Global error analysis of two-dimensional panel methods for dirichlet formulation

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A rigorous analytical study of the global error of panel methods is presented. The analysis is performed for a wide variety of body shapes and different panel geometries to fully understand their effect on the convergence of the method. In particular, we study the global error associated with panel methods applied to thin or thick bodies with purely convex parts or with both convex and concave parts, and with smooth or non-smooth boundaries. Most previous studies focused on the analysis of local error, considering only the influence of the nearest panels and excluding the rest. The difference is shown to be appreciable in many configurations. Generally, there is a lack of consensus concerning the order of magnitude of the error for panel methods even in the simplest case with flat panels and a constant distribution of doublets along them. This paper clarifies apparently different or inconsistent results obtained by other authors.

1. Introduction

The importance of the Laplace equation in aerodynamics (and many other fields of science) means that a great deal of effort has been directed toward developing analytical and numerical methods for its solution. Among the most popular numerical schemes are panel methods, or boundary element methods (BEMs) [1,2], which reduce the problem of finding the velocity potential for the entire fluid to the calculation of this potential on the surface of the body itself. Thus, the dimension of the problem is reduced from three to two (or, in the case of two-dimensional flows, from two to one) making BEMs very attractive for their low computational cost. Since the pioneering work of Hess and Smith there have been numerous publications and many numerical codes based on panel methods [3–14]; among these we emphasize the reviews of Hess [9], Erickson [10] and the book of Katz and Plotkin [12]. Boundary element methods are an active field of study, especially within the engineering community, with new applications being developed rapidly.

The panel method based on Green's formula was first introduced in the work of Morino and Kuo [4], in which the primary unknown was the velocity potential. There are two main formulations both based on Green's formula: Neumann and Dirichlet [12]. The Dirichlet formulation solves the Laplace equation numerically and provides the velocity potential. However, with the Neumann formulation, only differences of potential are obtained. The Dirichlet formulation is more stable and

more suitable to numerical computation than the Neumann formulation and leads to numerical errors of a smaller order of magnitude.

Initially, panel methods were developed using flat panels and a constant [3,5] or linear [6] distribution of singularities on each panel. Beginning in the 1970's however, singularity distributions were also modeled (on each panel) using quadratic [4,15,16] or cubic [17] functions. In a similar fashion, the panels themselves, which were initially taken to be flat, were generalized to include non-planar geometries [4,15]. However, in the last several decades many numerical codes returned to the original low-order approach, as indicated in [12]. The main reason for this is the more complicated implementation of higher-order methods compared to lower-order ones [11].

A lot of studies have addressed the question of error in these methods. Some carry out a numerical analysis of the error by comparing a numerical solution with a known analytical solution [18–20]. Others perform a local analysis of the error by using small curvature expansions to obtain local approximations to the velocity and potential integrals [21–24]. However, to the best of our knowledge, a rigorous analytical study of the global error of these methods that applies to thin or thick bodies, with purely convex parts or with both convex and concave parts, and with smooth or non-smooth boundaries, has not yet been performed. In this paper, we present such an analysis for a wide variety of body shapes and try to understand the effect of the body and panel geometry on the convergence of the method. This allows us to clarify several important questions about the convergence

rate for the velocity potential since, even for the simplest case of flat panels with a constant distribution of doublets along them, there is a lack of general consensus. Depending on the airfoil geometry, the panel geometry and the discretization, apparently different or inconsistent results are obtained by other authors, and differences between theoretical and numerical results exist as well [18–20,24,25].

This work presents a formal analytical and numerical analysis of the asymptotic global error in panel methods when applied to a Dirichlet formulation [12] for different body geometries. In addition, an analysis of the influence of the panel geometry on the global error is performed. The work is organized as follows. In Section 2 a brief description of panel methods is given. In Section 3 the global error analysis is performed analytically. In Section 4 the details of the error estimation are presented. Section 5 considers the numerical and analytical solutions for different body geometries in order to compare the actual and predicted errors in each case. Finally, in Section 6 the main conclusions are given.

2. Brief description of the panel method

The velocity potential around a body of known shape submerged in a potential flow satisfies the irrotational, incompressible continuity equation in the body's frame of reference [12]:

$$\nabla^2 \Phi = 0, \quad (1)$$

Boundary conditions require a vanishing normal velocity component on the body surface,

$$\nabla \Phi \cdot \mathbf{n} = 0, \quad (2)$$

and a constant velocity in the far field limit:

$$\lim_{r \rightarrow \infty} \nabla \Phi = \mathbf{V}_\infty, \quad (3)$$

where $\mathbf{V}_\infty$ denotes the imposed velocity far from the body.

Using Green's identity, the general solution to Eq. (1) can be written as:

$$\Phi(\mathbf{p}) = \int_{S_B} \left(\frac{\partial \Phi}{\partial n} - \frac{\partial \Phi_{\text{int}}}{\partial n} \right) \Phi_m ds - \int_{S_B} (\Phi - \Phi_{\text{int}}) \nabla \Phi_m \cdot \mathbf{n} ds - \int_{S_W} (\Phi^+ - \Phi^-) \nabla \Phi_m \cdot \mathbf{n} ds + \Phi_\infty, \quad (4)$$

which gives the velocity potential Φ at any point $\mathbf{p}$. This potential is considered to be caused by a distribution, on the surface of the body S_B , of point sources of intensity $\partial \Phi / \partial n - \partial \Phi_{\text{int}} / \partial n$ and doublets of intensity $\Phi - \Phi_{\text{int}}$ oriented along axes $\mathbf{n}$, and by a distribution, along the wake, of doublets of intensity $\Phi^+ - \Phi^-$ with axis of orientation $\mathbf{n}$. Fig. 1 shows the body and the relevant surfaces; on the body $\mathbf{n}$ is oriented outward while it points upward along the wake. Φ_m is the velocity potential produced at a point $\mathbf{p}$ by a point source of unit

Fig. 1. Sketch of the body and associated surfaces: body surface S_B , wake surface S_W , and a surface at infinity S_∞ .

strength located on ds , $\nabla \Phi_m \cdot \mathbf{n}$ gives the velocity potential at a point $\mathbf{p}$ produced by a doublet of unit strength located on ds and oriented along $-\mathbf{n}$, Φ_{int} is the so-called interior potential, which is required to satisfy the Laplace equation in the interior of the body, and the final term in Eq. (4) is the potential of the stationary flow far from the body, evaluated at $\mathbf{p}$: $\Phi_\infty = U_\infty (x \cos \alpha + z \sin \alpha)$, where $U_\infty = |\mathbf{V}_\infty|$, α is the angle between the incident flow and a reference line (angle of attack), and x and z are the coordinates of the point $\mathbf{p}$ in a fixed reference frame.

Imposing $\partial \Phi / \partial n = 0$ on the boundary of the body, the velocity potential at the point $\mathbf{p}$ can be written as

$$\Phi(\mathbf{p}) = \int_{S_B} \sigma \Phi_m ds - \int_{S_B} \mu \nabla \Phi_m \cdot \mathbf{n} ds - \int_{S_W} (\Phi^+ - \Phi^-) \nabla \Phi_m \cdot \mathbf{n} ds + \Phi_\infty, \quad (5)$$

where

$$\sigma = -\frac{\partial \Phi_{\text{int}}}{\partial n}, \quad \mu = \Phi - \Phi_{\text{int}}. \quad (6)$$

Taking the point $\mathbf{p}$ to be on the surface of the body reduces the problem to an integral equation for the unknown velocity potential on the surface.

