An Inequality for Algebroid Functions and Some Applications

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Abstract In this paper we present a version for algebroid functions of an inequality due to Barsegian (Math URSS Sbornik 42(2):155, 1982) and we get two consequences: first a kind of deficiency relation for curves (preimages of an algebroid function, the so-called Γ -lines) which is an extension of the well-known relation of Barsegian (Γ -lines. On the geometry of real and complex functions. Taylor & Francis, London, 2002) and secondly a proximity principle for the preimages of an algebroid function, also an extension of the principle for meromorphic functions.

Keywords Algebroid functions · Gamma-lines · Deficiency relations · Proximity principles

Mathematics Subject Classification 30C99

1 Algebroid Functions

An algebroid function is a multivaluated one $w = w(z)$ with a finite number of branches in $\mathbb{C}$ and whose only singularities are poles and algebraic branch points [10,11]. These functions satisfy an equation like

$$w^k + A_1(z) w^{k-1} + \cdots + A_{k-1}(z) w + A_k(z) = 0$$

with A_i , $i = 1, 2, \dots, k$ meromorphic functions.

k is the order of the algebroid function.

But we can also consider algebroid functions as holomorphic functions defined on a Riemann surface with k sheets [4]: we have the Riemann surface

$$\Xi = \{(z, w_i(z)), z \in \mathbb{C}, i = 1, 2, \dots, k\}$$

where $w_1, w_2, \dots, w_k$ are the branches of the algebroid function.

The canonical projection

$$\begin{aligned} p : \Xi &\rightarrow \mathbb{C} \\ (z, w_i(z)) &\rightarrow z \end{aligned}$$

is an holomorphic covering of $\mathbb{C}$ and

$$\begin{aligned} f : \Xi &\rightarrow \widehat{\mathbb{C}} \\ (z, w_i(z)) &\rightarrow w_i(z) \end{aligned}$$

is also holomorphic. It is the function associated to the algebroid function.

2 The Inequality

We can consider the relation

$$\begin{array}{ccc} & f & \\ \Xi & \rightarrow & \widehat{\mathbb{C}} \\ p \downarrow \nearrow & & \\ \mathbb{C} & & w_i \end{array}$$

between the function associated to the algebroid function and its branches. Ξ is now a Riemann surface of k sheets and p is locally the identity.

Then

$$w_i \circ p = f$$

but $p' = 1$ and $p'' = 0 \implies f'' = w_i'' \circ p$.

From now on $\Xi[r]$ is the part of the Riemann surface Ξ which projects by means of p onto $D[r]$, the closed disk of $\mathbb{C}$ centered in $(0, 0)$, with radius r .

$$\begin{aligned} p : \Xi &\rightarrow \mathbb{C} \\ p(\zeta) &= z \end{aligned}$$

and

$$\Phi = p^*(rdrd\varphi)$$

is the pull-back of the element of surface on the plane to the surface Ξ .

Proposition 1

$$\iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi \leq k \sum_{i=1}^k \iint_{D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi$$

with $z = r e^{i\varphi}$.

Proof Around any point of $D[r]$ there is a neighbourhood U_i with $p^{-1}(U_i \cap D[r]) \subset \Xi[r]$ and a w_i defined in $U_i \cap D[r]$.

p is locally an homeomorphism (it is the identity indeed) and applies every neighbourhood in $\Xi[r]$ onto a neighbourhood in $D[r]$.

Then, if around every point of $D[r]$, $z = p(\zeta)$ there is a neighbourhood U_i where a branch w_i of f is defined, $w_1, w_2, \dots, w_k$ are defined in $U_1 \cap D[r], U_2 \cap D[r], \dots, \text{in } U_k \cap D[r]$, in such a way that

$$\bigcup_{i=1}^k (U_i \cap D[r]) = D[r]$$

then

$$\sum_{i=1}^k \iint_{p^{-1}(U_i \cap D[r])} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi = \sum_{i=1}^k \iint_{U_i \cap D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi,$$

but

$$\iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi \leq \sum_{i=1}^k \iint_{p^{-1}(U_i \cap D[r])} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi$$

hence

$$\iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi \leq \sum_{i=1}^k \iint_{U_i \cap D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi$$

