', \tau)}{\partial \tau} + \frac{\phi(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] \frac{\mathbf{n}' \cdot \mathbf{e}(\mathbf{x}', \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} d\sigma. \end{aligned} \quad (5.94)$$

Applying (5.17) to (5.94) then an integral equation for the pressure field is obtained

$$\begin{aligned} p'(\mathbf{x}, t) = & -\frac{1}{4\pi} \int_{\Sigma} \frac{\mathbf{n}' \cdot \nabla'_{\tau} p'(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} d\sigma \\ & + \frac{1}{4\pi} \int_{\Sigma} \left[\frac{1}{a_0} \frac{\partial p'(\mathbf{x}', \tau)}{\partial \tau} + \frac{p'(\mathbf{x}', \tau)}{|\mathbf{x} - \mathbf{x}'|} \right] \frac{\mathbf{n}' \cdot \mathbf{e}(\mathbf{x}', \mathbf{x})}{|\mathbf{x} - \mathbf{x}'|} d\sigma. \end{aligned} \quad (5.95)$$

Chapter 6

Flow equations around a train in a tunnel

This chapter presents the theoretical development of a simplified model that can describe and predict the pressure, temperature and velocity of the air flow caused by a high-speed train inside a tunnel, particularly in long ones. The model parts from a one dimensional model, where the regions where three dimensional effects count, such as the head and tail of the train, are bypassed by means of algebraic equations that come from an integral analysis of the flow equations.

The aim is to obtain a model that can be solved numerically in the order of minutes for problems that require the general information of the flow inside the tunnel, such as the overall power dissipated by the train along the tunnel, or the general temperature rise on the air. Particularly, the problem of temperature rise on the tunnel wall along the years requires the calculation of hundreds of thousands of train runs; doing a complex computation for each passing would make the problem unsolvable, and that is where a simplified analysis such as the one proposed here can provide a general and robust tool for the computation of the flow inside the tunnel

6.1 Health and comfort limits

The health criterium establishes the maximum pressure variation that the train can be submitted during the whole time it stays inside the tunnel. The directive 96/48/CE of the Council of the European Union related to the operability of high-speed transeuropean trains considers that the passengers health must be ensured during any circumstance; this implies that criterium verification must be done for the less favorable case, considering that the trains are not sealed at all. For tunnels of single and double line, the maximum admissible pressure variation during the transit of the train inside the tunnel is 10 kPa. To accomplish such limit, the pressure variations experimented in the train exterior cannot overcome 10 kPa during the time that the train is inside the tunnel.

The comfort limit establishes that the interior pressure variation in the train cars, Δp

Figure 6.1: Diagram of waves generated by a train inside a tunnel and the temporal variation of the pressure for a fixed section in the tunnel.

does not reach a certain level in a time interval, Δt , during the whole time that the train is inside the tunnel.

Different criteria has been gathered, including the normative for a european level and the ones developed by railway administrations. There is a clear lack of homogeneity for the criteria (Δp and Δt), as well for the application conditions (single or double track with and without train crossing). Besides this, the normative does not especifices in a clear way all the conditions for the application of the criteria.

Table 6.1 summarizes the different criteria.

The lack homogeneity is evident. For example, for $\Delta t = 4$ s, the Δp values go from 850

Source	Δt (s)	Δp (Pa)	Single track	Double track	Train crossing	Sealed train	Unsealed train
Ministerio de Fomento	4	2000	x				x
	4	4000		x	x		x
	1	700	x			x	
	3	1000					
	10	1400					
prEN 14067-5	4	4500		x	x		x
	4	3500	x				x
	1	1000	x	x	x	x	
	4	1600					
	10	2000					
UIC 779-11	1	500	x	x		x	
	3	800					
	4	850					
	10	1000					
UIC 651	1	400	x	x	x	x	x
	10	1000	x	x	x	x	x
UIC 660	1	500	x	x		x	
	3	800					
	10	1000					
	>60	2000					

Table 6.1: Comfort limit criteria according to different norms.

Pa to 4500 Pa, depending on the source and if the train is sealed or not.

It would be recomendable to use the criteria from the source UIC 660 for values without train crossing, and the source from the Ministerio de Fomento, taking in account the train crossing. These creteria are reasonable since they consider the most conservative values (lower Δp).

In order to establish the health limit it is necessary to determine the interior pressure in the train cars. sedpressure depends on the exterior pressure and the sealeing degree of the cars. It is, as such, necessary to know the efective area of the licking in the cars. Since this is hard to determine, it is substituted by a parameter named τ_{din} : this parameter, expressed in seconds, measures the sealing degree of the car. With larger τ_{din} , larger sealing degree. Unsealed trains have a value of $\tau_{\text{din}} \approx 0$ s, while a perfectly sealed train would have a value $\tau_{\text{din}} \rightarrow \infty$. The AVE (Alta Velocidad Española) of the Madrid-Sevilla line has a $\tau_{\text{din}} \approx 4.5$ s. A high sealed train, as the ones being built nowadays (such as the Talgo 350) has $\tau_{\text{din}} \approx 7$ s. A very high sealed train (such as the german ICE III) has a $\tau_{\text{din}} \approx 12$ s.

Knowing the external pressure variation, the temporal evolution of the interior pressure in the cars can be known be solving the differential equation

$$\frac{d(p_{\text{int}} - p_a)}{dt} = \frac{(p_{\text{ext}} - p_a) - (p_{\text{int}} - p_a)}{\tau_{\text{din}}} \quad (6.1)$$

where t is the time, p_{int} is the interior pressure, p_{ext} is the exterior pressure, and p_a the atmospheric pressure. Once that the interior pressure is known, the internal pressure difference, $\Delta p_{\text{int}} = p_{\text{int}}(t + \Delta t) - p_{\text{int}}(t)$, can be obtained according to the chosen criterium.

6.2 Governing equations in the tunnel

In a fluid flow with two different characteristic dimensions D_T and L_T (D_T being the hydraulic tunnel diameter and L_T the tunnel length) so that $D_T \ll L_T$, the estimations of the order of magnitude in the Navier-Stokes equations (as it was shown in chapter 3) provide a characteristic transversal speed, v_t , very small compared with the characteristic longitudinal speed, u_c ($v_t/u_c \sim D_T/L_T \ll 1$); and the characteristic transversal pressure variation, $(\Delta p)_t$, is very small compared with the characteristic longitudinal pressure variation, $(\Delta p)_c$, independent of the Reynolds number. The practical conclusion is that it is possible to consider the unidirectional flow with speed u , and therefore the pressure is uniform in each tunnel section. Additionally, if the Reynolds number $\rho u_c D_T / \mu$ is large, the flow is turbulent and the velocity and temperature (u, T) profiles are almost uniform in each tunnel section. As the pressure and temperature are uniform, all the remaining thermodynamic variables, like density ρ or specific entropy S , are also uniform. In the above conditions, the motion equations in the tunnel are

$$\text{Continuity:} \quad \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0, \quad (6.2)$$

$$\text{Momentum:} \quad \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = -\frac{4\tau_{Tu}}{D_{Tu}}, \quad (6.3)$$

$$\text{Energy:} \quad \rho c_p \frac{\partial T}{\partial t} + \rho c_p u \frac{\partial T}{\partial x} = \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \frac{4u\tau_{Tu}}{D_{Tu}} + \frac{4q_{Tu}}{D_{Tu}}, \quad (6.4)$$

where t is time, x the axial coordinate along the tunnel measured from the tunnel entry, τ_{Tu} the friction stress at the tunnel wall, q_{Tu} the heat recieved, per unit area and time, by the air through the tunnel wall, and c_p the specific heat at constant pressure.

6.3 Order of magnitude of $(\Delta p)_c$, u_c and $(\Delta T)_c$

Ambient conditions are denoted with a subscript "a". In order to estimate the pressure increment inside the tunnel it should be taken into account that it is generated by the train motion.

The momentum equation in integral form to a control volume between two sections upstream and downstream the train and with a reference system fixed to the train (figure 6.2) is

Figure 6.2: Simplified scheme of the train inside the tunnel from a reference system that moves with the train.

$$\begin{aligned} \int_V \frac{\partial(\rho \vec{v})}{\partial t} d\Omega + \int_{\Sigma_{Tu}} \rho \vec{v}(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \rho \vec{v}(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \\ - \int_{\Sigma_{Tu}} p \vec{n} d\sigma - \int_{\Sigma_{Tr}} p \vec{n} d\sigma + \int_{\Sigma_{Tu}} \tau \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \tau \cdot \vec{n} d\sigma. \end{aligned} \quad (6.5)$$

Where $\vec{v}_c$ is the control volume velocity, which for the surfaces 3 and 4 is $-U\vec{i}$ and zero for the rest of the surfaces. Considering that the expansion wave created when the tail entered the tunnel has passed, the characteristic time for temporal changes will be the time it takes for the waves to travel along the tunnel back and forth, that is $t_c \sim L_{Tu}/a$; besides, the flow can be taken as incompressible, which will be demonstrated a little further on. As such

$$\int_{\Sigma_{Tu}} \rho \vec{v}(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \rho \vec{v}(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = 0, \quad (6.6)$$

$$\int_V \frac{\partial(\rho \vec{v})}{\partial t} d\Omega \sim \frac{\rho_a u_c a}{L_{Tu}} (A_{Tu} L_{Tu} - A_{Tr} L_{Tr}), \quad (6.7)$$

where u_c is the characteristic air flow speed in the tunnel. A_{Tu} is the transversal area of the tunnel, L_{Tu} the tunnel's length, A_{Tr} the transversal train area, L_{Tr} the train length, and $(A_{Tu}L_{Tu} - A_{Tr}L_{Tr})$ the volume on which the integral is being done. The pressure term integrated on the tunnel surface is

$$\int_{\Sigma_{Tu}} p \vec{n} d\sigma = (\Delta p)_c A_{Tu}. \quad (6.8)$$

The aerodynamic drag caused by the train, R , is due to the pressure difference between the head and the tail, and the friction stress on the train wall

$$R = - \int_{\Sigma_{Tr}} p \vec{n} d\sigma + \int_{\Sigma_{Tr}} \tau \cdot \vec{n} d\sigma = \frac{1}{2} C_D \rho_a U^2 A_{Tr}, \quad (6.9)$$

where C_D is the adimensional drag coefficient of the train and U the train speed. The friction term on the tunnel wall can be neglected when compared with the friction term on the train wall (this will also be shown further on).

Equation (6.5) can be approximated as

$$\int_V \frac{\partial(\rho \vec{v})}{\partial t} d\Omega + (\Delta p)_c A_{Tu} = R. \quad (6.10)$$

The pressure term on the left must be of importance, and as such

$$(\Delta p)_c A_{Tu} \sim R \sim C_D \rho_a U^2 A_{Tr}. \quad (6.11)$$

Defining $\beta = A_{Tr}/A_{Tu}$

$$(\Delta p)_c \sim \beta C_D \rho_a U^2, \quad (6.12)$$

which for typical values of β and C_D ,

$$(\Delta p)_c \sim \rho_a U^2. \quad (6.13)$$

At the same time, the pressure term must also be of the order of the unsteady term, that is

$$(\Delta p)_c A_{Tu} \sim \frac{\rho_a u_c a}{L_{Tu}} (A_{Tu} L_{Tu} - A_{Tr} L_{Tr}), \quad (6.14)$$

$$(\Delta p)_c \sim \frac{\rho_a u_c a}{L_{Tu}} \left(\frac{A_{Tu} L_{Tu}}{A_{Tu}} - \frac{A_{Tr} L_{Tr}}{A_{Tu}} \right) \sim \rho_a u_c a \left(1 - \beta \frac{L_{Tr}}{L_{Tu}} \right) \sim \rho_a u_c a \quad (6.15)$$

As such

$$\rho_a U^2 \sim \rho_a a_a u_c, \quad \frac{u_c}{U} \sim \frac{U}{a_a} \sim M. \quad (6.16)$$

Dividing $(\Delta p)_c$ by p_a , yields

$$\frac{(\Delta p)_c}{p_a} \sim \frac{\rho_a U^2}{p_a} \sim \frac{U^2}{a^2} \sim M^2, \quad (6.17)$$

Where the relation $\gamma p_a / \rho_a = a^2$ has been used, and $\gamma = 1.4 \sim 1$.

To estimate the value of $(\Delta T)_c$ imagine a scheme like the one in figure 6.2 with area 1 and 2 being the entry and exit tunnel portals, respectively, and has a system of reference fixed to the tunnel. The energy conservation equation in integral form would be

$$\begin{aligned} \frac{d}{dt} \int_V \rho \left(e + \frac{1}{2} v^2 \right) d\Omega + \int_{\Sigma_{Tu}} \rho \left(e + \frac{1}{2} v^2 \right) (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \rho \left(e + \frac{1}{2} v^2 \right) (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \\ - \int_{\Sigma_{Tu}} p \vec{v} \cdot \vec{n} d\sigma - \int_{\Sigma_{Tr}} p \vec{v} \cdot \vec{n} d\sigma + \int_{\Sigma_{Tu}} \vec{v} \cdot \tau \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \vec{v} \cdot \tau \cdot \vec{n} d\sigma \\ - \int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma - \int_{\Sigma_{Tr}} \vec{q} \cdot \vec{n} d\sigma + \int_V Q d\Omega, \end{aligned} \quad (6.18)$$

where the terms with $\vec{q}$ represent the heat transfer to the walls, and Q the heat released by the train elements such as electric engines and AC units.

Here $\vec{v}_c$ is zero for surfaces 1, 2, 3 and 4, and $U\vec{i}$ for 5, 6, 7 and 8. The mean velocity along the whole volume is of order u_c , so that the kinetic energy in the first and second terms of the left side of (6.18) can be neglected. This is

$$\frac{d}{dt} \int_V \rho \left(e + \frac{1}{2} v^2 \right) d\Omega \approx \frac{d}{dt} \int_V \rho e d\Omega \quad (6.19)$$

and

$$\int_{\Sigma_{Tu}} \rho \left(e + \frac{1}{2} v^2 \right) (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma \approx \int_{\Sigma_{Tu}} \rho e \vec{v} \cdot \vec{n} d\sigma, \quad (6.20)$$

remembering that $\vec{v}_c = 0$ in Σ_{Tu} . The third term is zero, since the air speed on Σ_{Tr} is equal to the surfaces speed, $\vec{v}_c = U\vec{i}$, $\vec{v} - \vec{v}_c = 0$ (there is no flow through the train), so

$$\int_{\Sigma_{Tr}} \rho \left(e + \frac{1}{2} v^2 \right) (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = 0. \quad (6.21)$$

adding the first term on the right to the second term on the left allows to use the entalpy, that is

$$\int_{\Sigma_{Tu}} \rho \left(e + \frac{p}{\rho} \right) \vec{v} \cdot \vec{n} d\sigma = \int_{\Sigma_{Tu}} \rho h \vec{v} \cdot \vec{n} d\sigma. \quad (6.22)$$

The friction term on the tunnel wall is of order

$$\int_{\Sigma_{Tu}} \vec{v} \cdot \tau \cdot \vec{n} d\sigma \sim \frac{L_{Tu} A_{Tu} \lambda_{Tu} \rho_a u_c^3}{D_{Tu}} \quad (6.23)$$

and the one on the train wall is

$$\int_{\Sigma_{Tr}} \vec{v} \cdot \tau \cdot \vec{n} d\sigma \sim \frac{L_{Tr} A_{Tr} \lambda_{Tr} \rho_a U^3}{D_{Tr}}, \quad (6.24)$$

(the order of these will be shown with more detail further on), so that the ratio (tunnel wall term over train wall term) is

$$\frac{D_{Tr} L_{Tu} A_{Tu} \lambda_{Tu} u_c^3}{D_{Tu} L_{Tr} A_{Tr} \lambda_{Tr} U^3} \sim \left(\frac{u_c}{U}\right)^3 \sim M^3 \ll 1, \quad (6.25)$$

as such the friction term on the tunnel wall can be neglected.

Remembering the definition of the aerodynamic drag, (6.9), and knowing that the air speed on Σ_{Tr} is $U\vec{i}$, then

$$-\int_{\Sigma_{Tr}} p \vec{v} \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \vec{v} \cdot \tau \cdot \vec{n} d\sigma = UR = \frac{1}{2} C_D \rho_a U^3 A_{Tr}. \quad (6.26)$$

With all this, (6.18) can be rewritten as

$$\begin{aligned} \frac{d}{dt} \int_V \rho e d\Omega + \int_{\Sigma_{Tu}} \rho h \vec{v} \cdot \vec{n} d\sigma = \\ \frac{1}{2} C_D \rho_a U^3 A_{Tr} - \int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma - \int_{\Sigma_{Tr}} \vec{q} \cdot \vec{n} d\sigma + \int_V Q d\Omega. \end{aligned} \quad (6.27)$$

For this analysis the characteristic time is the one that the train spends inside the tunnel, L_{Tu}/U , because it is in this time that the entire temperature rise is made, so that

$$\frac{d}{dt} \int_V \rho e d\Omega \sim \frac{\rho_a c_v (\Delta T)_c U A_{Tu} L_{Tu}}{L_{Tu}} \sim \rho_a c_v (\Delta T)_c U A_{Tu}, \quad (6.28)$$

where $e = c_v T$. The second term of (6.27) is of order

$$\int_{\Sigma_{Tu}} \rho h \vec{v} \cdot \vec{n} d\sigma = (\rho_2 h_2 u_2 - \rho_1 h_1 u_1) A_{Tu} \sim \rho_a c_p (\Delta T)_c u_c A_{Tu}, \quad (6.29)$$

where $h = c_p T$. By using the Reynolds analogy, the heat transfer to the tunnel wall term is of order

$$-\int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma = \frac{4 A_{Tu} L_{Tu} q_{Tu}}{D_{Tu}} \sim \frac{\lambda_{Tu} \rho_a u_c c_p (\Delta T)_c L_{Tu} A_{Tu}}{D_{Tu}}, \quad (6.30)$$

and the heat transfer to the train wall term is of order

$$-\int_{\Sigma_{Tr}} \vec{q} \cdot \vec{n} d\sigma = \frac{4 A_{Tr} L_{Tr} q_{Tr}}{D_{Tr}} \sim \frac{\lambda_{Tr} \rho_a U c_p (\Delta T)_c L_{Tr} A_{Tr}}{D_{Tr}}. \quad (6.31)$$

The unsteady term in (6.27) dividing (6.30) is of order

$$\frac{\lambda_{Tu}\rho_a u_c c_p (\Delta T)_c L_{Tu} A_{Tu}}{D_{Tu}\rho_a U c_v (\Delta T)_c A_{Tu}} \sim \frac{\lambda_{Tu}\gamma u_c L_{Tu}}{D_{Tu}U} \sim 1, \quad (6.32)$$

and the unsteady term diving (6.31) is of order

$$\frac{\lambda_{Tr}\rho_a U c_p (\Delta T)_c L_{Tr} A_{Tr}}{D_{Tr}\rho_a U c_v (\Delta T)_c A_{Tu}} \sim \frac{\lambda_{Tr}\gamma L_{Tr}\beta}{D_{ru}} \sim 1, \quad (6.33)$$

where the last two orders are obtained by using the aforementioned typical values. This means that while the train is inside the tunnel, the heat transfer to the tunnel and train walls are relevant for the rise in temperature.

It is safe to say that the heat released to the air in the tunnel by the passing of the train would be *at least* of the order of the aerodynamic drag multiplied by the train speed; this means the first term of equation (6.27) is of the order of UR , that is

$$\rho_a c_v (\Delta T)_c U A_{Tu} \sim C_D \rho_a U^3 A_{Tr}, \quad (\Delta T)_c \sim \frac{C_D U^2 A_{Tr}}{c_v A_{Tu}},$$

$$\frac{(\Delta T)_c}{T_a} \sim \frac{C_D U^2 \beta}{c_v T_a} \sim C_D \beta \frac{U^2}{a_a^2} \sim C_D \beta M^2,$$

remembering that $C_D \beta \sim 1$

$$\frac{(\Delta T)_c}{T_a} \sim M^2, \quad (6.34)$$

which is reasonable to think, since it is of the same order as $(\Delta p)_c/p_a$.

6.4 Simplification of equations far from the train

According to the orders of magnitude above mentioned, the ratio between the third and second terms of equation (6.2) is

$$\frac{u(\partial\rho/\partial x)}{\rho(\partial u/\partial x)} \sim \frac{u_c(\Delta\rho)_c/\ell_c}{\rho_a u_c/\ell_c} \sim \frac{(\Delta\rho)_c}{\rho_a} \sim M^2 \ll 1. \quad (6.35)$$

The ratio between the convective and pressure terms of equation (6.3) is

$$\frac{\rho u(\partial u/\partial x)}{\partial p/\partial x} \sim \frac{\rho_a u_c^2/\ell_c}{\rho_a u_c a/\ell_c} \sim \frac{u_c}{a} \sim M^2 \ll 1. \quad (6.36)$$

In the momentum equation (6.3), the ratio between the unsteady term and pressure term is given by

$$\frac{\rho(\partial u/\partial t)}{\partial p/\partial x} \sim \frac{\rho_a u_c/t_c}{\rho_a u_c a/\ell_c} \sim \frac{\ell_c}{at_c} \sim 1, \quad (6.37)$$

where the characteristic length associated to changes in flow variables, ℓ_c , is the one travelled by the initial wave, that propagates at the sound speed, a , so that $\ell_c \sim at_c$, being t_c the characteristic time. Estimations in equation (6.2) provide

$$\frac{\partial \rho / \partial t}{\rho(\partial u / \partial x)} \sim \frac{(\Delta \rho)_c / t_c}{\rho_a u_c / \ell_c} \sim \left[\frac{(\Delta \rho)_c}{\rho_a} \right] \left(\frac{a}{u_c} \right) \left(\frac{\ell_c}{at_c} \right) \sim 1. \quad (6.38)$$

According with the above estimations, the unsteady terms on both equations (6.2) and (6.3) are important. In the same way, the energy equation (6.4) can be also simplified because

$$\frac{u \partial T / \partial x}{\partial T / \partial t} \sim \frac{u_c (\Delta T)_c / \ell_c}{(\Delta T)_c / t_c} \sim \frac{u_c}{a} \frac{at_c}{\ell_c} \sim \frac{u_c}{a} \sim M^2 \ll 1. \quad (6.39)$$

and following the same reasoning with the pressure terms

$$\frac{u \partial p / \partial x}{\partial p / \partial t} \sim \frac{u_c (\Delta p)_c / \ell_c}{(\Delta p)_c / t_c} \sim \frac{u_c}{a} \frac{at_c}{\ell_c} \sim \frac{u_c}{a} \sim M^2 \ll 1. \quad (6.40)$$

The last two terms of the energy equation (6.4) are the friction dissipation and the heat conduction, whose ratio is

$$\frac{u \tau_{Tu}}{q_{Tu}} \sim \frac{\lambda_{Tu} \rho_a u_c^3}{\lambda_{Tu} \rho_a u_c c_p (\Delta T)_c} \sim \left(\frac{u_c}{a} \right)^2 \frac{T_a}{(\Delta T)_c} \sim \frac{M^4}{M^2} \sim M^2 \ll 1, \quad (6.41)$$

hence the dissipation is small compared with the conduction term. To estimate the order of magnitude of the heat conduction term, the Reynolds analogy has been considered, where λ_{Tu} is the Darcy-Weisbach friction coefficient.

The simplified equations reduce to

$$\text{Continuity:} \quad \frac{\partial \rho}{\partial t} + \rho_a \frac{\partial u}{\partial x} = 0, \quad (6.42)$$

$$\text{Momentum:} \quad \rho_a \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = -\frac{4\tau_{Tu}}{D_{Tu}}, \quad (6.43)$$

$$\text{Energy:} \quad \rho_a c_p \frac{\partial T}{\partial t} = \frac{\partial p}{\partial t} + \frac{4q_{Tu}}{D_{Tu}}. \quad (6.44)$$

Taking into account that the state equation provides $dp/p_a = d\rho/\rho_a + dT/T_a$, the energy equation (6.44) can be re-written as

$$\frac{\partial p}{\partial t} - a^2 \frac{\partial \rho}{\partial t} = \frac{4(\gamma - 1)q_{Tu}}{D_{Tu}}, \quad (6.45)$$

that represents the change of entropy. Combining (6.42) and (6.45), equation (6.42) can be re-written as

$$\frac{\partial p}{\partial t} + \rho_a a^2 \frac{\partial u}{\partial x} = \frac{4(\gamma - 1)q_{Tu}}{D_{Tu}}. \quad (6.46)$$

6.4.1 Solution for times of order $t_0 \sim L_{Tr}/U$

The entry time of a train of length L_{Tr} in a tunnel is of order $t_0 \sim L_{Tr}/U$, being U the train speed. During this time, the wave generated at the entry travels a distance $\ell_0 \sim at_0 \sim aL_{Tr}/U \sim L_{Tr}/M$. During this time, friction term in the momentum equation (6.43) compared with the pressure term is

$$\frac{\tau_{Tu}/D_{Tu}}{\partial p/\partial x} \sim \left(\frac{\lambda_{Tu}L_{Tr}}{D_{Tu}} \right) \left(\frac{\ell_0}{L_{Tr}} \right) \frac{\rho_a u_c^2}{\rho_a u_c a} \sim \left(\frac{\lambda_{Tu}L_{Tr}}{D_{Tu}} \right) \left(\frac{\ell_0}{L_{Tr}} \right) \left(\frac{u_c}{a} \right) \sim M. \quad (6.47)$$

In (6.47) it has been assumed that $\lambda_{Tu}L_{Tr}/D_{Tu} \sim 1$. In the same way, in equation (6.46) the conduction term is, related to the terms of the left hand side, as follows

$$\frac{q_{Tu}/D_{Tu}}{\rho_a a^2 (\partial u/\partial x)} \sim \frac{(\lambda_{Tu}/D_{Tu}) \rho_a u_c c_p (\Delta T)_c}{\rho_a a^2 u_c / \ell_0} \sim \left(\frac{\lambda_{Tu}L_{Tr}}{D_{Tu}} \right) \left(\frac{\ell_0}{L_{Tr}} \right) \frac{(\Delta T)_c}{T_a} \sim M. \quad (6.48)$$

The above result shows that for these times the friction effects are as small as the Mach number. According to this (specially when $M \ll 1$), neglecting the friction and heat transfer terms, and multiplying equation (6.43) by $\pm a$ and adding it to equation (6.46) the following characteristic equations are formed

$$\frac{dp}{dt} + \rho_a a \frac{du}{dt} = 0 \text{ along the lines } dx/dt = a, \quad (6.49)$$

$$\frac{dp}{dt} - \rho_a a \frac{du}{dt} = 0 \text{ along the lines } dx/dt = -a, \quad (6.50)$$

where the speed of sound a is treated as a constant. The change of entropy is propagated at a speed u , not at the speed of sound, so in order to capture that propagation the convective terms $u\partial T/\partial x$ and $u\partial p/\partial x$ that were neglected from the energy equation (6.4) must be included again into equation (6.44) leading to

$$\frac{dp}{dt} - \rho_a c_p \frac{dT}{dt} = 0 \text{ along the lines } dx/dt = u. \quad (6.51)$$

Integrating equations (6.49)-(6.51) leads to

$$p + \rho_a a u = I^+ \text{ along the lines } dx/dt = a, \quad (6.52)$$

$$p - \rho_a a u = I^- \text{ along the lines } dx/dt = -a, \quad (6.53)$$

$$p - \rho_a c_p T = I^0 \text{ along the lines } dx/dt = u, \quad (6.54)$$

where I^+ , I^- , and I^0 are all constants.

