Quantization Scheme for the Experiments with “Walking Droplets”

J. Montes, F. Revuelta, and F. Borondo

ABSTRACT: In this paper we explore the interest and feasibility of quantizing the macroscopic surface wave generated in the dynamics of walking droplets on a vertically vibrated liquid surface in the limit of high memory of the droplet trajectory, where an astonishing similarity with the quantum behavior has been experimentally observed.

1. INTRODUCTION

Quantization is a key issue in many wave phenomena at every scale, ranging from the macroscale in musical instruments, to the quantum microworld.¹

Indeed, it played a fundamental role in the early development of the quantum theory, after the introduction of the wave–particle dualism by de Broglie.² Then it was not clear what was the physical magnitude that should be primarily quantized, nor how this task should be performed, or even which was the value of the elementary quantum.³ Ehrenfest⁴ and Einstein,⁵ among others, were main actors at that time by recognizing the relevance of the action in this issue.⁶ Also Maslov⁷ played an important role by detecting a necessary topological contribution to the action, the now called Maslov index, which is not always easy to compute.⁸

In the late 1970s and 80s, when the computational power was still rather reduced, semiclassical quantization of the action became an attractive tool to compute quantum vibrational energy levels. A wide variety of different strategies, all based on different principles, were introduced to perform this task. Just to name a few, we have the following methods.

Probably, the most straightforward method is due to Noid and Marcus,⁹ who obtained the eigenenergies by imposing quantization conditions on numerically constructed invariant tori. Chapman, Garret and Miller¹⁰ obtained the generator for the transformation to good action-angle variable by iteratively solving the corresponding Hamilton–Jacobi equation using Fourier series. Other authors used classical perturbation theory to construct a normal form approximation to the Hamiltonian, which was subsequently quantized. An accelerated convergence was obtained by using Van Vleck¹¹ and Lie transformations in refs 12 and 13. Other quantization methods based on Fourier analysis of the trajectories¹⁴ were also proposed. Finally, we will mention the method developed by

Reinhardt et al.¹⁵ based on the adiabatic conservation of the action on perturbed Hamiltonian, also called “adiabatic switching”.

In connection to this it should be mentioned that some one decade ago Fort, Couder, and co-workers in Paris,¹⁶ followed by the group of Bush at MIT,¹⁷ carried out experiments in a macroscopic system showing properties highly reminiscent of the quantum behavior.¹⁸ In these experiments, a millimetric oil droplet, bouncing on a liquid bath vertically vibrated close to (but below) the Faraday wave threshold,¹⁹ was able to self-propel due to the interaction with the wave generated at the bounces (see video in ref 20). At each impact on the surface, the droplet creates a circular capillary wave, that in turn excites standing Faraday waves. These waves propagate, damping away with a time that is proportional to the extent to which the amplitude of the forcing vibration approaches the Faraday instability threshold. In this way, one can define a memory parameter M to account for the damping over the time interval between successive rebounds. Accordingly, the wave reshapes the surface, which results only from the last M impacts. When $M \gg 1$, the droplet suffers a small horizontal force, making it to start “walking” on the liquid surface, reason why this system is usually known as *walking droplets* (WD).

Moreover, in this system there is a strong coupling between wave and particle, since the former drives the latter, and the bounces of the latter sustain the former. The amazing fact is that for high values of the memory, the system shows properties very similar to those paradigmatic of quantum systems.¹⁸ In this sense, the fluid can be compared²¹ to the quantum pilot-wave theories proposed in the early days of

quantum mechanics by Louis de Broglie in his double solution conception,²² and further elaborated by Bohm.²³

