Faraday waves, streaming flow, and relaxation oscillations in nearly circular containers

M. Higuera and J. Porter

E.T.S.I. Aeronáuticos, Universidad Politécnica de Madrid, Plaza Cardenal Cisneros 3, 28040 Madrid, Spain

E. Knobloch

Department of Physics, University of California, Berkeley, California 94720, USA

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Faraday waves near onset in an elliptical container are described by a third-order system of ordinary differential equations with characteristic slow-fast structure. These equations describe the interaction of standing waves with a weakly damped streaming flow driven by Reynolds stresses in boundary layers at the free surface and the rigid walls, and capture the proliferation with decreasing damping of periodic and nonperiodic relaxation oscillations observed near onset in previous simulations. These structures are the result of slow drift through symmetry-related Hopf bifurcations.

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Surface gravity-capillary waves in a vertically vibrated elliptical container may exhibit relaxation oscillations near onset of the Faraday instability. These oscillations are the result of the coupling of the wave amplitudes to a weakly damped streaming flow driven by Reynolds stresses in boundary layers at the free surface and the rigid walls. In an appropriate limit, the properties of these waves are described by a three-dimensional set of ordinary differential equations that provide a description of both their origin and properties.

I. INTRODUCTION

In recent years, there has been considerable interest in the origin and properties of relaxation oscillations in systems of physical and biological interest. Much of this interest centers on problems arising in neurobiology, and in particular on the mechanisms, both chemical and mathematical, behind the onset and control of neural “spiking.” Mixed-mode oscillations, consisting of a certain number of small and large spikes, have been observed in both chemical experiments and in *in situ* neural activity measurements. Various models for these observations have been proposed, and it is now understood that there are several distinct mechanisms that lead to such oscillations. These include the breakup of an invariant torus,¹ the destruction or loss of stability of a Shil’nikov homoclinic orbit,² and proximity to a Shil’nikov–Hopf bifurcation.^{3,4} More recently, it has been recognized that a type of mathematical behavior known as the *canard* phenomenon, whereby a solution drifts for a time along an unstable slow manifold of the system, may occur in these systems and that it provides a natural explanation of the observed dynamics.⁵

In contrast to the systems described above, fluid systems undergo relaxation oscillations only rarely. One of the few systems where relaxation oscillations are predicted is the Faraday system. In this system, the vertical vibration of a liquid-filled container leads to a parametric instability that generates gravity-capillary waves on the free surface of the

liquid once the imposed acceleration exceeds a critical amplitude. If the viscosity of the liquid is low, its effects are largely confined to viscous boundary layers at the free surface and the rigid walls of the container. Reynolds stresses generated in these boundary layers then drive a mean flow called a *streaming* flow in the nominally inviscid bulk. Under appropriate conditions, this streaming flow can interact with the waves responsible for the viscous boundary layers in the first place, producing an interesting interaction between the waves and bulk flow. In this system, relaxation oscillations are possible when the time scale for the evolution of the streaming flow is much longer than the time scale for the evolution of the amplitude of the surface waves. This is precisely the case in nearly inviscid systems. As a consequence, the Faraday system offers an experimentally accessible fluid system in which relaxation oscillations may be observable, and this type of oscillation may be the *first* state that is observed beyond the onset of the Faraday instability.⁶ Experiments in slightly rectangular containers^{7,8} have indeed found (chaotic) relaxation oscillation-type behavior near onset, as well as other long time-scale dynamics further above criticality; see Ref. 9 for a qualitative explanation of these results. Other fluid systems exhibiting relaxation oscillations include binary fluid mixtures heated from below. In bounded containers, this system exhibits a state, also right at onset, that has been called a *repeated transient*.^{10,11} On the other hand, in periodic systems with a large spatial period, a different type of relaxation oscillation may be present, characterized by the spontaneous growth of almost stationary spatially localized dissipative structures called *convectons* due to an instability of the conduction state; these structures are then eroded on a much longer time scale until the system returns to the conduction state, and the process repeats.¹² In water-ethanol mixtures, these relaxation oscillations may involve a low amplitude spatiotemporally complex state called dispersive chaos, instead of the zero-amplitude conduction state.¹³

In the following, we briefly summarize the coupled equations used to describe the nearly inviscid Faraday system and summarize their essential properties. We then introduce the scaling that simplifies these equations to a set of three ordinary equations and show that these capture the essence of the relaxation oscillations in the Faraday system. Finally we use these equations to shed light on the origin of these oscillations and their properties.

II. COUPLED AMPLITUDE-STREAMING FLOW EQUATIONS IN SLIGHTLY ELLIPTICAL DOMAINS

The weakly nonlinear evolution of nearly inviscid Faraday waves in small aspect ratio domains is governed by a system of *coupled amplitude-streaming flow equations*.⁹ Of the finite geometries considered in Ref. 9, the (nearly) circular geometry is of special interest. In such geometries, the free oscillations may be described in terms of clockwise and counterclockwise rotating waves. If the container is circular, these waves are coupled through the parametric forcing and the primary state is a reflection-symmetric standing wave. Because of this reflection symmetry, only the phase of the standing wave couples with the streaming flow. However, as shown in Ref. 9, if the geometry of the container is perturbed from the perfectly circular case, both the phase *and* the amplitude of the standing waves couple to the streaming flow. In cylindrical coordinates, the governing coupled amplitude-streaming flow equations derived in Ref. 9 may be written in the form

$$A'_{\pm}(t) = -(1 + i\Gamma)A_{\pm} + i\Lambda A_{\mp} + i(\alpha_1|A_{\pm}|^2 + \alpha_2|A_{\mp}|^2)A_{\pm} + i\mu\bar{A}_{\mp} + i\Omega A_{\pm} \int_{-1}^0 \int_0^{2\pi} \int_0^R g(r,z) \mathbf{u} \cdot \mathbf{e}_{\theta} r dr d\theta dz, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} - [\mathbf{u} + \mathbf{G}(A_+, A_-)] \times (\nabla \times \mathbf{u}) = -\nabla p + \text{Re}^{-1} \Delta \mathbf{u}, \quad (2)$$