6.4.2 Solution for times for which the friction and heat conduction are important

When the characteristic time, t_1 , is large compared with the train entry time, t_0 , the travel length of the initial wave, ℓ_1 , is large compared with ℓ_0 . The length ℓ_1 is the appropriate, according with (6.47) and (6.48), to take into account the friction and heat conduction if

$$\left(\frac{\ell_1}{L_{Tr}}\right) M^2 \sim 1, \quad (6.55)$$

which means that $\ell_1 \sim L_{Tr}/M^2 \gg L_{Tr}$; $t_1 \sim t_0/M^2 \gg t_0$ and $\ell_1 \sim \ell_0/M \gg \ell_0$. Then, for times of order t_1 (or distances inside the tunnel of order ℓ_1) the equations (6.43), (6.46) and (6.44) written along the characteristics are

$$\frac{dp}{dt} + \rho_a a \frac{du}{dt} = -\frac{4a\tau_{Tu}}{D_{Tu}} + \frac{4(\gamma-1)q_{Tu}}{D_{Tu}} \text{ along the lines } dx/dt = a, \quad (6.56)$$

$$\frac{dp}{dt} - \rho_a a \frac{du}{dt} = \frac{4a\tau_{Tu}}{D_{Tu}} + \frac{4(\gamma-1)q_{Tu}}{D_{Tu}} \text{ along the lines } dx/dt = -a, \quad (6.57)$$

and

$$\frac{dp}{dt} - \rho_a c_p \frac{dT}{dt} = -\frac{4q_{Tu}}{D_{Tu}} \text{ along the lines } dx/dt = u. \quad (6.58)$$

As in the previous case, these equations solve pressure p , flow velocity u and temperature T upstream and downstream of the train inside the tunnel, and they need boundary conditions at the exit and entry portals as well as at the nose and tail of the train.

6.4.3 Times for which the pressure wave is damped

When the tunnel length, L_{Tu} , is very large compared with ℓ_1 or, equivalently, $L_{Tu}/L_{Tr} \gg 1/M^2$, being L_{Tr} the train length, the momentum (6.43) and energy (6.44) equations can be simplified. In this limit, the characteristic speed is different from the previous one ($u_c \sim UM$). It can be obtained from the momentum equation by requiring that the pressure gradient is of the same order than the turbulent friction, that is

$$\frac{\rho_a U^2}{L_{Tu}} \sim \frac{\lambda_{Tu}}{D_{Tu}} \rho_a \tilde{u}_c^2 \implies \frac{\tilde{u}_c}{U} \sim \sqrt{\left(\frac{D_{Tu}}{\lambda_{Tu} L_{Tr}}\right) \frac{L_{Tr}}{L_{Tu}}} \sim \sqrt{\frac{L_{Tr}}{L_{Tu}}} \ll M \ll 1. \quad (6.59)$$

Note that $\tilde{u}_c/u_c \sim \sqrt{L_{Tr}/L_{Tu}}/M \ll 1$, because $L_{Tr}/L_{Tu} \ll M^2$.

The relation between the characteristic time t_2 , associated to this limit, with the tunnel length L_{Tu} , is obtained from the continuity equation (6.42) where the two terms should be of the same order

$$\frac{(\Delta\rho)_c}{t_2} \sim \frac{\rho_a \tilde{u}_c}{L_{Tu}} \implies \frac{\tilde{u}_c t_2}{L_{Tu}} \sim \frac{(\Delta\rho)_c}{\rho_a} \sim M^2 \ll 1 \implies \frac{U t_2}{L_{Tu}} \sim M^2 \sqrt{\frac{L_{Tu}}{L_{Tr}}}. \quad (6.60)$$

Once the characteristic time t_2 is known, it is possible to evaluate the unsteady terms on the momentum (6.43) and energy (6.44) equations. The ratio between the unsteady and pressure terms in equation (6.43) is

$$\frac{\rho_a(\partial u/\partial t)}{\partial p/\partial x} \sim \left(\frac{\tilde{u}_c}{U}\right) \left(\frac{L_{Tu}}{Ut_2}\right) \sim \frac{L_{Tr}/L_{Tu}}{M^2} \ll 1, \quad (6.61)$$

because $L_{Tr}/L_{Tu} \ll M^2$. In the energy equation (6.44) the two unsteady terms are of the same order and if the tunnel wall is adiabatic ($q_{Tu} = 0$), the relation between density and pressure (see (6.44)) is $d\rho = dp/a^2$. When the tunnel wall is not adiabatic, the ratio between the unsteady terms and the heat conduction term is

$$\begin{aligned} \frac{\partial p/\partial t}{q_{Tu}/D_{Tu}} &\sim \frac{\rho_a U^2/t_2}{(\lambda_{Tu}/D_{Tu})\rho_a \tilde{u}_c c_p (\Delta T)_c} \\ &\sim \left[\frac{D_{Tu}}{\lambda_{Tu} L_{Tr}} \left(\frac{U}{a}\right)^2 \frac{T_a}{(\Delta T)_c} \right] \frac{L_{Tr} L_{Tu}}{L_{Tu} \tilde{u}_c t_2} \sim \frac{L_{Tr}/L_{Tu}}{M^2} \ll 1, \end{aligned} \quad (6.62)$$

and consequently the air temperature is equal to the tunnel wall temperature: $T = T_w$. If the tunnel wall temperature is constant, the state equation provides $d\rho = \gamma dp/a^2$. In both cases the relation between $d\rho$ and dp can be written as $d\rho = \alpha dp/a^2$ where $\alpha = 1$ or $\alpha = \gamma$, depending on the case considered. The continuity (6.42) and momentum (6.43) equations reduce to

$$\frac{\partial p}{\partial t} + \frac{\gamma p_a}{\alpha} \frac{\partial u}{\partial x} = 0, \quad (6.63)$$

$$\frac{\partial p}{\partial x} + \frac{\lambda_{Tu}}{2D_{Tu}} \rho_a u^2 = 0. \quad (6.64)$$

In the above equations, the wave structure of the flow has disappeared. Derivating (6.63) with respect to x and derivating (6.64) with respect to t and substracting them provides an equation for flow velocity

$$\frac{\partial^2 u}{\partial x^2} - \left(\frac{\alpha \lambda_{Tu}}{2D_{Tu} a^2} \right) \frac{\partial u^2}{\partial t} = 0. \quad (6.65)$$

The above equation must be solved with boundary conditions $\partial u/\partial x = 0$ and pressure equal to p_a at tunnel ends and the values of u at the front and rear of the train. Also, an initial condition for u is needed. With the dimensionless variables $\omega = u/\tilde{u}$, $\xi = x/L_{Tu}$ and $\tau = t/t_2$ being $t_2 = (L_{Tu}/U)M^2\sqrt{L_{Tu}/L_{Tr}}$ and $\tilde{u} = U(2/\alpha)(D_{Tu}/\lambda_{Tu}L_{Tr})\sqrt{L_{Tr}/L_{Tu}}$, equation (6.65) is reduced to

$$\frac{\partial^2 \omega}{\partial \xi^2} - \frac{\partial \omega^2}{\partial \tau} = 0. \quad (6.66)$$

6.4.4 Infinitely long tunnel

With the coordinate changes $z = x - Ut$ and $t' = t$, the first term of (6.65) is $\partial^2 u / \partial x^2 = \partial^2 u / \partial z^2$ and the second $\partial u^2 / \partial t = \partial u^2 / \partial t' - U(\partial u^2 / \partial z)$ and as $t' = t$, equation (6.65) can be written as

$$\frac{\partial^2 u}{\partial z^2} - \left(\frac{\alpha \lambda_{Tu}}{2D_{Tu}a^2} \right) \left(\frac{\partial u^2}{\partial t} - U \frac{\partial u^2}{\partial z} \right) = 0. \quad (6.67)$$

In equation (6.67) the terms $\partial u^2 / \partial t$ and $U(\partial u^2 / \partial z)$ are of the same order if Ut_2/L_{Tu} is of order unity. According to the estimations obtained in (6.60), $Ut_2/L_{Tu} \sim M^2 \sqrt{L_{Tu}/L_{Tr}} \sim 1$, that means tunnel lengths of the order $L_{Tu}/L_{Tr} \sim 1/M^4$.

If $L_{Tu}M^4/L_{Tr} \gg 1$, then $\partial u^2 / \partial t \ll U(\partial u^2 / \partial z)$, which corresponds to the infinitely long tunnel. In these conditions, equation (6.67) is reduced to

$$\frac{\partial^2 u}{\partial z^2} + \left(\frac{\alpha \lambda_{Tu} U}{2D_{Tu}a^2} \right) \frac{\partial u^2}{\partial z} = 0. \quad (6.68)$$

Equation (6.68) should be integrated, upstream the train, with the boundary conditions $u = 0$ at $z \rightarrow \infty$ and $u = u_N$ at $z = 0$, being u_N the air speed at the train nose. With the dimensionless variables $\tilde{w} = u/u_N$ and $\eta = z/\ell_c$ being $\ell_c = (2D_{Tu}/\alpha \lambda_{Tu})(a^2/Uu_N)$, equation (6.68) reduces to $\partial^2 \tilde{w} + \partial \tilde{w}^2 / \partial \eta = 0$, that can be integrated to give $\tilde{w} = 1/(1 + \eta)$. The solution downstream the train ($\eta < 0$) is $\tilde{w} = 1/(1 - \eta)$, where u_N should be substituted by the speed at the train tail u_T in the definition of $\tilde{w}$ and ℓ_c .

Once the speed is obtained, the pressure can be obtained from equation (6.64) that reads

$$\frac{\partial(p/p_a)}{\partial \eta} + \frac{\gamma}{\alpha} \frac{u_N}{U} \tilde{w}^2 = 0, \quad (6.69)$$

whose solution, with $p/p_a = 1$ when $z \rightarrow \infty$ is

$$\frac{p}{p_a} = 1 + \frac{\gamma}{\alpha} \frac{u_N}{U} \tilde{w}. \quad (6.70)$$

Downstream the train ($\eta < 0$) the pressure distribution is $p/p_a = 1 - (\gamma/\alpha)(u_T/U)\tilde{w}$.

6.5 Governing equations between train and tunnel

In the gap between train and tunnel the condition $L_{Tr}/D_g \gg 1$ holds, being L_{Tr} the train length and D_g the equivalent diameter of the gap region (this ratio is, at least, of order 70 for a high speed train). Under this hypothesis, the one-dimensional approximation for the equations between train and tunnel can be utilized. Taking a coordinate system relative to the train and being w the relative speed, the continuity, momentum and energy equations in the gap between train and tunnel are

$$\frac{\partial \rho}{\partial t} + w \frac{\partial \rho}{\partial s} + \rho_a \frac{\partial w}{\partial s} = 0, \quad (6.71)$$

$$\rho_a \frac{\partial w}{\partial t} + \rho_a w \frac{\partial w}{\partial s} + \frac{\partial p}{\partial s} = -\frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}} + \rho_a \frac{dU}{dt}, \quad (6.72)$$

$$\rho_a T_a \left(\frac{\partial S}{\partial t} + w \frac{\partial S}{\partial s} \right) = \frac{\dot{Q}}{(A_{Tu} - A_{Tr}) L_{Tr}}, \quad (6.73)$$

where s is the coordinate along the gap, positive from the nose to the tail of train, τ_f is the friction stress at the train walls, ℓ_p is the wet perimeter, S is the specific entropy, $\dot{Q}$ is the heat by unit time, released to the air by the friction dissipation, the heat transfer to the walls and the external heat released by the train (AC units, losses of heat from the electric engines, etc.); that is

$$\frac{\dot{Q}}{(A_{Tu} - A_{Tr}) L_{Tr}} = \frac{w \tau_f \ell_p}{A_{Tu} - A_{Tr}} + \frac{q_s \ell_p}{A_{Tu} - A_{Tr}} + Q_r, \quad (6.74)$$

where

$$\frac{q_s \ell_p}{A_{Tu} - A_{Tr}} = \frac{4}{D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) q_{Tr} + \frac{4}{D_{Tu}} \left(\frac{1}{1 - \beta} \right) q_{Tu}.$$

The term $\rho_a (dU/dt)$ in (6.72) takes into account for the body forces when the train speed is not constant. Note that the order of magnitude of w is U , but its variations are of order $u_c \ll U$. The friction term between train and tunnel is composed by the friction in the tunnel wall and the friction in the train wall, $\tau_p \ell_p = \tau_{pTu} \ell_{pTu} + \tau_{pTr} \ell_{pTr}$, where $\tau_{pTu} \sim \lambda_{Tu} \rho_a (U - w)^2 \sim \lambda_{Tu} \rho_a u_c^2$, and $\tau_{pTr} \sim \lambda_{Tr} \rho_a w^2 \sim \lambda_{Tr} \rho_a U^2$. Assuming $\lambda_{Tr} \sim \lambda_{Tu}$ and the perimeters $\ell_{pTu} \sim \ell_{pTr}$, the ratio between them is $(u_c/U)^2 \sim M^2 \ll 1$, and for this reason, the surviving friction term in (6.72) is

$$\frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}} = \frac{\lambda_{Tr}}{2D_{Tr}} \left(\frac{\beta}{\beta - 1} \right) \rho_a w |w|. \quad (6.75)$$

In the gap, the heat transfer to the tunnel wall is of order

$$\frac{4}{D_{Tu}} \left(\frac{1}{1 - \beta} \right) q_{Tu} \sim \frac{\lambda_{Tu} \rho_a u_c c_p (\Delta T)_c}{2D_{Tu}(1 - \beta)}, \quad (6.76)$$

and in the train wall is of order

$$\frac{4}{D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) q_{Tr} \sim \frac{\lambda_{Tr} \rho_a U c_p (\Delta T)_c \beta}{2D_{Tr}(1 - \beta)}, \quad (6.77)$$

so that the ratio between (6.76) over (6.77) is

$$\frac{D_{Tr}}{D_{Tu}} \frac{A_{Tu}}{A_{Tr}} \frac{u_c}{U} \sim \frac{u_c}{U} \sim M, \quad (6.78)$$

where for typical values it has been considered that $(D_{Tr} A_{Tu}) / (D_{Tu} A_{Tr}) \sim 1$. This means that for low Mach numbers the heat transfer to the tunnel wall can be neglected when compared with the heat transfer on the train wall, so that

$$\frac{q_s \ell_p}{A_{Tu} - A_{Tr}} = \frac{4}{D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) q_{Tr}. \quad (6.79)$$

6.5.1 Equations along the characteristics in the gap between train and tunnel

Equations (6.71) to (6.73) can be written in axis fixed to the tunnel. The absolute speed, u , is related to the relative speed, w , and the train speed, U , in the form $w = U - u$. The change in coordinate system gives

$$dx = Udt - ds; \quad dt' = dt, \quad (6.80)$$

and then $\partial/\partial s = -\partial/\partial x$ and $\partial/\partial t = \partial/\partial t' + U(\partial/\partial x)$. With these changes, equations (6.71) to (6.73) take the form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho_a \frac{\partial u}{\partial x} = 0, \quad (6.81)$$

$$\rho_a \frac{\partial u}{\partial t} + \rho_a u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = + \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}}, \quad (6.82)$$

$$\rho_a T_a \left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} \right) = \frac{\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}}, \quad (6.83)$$

where t' has been changed by t , because they are the same. Equations (6.81) to (6.94) are the governing equations in fixed axis for the flow in the gap between train and tunnel walls.

For the continuity (6.81) and momentum (6.82) equations the characteristic time is of the order of the time it takes for the waves to travel along the gap between train and tunnel, that is $t_c \sim L_{Tr}/a$, which leaves to $\partial \rho / \partial t \sim \rho_a (\partial u / \partial x)$ in (6.81) and $\rho_a (\partial u / \partial t) \sim \partial p / \partial x$ in (6.82); the unsteady terms must be maintained so that the compressibility effects are present and the wave structure can be formed. In the energy equation (6.94) the characteristic time is different, since the change of entropy is propagated at the air speed, that is $t_c \sim L_{Tr}/u_c$. In equation (6.81) the term $u \partial \rho / \partial x$ compared with $\rho_a (\partial u / \partial x)$ is

$$\frac{u \partial \rho / \partial x}{\rho_a (\partial u / \partial x)} \sim \frac{u_c \Delta \rho / L_{Tr}}{\rho_a u_c / L_{Tr}} \sim \frac{\Delta \rho}{\rho_a} \sim M^2 \ll 1. \quad (6.84)$$

The same applies for the term $\rho_a u (\partial u / \partial x)$ compared with $\partial p / \partial x$ in the momentum equation (6.82),

$$\frac{\rho_a u \partial u / \partial x}{\partial p / \partial x} \sim \frac{\rho_a u_c^2 / L_{Tr}}{\rho_a u_c a / L_{Tr}} \sim \frac{u_c}{a} \sim M^2 \ll 1. \quad (6.85)$$

and these two terms can be disregarded from their respective equations, yielding

$$\frac{\partial \rho}{\partial t} + \rho_a \frac{\partial u}{\partial x} = 0, \quad (6.86)$$

$$\rho_a \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = + \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}}. \quad (6.87)$$

In the energy equation (6.94) the term $u(\partial S/\partial x)$ over $\partial S/\partial t$ is

$$\frac{u(\partial S/\partial x)}{\partial S/\partial t} \sim \frac{u_c \rho_a T_a \Delta S / L_{Tr}}{\rho_a T_a \Delta S u_c / L_{Tr}} \sim 1. \quad (6.88)$$

Note that equation (6.94) can be written in terms of p and ρ by means of the relations $dS/c_v = dp/p - \gamma d\rho/\rho$,

$$\rho_a T_a \left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} \right) = \frac{1}{\gamma - 1} \left[\left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \right) - c_p T_a \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right) \right]. \quad (6.89)$$

Substituting $\partial \rho/\partial t + u(\partial \rho/\partial x)$ from equation (6.81) into (6.89), turns the energy equation (6.94) into

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho_a a^2 \frac{\partial u}{\partial x} = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}}. \quad (6.90)$$

Multiplying (6.87) by $\pm a$ and adding it to (6.90) gives

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} \pm a \frac{\partial p}{\partial x} \pm a \rho_a \frac{\partial u}{\partial t} + \rho_a a^2 \frac{\partial u}{\partial x} = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}} \pm a \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}}. \quad (6.91)$$

The term $u(\partial p/\partial x)$ over $a(\partial p/\partial x)$ in equation (6.91) is of order $u_c/a \sim M^2 \ll 1$, so that it can be neglected, leading to a set of equations

$$\frac{\partial p}{\partial t} + a \frac{\partial p}{\partial x} + a \rho_a \frac{\partial u}{\partial t} + \rho_a a^2 \frac{\partial u}{\partial x} = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}} + a \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}}, \quad (6.92)$$

$$\frac{\partial p}{\partial t} - a \frac{\partial p}{\partial x} - a \rho_a \frac{\partial u}{\partial t} + \rho_a a^2 \frac{\partial u}{\partial x} = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}} - a \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}}, \quad (6.93)$$

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = \frac{\dot{Q}}{\rho_a T_a (A_{Tu} - A_{Tr})L_{Tr}}, \quad (6.94)$$

that can be written in characteristic form

$$\frac{d}{dt}(p + \rho_a a u) = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}} + a \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}} \quad \text{along the lines} \quad \frac{dx}{dt} = a, \quad (6.95)$$

$$\frac{d}{dt}(p - \rho_a a u) = \frac{(\gamma - 1)\dot{Q}}{(A_{Tu} - A_{Tr})L_{Tr}} - a \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}} \quad \text{along the lines} \quad \frac{dx}{dt} = -a, \quad (6.96)$$

$$\frac{dS}{dt} = \frac{\dot{Q}}{\rho_a T_a (A_{Tu} - A_{Tr}) L_{Tr}}$$

$$\text{along the lines } \frac{dx}{dt} = u. \quad (6.97)$$

6.5.2 Flow around the nose and tail

In the vicinity of the train nose and tail the flow is three-dimensional. The waves travelling inside the tunnel reach the train nose and tail on times of order $t_c \sim L_{Tu}/a$; the speed variation in time caused by the passing of compression and expansion waves is of order u_c , but the speed variation in space, going from the train velocity to zero on the train head is of order U . Nevertheless, because the characteristic length ℓ_N of this regions is small compared with the tunnel length (for typical values $\ell_N/L_{Tu} \sim 0.002$, $M^2 \sim 0.04$, $\ell_N/L_{Tu} \ll M^2$), the flow is incompressible and quasi-steady in axis fixed to the train. That is

$$\frac{\vec{v} \cdot \nabla \rho}{\rho \nabla \cdot \vec{v}} \sim \frac{U \Delta p}{\rho_a U} \sim \frac{\Delta \rho}{\rho_a} \sim M^2 \ll 1,$$

and

$$\frac{\partial \rho / \partial t}{\rho \nabla \cdot \vec{v}} \sim \frac{\Delta \rho / t_c}{\rho U / \ell_N} \sim M^2 \frac{1}{M} \frac{\ell_N}{L_{Tu}} \ll 1. \quad (6.98)$$

With the above simplifications the continuity equation reduces to $\nabla \cdot \vec{v} = 0$. When no pressure waves are passing the train head and tail the unsteady term in the momentum equation, $\rho(\partial \vec{v} / \partial t) \sim \rho_a u_c / t_c \sim \rho_a u_c / (L_{Tu}/a)$, is negligible compared with the convective one, $\rho \vec{v} \cdot \nabla \vec{v} \sim \rho_a U^2 / \ell_N$,

$$\frac{\rho(\partial \vec{v} / \partial t)}{\rho \vec{v} \cdot \nabla \vec{v}} \sim \frac{\rho_a u_c a \ell_N}{\rho_a U^2 L_{Tu}} \sim M \frac{1}{M} \frac{\ell_N}{L_{Tu}} \ll 1.$$

Naturally, when the pressure waves pass over the nose, the characteristic time changes, $t_c \sim \ell_N/a$; the flow must be considered compressible in order to capture wave propagation, and the unsteady terms are high when compared with the convective terms; but this happens fast and sporadically. On the other hand, because the Reynolds number is very high, the viscous effects are also negligible. For a turbulent regime, during the passing of the waves, in order to obtain the scale where the diffusion effects are as relevant as the unsteady term, the diffusion and the unsteady terms must be of the same order, that is $\partial u / \partial t \sim \nu_T \partial^2 u / \partial y^2$. Since $\partial u / \partial t \sim u_c / t_c$ and $\nu_T \partial^2 / \partial y^2 \sim \nu_T u_c / \ell_c^2$, where ℓ_c is the characteristic length where diffusion effects are noticable, and $\nu_T \sim 10^{-1} \text{ m}^2/\text{s}$ (Schliting, 1987), then $(\ell_c / \ell_n)^2 \sim \nu_T / a \ell_n \sim 10^{-4}$. This also applies for the turbulent heat diffusion, since the turbulent Prandtl number is of order unity. In order to appreciate these effects the mesh would have to be so fine that the computational time required to calculate across the entire tunnel would be very large.

Figure 6.3: Control volume for the train nose.

By applying the continuity, momentum and energy equations to the control volume of figure 6.3, it is possible to obtain the relations needed to link the flow upstream the nose with the flow at the beginning of the gap between train and tunnel. The continuity equation applied to the control volume ABCDEFGA provides the relation

$$w = \frac{U - u_N}{1 - \beta}. \quad (6.99)$$

$$[-\rho_a(U - u_N)^2 A_{Tu} + \rho_a w^2 (A_{Tu} - A_{Tr})] \vec{i} = [p_N A_{Tu} - p_e (A_{Tu} - A_{Tr})] \vec{i} - \left[\vec{i} \cdot \int_{\Sigma_c} p \vec{n} dA \right] \vec{i}, \quad (6.100)$$

where $\vec{i}$ is the unit vector opposed to the train direction. The surface Σ_c is the open surface DEF of figure 6.3, where $p_e A_{Tr} \vec{i} = - \int_{\Sigma_e} p_e \vec{n} dA$, and therefore $- \int_{\Sigma_c} p \vec{n} dA - \int_{\Sigma_e} p_e \vec{n} dA$ represents the pressure integral to the closed nose body DEFHD that can be substituted by the nose drag D_N

$$D_N \vec{i} = \left[\vec{i} \cdot \int_{\Sigma_c} p \vec{n} dA + \vec{i} \cdot \int_{\Sigma_e} p_e \vec{n} dA \right] \vec{i} = C_{DN} \frac{1}{2} \rho_a (U - u_N)^2 A_{Tr} \vec{i}, \quad (6.101)$$

being C_{DN} the drag coefficient of the train nose. The momentum equation reduces to

$$\frac{2(p_N - p_e)}{\rho_a U^2} = \beta \left[C_{DN} + \frac{2}{1 - \beta} \right] \left(1 - \frac{u_N}{U} \right)^2. \quad (6.102)$$

Applying the integral energy equation for the same control volume, considering that the unsteady terms are small, as well as the friction and heat transfer effects, provides

$$\int_{\Sigma} \rho \left(e + \frac{p}{\rho} \right) \vec{v} \cdot \vec{n} dA = - \int_{\Sigma} \frac{1}{2} \rho v^2 \vec{v} \cdot \vec{n} dA, \quad (6.103)$$

$$\begin{aligned} -\rho_a c_p T_N (U - u_N) A_{Tu} + \rho_a c_p T_e \left(\frac{U - u_N}{1 - \beta} \right) (A_{Tu} - A_{Tr}) = \\ \frac{1}{2} \rho_a (U - u_N)^3 A_{Tu} - \frac{1}{2} \rho_a \frac{(U - u_N)^3}{(1 - \beta)^3} (A_{Tu} - A_{Tr}). \end{aligned} \quad (6.104)$$

By use of some algebra the following relation between the temperatures is obtained

$$\frac{T_N - T_e}{T_a} = \frac{\gamma - 1}{2} \beta M^2 \left[\frac{2 - \beta}{(1 - \beta)^2} \right] \left(1 - \frac{u_N}{U} \right)^2, \quad (6.105)$$

that corresponds to the conservation of the total enthalpy and, in first approximation, is $T_N = T_e$.

Equations (6.102) and (6.105), together with the invariant I^+ at C and the invariants I^- and I^0 at A (see figure 6.3) allow to calculate the five unknowns u_N , p_N , p_e , T_N and T_e .

Figure 6.4: Control volume for the train tail.

A similar analysis can be done for the train tail (see figure 6.4). The continuity equation provides

$$w_T = \frac{U - u_T}{1 - \beta}. \quad (6.106)$$

The momentum equation can be written as

$$\begin{aligned} [\rho_a (U - u_N)^2 A_{Tu} - \rho_a w_T^2 (A_{Tu} - A_{Tr})] \vec{i} = \\ [-p_T A_{Tu} + p_s (A_{Tu} - A_{Tr})] \vec{i} - \left[\vec{i} \cdot \int_{\Sigma_c} p \vec{n} dA \right] \vec{i}, \end{aligned} \quad (6.107)$$

and, as before, $-p_s A \vec{i} - \left[\vec{i} \cdot \int_{\Sigma_c} p \vec{n} dA \right] \vec{i} = -D_T \vec{i}$, and the momentum equation reduces to

$$\frac{2(p_s - p_T)}{\rho_a U^2} = \beta \left[C_{DT} - \frac{2}{1 - \beta} \right] \left(1 - \frac{u_T}{U} \right)^2, \quad (6.108)$$

where C_{DT} is the drag coefficient for the tail: $D_T = \frac{1}{2} C_{DT} \rho_a (U - u_T)^2 A_{Tr}$. The energy equation takes the form

$$\frac{T_T - T_s}{T_a} = \frac{\gamma - 1}{2} \beta M^2 \left[\frac{2 - \beta}{(1 - \beta)^2} \right] \left(1 - \frac{u_T}{U} \right)^2, \quad (6.109)$$

that in first approximation is $T_T = T_s$. Again, with the invariant I^+ downstream the train and the invariants I^- and I^0 at the end of the gap between train and tunnel, the five unknown u_T , p_T , p_s , T_T and T_s can be obtained.

