Analytical and simplified models for dynamic analysis of short skew bridges under moving loads

K. Nguyen¹, C. Velarde¹ and J.M. Goicolea¹

Abstract

Skew bridges induce inherently coupled bending and torsion response. The actual relevance of this coupling for the dynamic response under moving loads such as arise in high-speed railways is not well known, introducing uncertainties unless costly 3D dynamic models are performed. In this work models are developed based on beam theory and analytical extraction of vibration modes, which are both simple and fast. Firstly a general 3D beam model is derived, involving both bending and torsional modes. Following, a simplified beam model is proposed involving only bending modes. In both cases the eigenfrequencies, eigenmodes and orthogonality relationships are determined analytically from the boundary conditions, and the dynamic response is obtained numerically in the time domain. Both models are validated through representative realistic examples by comparing with two types of Finite Element models: a stick model with 3D beams, and a full 3D model with shell elements. Finally, parametric studies are performed with the simplified beam model for identifying parameters that influence the dynamic response under traffic loads. The results show that the degree of skewness has an important influence on the vertical displacement, but hardly on the vertical acceleration of the bridge. The torsional stiffness has a significant effect on the vertical displacement when the skew angle is large. Finally, the span length reduces the skewness effect on the dynamic behaviour of the bridge.

Keywords

skew bridge, bridge modelling, modal analysis, moving load

Introduction

The dynamic response of railway bridges subjected to traffic loads of high speed trains (above 200 km/h) is a significant design issue, which has been included in the recent eurocode requirements (CEN 2003). Background studies for this purpose have often been limited to simple bridges with line beam bending behaviour (Fryba 1996; ERRI 1999). Moreover, the High Speed Load Model (HSLM) used for design in (CEN 2003) is also based on this assumption. However, a substantial number of bridges in the new high speed lines have skew supports, developing coupled bending and torsion even for centred loads in single track bridges. As an example, in recent surveys we have identified 50 skewed underpasses out of 108 slab portal frames in the Madrid - Zaragoza HSL Soriano Muñoz (2016) and 9 skewed simply supported bridges out of 27 in the same line (Barrios Fragoso 2017).

The structural effect of the skewness is an additional torsion on the bridge deck (Kollbrunner and Basler 1969; Manterola 2006) which modifies the structural response. Several studies following analytical, numerical as well as experimental approaches have been made during the last decades in order to better understand the behaviour of skew bridges under static and dynamic loadings. Of special relevance is the research related to highway skew bridges subjected to earthquake loading: early work on this subject was reported by (Ghobarah and Tso 1974), in which a closed-form solution based on a beam model capable of capturing both flexural and torsional modes was proposed for skew bridges with intermediate supports. (Maragakis

and Jennings 1987) obtained the earthquake response of the skew bridge, modelling the bridge deck as a rigid body. The application of Finite Element (FE) *stick models* using beam elements was introduced by (Wakefield et al. 1991), and later further developed by (Meng and Lui 2000; Meng et al. 2001; Nielson and DesRoches 2007; Abdel-Mohti and Pekcan 2008; Kaviani et al. 2012; Yang et al. 2015; Serdar and Folić 2018). Despite their simplicity, stick models can provide reasonably good approximations for preliminary assessments. More detailed FE 3D models using shell and beam elements have been also applied for this problem (Meng and Lui 2000, 2002; Abdel-Mohti and Pekcan 2008; Nouri and Ahmadi 2012; Zakeri et al. 2014; Deng et al. 2015; Mallick and Raychowdhury 2015).

Regarding the dynamic response of skew bridges under moving traffic loads, most of the work has been performed on FE models using a combination of shell and beam elements and assisted by experimental testing (Bishara et al. 1993; Helba and Kennedy 1995; Khaloo and Mirzabozorg 2003; Menassa et al. 2007; Ashebo et al. 2007; He et al. 2012; Xue et al. 2018). FE models may provide good approximations, but require the end user significant effort for defining

¹Group of Computational Mechanics, School of Civil Engineering, Universidad Politécnica de Madrid, Spain

Corresponding author:

K. Nguyen, School of Civil Engineering, Universidad Politécnica de Madrid, Madrid, Spain

Email: kxanhnguyen.gia@upm.es

the model (element types and sizes, geometry, material properties, supports, connections etc.). Additionally they are costly in terms of engineering and computer time. Therefore, their use is limited to specific case studies and may not be practical for parametric studies such as Monte Carlo simulations or other situations requiring a large number of case studies.