with $\nabla \cdot \mathbf{u} = 0$. In these equations, $A_{\pm}$ are the complex amplitudes of clockwise and counterclockwise rotating waves, $\mathbf{u}$ is the streaming flow, Ω is a known constant, the nonlinear coefficients α_1 and α_2 are real, and the quantities Γ , μ , and Λ measure the detuning, forcing amplitude, and ellipticity of the container, respectively. The detuning Γ takes into account the mismatch between half the forcing frequency and the natural frequency of inviscid oscillations, and includes the frequency shift due to viscosity, while Λ can be considered as (half) the frequency difference between inviscid standing oscillations symmetric and antisymmetric about the major axes of the ellipse. The time has been scaled by the inverse surface wave damping rate $\delta \equiv \gamma_1 C_g^{1/2} + \gamma_2 C_g$, which depends on the fluid viscosity ν , fluid density ρ , surface tension T , depth h , and the acceleration due to gravity g through $C_g \equiv \nu(g h^3 + T h / \rho)^{-1/2} \ll 1$ and the known constants γ_1 and γ_2 , which depend on the excited mode.¹⁴ This scaling is responsible for the appearance of the Reynolds number $\text{Re} \equiv (\gamma_1 C_g^{1/2} + \gamma_2 C_g) / C_g$ in the Navier–Stokes-like Eq. (2) for the streaming flow.

Equation (2) contains a Stokes drift term $\mathbf{G}$ given by

$$\mathbf{G} = (|A_-|^2 - |A_+|^2) g(r, z) \mathbf{e}_{\theta}, \quad (3)$$

and is subject to the boundary conditions

$$\mathbf{u} = [\varphi_1 A_+ \bar{A}_- e^{2im\theta} + \text{c.c.} + \varphi_2 (|A_+|^2 + |A_-|^2)] \mathbf{n}_0 \times \mathbf{e}_{\theta} + [i\varphi_3 A_+ \bar{A}_- e^{2im\theta} + \text{c.c.} + \varphi_4 (|A_-|^2 - |A_+|^2)] \mathbf{e}_{\theta} \quad (4)$$

at $r=R$ and $z=-1$, and

$$\mathbf{u} \cdot \mathbf{e}_z = \frac{\partial \mathbf{u}}{\partial z} \cdot \mathbf{e}_r = 0, \quad \frac{\partial \mathbf{u}}{\partial z} \cdot \mathbf{e}_{\theta} = \varphi_5 (|A_-|^2 - |A_+|^2) \quad (5)$$

at the free surface $z=0$. Here $\mathbf{n}_0$ denotes the outward unit normal, m is the azimuthal wave number, and the scalar functions $\varphi_1, \dots, \varphi_5$ and g are all real, independent of θ and computable in terms of the components of the excited inviscid linear mode of the system.⁹

With $\Lambda \neq 0$ (finite ellipticity), the $O(2)$ symmetry of the circular container is broken, but Eqs. (1)–(5) remain equivariant under the transformations

$$\rho_1: A_{\pm} \rightarrow -A_{\pm}, \quad \theta \rightarrow \theta, \quad \mathbf{u} \cdot \mathbf{e}_{\theta} \rightarrow \mathbf{u} \cdot \mathbf{e}_{\theta}, \quad (6a)$$

$$\rho_2: A_{\pm} \leftrightarrow A_{\mp}, \quad \theta \rightarrow -\theta, \quad \mathbf{u} \cdot \mathbf{e}_{\theta} \rightarrow -\mathbf{u} \cdot \mathbf{e}_{\theta}, \quad (6b)$$

$$\rho_3: A_{\pm} \leftrightarrow (-1)^m A_{\pm}, \quad \theta \rightarrow \theta + \pi, \quad \mathbf{u} \cdot \mathbf{e}_{\theta} \rightarrow \mathbf{u} \cdot \mathbf{e}_{\theta}. \quad (6c)$$

These symmetries derive, respectively, from time translation through one period of the sinusoidal forcing (the minus sign indicating subharmonic instability), from spatial reflection about the major axis (taken to be $\theta=0$), and from rotation by π .

The interaction between surface waves and the streaming flow simplifies if the nonpotential term $-\mathbf{u} \times (\nabla \times \mathbf{u})$ in Eq. (2) can be neglected, which occurs whenever the magnitude of the streaming flow velocity remains small. In this case, Eqs. (2)–(5) for the streaming flow simplify dramatically and may be replaced by a one-dimensional linear eigenvalue problem for the azimuthal component of the streaming flow (the radial and axial components are slaved to the surface waves). This linear system can then be solved by expanding the azimuthal velocity in terms of the hydrodynamic modes. If only the most important of these modes is considered, we obtain a single-mode approximation in the form⁹

$$\dot{X} = \mu \bar{X} - X - i(\Gamma + \Lambda)X - 2\gamma Y + i(\alpha_2 - \alpha_1) \bar{X} Y^2 + i[(\alpha_1 + \alpha_2)|X|^2 + 2\alpha_1|Y|^2]X, \quad (7a)$$

$$\dot{Y} = \mu \bar{Y} - Y - i(\Gamma - \Lambda)Y + 2\gamma X + i(\alpha_2 - \alpha_1) \bar{Y} X^2 + i[(\alpha_1 + \alpha_2)|Y|^2 + 2\alpha_1|X|^2]Y, \quad (7b)$$

$$\dot{v} = \varepsilon[-v + i(\bar{X}Y - \bar{Y}X)], \quad (7c)$$

where the overdot denotes differentiation with respect to t . The new variables are defined by

$$X = \frac{e^{i/4}}{2}(A_+ - A_-), \quad Y = \frac{e^{-i/4}}{2}(A_+ + A_-), \quad v = -\frac{v_1}{2}, \quad (8)$$

where v_1 is the dominant hydrodynamic mode in the above-mentioned expansion for the azimuthal component of the streaming flow. The real coefficients α_1 and α_2 pertain to the circular case, γ results from the integral in Eq. (1), and $\hat{\varepsilon} = -\lambda_1/\text{Re} > 0$, where $\lambda_1 < 0$ is the first purely azimuthal hydrodynamic eigenvalue. Note that $\hat{\varepsilon} \propto 1/\text{Re} = C_g/(\gamma_1 C_g^{1/2} + \gamma_2 C_g)$ reflects the ratio of the surface wave dissipation time scale $1/|\gamma_1 C_g^{1/2} + \gamma_2 C_g|$ to the viscous time scale $1/C_g$. In shallow containers, the damping of the surface waves tends to be dominated by the Stokes boundary layers and $1/\text{Re} = O(\sqrt{C_g})$. In such cases, including water or silicon oil in centimeter-deep containers, the streaming flow is only weakly damped and is easily driven by time-averaged Reynolds stresses. The slow dynamics of this streaming flow (relative to the surface waves) leads to the possibility of relaxation oscillations.