To simplify calculations, in what follows we take $\Phi_{\text{int}} = 0$. In this case the point source distribution σ vanishes, and Eq. (5) reduces to

$$\Phi(\mathbf{p}) = - \int_{S_B} \Phi \nabla \Phi_m \cdot \mathbf{n} ds - \int_{S_W} (\Phi^+ - \Phi^-) \nabla \Phi_m \cdot \mathbf{n} ds + \Phi_\infty. \quad (7)$$

3. Global error estimate for dirichlet formulation

Here we derive an estimate for the expected numerical error upon solving Eq. (7) with the lower order panel method. This equation can be written as

$$\Phi(\mathbf{p}) = \frac{1}{2\pi} \int_{S_B} \frac{\Phi(s)(\mathbf{p} - \mathbf{s}) \cdot \mathbf{n}}{|\mathbf{p} - \mathbf{s}|^2} ds + \frac{\Gamma}{2\pi} \int_{S_W} \frac{(\mathbf{p} - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{p} - \boldsymbol{\xi}_w|^2} d\xi_w + \Phi_\infty(\mathbf{p}), \quad (8)$$

where, in the first integral, the variable of integration, s , is the arc length parameter along the body surface, $\mathbf{s} = \mathbf{s}(s)$ is a point on the body surface S_B , and $\mathbf{n} = \mathbf{n}(s)$ is the (unit) normal vector directed outward from that point. In the second integral the variable of integration is ξ_w , measuring distance along the wake panel S_W , while $\boldsymbol{\xi}_w = \boldsymbol{\xi}_w(\xi_w)$ is a point on the wake panel and $\mathbf{n}_w$ is a unit normal vector directed upwards. The prefactor $\Gamma = \Phi^+ - \Phi^-$ denotes the circulation around the body.

In what follows, the geometry of the body will be approximated with a collection of flat panels $\mathcal{C}_i$, $i = 1 \dots N$ of length l_i . We assume that the intensity of the doublet distribution is constant on each individual panel and that all panels are of comparable size, i.e, with a characteristic lengthscale $l = O(1/N)$; hereafter, we use the Landau notation " $O(\cdot)$ " for order of magnitude. The discretization of the body surface and the wake are illustrated in Fig. 2.

Enumeration of the panels begins at the point of attachment of the wake, with panel number 1, and continues clockwise around the body, ultimately reaching the starting point again after panel N (this time from above the wake panel). As illustrated in Fig. 2, the endpoints of these panels (which lie on the body surface) similarly divide the true body surface into N (curved) segments L_i . We may thus decompose the first integral term in Eq. (8) to get

Fig. 2. Discretization of the body surface.

Fig. 3. Relationship between p_i , x_i , and s_i in the concave (panel i) and convex (panel j) cases.

$$\Phi(\mathbf{p}) = \frac{1}{2\pi} \sum_{i=1}^N \int_{L_i} \frac{\Phi(s_i)(\mathbf{p} - \mathbf{s}_i) \cdot \mathbf{n}}{|\mathbf{p} - \mathbf{s}_i|^2} d\zeta_i + \frac{\Gamma}{2\pi} \int_{S_W} \frac{(\mathbf{p} - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{p} - \boldsymbol{\xi}_w|^2} d\zeta_w + \Phi_\infty(\mathbf{p}), \quad (9)$$

where we use the subscript i to indicate that s_i or s_i is restricted to panel i . The idea is to use this equation, evaluated at an appropriate point $\mathbf{p}$ that tends from the outside to the body surface (see Fig. 3), and to compare the value of $\Phi(\mathbf{p})$ with the “numerical” potential $\Phi^n(\mathbf{p})$, which is calculated by assuming a constant doublet distribution along each of the N panels ℓ_i . This latter potential can be written as

$$\Phi^n(\mathbf{p}) = \frac{1}{2\pi} \sum_{i=1}^N \Phi_i^n \int_{\ell_i} \frac{(\mathbf{p} - \boldsymbol{\xi}_i) \cdot \mathbf{n}_i}{|\mathbf{p} - \boldsymbol{\xi}_i|^2} d\zeta_i + \frac{\Gamma^n}{2\pi} \int_{S_W} \frac{(\mathbf{p} - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{p} - \boldsymbol{\xi}_w|^2} d\zeta_w + \Phi_\infty(\mathbf{p}), \quad (10)$$

where $\boldsymbol{\xi}_i$ is a point on panel i , with ξ_i measuring distance along it (see Fig. A.1 in Appendix A), $\mathbf{n}_i$ is an outward unit normal to this panel, and Γ^n is the numerically calculated circulation. The doublet strengths (potentials) Φ_i^n are determined by evaluating Eq. (10) at N control points $\mathbf{x}_i$, one located on each of the N control panels ℓ_i ,

$$\Phi_i^n = \frac{1}{2\pi} \sum_{j=1}^N \Phi_j^n \int_{\ell_j} \frac{(\mathbf{x}_i - \boldsymbol{\xi}_j) \cdot \mathbf{n}_j}{|\mathbf{x}_i - \boldsymbol{\xi}_j|^2} d\zeta_j + \frac{\Gamma^n}{2\pi} \int_{S_W} \frac{(\mathbf{x}_i - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{x}_i - \boldsymbol{\xi}_w|^2} d\zeta_w + \Phi_\infty(\mathbf{x}_i), \quad (11)$$

which generates an N -dimensional linear system of equations. The reference point $\mathbf{p}$, when associated with panel i , will be denoted by $\mathbf{p}_i$; the relationship between $\mathbf{p}_i$ and $\mathbf{x}_i$ is depicted in Fig. 3.

In the convex case (the typical situation) we let the point $\mathbf{p}_i$ tend to the true body surface. On the other hand, if the body is concave at panel i then we let $\mathbf{p}_i$ tend to the panel itself (in practice, and for simplicity, we take $\mathbf{p}_i = \mathbf{x}_i$). This distinction is necessary so that both Φ and Φ^n are well defined at the point of comparison $\mathbf{p}_i$.

We define the error as $\varepsilon = \Phi(\mathbf{p}) - \Phi^n(\mathbf{p})$, which, for panel i , becomes

$$\varepsilon_i = \Phi(\mathbf{p}_i) - \Phi^n(\mathbf{p}_i). \quad (12)$$

To estimate the error ε_i we take an appropriate point $\mathbf{p}_i$ on each panel and subtract Eq. (10) from Eq. (9) to get

$$2\pi\varepsilon_i = \sum_{j=1}^N \int_{L_j} \frac{\Phi(s_j)(\mathbf{p}_i - \mathbf{s}_j) \cdot \mathbf{n}}{|\mathbf{p}_i - \mathbf{s}_j|^2} d\zeta_j - \sum_{j=1}^N \Phi_j^n \int_{\ell_j} \frac{(\mathbf{p}_i - \boldsymbol{\xi}_j) \cdot \mathbf{n}_j}{|\mathbf{p}_i - \boldsymbol{\xi}_j|^2} d\zeta_j + (\Gamma - \Gamma^n) \int_{S_W} \frac{(\mathbf{p}_i - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{p}_i - \boldsymbol{\xi}_w|^2} d\zeta_w. \quad (13)$$

Next we employ the mean value theorem, writing for panels in the convex case

$$\begin{aligned} \Phi(s_i) &= \Phi(\mathbf{p}_i) + \frac{\partial\Phi}{\partial s}(\mathbf{z}_s)[s(s_i) - s(\mathbf{p}_i)] = \Phi^n(\mathbf{p}_i) + \varepsilon_i \\ &+ \frac{\partial\Phi}{\partial s}(\mathbf{z}_s)[s(s_i) - s(\mathbf{p}_i)] = \Phi^n(\mathbf{x}_i) + \varepsilon_i + \psi_i(s_i), \end{aligned} \quad (14)$$

with

$$\psi_i(s_i) = \frac{\partial\Phi}{\partial s}(\mathbf{z}_s)[s(s_i) - s(\mathbf{p}_i)] + \nabla\Phi^n(\mathbf{z}_i) \cdot (\mathbf{p}_i - \mathbf{x}_i), \quad (15)$$

where s is the aforementioned arc parameter but could, in principle, be any scalar variable that parameterizes a path tending to the body surface from outside (where Φ is well-defined) connecting s_i and $\mathbf{p}_i$. The point $\mathbf{z}_s$ is located along that path, while $\mathbf{z}_i$ is another point (on a line segment) between $\mathbf{p}_i$ and $\mathbf{x}_i$.