Indeed, several characteristic quantum phenomena have been emulated in WD experiments. For example, diffraction by single and double slits interference,²⁴ where the usual quantum pattern is obtained from the angular distribution of droplet trajectories, was one of them. Tunneling,²⁵ where a droplet overcomes the barrier formed by different depths in the liquid container, has also been observed in WD. Similarly, the existence of orbit quantization-like conditions,^{26,27} where the fluid container is subjected to a constant velocity rotating motion and then only droplet orbits corresponding to specific values of the radius are stable, has also been reported. Other quantum effects have been reproduced, such as, Zeeman-like level splitting^{28,29} in experimental situations in which a harmonic potential is implemented in the hydrodynamical system either by means of a ferromagnetic material added to the droplet and an external magnetic field or by using a rotating container. Also, the emergence of a coherent statistics pattern due to interference of the orbits “bouncing” on the boundaries of a closed container,³⁰ which are the macroscopic analogue of the electronic quantum density existing in nanometric corrals on metallic surfaces³¹ or entangled bound states consisting of a number of droplets simultaneously performing complicated choreographies,³² has been experimentally observed in this macroscopic system.

Recent advances in the theory and experiments of WD can be found in ref 33.

Due to the close resemblance between the WD phenomenology and the quantum world, it seems interesting to study quantization in this context, as done in this work.

The organization of the paper is as follows. In section 2 we describe the theoretical framework. First, we discuss the dynamics of the WD in its sojourns on the liquid surface and the corresponding equations of motion. Second, we briefly describe the main characteristics of the chaotic potential used in our calculations. In section 3 we present our numerical results concerning WD orbits and generated waves, and a way to introduce quantization on them is proposed. Finally, we sum up the paper in section 4 with the conclusions of our work.

2. THEORY

2.1. Walking Droplets. In a typical WD experiment a cylindrical container filled with silicon oil is bolted to a vibrating electromagnetic shaker, which oscillates vertically with a frequency $\omega = 2\pi f$ and an amplitude A_0 . Then, the effective gravity is $g + A_0\omega^2 \sin \omega t = g + \gamma \sin \omega t$, γ being a parameter that is easily controlled in the experiment. Below the so-called Faraday’s threshold, γ_F , the oil surface is at rest, despite the fact that it is being excited by the shaker. However, in the $\gamma \lesssim \gamma_F$ regime, small droplets of the same oil are able to bounce indefinitely without coalescing on the surface. In this way, long-lived waves are created whose ripples are able to impel the droplet, making it a WD. Then, the vertical and horizontal motions of the droplet can be assumed to be uncoupled, and the trajectory of the WD can be studied theoretically in a stroboscopic fashion by following only the position at the bounces. The corresponding two-dimensional dynamics for $\mathbf{q} = (X, Y)$, in which the vertical Z motion is averaged out, can be followed using Newton’s second law, which in this case reads

$$m\ddot{\mathbf{q}}(t) + D\dot{\mathbf{q}}(t) = -\nabla U(\mathbf{q}(t)) - mg\nabla\psi(\mathbf{q}(t)) \quad (1)$$

where m is the mass of the particle and D is a parameter giving account of the friction. Obviously, the magnitude of these parameters are related to the characteristics of the fluid. In eq 1, $\mathbf{q}(t)$ is the position vector of the particle on the liquid surface at time t , U is the external potential acting on the particle (see section 2.2 below), and $-mg\nabla\psi$ is the term that accounts for the force exerted by the wave formed on the liquid surface, as a result of the addition of the different dispersing ripples originated at each bounce. The height of this wave at a given point of the surface, $\mathbf{q}$, and a given time, t , can be modeled as

$$\psi_t(\mathbf{q}) = \int_{-\infty}^t J_0(k_F|\mathbf{q}(t) - \mathbf{q}_p(s)|)e^{-(t-s)} ds \quad (2)$$

where $\mathbf{q}_p(s)$ represents the vector position of the WD trajectory at time s , and the exponential term accounts for the time decay of the ripple. Notice that eq 2 assumes that the bounces of the droplet deform the surface with the shape of a Bessel function of the first kind, J_0 . This function is characterized by the Faraday’s wavenumber, k_F (or alternatively the Faraday’s wavelength $\lambda_F = 2\pi/k_F$), which plays here the same role as the de Broglie’s wavelength in the quantum theory.²¹ The integral in eq 2 accounts for the influence of the history of previous bounces on the shape of the liquid surface, also introducing at the same time a coupling between the corpuscular droplet and the wave on the surface that it creates. Notice the similarity with de Broglie’s pilot wave theory.