Equations (7a)–(7c) are equivariant with respect to the group D_2 generated by the reflections

$$R_1: (X, Y, v) \rightarrow (-X, Y, -v), \quad (9a)$$

$$R_2: (X, Y, v) \rightarrow (X, -Y, -v), \quad (9b)$$

corresponding to ρ_2 and $\rho_1 \circ \rho_2$ in Eqs. (6a)–(6c), respectively. Note that these are not the only relations to Eqs. (6a)–(6c). In particular, if the azimuthal wave number m is even, then R_1 also corresponds to $\rho_3 \circ \rho_2$, which represents a reflection about the minor axis of the ellipse, while if m is odd, this same reflection corresponds to R_2 . These symmetries are of crucial importance for the behavior of the system. Of the four cases considered in Ref. 6, case IV is of special interest since it results in complex dynamics close to the primary bifurcation, where the magnitude of the streaming flow remains small as required for the validity of Eqs. (7a)–(7c). This case forms the focus of the present paper.

Although the basic solutions of Eqs. (7a)–(7c) were described in Ref. 6, we summarize them here for convenience. To begin with, there are two types of steady states: pure modes (P), which are associated with vanishing azimuthal flow, and mixed modes (M), which do exhibit azimuthal flow.

Pure modes come in two varieties, one symmetric about the major axis of the container and the other antisymmetric. We denote these by $P_+ : (0, Y, 0) = (0, R_+ e^{i\phi_+}, 0)$ and $P_- : (X, 0, 0) = (R_- e^{i\phi_-}, 0, 0)$; these states are invariant under R_1 and R_2 , respectively, and their amplitudes satisfy

$$[\Gamma \mp \Lambda - (\alpha_1 + \alpha_2)R_{\pm}^2]^2 = \mu^2 - 1. \quad (10)$$

The $P_{\pm}$ solutions bifurcate from the trivial state at

$$\mu_{\pm} = [1 + (\Gamma \mp \Lambda)^2]^{1/2}, \quad (11)$$

and do so subcritically if $(\Gamma \mp \Lambda)(\alpha_1 + \alpha_2) > 0$. The instabilities of $P_{\pm}$ are of two types: either the unstable eigenvector respects the symmetry of the solution, or it does not. Steady-state bifurcations of the first type lead to saddle-node (SN) bifurcations, while steady-state bifurcations of the second type are symmetry-breaking (SB) bifurcations and generate

FIG. 1. (Color online) Bifurcation diagram showing the Euclidean norm $\sqrt{|X|^2 + |Y|^2 + v^2}$ of the pure modes $P_{\pm}$ and the mixed modes M in Eqs. (7a)–(7c) as a function of μ . Solid (dashed) lines indicate stable (unstable) states. Saddle-node ($\triangle$), symmetry-breaking, and Hopf ($\circ$) bifurcations are labeled by SN, SB, and H, respectively. The parameters are $\Gamma=0.15$, $\Lambda=0.18$, $\alpha_1=0.48$, $\alpha_2=-0.58$, $\gamma=-0.35$, and $\hat{\varepsilon}=0.001$.

(pairs of) mixed modes. The pure modes $P_{\pm}$ may also undergo Hopf bifurcation if $\gamma\Lambda \neq 0$, indicating that this oscillatory instability involves the excitation of the streaming flow ($\gamma \neq 0$) and requires an elliptical container ($\Lambda \neq 0$). The oscillating solutions that result resemble trapped direction-reversing traveling waves, oscillating about the major or minor axes, but exhibit no net drift.

Mixed modes $M : (X, Y, v) = (R_- e^{i\phi_-}, R_+ e^{i\phi_+}, v)$ satisfy

$$E \equiv R_+^2 + R_-^2 = \frac{\Gamma}{2(\alpha_1 + \gamma)}, \quad (12a)$$

$$[E(\alpha_1 + \alpha_2) - \Gamma]^2 + [1 + \Lambda^2 + \Lambda L(\alpha_2 - \alpha_1 - 2\gamma)]^2 = (1 + \Lambda^2)\mu^2, \quad (12b)$$

$$\tan(\phi_+ - \phi_-) = \Lambda, \quad (12c)$$

$$\cos^2(\phi_+ + \phi_-) = \frac{[E(\alpha_1 + \alpha_2) - \Gamma]^2}{\mu^2(1 + \Lambda^2)}, \quad (12d)$$

$$v = 2R_+R_- \sin(\phi_- - \phi_+), \quad (12e)$$

where $L \equiv R_+^2 - R_-^2$. These mixed modes can undergo SN or Hopf bifurcations. In contrast to $P_{\pm}$, a Hopf bifurcation is possible even when $\gamma=0$, but in that case it occurs only if P_+ and P_- bifurcate in opposite directions. If $\gamma \neq 0$, a Hopf bifurcation may occur along the mixed mode branch even when it connects two supercritical pure mode branches.

Figure 1 shows the steady solutions $P_{\pm}$ and M and their stability for the set of coefficients used in case IV of Ref. 6. The first instability of the trivial state, which occurs at $\mu \approx 1.00045$, produces unstable P_+ in a subcritical bifurcation. The second primary instability at $\mu \approx 1.053$ produces unstable P_- via a supercritical bifurcation. The P_+ branch quickly turns around in a SN bifurcation at $\mu=1$ but not before suffering a Hopf bifurcation at $\mu \approx 1.0001$; hence it remains unstable even after the SN. A SB bifurcation at $\mu \approx 1.00038$ on the upper P_+ branch gives rise to mixed modes M , which are unstable until a stabilizing subcritical Hopf bifurcation at $\mu \approx 1.034$, roughly midway between the initial

SB bifurcation on P_+ and a terminating SB bifurcation on P_- at $\mu \approx 1.073$. This final SB bifurcation transfers stability from M to P_- .