In the concave case we set $\mathbf{p}_i = \mathbf{x}_i$ and write

$$\Phi(s_i) = \Phi(\mathbf{x}_i) + \psi_i(s_i) = \Phi^n(\mathbf{x}_i) + \varepsilon_i + \psi_i(s_i), \quad (16)$$

with

$$\psi_i(s_i) = \nabla\Phi(\mathbf{z}_s) \cdot (\mathbf{s}_i - \mathbf{x}_i). \quad (17)$$

In the above, $\mathbf{z}_s$ is a point (on a line segment) between s_i and $\mathbf{x}_i$. Note that the only difference between the convex and concave cases is in the definition of ψ_i .

By introducing Eqs. (14) and (16) in Eq. (13), we obtain

$$\begin{aligned} 2\pi\varepsilon_i &= \sum_{j=1}^N \int_{L_j} \frac{[\Phi_j^n + \varepsilon_j + \psi_j(s_j)](\mathbf{p}_i - \mathbf{s}_j) \cdot \mathbf{n}}{|\mathbf{p}_i - \mathbf{s}_j|^2} d\zeta_j \\ &- \sum_{j=1}^N \Phi_j^n \int_{\ell_j} \frac{(\mathbf{p}_i - \boldsymbol{\xi}_j) \cdot \mathbf{n}_j}{|\mathbf{p}_i - \boldsymbol{\xi}_j|^2} d\zeta_j + (\Gamma - \Gamma^n) \delta_{i\infty}. \end{aligned} \quad (18)$$

Note the fact that

$$\int_C \frac{(\mathbf{p}_i - \mathbf{s}_j) \cdot \mathbf{n}}{|\mathbf{p}_i - \mathbf{s}_j|^2} d\zeta_j = \int_C d\theta_{p_i s_j} = \theta_{p_i s_j}^F - \theta_{p_i s_j}^I = \delta_{ij}, \quad (19)$$

where $\theta_{p_i s_j}$ is the (polar) angle associated with the vector $\mathbf{s}_j - \mathbf{p}_i$ (see Fig. 4), a function of the position s_j along a path C , $\theta_{p_i s_j}^I$, corresponds to the node $\mathbf{x}_j^I$ of panel j , and $\theta_{p_i s_j}^F$, corresponds to the node $\mathbf{x}_j^F$ (see Fig. 2) and δ_{ij} is the angle subtended by panel j as viewed from $\mathbf{p}_i$. The reference axis for $\theta_{p_i s_j}$ is taken parallel to the wake panel.

Eq. (19) shows that the value of the integral on the left-hand side only depends on the relative position between the endpoints of the path L_j , and therefore

$$\int_{L_j} \frac{(\mathbf{p}_i - \mathbf{s}_j) \cdot \mathbf{n}}{|\mathbf{p}_i - \mathbf{s}_j|^2} d\zeta_j = \int_{\ell_j} \frac{(\mathbf{p}_i - \boldsymbol{\xi}_j) \cdot \mathbf{n}_j}{|\mathbf{p}_i - \boldsymbol{\xi}_j|^2} d\zeta_j = \delta_{ij}. \quad (20)$$

Note that in Eq. (11), used to calculate the numerical values of the velocity potential, δ_{ij} is the angle subtended by panel j as viewed from $\mathbf{x}_i$, so the only difference in the numerical calculations if curved panels are used instead of flat panels, is the position of $\mathbf{x}_i$, which is placed on the curved panel instead of the flat one (see Fig. 4).

Fig. 4. Definition of $\theta_{p_i s_j}$ and δ_{ij} .

By introducing Eq. (20) into Eq. (18) we have

$$2\pi\epsilon_i = \sum_{j=1}^N \delta_{ij}\epsilon_j + \sum_{j=1}^N \beta_{ij} + (\Gamma - \Gamma^n)\delta_{i\infty}, \quad (21)$$

with

$$\beta_{ij} = \int_{L_j} \frac{\psi_j(s_j)(\mathbf{p}_i - \mathbf{s}_j) \cdot \mathbf{n}}{|\mathbf{p}_i - \mathbf{s}_j|^2} ds_j, \quad (22)$$

and

$$\delta_{i\infty} = \int_{S_W} \frac{(\mathbf{p}_i - \boldsymbol{\xi}_w) \cdot \mathbf{n}_w}{|\mathbf{p}_i - \boldsymbol{\xi}_w|^2} dS_w, \quad (23)$$

where $\delta_{i\infty}$ is the angle subtended by the wake panel as viewed from $\mathbf{p}_i$.

Finally, we can use the Kutta condition to express the circulation in terms of the velocity potential at the trailing edge

$$\Gamma = \Phi(x_N^F) - \Phi(x_1^I), \quad (24a)$$

$$\Gamma^n = \Phi_N^n - \Phi_1^n, \quad (24b)$$

where x_N^F (the final point on the N th panel) and x_1^I (the initial point on the first panel) should be thought of as tending to the trailing edge from above and below, respectively. Using (24) in Eq. (21) gives

$$\begin{aligned} 2\pi\epsilon_i &= \sum_{j=1}^N \delta_{ij}\epsilon_j + \sum_{j=1}^N \beta_{ij} + [\Phi(x_N^F) - \Phi_N^n - \Phi(x_1^I) + \Phi_1^n]\delta_{i\infty} \\ &= \sum_{j=1}^N \delta_{ij}\epsilon_j + \sum_{j=1}^N \beta_{ij} + [\epsilon_N + \psi_N(x_N^F) - \epsilon_1 - \psi_1(x_1^I)]\delta_{i\infty}. \end{aligned} \quad (25)$$

Applying Eq. (25) to each of the N panels yields a linear system of equations for the “unknowns” ϵ_i . Eq. (25) can be expressed as a matrix equation,

$$\begin{pmatrix} \tilde{\delta}_{11} - \delta_{1\infty} & \delta_{12} & \delta_{13} & \cdots & \delta_{1N} + \delta_{1\infty} \\ \delta_{21} - \delta_{2\infty} & \tilde{\delta}_{22} & \delta_{23} & \cdots & \delta_{2N} + \delta_{2\infty} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta_{N1} - \delta_{N\infty} & \delta_{N2} & \delta_{N3} & \cdots & \tilde{\delta}_{NN} + \delta_{N\infty} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} + \Delta_{1N} \begin{pmatrix} \delta_{1\infty} \\ \delta_{2\infty} \\ \vdots \\ \delta_{N\infty} \end{pmatrix}, \quad (26)$$

where $\tilde{\delta}_{ii} = \delta_{ii} - 2\pi$, $\beta_i = -\sum_{j=1}^N \beta_{ij}$ and $\Delta_{1N} = \psi_1(x_1^I) - \psi_N(x_N^F)$. Note that, for the coefficients δ_{ij} and $\delta_{i\infty}$ in Eq. (25), the integration path is irrelevant; they do not depend on the panel shapes, only on the endpoints of the panels. So, if there is any influence on the error magnitude of the panel shapes, it can only be through the coefficients β_i or Δ_{1N} .

To estimate the error ϵ_i we need to estimate the right-hand side (RHS) of Eq. (26) and to establish that the $N \times N$ matrix is invertible and that its eigenvalues are well-behaved (i.e., not too large or too small). In the next section we estimate the order of the RHS of Eq. (26) analytically. Then, in Section 5 we analytically obtain the order of magnitude of the error for different body configurations and validate these against several numerical experiments.

4. Details of error estimate

We estimate here the size of the RHS of Eq. (26). First, recall that $\beta_i = -\sum_{j=1}^N \beta_{ij}$, where β_{ij} was defined in (22). By using the polar angle $\theta_{p_i s_j}$, defined in Eq. (19), we have

$$\beta_{ij} = \int_{L_j} \psi_j(s_j) d\theta_{p_i s_j}, \quad (27)$$

where ψ_i is defined by Eq. (15) in the convex case and Eq. (17) in the concave case.