Steady and incompressible limit

The flow in the gap between train and tunnel can be solved with the system of equations (6.95) to (6.97), but for small train Mach numbers, the problem in the gap can be reduced to an incompressible and steady flow; this can be done acknowledging that, for an observer mounted on the train, the pressure waves reach the train between periods of time of order $t_c \sim L_{Tu}/a$, as it was done in a similar fashion for the nose and the train tail¹. The orders of the relations between terms on the continuity equation are

$$\begin{aligned} \frac{\partial \rho / \partial t}{\rho_a (\partial w / \partial s)} &\sim \frac{\Delta \rho a / L_{Tu}}{\rho_a u_c / L_{Tr}} \sim \frac{\Delta \rho}{\rho_a} \frac{a}{u_c} \frac{L_{Tr}}{L_{Tu}} \sim M^2 \frac{1}{M^2} \frac{L_{Tr}}{L_{Tu}} \sim M^2 \ll 1, \\ \frac{w (\partial \rho / \partial s)}{\rho_a (\partial w / \partial s)} &\sim \frac{U \Delta \rho / L_{Tr}}{\rho_a u_c / L_{Tr}} \sim \frac{U}{u_c} \frac{\Delta \rho}{\rho_a} \sim \frac{1}{M} M^2, \end{aligned}$$

so that for small Mach numbers the continuity equation reduces to $\partial w / \partial s = 0$, or equivalently, $\partial u / \partial x = 0$, which implies that $u_N = u_T$.

On the momentum equation the pressure term must be of the same order as the friction term, since it will be the friction stress on the train wall what causes the major pressure rise, so

$$\frac{\partial p}{\partial s} \sim \frac{\tau_f \ell_p}{A_{Tu} - A_{Tr}} \sim \frac{\lambda_{Tr}}{D_{Tr}} \rho_a U^2, \quad \Delta p \sim \frac{\lambda_{Tr} L_{Tr}}{D_{Tr}} \rho_a U^2 \sim \rho_a U^2,$$

where for typical values $\lambda_{Tr} L_{Tr} / D_{Tr} \sim 1$. This implies that

$$\frac{\rho_a (\partial w / \partial t)}{\partial p / \partial s} \sim \frac{\rho_a u_c a / L_{Tu}}{\rho_a U^2 / L_{Tr}} \sim M \frac{1}{M} M^2 \sim M^2 \ll 1,$$

¹Again, when the pressure waves pass along the train and tunnel the characteristic time changes, $t'_c \sim L_{Tr}/a$, and the unsteady and compressible effects of the flow must be taken into account, but since the ratio of these two times is $t'_c/t_c \sim L_{Tr}/L_{Tu} \sim M^2$ (for typical values $L_{Tr}/L_{Tu} \sim 0.04 \sim M^2$), then it is reasonable to consider an instantaneous change of flow properties between head and tail of the train for small Mach numbers.

$$\frac{\rho_a w(\partial w / \partial s)}{\partial p / \partial s} \sim \frac{\rho_a U u_c / L_{Tr}}{\rho_a U^2 / L_{Tr}} \sim M.$$

Then, (for small Mach) the momentum equation reduces to $\partial p / \partial s = -\tau_f \ell_p / (A_{Tu} - A_{Tr})$, or equivalently, $\partial p / \partial x = \tau_f \ell_p / (A_{Tu} - A_{Tr})$, that can be integrated to obtain

$$\frac{2(p_e - p_s)}{\rho_a U^2} = \frac{\lambda_{Tr} L_{Tr}}{D_{Tr}} \frac{\beta}{(1 - \beta)^3} \left(1 - \frac{u_N}{U}\right)^2, \quad (6.110)$$

being p_e and p_s the pressures shown in figures 6.3 and 6.4 respectively.

Combining equations (6.102) and (6.110) it is obtained

$$\frac{2(p_N - p_s)}{\rho_a U^2} = \beta \left[C_{DN} + \frac{2}{1 - \beta} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right] \left(1 - \frac{u_N}{U}\right)^2, \quad (6.111)$$

and combining (6.108) and (6.111)

$$\frac{2(p_N - p_T)}{\rho_a U^2} = \beta \left[C_{DN} + C_{DT} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right] \left(1 - \frac{u_N}{U}\right)^2. \quad (6.112)$$

At instant $t^- = (L_{Tr}/U)^-$ the tail is just before the tunnel entry. The boundary condition provides $p_s = p_a$, and from (6.111)

$$\left[\frac{2(p_N - p_a)}{\rho_a U^2} \right]^- = \beta \left[C_{DN} + \frac{2}{1 - \beta} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right] \left(1 - \left[\frac{u_N}{U} \right]^- \right)^2 \quad (6.113)$$

At instant $t^+ = (L_{Tr}/U)^+$, the tail has just entered in the tunnel. The boundary condition in this case is $p_T = p_a$, and from (6.112)

$$\left[\frac{2(p_N - p_a)}{\rho_a U^2} \right]^+ = \beta \left[C_{DN} + C_{DT} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right] \left(1 - \left[\frac{u_N}{U} \right]^+ \right)^2. \quad (6.114)$$

Upstream the nose train, the pressure p_N and the velocity u_N are related between them by the invariant (6.53). Equations (6.113) and (6.114) together with the invariant upstream the train $p_N - \rho_a a u_N = p_a$, that provides $2(p_N - p_a) / \rho_a U^2 = 2a u_N / U^2$, determines $(u_N/U)^-$ and $(u_N/U)^+$. Taking into account that $u_N/U \ll 1$, a first approximation is

$$\left[\frac{u_N}{U} \right]^- = \frac{\beta U}{2a} \left[C_{DN} + \frac{2}{1 - \beta} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right], \quad (6.115)$$

$$\left[\frac{u_N}{U} \right]^+ = \frac{\beta U}{2a} \left[C_{DN} + C_{DT} + \frac{\lambda_{Tr} L_{Tr} / D_{Tr}}{(1 - \beta)^3} \right]. \quad (6.116)$$

The above results represent an instantaneous change on pressure immediately upstream the train. In a first approximation this change in pressure and velocity is

$$\begin{aligned}
\frac{2\Delta p_T}{\rho_a U^2} &= \left[\frac{2(p_N - p_a)}{\rho_a U^2} \right]^- - \left[\frac{2(p_N - p_a)}{\rho_a U^2} \right]^+ \\
&= \beta \left(\frac{2}{1 - \beta} - C_{DT} \right) = \frac{2a}{U} \left\{ \left[\frac{u_N}{U} \right]^- - \left[\frac{u_N}{U} \right]^+ \right\}. \quad (6.117)
\end{aligned}$$

These results are linked with the aerodynamic properties of a train in a tunnel. The so called pressure signature of a train, sketched in figure 6.5, where $\Delta p_N = p_N - p_a$ is obtained from (6.102) with $p_e = p_a$; $\Delta p_{FR} = p_e - p_s$ is obtained from (6.110); and Δp_T is obtained from (6.117). When the pressure signature is measured for a given train, the coefficients λ_{Tr} , C_{DN} and C_{DT} can be obtained.

Figure 6.5: Train pressure signature with constant U .

Aerodynamic Drag

The aerodynamic drag of the train can be obtained from the momentum integral equation applied to a control volume linked to the train between two tunnel sections upstream the nose and downstream the tail. The result is $R = (p_N - p_T)A_{Tu}$. The aerodynamic drag R , as mentioned before, can be written as $R = \frac{1}{2}C_D\rho_a U^2 A_{Tr}$ and the drag coefficient, according with (6.112) is given by

$$C_D = \left[C_{DN} + C_{DT} + \frac{\lambda_{Tr} L_{Tr}/D_{Tr}}{(1 - \beta)^3} \right] \left(1 - \frac{u_N}{U} \right)^2. \quad (6.118)$$

The drag coefficient is, in first approximation, the quantity inside the square bracket, that can easily duplicate or triplicate the C_D value in open air (Vardy and Reinke, 1999) and its dependence with the train speed is weak (Zhu et al., 2011), as shown in (6.118).

The energy equation in the gap

In the approximation of steady and incompressible flow around the train, continuity and momentum equations determine the pressure and velocity fields, which are uncoupled from the energy equation. Using the same control volume as for (6.118), and applying the integral energy equation to this volume (axis fixed on the train)

$$\int_{\Sigma} \rho(c_p T + \frac{1}{2}w^2)(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \int_{\Sigma} \vec{v} \cdot \tau' \vec{n} d\sigma - \int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma + \int_{\Omega} Q_r d\Omega, \quad (6.119)$$

where the kinetic terms from the left side of the total energy equations, $\frac{1}{2}w^2$, are equal in the entrance and the exit of the gap, so that the difference is equal to zero. Each term of equation (6.119) is

$$\begin{aligned} \int_{\Sigma} \rho(c_p T + \frac{1}{2}w^2)(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma &= \rho_a c_p (T_s - T_e) w (A_{Tu} - A_{Tr}), \\ \int_{\Sigma} \vec{v} \cdot \tau' \vec{n} d\sigma &= -U \frac{\tau_w T_u}{r_{hTu}} A_{Tu} L_{Tr}, \\ - \int_{\Sigma} \vec{q} \cdot \vec{n} d\sigma &= \frac{q_{Tu}}{r_{hTu}} A_{Tu} L_{Tr} + \frac{q_{Tr}}{r_{hTr}} A_{Tr} L_{Tr}, \\ \int_{\Omega} Q_r d\Omega &= Q_r (A_{Tu} - A_{Tr}). \end{aligned}$$

The work by viscosity forces, $-U \tau_w T_u A_{Tu} L_{Tr} / r_{hTu}$, and the heat transfer to the tunnel wall, $q_{Tu} A_{Tu} L_{Tr} / r_{hTu}$ can be neglected when compared with the heat transfer on the train wall, $q_{Tr} A_{Tr} L_{Tr} / r_{hTr}$. As such (6.119) takes the form

$$\rho_a c_p (T_s - T_e) w (A_{Tu} - A_{Tr}) = \frac{q_{Tr}}{r_{hTr}} A_{Tr} L_{Tr} + Q_r (A_{Tu} - A_{Tr}) L_{Tr},$$

with $q_{Tr} A_{Tr} L_{Tr} / r_{hTr} = \rho_a (\lambda_{Tr} A_{Tr} L_{Tr} / 8r_{hTr}) w c_p (T_{wTr} - T)$, so that

$$\frac{T_e - T_s}{T_a} = -\frac{\lambda_{Tr} L_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) \frac{T_{wTr} - T}{T_a} - \frac{L_{Tr}}{\rho_a c_p T_a} \left(\frac{1 - \beta}{U - u_N} \right) Q_r, \quad (6.120)$$

where the relation $w = (U - u_N) / (1 - \beta)$ has been used. If the train wall is higher than the air temperature, $T_{wTr} > T$, then the temperature at the exit of the gap will be higher than at the entry, $T_s > T_e$.

Subtracting equation (6.109) from (6.105) leads to

$$\begin{aligned} \frac{T_N - (T_e - T_s) - T_T}{T_a} &= \frac{\gamma - 1}{2} \beta M^2 \left[\frac{2 - \beta}{(1 - \beta)^2} \right] \left(1 - \frac{u_N}{U} \right)^2 \\ &\quad - \frac{\gamma - 1}{2} \beta M^2 \left[\frac{2 - \beta}{(1 - \beta)^2} \right] \left(1 - \frac{u_T}{U} \right)^2 = 0, \end{aligned} \quad (6.121)$$

since $u_N = u_T$. This means that

$$\frac{T_N - T_T}{T_a} = \frac{T_e - T_s}{T_a} = -\frac{\lambda_{Tr} L_{Tr}}{8r_{hTr}} \left(\frac{\beta}{1 - \beta} \right) \frac{T_{wTr} - T}{T_a} - \frac{L_{Tr}}{\rho_a c_p T_a} \left(\frac{1 - \beta}{U - u_N} \right) Q_r, \quad (6.122)$$

which means that the temperature increment between the nose and the tail of the train is mainly caused by the heat transfer with the train wall and the external heat coming from the train (AC, heat losses from the electric engines, etc.).

6.6 Discretized equations for the numerical model

It is possible to make a computer program that take into account all of the effects (compressibility, propagation along the characteristic lines $dx/dt = u \pm a$ and $dx/dt = u$, etc.) by using numerical methods such as finite volumes with TVD schemes; nevertheless a more simple program that takes into account the most relevant physical trends, as explained in the precedent sections, can be done. The characteristic equations to be solved for the tunnel alone are (6.56) to (6.58) with

$$q_{Tu} = \frac{\lambda_{Tu}}{8} \rho_a |u| [c_p (T_{wTu} - T)], \quad (6.123)$$

and

$$\tau_{Tu} = \frac{\lambda_{Tu}}{8} \rho_a u |u|. \quad (6.124)$$

The characteristic equations to be solved for the gap between train and tunnel are (6.95) to (6.97) with

$$\frac{a \tau_f \ell_p}{A_{Tu} - A_{Tr}} = \frac{\lambda_{Tr}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \rho_a a (U - u) |U - u|, \quad (6.125)$$

and

$$\frac{\dot{Q}}{(A_{Tu} - A_{Tr}) L_{Tr}} = \frac{\lambda_{Tr}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \rho_a |U - u| [c_p (T_{wTr} - T) + (U - u)^2] + Q_r, \quad (6.126)$$

which takes into account that the friction effects and heat transfer on the tunnel wall can be neglected when compared with the effects on the train wall.

Using the following non-dimensional variables

$$\tau = \frac{ta}{L_{Tu}}, \quad \xi = \frac{x}{L_{Tu}}, \quad \varphi = \frac{p - p_a}{p_a}, \quad V = \frac{u}{a}, \quad \theta = \frac{T - T_a}{T_a}, \quad M_T = \frac{U}{a}, \quad (6.127)$$

and considering that $T_{wTr} = T_{wTu} = T_a$, turns equations (6.56)-(6.58) into

$$\frac{d}{d\tau}(\varphi + \gamma V) = -\frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V + \theta) \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = 1, \quad (6.128)$$

$$\frac{d}{d\tau}(\varphi - \gamma V) = \frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V - \theta) \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = -1, \quad (6.129)$$

$$\frac{d}{d\tau} \left(\varphi - \frac{\gamma}{\gamma - 1} \theta \right) = \frac{\gamma \lambda_{Tu} L_{Tu}}{2(\gamma - 1)D_{Tu}} |V|\theta \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = V, \quad (6.130)$$

and turns equations (6.95)-(6.97) into

$$\begin{aligned} \frac{d}{d\tau}(\varphi + \gamma V) = & \frac{\lambda_{Tr} L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \gamma |M_T - V| \{ (\gamma - 1)(M_T - V)^2 + (M_T - V) - \theta \} \\ & + \frac{(\gamma - 1)L_{Tu}Q_r}{ap_a} \end{aligned} \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = 1, \quad (6.131)$$

$$\begin{aligned} \frac{d}{d\tau}(\varphi - \gamma V) = & \frac{\lambda_{Tr} L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \gamma |M_T - V| \{ (\gamma - 1)(M_T - V)^2 - (M_T - V) - \theta \} \\ & + \frac{(\gamma - 1)L_{Tu}Q_r}{ap_a} \end{aligned} \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = -1, \quad (6.132)$$

$$\begin{aligned} \frac{d}{d\tau} \left(\varphi - \frac{\gamma}{\gamma - 1} \theta \right) = & \frac{\lambda_{Tr} L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \gamma |M_T - V| \left\{ \frac{\theta}{\gamma - 1} - (M_T - V)^2 \right\} \\ & - \frac{L_{Tu}Q_r}{ap_a} \end{aligned} \quad \text{along the lines} \quad \frac{d\xi}{d\tau} = V. \quad (6.133)$$

For discretization of equations (6.128)-(6.133) an explicit Euler method is used, with the grid shown in figure 6.6. The time integration is done with two substeps which ads

stability to the code. The steepness of C^+ (going from $(i-1, n)$ to $(i, n+1)$ and from (i, n) to $(i, n+1/2)$) and C^- (going from $(i+1, n)$ to $(i, n+1)$, from (i, n) to $(i-1, n+1/2)$) is constant and equal to 1 and -1, respectively; but for C^0 (going from (i', n) to $(i, n+1)$) the steepness varies with V and, as such, the values in (i', n) and $(i', n+1/2)$ are linearly interpolated along space using the values at nodes $i-1$ and i (to start the interpolation an iterative method of two iterations is used assuming in the first iteration that $V_{i',n} = V_{i,n}$).

Figure 6.6: Scheme of the grid used for the numerical model.

To obtain the values at $(i, n+1/2)$ for the tunnel without train the discretized equations are

$$\varphi_{i,n+1/2} + \gamma V_{i,n+1/2} = \varphi_{i,n} + \gamma V_{i,n} - \frac{\Delta\tau}{2} \left[\frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V + \theta) \right]_{i,n}, \quad (6.134)$$

$$\varphi_{i,n+1/2} - \gamma V_{i,n+1/2} = \varphi_{i+1,n} - \gamma V_{i+1,n} + \frac{\Delta\tau}{2} \left[\frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V - \theta) \right]_{i+1,n}, \quad (6.135)$$

To obtain the values at $(i, n+1)$ the discretized equations for the tunnel are

$$\begin{aligned} \varphi_{i,n+1} + \gamma V_{i,n+1} &= \varphi_{i-1,n} + \gamma V_{i-1,n} - \Delta\tau \left[\frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V + \theta) \right]_{i-1,n+1/2} \\ &\text{along the lines } \xi_i - \xi_{i-1} = \Delta\tau, \end{aligned} \quad (6.136)$$

$$\begin{aligned} \varphi_{i,n+1} - \gamma V_{i,n+1} &= \varphi_{i+1,n} - \gamma V_{i+1,n} + \Delta\tau \left[\frac{\gamma \lambda_{Tu} L_{Tu}}{2D_{Tu}} |V|(V - \theta) \right]_{i,n+1/2} \\ &\text{along the lines } \xi_i - \xi_{i+1} = -\Delta\tau, \end{aligned} \quad (6.137)$$

$$\varphi_{i,n+1} - \frac{\gamma}{\gamma-1}\theta_{i,n+1} = \varphi_{i',n} - \frac{\gamma}{\gamma-1}\theta_{i',n} + \Delta\tau \left[\frac{\gamma\lambda_{Tu}L_{Tu}}{2(\gamma-1)D_{Tu}}|V|\theta \right]_{i',n+1/2} \quad \text{along the lines} \quad \xi_i - \xi_{i'} = V_{i',n}\Delta\tau. \quad (6.138)$$

The intermediate values, $(i, n + 1/2)$, for the discretized equations on the gap between train and tunnel are computed analogously to (6.134) and (6.135), and the values at $(i, n + 1)$ are obtained using

$$\begin{aligned} \varphi_{i,n+1} + \gamma V_{i,n+1} &= \varphi_{i-1,n} + \gamma V_{i-1,n} \\ &+ \Delta\tau \left[\frac{\lambda_{Tr}L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1-\beta} \right) \gamma |M_T - V| \{ (\gamma-1)(M_T - V)^2 \right. \\ &\quad \left. + (M_T - V) - \theta \} + \frac{(\gamma-1)L_{Tu}Q_r}{ap_a} \right]_{i-1,n+1/2} \quad \text{along the lines} \quad \xi_i - \xi_{i-1} = \Delta\tau, \end{aligned} \quad (6.139)$$

$$\begin{aligned} \varphi_{i,n+1} - \gamma V_{i,n+1} &= \varphi_{i+1,n} - \gamma V_{i+1,n} \\ &+ \Delta\tau \left[\frac{\lambda_{Tr}L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1-\beta} \right) \gamma |M_T - V| \{ (\gamma-1)(M_T - V)^2 \right. \\ &\quad \left. - (M_T - V) - \theta \} + \frac{(\gamma-1)L_{Tu}Q_r}{ap_a} \right]_{i,n+1/2} \quad \text{along the lines} \quad \xi_i - \xi_{i+1} = -\Delta\tau, \end{aligned} \quad (6.140)$$

$$\begin{aligned} \varphi_{i,n+1} - \frac{\gamma}{\gamma-1}\theta_{i,n+1} &= \varphi_{i',n} - \frac{\gamma}{\gamma-1}\theta_{i',n} \\ &+ \Delta\tau \left[\frac{\lambda_{Tr}L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1-\beta} \right) \gamma |M_T - V| \left\{ \frac{\theta}{\gamma-1} \right. \right. \\ &\quad \left. \left. - (M_T - V)^2 \right\} - \frac{L_{Tu}Q_r}{ap_a} \right]_{i',n+1/2} \quad \text{along the lines} \quad \xi_i - \xi_{i'} = V_{i',n}\Delta\tau. \end{aligned} \quad (6.141)$$

The subindex in the last term on the right of equations (6.136)-(6.141) means that the variables must be evaluated in that node.

For the pressure and velocity jumps on the nose of the train

$$\varphi_e = \varphi_N - \frac{\gamma}{2}\beta \left[C_{DN} + \frac{2}{1-\beta} \right] (V_N - M_T)^2, \quad (6.142)$$

$$V_e = M_T + \frac{V_N - M_T}{1 - \beta}, \quad (6.143)$$

and for the jumps on the tail

$$\varphi_T = \varphi_s + \frac{\gamma}{2}\beta \left[\frac{2}{1 - \beta} - C_{DT} \right] (V_T - M_T)^2, \quad (6.144)$$

$$V_T = M_T + (V_s - M_T)(1 - \beta). \quad (6.145)$$

In equations (6.142) and (6.143) the subindex e indicates the values on the node closest to the nose from the left, and the subindex N indicates the values on the node closest to the right. In a similar way, in equations (6.144) and (6.145) the subindex T indicates the values on the node closest to the tail from the left, and the subindex s indicates the values on the node closest to the right.

For the case where the flow in the gap between train and tunnel is considered incompressible equations (6.139)-(6.141) are not computed. Instead the following expressions are used

$$\varphi_e - \varphi_s = \frac{\gamma \lambda_{Tr} L_{Tr}}{2D_{Tr}} \left[\frac{\beta}{(1 - \beta)^3} \right] (M_T - V_N)^2, \quad (6.146)$$

$$V_e = V_s, \quad (6.147)$$

as if the information traveled instantaneously through the gap, which is a reasonable approximation if the train length is very small when compared with the tunnel length.

For the first instant of the entrance of the train the pressure on the second spatial node (the immediate to the right of the train nose) is calculated with (6.142)

$$\varphi_{2,2} = \varphi_e + \frac{\gamma}{2}\beta \left[C_{DN} + \frac{2}{1 - \beta} \right] (V_N - M_T)^2, \quad (6.148)$$

where $\varphi_e = 0$, since the gap between train and tunnel is open to the atmosphere, and $V_N = 0$, which is an approximation to start the program since $V_N \ll M_T$. The velocity at the exit is obtained from (6.143) with $V_e = V_{1,2}$. When the train is entering the flow between train and tunnel exits at the entrance tunnel, so that $\varphi_{1,n+1} = 0$. The velocity $V_{1,n+1}$ is found by using C^- , that is from (6.140),

$$\begin{aligned} V_{1,n+1} = & \frac{1}{\gamma}(\varphi_{1,n+1} - \varphi_{2,n}) + V_{2,n} \\ & - \frac{\Delta\tau}{\gamma} \left[\frac{\lambda_{Tr} L_{Tu}}{2D_{Tr}} \left(\frac{\beta}{1 - \beta} \right) \gamma |M_T - V| \{ (\gamma - 1)(M_T - V)^2 \right. \\ & \left. - (M_T - V) - \theta \} + \frac{(\gamma - 1)L_{Tu}Q_r}{ap_a} \right]_{1,n+1/2}. \end{aligned} \quad (6.149)$$

The moment the tail enters the pressure in the immediate node to the right of the tail is obtained from (6.144)

$$\varphi_{2,n+1} = \varphi_T - \frac{\gamma}{2}\beta \left[\frac{2}{1-\beta} - C_{DT} \right] (V_T - M_T)^2, \quad (6.150)$$

where in first approximation $\varphi_T = \varphi_{1,n+1} \approx 0$ and $V_T = V_{1,n+1} \approx M_T + (V_{2,n} - M_T)(1-\beta)$. The exit of the train from the tunnel is obtained in an analogous way.

When the train is inside the tunnel the boundary conditions at the entrance portal are in first approximation

$$\begin{aligned} \varphi_{1,n+1} &= -\frac{1}{2}\gamma V_{1,n}^2 \quad \text{if } V_{1,n} > 0 \\ &= 0 \quad \text{if } V_{1,n} < 0, \\ V_{1,n+1} &= \frac{1}{\gamma}(\varphi_{1,n+1} - \varphi_{2,n}) + V_{2,n} - \frac{\Delta\tau}{\gamma} \left[\frac{\gamma\lambda_{Tr}L_{Tr}}{2D_{Tr}}|V|(V - \theta) \right]_{1,n+1/2}, \\ \theta_{1,n+1} &= 0 \text{ if } V_{1,n} > 0, \text{ and } \theta_{1,n+1} = \theta_{1,n} \text{ if } V_{1,n} < 0 \end{aligned} \quad (6.151)$$

and assuming that the exit portal is higher than the entry portal by a height H_{Tu} , the at the exit portal

$$\begin{aligned} \varphi_{L,n+1} &= -\gamma g \frac{H_{Tu}}{a^2} \quad \text{if } V_{L,n} > 0 \\ &= -\frac{1}{2}\gamma V_{L,n}^2 - \gamma g \frac{H_{Tu}}{a^2} \quad \text{if } V_{L,n} < 0, \\ V_{L,n+1} &= -\frac{1}{\gamma}(\varphi_{L,n+1} - \varphi_{L-1,n}) + V_{L-1,n} - \frac{\Delta\tau}{\gamma} \left[\frac{\gamma\lambda_{Tr}L_{Tr}}{2D_{Tr}}|V|(V + \theta) \right]_{L-1,n+1/2} \\ \theta_{L,n+1} &= 0 \text{ if } V_{L,n} < 0, \text{ and } \theta_{L,n+1} = \theta_{L,n} \text{ if } V_{L,n} > 0. \end{aligned} \quad (6.152)$$

6.7 Comparison with experimental data

The results obtained with this program have been compared with field measurements from Woods and Pope (1981) in figures 6.7 and 6.8, William-Louis and Tournier (2005) in figures 6.9 and 6.10, and scale model experiment from Yoon et al. (2001) in figure 6.11 and Ricco et al. (2007) in figures 6.12 and 6.13. In figure 6.7 the velocity signal agrees well with the numerical data of Woods and Pope (1981) and presents some slight variations with respect to the experimental data. In figure 6.8 the pressure signal agrees well with the numerical and experimental data until the eight second. The variation with the experimental data was acknowledged by Woods and Pope (1981). They noticed that with longer trains there was a slight improvement in the theoretical results, concluding

that the increase in the train length made the three-dimensional effects produced by the nose and tail less important in relation to the train as a whole, so that the flow conditions were more faithfully reproduced by a one-dimensional model. It must be remarked that the theoretical model from Woods and Pope (1981) is more complex and complicated to implement than the one proposed by the theoretical approach and numerical scheme of the present work, providing very similar results, which gives a validation for it. The values of λ , C_{DN} and C_{DT} were obtained from the reference using their equivalent relations for the pressure loss on the head and tail of the train. Figures 6.9 and 6.10 show good agreement with the experimental data, particularly for the peaks and troughs of the pressure and the damping of the waves, proving that it can capture the general information of the system. Figure 6.11 presents a reasonable agreement with the experimental data. Figures 6.12 and 6.13 show a good correlation with the peaks and troughs of the data of Ricco et al. (2007), however, at instants 0.025 s and 0.06 s of figure 6.12 there can be seen a peak that the present model does not accurately reproduce. This is caused by massive flow separation for cases where the train model has a steep front, acknowledged by Ricco et al. (2007). Since the model proposed tries to be as simple as it can, local behavior around head and tail of the train, or the steepness of the waves are out of the capacities of a scheme based on the characteristics method. Nevertheless, in cases where the train has a streamlined head, flow separation is almost negligible (Schetz, 2001; Shuanbao et al., 2014; Ji-qiang et al., 2017). As such, the data where the train generates notorious flow separation is not representative of what it is intended to model.