To study the motion of the walkers we scale eq 1 in order to make it adimensional, thus obtaining

$$\kappa\ddot{\mathbf{x}} + \dot{\mathbf{x}} = -\nabla V(\mathbf{x}) - \beta\nabla\psi(\mathbf{x}) \quad (3)$$

where now $\mathbf{x}(t)$ is the adimensional position at the scaled time t , κ is the scaled mass, V the corresponding scaled potential, and β is a *memory* parameter, which modulates, to a greater or lesser degree, the effect of the dispersing waves created at the previous bounces in the wave field ψ at time t . Full details of this adimensionality procedure can be found in ref 29.

We conclude this subsection by noting that although the actual values of κ and β are related for any specific liquid used in a given WD experiment through the magnitude of the Faraday’s threshold, γ_F , we will consider them in the rest of the paper as independent parameters in order to be able to explore the full range of dynamical possibilities that they allow.

2.2. External Potential. In this work we use an external potential, $V(\mathbf{q})$, which ensures that in the absence of forces due to the vibrating liquid surface, the trajectories of the walkers are very chaotic. Actually, the quantization of classical chaotic systems is an area of much interest, usually known as *quantum chaos*,³⁴ and many interesting phenomena^{35,36} fall within the scope of this area.

A very popular choice for this type of potentials are the chaotic billiards, such as the Bunimovich stadium, which has been often used both in theoretical and experimental studies. In our case, we prefer instead to use a coupled quartic oscillator potential in two dimensions

$$V(x,y) = \frac{\alpha_1}{2}x^2y^2 + \frac{\alpha_2}{4}(x^4 + y^4) \quad (4)$$

for a number of reasons. First, it is a continuous, smooth potential easy to be treated by numerical procedures. Second, this potential is homogeneous, and then, in the Hamiltonian case, the trajectories exhibit mechanical similarity. That is, the

classical motion at any value of the energy can be obtained by scaling the results from $E = 1$. This avoids the hassles derived from the dependence/evolution of the phase space structure with the energy. Third, using this potential avoids the complications (discontinuities) due to the bouncing of trajectories at hard walls (although in the case of the droplets paths, this is alleviated by the effect of the wave also bouncing at the walls). Fourth, the dynamical behavior of this system is extremely chaotic and free of the marginally stable (perpendicular to the straight borders and whispering gallery) motions that complicate the dynamics of the Bunimovich stadium. And fifth, this potential has been extensively used in different studies of classical and quantum chaos.³⁷ At this point it should be remarked that, although creating such a potential in an actual WD experimental setup may not be easy, it is, however, not impossible either, since it can be obtained, at least to some extent, by adequately shaping the bottom of the fluid container or by using suitably shaped magnetic fields and a ferromagnetic fluid.

The quartic potential (4) includes two parameters, α_1 and α_2 , whose values determine the degree of the chaos of the particle dynamics. For $\alpha_1 = 1$ and $\alpha_2 = 1/100$ the corresponding trajectories present a very high degree of chaos. Actually, for a long time it was thought to be totally ergodic (in the limit $\alpha_2 \rightarrow 0$) until Dahlqvist and Russberg³⁸ found a family of tiny islands of regularity. As the value of α_1 is increased, the potential becomes more and more regular. Indeed, for especial cases, such as $\alpha_1 = 0$ or $\alpha_1 = 1$, the behavior is completely regular.

In Figure 1 we present the quartic potential (4) for $\alpha_1 = 1$ and $\alpha_2 = 1/100$ in the form of a contour plot for $E = 1$ (recall that others contours can be obtained by a simple scaling), together with the $C := \{x = 0, p_x > 0\}$ Poincaré surface of section at the same energy, showing the very chaotic character of the corresponding Hamiltonian dynamics, i.e., without considering the influence of the wave created by the WD.

3. RESULTS

In this section, we present some results corresponding to the dynamics of WDs when governed by the adimensional eq 3 including (unless otherwise stated) the external quartic potential (4). We also analyze the wave originated in the bouncing of the WD, when it sojourns over the liquid surface. All calculations presented here have been calculated for $\kappa = 1$ and 8.