We first consider the convex case and focus on the second term in Eq. (15). We can use the mean value theorem, applied to each component of $\nabla\Phi^n$, to establish

$$\nabla\Phi^n(\mathbf{z}_i) = \nabla\Phi^n(\mathbf{x}_i) + O(l), \quad (28)$$

and may write

$$\nabla\Phi^n(\mathbf{x}_i) \cdot (\mathbf{p}_i - \mathbf{x}_i) = [\nabla\Phi^n(\mathbf{x}_i) \cdot \mathbf{t}_i][(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{t}_i] + \quad (29)$$

$$[\nabla\Phi^n(\mathbf{x}_i) \cdot \mathbf{n}_i][(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{n}_i]. \quad (30)$$

In general, we have $|\nabla\Phi^n(\mathbf{x}_i)| = O(1)$, $(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{n}_i = O(l^2)$ and $(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{t}_i = O(l)$, so we expect an $O(l)$ contribution from this term in Eq. (15) unless $(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{t}_i = O(l^2)$, i.e., unless the point of comparison $\mathbf{p}_i$ on the surface is taken to be “directly above” the control point $\mathbf{x}_i$. We hereafter assume this to be the case, and write

$$\psi_i(s_i) = \frac{\partial\Phi}{\partial s}(\mathbf{z}_i)[s(s_i) - s(\mathbf{p}_i)] + O(l^2). \quad (31)$$

Note that if we use curved panels this term could be $o(l^2)$.

The remaining term is $O(l)$ but makes a contribution to β_{ij} of lower order if the control points are taken to be the midpoints of the panels $\mathbf{t}_i$, as they usually are. To prove this we appeal to the mean value theorem to write

$$\frac{\partial\Phi}{\partial s}(\mathbf{z}_s) = \frac{\partial\Phi}{\partial s}(\mathbf{p}_i) + \nabla \frac{\partial\Phi}{\partial s}(\tilde{\mathbf{z}}_s) \cdot [\mathbf{z}_s - s(\mathbf{p}_i)] = \frac{\partial\Phi}{\partial s}(\mathbf{p}_i) + O(l), \quad (32)$$

where $\tilde{\mathbf{z}}_s$ is a point on the surface between $\mathbf{p}_i$ and $\mathbf{z}_s$. Thus

$$\begin{aligned} \psi_i(s_i) &= \left[\frac{\partial\Phi}{\partial s}(\mathbf{p}_i) + O(l) \right] [\xi_i(s_i) - \xi_i(\mathbf{p}_i)] [1 + O(l^2)] + O(l^2) \\ &= \frac{\partial\Phi}{\partial s}(\mathbf{p}_i) [\xi_i(s_i) - \xi_i(\mathbf{x}_i)] + O(l^2), \end{aligned} \quad (33)$$

where we have made use of Eq. (A.3) and the fact that $\xi_i(\mathbf{p}_i) = \xi_i(\mathbf{x}_i)$ if $(\mathbf{p}_i - \mathbf{x}_i) \cdot \mathbf{t}_i = 0$. Note that if we use curved panels the estimate of the first term of the RHS would be the same.

Using Eq. (33), Eq. (27) simplifies to

$$\beta_{ij} = \frac{\partial\Phi}{\partial s}(\mathbf{p}_i) \int_{L_j} [\xi_j(s_j) - \xi_j(\mathbf{x}_i)] d\theta_{p_i s_j} + O(l^3). \quad (34)$$

It remains to estimate the integral in Eq. (34), which is a matter of straightforward geometry. For $i \neq j$, using the definitions in Fig. 5 we write $\xi_j(s_j) - \xi_j(\mathbf{x}_i) = l_1 + l_2$ and immediately conclude, since $h(\xi_j) = O(l^2)$ (see Section Appendix A) that $l_2 = O(l^2)$. Then, we have

Fig. 5. Definitions used in evaluating the integral in Eq. (34) in the convex case. For simplicity, we drop the subscripts on the angles here.

$$\begin{aligned}\xi_j(s_j) - \xi_j(x_j) &= l_1 + O(l^2) = -\rho \cos \bar{\theta} + \rho \frac{\sin \bar{\theta} \cos \theta}{\sin \theta} + O(l^2) \\ &= \frac{\rho \sin \theta}{\sin \bar{\theta}} + O(l^2),\end{aligned}\quad (35)$$

where we have dropped the subscripts for convenience. Then, if we use $\theta = \bar{\theta} + O(l)$,

$$\xi_j(s_j) - \xi_j(x_j) = \frac{\rho}{\sin \bar{\theta}}(\bar{\theta} - \theta) + O(l^2). \quad (37)$$

By using Eq. (37), the integral in Eq. (34) is easily evaluated,

$$\int_{\theta^I}^{\theta^F} \left[\frac{\rho}{\sin \bar{\theta}}(\bar{\theta} - \theta) + O(l^2) \right] d\theta = \frac{\rho}{2 \sin \bar{\theta}}(\alpha^I - \alpha^F)(\alpha^I + \alpha^F) + O(l^3), \quad (38)$$

where we have defined the $O(l)$ quantities $\alpha^I = \theta^I - \bar{\theta}$ and $\alpha^F = \bar{\theta} - \theta^F$. Note that $\alpha^I + \alpha^F = \delta_{ij}$ is the angle subtended by panel j as viewed from p_i while $\alpha^I - \alpha^F$ is the difference between the angles subtended by the portions of l_j before and after (in a clockwise sense) x_j . In general this difference will be $O(l)$ as well, but in the case where x_j is located at the center of the panel (total length l_j) the leading order terms cancel. To see this we use

$$\frac{\sin(\bar{\theta} + \alpha^I)}{\rho} = \frac{2 \sin \alpha^I}{l_j}, \quad (39)$$

$$\frac{\sin(\bar{\theta} - \alpha^F)}{\rho} = \frac{2 \sin \alpha^F}{l_j}, \quad (40)$$

and expand in l_j to get,

$$\alpha^I = \frac{l_j}{2\rho} \sin \bar{\theta} (1 + \frac{l_j}{2\rho} \cos \bar{\theta}) + O(l^3), \quad (41)$$

$$\alpha^F = \frac{l_j}{2\rho} \sin \bar{\theta} (1 - \frac{l_j}{2\rho} \cos \bar{\theta}) + O(l^3). \quad (42)$$

It then follows that

$$\alpha^I - \alpha^F = \left(\frac{l_j}{2\rho} \right)^2 \sin(2\bar{\theta}) + O(l^3), \quad (43)$$

and, returning to Eq. (34), that

$$\beta_{ij} = \frac{l_j^2 \delta_{ij} \cos \bar{\theta}}{4\rho} \frac{\partial \Phi}{\partial s}(p_i) + O(l^3) = O(l^3). \quad (44)$$

For $i=j$, considering p_i "directly above" x_i we have:

$$\xi_j(s_j) - \xi_j(x_j) = \rho \cos \theta, \quad (45)$$

so the integral in Eq. (34) is,

$$\int_{\theta^I}^{\theta^F} \rho \cos \theta d\theta = \rho (\sin \theta^F - \sin \theta^I) = O(l^3). \quad (46)$$

In the concave case we have

$$\psi_i(s_i) = \nabla \Phi(\tilde{z}_i) \cdot (s_i - x_i). \quad (47)$$

We again make use of the mean value theorem, and write

$$\nabla \Phi(\tilde{z}_i) = \nabla \Phi(s_i) + O(l) = \frac{\partial \Phi}{\partial s}(s_i) \mathbf{t} + O(l), \quad (48)$$

and also

$$\frac{\partial \Phi}{\partial s}(s_i) = \frac{\partial \Phi}{\partial s}(x_i) + \nabla \frac{\partial \Phi}{\partial s}(\tilde{z}_i) \cdot (s_i - x_i) = \frac{\partial \Phi}{\partial s}(x_i) + O(l), \quad (49)$$

where $\tilde{z}_i$ is a point between s_i and x_i . Together Eqs. (48), (49), and (A.2) imply

Fig. 6. Definitions used in evaluating the integral in Eq. (34) in the concave case.

$$\psi_i(s) = \left[\frac{\partial \Phi}{\partial s}(x_i) t_i + O(l) \right] \cdot (s_i - x_i) = \frac{\partial \Phi}{\partial s}(p_i) [\xi_i(s_i) - \xi_i(x_i)] + O(l^2), \quad (50)$$

where we have used $p_i = x_i$ to conform with Eq. (33). The expression for β_{ij} , which is again given by Eq. (34), is obtained much as before. The geometric situation in the concave case for $i \neq j$ is depicted in Fig. 6.