Even though the present approach considers abrupt changes in pressure that really are smoother (hence the discrepancies with the real data) the peaks and troughs of the pressure waves are computed with a good approximation, making this model valid to obtain general data on real train tunnels with lengths of kilometers, as well as on scale models with tunnel lengths of the order of meters. Discrepancies in figures 6.9, 6.10, 6.11, 6.12 and 6.13 are due to the models given to the nose and tail in section 6.5.2 that reduces these interfaces to one discontinuity with abrupt pressure changes, as it is shown in equations 6.102 and 6.108.

The aim of our approach is to be able to capture the general information of the flow inside the tunnel for problems such as the temperature rise in long tunnel walls over long periods. For problems like these there is a huge amount of train runs, with each one requiring a computation to obtain pressure, velocity and temperature. As such, fast calculations made with the robust theoretical approach in the present work can be very useful. As mentioned in section 6.6, the characteristic method was used to solve the equations, but other methods can be used. The aim of these comparisons is to validate the theoretical approach taken, not a particular numerical scheme.

Figure 6.7: Velocity signal at 300 m from the entry portal compared with experimental data from Woods and Pope (1981). Train entry velocity 124 km/h.

Figure 6.8: Pressure signal at 500 m from the entry portal compared with experimental data from Woods and Pope (1981). Train entry velocity 124 km/h.

Figure 6.9: Pressure signal at a fixed section compared with experimental data from William-Louis and Tournier (2005). Tunnel length of 1500 m. Train velocity of 298 km/h.

Figure 6.10: Pressure signal 72 m after the train head compared with experimental data from William-Louis and Tournier (2005). Tunnel length of 1500 m. Train velocity of 298 km/h.

Figure 6.11: Pressure signal at a fixed section compared with experimental data from Yoon et al. (2001). Measurement at 1.8 m from the entry portal. Tunnel length of 7.6 m. Train velocity of 300 km/h.

Figure 6.12: Pressure signal at a fixed section compared with experimental data from Ricco et al. (2007). Measurement at 0.9 m from the entry portal. Tunnel length of 6 m. Train velocity of 110 km/h.

Figure 6.13: Pressure signal at a fixed section compared with experimental data from Ricco et al. (2007). Measurement at 0.9 m from the entry portal. Tunnel length of 6 m. Train velocity of 153 km/h.

6.8 Comparison with the infinitely long tunnel solution

Using this numerical code for the case where the parameter $\ell_c = (2D_{Tu}/\alpha\lambda_{Tu})$ (a^2/Uu_N) is very small when compared with L_{Tu} leads for the asymptotic solution of the infinitely long tunnel explained on section 6.4.4. Figures 6.14 and 6.15 show the numerical data compared with the asymptotic pressure solution for an infinitely long tunnel from equation (6.70), $p/p_a = 1 + \frac{\gamma}{\alpha} \frac{u_N}{U} \tilde{\omega}$, for the adiabatic case ($\alpha = 1$) and a value $\ell_c/L_{Tu} = 0.02$ and $\ell_c/L_{Tu} = 0.002$ respectively. It can be seen that the present numerical code and the asymptotical behavior of the infinitely long tunnel solution tend to the same form as ℓ_c/L_{Tu} gets smaller. The difference between the numerical code and the asymptotic solution will depend on the value of ℓ_c/L_{Tu} ; larger values of this parameter means that the numerical code computes a scenario where the asymptotic solution of (6.70) does not apply; on the other hand, smaller values of ℓ_c/L_{Tu} means that the data from the numerical code gets closer to a case where the approximation of the infinitely long tunnel is valid. As can be seen from the definition of ℓ_c this asymptotical behavior depends not only on the length of the tunnel but also on other parameters such as the tunnel diameter, the friction coefficient at the tunnel wall, the train velocity, etc. In this scenario the waves have been damped and pressure changes are only gradual. For both figures 6.14 and 6.15 the data is presented dimensionless and along a spatial axis that moves with the train.

Figure 6.14: Comparison of the numerical dimensionless pressure with the asymptotical solution of equation (6.70) for $\ell_c/L_{Tu} = 0.02$.

Figure 6.15: Comparison of the numerical dimensionless pressure with the asymptotical solution of equation (6.70) for $\ell_c/L_{Tu} = 0.002$.

6.9 Temperature distribution inside the tunnel

The numerical code proposed can be utilized to obtain results to study the heating with time of long tunnels, an increasingly important problem.

In figures 6.16, 6.17 and 6.18 the pressure, velocity and temperature distributions are represented along a tunnel of 30 km in length, at the moment the train leaves the tunnel. At the end of the tunnel there are abrupt changes in pressure, velocity and temperature due to the interaction between the train and the exit portal that generates a compression wave (when the head exits the tunnel) and an expansion wave (when the tail exits the tunnel) which travel along the tunnel back and forth at the speed of sound. As it can be seen in the figures, the difference between isotherm and adiabatic tunnel wall temperature are inappreciable for pressure and velocities because the continuity and momentum equations are uncoupled from the energy equation. Nevertheless, for the temperature there is an important difference between the adiabatic and the isotherm tunnel wall cases; the isotherm tunnel wall temperature implies that the air in the tunnel reaches the wall temperature, but in the case of adiabatic tunnel wall temperature, there is a variation of air temperature along the tunnel, $\frac{\Delta T}{T_a} \approx \frac{\gamma}{\gamma-1} \frac{\Delta p}{p_a}$, as can be seen in figure 6.18.

When the external air is entering from the tunnel entry section, air at ambient temperature is moving into the tunnel. A contact surface, moving with the air velocity, separates the air at ambient temperature from the air heated by the train (because the majority of the cooling is caused by convection, not by diffusion, hence the abrupt change). The contact surface arrives up to 2150 m at the instant $t = 1087.22$ s (the train is located at 60 km away from the tunnel exit), as can be seen in figure 6.19; abrupt changes can

be seen on the right hand as well. These are the temperature increments caused by the compression wave generated when the train nose leaves the tunnel. For each time shown, the wave has traveled to the tunnel entry, reflected back to the exit and then reflected again, traveling more than 61.6 km (between every time registered there are 181.2 s), so that it appears a little bit moved from the position where it was last registered and appearing more damped as time passes.

Figures 6.18 and 6.19 show small temperature perturbations near positions 12000 m, 20000 m and 24000 m, which, as figure 6.19 shows, are being propagated at the speed of the air velocity. These small jumps are due to entropy changes. The entropy is altered by the interaction of compression and expansion waves with the train, and this changes propagate at the air velocity u . This behaviour appears regardless of the different number of nodes, as figure 6.20 reveals for the particular spatial distribution of temperature at $t = 362$ s. It can be seen that despite the number of nodes in space the position of the perturbation is the same. The position of these changes coincide with the points where the reflected compression and expansion waves reach the train, that is 11971 m, 19744 m and 23782 m, at times 144 s, 237 s and 285 s, respectively, as shown in figure 6.21.

The air temperature in the adiabatic tunnel wall case decreases along the tunnel. This is because the entropy increment is zero upstream the train and constant downstream the train. The temperature is given by

$$\frac{T - T_a}{T_a} = \frac{1}{\gamma} \left[\frac{\Delta S}{c_v} + (\gamma - 1) \frac{p - p_a}{p_a} \right],$$

and, decreases for increasing negative values of $p - p_a$, as given in figure 6.16. When time increases, the temperature also increases because the entropy increment ΔS remains constant but the pressure recovers the ambient value slowly. This effect can be seen in figure 6.19 where the temperature distribution along the tunnel is plotted for several instants once the train leaves the tunnel. External radiated heat from the AC units, electric engines losses, etc., was not considered for these cases. Note that when the train leaves the tunnel, the air and tunnel wall temperatures (because the wall is considered adiabatic) approach to a nearly constant value (see figure 6.19) except for a distance from the entry that is small when compared to the tunnel length.

This new level of tunnel temperature is the one seen by the next train to cross the tunnel. This new train increases the tunnel temperature and the process is repeated with the following trains travelling inside the tunnel. It is clear that there is a conduction process in the solid material (rock) that surrounds the tunnel wall. The thermal penetration in the tunnel rock, $\ell_R \sim \sqrt{\alpha_R t}$, is of the order of centimeter during the time when a train passes, t (in the last expression α_R is the thermal diffusivity of the rock). This justifies the assumption of an adiabatic wall during the passing of trains; nevertheless, the conduction in the rock takes place over longer times, and the problem has two time scales that will be considered on chapter 8.

Figure 6.16: Pressure distribution along the tunnel, when the train is at the exit portal. Train velocity is 300 km/h.

Figure 6.17: Air velocity distribution along the tunnel, when the train is at the exit portal. Train velocity is 300 km/h.

6.10 A chimney in a long tunnel

The effects of a vertical chimney located at the middle of a long tunnel are studied, looking to find a passive measure for removing some of the heat accumulated during the pass of a high-speed train in the tunnel. The governing equations for the chimney are

Figure 6.18: Temperature distribution along the tunnel, when the train is at the exit portal. Train velocity is 300 km/h.

Figure 6.19: Temperature distribution along the tunnel at different times measured from the instant of the train entering the tunnel. Train velocity is 300 km/h.

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial z} + v \frac{\partial \rho}{\partial z} = 0, \quad (6.153)$$

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial z} + \frac{\partial p}{\partial z} = -\rho g - \frac{4\tau_{ch}}{D_{ch}}, \quad (6.154)$$

Figure 6.20: Spatial distributions of temperature at $t = 362$ s for different number of nodes in the simulation.

Figure 6.21: Schematic xt plane of the initial compression and expansion waves, their reflections and the train trajectory.

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p v \frac{\partial T}{\partial z} = \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial z} + \frac{4v\tau_{ch}}{D_{ch}} + \frac{4q_{ch}}{D_{ch}}, \quad (6.155)$$

where z is the distance on the vertical direction, D_{ch} the hydraulic diameter at the chimney, and τ_{ch} and q_{ch} are friction and heat transfer at the chimney wall, respec-

tively. In appendix B is shown that the characteristic velocity at the chimney is of order $v_c \sim \sqrt{gH\Delta T/T}$, where H is the height of the chimney, so that the order of the work done by viscous forces over the heat transfer term on the energy equation (6.155) is

$$\frac{v_c \tau_{ch}}{q_{ch}} \sim \frac{v_c^3}{v_c c_p \Delta T} \sim \frac{v_c^2}{a^2 \Delta T/T} \sim \frac{gH\Delta T/T}{a\Delta T/T} \sim \frac{gH}{a^2} \ll 1,$$

so that the viscous work can be neglected from the energy equation. For this approximation the speed of sound, a , is considered constant, with the air flow velocity being much smaller than a . The density is considered constant everywhere except at the gravitational term in the momentum equation, since this will be the cause for movement when temperature rises (Boussinesque approximation). With that in mind, the governing equations can be transformed into

$$\frac{dp}{dt} + \rho_a a \frac{dv}{dt} = \frac{4(\gamma - 1)q_{ch}}{D_{ch}} - \rho_a g - \frac{4a\tau_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = a, \quad (6.156)$$

$$\frac{dp}{dt} - \rho_a a \frac{dv}{dt} = \frac{4(\gamma - 1)q_{ch}}{D_{ch}} + \rho_a g + \frac{4a\tau_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = -a, \quad (6.157)$$

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = \frac{4v\tau_{ch}}{D_{ch}} + \frac{4q_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = v, \quad (6.158)$$

where the density for the gravitational term is obtained by use of the state equation $\rho = p/(R_g T)$.

In order to couple the chimney with the tunnel, the control volume shown in figure 6.22 is used to analyze the integral equations of continuity, momentum and energy. Section 1 and 2 are in the tunnel, while section three represents the base of the chimney.

Given that the Boussinesq approximation is being used, as well as a very small size compared to the total tunnel volume, the flow can be considered as almost incompressible in this region, so that the integral continuity equation turns into

$$u_1 A_{Tu} = u_2 A_{Tu} + v_{ch} A_{ch}, \quad (6.159)$$

where u_1 and u_2 are flow velocities in the tunnel in sections 1 and 2 respectively, and v_{ch} is the flow velocity at the base of the chimney (section 3).

From the integral momentum equation, the temporal term is of order

$$\frac{d}{dt} \int_{\Omega} \rho \vec{v} d\Omega \sim \frac{\rho_a \tilde{u}_c}{t_c} A_{Tu} D_{ch} \sim \frac{\rho_a \tilde{u}_c U^2}{L_{Tu} \tilde{u}_c} A_{Tu} D_{ch},$$

where $\tilde{u}_c \sim U \sqrt{L_{Tr}/L_{Tu}}$, as mentioned before in section 6.4.3, while the convective term is of order

$$\int_{\Sigma_{Tu}} \rho \vec{v} (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma \sim \rho_a \tilde{u}_c^2 A_{Tu},$$

the pressure term is

Figure 6.22: Sketch of the control volume for the coupling between the chimney and the tunnel.

$$\int_{\Sigma_{Tu}} p \vec{n} d\sigma \sim \Delta p A_{Tu},$$

and the friction term is

$$\int_{\Sigma_{Tu}} \tau' \cdot \vec{n} d\sigma \sim \frac{\lambda A_{Tu} D_{ch} \rho_a \tilde{u}_c^2}{D_{Tu}}.$$

The temporal term over the convective term is of order

$$\frac{\rho_a U^2 A_{Tu} D_{ch}}{L_{Tu} \rho_a \tilde{u}_c^2 A_{Tu}} \sim \frac{U^2 D_{ch}}{\tilde{u}_c^2 L_{Tu}} \sim \frac{L_{Tu}}{L_{Tr}} \frac{D_{ch}}{L_{Tu}} \sim \frac{D_{ch}}{L_{Tr}} \ll 1.$$

The pressure term must be of the same order as the convective, $\Delta p \sim \rho_a \tilde{u}_c^2$, and the friction term over the convective one is of order

$$\frac{\lambda A_{Tu} D_{ch} \rho_a \tilde{u}_c^2}{D_{Tu} \rho_a \tilde{u}_c^2 A_{Tu}} \sim \frac{\lambda D_{ch}}{D_{Tu}} \ll 1.$$

As such, the integral momentum equation projected in the axial direction of the tunnel leads to

$$p_1 - p_2 = \rho_a (u_2^2 - u_1^2) + \rho_a v_{ch} u^* \frac{A_{ch}}{A_{Tu}}, \quad (6.160)$$

where u^* is the axial velocity at the chimney's base, and can be approximated as the mean of u_1 and u_2 .

The integral momentum equation projected in the z direction leads to

$$p_3 - p_1 = p_b - p_1 - \rho v_{ch}^2, \quad (6.161)$$

where p_b is the pressure at the bottom of the tunnel, contrary to section 3, and it can be approximated by the mean of p_1 and p_2 .

The value of the temperature in the sections will depend on the direction of the flow velocity. If one flow is entering in the control volume and two are exiting, then the value of the temperature in the three sections is the same as the one where the flow is entering; if two flows are entering the control volume and one is exiting it, then the temperature in the section where the flow is leaving is found by means of the equation

$$T_1 u_1 A_{Tu} = T_2 u_2 A_{Tu} + T_3 v_{ch} A_{ch}, \quad (6.162)$$

which comes from the integral energy equation by neglecting the temporal, pressure, viscous and heat transfer terms, and the kinetic energy when compared with the internal energy. The numerical scheme for solving the governing equations in characteristic form, together with the coupling equations is explained in appendix A.

For the exit of the chimney, the boundary conditions are analogous to what was done with the tunnel portals, except here the atmospheric pressure is $p_a - \rho_a g H$.

Figures 6.23, 6.24 and 6.25 show the mean velocity, pressure, and temperature, respectively, against time for a long tunnel (30 km). Each figure compares a case with and without chimney; the chimney has 100 m of height and 15 m² of area. The overall pressure diminishes with the presence of a chimney (figure 6.24). The velocity is less when there is a chimney until around the second 400, when the train leaves the tunnel, and the behaviour reverts once the chimney starts to generate a convective flow that increases the value of the velocity (figure 6.23).

Even though the velocity reduces with a chimney, which means less heat transfer to the wall, the convective flow caused by it reduces the mean temperature inside the tunnel. This means that the use of a chimney could be a passive solution not only for the reduction of the temperature rise, but also as a damper of the overall pressure peaks.

Figure 6.23: Comparison of the mean velocity against time inside a long tunnel for a case without chimney, and a case with a chimney at the center of the tunnel of 100 m of length and 15 m^2 of area.

Figure 6.24: Comparison of the mean pressure against time inside a long tunnel for a case without chimney, and a case with a chimney at the center of the tunnel of 100 m of length and 15 m^2 of area.

Figure 6.25: Comparison of the mean temperature against time inside a long tunnel for a case without chimney, and a case with a chimney at the center of the tunnel of 100 m of length and 15 m² of area.

Chapter 7

Prediction of the Sonic Boom

The present chapter focuses on the propagation and deformation of the first compression wave generated when a high speed train enters a tunnel. This first wave is a determining factor in shock wave formation inside the tunnel, and therefore in sonic boom phenomena. This is modelled with the one-dimensional equations for the flow in a tube, and an analogy that relates a piston inside a tube with the entering train, both opening the door to an algebraic formulation that delimits the problem of shock formation. The model is represented in a characteristic form to find the relation between the distance where the pressure wave becomes a shock wave (named the regression distance) and the parameters that define the initial wave profile, namely the maximum pressure increment and the maximum pressure gradient downstream the entry portal. Steady and unsteady wall friction and heat transfer effects are analysed. Differences between the regression distance for the cases with friction and heat transfer, and without them, seems to be delimited, which makes the algebraic formulation suited for fast decisions at the time of conceptual design of high-speed lines. Model validation with more complex models and experimental data is provided.

7.1 One-dimensional flow equations

The one- dimensional governing equations of continuity, momentum and energy for the flow in a pipe (characterized by $L \ll D$, being L the pipe length and D the equivalent diameter) without a radiation heat source and body forces (Shapiro, 1953; Woods and Pope, 1981; Raghunathan et al., 2002; Rivero et al., 2018)

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0, \quad (7.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{4\tau_w}{\rho D_{Tu}}, \quad (7.2)$$

$$c_p \rho \frac{\partial T}{\partial t} + c_p \rho u \frac{\partial T}{\partial x} - \frac{\partial p}{\partial t} - u \frac{\partial p}{\partial x} = \frac{4u\tau_w}{D_{Tu}} + \frac{4q_T}{D_{Tu}}, \quad (7.3)$$

where t is the time, x is the axial coordinate, u , ρ , p , and T are the velocity, density, pressure, and temperature of the fluid respectively, c_p is the specific heat at constant pressure of air, D_{Tu} is the hydraulic diameter of the tunnel, τ_w is the friction on the wall that is given by

$$\tau_w = \frac{\lambda}{8} \rho u |u| + \frac{4\rho\nu}{D_{Tu}} \int_0^t W(t-t^*) \frac{\partial u}{\partial t^*} dt^*, \quad (7.4)$$

and q_T is the heat transfer with the tunnel wall, according to the Reynolds analogy¹, which is given by

$$q_T = \frac{\lambda}{8} \rho c_p |u| (T_w - T), \quad (7.5)$$

where λ is the Darcy-Weisbach friction coefficient for steady flow, and T_w is the wall temperature, which for the present work is assumed to be constant, (Sadokierski and Thiffeault, 2008; Rivero et al., 2018). The convoluted integral in (7.4) represents the effect of the unsteady friction, a model first proposed by Zielke (1968) for laminar flow, and adapted for turbulent flow by Vardy and Brown (2002). The weight function shown here, $W(t)$, is for smooth pipes, which is valid since the case for validation has slab track in the tunnel, and is expressed as

$$W(\tau) = \frac{Ae^{-B\tau}}{\sqrt{\tau}}, \quad (7.6)$$

where $A = 1/(2\sqrt{\pi})$ and $B = \text{Re}^k/12.86$, with $k = \log_{10}(15.29\text{Re}^{-0.0567})$. To implement this model numerically consumes much memory and computational time, but Vitkovsky et al. (2004) approximated the weight function by using a sum of exponentials, which eliminates the problem caused by the convolution in the integral. They approximated the function by

$$W(\tau) \approx \sum_{k=1}^N A m_k^* e^{-(n_k^* - B)\tau}, \quad (7.7)$$

where the coefficients m_k^* and n_k^* are written in table 7.1. Equations (7.1) to (7.3) need to be supplemented with

$$p = (c_p - c_v)\rho T; \quad a^2 = \gamma \frac{p}{\rho}, \quad (7.8)$$

which are the state equation for ideal gases, and the relation to determine the speed of propagation, where c_v is the specific heat at constant volume, γ is the specific heat ration and a is the speed of sound. Equations (7.1) to (7.3) need three initial and boundary conditions, namely two independent thermodynamic variables, for example ρ and p , and the axial velocity u .

¹The Reynolds analogy was chosen for its simplicity and effectivity (Fukuda et al., 2006; Rivero et al., 2018), but other relations can be used, such as the Dittus-Boelter correlation.

k	n_k^*	m_k^*	τ_{mk}
1	4.78793	5.03362	-
2	51.0897	6.48760	3.20×10^{-2}
3	210.868	10.7735	8.70×10^{-3}
4	765.030	19.9040	2.44×10^{-3}
5	2731.01	37.4754	6.84×10^{-4}
6	9731.44	70.7117	1.92×10^{-4}
7	34668.5	133.460	5.39×10^{-5}
8	123511	251.933	1.51×10^{-5}
9	440374	476.597	4.20×10^{-6}
10	1590300	932.860	1.02×10^{-6}

Table 7.1: Coefficients for the Vitkovsky et al. (2004) approximation of the Vardy and Brown (2002) function for turbulent flow in smooth pipes. τ_{mk} is the minimum time the function can be computed with before it starts to diverge from the Vardy-Brown function.

The momentum equation in integral form applied to a control volume between two sections upstream and downstream the train provides (Rivero et al., 2018) that the increment on pressure Δp , multiplied by the tunnel area A_{Tu} is of the same order as the aerodynamic drag of the train: $\Delta p A_{Tu} \sim C_D \rho U^2 A_{Tr}$, where C_D is the drag coefficient, U is the train speed and A_{Tr} the transversal train area. From the above estimation $\Delta p \sim \rho U^2$, and then $\Delta p/p \sim (U/a)^2 \sim M^2 \ll 1$, where $M = U/a$ is the train Mach number, because $C_D(A_{Tr}/A_{Tu}) = \beta C_D \sim 1$ (Vardy and Reinke, 1999; Cross et al., 2015). At the same time, the pressure increment behind the wave is (Rivero et al., 2018) $\Delta p \sim \rho a u_c$ with u_c being the characteristic speed of air inside the tunnel. Then the characteristic value of the air speed is $u_c \sim U(U/a) \ll U$ or $u_c/a \sim (U/a)^2 \ll 1$. The increment of air temperature over the temperature is of order $\Delta T/T \sim \Delta p/p \ll 1$ (Rivero et al., 2018), so that on the right side of (7.3), the work produced by the viscous forces compared with the heat transfer on the wall is of order $u\tau_w/q_T \sim u^3/(uc_p\Delta t) \sim u^2/(a^2\Delta T/T) \sim u^2/U^2 \sim M^2 \ll 1$, since $c_p T \sim a^2$; this means that the work done by viscous forces can be neglected from (7.3). Equations (7.1)-(7.3) can be rearranged to be written in characteristic form as (Rotty, 1962; Courant and Friedrichs, 1976)

$$\frac{dp}{dt} + \rho a \frac{du}{dt} = -\frac{4a\tau_w}{D_{Tu}} + (\gamma - 1) \frac{4q_T}{D_{Tu}} \text{ along the lines } C^+: \frac{dx}{dt} = u + a, \quad (7.9)$$

$$\frac{dp}{dt} - \rho a \frac{du}{dt} = \frac{4a\tau_w}{D_{Tu}} + (\gamma - 1) \frac{4q_T}{D_{Tu}} \text{ along the lines } C^-: \frac{dx}{dt} = u - a, \quad (7.10)$$

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = \frac{4q_T}{D_{Tu}} \text{ along the lines } C^0: \frac{dx}{dt} = u. \quad (7.11)$$

7.2 Boundary condition at the entry section

The time required for the head of a high-speed train to fully penetrate the tunnel is of order of hundredths of a second and, during this time, the generated wave has propagated upstream a distance ℓ equivalent to several tunnel diameters. In this distance the effects of friction and heat conduction in the tunnel are negligible because $\lambda\ell/D_{Tu}$ is small. The wave pressure distribution can be assumed that it is generated by a piston with speed $u_p(t)$ along the x -axis. Because no friction or heat transfer effects are being considered at this stage, the right hand side of (7.11) is zero, and this equation is reduced to the entropy conservation that provides the homentropic relations

$$\frac{p}{p_a} = \left(\frac{\rho}{\rho_a}\right)^\gamma = \left(\frac{T}{T_a}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{a}{a_a}\right)^{\frac{2\gamma}{\gamma-1}}. \quad (7.12)$$

Equations (7.9) and (7.10) without the friction terms and with the help of equation (7.12), can be integrated to give (Shapiro, 1953; Courant and Friedrichs, 1976)

$$\frac{2a}{\gamma-1} + u = R^+; \text{ along the lines } C^+ : \frac{dx}{dt} = u + a, \quad (7.13)$$

$$\frac{2a}{\gamma-1} - u = R^-; \text{ along the lines } C^- : \frac{dx}{dt} = u - a, \quad (7.14)$$

where R^+ and R^- are constants along the lines C^+ and C^- respectively. The characteristic C^- starts from the initial condition where $a = a_a$ and $u = 0$ and therefore $R^- = 2a_a/(\gamma-1)$. When the line C^- arrives to the piston, where the velocity is the piston speed $u_p(t)$ and the sound speed at the piston face is $a_p(t)$, the equation (7.14) provides the relation: $2a_p/(\gamma-1) - u_p = 2a_a/(\gamma-1)$. Then, the relation between the piston velocity and the speed of sound is

$$u_p = \frac{2a_a}{\gamma-1} \left(\frac{a_p}{a_a} - 1 \right). \quad (7.15)$$

According to (7.12), the relation between the sound speed a_p and the pressure p_p at the piston face is $a_p/a_a = (p_p/p_a)^{(\gamma-1)/2\gamma}$, that substituted into (7.15) provides the relation between the piston velocity and the pressure at the piston face

$$u_p = \frac{2a_a}{\gamma-1} \left[\left(\frac{p_p}{p_a} \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right]. \quad (7.16)$$

Equations (7.12) and (7.16) represent the boundary conditions at the piston face as function of the pressure, which is assumed to be known.

7.3 Approximation of the initial pressure profile

In the system train-tunnel the initial pressure rise is caused by the entering of the train; its shape depends essentially of two parameters (Bellenoue et al., 2002; Anthoine, 2009): the

Figure 7.1: Sketch of the wave pressure distribution vs time.

initial maximum pressure increment, $(\Delta p)_{max}$, and the initial maximum pressure gradient, $(dp/dt)_{max}$, as can be seen in figure 7.1.