The most interesting dynamics in Eqs. (7a)–(7c) are associated with the relaxation oscillations described in Ref. 6. These dynamics result from the disparate time scales characterizing the surface waves and streaming flow, and occur very close to the initial instability. In the remainder of this article, we investigate in more detail the nature of these relaxation oscillations. We begin, in the following section, by deriving a simpler three-dimensional system, taking advantage of the fact that the dynamics of interest occur close to the onset of $P_\pm$ and that Γ and Λ are both small.

III. DERIVATION OF THE THREE-DIMENSIONAL SYSTEM

Equations (7a)–(7c) simplify substantially in the limit

$$\Gamma, \Lambda \ll 1. \quad (13)$$

In this limit, the points $\mu_\pm$ where the $P_\pm$ solutions bifurcate from the trivial state [see Eq. (11)] can be written

$$\mu_\pm = 1 + \frac{1}{2}(\Gamma \mp \Lambda)^2 + \dots \quad (14)$$

Likewise, from Eq. (10), the amplitudes $R_\pm$ of these modes may be expanded,

$$\mu - 1 = \frac{1}{2}[\Gamma \mp \Lambda - (\alpha_1 + \alpha_2)R_\pm^2]^2 + \dots \quad (15)$$

Equations (13)–(15) determine the distinguished limit considered in this section,

$$X \sim \eta, \quad Y \sim \eta, \quad \Gamma \sim \eta^2, \quad \Lambda \sim \eta^2, \quad \mu - 1 \sim \eta^4, \quad (16)$$

where $\eta \ll 1$. It follows that this limit focuses on small-amplitude states near the threshold of the primary instability. To obtain the corresponding evolution equations for the amplitudes X , Y , and v , we write

$$\Gamma = \eta^2 \Gamma_0, \quad \Lambda = \eta^2 \Lambda_0, \quad (17)$$

introduce the slow time $\tau = \eta^4 t$, and assume asymptotic expansions of the form

$$(X, Y) = \eta(X_0, Y_0) + \eta^3(X_1, Y_1) + \eta^5(X_2, Y_2) + \dots, \quad (18)$$

$$v = \eta^4 v_1 + \dots, \quad \mu - 1 = \eta^4 \mu_1 + \dots$$

Substituting these expansions into Eqs. (7a) and (7b) and collecting terms of like order in η , we obtain

$$O(\eta): \quad X_0 = \bar{X}_0, \quad Y_0 = \bar{Y}_0, \quad (19)$$

$$\begin{aligned} O(\eta^3): \quad X_1 - \bar{X}_1 = & -i(\Gamma_0 + \Lambda_0)X_0 + i(\alpha_2 - \alpha_1)\bar{X}_0 Y_0^2 \\ & + i[(\alpha_1 + \alpha_2)|X_0|^2 + 2\alpha_1|Y_0|^2]X_0, \\ Y_1 - \bar{Y}_1 = & -i(\Gamma_0 - \Lambda_0)Y_0 + i(\alpha_2 - \alpha_1)\bar{Y}_0 X_0^2 \\ & + i[(\alpha_1 + \alpha_2)|Y_0|^2 + 2\alpha_1|X_0|^2]Y_0, \end{aligned} \quad (20)$$

$$\begin{aligned} O(\eta^5): \quad X_2 - \bar{X}_2 = & \mu_1 \bar{X}_0 - X_0' - i(\Gamma_0 + \Lambda_0)X_1 - 2\gamma v_1 Y_0 \\ & + i(\alpha_2 - \alpha_1)(Y_0^2 \bar{X}_1 + 2Y_0 \bar{X}_0 Y_1) \\ & + i(\alpha_1 + \alpha_2)(2|X_0|^2 X_1 + X_0^2 \bar{X}_1) \\ & + 2i\alpha_1(|Y_0|^2 X_1 + Y_0 X_0 \bar{Y}_1 + \bar{Y}_0 X_0 Y_1), \\ Y_2 - \bar{Y}_2 = & \mu_1 \bar{Y}_0 - Y_0' - i(\Gamma_0 - \Lambda_0)Y_1 + 2\gamma v_1 X_0 \\ & + i(\alpha_2 - \alpha_1)(X_0^2 \bar{Y}_1 + 2X_0 \bar{Y}_0 X_1) \\ & + i(\alpha_1 + \alpha_2)(2|Y_0|^2 Y_1 + Y_0^2 \bar{Y}_1) \\ & + 2i\alpha_1(|X_0|^2 Y_1 + X_0 Y_0 \bar{X}_1 + \bar{X}_0 Y_0 X_1), \end{aligned} \quad (21)$$

where the prime signifies a derivative with respect to τ . Finally, Eq. (7c) for the streaming flow becomes, at leading order,

$$v_1' = \frac{\hat{\varepsilon}}{\eta^4}[-v_1 + i(\bar{X}_0 Y_1 + \bar{X}_1 Y_0 - X_0 \bar{Y}_1 - X_1 \bar{Y}_0)]. \quad (22)$$

Thus relaxation oscillations, in which the streaming flow evolves slowly compared to the wave amplitudes, require $\varepsilon \equiv \hat{\varepsilon}/\eta^4 \ll 1$.

Equations describing the evolution of the surface wave amplitudes X_0 and Y_0 in the presence of the streaming flow v_1 are readily obtained from the real part of Eqs. (21) together with Eq. (22) using Eqs. (19), which imply that X_0 and Y_0 are real, and Eqs. (20), which imply that X_1 and Y_1 have nonzero imaginary parts. The final system can be written (after dropping subscripts) as

$$\begin{aligned} X' = & \mu X - \beta_1^+ X - 2\gamma v Y + \beta_2^+ X^3 + \beta_3^+ Y^2 X \\ & - \beta_4(X^4 + 2X^2 Y^2 + Y^4)X, \end{aligned} \quad (23a)$$

$$\begin{aligned} Y' = & \mu Y - \beta_1^- Y + 2\gamma v X + \beta_2^- Y^3 + \beta_3^- X^2 Y \\ & - \beta_4(X^4 + 2X^2 Y^2 + Y^4)Y, \end{aligned} \quad (23b)$$

$$v' = -\varepsilon(v + 2\Lambda XY). \quad (23c)$$

The coefficients $\beta_1^\pm, \dots, \beta_4$ are given by

$$\begin{aligned} \beta_1^\pm = & (\Gamma \pm \Lambda)^2/2, \quad \beta_2^\pm = (\alpha_1 + \alpha_2)(\Gamma \pm \Lambda), \\ \beta_3^\pm = & (\alpha_1 + \alpha_2)\Gamma \pm (3\alpha_1 - \alpha_2)\Lambda, \quad \beta_4 = (\alpha_1 + \alpha_2)^2/2. \end{aligned} \quad (24)$$