We write $\xi_i(s_i) - \xi_i(x_i) = l_1 - l_2$ and observe that $l_2 = O(l^2)$, as before. We have

$$l_1 = \frac{\rho \sin \theta}{\sin \bar{\theta}}, \quad (51)$$

and thus

$$\xi_i(s_i) - \xi_i(x_i) = \frac{\rho \sin \theta}{\sin \bar{\theta}} + O(l^2) = \frac{\rho}{\sin \bar{\theta}}(\bar{\theta} - \theta) + O(l^2). \quad (52)$$

The remaining steps in estimating the integral in Eq. (34) are exactly as in the convex case, the result being Eq. (44).

Thus, under the assumptions that the control points x_i are located at the center of the panels and that the points of (error) comparison p_i are taken to be directly outside of (or, in the concave case, coincident with) the x_i , such that $(p_i - x_i) \cdot t_i = O(l^2)$, we have $\beta_{ij} = O(l^3)$. This implies, since $l = O(N^{-1})$, that $\beta_i = -\sum_{j=1}^N \beta_{ij} = O(l^2)$. Note that the use of curved panels would not affect the order of β_i .

It only remains to show that the second term on the RHS of Eq. (26), which is proportional to $\Delta_{1N} = \psi_1(x_1^I) - \psi_N(x_N^F)$ is not larger. In the case of a body whose upper and lower surfaces meet with a finite angle at the trailing edge, we know from continuity of the fluid velocity that the trailing edge is a stagnation point. We thus expect $\frac{\partial \Phi}{\partial s}(p_i) = O(l)$, $i = 1, N$, which implies via Eqs. (33) and (50) that $\psi_1(x_1^I)$ and $\psi_N(x_N^F)$ are both $O(l^2)$, the same as β_i . If the upper and lower surfaces are tangent at the trailing edge (i.e., there is a cusp) then we have $\frac{\partial \Phi}{\partial s}(p_1) = -\frac{\partial \Phi}{\partial s}(p_N) + O(l)$, i.e., the velocities are (nearly) equal while the sense of ds is opposite. This means, assuming that $\ell_N = \ell_1 + O(l^2)$, that

$$\begin{aligned}\Delta_{1N} &= \psi_1(x_1^I) - \psi_N(x_N^F) = \left[-\frac{l_1}{2} \frac{\partial \Phi}{\partial s}(p_1) + O(l^2) \right] - \left[\frac{l_1}{2} \frac{\partial \Phi}{\partial s}(p_N) + O(l^2) \right] \\ &= -\frac{l_1}{2} \left[\frac{\partial \Phi}{\partial s}(p_1) + \frac{\partial \Phi}{\partial s}(p_N) \right] + O(l^2) = O(l^2).\end{aligned}\quad (53)$$

Thus the second term on the RHS of Eq. (26) is, like β_i , $O(l^2)$.

Fig. 7. Sketch of trailing edge in a sharp body.

4.1. Details of error estimate for sharp trailing edge

In Fig. 7 a sketch of the panels near the trailing edge is shown. In this case, for the panels near the trailing edge, we cannot use the approximation of Eq. (36). To estimate the leading term of the integral in Eq. (34) we use Eq. (35), obtaining:

$$\int_{\theta^F}^{\theta^I} \left[-\rho \cos \bar{\theta} + \frac{\rho \sin \bar{\theta} \cos \theta}{\sin \theta} \right] d\theta = -\rho \cos \bar{\theta} (\theta^F - \theta^I) + \rho \sin \bar{\theta} (\log |\sin \theta^F| - \log |\sin \theta^I|). \quad (54)$$

For $i = 1, j = N$ we have

$$\rho = l \sin(\gamma/2), \quad \bar{\theta}_{1N} = \frac{\pi + \gamma}{2}, \quad \theta_{1N}^F = \gamma, \quad \theta_{1N}^I = \delta_{1N} + \gamma, \quad (55)$$

with

$$\cos \delta_{1N} = \frac{4 \sin^2(\gamma/2) - 1}{\sqrt{1 + 8 \sin^2(\gamma/2)}}. \quad (56)$$

Note that when γ tends to π then δ_{1N} tends to zero, and when γ tends to zero δ_{1N} tends to π . The integral in Eq. (54) for panels 1 and N is then

$$\int_{\theta^F}^{\theta^I} \left[\rho \cos \bar{\theta} + \frac{\rho \sin \bar{\theta} \cos \theta}{\sin \theta} \right] d\theta = l \sin \left(\frac{\gamma}{2} \right) \left[\delta_{1N} \cos \left(\frac{\pi + \gamma}{2} \right) + \sin \left(\frac{\pi + \gamma}{2} \right) \log \left| \frac{\sin \gamma}{\sin(\gamma + \delta_{1N})} \right| \right]. \quad (57)$$

Thus, not only for panels 1 and N , but also for other panels near the trailing edge, the integral is not $O(l^3)$ as obtained in (46), but $O(l)$. However, as this occurs only for the panels near the trailing edge, $\partial \Phi(\mathbf{p}_j)/\partial s \approx O(l)$, so we have $\beta_{ij} \approx O(l^2)$ for these panels (see Eq. (34)) and $\beta_{ij} \approx O(l^3)$ for panels sufficiently far from the trailing edge. Depending on the shape of the profile, the order of $\beta_i = -\sum_{j=1}^N \beta_{ij}$ is $O(l^p)$ with $p \in [1, 2]$.

5. Results

In this section the errors ε_i are estimated analytically for bodies with and without sharp corners and with concave or convex parts. In order to validate the theoretical results several numerical experiments are performed for which an accurate analytical solution is known. A comparison of the numerical and analytical solutions determines the true numerical error and we check whether this result agrees with the theoretical prediction.

For all numerical experiments described in this section, the wake is oriented parallel to the x axis, and the angle of attack α is defined as the angle between this axis and the incident flow, which is given in each case. The modulus of the far field velocity is set to unity: $U_\infty = 1$.

5.1. Numerical error for thick bodies without sharp corners

Appendix A shows that $\delta_{ii} = \pi + O(l)$, so $\tilde{\delta}_{ii} = -\pi + O(l)$ and $\tilde{\delta}_{ij}$, with $i \neq j$, ∞ , is $O(l)$ or smaller for these bodies. Moreover we have $\delta_{1\infty} = -\delta_{N\infty} = \pi/2 + O(l)$ and $\delta_{1N} = \delta_{N1} = O(l)$, so the $N \times N$ matrix in Eq. (26) can be expressed as

$$\begin{pmatrix} -3\pi/2 + O(l) & O(l) & O(l) & \dots & \pi/2 + O(l) \\ -\delta_{2\infty} + O(l) & -\pi & O(l) & \dots & \delta_{2\infty} + O(l) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi/2 + O(l) & O(l) & O(l) & \dots & -3\pi/2 + O(l) \end{pmatrix}. \quad (58)$$

The matrix is, therefore, well-behaved since the off-diagonal terms are $O(l)$ or smaller except for the first and the last column while the diagonal entries are, in general, close to $-\pi$. We thus expect the errors ε_i to be of the same order as the RHS. Section 4 shows the RHS of Eq. (26) to be $O(l^2)$ provided two conditions hold:

- The control points $\mathbf{x}_i$ are located at the center of the panels ℓ_i .
- The points of comparison $\mathbf{p}_i$ are located "directly above" (or coincident with, in the concave case) the $\mathbf{x}_i$ such that $\ell_i(\mathbf{p}_i - \mathbf{x}_i) = O(l^2)$ where ℓ_i is a unit vector tangent to panel i .