The magnitudes that define the tunnel and the train are included in these two parameters. For example, the $(\Delta p)_{max}$ can be approximated by the expression (Rivero et al., 2018)

$$\frac{(\Delta p)_{max}}{\rho_a U^2} \approx \frac{\beta}{2} \Gamma \left(1 - \beta \Gamma \frac{U}{a_a} \right),$$

where β is the blockage ratio (train to tunnel transverse areas) and $\Gamma = C_{DN} + 2/(1 - \beta)$, being C_{DN} the drag coefficient of the train nose. In the above expression, it can be seen that the value of $(\Delta p)_{max}$ depends on the train speed U , the train transversal area and the tunnel transversal area through β , the Mach number, U/a_a , of the train and the drag coefficient of the train nose. The value of the $(dp/dt)_{max}$ can be approximated by the expression (Vardy, 2008)

$$\left(\frac{\partial p}{\partial t} \right)_{max} = \frac{(\Delta p)_{max}}{T_{wave}},$$

where $T_{wave} \approx \phi D_{Tu}/U$ is the period that corresponds to the train entering a distance of the order of the tunnel diameter, D_{Tu} , and ϕ and empirical coefficient that ranges between $0.75 < \phi < 1.25$, and depends upon the level of the streamlining of the train nose and the tunnel entrance.

With these two values the pressure profile can be approximated with functions that come

close to the experimental (or numerically obtained). A profile with a hyperbolic tangent has been chosen (Miyachi et al., 2013), given its simplicity and good adjustment with the measured or CFD obtained profiles. Nevertheless, other mathematical functions can be used or even tabulated values taken from experimental or CFD results. The latter approach, tabulated profile from experimental data, will be compared with results from the hyperbolic tangent in the following sections. What is relevant to the initial pressure wave is the maximum pressure rise and the maximum pressure gradient, the hyperbolic tangent is only a way to write these parameters. Proposing

$$Z = \tanh(\xi) + 1; \text{ with } \xi = t/t_0 - \alpha, \quad (7.17)$$

where t_0 is the characteristic growing time of the pressure, and is determined with the value of the maximum pressure gradient $(dp/dt)_{max}$, and α is a parameter whose value allows to adjust the position of the profile with respect to the measured one or the one obtained with CFD.

The pressure profile can be approximated by the use of

$$\Delta p = p - p_a = \frac{1}{2}(\Delta p)_{max} Z(t). \quad (7.18)$$

Giving that

$$\frac{dp}{dt} = \frac{1}{2}(\Delta p)_{max} \frac{dZ}{dt}, \quad (7.19)$$

and since $(dZ/dt)_{max} = 1/t_0$, then

$$\left(\frac{dp}{dt}\right)_{max} = \frac{(\Delta p)_{max}}{2t_0}; \quad t_0 = \frac{(\Delta p)_{max}}{2(dp/dt)_{max}}. \quad (7.20)$$

As such, given $(\Delta p)_{max}$ and $(dp/dt)_{max}$ the value of t_0 is obtained from equation (7.20) and the pressure profile from (7.18).

Figure 7.2 shows the comparison between the pressure wave generated by a high-speed train entering a tunnel in Hieke et al. (2011) and the approximation previously given by equation (7.18). Although the profile of the hyperbolic tangent does not exactly reproduce the experimental results, it suffices that the two parameters, $(\Delta p)_{max}$ and $(dp/dt)_{max}$, are well represented, in order to obtain good results of the propagation of the wave and the formation of the weak shock wave. In this way, a linear profile that connects the minimum and the maximum pressures with the maximum slope, also gives good results, as will be seen later.

7.4 Isentropic algebraic solution

The friction and heat transfer terms of equations (7.9) to (7.11) could be negligible if $(\lambda L/D_{Tu})(u_c/a_a)$ were small. But in fact, for tunnels of length of the order of 10 kilometers (in which the sonic boom normally occurs), the friction terms in (7.9) and (7.10) and the heat transfer term in (7.11) are comparable to those of the first member and,

Figure 7.2: Comparison of the initial pressure profile of Hieke et al. (2011) with an approximate hyperbolic tangent profile.

therefore, they cannot be neglected. However, the solution obtained without these effects is mathematically simple and allows to physically explain the phenomenon of the generation of the weak shock wave.

The motion equations without friction and heat transfer terms were given by the system of differential equations (7.13) and (7.14) that require initial and boundary conditions in terms of a and u . As initial condition we have $u = 0$ and $a = a_a$ for all tunnel length. As boundary condition, at the entry section, the wave pressure distribution is supposed to be known. With the pressure, $p_p(t)$, known, the a_p and u_p values are known through the relations (7.12) and (7.16). As the problem is hyperbolic, no more conditions are needed to propagate the wave along the tunnel.

From equations (7.13) and (7.14) the air velocity u and the speed of sound a can be obtained

$$u = \frac{1}{2}(R^+ - R^-); \quad a = \frac{\gamma - 1}{4}(R^+ + R^-). \quad (7.21)$$

From the initial condition, the invariant $R^- = 2a_a/(\gamma - 1)$ is constant throughout the fluid field and as R^+ is constant along the lines C^+ , the velocity u and sound speed a are constant along these lines, and take the value from the boundary condition at the piston face, that is: $u(t) = u_p(t_p)$ and $a(t) = a_p(t_p)$ along the straight lines C^+ of equation

$$x = x_p(t_p) + [u_p(t_p) + a_p(t_p)](t - t_p). \quad (7.22)$$

Note that x_p , $u_p = dx_p/dt_p$ and a_p are the values of x , u and a at the piston where the

Figure 7.3: Sketch of the u_p trajectory, the C^+ lines, and the regression point over the $x - t$ plane.

time is t_p different from t , as shown in figure 7.3.

This simple wave theory can be applied to determine the propagation of the initial compression wave and, eventually, determine where the shock wave will be formed. This allows to know the maximum length that the tunnel must have so that, with a given pressure profile, the sonic boom is avoided. The friction effect would be conservative in the sense that the weak shock wave would be formed at a larger distance than the one predicted by the non-friction model.

It is possible to find a simple expression for the regression point and hence for the point where the shock wave takes place. This information is useful since it will tell us if some countermeasures should be considered while the high speed line is at the conceptual design stage.

The equation of the characteristics C^+ given in (7.22) can be written in terms of u_p by using the $a_p(u_p)$ relation given in (7.16)

$$x - x_p(t_p) = \left[a_a + \frac{\gamma + 1}{2} u_p(t_p) \right] (t - t_p). \quad (7.23)$$

The envelope line (see figure 7.3) of coordinates (x_e, t_e) is obtained in parametric form with parameter t_p , deriving equation (7.23) with respect to t_p and substituting t by t_e

$$t_e - t_p = \frac{2}{\gamma + 1} \left[a_a + \frac{\gamma - 1}{2} u_p(t_p) \right] \left(\frac{du_p}{dt_p} \right)^{-1}. \quad (7.24)$$

The position $x = x_e$ corresponding to $t = t_e$ is obtained from (7.23) with $t - t_p = (t_e - t_p)$, given in (7.24), to obtain

$$x_e - x_p(t_p) = \frac{2}{\gamma + 1} \left[a_a + \frac{\gamma + 1}{2} u_p(t_p) \right] \left[a_a + \frac{\gamma - 1}{2} u_p(t_p) \right] \left(\frac{du_p}{dt_p} \right)^{-1}. \quad (7.25)$$

The straight lines (7.23) are convergent if $du_p/dt_p > 0$ (as it is the case, because $dp/dt > 0$) so that the characteristics C^+ cross each other. When two (or more) characteristic lines C^+ arrive to the same point of the envelope, each one arrives with a different value of the speed (and pressure) and then the fluid variables at this point are multivalued and the solution is not valid. The point in the $x - t$ plane where this happens is called the regression point of coordinates (x_r, t_r) . This point is the minimum of the envelope line and it is obtained deriving (7.24) with respect to t_p and making $dt_e/dt_p = 0$, to obtain

$$\left[a_a + \frac{\gamma - 1}{2} u_p(t_p) \right] \frac{d^2 u_p / dt_p^2}{(du_p / dt_p)^2} = \gamma. \quad (7.26)$$

Equation (7.26) provides the $t_p = t_p^*$ value that corresponds to the coordinates of the regression point. The t_r value is the t_e obtained from (7.24) and the one of x_r is the value of x_e obtained from (7.25), by substituting $t_p = t_p^*$ in both equations.

Figure 7.4 presents the pressure profiles vs time for different positions along the tunnel. In this figure the initial profile is the one labeled x_1 and the corresponding to the regression point is labeled as x_3 , being x_2 an intermediate position. The profile corresponding to x_4 is not real because the pressure has three values (multivalued) at the same time.

The regression point as a function of the maximum pressure gradient for different pressure increments $(\Delta p)_{max}$ is given in figure 7.5. As it can be seen in this figure, the influence of the pressure increment on the regression point is not as important as the maximum pressure gradient.

7.5 The numerical model

With the initial wave known, the values of the maximum increment of pressure $(\Delta p)_{max}$ and of the maximum gradient $(dp/dt)_{max}$ are also known. With these two parameters, together with the ambient pressure and speed of sound, the following dimensionless variables can be constructed:

$\tau = t[(dp/dt)_{max}/(\Delta p)_{max}]$; $\xi = x[(dp/dt)_{max}/\{a_a(\Delta p)_{max}\}]$; $V = u/a_a$; $\varphi = (p - p_a)/(\Delta p)_{max}$; $\theta = (T - T_a)/T_a$; $\theta_w = (T_w - T_a)/T_a$; $\rho^+ = \rho/\rho_a$ and $a^+ = a/a_a$.

In terms of the dimensionless variables, equations (7.9) to (7.11) are rewritten as

$$\frac{d\varphi}{d\tau} + \gamma \frac{p_a}{(\Delta p)_{max}} \rho^+ a^+ \frac{dV}{d\tau} = -\tau_s^+ - \tau_u^+ + (\gamma - 1)q_T^+; \quad \text{along the lines } C^+: \frac{d\xi}{d\tau} = V + a^+, \quad (7.27)$$

Figure 7.4: Comparison of pressure profiles at different distances of $x_1 = 0$ m, $x_2 = 2000$ m, $x_3 = 4092.9$ m and $x_4 = 6100$ m, with values of $\Delta p_{max} = 2000$ Pa, $(dp/dt)_{max} = 10000$ Pa/s. Isentropic case.

$$\frac{d\varphi}{d\tau} - \gamma \frac{p_a}{(\Delta p)_{max}} \rho^+ a^+ \frac{dV}{d\tau} = \tau_s^+ + \tau_u^+ + (\gamma - 1)q_T^+;$$

$$\text{along the lines } C^-: \frac{d\xi}{d\tau} = V - a^+, \quad (7.28)$$

$$\frac{d\varphi}{d\tau} - \frac{\gamma}{\gamma - 1} \frac{p_a}{(\Delta p)_{max}} \rho^+ \frac{d\theta}{d\tau} = -q_T^+;$$

$$\text{along the lines } C^0: \frac{d\xi}{d\tau} = V, \quad (7.29)$$

where the dimensionless friction and heat transfer terms are $\tau_s^+ = f_s a^+ \rho^+ V |V|$, $\tau_u^+ = f_u \rho^+ a^+ \int_0^\tau W(\tau - \tau^*) \left(\frac{\partial V}{\partial \tau^*} \right) d\tau^*$ (where the time derivative inside the integral is done at constant ξ), $q_T^+ = f_s \rho^+ |V| (\theta_w - \theta) / (\gamma - 1)$, and the dimensionless coefficients are $f_s = [\lambda \gamma a_a p_a] / [2 D_{Tu} (dp/dt)_{max}]$ and $f_u = [16 \nu \gamma p_a] / [D_{Tu}^2 (dp/dt)_{max}]$. The dimensionless density, ρ^+ , and speed of sound a^+ , are obtained from the state equation and speed of sound relation from (7.8) as

$$\rho^+ = \frac{(\Delta p)_{max} \varphi + p_a}{p_a (\theta + 1)}; \quad a^+ = \sqrt{\frac{(\Delta p)_{max} \varphi + p_a}{p_a \rho^+}}. \quad (7.30)$$

Figure 7.5: Regression point as a function of the maximum pressure gradient for different values of $(\Delta p)_{max}$. Notice that they barely change with the pressure increment $(\Delta p)_{max}$. Isentropic case.

The initial conditions are

$$\varphi = 0, V = 0, \theta = 0, \rho^+ = 1 \text{ and } a^+ = 1, \text{ at } \tau = 0, \quad (7.31)$$

$$\begin{aligned} \varphi &= (p_p - p_a)/(\Delta p)_{max}, V = u_p/a_a, \theta = (T_p - T_a)/T_a. \\ \rho^+ &= \rho_p/\rho_a, \text{ and } a^+ = a_p/a_a \\ &\text{along the piston trajectory } d\xi_p/d\tau = V_p, \end{aligned} \quad (7.32)$$

where $V_p = u_p/a_a$ and ξ_p are the dimensionless values of the piston velocity, u_p , and position, x_p , respectively. The u_p value can be obtained from (7.16), and ρ_p , T_p and a_p from the nomentropic relations (7.12), since we have considered that the friction and heat transfer effects can be neglected at the piston face.

To solve numerically, a grid as shown in figure 7.6 is used; such grid is built as the variables V and a^+ are calculated, spacing accordingly with the intersection of the characteristic lines C^+ and C^- whose slopes are $V + a^+$ and $V - a^+$ respectively. Discretizing equations (7.27) to (7.29):

$$\begin{aligned}
& (\varphi_{n+1,i} - \varphi_{n+1,i-1}) + \gamma \frac{p_a}{(\Delta p)_{max}} \rho_{n+1,i-1}^+ a_{n+1,i-1}^+ (V_{n+1,i} - V_{n+1,i-1}) = \\
& \quad (-\tau_{s\ n+1,i-1}^+ - \tau_{u\ n+1,i-1}^+ + (\gamma - 1)q_{T\ n+1,i-1}^+)(\tau_{n+1,i} - \tau_{n+1,i-1}); \\
& \text{along the lines } C^+: (\xi_{n+1,i} - \xi_{n+1,i-1}) = (\tau_{n+1,i} - \tau_{n+1,i-1})(V_{n+1,i-1} + a_{n+1,i-1}^+), \quad (7.33)
\end{aligned}$$

$$\begin{aligned}
& (\varphi_{n+1,i} - \varphi_{n,i+1}) - \gamma \frac{p_a}{(\Delta p)_{max}} \rho_{n,i+1}^+ a_{n,i+1}^+ (V_{n+1,i} - V_{n,i+1}) = \\
& \quad (\tau_{s\ n,i+1}^+ + \tau_{u\ n,i+1}^+ + (\gamma - 1)q_{T\ n,i+1}^+)(\tau_{n+1,i} - \tau_{n,i+1}); \\
& \text{along the lines } C^-: (\xi_{n+1,i} - \xi_{n,i+1}) = (\tau_{n+1,i} - \tau_{n,i+1})(V_{n,i+1} + a_{n,i+1}^+), \quad (7.34)
\end{aligned}$$

$$\begin{aligned}
& (\varphi_{n+1,i} - \varphi_{n,i'}) - \frac{\gamma}{\gamma - 1} \frac{p_a}{(\Delta p)_{max}} \rho_{n,i'}^+ (\theta_{n+1,i} - \theta_{n,i'}) = -q_{T\ n,i'}^+ (\tau_{n+1,i} - \tau_{n,i'}); \\
& \text{along the lines } C^0: (\xi_{n+1,i} - \xi_{n,i'}) = (\tau_{n+1,i} - \tau_{n,i'})V_{n+1,i'}, \quad (7.35)
\end{aligned}$$

where the values at node (n, i') are found by a linear interpolation between nodes (n, i) and $(n, i+1)$. The unknowns $\varphi_{n+1,i}$ and $V_{n+1,i}$ are found with equations (7.33) and (7.34), the value of $\theta_{n+1,i}$ is found with (7.35), and $\rho_{n+1,i}^+$ and $a_{n+1,i}^+$ are found with (7.30). To obtain the air velocity at the face of the piston the node $(n+1, 1)$ must first be found: a line with a slope $1/V_{n,1}$ departs from the node $(n, 1)$ and is intersected by a line with a slope $1/(V_{n,2} - a_{n,2}^+)$ that departs from the node $(n, 2)$; this is the node $(n+1, 1)$, the piston position in time and space. The air velocity is then found by using the dimensionless form of equation (7.16), that is

$$V_{n+1,1} = \frac{2}{\gamma - 1} \left[\left(\frac{(\Delta p)_{max}}{p_a} \varphi_{n+1,1} + 1 \right)^{\frac{\gamma-1}{2\gamma}} - 1 \right].$$

To get the value of the regression point, the coordinates in the (ξ, τ) plane where the characteristics cross each other were obtained (when $\tau_{n+1,i} < \tau_{n,i+1}$); with the smallest value of them being the regression point, ξ_r .

7.6 Validation of the model

To begin with, the numerical model is compared with the algebraic solution for the case where friction and heat transfer are neglected; figure 7.7 presents this comparison, and it can be seen that both are in well agreement, as it would be expected.

For the cases with friction and heat transfer terms, the characteristics numerical scheme was compared with the numerical method proposed by Harten (1982). This model is a finite volume scheme of second order implemented for hyperbolic systems of conservation laws

Figure 7.6: Grid used in the characteristic numerical method. The only fixed value is $d\xi$ which is used to build the first C^+ characteristic line. Starting from there, the mesh spacing is determined by the values of the speed V and φ .

$$w_t + f(w)_x = q, \quad w(x, 0) = \phi(x)$$

where w is a column vector of m components, f is the flux function vector of m components, q a vector volume of sources and $\phi(x)$ the initial conditions. These, for our problem, are

$$w = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ uE + up \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ -\frac{4\tau_w}{D_{T_u}} \\ \frac{4q_T}{D_{T_u}} \end{pmatrix}, \quad \phi(x) = \begin{pmatrix} \rho_a \\ 0 \\ p_a/(\gamma - 1) \end{pmatrix},$$

with $E = p/(\gamma - 1) + (1/2)\rho u^2$. The Euler and explicit numerical scheme of second order is

$$w_j^{n+1} = w_j^n - \lambda(\bar{f}_{j+\frac{1}{2}}^n - \bar{f}_{j-\frac{1}{2}}^n) + q_j^n,$$

where $\lambda = \Delta t/\Delta x$, the subindex j refers to the node in space, the superindex n to the

Figure 7.7: Comparison of the algebraic and numerical solutions. The regression distance, ξ_r , is plotted against the dimensionless pressure increment $(\Delta p)_{\max}/p_a$. The friction and heat transfer are neglected.

node in time and the bar over the flux indicates the approximate value of f on the faces of the cell. To obtain it

$$\bar{f}_{j+\frac{1}{2}} = \frac{1}{2} \left[f(w_j) + f(w_{j+1}) - \frac{1}{\lambda} \sum_{k=1}^m Q(\lambda \bar{a}_{j+\frac{1}{2}}^k) \Delta_{j+\frac{1}{2}} w \right],$$

where $\Delta_{j+\frac{1}{2}} w = w_{j+1} - w_j$; and the values of $\bar{a}_{j+\frac{1}{2}}^k$ are the eigenvalues of the mean value Jacobian, $A(w_j, w_{j+1}) = f_u(w_j, w_{j+1})$, and indicate the steepness of the characteristics of the system, which is the velocity in which the information propagates along with the numerical instabilities, and $Q(x)$ is a function often referred to as the coefficient of numerical viscosity. For this case, using the one proposed by Harten (1982), $Q(x) = |x|$. The main idea of this function is to be grand enough to contrarest the instabilities caused by spurious solutions, but small enough to not act as a relevant viscosity in the final solution; and is particularly active when there are notorious changes on w , which would cause for numerical instabilities. It can be seen that this is a much more complex scheme than the characteristic scheme proposed, requires more time to implement and more computational time to run.

To compare the characteristics numerical method proposed on this chapter, and the TVD method, figures 7.8, 7.9 and 7.10 show the distortion of a pressure profile for distances of 2000, 3000 and 4000 m respectively. A hyperbolic tangent is used as the initial pressure

profile, with $(\Delta p)_{max} = 2000$ Pa, $(dp/dt)_{max} = 8000$ Pa/s and $\lambda_{Tu} = 0.02$. There is well agreement in all figures.

Figure 7.8: Comparison of the characteristics numerical method with the TVD method by Harten (1982) for a case with $(\Delta p)_{max} = 2000$ Pa, $(dp/dt)_{max} = 8000$ Pa/s and $\lambda = 0.04$ at 2000 m from the tunnel's entrance.

The numerical data from Miyachi et al. (2013) showing the progression of the maximum pressure gradient of the pressure wave as it travels along the tunnel was compared with our numerical results for the cases without friction (figure 7.11) and with friction (figure 7.12); both cases are in well agreement.

Hieke et al. (2011) performed measurements of pressure in the high-speed line tunnel of Euerwang in Germany at different distances. We introduced their initial pressure profile, shown in figure 7.13, which was measured at 200 m from the tunnel's entrance, and compared their measurements at 1660 (figure 7.14), 4580 (figure 7.15) and 6040 m (figure 7.16), with our simulations for cases with all effects of friction and heat transfer, without unsteady friction and without both friction and heat transfer. At figure 7.15 it can be seen that the shock wave is close to formation; our calculations determined the regression distance for this case to be near the 5000 m. At figure 7.16 the shock wave has been formed, since our simulation presents a solution that has no physical meaning (a single point in time and space presents a multivalued solution, which is impossible). In all these comparisons it can be seen that the difference between the cases with and without unsteady friction differ slightly from the amplitude of the wave. Curiously enough, the cases without friction, although presenting an incorrect pressure amplitude, show a very similar maximum pressure gradient, which in the long road is the most important factor for the determination of the regression distance; as such, the algebraic model could be used for

Figure 7.9: Comparison of the characteristics numerical method with the TVD method by Harten (1982) for a case with $(\Delta p)_{max} = 2000$ Pa, $(dp/dt)_{max} = 8000$ Pa/s and $\lambda = 0.04$ at 3000 m from the tunnel's entrance.

a quick estimation of the regression distance, without introducing a considerable error in the final calculation. Even more, using the algebraic model and finding a regression distance larger than the tunnel in design adds a factor of security, since the friction effects would only make the regression distance to be even larger, *i. e.* if the algebraic model predicts that no shock wave will be formed, then it is on the safe side. As a proof of this, an indirect validation is the case of the La Cabrera tunnel, on the Madrid-Valencia high-speed line. This tunnel has a length of 7250 m and a section of 53 m². The tunnel has the peculiarity that it is descending towards Valencia. In the video referenced in 54otrebtor (2013), the sonic boom generated by a train entering through the portal opposite to the portal shown in the image can be heard. It passes 66 seconds from the moment the sonic boom is heard in the video until the train appears in the image. The wave travels the whole tunnel in 21 seconds. During this time the train, at 300 km/h, covers 1750 m. The remaining 5500 m are covered by the train in the aforementioned 66 seconds. The maximum pressure increment $(\Delta p)_{max}$ and the maximum pressure gradient $(dp/dt)_{max}$ were measured for a similar train in a tunnel with similar entry portal geometry in the Madrid-Barcelona-La Junquera (Spanish-French border) high-speed line. With the analysis developed in this paper and using the values of $(\Delta p)_{max}$ and $(dp/dt)_{max}$ that corresponds to 300 km/h, we obtain that a weak shock wave is generated at a distance close to 5000 m, obviously lower than the length of the tunnel, that in turn generates the sonic boom heard in the video. When the train is traveling in the opposite direction, there is no sonic boom. This is because the speed uphill is lower than the 300 km/h downhill.

Figure 7.10: Comparison of the characteristics numerical method with the TVD method by Harten (1982) for a case with $(\Delta p)_{max} = 2000$ Pa, $(dp/dt)_{max} = 8000$ Pa/s and $\lambda = 0.04$ at 4000 m from the tunnel's entrance.

The speed uphill should be about 260 km/h, so that the maximum pressure gradient and the maximum pressure increment are reduced around a 30% and the generation of the weak shock wave occurs at a distance greater than the length of the tunnel. In fact, the speed uphill is of 261 km/h at the tunnel entry, and of 252 km/h at the exit. The speed downhill is 300 km/h constant.

The results in figures 7.14 to 7.16 were done introducing a profile that closely resembles the measured data by Hieke et al. (2011) at 200 m (figure 7.13), but a hyperbolic tangent profile, or even a simpler linear profile could have been used as long as they maintain the same maximum pressure amplitude and gradient. In figure 7.17 a comparison between the pressure distributions obtained with the three profiles is given at the same tunnel position of 4580 m. The conclusion of this comparison is that the wave propagation and the weak shock formation basically depends only on the maximum pressure increment and on the maximum pressure gradient, as was pointed out before.

7.7 Parametric analysis

The dimensionless regression distance depends on the maximum pressure gradient over the ambient pressure, $(\Delta p)_{max}/p_a$, the steady friction coefficient, f_s , the unsteady friction coefficient, f_u , and the air specific heat ratio, γ . But in comparison with the importance of the steady friction, the unsteady friction term could be neglected. Figure 7.18 presents

Figure 7.11: Comparison with the Miyachi et al. (2013) numerical data showing the progression of the maximum pressure gradient as the pressure wave progresses along the tunnel. Case without friction.

the comparison of the dimensionless regression distance, ξ_r , vs the dimensionless pressure increment, $(\Delta p)_{max}/p_a$, for cases with and without unsteady friction. Notice the similitude between the two of them. For typical values our calculations determined that the weight function on the unsteady term at its highest value is of order $W \sim 100$, so that the ratio of the unsteady friction over the steady friction at its highest point is of order

$$\frac{\rho \nu W u_c / D_{Tu}}{\lambda_{Tu} \rho u_c^2} \sim \left(\frac{\nu}{\lambda_{Tu}} \right) \frac{W}{D_{Tu} u_c} \ll 1.$$

As such, for the parametric analysis the unsteady friction term can be neglected, and the dimensionless regression distance will depend on only three parameters, that is $\xi_r = \xi_r(\gamma, (\Delta p)_{max}/p_a, f_s)$. Since we are dealing with air γ remains constant and unchanged. The regression distance, ξ_r , was obtained for different values of $(\Delta p)_{max}/p_a$ and f_s , and plotted as shown in figure 7.19. It can be seen that from the two of them, the most influential is $(\Delta p)_{max}/p_a$. A larger friction coefficient f_s generates a larger regression distance, as it was expected, and it is most relevant at high values of $(\Delta p)_{max}/p_a$, since at higher $(\Delta p)_{max}/p_a$ the air flow velocity is larger, and the friction effects become much more relevant.

It is worth mentioning that thanks to the way the dimensionless variables are constructed, the maximum pressure gradient does not appear explicitly in the parametric analysis, but it is hidden inside ξ and τ . This gradient, as shown in figure 7.5, still remains as the most

Figure 7.12: Comparison with the Miyachi et al. (2013) numerical data showing the progression of the maximum pressure gradient as the pressure wave progresses along the tunnel. Case with friction.

relevant factor for the determination of the regression distance.

Notice that although there is a difference between the case without friction and heat transfer, $f_s = 0$, and the cases with friction, the difference is not very large. This supports the fact that the isentropic solution could well be used for a first fast calculation, as mentioned before, which, in case it yields a larger regression distance than the tunnel length, secures the lack of a shock wave, and as such, of a sonic boom when the micro-pressure wave is released at the tunnel's exit portal.