All of the variables and parameters in Eqs. (23a)–(23c) are expected to be $O(1)$ except ε , which is taken to be small. However, in the numerical work described below, we sacrifice this property for the sake of an easy comparison with Ref. 6 by performing the transformations $(X, Y) \mapsto (X, Y)/\eta$, $(\Gamma, \Lambda) \mapsto (\Gamma, \Lambda)/\eta^2$, $(v, \mu, \varepsilon) \mapsto (v, \mu, \varepsilon)/\eta^4$, and $\tau \mapsto \eta^4 \tau$, which restores the “original” scaling of Eqs. (7a)–(7c).

FIG. 2. (Color online) Bifurcation diagram showing the Euclidean norm $\sqrt{|X|^2 + |Y|^2 + v^2}$ of the pure modes $P_{\pm}$ and mixed modes M in Eqs. (23a)–(23c) as a function of μ . Solid (dashed) lines indicate stable (unstable) states. Saddle-node (Δ), symmetry-breaking, and Hopf ($\circ$) bifurcations are labeled by SN, SB, and H, respectively. The parameters are as in Fig. 1 with $\varepsilon(=\hat{\varepsilon})=0.001$.

IV. COMPARISON WITH THE FIVE-DIMENSIONAL SYSTEM

In this section, we establish the validity of the reduced three-dimensional system (23a)–(23c) by comparing its dynamics with those observed in the five-dimensional system (7a)–(7c).

In Fig. 2, the steady-state solutions of Eqs. (23a)–(23c) are shown for the same set of coefficients as used in Fig. 1. Aside from the fact that the μ in Fig. 2 differs by 1 from the μ in Fig. 1 [a consequence of Eq. (18)], the two figures are virtually indistinguishable: over this range of forcing, the same bifurcations occur in both Eqs. (7a)–(7c) and (23a)–(23c), and in approximately the same locations.

We may also compare the relaxation oscillation solutions exhibited by these two systems. Figure 3 shows the period of several distinct periodic relaxation oscillations in Eqs. (7a)–(7c) as a function of μ . The figure reveals a series of periodic solutions, distinguished by their symmetry (R_1 or R_2) and by the number of fast oscillations executed while the solution drifts along the slow manifold. The R_1 -symmetric solutions are the easiest to find and exist over a wider range of μ than the R_2 -symmetric solutions. The region of stability of the simplest solutions of the former type is bounded by torus (TR) bifurcations at either end, while the more complex solutions of this type all lose stability at SB bifurcations with increasing μ . In contrast, the stability regions of the R_2 -symmetric solutions are bounded by SN bifurcations on the left and SB bifurcations on the right.

Figure 4 shows the period of the corresponding relaxation oscillations as a function of the μ in Eqs. (23a)–(23c). The types of periodic solutions observed and the bifurcations they exhibit are similar to those in Fig. 3, although the two figures do not correspond precisely—the numerically located R_2 -symmetric solutions, in particular, are characterized here by different numbers of loops (fast oscillations) than those in Fig. 3. On a qualitative level, however, the relaxation oscillations in Eqs. (23a)–(23c) are clearly the same as those in Eqs. (7a)–(7c).

FIG. 3. (Color online) The period of several branches of periodic relaxation oscillations in Eqs. (7a)–(7c) as a function of μ when $\hat{\varepsilon}=0.001$. The five lower branches are R_1 -symmetric, while the two upper branches are R_2 -symmetric. Solid (dashed) lines indicate stable (unstable) solutions. Saddle-node (SN), symmetry-breaking (SB), and torus (TR) bifurcations are indicated. Note that the μ in Eqs. (7a)–(7c) is related to that in Eqs. (23a)–(23c) by adding 1. The remaining parameters are as in Fig. 1.

FIG. 4. (Color online) The period of several branches of periodic relaxation oscillations in Eqs. (23a)–(23c) as a function of μ when $\varepsilon(=\hat{\varepsilon})=0.001$. The seven lower branches are R_1 -symmetric, while the four upper branches are R_2 -symmetric. Solid (dashed) lines indicate stable (unstable) solutions. Saddle-node (SN), symmetry-breaking (SB), and torus (TR) bifurcations are indicated. The parameters are as in Fig. 3.

V. NUMERICAL INVESTIGATION AND ANALYSIS

In this section, we take the simplified three-dimensional system (23a)–(23c) and investigate in more detail the relaxation oscillations mentioned above and described in Ref. 6. These oscillations are easier to visualize in three dimensions and more amenable to analysis, although much of this analysis must still be carried out numerically. This is because, as we shall see, the features of the relaxation oscillations are determined, to a large extent, by global properties of the fast system [i.e., Eqs. (23a) and (23b) with v constant].

To emphasize the separation of time scales that gives rise to relaxation oscillations, we rewrite Eqs. (23a)–(23c) in the general form

$$X' = F_1(X, Y, v), \quad Y' = F_2(X, Y, v), \quad v' = \varepsilon G(X, Y, v).$$

The one-dimensional nullclines Σ defined by $F_1(X, Y, v) = F_2(X, Y, v) = 0$ are of particular interest in this system. In the limit $\varepsilon = 0$, these nullclines are precisely the set of equilibria. For $0 < \varepsilon \ll 1$, there exist nearby invariant slow manifolds $\tilde{\Sigma}$ that either attract or repel (fast) trajectories. Relaxation oscillations may be characterized as a sequence of slow drifts along $\tilde{\Sigma}$ separated by fast transition events. The various nullclines in Eqs. (23a)–(23c) that make up the set Σ can be seen in Fig. 5 (compare with Fig. 25 of Ref. 6), which is merely a bifurcation diagram of the fast system $(X', Y') = (F_1, F_2)$ with v in the role of a bifurcation parameter.