If either of these conditions is violated the error is $O(l)$.

To confirm these results we obtain the numerical error in known cases, beginning with the flow around a thick ellipse with major axis of length 5 and minor axis of length 3; see Fig. 8. The ellipse and the analytical expression for the velocity potential around it have been obtained from a Joukowski transformation:

$$\tau = t + \frac{a^2}{t}, \quad (59)$$

of a circle centered at $(x_0, y_0) = (0, 0)$ with radius $R=2$ and $a=1$. In Eq. (59) t is a complex variable in the original domain, τ is a complex variable in the transformed space and a is a constant. Since the complex potential around a circle has the analytical expression,

$$f(t) = U_\infty(t - t_0)\exp(-i\alpha) + U_\infty \frac{R^2}{t - t_0} \exp(i\alpha) + \frac{i\Gamma}{2\pi} \log(t - t_0), \quad (60)$$

with $t_0 = x_0 + iy_0$, we can express (see [12]) the complex potential around the body in the transformed space as $f(t(\tau))$; the velocity potential Φ is the real part of $f(t(\tau))$.

Fig. 9 shows the numerical error obtained with this body by comparing the analytical and numerical solutions with $\alpha = 2^\circ$. This is calculated as a function of the number of panels, with the control points placed at the center of the panels, and at a point $\mathbf{p}$ placed (see Fig. 8) at $(x, y) = (-0.947, -1.388)$. A (base 10) logarithmic scale has been used on both axis to show the order of the error. This figure shows that the numerical error is $O(1/N^2)$ when the control points are placed at the center of the panels, as the theoretical analysis predicts. Similar results

Fig. 8. Ellipse with axis length 5 and 3.

Fig. 9. Comparison of the numerical error for a thick ellipse with the control points placed at the center of the panels lower curve and at the position $l/3$ (upper curve).

are found at other points of the body. This figure also shows the numerical error for this body when the control points are placed, instead, at the position $l/3$; the order of the error in this case is $O(1/N)$, as predicted by the theory. Hereafter, the numerical error will be calculated with the control points placed at the center of the panels.

To validate the theoretical results for bodies with both convex and concave parts, we calculate the numerical error for the body of Fig. 10. The body and the analytical solution for the potential velocity around it have been obtained from a Joukowski transformation with $a=1$ of a

Fig. 10. Nonsymmetric body with concave and convex parts.

Fig. 11. Numerical error for a nonsymmetric body with concave and convex parts.

circle with radius $R=2$ centered at $(x_0, y_0) = (0, 1)$. Fig. 11 shows the numerical error at two points placed at $(x, y) = (-0.358, 2.643)$ (on the convex part of the profile) and $(x, y) = (-0.727, -0.0653)$ (the concave part of the profile) for $\alpha = 2^\circ$ as a function of the number of panels. This figure shows that the numerical error is $O(1/N^2)$ for both points, which is as the theoretical analysis predicts. Similar results are found at other points of the body.

5.2. Numerical error for thin bodies without sharp corners

For these types of bodies (see, e.g., the ellipse of Fig. 12, it is necessary to take $l \ll h$, with h being a characteristic thickness of the body, such as the length of the vertical axis for the ellipse in Fig. 12, to avoid the following problems:

- The $N \times N$ matrix could be close to singular. To illustrate this problem we suppose all panels are the same length. If $h \ll l$ the $N \times N$ matrix in Eq. (26) would be as follows for rows k and $\bar{k} = N - k + 1$, with panel k on the lower surface and $\bar{k}$ on the upper surface, both sufficiently far from the leading and trailing edges:

Fig. 12. Thin ellipse.

Fig. 13. Detail of the panels near the trailing edge of a thin ellipse with $h \ll l$.

$$\begin{pmatrix} \tilde{\delta}_{11} - \delta_{1\infty} & \delta_{12} & \cdots & \delta_{1k} & \cdots & \delta_{1\ell} & \cdots & \delta_{1N} + \delta_{1\infty} \\ \delta_{21} - \delta_{2\infty} & \tilde{\delta}_{22} & \cdots & \delta_{2k} & \cdots & \delta_{2\ell} & \cdots & \delta_{2N} + \delta_{2\infty} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\delta_{k\infty} + O(l) & O(l) & \cdots & -\pi + O(l) & \cdots & \pi + O(l) & \cdots & \delta_{k\infty} + O(l) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \delta_{\ell\infty} + O(l) & O(l) & \cdots & \pi + O(l) & \cdots & -\pi + O(l) & \cdots & -\delta_{\ell\infty} + O(l) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \delta_{N1} - \delta_{N\infty} & \delta_{N2} & \cdots & \delta_{Nk} & \cdots & \delta_{NN-k+1} & \cdots & \tilde{\delta}_{NN} + \delta_{N\infty} \end{pmatrix} \quad (61)$$

Here we have used the fact that, if $h \ll l$, then $\delta_{k\ell} = \pi + O(l)$, $\delta_{\ell k} = \pi + O(l)$ and $\delta_{\ell\infty} = -\delta_{k\infty} + O(l)$. The matrix eigenvalues could thus be ill-behaved.

ii. Besides the problem above, when $h \ll l$ the configuration of the panels near the trailing edge is as in Fig. 13, which shows the trailing edge of the ellipse of Fig. 12 in detail. In this case, the angle of the panels at the leading and trailing edges is not close to π , so the body would behave as if it had a sharp corner. In this situation we cannot use the approximation in Eq. (36), and the order of β_i may not be $O(l^2)$, as explained in Section 4.1 and discussed in the next section.

To validate the analytical results we calculate the numerical error for the ellipse of Fig. 12, with major axis of length 7.61 and minor axis of length 0.39. The ellipse and the analytical solution for the velocity potential around it have been obtained from a Joukowski transformation with $a=1.9$ of a circle centered at $(x_0, y_0) = (0, 0)$ with radius $R=2$. Fig. 14 shows the numerical error for $\alpha = 2^\circ$ at two points: p_1 , at $(x, y) = (0.127, -0.195)$ and p_2 , at $(x, y) = (3.439, -0.083)$ (near the trailing edge), as a function of the number of panels. This figure shows how the order of the numerical error decreases when the length of the

Fig. 14. Numerical error for a thin ellipse.

panel decreases. It is $O(1/N^2)$ when the length of the panel is sufficiently small compared to the length of the vertical axis of the ellipse, h .

A comparison of this figure with Fig. 9 shows that the slope of the error curve changes in the case of the thin ellipse and remains nearly constant for the thick one. Thus, if the length of the panels, which is $O(1/N)$, is sufficiently small compared to the thickness of the body, the order of the error is $O(1/N^2)$ with flat panels in both cases, including near the points of higher curvature, as the theoretical analysis predicts. In particular, the error is not $O(1/N)$ for flat panels and $O(1/N^2)$ for curved panels, as other authors [19] suggest. In [24] it is found that thickness does not affect the order of magnitude of the error if the number of panels is large enough. However, in that reference, the authors do not give an analytical explanation for this fact but, instead, present calculations with different Karman-Trefft bodies, situations where the effect of the trailing edge angle may be more important than the effect of the thickness.