7.8 Micro-pressure wave emission

The most common model for the prediction of the micro-pressure wave emitted to the open air at the tunnel exit is the Kirchhoff integral formula, shown in chapter 5, which is related with the pressure increment value, the temporal gradient and the spatial gradient of the pressure wave at the tunnel's exit,

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\Sigma} \left[\frac{\cos \theta}{r^2} p' - \frac{1}{r} \frac{\partial p'}{\partial n} + \frac{\cos \theta}{a_0 r} \frac{\partial p'}{\partial \tau} \right]_{(\mathbf{x}', \tau)} d\sigma, \quad (7.36)$$

where $\mathbf{x}$ is the position where the observer is standing, $\mathbf{x}'$ is the position on emitting pressure surface, and n is the normal direction to the surface emitting the pressure, which

Figure 7.13: Comparison with the Hieke et al. (2011) data at 200 m from the tunnel's entrance, and the initial pressure profile introduced for the numerical simulation.

in this case is in the axial direction of the tunnel. This formula can be simplified for cases where $r \gg D_{Tu}$, being D_{Tu} the hydraulic diameter of the tube, and $r \gg \ell_c$, where ℓ_c is the characteristic length of the pressure wave reaching the tunnel's exit. When $D_{Tu}/r \ll 1$, $\cos \theta \approx 1$. Also, given that $\Delta a/a_0 \ll 1$ along the compression wave, the spatial pressure derivative can be approximated by

$$\frac{\partial p'}{\partial n} \approx -\frac{1}{a_0} \frac{\partial p'}{\partial \tau}. \quad (7.37)$$

To illustrate this, using the algebraic model of section 7.4, figure 7.20 shows a comparison between values of the spatial derivative at point $x = 6800$ m, obtained with pressure distributions for different instants, and values obtained with the temporal pressure signal at that distance by use of equation (7.37); it can be seen how similar they are, having a maximum variation of around 5%.

With this in mind, the first term on the integral can be neglected when compared with the other two

$$\frac{p'(\mathbf{x}', \tau)}{r^2} \sim \frac{p'_c}{r^2}, \quad \frac{1}{a_0 r} \frac{\partial p'}{\partial \tau} \sim \frac{p'_c}{a_0 r t_c} \sim \frac{p'_c}{r \ell_c},$$

since $t_c a_0 \sim \ell_c$, which means that

Figure 7.14: Comparison with the Hieke et al. (2011) data at 1660 m from the tunnel's entrance, with numerical simulations that incorporate all friction terms, without unsteady friction, and without any friction at all.

$$\frac{\frac{p'}{r^2}}{\frac{1}{a_0 r} \frac{\partial p'}{\partial \tau}} \sim \frac{\ell_c}{r} \ll 1,$$

leading to

$$p'(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\Sigma} \frac{2}{a_0 r} \frac{\partial p'}{\partial \tau} d\sigma.$$

Since $\cos \theta \approx 1$, the value of $\partial p' / \partial \tau$ can be assumed constant along the surface so that

$$p'(\mathbf{x}, t) = \frac{1}{2\pi a_0 r} \frac{\partial p'}{\partial \tau} \int_{\Sigma} d\sigma = \frac{A_{Tu}}{\pi a_0 r} \frac{\partial p'}{\partial \tau}, \quad (7.38)$$

where the surface Σ is $2A_{Tu}$ to account for the reflection with the floor. This simplified expression has been used in literature many times (Kashimura et al., 2000; Raghunathan et al., 2002; Vardy, 2008).

Using the complete Kirchhoff integral expression of equation (7.36) some micro-pressure wave profiles were obtained using a tangent hyperbolic pressure profile at the tunnel exit; the distance was fixed (30 m from the exit aligned with the tunnel axis), as well as the tunnel cross sectional area (70 m²). Figure 7.21 shows the relation between the

Figure 7.15: Comparison with the Hieke et al. (2011) data at 4580 m from the tunnel's entrance, with numerical simulations that incorporate all friction terms, without unsteady friction, and without any friction at all. Notice that the shock wave is close to formation.

maximum pressure gradient of the compression wave at the tunnel's exit, $(dp/dt)_{\max}$, and the maximum pressure of the resulting micro-pressure wave, varying the value of the maximum pressure increment of the compression wave, $\Delta p_{\max}$.

It can be seen how the variation of $\Delta p_{\max}$ is much less important than the variation of $(dp/dt)_{\max}$, which agrees with what was mentioned for the simplification of the Kirchhoff expression; this will be even more dramatic at a larger distance from the tunnel's exit.

Figure 7.22 groups the same results but with decibels instead of Pascals for the maximum value of the micro-pressure wave, using the formula

$$I_{DB} = 10 \log_{10} \frac{I_n (\text{W/m}^2)}{10^{-12} (\text{W/m}^2)}, \quad (7.39)$$

where the acoustic intensity, as mentioned in section 5.2, is $I_n = p' u'_n = p' u_e$.

As mentioned in chapter 5 the pain threshold lies around 150 DB, which is the order of values that the micro-pressure waves are emitting when the compression wave at the exit has a $(dp/dt)_{\max} > 60000$ [Pa/s], and is easily reached in long tunnels with slab track (around the order of 6 km of length) and train speeds of 300 km/h.

Although this gives an idea of the magnitudes that a sonic boom can reach, Kirchhoff integral formula becomes invalid for shock waves, since $dp/dt \rightarrow \infty$. For that case Kashimura et al. (2000) obtained an experimental correlation that relates the magnitude

Figure 7.16: Comparison with the Hieke et al. (2011) data at 6040 m from the tunnel's entrance, with numerical simulations that incorporate all friction terms, without unsteady friction, and without any friction at all. The shock wave has been formed, and as such, the numerical data has no physical meaning (a single point in time and space presents a multivalued solution).

of the shock wave, with the magnitude of the emitted micro-pressure wave and the distance from the tunnel's exit and the observer. This expression is only valid for the case where $\theta \approx 0$,

$$\frac{p'}{\Delta p_{\max}} = \frac{0.554}{(r/D_{Tu})^{1.4}} \text{ for } 1 \leq \frac{r}{D_{Tu}} \leq 5, \quad (7.40)$$

where p' is the magnitude of the micro-pressure wave, and $\Delta p_{\max}$ the magnitude of the shock wave. It can be seen that it is limited to a small range of r/D_{Tu} .

Figure 7.17: Comparison of the pressure profile at a distance of 4580 m using as an initial profile the one measured by Hieke et al. (2011), a hyperbolic tangent, and a linear profile. Notice how the three of them present the same maximum pressure gradient and maximum pressure increment.

Figure 7.18: Comparison between the dimensionless regression point ξ_r , with and without the unsteady friction term; notice the similitude of both.

Figure 7.19: Regression distance ξ_r against $(\Delta p)_{max}/p_a$ for different values of the dimensionless friction coefficient f_s .

Figure 7.20: Comparisson of values of the pressure spatial derivative at the tunnel exit using pressure distributions at different instants, and using the equation (7.37).

Figure 7.21: Relation between the $(dp/dt)_{\max}$ of the compression wave at the tunnel's exit and the maximum pressure increment of the micro-pressure wave, varying $\Delta p_{\max}$ of the compression wave, for a fixed distance of 30 m from the tunnel's exit, with a tunnel's cross sectional area of 70 m^2 .

Figure 7.22: Relation between the $(dp/dt)_{\max}$ of the compression wave at the tunnel's exit and the maximum acoustic intensity of the micro-pressure wave in decibels, varying $\Delta p_{\max}$ of the compression wave, for a fixed distance of 30 m from the tunnel's exit, with a tunnel's cross sectional area of 70 m².

Chapter 8

Wall temperature in long tunnels

The following chapter describes a simplified model for the prediction of the temperature inside a long tunnel in a high speed railway line after years of operation. It parts from the integral conservation equations of continuity, momentum and energy for the temperature of the air, coupled with the heat equation for the rock temperature surrounding the tunnel. It is compared with numerical results from the model of chapter 6 to understand its capacities and limitations.

It is separated in two stages, one when the train is inside the tunnel, and the other when the train is outside the tunnel. These stages are followed one after the other in cycles that depend on the operation times of the line.

For tunnels with lengths of the order of tenths of kilometers, after years of operation, the model predicts a rise in the air temperature and tunnel wall, which points to the use of a cooling system in order to maintain the tunnel temperature in safe and comfortable levels for the workers that deliver maintenance to the tunnel and for the train itself. As such, this model can be used as a fast calculation tool to predict if certain changes must be done to the tunnel in the design stage.

8.1 Period with the train inside the tunnel

During the period with the train inside the tunnel, which will be called *first stage*, the integral conservation of momentum equation projected in the axial direction is

$$\begin{aligned} \vec{i} \cdot \left\{ \frac{d}{dt} \int_{\Omega} \rho \vec{v} d\Omega + \int_{\Sigma_{Tu}} \rho [\vec{v}(\vec{v} - \vec{v}_c)] \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \rho [\vec{v}(\vec{v} - \vec{v}_c)] \cdot \vec{n} d\sigma \right\} \\ = \vec{i} \cdot \left\{ - \int_{\Sigma_{Tu}} p \vec{n} d\sigma + \int_{\Sigma_{Tu}} \tau' \cdot \vec{n} d\sigma \right. \\ \left. - \int_{\Sigma_{Tr}} p \vec{n} d\sigma + \int_{\Sigma_{Tr}} \tau' \cdot \vec{n} d\sigma + \int_{\Omega} \rho \vec{f}_m d\Omega \right\}, \quad (8.1) \end{aligned}$$

where the subscripts Tu and Tr indicate tunnel and train respectively, $\vec{v}_c = 0$ on the tunnel surface, and $\vec{v}_c = U\vec{i}$ on the train surface; $\vec{f}_m$ is a body force which could be

gravity, but since the tunnel is assumed to be horizontal $\vec{f}_m \cdot \vec{i} = 0$

It has been seen previously (section 6.4.3) in a very long tunnel the friction terms are the most important, because $\lambda_{Tu}L_{Tu}/D_{Tu} \gg 1$, and they must be compensated by the over pressure generated by the train, and this provides the order of magnitude of the air velocity u_c

$$\rho_a U^2 \sim \frac{\lambda_{Tu}L_{Tu}}{D_{Tu}} u_c^2 \rightarrow \frac{u_c}{U} \sim \sqrt{\frac{D_{Tu}}{\lambda_{Tu}L_{Tu}}} \ll 1. \quad (8.2)$$

There is no flow through the train surface because $\vec{v} = \vec{v}_c = U\vec{i}$ and then

$$\vec{i} \cdot \int_{\Sigma_{Tr}} \rho[\vec{v}(\vec{v} - \vec{v}_c)] \cdot \vec{n} d\sigma = 0.$$

The integral

$$\vec{i} \cdot \left\{ - \int_{\Sigma_{Tr}} p \vec{n} d\sigma + \int_{\Sigma_{Tr}} \tau' \cdot \vec{n} d\sigma \right\} = \frac{1}{2} C_D \beta \rho_a U^2 A_{Tu}, \quad (8.3)$$

represents the aerodynamic drag of the train.

The pressure term at the tunnel walls is

$$-\vec{i} \cdot \int_{\Sigma_{Tu}} p \vec{n} d\sigma = \vec{i} \cdot \left\{ - \int_{A_{ETu} + A_{STu}} p \vec{n} d\sigma \right\} \sim \rho_a u_c^2 A_{Tu}, \quad (8.4)$$

because the pressure difference at the entry (A_{ETu}) and exit (A_{STu}) sections is of the order of the dynamic pressure. The integral extended to the tunnel lateral walls is zero because $\vec{i} \cdot \vec{n} = 0$. The convective term applied to the entry and exit areas of the tunnel is also of the same order as the pressure one

$$\vec{i} \cdot \int_{\Sigma_{Tu}} \rho \vec{v} \vec{v} \cdot \vec{n} d\sigma = \vec{i} \cdot \int_{A_{ETu} + A_{STu}} \rho \vec{v} \vec{v} \cdot \vec{n} d\sigma \sim \rho_a u_c^2 A_{Tu}, \quad (8.5)$$

and the integral extended to the tunnel walls do not contribute to the axial momentum equation because $\vec{v} = \vec{v}_c = 0$ there.

The friction term extended to the tunnel walls is given by

$$\vec{i} \cdot \int_{\Sigma_{Tu}} \tau' \cdot \vec{n} d\sigma \sim \frac{\lambda_{Tu}L_{Tu}}{D_{Tu}} \rho_a u_c^2 A_{Tu}. \quad (8.6)$$

The integrals extended to the tunnel entry and exit areas are of the order of $\lambda_{Tu} \rho_a u_c^2 A_{Tu}$, which compared to the previous one are of order $D_{Tu}/L_{Tu} \ll 1$.

Note that (8.4) and (8.5) are of the same order, $\rho_a u_c^2 A_{Tu}$, very small compared with (8.3), and the ratio is $u_c^2/(\beta C_D U^2) \sim (u_c/U)^2 \ll 1$. Nevertheless, with the order of magnitude given in (8.2), the terms (8.3) and (8.6) are of the same order.

According with the above simplifications, the momentum equation reads

$$\vec{i} \cdot \frac{d}{dt} \int_{\Omega} \rho \vec{v} d\Omega = \frac{1}{2} C_D \rho_a U^2 \beta A_{Tu} + \vec{i} \cdot \int_{\Sigma_{Tu}} \tau' \cdot \vec{n} d\sigma. \quad (8.7)$$

Taking the spatial average of the velocity throughout the volume, equation (8.7) can be approximated as the following differential equation¹

$$\frac{du}{dt} = \frac{1}{2}C_D\beta\frac{U^2}{L_{Tu}} - \frac{\lambda_{Tu}}{2D_{Tu}}|u|u. \quad (8.8)$$

For that same control volume, the integral conservation of energy equation is

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \rho \left(e + \frac{1}{2}v^2 \right) d\Omega + \int_{\Sigma_{Tu}} \rho \left(h + \frac{1}{2}v^2 \right) \vec{v} \cdot \vec{n} d\sigma \\ + \int_{\Sigma_{Tr}} \rho \left(e + \frac{1}{2}v^2 \right) (\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = \int_{\Sigma_{Tu}} \vec{v} \cdot \vec{\tau}' \cdot \vec{n} d\sigma \\ - \int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} (-p\vec{n} + \vec{\tau}' \cdot \vec{n}) \cdot \vec{v} d\sigma - \int_{\Sigma_{Tr}} \vec{q} \cdot \vec{n} d\sigma \\ + \int_{\Omega} Q_R d\Omega + \int_{\Omega} \rho \vec{f}_m \cdot \vec{v} d\Omega, \end{aligned} \quad (8.9)$$

where Q_R is a heat added provided by the energy losses on the train (the heat of the electric engines, the air conditioning, the mechanical inefficiencies, etc.). For this first approach, it will be considered zero. The same happens with $\vec{f}_m \cdot \vec{v} = 0$, since the tunnel is considered horizontal. As such, the power of the train will be the aerodynamic drag multiplied by the train velocity, that is

$$- \int_{\Sigma_{Tr}} p\vec{v} \cdot \vec{n} d\sigma + \int_{\Sigma_{Tr}} \vec{v} \cdot \vec{\tau}' \cdot \vec{n} d\sigma = \frac{1}{2}C_D\rho_a U^3 \beta A_{Tu}. \quad (8.10)$$

The temporal term of equation (8.9) is of order

$$\frac{d}{dt} \int_{\Omega} \rho \left(e + \frac{1}{2}v^2 \right) d\Omega \sim \rho_a u_c A_{Tu} c_v \Delta T \left(\frac{U}{u_c} \right)^2, \quad (8.11)$$

because the characteristic time is $t_c \sim (L_{Tu}/U)/(u_c/U)$, obtained previously. In the above estimation the kinetic energy is negligible compared with the internal energy. As the temporal term must be of the same order as the power of the train, the temperature increment should be

$$c_v \Delta T \sim \beta C_D u_c U \sim u_c U, \quad (8.12)$$

¹Note that the characteristic time, from equation (8.8) is of the order $t_c \sim (L_{Tu}/U)(u_c/U)$, which is small compared to the transit time of the train in the tunnel, $Ut_c/L_{Tu} \sim u_c/U \ll 1$.

The continuity equation in integral form provides $\frac{d}{dt} \int_{\Omega} \rho d\Omega + \int_{\Sigma} \rho(\vec{v} - \vec{v}_c) \cdot \vec{n} d\sigma = 0$, which leads to $L_{Tu} A_{Tu} \frac{d\bar{\rho}}{dt} \sim \rho_a u A_{Tu}$, where $\bar{\rho}$ is the spatial average of the density. The left side term of equation (8.7) can be expressed as $\frac{d(\bar{\rho}u)}{dt} A_{Tu} L_{Tu} = \left(\bar{\rho} \frac{du}{dt} + u \frac{d\bar{\rho}}{dt} \right) A_{Tu} L_{Tu}$, where u is the velocity averaged through space, $\bar{\rho} \frac{du}{dt} \sim \frac{\rho_a u}{t_c}$, and $u \frac{d\bar{\rho}}{dt} \sim \frac{\rho_a u^2}{L_{Tu}}$, and the ratio $\frac{u(d\bar{\rho}/dt)}{\bar{\rho}(du/dt)} \sim \frac{u_c t_c}{L_{Tu}} \sim \left(\frac{u_c}{U} \right)^2 \ll 1$. For this reason the term $\vec{i} \cdot \frac{d}{dt} \int \rho \vec{v} d\Omega$ of equation (8.7) is approximated by $\rho_a (du/dt) A_{Tu} L_{Tu}$.

and the ratio of the kinetic energy with $c_v\Delta T$ is

$$\frac{u_c^2}{c_v\Delta T} \sim \frac{u_c^2}{u_c U} \sim \frac{u_c}{U} \ll 1.$$

The convective term in the tunnel is of order

$$\int_{\Sigma_{Tu}} \rho \left(h + \frac{1}{2}v^2 \right) \vec{v} \cdot \vec{n} d\sigma = \int_{A_{ETu} + A_{STu}} \rho \left(h + \frac{1}{2}v^2 \right) \vec{v} \cdot \vec{n} d\sigma \sim \rho_a u_c A_{Tu} c_p \Delta T,$$

and the integral extended to the tunnel walls is zero because $\vec{v} = 0$.

The ratio of this last term with the temporal one given in (8.11) is

$$\frac{\rho_a u_c A_{Tu} c_p \Delta T}{\rho_a u_c A_{Tu} c_v \Delta T (U/u_c)^2} \sim \left(\frac{u_c}{U} \right)^2 \ll 1.$$

The convective term in the train walls is zero, because $\vec{v} = \vec{v}_c = U\vec{i}$, so that $\vec{v} - \vec{v}_c = 0$.

The work of the friction forces is zero at the tunnel walls, as such

$$\int_{\Sigma_{Tu}} \vec{n} \cdot \tau' \cdot \vec{v} d\sigma = \int_{A_{ETu} + A_{STu}} \vec{n} \cdot \tau' \cdot \vec{v} d\sigma \sim \lambda_{Tu} \rho_a u_c^3 A_{Tu}$$

and the comparison with the train power (8.10) is

$$\lambda_{Tu} \frac{\rho_a u_c^3 A_{Tu}}{\rho_a U^3 A_{Tu}} \sim \lambda_{Tu} \left(\frac{u_c}{U} \right)^3 \ll 1,$$

since λ_{Tu} is also small.

The heat transfer through the tunnel walls is of order

$$\int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma \sim \frac{\lambda_{Tu} L_{Tu}}{D_{Tu}} \rho_a u_c A_{Tu} c_p \Delta T \sim \rho_a U^3 A_{Tu}, \quad (8.13)$$

of the same order as the train power (8.10). That is because $\lambda_{Tu} L_{Tu}/D_{Tu} \sim (U/u)^2$ and $c_p \Delta T \sim u_c U$, as shown before. Analogously to what happens with the momentum equation, the integrals extended to the tunnel entry and exit over (8.13), which is of the order of the train power, lead to the ratio of order $D_{Tu}/L_{Tu} \ll 1$, and can be neglected.

The heat transfer through the train is of order

$$\int_{\Sigma_{Tr}} \vec{q} \cdot \vec{n} d\sigma \sim \frac{\lambda_{Tr} L_{Tr}}{D_{Tr}} \rho_a U A_{Tr} c_p \Delta T \sim \rho_a U^2 u_c A_{Tr},$$

because $\lambda_{Tr} L_{Tr}/D_{Tr} \sim 1$ and $c_p \Delta T \sim u_c U$. The ratio of this last term with the train power (8.10) is

$$\frac{\rho_a U^2 u_c A_{Tr}}{\rho_a U^3 A_{Tu}} \sim \beta \frac{u_c}{U} \ll 1,$$

since $\beta < 1$ and $u_c/U \ll 1$.

With the above simplifications, the energy equation (8.9) takes the form

$$\frac{d}{dt} \int_{\Omega} \rho e d\Omega = \frac{1}{2} \beta C_D \rho_a U^3 A_{Tu} - \int_{\Sigma_{Tu}} \vec{q} \cdot \vec{n} d\sigma, \quad (8.14)$$

which, as it was considered with (8.8), can be approximated by a differential equation, taking the spatial average of the internal energy,

$$\frac{dT}{dt} = \frac{1}{2} \frac{C_D U^3 \beta}{c_v L_{Tu}} + \frac{4q_{Tu}}{\rho_a c_v D_{Tu}}. \quad (8.15)$$

Equations (8.8) and (8.15) are coupled with the heat equation of the rock surrounding the tunnel.

8.2 The heat equation on the rock

The rock surrounding the tunnel has different temperatures at different heights; this is caused by the geothermal gradient, which for regular values presents variations of the order of 1°C for each 30 m (Lovering and Goode, 1963). This means that the rock temperature will be a function of time, the radial direction and the angle. Neglecting the changes in the axial direction, the heat equation must be solved in polar coordinates

$$\frac{\partial T_R}{\partial t} = \alpha_R \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_R}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T_R}{\partial \theta^2} \right], \quad (8.16)$$

where T_R is the temperature of the rock, and α_R the heat diffusion coefficient in the rock. The Laplacian in polar coordinates with a value that depends of θ at $r \rightarrow \infty$ admits solutions of the type $f_n(r) \cos(n\theta)$, so assuming that

$$T_R = T_{R_0}(t, r) + \sum_{n=1}^{\infty} T_{R_n}(t, r) \cos(n\theta), \quad (8.17)$$

by introducing (8.17) in equation (8.16), T_{R_0} is found through

$$\frac{\partial T_{R_0}}{\partial t} = \alpha_R \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_{R_0}}{\partial r} \right), \quad (8.18)$$

and T_{R_n} through

$$\frac{\partial T_{R_n}}{\partial t} = \alpha_R \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_{R_n}}{\partial r} \right) - \frac{n^2}{r^2} T_{R_n} \right]. \quad (8.19)$$

Nevertheless, to obtain the air temperature, with equation (8.15), requires the temperature at the tunnel wall to be axisymmetrical, so that an azimuthal average is performed on T_R ,

$$\frac{1}{2\pi} \int_0^{2\pi} T_R d\theta = \frac{1}{2\pi} \int_0^{2\pi} T_{R_0}(t, r) d\theta + \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} T_{R_n}(t, r, \theta) \cos(n\theta) d\theta,$$

and since

$$\int_0^{2\pi} \cos(n\theta) d\theta = 0,$$

then

$$\frac{1}{2\pi} \int_0^{2\pi} T_R d\theta = \bar{T}_R = T_{R_0}(t, r),$$

where $\bar{T}_R$ is the azimuthal averaged rock temperature, which for simplicity, from here on, will be written simply as T_R . As such, the rock temperature in this model will be considered axisymmetric, and the initial condition on the rock temperature will be an azimuthal averaged of the geothermic gradient.

With all this in mind, equations (8.8) and (8.15) must be coupled with

$$\frac{\partial T_R}{\partial t} = \alpha_R \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_R}{\partial r} \right). \quad (8.20)$$

Equation (8.20) is coupled through the use of the boundary condition

$$k_R \frac{\partial T_R}{\partial r} \Big|_{r=D_{Tu}/2} = q_{Tu} = k_f \frac{Nu_D}{D_{Tu}} (T_{R_w} - T), \quad (8.21)$$

where D_{Tu} is the tunnel diameter, k_R and k_f are the heat conduction coefficient of the rock and air respectively, Nu_D the Nusselt number based on the tunnel diameter, and T_{R_w} the azimuthal averaged tunnel wall temperature. For this case the Nusselt number is obtained from the Dittus-Boelter correlation, which states

$$Nu_D = 0.023 Re_D^{0.8} Pr^n, \quad (8.22)$$

where Re_D is the Reynolds number based on the tunnel diameter and the air velocity u , Pr is the Prandtl number, $Pr = \mu c_p / k_f$, and $n = 0.3$ when the air is getting cooled, and $n = 0.4$ when the air is getting heated.

Equations (8.8), (8.15), (8.20) and (8.21) compose the model for the stage when the train is inside the tunnel.

8.3 Period with the train outside the tunnel

Once the train has leaved the tunnel, which will be defined as *second stage*, the model is altered by removing the terms that represent the effect of the train, that is the aerodynamic drag in equation (8.8) and the train power in equation (8.15). This leaves to equations

$$\frac{du}{dt} = -\frac{\lambda_{Tu}}{2D_{Tu}} |u|u, \quad (8.23)$$

and

$$\frac{dT}{dt} = \frac{4q_{Tu}}{\rho_a c_v D_{Tu}}, \quad (8.24)$$

which must also be coupled with equations (8.20) and (8.21).

The characteristic time for this stage can be obtained either from equation (8.23) or from (8.24). Using the latter, the temporal term must be of the same order as the heat transfer term, that is

$$\frac{dT}{dt} \sim \frac{\Delta T}{t_c} \sim \frac{4q_{Tu}}{\rho_a c_v D_{Tu}} \sim \frac{\lambda_{Tu}}{D_{Tu}} u_c \Delta T,$$

so that

$$t_c \sim \frac{D_{Tu}}{\lambda_{Tu} u_c} \sim \frac{D_{Tu}}{\lambda_{Tu} L_{Tu}} \frac{L_{Tu}}{u_c} \sim \left(\frac{u_c}{U} \right)^2 \cdot \frac{L_{Tu}}{u_c} \sim \frac{L_{Tu}}{U} \cdot \frac{u_c}{U} \ll \frac{L_{Tu}}{U},$$

which is the same as the characteristic time of the first stage.

Defining the schedule time between the passing of two trains as t_{c0} , if $t_{c0} > t_c$ (which tends to be usually the case) it will mean that the air temperature will reach the wall temperature and the air flow will become null before the next train arrives. On the other hand, if $t_{c0} < t_c$ the air temperature will be above the wall temperature, and the air flow will not stop as long as they are trains passing through the tunnel.

There is also a nocturnal period when no trains pass by, and is of the order of hours. In this time, the air flow is non-existent, and there is no difference between air temperature and wall temperature, so that no significant natural convection can happen. As such, during this period the wall acts as an adiabatic barrier, until the trains start circulating again.

During the time t_c the order of the heat penetration distance in the rock, ℓ_{cR} , can be found from equation (8.20) by comparing the orders of the temporal term with the diffusion term, that is

$$\frac{\Delta T_R}{t_c} \sim \alpha_R \frac{\Delta T_R}{\ell_{cR}^2} \rightarrow \ell_{cR}^2 \sim \alpha_R t_c.$$

This distance, compared with the tunnel diameter is of order

$$\left(\frac{\ell_{cR}}{D_{Tu}} \right)^2 \sim \frac{\alpha_R t_c}{D_{Tu}^2} \sim \frac{\alpha_R}{\lambda_{Tu} u_c D_{Tu}} \ll 1.$$

On the other hand, the time it takes for the heat to penetrate in the rock a distance of the order of D_{Tu} is

$$\frac{\Delta T_R}{t_{cR}} \sim \alpha_R \frac{\Delta T_R}{D_{Tu}^2} \rightarrow t_{cR} \sim \frac{D_{Tu}^2}{\alpha_R},$$

and when comparing this time with t_c for typical values

$$\frac{t_c}{t_{cR}} \sim \frac{\alpha_R}{\lambda D_{Tu} u_c} \ll 1.$$

As such, if only a few train passings are being studied, then the tunnel wall can be treated as an isothermal wall, since the characteristic times for the heating of the rock and the air are very different. But, if a period of time of the order of years is analyzed, in which hundreds of thousands of trains pass, then the heating of the rock will become relevant, and the variation of the tunnel wall temperature must be considered.