But this argument only is valid with only one sheet in the surface Ξ , p being a homeomorphism only locally because it projects points ζ in the different sheets of Ξ onto the same point $z \in \mathbb{C}$, hence we must consider the totality of the sheets and we get

$$\iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi \leq k \sum_{i=1}^k \iint_{U_i \cap D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi$$

finally, we observe that

$$\iint_{U_i \cap D[r]} \left| \frac{w_i''(\zeta)}{w_i'(\zeta)} \right| r dr d\varphi \leq \iint_{D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi$$

and we finally get

$$\iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi \leq k \sum_{i=1}^k \iint_{D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi$$

Observation: arguing in the same way for the integral

$$A(r) = \frac{1}{\pi} \iint_{\Xi[r]} \frac{|f'(\zeta)|^2}{(1 + |f'(\zeta)|^2)^2} \Phi$$

which is the density of the well-known Ahlfors–Shimizu characteristic, we get

$$A(r) \leq k \sum_{i=1}^k \iint_{D[r]} \frac{|w_i'(z)|^2}{(1 + |w_i'(z)|^2)^2} r dr d\varphi$$

and we shall call, from now on

$$A_i(r) = \iint_{D[r]} \frac{|w_i'(z)|^2}{(1 + |w_i'(z)|^2)^2} r dr d\varphi$$

$A(r)$ is the Ahlfors–Shimizu density of the Riemannian image of $\Xi[r]$ under f and $A_i(r)$ those of $D[r]$ under the branches of the algebroid function.

3 The Defect Relation

Let Γ_ν , $\nu = 1, 2, \dots, q$ be q Jordan, smooth, disjoint, bounded curves, with $\nu(\Gamma_\nu) < \infty$, where $\nu(\Gamma) = \text{Var}_{\zeta \in \Gamma} \alpha_\Gamma(\zeta)$, being $\alpha_\Gamma(\zeta)$ the angle between the tangent to $p(\Gamma)$ in $p(\zeta)$ and the OX axis. We add the hypothesis that $f^{-1}(\Gamma_\nu)$ does not contain poles nor branch points of Ξ .

Def.- $\Delta(\Gamma) = \liminf_{r \rightarrow \infty} \frac{\Lambda(r, \Gamma)}{\sum_{i=1}^k A_i(r)}$, where $\Lambda(r, \Gamma)$ is the length of the preimages of Γ in $\Xi[r]$, under the algebroid function f .

$\Delta(\Gamma)$ is the deficiency of Γ and we shall call Γ a deficient curve if and only if $\Delta(\Gamma) \neq 0$.

With this notations, we have the

Theorem 1 Let $\Gamma_v, v = 1, 2, \dots, q$ be a collection of curves as above and let $\Delta(\Gamma_v)$ be the deficiency of every Γ_v , then:

$$\sum_{v=1}^q \Delta(\Gamma_v) \leq Ck$$

where C is an absolute constant and k is the order of the algebroid function.

Proof We begin with

$$\sum_{v=1}^q \Lambda(r, \Gamma_v) \leq K \iint_{\Xi[r]} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \Phi + h\pi r \sqrt{kA(r)} + 2\sqrt{2}k\pi r$$

K and h being independent of the function f . [1]

Then

$$\sum_{v=1}^q \Lambda(r, \Gamma_v) \leq K k \iint_{D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi + h\pi r \sqrt{kA(r)} + 2\sqrt{2}k\pi r$$

but

$$\iint_{D[r]} \left| \frac{w_i''(z)}{w_i'(z)} \right| r dr d\varphi \leq \widehat{K} r A_i(r)$$

for $r \in E_i \subset \mathbb{R}$ such that $\liminf_{r \rightarrow \infty} \frac{1}{r} \int_{E_i \cap [0, \infty)} d \ln t = C_i \in (0, 1)$, i. e. E_i being a set of lower logarithmic density C_i . (Branches w_i are meromorphic functions) [3].