5.3. Numerical error for bodies with sharp corners

For bodies of this type the order of β_i may not be $O(l^2)$, as explained in Section 4.1. From Eq. (26) and the approximation of the $N \times N$ matrix in Eq. (58) we can obtain the leading order of the error for panels 1, N , and for a generic panel k :

$$i = 1: -\frac{3\pi}{2}\epsilon_1 + \frac{\pi}{2}\epsilon_N = \beta_1 + O(l^2), \quad (62)$$

$$i = k: -\delta_{k\infty}\epsilon_1 - \pi\epsilon_k + \delta_{k\infty}\epsilon_N = \beta_k + O(l^2), \quad (63)$$

$$i = N: \frac{\pi}{2}\epsilon_1 - \frac{3\pi}{2}\epsilon_N = \beta_N + O(l^2), \quad (64)$$

so

$$\epsilon_1 = -\frac{3\beta_1 + \beta_N}{4\pi} + O(l^2), \quad (65)$$

$$\epsilon_N = -\frac{\beta_1 + 3\beta_N}{4\pi} + O(l^2), \quad (66)$$

$$\epsilon_k \simeq \frac{1}{\pi} \left(\delta_{k\infty} \frac{\beta_1 - \beta_N}{2\pi} - \beta_k \right) + O(l^2). \quad (67)$$

If panel k is far enough from the trailing edge, the term β_k would be of order $O(l^2)$. However, the error is also influenced by the terms β_1 and β_N , as we show in Section 4.1, for bodies with a sharp trailing edge; depending on the shape of the profile, they could be $O(l^p)$ with $p \in [1, 2]$. The numerical error could thus be $O(1/N^p)$ with $p \in [1, 2]$ even though the point in question is far from the trailing edge.

To validate the analytical results we consider different bodies, both symmetric and asymmetric. Fig. 15 shows several symmetric bodies obtained from a Karman-Trefft transformation:

$$\frac{\tau - k_T a}{\tau + k_T a} = \left(\frac{t - a}{t + a} \right)^{k_T}, \quad (68)$$

with different values of k_T ; the constant a is given by:

$$(a - x_0)^2 = R^2 - y_0^2, \quad (69)$$

with $(x_0, y_0) = (-0.2, 0)$ and $R=1$. Fig. 16 shows the numerical error for $\alpha = 2^\circ$ as a function of the number of panels. As this figure shows, the numerical error is $O(1/N^p)$ with $p \in [1, 2]$, and p decreases when k_T increases because the angle at the trailing edge, γ , decreases. Similar behaviour for the error is obtained with other panels.

There is an exception for symmetric bodies when $\alpha = 0^\circ$. In this particular case, with symmetric geometry and symmetric flow around the body we have $\partial\Phi(p_k)/\partial s = \partial\Phi(p_{N-k+1})/\partial s$. Since the integral in Eq. (34) depends only on geometry we conclude that $\beta_k = \beta_{N-k+1}$ and, in particular, $\beta_1 = \beta_N$. In this case, from Eq. (67):

Fig. 15. Symmetric bodies with a sharp corner obtained from a Karman-Trefft transformation with $k_T = 1.1, 1.3$ and 1.6 .

Fig. 16. Numerical error for bodies with sharp corners for $\alpha = 2^\circ$.

$$\epsilon_1 = \epsilon_N \simeq -\frac{\beta_1}{\pi}, \quad (70)$$

$$\epsilon_k \simeq -\frac{\beta_k + \Delta_{1N}}{\pi}, \quad (71)$$

so, for panels far from the trailing edge, the error is $O(1/N^2)$ as Fig. 17 shows.

For asymmetric bodies the results are similar although, in this case, there is no significant difference between the order of the error for $\alpha = 0^\circ$ and for $\alpha \neq 0^\circ$ because symmetric configurations are excluded. Fig. 18 shows asymmetric bodies obtained from a Karman-Trefft transformation with different values of k_T and $(x_0, y_0) = (-0.2, 0.3)$, $R=1$, while Fig. 19 shows the numerical error for $\alpha = 0^\circ$. Results for $\alpha = 2^\circ$ are very similar, as illustrated in Fig. 20.

Thus, the error is $O(1/N^p)$ with $p \in [1, 2]$ and depends on the angle γ at the sharp corner. Figs. 21 and 22 show the evolution of p with γ and k_T , respectively, for different values of α with symmetric bodies. When $\gamma \rightarrow 180^\circ$ the error tends to $O(1/N^2)$ and when $\gamma \rightarrow 0^\circ$ the error tends to $O(1/N)$. Note that the exponent p can take any value between 1 and 2; it is not required that $p=1$ or $p=2$, as several authors suggest [18,19].

In [20] the error for panels near the trailing edge was also analyzed, and it was found that panels placed on the same side contribute an $O(l^2)$ amount to the error while panels on the opposite side contribute

Fig. 17. Numerical error for symmetric bodies with sharp corners with $\alpha = 0^\circ$.

Fig. 18. Asymmetric bodies with sharp corners obtained from a Karman-Trefft transformation with $k_T = 1.3, 1.5$ and 1.8 .

an $O(l)$ amount. The authors associate this last error with the convergence of the method: $O(1/N)$. However, we have shown here that panels placed on the same side contribute an $O(l^3)$ amount to the error while those on the opposite side contribute an $O(l^2)$ amount. Here we calculate the global error, considering the influence of all panels, the shape of the body, and the angle at the trailing edge, and find that the exponent p can take any value between 1 and 2, as stated above.

In [24] the authors obtain a numerical estimate of the error and find $p=1.4$ for a particular Karman-Trefft body; the asymptotic order of the accuracy, however, is higher than the theoretical one they obtain, probably because they perform a local error analysis, not a global error analysis as done here. In [13] the convergence of the panel method in problems with non-smooth boundaries is analyzed. The authors use polygons and analyze the convergence at the corners, and also find that the error at the corners is the main factor that reduces the overall convergence of the method.

Fig. 19. Numerical error for asymmetric bodies with sharp corners with $\alpha = 0^\circ$.

Fig. 20. Numerical error for asymmetric bodies with sharp corners with $\alpha = 2^\circ$.

Fig. 21. Evolution of p with the angle of the trailing edge γ (in degrees) for different α .

Fig. 22. Evolution of p with k_T for different α .

5.4. Numerical error for curved panels

Section 4 shows that the use of curved panels does not modify the order of β_i or Δ_{IN} , so there is no benefit to using curved panels instead of flat panels. To validate this theoretical conclusion, we consider the thin ellipse of Fig. 12. Fig. 23 compares the numerical error for this ellipse when approximated by flat panels with that found using two types of curved panels. The curved panels are placed so that the control points are 1/3 or 2/3 of the distance between x_i and p_i . The figure shows that there is no difference in the order of the error when curved panels are used. This verifies that the curvature of the panels has no influence on the order of magnitude of the error, as suggested in [16] and by other authors, at least for the case analyzed here with a doublet distribution of constant strength along the panel. Generally, for methods using a constant strength singularity distribution, it does not reduce the order of the error to use more complex panel geometries by including, e.g., quadratic or cubic terms.

Fig. 23. Comparison of numerical error for flat panels and curved panels. The results are so close that it is hard to distinguish them on the scale of the plot.

6. Conclusions

In this work a formal analytical and numerical analysis of the asymptotic global error in panel methods using flat panels is presented for bodies with the possibility of convex and concave portions. The analysis shows that there are several factors affecting the order of the error.

(i) Regarding the geometry of the body, the main factor affecting the order of the error is the existence of sharp corners. If there are no sharp corners, the error is $O(1/N^2)$, even for flat panels and bodies with strong curvature, provided the length of the panels, which is $O(1/N)$, is sufficiently small compared to the thickness of the body. If there are sharp corners, the error is $O(1/N^p)$ with $p \in [1, 2]$, and this factor depends on the angle γ at the sharp corner. When $\gamma \rightarrow \pi$ the leading error tends to $O(1/N^2)$ and when $\gamma \rightarrow 0$ the leading error tends to $O(1/N)$. The exponent p can take any value between 1 and 2, and not only those two particular values, as several authors suggest.