A possible alternative is to develop models following analytical solutions based on beam theory, which capture the behaviour of the skew bridge and yield sufficient accuracy. The advantage of such analytical models is that the data input is much simpler (only general structural parameters such as mass, span length, flexural and torsional stiffness) facilitating their use for the end user and, of course, enabling massive parametric studies. However, this approach implies limitations which must also be understood. It requires line beam-type bridges in which the deck cross-section behaves rigidly enough and may be represented by beam models with bending and torsion. Open cross-sections which may experience significant distortion would generally need more detailed 3D FE models.

In this work we have developed numerical models based on beam theory and closed-form modal analysis for the response of skew bridges subjected to moving traffic loads. Firstly general model is described which involves bending and torsion modes of the beam. Following, a simpler model is presented which accepts some approximations and involves only vertical bending of the beam. The dynamic response is obtained in the time domain using a piecewise exact algorithm. Both models are validated on representative examples and compared against results obtained by 3D FE models. Finally, several parametric studies are performed with the simplified model in order to identify variables that influence significantly the vertical dynamic response of skew bridges under traffic loads.

General beam model

Although the theory may be applied to more general settings, for the sake of simplicity a simply-supported skew bridge as shown in Fig. 1 will be considered within this work. The skewness angle α is defined as that formed between the line of abutment support (y') and the normal (y) to the longitudinal axis (deck centreline x). The length of bridge is taken as the clear-span length L . The model assumptions are summarised as follows.

- The bridge deck is modelled as a linear elastic 3D Euler-Bernoulli beam supported at the ends.
- The bridge deck is very stiff in the horizontal xy plane, so transverse deflections in the y direction will be neglected.
- The bending stiffness EI , torsional stiffness GJ and mass per unit length $\bar{m}$ are constant over the length L .
- Warping and distortion effects for the cross-section of the bridge deck may be neglected. It's noted that with this assumption application is limited to closed-type cross sections, such as hollow slabs or prismatic box-girders bridges for which the warping and distortion are small (Kollbrunner and Basler (1969); Waldron (1988); Menn (1990); Mo et al. (2000); Nallasivam et al. (2007)).

Figure 1. A simply-supported skew bridge: in plane view and bridge model's sketch

Under these assumptions, the bending of the bridge in the xz plane and its twisting about the x axis are the only deformations of the bridge deck. The governing equations of motion for transverse and torsional vibration are:

$$\bar{m}\ddot{u} + c_f\dot{u} + EI\frac{\partial^4 u}{\partial x^4} = p(x, t) \quad (1a)$$

$$\bar{m}r^2\ddot{\theta} + c_t\dot{\theta} - GJ\frac{\partial^2 \theta}{\partial x^2} = m_t(x, t) \quad (1b)$$

where r is the radius of gyration; $u(x, t)$ and $\theta(x, t)$ are the vertical deflection and torsional rotation of the bridge deck; $p(x, t)$ and $m_t(x, t)$ are the vertical and torsional loads applied on the bridge at position x and at time t , respectively. The external damping terms are defined by $c_f\dot{u}$ and $c_t\dot{\theta}$ and are assumed to be proportional to the mass ($c_f = 2\bar{m}\zeta_n\omega_n$ for flexural vibration and $c_t = 2\bar{m}r^2\zeta_n\omega_n$ for torsional vibration).

Natural frequencies and mode shapes

Using modal analysis, the solution for free vibrations of the bridge can be expressed decoupled with an infinite sum of modal coordinates and mode shapes as:

$$\begin{Bmatrix} u(x, t) \\ \theta(x, t) \end{Bmatrix} = \sum_{n=1}^{\infty} q_n(t) \begin{Bmatrix} \phi_n(x) \\ \varphi_n(x) \end{Bmatrix} \quad (2)$$

where $\phi_n(x)$ and $\varphi_n(x)$ are the n^{th} flexural and torsional mode shapes, and $q_n(t)$ the generalised coordinates of the coupled flexural-torsional amplitude. The governing equations for undamped free vibrations can be rewritten for each mode as:

$$\frac{1}{\phi_n(x)} \frac{d^4 \phi_n(x)}{dx^4} = \frac{\bar{m}\omega_n^2}{EI} \quad (3a)$$

$$\frac{1}{\varphi_n(x)} \frac{d^2 \varphi_n(x)}{dx^2} = -\frac{\bar{m}r^2\omega_n^2}{GJ} \quad (3b)$$