3.1. Walking Droplets Orbits. We begin by considering the dynamics of the orbits described by the WD. This dynamics is governed by eq 3, where the friction term dissipates WD energy. However, the WD can also gain energy from the vibrating surface at the bounces. When these two terms balance, the WD comes to a stable stationary situation with constant or alternatively time-periodic final energy. This is indeed what happens to all trajectories in the regime defined by the set of parameter values considered in this paper.

Some results for $\kappa = 1$ are shown in Figure 2, where we first consider the case in which the external potential is eliminated, i.e., $\alpha_1 = \alpha_2 = 0$ in eq 4. In this case, the trajectories maintain the initial direction, and the velocity always converges to the same final value, regardless of the value of the initial conditions. This is illustrated in panel a with three examples for $\beta = 100$. Panel b shows the dependence of the final velocity with β , which is given by²⁹

Figure 1. (a) Contour line corresponding to $E = 1$ for the quartic potential (4) with $\alpha_1 = 1$ and $\alpha_2 = 1/100$. (b) $C := \{x = 0, p_x > 0\}$ Poincaré surface of section at the same energy, showing the very chaotic character of the corresponding Hamiltonian dynamics.

$$v_f = \frac{1}{\sqrt{2}}(-1 + 2\beta - \sqrt{1 + 4\beta})^{1/2} \quad (5)$$

which is proportional to $\beta^{1/2}$ in the high memory limit, i.e., when $\beta \rightarrow \infty$.

Let us discuss now what happens when the quartic potential (4), introducing a chaotic force exerted on the droplet, is included in the dynamics. Here, the dissipative character of the dynamics, dictated by eq 3, makes all trajectories attracted to orbits along either the horizontal or the vertical axis, following after a certain time a straight line. However, in this case the final asymptotic portion of the trajectories exhibit a more complex time dependence on the final energy, which in our case is periodic. Furthermore, the final mean total energy, $\langle E \rangle$, approximately increases linearly with the memory parameter β , as $\langle E \rangle \sim v_f^2 \sim \beta$. In Figure 3, we illustrate these results using the same three initial conditions of Figure 2a. In panel a, we present the phase space view (x, v_x) of the three trajectories, which are quickly attracted to a figure approximately corresponding to a deformed tilted rectangle. In panel b, we show the corresponding values of the energy, which are seen to exhibit a three-periodic asymptotic behavior. Finally, panel c shows the approximately linear dependence of the final energy with the value of β .

We close this section by summarizing two main conclusions, which are relevant for the rest of the discussion in this work. First, in the dynamical regime that we are considering, i.e., $\kappa = 1$, the asymptotic dynamics of the WD orbits are governed by

Figure 2. Dynamics of the walking droplet governed by eq 3 for $\kappa = 1$ and no external potential [$\alpha_1 = \alpha_2 = 0$ in eq 4]. (a) All initial conditions reach the same stationary final velocity for sufficiently long times, $v \simeq 9.7$ in this case, where $\beta = 100$. (b) Value of the stationary final velocity as a function of the memory parameter β .

an attractor, which always drives them to one axis. Second, in this dynamics the memory parameter β and the WD energy are both related in such a way that the latter increases with the former. The corresponding functional form depends on the characteristics of the external potential, as shown in the examples of Figures 2 and 3.

3.2. Instant Wave vs Main Wave. Let us now consider the second “force” appearing in the equation of motion (3) for the WD, which comes from the ripples in the liquid surface induced by the different bounces of the WD on it.

For this purpose, it is important to distinguish between two different surface waves. On the one hand, we have the instant wave at a given moment of time t , which results from the addition of all dispersing ripples originated by the different bounces of the WD on the surface. Assuming a Bessel form and in the adimensional version of the equation of motion, this wave simply reads [cf. eq 2]

$$\psi_t(\mathbf{x}) = \int_{-\infty}^t J_0(|\mathbf{x} - \mathbf{x}_p(s)|) e^{-(t-s)} ds \quad (6)$$

Notice that this term is modulated by the memory parameter β in the dynamical eq 3 and also implicitly depends on it through the term $\mathbf{x}_p(s)$.