(ii) To minimize error, the control points x_i should be located at the center of the panels ℓ_i , and the points of comparison p_i should be located "directly above" the control points x_i in the convex case and coincident with x_i in the concave case. If this condition is violated the error is $O(1/N)$.

(iii) There is no difference in the order of the error for bodies with only convex parts compared to bodies with both convex and concave

Fig. A.2. Illustration of intermediate value theorem.

parts.

(iv) Concerning the shape of the panels, the analysis shows that the use of curved panels rather than flat ones does not improve the order of the error.

The work presented here suggests several promising directions for future research, including application of this type of analysis to the study of error for panel methods using doublet distributions of variable strength along the panels, extension of the analysis to three-dimensional problems, and extension to other applications of BEM such as, for example, adaptive BEM computational schemes (see [26]).

Appendix

Here we give the details behind some of the previous results.

Appendix A. Local geometric considerations

It is useful to consider a system of local coordinates centered on a particular panel, as shown in Fig. A.1. The origin is taken to be at the center of the panel ℓ_i and the ξ_i and η_i axes are oriented parallel and perpendicular, respectively, to this panel. The body surface may be described in terms of these local coordinates by a function $\eta_i = h(\xi_i)$, $\xi_i \in [-l_i/2, l_i/2]$. An arbitrary point $s_i = (\xi_i, h(\xi_i))$ on this surface is associated with a unit vector $\mathbf{n}$, normal to the body surface at that point, and a tangential unit vector $\mathbf{t}$. Explicitly, these vectors are

$$\mathbf{n} = \frac{(-h'(\xi_i), 1)}{\sqrt{1 + h'(\xi_i)^2}} = \mathbf{n}_i + O(h'(\xi_i)), \quad (\text{A.1})$$

$$\mathbf{t} = \frac{(1, h'(\xi_i))}{\sqrt{1 + h'(\xi_i)^2}} = \mathbf{t}_i + O(h'(\xi_i)), \quad (\text{A.2})$$

where $\mathbf{n}_i = (0, 1)$ and $\mathbf{t}_i = (1, 0)$

The intermediate value theorem assures us that there exists a point $x_0 = (\xi_0, \eta_0)$ on the surface L_i such that the tangent at x_0 is parallel to the panel ℓ_i , as shown in Fig. A.2. Thus, we may express the curve as $h(\xi_i) = h(\xi_0) - k(\xi_i - \xi_0)^2 + O(l^3)$, with k a constant. Evaluating this expression at the endpoints $h(\pm l_i/2) = 0$ immediately yields $h(\xi_0) = O(l^2)$ while differentiation implies that $h'(\xi_i) = -2k(\xi_i - \xi_0) = O(l)$. Furthermore, one can see that to differentiate with respect to ξ_i or with respect to s_i is equivalent up to $O(l^2)$,

$$ds_i = \sqrt{1 + h'^2} d\xi_i = d\xi_i (1 + O(l/2)) = d\xi_i (1 + O(l^2)). \quad (\text{A.3})$$

The result used in Eq. (A.4) follows immediately from Eq. (A.1),

Fig. A.1. Definition of local coordinate system.

Fig. A.3. Angle δ_{ii} subtended at p_i by panel i itself, in the convex case.

$$n = n_i + O(l). \quad (\text{A.4})$$

Another result, used first in the context of Eq. (26), follows immediately from the fact that $h(\xi_i) = O(l^2)$ while the panel itself is $O(l)$. As seen in Fig. A.3, the angle subtended at the point p_i by panel i itself is $\pi - \alpha - \beta$ in the convex case and π in the concave case (when p_i tends to the panel from outside). So, we can write

$$\delta_{ii} \simeq \pi - h(p_i) \left(\frac{1}{l_{ia}} + \frac{1}{l_{ip}} \right) = \pi + O(l). \quad (\text{A.5})$$

References

- [1] Brebbia CA, Dominguez J. Boundary element methods for potential problems. Appl Math Model 1977;1:372–8.
- [2] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques: theory and applications in engineering. Berlin: Springer; 1984.
- [3] Hess J, Smith A. Calculation of non-lifting potential flow about arbitrary three-dimensional bodies, Douglas Report ES40622, Douglas Aircraft Company; 1962.
- [4] Morino L, Kuo CC. Subsonic potential aerodynamics for complex configurations: a general theory. AIAA J 1974;12(2):191–7.
- [5] Rubbert PE, Saaris GR. A general three-dimensional potential flow method applied to V/STOL aerodynamics, SAE Paper 680304; 1968.
- [6] Hess J. Calculation of potential flow about arbitrary three-dimensional lifting bodies, Douglas Report MDC-J5679-01, Douglas Aircraft Company; 1972.
- [7] Morino L, Chen LT, Suci E. Steady and oscillatory subsonic and supersonic aerodynamics around complex configurations. AIAA J 1975;13(3):368–74.
- [8] Ehlers FE, Rubbert PE. A mach line panel method for computing the linearized supersonic flow. NASA 1979:152126, [CR-152126].
- [9] Hess JL. Panel methods in computational fluid dynamics. Annu Rev Fluid Mech 1990;22:255–74.
- [10] Erickson L. Panel Methods—An Introduction, TP 2995, NASA; 1990.
- [11] Hwang WS. A boundary node method for airfoils based on the Dirichlet condition. Comput Methods Appl Mech Eng 2003;190:1679–88.
- [12] Katz J, Plotkin A. Low-speed aerodynamics. Cambridge University Press; 2001.
- [13] Ye W, Fei Y. On the convergence of the panel method for potential problems with non-smooth domains. Eng Anal Bound Elem 2009;33(6):837–44.
- [14] Yu W, Wang Z, Hong X. Boundary element techniques: theory and applications in engineering. Eng Anal Bound Elem 2004;28:1035–44.
- [15] Johnson FT, Rubbert PE. Advanced panel-type influence coefficient methods applied to subsonic flow, AIAA Paper 75-50; 1975.
- [16] Ehlers FE, Epton MA, Johnson FT, Magnus AE, Rubbert PE. A higher order panel method for linearized flow, CR 3062, NASA; 1979.
- [17] Mercer JE, Weber JA, Lesford EP. Aerodynamic influence coefficient method using singularity splines, CR 2423, NASA; 1974.
- [18] Oskam B. Asymptotic convergence of higher-Order accurate panel methods. J Math Anal Appl 1985;23:126–30.
- [19] Bellamy-Knights PG, Benson MG, Gerrard JH, Gladwell I. Convergence properties of panel methods. Comput Methods Appl Mech Eng 1989;76:171–8.
- [20] Kinnas SA, Hsin CY. The local error of a low-order boundary element method at the trailing edge of a hydrofoil and its effect on the global solution. Comput Fluids 1994;23:63–75.
- [21] Hess J. Higher order numerical solution of the integral equation for the two-dimensional Neumann problem. Comput Methods Appl Mech Eng 1972;2:1–15.
- [22] Hess J. Improved solution for potential flow about arbitrary axisymmetric bodies by the use of a higher order surface source method. Comput Methods Appl Mech Eng 1975;5:297–308.
- [23] Romate JE. Local Error Analysis in 3D Panel Methods. J Eng Math 1988;4:123–42.
- [24] Vazquez G, Falcão de Campos JAC, Eca L. A numerical study on low and higher-order potential based BEM for 2D inviscid flows. Comput Mech 2003;32:27–335.
- [25] Falcão de Campos JAC, Ferreira de Sousa PJA, Bosschers J. A verification study on low-order three-dimensional potential-based panel codes. Comput Fluids 2006;35:61–73.
- [26] Zhang W, Yu W, Liu H, Wang Z. Hierarchical h-adaptive computation of VLSI interconnect capacitance with QMM acceleration. In Proceedings of the 8th international conference on solid-state and integrated circuit technology; 2006:1438–1440.