Then

$$\sum_{v=1}^q \Lambda(r, \Gamma_v) \leq Kkr \sum_{i=1}^k A_i(r) + h\pi r \sqrt{kA(r)} + 2\sqrt{2}k\pi r$$

but

$$\sqrt{kA(r)} \leq kA(r) \leq k^2 \sum_{i=1}^k A_i(r)$$

and we get

$$\sum_{v=1}^q \Lambda(r, \Gamma_v) \leq Kkr \sum_{i=1}^k A_i(r) + h\pi r k^2 \sum_{i=1}^k A_i(r) + 2\sqrt{2}k\pi r$$

and finally

$$\sum_{v=1}^q \Delta(r, \Gamma_v) \leq kr \left[C_1 \sum_{i=1}^k A_i(r) + C_2 \right]$$

with $r \in E$ as above.

Then, with the definition of $\Delta(\Gamma)$ we get the proposition we were looking for.

Now, if we have $q \rightarrow \infty$ we get

$$\sum_{v=1}^{\infty} \Delta(\Gamma_v) \leq Ck$$

and it remains bounded, so that we must have $\Delta(\Gamma) = 0$ for almost all curves and only a numerable quantity of curves are such that $\Delta(\Gamma) > 0$. Those are the deficient curves.

4 The Proximity Principle

We are now speaking about the problem of the distribution of preimages of a function (an algebroid one, in our case) and about an estimation of their proximity. If we call a -points corresponding to z those that satisfies $f(z) = a$, we will try to progress in the direction of the so-called Littlewood Property [6] which this author has asserted for entire functions and says, roughly speaking, that almost all a -points of those functions, for almost all values of a , belong to a subset of $\mathbb{C}$ of small area. Littlewood has proved that the property is true under the hypothesis that the bound

$$\iint_{D(1)} \frac{|p'(z)|}{1 + |p(z)^2|} r dr d\varphi \leq cn^{\frac{1}{2}-\alpha}$$

is fulfilled by a polynomial p of degree less or equal than n and $\alpha > 0$.

This is the conjecture of Littlewood, proved by Lewis and Wu [7].

In this direction, we extend a result of Barsegian [Bar1] for meromorphic functions to algebroid functions.

Let $a_v, b_v, v = 1, 2, \dots, q$ be values in $\mathbb{C}$, f an algebroid function, $\zeta_i(z)$ are the preimages of z , i. e. $f(\zeta_i(z)) = z$. $n(r, z)$ is the number of preimages of z in $\Xi[r]$.

Let Γ_v be Jordan, bounded, smooth, disjoint, bounded curves, with $v(\Gamma_v) < \infty$, and the extreme points of Γ_v are a_v and b_v .

$f^{-1}(\Gamma_v) \cap \Xi[r]$ contains $n(r, a_v)$ curves $\gamma_i, i = 1, 2, \dots, n(r, a_v)$ with only ordinary points, i. e. they do not pass through singular points.

We assign to the preimages of the extreme points a_v and b_v the same index, $\zeta_i(a_v), \zeta_i(b_v)$ respectively. The other points from $n(r, a_v)$ to $n(r, b_v)$ remains arbitrary.

So, we define

$$B(r, a, b) = \sum_{i=1}^{n(r, a)} \text{dist}(\zeta_i(a), \zeta_i(b))$$

where

$$\text{dist}(\zeta_i(a), \zeta_i(b)) = \inf_{\gamma(\zeta_i(a), \zeta_i(b))} \int ds$$

being ds the metric on the surface Ξ and $\gamma(\alpha, \beta)$ those regular curves joining the points α and β and

$$A(r, a, b) = \sum_{i=n(r, a)+1}^{n(r, b)} |z_i(b)|$$

with $z_i(b) = p(\zeta_i(b))$.

The proximity principle is given by the next

Theorem 2 *There exist K , and $C \in (0, 1)$ absolute constants such that for every $r \in E \subset [0, \infty)$ of lower logarithmic density C*

$$\sum_{v=1}^q |A(r, a_v, b_v) + B(r, a_v, b_v)| \leq Kr \left[A(r) + \sum_{i=1}^k A_i(r) \right] + O(rA(r))$$

where $A(r)$ and $A_i(r)$ are as above.