On the other hand, we have the main (averaged) wave, which is given by

Figure 3. Same as Figure 2 when the quartic potential (4) is added in the dynamics of eq 3. (a) Phase space view and (b) time variation of their energies for the same trajectories in Figure 2a. (c) Value of the stationary final energy as a function of the memory parameter β .

$$\psi(\mathbf{x}) = \int_{-\infty}^{\infty} J_0(|\mathbf{x} - \mathbf{x}_p(s)|) ds \quad (7)$$

where the exponential term responsible for the wave dispersion has been eliminated. Let us remark that the wave defined in this way needs to be properly normalized.

Some results for these two waves, computed for a representative trajectory (recall that, as previously discussed, our results are independent of the initial conditions), propagated for different values of the memory parameter β are shown in Figure 4 in blue and red, respectively. As can be seen, the β parameter plays the role here of the excitation

Figure 4. Squared instant (blue) wave for $t > 1000$ when the droplet is at $x = 0$ with $v_x > 0$ and corresponding main wave (red) for different values of β : (a) 5.05, (b) 89.59, (c) 242.33, and (d) 370.58.

Figure 5. Quantization of the walking droplet main wave. It is shown here how a “quantized” wave with a clear, regular nodal pattern is obtained [panel b] by adding two Bessel functions centered at the turning points, when they are adequately separated [panel a]. As can be easily ascertained, this “quantized state” corresponds to $n = 8$. The positions of the corresponding turning points, at $x_{1,2} = \pm 13.32$ are marked with vertical dashed lines. In panel d we show how a nonquantized situation takes place when we move the position of the turning points to $x_{1,2} = \pm 11.00$ [panel c].

Figure 6. (a) Squared main wave given by eq 6 as a function of the position x , for $\kappa = 8$ and different values of the memory parameter β , 250 (b), 160 (c), and 90 (d), respectively, playing here the role of the energy in a regular mechanical system. When the corresponding turning points, whose positions depend on β , are adequately separated, a maximal constructive interference takes place, this indicating the position of a quantization condition. (b)–(d) Three “quantized states” corresponding to $n = 8, 7$, and 6 , also shown at the colored horizontal lines in panel a.

energy in quantum systems because the number of nodes, n , i.e., the vibrational excitation, of the main wave function (red) depends on it: for $\beta = 5.05$ we have $n = 2$ (a); for $\beta = 89.59$, $n = 6$ (b); for $\beta = 242.33$, $n = 8$ (c); and for $\beta = 370.58$, $n = 9$ (d).

Moreover, the shape of these main waves looks very similar to that appearing in the standard quantum mechanical harmonic oscillator,³⁹ the “probability” being more highly localized at the positions of the two symmetrically located turning points, $\pm x_{TP}$, of the WD orbits.

More interesting is that the instant wave (blue) tends to the main wave as the value of β increases. To gauge this result, one should take into account that the instant wave is somehow only half of the main one, since it is computed at $x = 0$ with $\nu_x > 0$, and then the intensity into the $x > 0$ region has not had a chance to develop. Likewise, as can be seen by examining the different panels in Figure 4, we observe that as the memory parameter β increases, more and more peaks of the main wave (red) are well reproduced by the ripples in the instant wave (blue). In this way, while the agreement between both waves in panel a is very poor, in panel b both waves agree well in the vicinity of the left turning point at $x_{TP} \simeq -20$ (although the intensities are still very different). However, in panel c the agreement is much better around that point; also a good agreement exists at the maximum next to it. And finally, in panel d we observe good agreement (again leaving aside the intensity) with the five leftmost maxima of the main wave. We could continue increasing the memory of the wave created

along the WD trajectory, i.e., β , and get better and better agreement between the instant and the main waves.

In this way, we can conclude that in the high memory limit the instant wave tends very accurately to reproduce the shape of the main wave, obviously putting aside the fact that the latter is symmetric with respect to the origin $x = 0$. This limiting result is reasonable and can be easily understood if one takes into account that when the memory is high, the ripples originating at each consecutive bounce do not have the opportunity to decay, and thus they effectively contribute to the main wave. This fact enables the experimental verification of the quantization phenomenon reported here.