8.4 Initial and boundary conditions

Time starts to count when the first train enters the tunnel. At this moment the conditions are

$$u(0) = 0, \quad T(0) = T_a, \quad T_R(0, r) = f(r), \quad (8.25)$$

where $f(r)$ is a function of the distance r . This could be as simple as a constant temperature or a linear distribution on the r direction (like the one showed in figure 8.1).

Figure 8.1: Sketch of a generic and lineal temperature gradient as initial condition for the rock temperature, where the difference in color represents a difference in temperature. The distribution of temperature can be as desired, as long as it is axisimetric.

Once the first train has exit the tunnel, the final conditions left on this first stage will become the initial conditions for the second stage (empty tunnel); when the next train arrives, the final conditions of the second stage will become the initial conditions of this new first stage, and so on.

The boundary conditions imposed in the heat equation (8.20) are the coupling equation

$$k_R \frac{\partial T_R}{\partial r} = k_f \frac{Nu_D}{D_{Tu}} (T_{R_w} - T), \text{ at } r = D_{Tu}/2,$$

where $T_R(t, R) = T_{R_w}$, and

$$T = T_{R_\infty}, \text{ at } r \rightarrow \infty, \quad (8.26)$$

where T_{R_∞} is the azimuthal averaged rock temperature at a very large distance compared to the radius.

8.5 Numerical scheme

Using non dimensional variables

$$\tau = \frac{t}{t_c}, \eta = \frac{r}{D_{Tu}}, \theta_R = \frac{T_R - T_a}{\Delta T}, \theta = \frac{T - T_a}{\Delta T}, V = \frac{u}{U} \quad (8.27)$$

where $t_c = D_{Tu}^2/\alpha_R$ and $\Delta T = \beta C_D U^2/2c_v$, equations (8.20) and (8.21) turn into

$$\frac{\partial \theta_R}{\partial \tau} = \frac{\partial^2 \theta_R}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial \theta_R}{\partial \eta}, \quad (8.28)$$

$$\left. \frac{\partial \theta_R}{\partial \eta} \right|_{\eta=1/2} = \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta), \quad (8.29)$$

respectively. For the first stage, equations (8.8) and (8.15) turn into

$$\frac{dV}{d\tau} = \frac{1}{2} C_D \beta \frac{D_{Tu}^2 U}{\alpha_R L_{Tu}} - \frac{\lambda_{Tu} D_{Tu} U}{\alpha_R} |V|V, \quad (8.30)$$

and

$$\frac{d\theta}{d\tau} = \frac{D_{Tu}^2 U}{\alpha_R L_{Tu}} + \frac{k_f Nu_D}{\rho_a c_v \alpha_R} (\theta_{R_w} - \theta), \quad (8.31)$$

and for the second stage equations (8.23) and (8.24) turn into

$$\frac{dV}{d\tau} = -\frac{\lambda_{Tu} D_{Tu} U}{\alpha_R} |V|V, \quad (8.32)$$

and

$$\frac{d\theta}{d\tau} = \frac{k_f Nu_D}{\rho_a c_v \alpha_R} (\theta_{R_w} - \theta). \quad (8.33)$$

The initial conditions are

$$V = 0, \quad \theta = 0, \text{ and } \theta_R = F(\eta), \text{ at } \tau = 0, \quad (8.34)$$

being $F(\eta)$ an axisimetric function. The boundary conditions for equation (8.28) are (8.29), remembering that $\theta_R|_{\eta=1/2} = \theta_{R_w}$, and

$$\theta_R \rightarrow 0, \text{ when } \eta \rightarrow \infty. \quad (8.35)$$

The numerical scheme here proposed is a simple Euler explicit, so that equation (8.28), is discretized as

$$\frac{\theta_{R_j}^{n+1} - \theta_{R_j}^n}{\Delta\tau} = \frac{\theta_{R_{j+1}}^n - 2\theta_{R_j}^n + \theta_{R_{j-1}}^n}{\Delta\eta^2} + \frac{1}{\eta_j} \frac{\theta_{R_{j+1}}^n - \theta_{R_{j-1}}^n}{2\Delta\eta}, \quad (8.36)$$

where n represents the node in time, and j the node in space. It is important to consider that, according to the Von Nuemann stability criterium, $\Delta\tau/\Delta\eta^2 < 0.5$ (Hirsch, 2007).

The coupling equation (8.29) is discretized as

$$\frac{\theta_{R_2}^{n+1} - \theta_{R_1}^{n+1}}{\Delta\eta} = \frac{k_f}{k_R} Nu_D (\theta_{R_1}^{n+1} - \theta^n). \quad (8.37)$$

For the first stage, with the train inside the tunnel, equations (8.30) and (8.31) are discretized as

$$\frac{V^{n+1} - V^n}{\Delta\tau} = \frac{1}{2} C_D \beta \frac{D_{Tu}^2 U}{\alpha_R L_{Tu}} - \frac{\lambda_{Tu} D_{Tu} U}{\alpha_R} |V^n| V^n, \quad (8.38)$$

and

$$\frac{\theta^{n+1} - \theta^n}{\Delta\tau} = \frac{D_{Tu}^2 U}{\alpha_R L_{Tu}} + \frac{4k_f Nu_D}{\rho_a c_v \alpha_R} (\theta_{R_1}^n - \theta^n). \quad (8.39)$$

For the second stage, with an empty tunnel, equations (8.32) and (8.33) are discretized as

$$\frac{V^{n+1} - V^n}{\Delta\tau} = - \frac{\lambda_{Tu} D_{Tu} U}{\alpha_R} |V^n| V^n, \quad (8.40)$$

and

$$\frac{\theta^{n+1} - \theta^n}{\Delta\tau} = \frac{4k_f Nu_D}{\rho_a c_v \alpha_R} (\theta_{R_1}^n - \theta^n). \quad (8.41)$$

8.6 Comparison with the model from chapter 6

The simple model formed by equations (8.30)-(8.33) is compared with results from the model of chapter 6 averaged in space, so that they appeared as a function of time only. For this comparison a specific case was proposed with a tunnel length of 30 km, a train speed of 61.22 m/s and a ratio area, β , of 0.21, first with an adiabatic wall and then with an isothermal wall.

Figure 8.2 shows the comparison of the air velocity and air temperature for the case of an adiabatic wall, while figure 8.3 shows the comparison for the case of an isothermal wall. It can be seen that the spatial averaged temperature of the model of chapter 6

has fluctuations that diminish with time (figures 8.2b and 8.3b), these are caused by the pressure waves that travel back and forth inside the tunnel, and get damped along time; in this same model, air velocity experiences changes caused by the pressure waves, but it is less affected than temperature is, and when averaged in space it is almost unnoticeable.

Figure 8.2: Comparison of the simple model and the model of chapter 6 for, a), the averaged air velocity on space, and b), the averaged air temperature on space, with an adiabatic wall

Figure 8.3: Comparison of the simple model and the model of chapter 6 for, a), the averaged air velocity on space, and b), the averaged air temperature on space, with an isothermal wall

8.7 Results of the temperature rise

Using the numerical model formed by equations (8.36)-(8.41), for the same case of a tunnel length of 30 km, train speed of 61.22 m/s, area ratio of 0.21, a period between trains of 15 minutes and a nocturnal resting period of 6 hours, the wall and air temperature were calculated for a period of 25 years, as shown in figure 8.4. The initial rock and air temperature were considered uniform and equal to the ambient temperature. It can be

seen how the temperature rises faster at the begining, to later tend to an asymptotic limit, and how the difference between air and wall temperature is maintained at a mean constant value. The general increment is in the order of 20 °C, which could led to a problematic situation for the workers that provide maintenance to the tunnel.

Figure 8.4: Temperature increment on the air and tunnel wall throughout the years, with $L_{Tu} = 30$ km, $U = 61.22$ m/s, $\beta = 0.21$, a period between trains of 15 minutes, and a nocturnal period of six hours.

To see with more detail the temperature increment caused by the passing of a train, and the period between trains, figure 8.5 shows the passing of seven trains. Note the similarity to the air temperature behaviour of figure 8.3b and how the temperature increments are in the order mentioned in (8.12); it can also be seen how the air changes more easily its temperature, while the rock maintains a more stable and less oscillating value.

Figure 8.6 shows a distance of heat penetration in the rock, η_p , defined as

$$\eta_p = \frac{1}{\theta_{R_w}} \int_{1/2}^{\infty} \theta_R d\eta, \quad (8.42)$$

where θ_{R_w} is the dimensionless temperature at the tunnel wall. It can be seen how the asymptotic behaviour resembles the one of the wall and air temperatures of figure 8.4.

For comparison, figure 8.7 shows the temperature spatial distributions of the rock for different times, 5, 15 and 25 years. It can be seen how as times goes by, the temperature increment advances slower.

8.8 A pseudo-similarity solution

To prove the mentioned behaviour, for large times when $\alpha_R t \gg D^2$, an approximate similarity solution can be found for equation (8.28) with conditions (8.29) and (8.35), and

Figure 8.5: Increment on the tunnel wall and air temperature due to the passing of seven trains. $L_{Tu} = 30$ km, $U = 61.22$ m/s, $\beta = 0.21$, and a period between trains of 15 minutes.

Figure 8.6: Distance of heat penetration in the rock η_p . $k_R = 2$ W/(m·K), $\alpha_R = 1.01 \times 10^6$ m²/s, and $T_{R\infty} = T_a$ with an initial uniform temperature in the rock.

an initial condition of $\theta_R = 0$ at $t = 0$.

First of, lets consider the variables $\tau' = \tau$ and $\zeta = \eta/\sqrt{\tau}$, which give the derivate

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau'} - \frac{1}{2} \frac{\zeta}{\tau} \frac{\partial}{\partial \zeta} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{1}{\sqrt{\tau}} \frac{\partial}{\partial \zeta},$$

Figure 8.7: Temperature spatial distribution in the rock for different periods of time. $k_R = 2 \text{ W/(m}\cdot\text{K)}$, $\alpha_R = 1.01 \times 10^6 \text{ m}^2/\text{s}$, and $T_{R\infty} = T_a$ with an initial uniform temperature in the rock.

so that equation (8.28) becomes²

$$\tau \frac{\partial \theta_R}{\partial \tau} = \frac{\partial^2 \theta_R}{\partial \zeta^2} + \left(\frac{1}{2} \zeta + \frac{1}{\zeta} \right) \frac{\partial \theta_R}{\partial \zeta}, \quad (8.43)$$

with condition (8.35) becoming

$$\theta_R \rightarrow 0 \text{ when } \zeta \rightarrow \infty, \quad (8.44)$$

and (8.29) becoming

$$\frac{\partial \theta_R}{\partial \zeta} = -q_\zeta \text{ at } \zeta = \zeta_w, \quad (8.45)$$

where $-q_\zeta = \sqrt{\tau} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta)$ and $\zeta_w = \frac{1}{2\sqrt{\tau}}$.

Equation (8.28) could be transformed into an ordinary differential equation by using just the variable $\zeta = \eta/\sqrt{\tau}$, that yields the derivatives

$$\frac{\partial}{\partial \tau} = -\frac{1}{2} \frac{\zeta}{\tau} \frac{\partial}{\partial \zeta} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{1}{\sqrt{\tau}} \frac{\partial}{\partial \zeta},$$

which turn equation (8.28) into

$$2\zeta \frac{d^2 \theta_R}{d\zeta^2} + (\zeta^2 + 2) \frac{d\theta_R}{d\zeta} = 0, \quad (8.46)$$

²Bear in mind that, since $\tau' = \tau$, the difference between them will be forgotten when writing them.

but condition (8.45) introduces a dependency with τ . Lets assume for the moment that for $\tau \gg 1$ the dependency of θ_R with respect to τ can be neglected, and that the ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2) \ll 1$, so that we can use equation (8.46) to find a solution³. This leads to

$$\frac{d\theta_R}{d\zeta} = K_1 \frac{e^{-\frac{1}{4}\zeta^2}}{\zeta}, \quad (8.47)$$

where K_1 is found by means of (8.45),

$$K_1 = -\zeta_w e^{\frac{1}{4}\zeta_w^2} q_\zeta = \frac{1}{2} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta) e^{\frac{1}{16\tau}} \approx \frac{1}{2} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta).$$

This approximation is due to the fact that for $\tau \gg 1$, $e^{1/16\tau} \rightarrow 1$, so that the dependency of $\theta_R(\tau, \zeta) \approx \theta_R(\zeta)$.

Integrating (8.47) leads to

$$\theta_R = K_1 \int_{\zeta_w}^{\zeta} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' + K_2,$$

where K_2 is found by means of (8.44). With this,

$$\theta_R = K_1 \left\{ \int_{\zeta_w}^{\zeta} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' - \int_{\zeta_w}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' \right\},$$

so that finally

$$\theta_R \approx \frac{1}{2} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta) \int_{\infty}^{\zeta} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta'. \quad (8.48)$$

Solution (8.48) can be compared with the numerical data obtained for the heat transfer inside the rock, as shown in figure 8.8 where these are compared for fixed times of 5, 15 and 25 years. It can be seen how the pseudo-similarity solution resembles closely the behaviour of the numerical results.

³This assumptions are justified in the Appendix C.

Figure 8.8: Temperature spatial distribution in the rock for the numerical and the similarity solutions at 5, 15 and 25 years. $k_R = 2 \text{ W}/(\text{m}\cdot\text{K})$, $\alpha_R = 1.01 \times 10^6 \text{ m}^2/\text{s}$, and $T_{R\infty} = T_a$ with an initial uniform temperature in the rock.

Chapter 9

Conclusions

Throughout the years, more and more countries are adding high-speed railway lines to their infrastructure. Countries in South-East Asia, Europe and America require an extensive use of bridges and tunnels to develop a route with the least amount of curves, so that the trains can circulate at the highest speed possible. The movement of high-speed trains inside tunnels generates many fluidynamic phenomena; some of them have been addressed in this thesis.

The CFD methods are impossible to perform, given that the computational capacity and time for the calculus would be enormous, because of the high resolution needed to capture the waves inside the tunnel. As such, it is important to generate models that are as simple as possible, while maintaining a good level of agreement with real effects

In chapter 6 a description of the flow equations around a high speed train inside a tunnel is presented. The formulation is simplified according to the region considered: vicinity of the nose and tail; gap between train and tunnel; only tunnel, etc. In each of these regions the most relevant effects are maintained. This allows the creation of a faster computational program that focuses on the important subjects of the physical problem while maintaining an appropriate degree of precision in the calculations, so that it can be sure that safety and comfort standards will be met once the tunnel has been built, specially the safety one that relates with health of the passengers and the workers inside the tunnel.

Three dimensional effects like the ones present in the nose and tail can be solved using the motion equations in integral form to connect the flow variables upstream the nose with the flow downstream in the gap between train and tunnel. The same is applied to the tail. To close the problem experimental coefficients are needed. These coefficients can be obtained by interpreting physically the so called train signature. The use of a chimney that connects the tunnel with the exterior was found to be a passive solution for reducing the pressure and temperature of the flow in the tunnel.

A one dimensional model for the prediction of a shock wave caused by the first compression pressure wave generated when a high-speed train enters a tunnel was proposed in chapter 7; this is caused by the difference on the speed of propagation ($u + a$) between downstream and upstream the wave: the more the wave advances, the more it becomes steeper until eventually could become a weak shock wave. It is fundamental to determine if the entrance of a high-speed train inside a tunnel will generate the aforementioned sonic

boom, because it can be, not only uncomfortable, but even risky for the people, animals and buildings that are surrounding the tunnel exit portal; if such sonic boom is predicted to appear, certain countermeasures can be adopted, such as reducing the train speed, adding tunnel hoods with windows, or portal shafts.

To simulate the distortion of this first compression wave the small changes in the speed of sound, which are of the order of the flow velocity, $\Delta a \sim u \ll a$, must be retained in order to appreciate the difference on speed of propagation above mentioned. To determine this, an algebraic solution was found for the case without friction and heat transfer, in which the flow can be considered isentropic. For the case with friction, steady and unsteady, and heat transfer, a numerical model of simple application based on the characteristics method was proposed. The numerical results were compared with the mentioned algebraic solution, a numerical finite volume TVD method, the numerical data of Miyachi et al. (2013) and the field measurements of Hieke et al. (2011); it was found that the numerical model agrees well with all of them.

It was shown that the most important factors to determine the distance where the shock wave is formed are the maximum initial pressure gradient and the maximum initial pressure amplitude, so that profiles as simple as a hyperbolic tangent, or even a linear profile, can be used instead of the full detailed initial pressure wave (measured or obtained by CFD), as long as the aforementioned factors are the same.

Furthermore, it was found that the unsteady friction term has very little influence on the distance where the shock wave is formed, the regression distance, when compared with the steady friction term. As such, a parametric analysis of a dimensionless form of the regression distance was performed changing two parameters only, the initial maximum pressure amplitude of the pressure wave over the ambient pressure, and a dimensionless coefficient for the steady friction term and heat transfer. It was shown that the dimensionless amplitude has a larger impact than the friction coefficient, and that above all, the initial maximum pressure gradient, which was embedded in the dimensionless regression distance and time, is the most relevant factor.

It was also seen that, even though the friction and heat transfer terms can modify the distance where the shock wave is formed, the difference with respect to a hypothetical case without friction is not very large, and as such the algebraic solution can be used as a fast tool to determine in first instance if the shock wave will be formed; *i.e.* if the regression distance found is larger than the tunnel length, then it is certain that no shock wave will be formed, since the friction terms are conservative and produce an even larger regression distance.

The simple model from chapter 8 presents the possibility to predict an estimate of the temperature rise that would occur inside a long tunnel that lacks a cooling system. The continuous passing of trains through long tunnels along the years is a subject that has not been vastly addressed, and could incur into dangerous situations for the people that walk inside the tunnel; short tunnels are ventilated by means of the currents generated with each passing of trains on both directions, but for long tunnels (which are of the order of tenths of kilometers with only one track) that ventilation is not effective, because the air temperature rises to the wall temperature in the first kilometers inside the tunnel, unable to cool down the rest of the tunnel. In the solutions that were found, a temperature rise

of the order of 20 °C appears in the tunnel wall and the air flow after a period of about 20 years.

Results show that as times goes by, the dependence on time diminishes for both the air and rock temperature, and the rise on temperature and the distance of penetration inside the rock present an asymptotical behaviour. The implementation of a cooling system as a heat sink could easily be implemented along equations (8.15) and (8.24). The magnitude of a project such as a tunnel with these dimensions needs to consider many risks and possibilities that could arise in the future, so that there can be changes in the design stage that avoid these problems. Simple tools as this one allows to predict in a fast way the state of the tunnel throughout the years, so that, by implementing the optimal measures, the temperature rise in the tunnel wall and air flow can be avoided, improving the working conditions for the workers that mantain the tunnel, or for the people that could be in the need to cross the tunnel by foot in case of an emergency.

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Appendix A

Numerical scheme for the chimney

A.1 Governing equations

The governing equations in characteristic form for the chimney are

$$\frac{dp}{dt} + \rho_a a \frac{dv}{dt} = \frac{4(\gamma - 1)q_{ch}}{D_{ch}} - \rho_a g - \frac{4a\tau_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = a, \quad (\text{A.1})$$

$$\frac{dp}{dt} - \rho_a a \frac{dv}{dt} = \frac{4(\gamma - 1)q_{ch}}{D_{ch}} + \rho_a g + \frac{4a\tau_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = -a, \quad (\text{A.2})$$

$$\rho c_p \frac{dT}{dt} - \frac{dp}{dt} = \frac{4v\tau_{ch}}{D_{ch}} + \frac{4q_{ch}}{D_{ch}} \text{ along the lines } \frac{dz}{dt} = v, \quad (\text{A.3})$$

where $\tau_{ch} = \frac{\lambda_{ch}}{8} \rho_a |v|v$, and, by using the Reynolds analogy, $q_{ch} = \frac{\lambda_{ch}}{8} \rho_a |v|(T_w - T)$, with T_w being the temperature at the chimney wall.

Using the following non-dimensional variables

$$\begin{aligned} \tau &= \frac{ta}{L_{Tu}}, \quad \eta = \frac{z}{L_{Tu}}, \quad \varphi = \frac{p - p_a}{p_a}, \quad V = \frac{v}{a}, \\ \theta &= \frac{T - T_a}{T_a}, \quad \theta_w = \frac{T_w - T_a}{T_a}, \quad M_T = \frac{U}{a}, \quad \rho^+ = \frac{\rho}{\rho_a}, \end{aligned}$$

turns equation (A.1) to (A.3) into

$$\begin{aligned} \frac{d}{d\tau}(\varphi + \gamma V) &= \frac{\gamma \lambda_{ch} L_{Tu}}{2D_{ch}} |V| [(\theta_w - \theta) - V] - \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \\ &\text{along the lines } \frac{d\eta}{d\tau} = 1, \quad (\text{A.4}) \end{aligned}$$

$$\begin{aligned} \frac{d}{d\tau}(\varphi - \gamma V) &= \frac{\gamma \lambda_{ch} L_{Tu}}{2D_{ch}} |V| [(\theta_w - \theta) + V] + \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \\ &\text{along the lines } \frac{d\eta}{d\tau} = -1, \quad (\text{A.5}) \end{aligned}$$

$$\frac{d}{d\tau} \left(\varphi - \frac{\gamma}{\gamma - 1} \theta \right) = - \frac{\gamma \lambda_{ch} L_{Tu}}{2(\gamma - 1) D_{ch}} |V| (\theta_w - \theta) \quad \text{along the lines} \quad \frac{d\eta}{d\tau} = V, \quad (\text{A.6})$$

respectively. To obtain the value of ρ^+ the state equation with non-dimensional variables is used

$$\rho^+ = \frac{p_a \varphi + p_a}{T_a \theta + T_a}. \quad (\text{A.7})$$

Equations (A.4) to (A.6) will be discretized using the same scheme as for the tunnel (figure A.1).

Figure A.1: Scheme of the grid used for the numerical model on the chimney.

To obtain the values at $(i, n + 1/2)$ the discretized equations are

$$\begin{aligned} \varphi_{i,n+1/2} + \gamma V_{i,n+1/2} &= \varphi_{i,n} + \gamma V_{i,n} \\ &+ \frac{\Delta\tau}{2} \left[\frac{\gamma \lambda_{ch} L_{Tu}}{2 D_{ch}} |V| [(\theta_w - \theta) - V] - \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \right]_{i,n}, \quad (\text{A.8}) \end{aligned}$$

$$\begin{aligned} \varphi_{i,n+1/2} - \gamma V_{i,n+1/2} &= \varphi_{i+1,n} - \gamma V_{i+1,n} \\ &+ \frac{\Delta\tau}{2} \left[\frac{\gamma \lambda_{Tr} L_{Tr}}{2 D_{Tr}} |V| [(\theta_w - \theta) + V] + \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \right]_{i+1,n}, \quad (\text{A.9}) \end{aligned}$$

and the value of $\rho_{i,n+1/2}^+$ is assumed to be $\frac{1}{2}(\rho_{i,n}^+ + \rho_{i+1,n}^+)$.

To obtain the values at $(i, n + 1)$ the discretized equations are

$$\begin{aligned} \varphi_{i,n+1} + \gamma V_{i,n+1} &= \varphi_{i-1,n} + \gamma V_{i-1,n} \\ &+ \Delta\tau \left[\frac{\gamma \lambda_{ch} L_{Tu}}{2D_{ch}} |V| [(\theta_w - \theta) - V] - \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \right]_{i-1, n+1/2}, \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} \varphi_{i,n+1} - \gamma V_{i,n+1} &= \varphi_{i+1,n} - \gamma V_{i+1,n} \\ &+ \Delta\tau \left[\frac{\gamma \lambda_{ch} L_{Tu}}{2D_{ch}} |V| [(\theta_w - \theta) + V] + \frac{\rho_a g L_{Tu}}{p_a} \rho^+ \right]_{i, n+1/2}, \end{aligned} \quad (\text{A.11})$$

$$\varphi_{i,n+1} - \frac{\gamma}{\gamma - 1} \theta_{i,n+1} = \varphi_{i',n} - \frac{\gamma}{\gamma - 1} \theta_{i',n} - \Delta\tau \left[\frac{\gamma \lambda_{ch} L_{Tu}}{2(\gamma - 1)D_{ch}} |V| (\theta_w - \theta) \right]_{i', n+1/2}. \quad (\text{A.12})$$

A.2 Coupling with the tunnel

The nodes for the coupling are as shown in figure A.2, so that the coupling equations are

$$u_{n_{ch1}} A_{Tu} = u_{n_{ch2}} A_{Tu} + v_1 A_{ch}, \quad (\text{A.13})$$

$$p_{n_{ch1}} - p_{n_{ch2}} = \rho_a (u_{n_{ch2}}^2 - u_{n_{ch1}}^2) + \frac{1}{2} \rho_a v_1 (u_{n_{ch1}} + u_{n_{ch2}}) \frac{A_{ch}}{A_{Tu}}, \quad (\text{A.14})$$

$$p_1 - p_{n_{ch1}} = \frac{1}{2} (p_{n_{ch2}} - p_{n_{ch1}}) - \rho v_1^2, \quad (\text{A.15})$$

$$T_{n_{ch1}} u_{n_{ch1}} A_{Tu} = T_{n_{ch2}} u_{n_{ch2}} A_{Tu} + T_1 v_1 A_{ch}. \quad (\text{A.16})$$

In order to find these nine unknowns, equations (A.13) to (A.16) are complemented with the C^+ that comes from n_{ch1} , the C^- that comes from n_{ch2} , and the C^- that comes from the chimney at 1. To close the system, the way to find the temperatures is as mentioned in section 6.10: if one flow is entering in the control volume that conforms the coupling of tunnel and chimney, and two flows are exiting, then the value of the temperature in the three sections is the same as the one where the flow is entering; if two flows are entering the control volume and one is exiting it, then the temperature in the section where the flow is leaving is found by means of the equation (A.16). For simplification, the value of the temperature of the flow (or flows) that is entering will be the value it has in the adjacent node, which for n_{ch1} would be $n_{ch1} - 1$, for n_{ch2} would be $n_{ch2} + 1$ and for 1 would be the node 2 in the chimney¹.

For the exit of the chimney, the boundary conditions are analogous to what was done with the tunnel portals, except here the atmospheric pressure is $p_a - \rho_a g H$, where H is the height of the chimney.

¹The C^0 could be used to find the temperature in either of the nodes where the flow is entering the control volume, but it was found that this provides very similar results, and it is more complex to implement.

Figure A.2: Scheme of the nodes for the coupling between chimney and tunnel.

Appendix B

Analytic solution for the chimney

The quasi-steady analytic solution for the flow in a chimney inside a tunnel is presented here.