Proof The extreme points of γ_i are a -points and b -points, $\zeta_i(a)$ and $\zeta_i(b)$ respectively, then there are $n(r, a_v)$ pairs $(\zeta_i(a_v), \zeta_i(b_v))$, $i = 1, 2, \dots, n(r, a_v)$ and the remain are summed in an arbitrary manner.

Let F_r be the riemannian image of $\Xi[r]$ under f . There are $n_0(r, \Gamma)$ simple islands of F_r over Γ [5] and $\tilde{n}(r, \Gamma)$ multiple islands (any of them counted only once).

Now, if $n(r, \Gamma)$ is the sum of the multiplicities of all islands over Γ , we have

$$n(r, \Gamma) = n_0(r, \Gamma) + n_1(r, \Gamma) + \tilde{n}(r, \Gamma)$$

where $n_1(r, \Gamma)$ is the sum of the orders of all islands over Γ .

Then

$$A(r) - n(r, \Gamma) = A(r) - n_0(r, \Gamma) - n_1(r, \Gamma) - \tilde{n}(r, \Gamma)$$

and so

$$A(r) - n(r, \Gamma) + n_1(r) = A(r) - n_0(r, \Gamma) - \tilde{n}(r, \Gamma)$$

but, by the second fundamental theorem of Ahlfors [9]

$$\sum_{\nu=1}^q n(r, \Gamma_\nu) - \sum_{\nu=1}^q n_1(r, \Gamma_\nu) \geq (q-2)A(r) - hL(r)$$

where h is a constant depending on the curves and $L(r)$ is the length of the relative boundary of F_r and consequently

$$hL(r) = o(A(r))$$

then

$$\sum_{\nu=1}^q (A(r) - n_0(r, \Gamma_\nu) - \tilde{n}(r, \Gamma_\nu)) \leq 2A(r) + hL(r)$$

but, from the first fundamental theorem of Ahlfors [9] we can deduce

$$\sum_{\nu=1}^q n(r, \Gamma_\nu) \leq qA(r) + hL(r)$$

then

$$\sum_{\nu=1}^q n_0(r, \Gamma_\nu) + \sum_{\nu=1}^q n_1(r, \Gamma_\nu) + \sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) \leq qA(r) + hL(r)$$

and consequently

$$\sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) \leq qA(r) - \sum_{\nu=1}^q n_0(r, \Gamma_\nu) - \sum_{\nu=1}^q n_1(r, \Gamma_\nu) + hL(r)$$

but $\tilde{n} < n_1$, hence

$$\sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) \leq qA(r) - \sum_{\nu=1}^q n_0(r, \Gamma_\nu) - \sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) + hL(r)$$

and

$$qA(r) - \sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) - \sum_{\nu=1}^q n_0(r, \Gamma_\nu) \leq 2A(r) + hL(r)$$

so that

$$\sum_{\nu=1}^q \tilde{n}(r, \Gamma_\nu) \leq 2A(r) + hL(r)$$

with different constant h and, by

$$\sum_{\nu=1}^q (A(r) - n(r, \Gamma_\nu)) = \sum_{\nu=1}^q (A(r) - n_0(r, \Gamma_\nu) - n_1(r, \Gamma_\nu) - \tilde{n}(r, \Gamma_\nu))$$

we get

$$\sum_{\nu=1}^q (A(r) - n_0(r, \Gamma_\nu)) \leq 4A(r) + hL(r)$$

with different h again and

$$\sum_{\nu=1}^q (n_1(r, \Gamma_\nu) + \tilde{n}(r, \Gamma_\nu)) \leq 4A(r) - \sum_{\nu=1}^q (A(r) - n(r, \Gamma_\nu)) + hL(r)$$