3.3. Quantization. Despite the similarities of the main wave shapes obtained in the previous subsection and the true quantum case, it cannot be forgotten that the WD problem is macroscopic and then classical. Here the energy is continuous and it is not quantized, and then there is no immediate way to claim the existence of a physically based quantization rule in it. However, driven by the striking similarity described above, we will further investigate this issue in this part of the paper.

The idea is to make an *ansatz* that is, at the same time, coherent with the nature of the problem, and able to satisfactorily explain our numerical findings. For this purpose, we will start by considering that since the trajectories spend a great deal of time at the turning points, where the velocity becomes zero in the change of sign, these regions will be the most important ones in the main wave construction by the successive bounces of the droplet on the liquid surface. In fact, as the problem is symmetric with respect to the origin, two such waves should exist, one at each of the symmetrical turning points. Moreover, these two waves should have a constructive interference, in order to be able to construct a wave that looks like some sort of “quantum-like” wave function. Obviously, this situation will only happen for some particular values of the memory parameter β . In this way, a reasonable condition for the quantization of β can be obtained.

This argument is illustrated in the results presented in Figure 5, where panels a and c represent two squared Bessel functions, J_0^2 , centered at each of the turning points, $x_{TP} = x_{1,2}$, of the WD trajectories associated with two different values of β . The corresponding normalized results obtained by adding these two functions are also given in panels b and d, respectively. As can be seen, in the first case, where $x_{TP} = \pm 13.32$ [see panel b], the two partial wave are in phase, and then there is a constructive interference, which results in a final wave that resembles a quantum state. However, for the second value of $x_{TP} = \pm 11.00$ [see panel d], the interference is destructive at intermediate values of x ; therefore, there is not a regular nodal pattern that can be assigned to any quantized situation.

We conclude our study by showing next how our results discussed in subsection 3.1, where we showed the dependence of the final energy of the orbits with β , can be used to obtain suitable quantization conditions for the main “pilot” wave in the WD problem.

The idea is quite simple. Since the WD energy, and thus the separation between turning points, increases with the memory constant β , we can continuously vary this parameter to find the different consecutive excited vibrational “states” at the points where the constructive interference between left and right waves is a maximum. The results for the high memory limit are presented in Figure 6, where we show in panel a the location of the turning points, as a function of the position x , and the corresponding (squared) main waves obtained for $\kappa = 8$ at

different values of β (which plays the role of the energy). As can be seen at specific value of β clear quantized “states” are obtained, due to a maximal constructive interference. Three of these cases have been marked with horizontal cyan, pink, and green lines, respectively. Furthermore, the corresponding squared main waves are also presented, separately for the sake of clarity, in panels d, c, and b, respectively. In these three we see a clear, regular nodal pattern corresponding to the consecutive “vibrational quantum numbers” (from bottom to top) $n = 6, 7$, and 8 , with a great accumulation of intensity/“probability” in the turning points, similar to what happens in the semiclassical high n limit of quantum mechanics.³⁹

4. SUMMARY AND DISCUSSION

Summarizing, in this paper we have studied the dynamics of a WD under the influence of a two-dimensional coupled quartic potential in the high memory limit. We have demonstrated the emergence of nodal patterns in the surface wave that it creates, as a result of a coherent interference. The tuning parameter in this case is the memory parameter β , which plays the role of an energy. Likewise, we have proposed a simple *ansatz* to explain where the quantization condition of the wave field is fulfilled.

We believe that this work opens a door to understand the similarities found between WD experiments and the quantum theory, in such a way that the former can be an adequate playground to study certain aspects of the latter, for example, to deepen the understanding of the unfinished de Broglie double solution theory, especially as revived by Bohm’s work. A more quantitative analysis of the quantization condition reported in this paper will be reported elsewhere.⁴⁰

AUTHOR INFORMATION

Corresponding Authors

*E-mail: fabio.revuelta@upm.es.

*E-mail: f.borondo@uam.es.

ORCID

F. Revuelta: 0000-0002-2410-5881

F. Borondo: 0000-0003-3094-8911

Notes

The authors declare no competing financial interest.

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