Figure 5 shows that two branches of mixed modes [each branch itself representing two symmetry-related states $\pm(X, Y)$] appear at $|v| \approx 0.172$ in a SN bifurcation. The large amplitude branch, which is always unstable, passes through its point of maximum amplitude at $v = 0$ and is at that point a pure mode because $X = 0$; moreover, since the relaxation oscillations of interest occur for small $|v|$, the relevant states on this branch satisfy $|X| \ll |Y|$ and we therefore denote this branch by Σ_P . The smaller amplitude branch of mixed modes Σ_M , followed from the SN on the left, say, is unstable until a subcritical Hopf bifurcation at $v \approx -0.0032$. It then remains stable until it terminates on the trivial state (O) at $v \approx 0.0342$. The Hopf bifurcations at $|v| \approx 0.0032$ produce unstable periodic states that grow in amplitude until they collide with the larger amplitude mixed mode branch Σ_P in homoclinic bifurcations at $|v| = 3.54 \times 10^{-6}$. Due to the symmetries (9a) and (9b), the entire diagram is symmetric about $v = 0$.

A comment on notation is appropriate here. The nullclines Σ that exist when $\mu = 0.015$ can be divided into Σ_0 , corresponding to the trivial state of the fast system, Σ_P , associated with the pure mode at $v = 0$ and Σ_M , associated with mixed states where $|X| \sim |Y|$. When necessary, we will refer to the two symmetry-related branches of Σ_P as $\Sigma_P^\pm$, where the $\pm$ reflects the sign of Y . Similarly, the four symmetry-related branches of Σ_M will sometimes be distinguished using the notation $^f\Sigma_M^\pm$ and $^b\Sigma_M^\pm$, where the $\pm$ again carries the sign of Y while the left superscript reflects the sign of v' along this nullcline, and hence the sense of the drift [forward (f) for increasing v and backward (b) for decreasing v] when ε is small.

FIG. 5. (Color online) (a) Bifurcation diagram showing the steady states of the two-dimensional fast system ($\varepsilon = 0$) as a function of v when $\mu = 0.015$. (b) An enlargement showing the smaller amplitude solutions for $-0.02 < v < 0.02$, the relevant range for relaxation oscillations. The different parts of the nullclines Σ are labeled; solid (dashed) lines denote stable (unstable) solutions. The unstable periodic solutions (thick dashed lines) generated in the small $|v|$ Hopf bifurcations are shown in (b).

Figure 6 shows the structure of the two-dimensional fast system for various values of v . Note that between (a) and (b), a heteroclinic connection between Σ_0 and $\Sigma_P^\pm$ must occur, while between (b) and (c) the subcritical Hopf bifurcation takes place (at $v \approx 0.0032$). The breaking of the heteroclinic connection in (b) creates a small region of initial conditions (near the repelling $^f\Sigma_M^+$ in quadrant II) between $W^s(\Sigma_0)$ and $W^s(\Sigma_P^\pm)$ that pass from $Y > 0$ to $Y < 0$ through a small gap near O , although most such trajectories remain in $Y > 0$. Figures 6(c) and 6(d) show that this gap increases with increasing v but continues to remain small. It is this phase space structure that is responsible for the comparative rarity of R_2 -symmetric solutions of Eqs. (23a)–(23c).

The stable relaxation oscillations observed in Eqs. (23a)–(23c) occur only over a small range of v [the region of relevance, $|v| \lesssim 0.02$, is shown in Fig. 5(b)] and are characterized by alternating drifts along the symmetry-related slow manifolds $^f\tilde{\Sigma}_M^\pm$ and $^b\tilde{\Sigma}_M^\pm$ corresponding to the small-amplitude mixed mode branches in Fig. 5(b). A typical trajectory in this regime [with $v(0) \sim -0.02$, say] is attracted to (one of) the stable slow manifolds $^f\tilde{\Sigma}_M^\pm$, then drifts slowly along in the direction of increasing v , eventually drifting past the Hopf bifurcation at $v \approx 0.0032$ after which $^f\tilde{\Sigma}_M^\pm$ become

FIG. 6. (Color online) Series of phase portraits from the two-dimensional fast system showing the changes in the stable and unstable manifolds of $\Sigma_P^\pm$ and Σ_O as v increases: (a) $v=0.001$, (b) $v=0.003$, (c) $v=0.004$, and (d) $v=0.01$. Parameters are as in Fig. 1.

repelling. Due to the phenomenon of *delayed loss of stability*,^{15–19} the trajectory does not immediately jump to (one of) the stable slow manifolds ${}^b\Sigma_M^\pm$, but lingers near the repelling manifold until $v \sim 0.02$, at which point it finally jumps to ${}^b\Sigma_M^+$ or ${}^b\Sigma_M^-$, where the story repeats, but now for v

decreasing. This process may result in both periodic and chaotic relaxation oscillations; an example of the latter is shown in Fig. 7.

For small ε , it is very easy to find chaotic relaxation oscillation solutions like that shown in Fig. 7, in part because

FIG. 7. (Color online) Chaotic relaxation oscillation dynamics in Eqs. (23a)–(23c). Projection onto (a) the (X, Y) plane and (b) the (v, Y) plane. The nullclines $f_{\Sigma_M}^\pm$ and $b_{\Sigma_M}^\pm$ are shown for reference, as are the Hopf (H) bifurcations (in v) occurring along them. The drift along the nearby slow manifolds is evident. We use $\mu=0.015$ and $\varepsilon=0.0008$ with the remaining parameters as in Fig. 1.

FIG. 8. (Color online) The period of several branches of R_1 -symmetric periodic oscillations as a function of ε when $\mu=0.015$. Solid (dashed) lines indicate stable (unstable) solutions. Symmetry-breaking (SB) and torus (TR) bifurcations are indicated.

there are no stable steady states over the relevant range of v . For moderate values of ε such as, for example, $\varepsilon \sim 0.002$ stable R_1 -symmetric periodic solutions are easy to locate while, for reasons already explained, stable R_2 -symmetric solutions are much harder to find. Furthermore, there is typically a competition, at a given set of parameter values, between many different relaxation oscillation solutions, distinguished by the number of times they wind around the slow manifolds $f\tilde{\Sigma}_M^\pm$ and $b\tilde{\Sigma}_M^\pm$, and by the manner in which they jump between them. A numerical study of the R_1 -symmetric periodic solutions (see Fig. 4) shows that for fixed ε , these solutions are present in a finite interval of μ , between two values of μ at which the oscillation period appears to diverge. This divergence indicates the formation of homoclinic connections, apparently resulting from collision with the pure mode on the left (smaller μ) and with the origin on the right (larger μ).