B.1 Continuity

The continuity equation in the z direction is

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial z} + v \frac{\partial \rho}{\partial z} = 0. \quad (\text{B.1})$$

The relation between the spatial terms is

$$\frac{v(\partial \rho / \partial z)}{\rho(\partial v / \partial z)} \sim \frac{\Delta \rho}{\rho} \ll 1,$$

and the order of the temporal term over $\rho(\partial v / \partial z)$ is

$$\frac{\partial \rho / \partial t}{\rho(\partial v / \partial z)} \sim \frac{\Delta \rho}{\rho} \frac{H / v_c}{t_c}.$$

It will be assumed that $(H/v_c)/t_c$ is small, which will be proved further on. This implies that the flow movement is quasi-steady. According to this, the continuity equation is reduced to the incompressible case

$$\frac{\partial v}{\partial z} = 0. \quad (\text{B.2})$$

B.2 Momentum

With the above simplifications, and considering that the velocity will be in the upward direction, the momentum equation becomes

$$0 = -\frac{\partial p}{\partial z} - \rho g - \frac{\lambda_{ch}}{2D_{ch}} \rho v^2, \quad (\text{B.3})$$

where λ_{ch} and D_{ch} are the friction coefficient and the hydraulic diameter of the chimney, respectively. In order to retain the flotability effects this will be written as

$$0 = -\frac{\partial(p + \rho_a g z)}{\partial z} - (\rho - \rho_a)g - \frac{\lambda_{ch}}{2D_{ch}}\rho_a v^2,$$

where the density in the friction term has been changed from ρ to ρ_a given that $(\rho - \rho_a)/\rho_a \ll 1$. However, it is necessary to retain the flotability term $(\rho - \rho_a)g$, which in some cases will be the cause of the movement.

The pressure can be written as $p = p_h + p_m$, where p_h is the pressure variation caused by the fluid static in the exterior of the chimney, that is

$$-\frac{\partial p_h}{\partial z} - \rho_a g = 0 \rightarrow p_h = p_a - \rho_a g z,$$

being p_a the atmospheric pressure at $z = 0$ (the level of the tunnel's portals). According to this

$$p + \rho_a g z = (p_h + \rho_a g z) + p_m = p_a + p_m.$$

With this being said, equation (B.3) becomes

$$0 = -\frac{\partial p_m}{\partial z} - (\rho - \rho_a)g - \frac{\lambda_{ch}}{2D_{ch}}\rho_a v^2, \quad (\text{B.4})$$

where p_m is what differs the pressure p from the fluid static and is caused by the movement of the flow generated by the flotability (if there is no pressure difference imposed at the base of the chimney).

In order to evaluate the density difference term, the state equation can be written as

$$\frac{dp}{p_a} = \frac{d\rho}{\rho_a} + \frac{dT}{T_a}, \quad (\text{B.5})$$

where $dp = dp_h + dp_m$, but dp_h will not contribute to the density or temperature variation in this approximation¹, so that

$$\frac{\Delta p_m}{p_a} = \frac{\rho - \rho_a}{\rho_a} + \frac{T - T_a}{T_a}, \quad (\text{B.6})$$

and as it will be seen further on $(\Delta p_m/p_a) \ll (T - T_a)/T_a$, so that equation (B.4) is reduced to

$$0 = -\frac{\partial p_m}{\partial z} + g \frac{T - T_a}{T_a} - \frac{\lambda_{ch}}{2D_{ch}}\rho_a v^2, \quad (\text{B.7})$$

which has to be integrated with the boundary condition

¹Notice that using the standard atmosphere, in a height of 100 m the temperature is reduced around 0.5 °K and the density around 0.012 kg/m³, against the 288 °K and 1.22 kg/m³ of the respective values of temperature and density at ground level.

$$z = 0: \quad p(0) = p_0 - \frac{1}{2}\rho_a v^2 = p_a + p_m(0),$$

so that

$$p_m(0) = (p_0 - p_a) - \frac{1}{2}\rho_a v^2.$$

It can be observed in this last equation that the changes in p_m are going to be of the order $\rho_a v^2$ or of the order of the imposed pressure difference $p_0 - p_a$. In the case that both are of the same order, $(p_0 - p_a) \sim \rho_a g H (T - T_a)/T_a$, the flotability effects count as much as the imposed pressure difference. If the imposed pressure difference is large enough, then the flotability effects can be neglected.

To obtain velocity v , an additional condition in $z = H$ is required. At the chimney exit the pressure is $p(H) = p_a - \rho_a g H$, which translates to

$$p_a - \rho_a g H = p_a - \rho_a g H + p_m(H),$$

and as such,

$$z = H: \quad p_m(H) = 0.$$

The gradient of p_m at equation (B.7) is of order

$$\frac{1}{\rho_a} \frac{\partial p_m}{\partial z} \sim \frac{\Delta p_m}{\rho_a H},$$

the flotability term is of order

$$\frac{T - T_a}{T_a} g \sim \epsilon g,$$

where $\epsilon = (T_0 - T_a)/T_a$, being T_0 the temperature in the interior of the tunnel, at the base of the chimney. When this term forces the movement, the pressure term must be at most of the same order, which is $\Delta p_m \sim \epsilon \rho_a g H$. Choosing a dimensionless variable

$$\varphi = \frac{p_m}{\epsilon \rho_a g H} \sim 1,$$

the boundary condition at $z = 0$ might be written as

$$\varphi(0) = \frac{p_0 - p_a}{\epsilon \rho_a g H} - \frac{v^2}{2\epsilon g H} = \varphi_0 - \frac{1}{2}u^2, \quad (\text{B.8})$$

where $\varphi_0 = (p_0 - p_a)/(\epsilon \rho_a g H)$ and $u^2 = v^2/(\epsilon g H)$, which indicates that the velocity caused by flotability is of order $\sqrt{\epsilon g H}$ and the pressure variations are of order $\epsilon \rho_a g H$, so that $\varphi_0 \sim 1$. If the imposed pressure difference $p_0 - p_a$ is large compared with $\epsilon \rho_a g H$, the value of φ_0 is large. In this case, the flotability term can be neglected and the velocity will be such that $v \sim \sqrt{(p_0 - p_a)/\rho_a}$. Going back to the case where $\varphi_0 \sim 1$, the moment equation (B.7) can be written as

$$0 = -\frac{\partial\varphi}{\partial\xi} + \theta - \frac{\lambda_{ch}H}{2D_{ch}}u^2, \quad (\text{B.9})$$

being $\theta = (T - T_a)/(T_p - T_a)$ and $\xi = z/H$. The boundary conditions become

$$\xi = 0: \quad \varphi(0) = \varphi(0) - \frac{1}{2}u^2,$$

$$\xi = 1: \quad \varphi(1) = 0.$$

B.3 Energy

The equation for the internal energy is

$$\rho_a v c_v \frac{\partial T}{\partial z} = \frac{\lambda_{ch}}{2D_{ch}} \rho_a v^3 + \frac{\lambda_{ch}}{2D} \rho_a c_p (T_p - T), \quad (\text{B.10})$$

where the term $-p \frac{\partial v}{\partial z}$ of the complete internal energy equation is null, since v does not depend on z . The relation between the friction and the conduction term is²

$$\frac{(\lambda_{ch}/D_{ch})\rho_a v^3}{(\lambda_{ch}/D_{ch})\rho_a v c_p (T_p - T)} \sim \frac{v^2}{c_p \Delta T} \sim \frac{gH}{c_p T} \ll 1,$$

so that equation (B.10) is reduced to

$$\frac{\partial T}{\partial z} = \frac{\lambda_{ch}}{2D_{ch}} \gamma (T_p - T). \quad (\text{B.11})$$

Naming $\theta = (T - T_a)/(T_0 - T_a)$; $\theta_p = (T_p - T_a)/(T_0 - T_a)$; $\eta = \gamma \lambda_{ch} z / (2D_{ch}) = \gamma \lambda_{ch} H \xi / (2D_{ch})$, equation (B.11) becomes

$$\frac{\partial \theta}{\partial \eta} = \theta_p - \theta, \quad (\text{B.12})$$

which must be integrated with condition $\theta(0) = 1$ and $\theta_p(\eta)$ should be given. The problem to solve then is

$$\frac{\partial \varphi}{\partial \xi} = \theta - \frac{\lambda_{ch}H}{2D_{ch}}u^2, \quad (\text{B.13})$$

$$\frac{\partial \theta}{\partial \eta} = \theta_p - \theta, \quad (\text{B.14})$$

$$\theta_p = \theta_p(\eta), \quad (\text{B.15})$$

with boundary conditions

²When $p_0 - p_a \sim \rho g H$ ($\varphi_0 \gg 1$) the ratio $v^2/c_p \Delta T \sim gH/c_p \Delta T$ is also small to the unity, with $(p_0 - p_a/p_a \ll 1)$.

$$\xi = 0, (\eta = 0) : \varphi = \varphi_0 - \frac{1}{2}u^2 ; \theta = 1,$$

$$\xi = 1, \left(\eta = \frac{\gamma\lambda_{ch}H}{2D_{ch}} \right) : \varphi = 0.$$

The solution will depend on the value of θ_p and as such, different cases are presented.

B.3.1 Adiabatic case

The energy equation (B.11) is reduced to $\partial T / \partial z = 0$, since there is no heat flux through the wall. Solution of (B.14) is $\theta = 1$. Introducing this value in equation (B.13) then

$$\frac{\partial \varphi}{\partial \xi} = 1 - \frac{\lambda_{ch}}{2D_{ch}}u^2, \quad (\text{B.16})$$

to be integrated with conditions

$$\xi = 0 : \varphi = \varphi_0 - \frac{1}{2}u^2$$

$$\xi = 1 : \varphi = 0.$$

Solution, with the condition at $\xi = 0$, is

$$\varphi = \varphi_0 + \xi - \left(1 + \frac{\lambda_{ch}}{D_{ch}} \right) \frac{1}{2}u^2.$$

The velocity u is obtained with the condition at $\xi = 1$

$$u^2 = \frac{2(1 + \varphi_0)}{1 + \lambda_{ch}H/D_{ch}}.$$

Notice that the velocity is zero when $\varphi_0 = -1$, which is equivalent to

$$p_0 - p_a = -\rho_a g H \left(\frac{T_0 - T_a}{T_a} \right),$$

this means that a depression ($p_0 - p_a < 0$) compensates the flotability ($T_0 - T_a > 0$), or an over pressure ($p_0 - p_a > 0$) is compensated by cooling ($T_0 - T_a < 0$).

B.3.2 Isotherm case

Here θ_p is a constant, so that the energy equation (B.14) is integrated to give

$$\theta = \theta_p + (1 - \theta_p)e^{-\eta},$$

which substituted at (B.13) provides

$$\frac{\gamma\lambda_{ch}H}{2D_{ch}} \frac{\partial\varphi}{\partial\eta} = \theta_p + (1 - \theta_p)e^{-\eta} - \frac{\lambda_{ch}H}{2D_{ch}} u^2,$$

whose solution, with the condition at $\eta = 0$, is

$$\frac{\gamma\lambda_{ch}H}{2D_{ch}} \varphi = \frac{\gamma\lambda_{ch}H}{2D_{ch}} \left(\varphi_0 - \frac{1}{2} u^2 \right) + \left(\theta_p - \frac{\lambda_{ch}H}{2D_{ch}} u^2 \right) \eta + (1 - \theta_p)(1 - e^{-\eta}).$$

The condition in $\eta = \gamma\lambda_{ch}H/2D_{ch}$ ($\xi = 1$) provides the value of u

$$u^2 = \frac{2 \left\{ \varphi_0 + \theta_p + \frac{2D_{ch}}{\gamma\lambda_{ch}H} (1 - \theta_p)(1 - e^{-\frac{\gamma\lambda_{ch}H}{2D_{ch}}}) \right\}}{1 + \lambda_{ch}H/D_{ch}}.$$

B.3.3 Wall temperature with a lineal distribution

For this case $\theta_p = a - b\eta$, where a and b are known constants. The energy equation becomes

$$\frac{\partial\theta}{\partial\eta} = a - b\eta - \theta,$$

whose solution, with the condition of $\theta = 1$ at $\eta = 0$, is

$$\theta = a + b - b\eta + [1 - (a + b)]e^{-\eta}.$$

With this, equation (B.13) becomes

$$\frac{\gamma\lambda_{ch}H}{2D_{ch}} \frac{\partial\varphi}{\partial\eta} = \left(a + b - \frac{\lambda_{ch}H}{2D_{ch}} \right) - b\eta + [1 - (a + b)]e^{-\eta},$$

which, with the condition at $\eta = 0$, has the solution

$$\frac{\gamma\lambda_{ch}H}{2D_{ch}} \varphi = \frac{\gamma\lambda_{ch}H}{2D_{ch}} \left(\varphi_0 - \frac{u^2}{2} \right) + \left(a + b - \frac{\lambda_{ch}H}{2D_{ch}} u^2 \right) \eta - \frac{1}{2} b\eta^2 + [1 - (a + b)](1 - e^{-\eta}).$$

The condition at $\eta = \frac{\gamma\lambda_{ch}H}{2D_{ch}}$ ($\xi = 1$) yields the velocity u

$$u^2 = \frac{2 \left\{ \varphi_0 + a + b - \frac{\gamma\lambda_{ch}H}{D_{ch}} \frac{b}{4} + \frac{2D_{ch}}{\gamma\lambda_{ch}H} [1 - (a + b)] \left(1 - e^{-\frac{\gamma\lambda_{ch}H}{2D_{ch}}} \right) \right\}}{1 + \lambda_{ch}H/D_{ch}}.$$

B.4 Quasi-steady assumption

In order for the flow to be considered quasi stationary it is necessary that $(H/v_c)/t_c$ be small.

The characteristic time is given by the evolution of the flow in the tunnel, and as such is of order

$$t_c \sim \frac{\epsilon}{M^2} \frac{L_{Tu}}{U},$$

where L_{Tu} is the length of the tunnel, U is the train velocity and M is the Mach number of the train. As such

$$\frac{H/v_c}{t_c} \sim \frac{H}{\sqrt{\epsilon g H}} \frac{M^2}{\epsilon} \frac{U}{L_{Tu}} \sim \frac{H}{L_{Tu}} \frac{M^2}{\epsilon} \frac{U}{\sqrt{\epsilon g H}}.$$

For typical values such as $H = 100$ m; $L_{Tu} = 30000$ m; $U = 80$ m/s and $\epsilon = 1/30$ the Strouhal number is obtained, $(H/v_c)/t_c \sim 8 \times 10^{-2}$.

Condition $(H/v_c)/t_c \ll 1$ is necessary so that the temporal term of the momentum equation, $\rho(\partial v/\partial t)$, and of the energy equation, $\rho c_v(\partial T/\partial t)$, can be neglected. In the continuity equation (B.1) the temporal term can be neglected if $\epsilon(H/v_c)/t_c$ is small, which is secured with the condition $St \ll 1$, where St is the Strouhal number.

B.5 Comparison with the numerical solution

These analytic solutions are compared with the numerical ones obtained with the scheme from appendix A.

Figures B.1 to B.3 show the adiabatic wall case for $\varphi_0 = 0$, 1 and 50, respectively; with the temperature at the base being 10 °C above the ambient temperature.

Figures B.4 to B.6 show the isotherm wall case for $\varphi_0 = 0$, 1 and 50, respectively; with the wall being at the ambient temperature, and the temperature at the base 10 °C above it.

Figures B.7 to B.9 show the lineal distribution temperature wall case for $\varphi_0 = 0$, 1 and 50, respectively; with the wall being 4 °C higher at the base than at the top, and a base temperature 10 °C above the ambient temperature.

Figure B.1: Comparison of the analytic and the numerical solution of the chimney flow for an adiabatic case with $\varphi_0 = 0$ for, a), the pressure, and b), the temperature. Flow velocity has a difference of 2.03 %.

Figure B.2: Comparison of the analytic and the numerical solution of the chimney flow for an adiabatic case with $\varphi_0 = 1$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 0.65 %.

Figure B.3: Comparison of the analytic and the numerical solution of the chimney flow for an adiabatic case with $\varphi_0 = 50$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 0.4 %.

Figure B.4: Comparison of the analytic and the numerical solution of the chimney flow for an isotherm wall case with $\varphi_0 = 0$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 1.3 %.

Figure B.5: Comparison of the analytic and the numerical solution of the chimney flow for an isotherm wall case with $\varphi_0 = 1$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 0.04 %.

Figure B.6: Comparison of the analytic and the numerical solution of the chimney flow for an isotherm wall case with $\varphi_0 = 50$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 0.91 %.

Figure B.7: Comparison of the analytic and the numerical solution of the chimney flow for a lineal distribution wall temperature case with $\varphi_0 = 0$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 1.19 %.

Figure B.8: Comparison of the analytic and the numerical solution of the chimney flow for a lineal distribution wall temperature case with $\varphi_0 = 1$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 0.29 %.

Figure B.9: Comparison of the analytic and the numerical solution of the chimney flow for a lineal distribution wall temperature case with $\varphi_0 = 50$ for, a), the pressure, and b), the temperature. Flow velocity has a discrepancy of 1.02 %.

It can be seen how the air temperature is not affected by the change of value of φ_0 , and how when $\varphi_0 \ll 1$ the temperature at the wall is irrelevant for the value of the pressure, as it was mentioned previously.

For all cases, it can be seen that the numerical solution resembles quite closely the analytic solutions in pressure and temperature, while the air flow velocity has discrepancies of around 1%. With this in mind, we consider the numerical scheme suitable to be coupled to the simulation of the high-speed train inside the tunnel.

Appendix C

The value of $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ when $\tau \ll 1$

In order to prove that the solution

$$\theta_R = \frac{1}{2} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta) e^{\frac{1}{16\tau}} \int_{\infty}^{\zeta} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta'. \quad (C.1)$$

found by means of

$$2\zeta \frac{d^2\theta_R}{d\zeta^2} + (\zeta^2 + 2) \frac{d\theta_R}{d\zeta} = 0, \quad (C.2)$$

is valid for $\tau \gg 1$, the term $\tau(\partial\theta_R)/(\partial\tau)$ must be very small when compared with one of the two terms of the right hand side of

$$\tau \frac{\partial\theta_R}{\partial\tau} = \frac{\partial^2\theta_R}{\partial\zeta^2} + \left(\frac{1}{2}\zeta + \frac{1}{\zeta}\right) \frac{\partial\theta_R}{\partial\zeta}. \quad (C.3)$$

Let $\partial^2\theta_R/\partial\zeta^2$ be chosen and these terms evaluated using (C.1), which for comodity can be written as

$$\theta_R = -K e^{\frac{1}{16\tau}} \int_{\zeta}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta', \quad (C.4)$$

where $K = \frac{1}{2} \frac{k_f}{k_R} Nu_D (\theta_{R_w} - \theta)$, has a negative value, and can be considered a constant¹. With this in mind

$$\tau \frac{\partial\theta_R}{\partial\tau} = \frac{K}{16\tau} e^{\frac{1}{16\tau}} \int_{\zeta}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta', \quad (C.5)$$

¹It is important to notice that strictly speaking K is not a constant because it depends on the value of the heat transfer that fluctuates during the passing of a train and the absence of it, but, as it was established before, these times are very small when compared to the characteristic time of heat diffusion in the rock and in the long term the value of the heat transfer fluctuates around a constant value.

$$\frac{\partial\theta_R}{\partial\zeta} = Ke^{\frac{1}{16\tau}} \frac{e^{-\frac{1}{4}\zeta^2}}{\zeta}, \quad (\text{C.6})$$

and

$$\frac{\partial^2\theta_R}{\partial\zeta^2} = -Ke^{\frac{1}{16\tau}} \left[\frac{1}{2} + \frac{1}{\zeta^2} \right] e^{-\frac{1}{4}\zeta^2}. \quad (\text{C.7})$$

To evaluate the ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ the integral in (C.4) will be expanded using integration by parts, that is

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Applying a change of variable

$$z = \frac{1}{4}\zeta'^2, \quad \zeta' = 2\sqrt{z}, \quad d\zeta' = \frac{1}{\sqrt{z}}dz,$$

then the integral in (C.4) becomes

$$\int_{\zeta}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' = \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{e^{-z}}{2z} dz.$$

For the case where $\eta \sim 1$, and hence, $\zeta \ll 1$,

$$u = \frac{1}{2}e^{-z}, \quad du = -\frac{1}{2}e^{-z}dz; \quad dv = \frac{1}{z}dz, \quad v = \ln(z),$$

which leads to

$$\int_{\frac{1}{4}\zeta^2}^{\infty} \frac{e^{-z}}{2z} dz = \frac{1}{2}e^{-z} \ln(z) \Big|_{\frac{1}{4}\zeta^2}^{\infty} + \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2}e^{-z} \ln(z) dz. \quad (\text{C.8})$$

The first term on the right, when $\zeta \rightarrow \infty$, tends to zero, since by use of l'Hôpital rule

$$\lim_{z \rightarrow \infty} \frac{\ln(z)}{e^z} = \lim_{z \rightarrow \infty} \frac{1/z}{e^z} = 0.$$

The integral on the right hand side of (C.8) is evaluated in a similar fashion as before, that is

$$u = \frac{1}{2}e^{-z}, \quad du = -\frac{1}{2}e^{-z}dz; \quad dv = \ln(z)dz, \quad v = z \ln(z) - z,$$

$$\int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2}e^{-z} \ln(z) dz = \frac{1}{2}e^{-z} [z \ln(z) - z] \Big|_{\frac{1}{4}\zeta^2}^{\infty} + \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2}e^{-z} [z \ln(z) - z] dz. \quad (\text{C.9})$$

Same as before, the first term on the right side of (C.9) tends to zero when $z \rightarrow \infty$. This yields

$$\int_{\zeta}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' = -\frac{1}{2}e^{-\frac{1}{4}\zeta^2} \left[\ln\left(\frac{1}{4}\zeta^2\right) + \frac{1}{4}\zeta^2 \ln\left(\frac{1}{4}\zeta^2\right) - \frac{1}{4}\zeta^2 + o\{\zeta^4 \ln(\zeta^2)\} \right]. \quad (\text{C.10})$$

The ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ is

$$\begin{aligned} \frac{\tau\partial\theta_R/\partial\tau}{\partial^2\theta_R/\partial\zeta^2} &= -\frac{1}{16\tau} \frac{Ke^{\frac{1}{16\tau}} \frac{1}{2} e^{-\frac{1}{4}\zeta^2} \left[\ln\left(\frac{1}{4}\zeta^2\right) + \frac{1}{4}\zeta^2 \ln\left(\frac{1}{4}\zeta^2\right) - \frac{1}{4}\zeta^2 + o\{\zeta^4 \ln(\zeta^2)\} \right]}{-Ke^{\frac{1}{16\tau}} e^{-\frac{1}{4}\zeta^2} \left[\frac{1}{2} + \frac{1}{\zeta^2} \right]} \\ &= \frac{1}{16\tau} \frac{1}{2} \left[\frac{\ln\left(\frac{1}{4}\zeta^2\right) + \frac{1}{4}\zeta^2 \ln\left(\frac{1}{4}\zeta^2\right) - \frac{1}{4}\zeta^2 + o\{\zeta^4 \ln(\zeta^2)\}}{\frac{1}{2} + \frac{1}{\zeta^2}} \right]. \quad (\text{C.11}) \end{aligned}$$

Focusing on the term inside the brackets of (C.11), neglecting terms of order $o\{\zeta^4 \ln(\zeta^2)\}$, and using again the variable $z = \frac{1}{4}\zeta^2$ for clarity, it can be seen that

$$\lim_{z \rightarrow 0} \frac{\ln(z) + z \ln(z) - z}{\frac{1}{2} + \frac{1}{4z}} = \lim_{z \rightarrow 0} \frac{\frac{z}{2} + \ln(z)}{-\frac{1}{4z^2}} = \lim_{z \rightarrow 0} \frac{-\frac{1}{z^2} + \frac{1}{z}}{\frac{1}{2z^3}} = \lim_{z \rightarrow 0} -2z + 2z^2 = 0,$$

which proves that the ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ tends to zero when $\zeta \rightarrow 0$.

For the case when $\zeta \sim 1$ it can be seen that the term inside brackets of (C.11) is of order unity, and as such

$$\frac{\tau\partial\theta_R/\partial\tau}{\partial^2\theta_R/\partial\zeta^2} \sim \frac{1}{\tau},$$

which means that for $\tau \gg 1$, the ratio can be neglected.

Now, for the case when $\zeta \gg 1$, the integral in (C.4) will be expanded again using integration by parts, but this time

$$u = \frac{1}{z}, \quad du = -\frac{1}{z^2} dz; \quad dv = \frac{1}{2} e^{-z} dz, \quad v = -\frac{1}{2} e^{-z},$$

so that

$$\int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2} \frac{e^{-z}}{z} dz = -\frac{1}{2} \frac{e^{-z}}{z} \Big|_{\frac{1}{4}\zeta^2}^{\infty} - \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2} \frac{e^{-z}}{z^2} dz = 2 \frac{e^{-\frac{1}{4}\zeta^2}}{\zeta^2} - \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2} \frac{e^{-z}}{z^2} dz. \quad (\text{C.12})$$

Repeating the steps for the integral on the right hand side of (C.12),

$$u = \frac{1}{z^2}, \quad du = -\frac{2}{z^3} dz; \quad dv = \frac{1}{2} e^{-z} dz, \quad v = -\frac{1}{2} e^{-z},$$

which yields

$$-\int_{\frac{1}{4}\zeta^2}^{\infty} \frac{1}{2} \frac{e^{-z}}{z^2} dz = \frac{1}{2} \frac{e^{-z}}{z^2} \Big|_{\frac{1}{4}\zeta^2}^{\infty} + \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{e^{-z}}{z^3} dz = -8 \frac{e^{-\frac{1}{4}\zeta^2}}{\zeta^4} + \int_{\frac{1}{4}\zeta^2}^{\infty} \frac{e^{-z}}{z^3} dz. \quad (\text{C.13})$$

For the integral on the right side of (C.13)

$$u = \frac{1}{z^3}, \quad du = -\frac{3}{z^4}dz; \quad dv = e^{-z}dz, \quad v = -e^{-z},$$

which yields

$$\int_{\frac{1}{4}\zeta^2}^{\infty} \frac{e^{-z}}{z^3} dz = -\frac{e^{-z}}{z^3} \Big|_{\frac{1}{4}\zeta^2}^{\infty} - \int_{\frac{1}{4}\zeta^2}^{\infty} 3\frac{e^{-z}}{z^4} dz = 64\frac{e^{-\frac{1}{4}\zeta^2}}{\zeta^6} - \int_{\frac{1}{4}\zeta^2}^{\infty} 3\frac{e^{-z}}{z^4} dz, \quad (\text{C.14})$$

and so on, so that

$$\int_{\zeta}^{\infty} \frac{e^{-\frac{1}{4}\zeta'^2}}{\zeta'} d\zeta' = e^{-\frac{1}{4}\zeta^2} \left[\frac{2}{\zeta^2} - \frac{8}{\zeta^4} + \frac{64}{\zeta^6} - o\left(\frac{1}{\zeta^8}\right) \right]. \quad (\text{C.15})$$

With this, the ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ can be written as

$$\frac{\tau\partial\theta_R/\partial\tau}{\partial^2\theta_R/\partial\zeta^2} = -\frac{1}{16\tau} \left[\frac{\frac{2}{\zeta^2} - \frac{8}{\zeta^4} + \frac{64}{\zeta^6} - o\left(\frac{1}{\zeta^8}\right)}{\frac{1}{2} + \frac{1}{\zeta^2}} \right], \quad (\text{C.16})$$

so that

$$\lim_{\zeta \rightarrow \infty} \frac{\tau\partial\theta_R/\partial\tau}{\partial^2\theta_R/\partial\zeta^2} = 0.$$

To illustrate this, figure C.1 shows the absolute value of $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ against ζ for different values of τ . Notice how, when $\zeta \rightarrow 0$, the value of the ratio decays, when $\zeta \sim 1$ the ratio increases, and its value decays again when $\zeta \rightarrow \infty$; also, for any fixed ζ , when τ increases, the absolute value of the ratio decreases.

This proves that for $\tau \gg 1$, the solution (C.1) can be treated as an approximate similarity solution

Figure C.1: Absolute value of the ratio $\tau(\partial\theta_R/\partial\tau)/(\partial^2\theta_R/\partial\zeta^2)$ from the pseudo-similarity solution of θ_R against ζ for different values of τ .