but Miles [8] has proved that

$$\sum_{\nu=1}^q |A(r) - n(r, a_\nu)| \leq KA(r), \quad r \rightarrow \infty$$

with $r \in E$, a subset of $[0, \infty)$ with lower logarithmic density $C \in (0, 1)$ and every a_ν belongs to a Γ_ν so that

$$\sum_{\nu=1}^q |A(r) - n(r, \Gamma_\nu)| \leq KA(r), \quad r \rightarrow \infty \quad r \in E$$

and the same is true for $n(r, b_\nu)$, hence

$$\sum_{\nu=1}^q (n_1(r, a_\nu) + \tilde{n}(r, a_\nu)) \leq 4A(r) - \sum_{\nu=1}^q (A(r) - n(r, a_\nu)) + hL(r)$$

$$\sum_{v=1}^q (n_1(r, b_v) + \tilde{n}(r, b_v)) \leq 4A(r) - \sum_{v=1}^q (A(r) - n(r, b_v)) + hL(r)$$

and we get

$$\sum_{v=1}^q (n_1(r, a_v) + n_1(r, b_v) + \tilde{n}(r, a_v) + \tilde{n}(r, b_v)) \leq 8A(r) + 2KA(r) + hL(r)$$

always with $r \rightarrow \infty$, $r \in E$.

Over Γ_v there are $n(r, \Gamma_v)$ simple islands of F_r . Their preimages are $\gamma_i(v)$, $i = 1, 2, \dots, n_0(r, \Gamma_v)$ and $n_0(r, \Gamma_v) \leq n(r, a_v)$.

The extreme points of $\gamma_i(v) \subset \Xi[r]$ are not multiple points, but ordinary points, namely $\zeta_i(a_v)$ and $\zeta_i(b_v)$ and there are $n_0(r, \Gamma_v)$ pairs of points of this kind and

$$\sum_{i=1}^{n_0(r, \Gamma_v)} \text{length}(\gamma_i(v)) \leq \Lambda(r, \Gamma_v)$$

where $\Lambda(r, \Gamma_v)$ is the sum of the lengths of preimages of f .

And so

$$\sum_{i=1}^{n_0(r, \Gamma_v)} \text{dist}(\zeta_i(a_v), \zeta_i(b_v)) \leq \Lambda(r, \Gamma_v)$$

It is obvious that

$$\sum_{i=n_o(r, \Gamma_v)+1}^{n(r, a_v)} \text{dist}(\zeta_i(a_v), \zeta_i(b_v)) + A(r, a_v, b_v) \leq$$

$$\leq r(n(r, a_v) - n_0(r, \Gamma_v) + n(r, b_v) - n_0(r, \Gamma_v))$$

hence

$$\sum_{i=n_o(r, \Gamma_v)+1}^{n(r, a_v)} \text{dist}(\zeta_i(a_v), \zeta_i(b_v)) + A(r, a_v, b_v) \leq 2r(n(r, \Gamma_v) - n_0(r, \Gamma_v))$$

and

$$A(r, a_v, b_v) + B(r, a_v, b_v) = \sum_{i=1}^{n_0(r, \Gamma_v)} \text{dist}(\zeta_i(a_v), \zeta_i(b_v))$$

$$+ \frac{n(r, \Gamma_v)}{n_0(r, \Gamma_v) + 1} \text{dist}(\zeta_i(a_v), \zeta_i(b_v)) + A(r, a_v, b_v)$$

$$\Rightarrow \sum_{v=1}^q |A(r, a_v, b_v) + B(r, a_v, b_v)| \leq \sum_{v=1}^q \Lambda(r, \Gamma_v) + (8 + 2K) r A(r) h r L(r)$$

but we saw previously that

$$\sum_{v=1}^q \Lambda(r, \Gamma_v) \leq k r \left[C_1 \sum_{i=1}^k A_i(r) + C_2 \right]$$

and finally

$$\begin{aligned} & \sum_{v=1}^q |(r, a_v, b_v) + B(r, a_v, b_v)| \\ & \leq k r \left(C_1 \sum_{i=1}^k A_i(r) + C_2 \right) + (8 + 2K) r A(r) + h r L(r) \end{aligned}$$

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