It is also of interest to examine what happens for fixed μ as ε decreases. Since the limit $\varepsilon=0$ is singular, this case is much more subtle than the one just described. Figure 8 shows that R_1 -symmetric periodic relaxation oscillations appear in SN bifurcations with decreasing ε . The solutions with the least number N of twists around the slow manifolds $f\tilde{\Sigma}_M^\pm$

FIG. 9. Scaling of saddle-node (SN) bifurcations in Fig. 8. The product of N and the location $\varepsilon_{SN}(N)$ of the saddle-node bifurcations approaches a constant value as N increases.

FIG. 10. R_1 -symmetric periodic solutions on the fourth branch from the bottom in Fig. 8. Shown are a stable solution at (a) $\varepsilon=0.0015$ and unstable large-period solutions at (b) $\varepsilon=0.000\ 125$ (near the global bifurcation on the left), and (c) $\varepsilon=0.000\ 213$ (near the global bifurcation on the right).

and $b\tilde{\Sigma}_M^\pm$ appear first, followed monotonically by solutions with larger and larger values of N . These SN bifurcations, like other properties of the solutions, appear to accumulate at $\varepsilon=0$ as $1/N$ (see Fig. 9). Moreover, the homoclinic bifurcations at which a solution with twist N has infinite period also appear to accumulate at $\varepsilon=0$ as $1/N$, although this is difficult to establish with certainty due to the very long periods involved, large proportions of which come from the long times spent drifting along unstable slow manifolds that now include $\tilde{\Sigma}_P$ or $\tilde{\Sigma}_O$, in addition to $\tilde{\Sigma}_M^\pm$. Additional numerical complications relate directly to the heteroclinic connection between Σ_P and Σ_O in the fast system (see Fig. 6). We remark that the details of this behavior repeat for nearby values of μ , which serve only to determine (inessential) constants but do not affect the ultimate scaling as $\varepsilon \rightarrow 0$. A possible explanation for this behavior is outlined in the next section.

Figure 10 shows a series of periodic solutions along one of the branches in Fig. 8, including the low-period stable section (a), the high-period portion on the left (b), and the high-period portion on the right (c). Figure 10(b) reveals a solution close to a homoclinic connection with the origin, while Fig. 10(c) shows a solution close to a homoclinic connection with the pure mode P_+ . All of these solutions are characterized by the same value of N and are R_1 -symmetric.

FIG. 11. Additional phase portraits [cf. Figs. 10(b) and 10(c)] of large-period R_1 -symmetric solutions on the fourth branch from the bottom in Fig. 8. (a) Near the global bifurcation on the left at $\varepsilon=0.000125$, and (b) near the global bifurcation on the right at $\varepsilon=0.000213$. The slow manifolds $\tilde{\Sigma}_O$ and $\tilde{\Sigma}_P$ are labeled.

Although the high-period solutions in (b) and (c) come very close to both $\tilde{\Sigma}_P$ and $\tilde{\Sigma}_O$ (due to the corresponding heteroclinic connection in the two-dimensional fast system), the corresponding time series indicate that they are in fact homoclinic in nature. Furthermore, Fig. 11 suggests that these homoclinic connections can be identified as gluing bifurcations. In Fig. 11(a), which is at $\varepsilon=0.000125$, we see that the high-period limit cycle (approximate period 49600) nearly collides with the origin, and presumably does so at a slightly lower value of ε , pinching off to produce a pair of R_1 symmetry-related asymmetric periodic orbits. In Fig. 11(b), which is at $\varepsilon=0.000213$, we observe a high-period limit cycle (approximate period 362000) that apparently collides with a pure mode, producing another pair of asymmetric periodic orbits. All of these asymmetric periodic orbits are unstable since they must drift along unstable portions of $\tilde{\Sigma}$, and hence are not seen.

VI. MAP AND SCALING

Although the basic character of the relaxation oscillation dynamics can be seen in Figs. 4–10, it is conceptually useful to simplify the slow drifting part of the dynamics and to model the full system (23a)–(23c) by a Poincaré map. To this end, we focus on R_1 -symmetric solutions and introduce an appropriate “normal form” coordinate system (r, θ, z) oriented along the slow manifold $^f\tilde{\Sigma}_M^+$ such that the drift speed

and angular frequency are constant (time may have to be rescaled as well) and the Hopf bifurcation (in the fast system) occurs at $z=0$. Specifically, we write

$$r' = zr, \quad \theta' = \omega, \quad z' = \epsilon, \quad (25)$$

to describe the quasilinear dynamics of the drift along $^f\tilde{\Sigma}_M^+$. Analogous coordinates are used near $^b\tilde{\Sigma}_M^+$ and, for convenience, we absorb the reflection R_1 into their definition so that R_1 maps a point (r, θ, z) near $^f\tilde{\Sigma}_M^+$ to the same point near $^b\tilde{\Sigma}_M^+$. Equations (25) are easily integrated to yield

$$r(t) = r_0 e^{z_0 t + \epsilon t^2/2}, \quad \theta(t) = \theta_0 + \omega t, \quad z(t) = z_0 + \epsilon t, \quad (26)$$

where (r_0, θ_0, z_0) are the initial conditions. We next introduce the local Poincaré sections

$$\Gamma_{f,b}: \{(r, \theta, z) | r = \delta\} \quad (27)$$

centered around $^f\tilde{\Sigma}_M^+$ and $^b\tilde{\Sigma}_M^+$, respectively, and use Eq. (26) to obtain the local map from $\Gamma_{f,b}$ to itself:

$$T_f: (\delta, \theta, -z) \mapsto (\delta, \theta + 2\omega z/\epsilon, z), \quad z > 0. \quad (28)$$

Note that this map, which describes “delayed loss of stability” (see, e.g., Refs. 15, 16, and 18), is symmetric in z (the image has the same value of $|z|$ as the pre-image), meaning that trajectories drift the same distance past the Hopf bifurcation as their initial distance from it.

The global map from Γ_f to Γ_b takes the form

$$T_g: (\delta, \theta, z) \mapsto (\delta, f(\theta, z), z + g(\theta, z)), \quad z > 0, \quad (29)$$

where the unknown functions $f(\theta, z)$ and $g(\theta, z)$ reflect the fact that there is a continuum of connections between $^f\tilde{\Sigma}_M^+$ and $^b\tilde{\Sigma}_M^+$ in the fast system. All R_1 -symmetric periodic solutions satisfy $T_g \circ T_f(\delta, \theta, z) = (\delta, \theta, z)$, i.e.,

$$\tilde{f}(\theta, |z|) \equiv f(\theta + 2\omega|z|/\epsilon, |z|) - \theta = 0, \quad (30a)$$

$$\tilde{g}(\theta, |z|) \equiv g(\theta + 2\omega|z|/\epsilon, |z|) = 0. \quad (30b)$$

These equations depend in an essential way on the global structure of the two-dimensional fast system, and must therefore be determined numerically. In particular, we expect the functions $f(\theta, z)$ and $g(\theta, z)$ to exhibit singularities corresponding to the various global bifurcations already identified. Figure 12 shows that such singularities are indeed present. However, for reasons already mentioned, we expect these results to depend only weakly on ϵ since they reflect the fast, and not the slow, dynamics. This means that the ϵ -dependence of Eqs. (30a) and (30b) is simple, which in turn allows us to obtain a simple scaling law. Suppose, for example, that a solution to Eqs. (30a) and (30b) exists for some value $\epsilon = \epsilon_N$ and let this solution be characterized by the number of loops (sufficiently large) it makes around $^f\tilde{\Sigma}_M^+$: $2\omega|z|/\epsilon_N = 2\pi N + \alpha$, $0 < \alpha < 2\pi$. Then for $\epsilon_M = \omega|z|/\epsilon_N / [(M-N)\pi\epsilon_N + \omega|z|]$ we have $2\omega|z|/\epsilon_M = 2\pi M + \alpha$. In other words, the leading-order effect of changing ϵ is to alter the drift speed and hence the number of loops (for fixed drift distance) around $^f\tilde{\Sigma}_M^+$. Adding or subtracting integer numbers of loops, while keeping everything else constant, generates additional solutions. This argument is more general, in fact,

FIG. 12. (Color online) The functions (a) $\tilde{f}$ and (b) $\tilde{g}$, defined in Eqs. (30a) and (30b), at $|z|=0.01$. The small gap for $0.28 \leq \theta/2\pi \leq 0.38$ reflects the fact that over this range, the global map T_g approaches $b\tilde{\Sigma}_M^-$, the R_2 symmetry-related partner of $f\tilde{\Sigma}_M^+$ (indicating the possibility of R_2 -symmetric orbits), rather than its R_1 symmetry-related partner $b\tilde{\Sigma}_M^+$ as assumed in Eqs. (30a) and (30b).

than Eqs. (30a) and (30b) and applies to any type of periodic relaxation oscillation solution including those, near global bifurcations, that visit other slow manifolds besides $f\tilde{\Sigma}_M^\pm$ and $b\tilde{\Sigma}_M^\pm$. The existence, for sufficiently small ϵ , of any periodic solution (or homoclinic cycle) with an interval of drift along $f\tilde{\Sigma}_M^\pm$ implies an infinite cascade of similar solutions with increasing numbers of loops around $f\tilde{\Sigma}_M^\pm$. The cascade accumulates on $\epsilon=0$ and satisfies $\epsilon_N \approx \omega|z|/(\pi N)$. This prediction is verified for the SN bifurcations at which R_1 -symmetric periodic solutions are created with decreasing ϵ (see Fig. 8) in Fig. 9.

VII. CONCLUSION

In this paper, we have shown that the multiplicity of periodic relaxation oscillations observed near the onset of the nearly inviscid Faraday instability in elliptical containers⁶ can be modeled by a simple third-order set of ordinary differential equations. These equations describe the interaction of standing waves with a weakly damped azimuthal streaming flow that is generated in viscous boundary layers at the rigid walls and the free surface produced by the waves. We have shown that this simple system may be derived from the more general formulation of the problem using an appropriate asymptotic analysis that focuses on the slow dynamics close to threshold of the primary instability. We have seen that the behavior of this third-order system is almost identical to that of the higher-dimensional equations studied in Ref. 6. Moreover, in the nearly inviscid regime, these equa-

tions inherit the slow-fast structure of the physical system, in that the amplitudes respond on a faster time scale than the weakly damped streaming flow, a fact that allows one to make substantial progress in understanding the observed dynamics.

Physically, the (stable parts of the) periodic solutions we describe here represent a long time scale periodic switching between generalized traveling waves related by reflection symmetry. Each generalized traveling wave (a combination of left- and right-moving waves) is coupled to a streaming flow whose magnitude slowly alternates between positive and negative values during the course of the relaxation oscillation; the extrema of this slow streaming flow oscillation correspond to the comparatively abrupt changes in the direction of the surface waves. It is important to emphasize that the weakly damped streaming flow with its slow time scale is the essential ingredient of these relaxation oscillations. Provided the symmetry of the container permits a strong interaction between streaming flow and surface waves, relaxation oscillations are a relatively generic phenomenon. Other types of relaxation oscillations have been observed with different parameter values.⁶

Of particular interest, and relevance to the dynamics associated with mixed mode oscillations, is the proliferation of these different periodic orbits in the singular limit $\epsilon \rightarrow 0$. Somewhat similar behavior has been noted in other systems, although generally such systems are obtained by small (dissipative) perturbations of Hamiltonian systems (see, for example, Refs. 20 and 21 and references therein). We surmise that a singular perturbation analysis of Eqs. (23a)–(23c) along the lines of Ref. 5, in which μ is scaled with ϵ , may allow one to establish the presence of a heteroclinic connection in the singular limit $\epsilon \rightarrow 0$; it is possible that the unfolding of such a bifurcation may then capture much of the behavior reported in the present paper.

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