The eigenvalue solution to the above equations may be expressed using standard theory, e.g. (Clough and Penzien 1993):

$$\phi_n(x) = C_{n,1} \sin(\beta_n x) + C_{n,2} \cos(\beta_n x) + C_{n,3} \sinh(\beta_n x) + C_{n,4} \cosh(\beta_n x) \quad (4a)$$

$$\varphi_n(x) = C_{n,5} \sin(\lambda_n x) + C_{n,6} \cos(\lambda_n x) \quad (4b)$$

where $\beta_n^4 = \bar{m}\omega_n^2/EI$; $\lambda_n^2 = \bar{m}r^2\omega_n^2/GJ$; and $C_{n,1}$, $C_{n,2}$, $C_{n,3}$, $C_{n,4}$, $C_{n,5}$, $C_{n,6}$ are constants which will be determined by the boundary conditions.

The boundary conditions are shown in Fig. 1, with the bridge simply-supported at the abutment ends. At the support lines, there are zero vertical displacements ($u(0,t) = u(L,t) = 0$), zero rotation about the x' axis ($\theta_{x'}(0,t) = \theta_{x'}(L,t) = 0$) and zero bending moment along x' axis ($M_{y'}(0,t) = M_{y'}(L,t) = 0$). Using the change of coordinates shown in Fig. 2, the following relationships are obtained:

Figure 2. Coordinate systems: M_x and $M_{x'}$ are the torsional moment in the axis x and x' respectively; M_y and $M_{y'}$ are the vertical bending moment in the axis y and y' respectively.

$$\theta_{x'} = \theta \cos(\alpha) - u' \sin(\alpha) \quad (5a)$$

$$\begin{aligned} M_{y'} &= M_x \sin(\alpha) + M_y \cos(\alpha) \\ &= GJ\theta' \sin(\alpha) + EIu'' \cos(\alpha) \end{aligned} \quad (5b)$$

Hence, the boundary conditions for the problem can be written as:

$$\phi(0) = \phi(L) = 0 \quad (6a)$$

$$\varphi(0) \cos(\alpha) - \phi'(0) \sin(\alpha) = 0 \quad (6b)$$

$$\varphi(L) \cos(\alpha) - \phi'(L) \sin(\alpha) = 0 \quad (6c)$$

$$GJ\varphi'(0) \sin(\alpha) + EI\phi''(0) \cos(\alpha) = 0 \quad (6d)$$

$$GJ\varphi'(L) \sin(\alpha) + EI\phi''(L) \cos(\alpha) = 0 \quad (6e)$$

Developing these six conditions, a homogeneous system of equations is obtained as:

$$\mathbf{A}\mathbf{c} = \mathbf{0} \quad (7)$$

where $\mathbf{c} = [C_{n,1}, C_{n,2}, C_{n,3}, C_{n,4}, C_{n,5}, C_{n,6}]^T$ is a vector of six constants to be determined, and the matrix $\mathbf{A}$ is formulated as shown in Appendix 1. The eigenvalues are calculated by solving $\det(\mathbf{A}) = 0$. It is noted that the determinant of $\mathbf{A}$ can be expressed as a function of a single variable, β ($\lambda = r\beta^2\sqrt{EI/GJ}$). The extraction of the eigenvalues can be performed by using any symbolic mathematical program (e.g. Maple or Matlab) or the approximate formulation proposed by [Newmark and Veletsos \(1952\)](#). In fact, in this study the symbolic calculation implemented in Matlab is used to extract the values of β for desired modes used in the dynamic calculation. The eigenvectors are then obtained by applying singular value decomposition to the matrix $\mathbf{A}$.

Orthogonality conditions

In order to apply the modal superposition, it is necessary to ensure the orthogonality relationship between the mode shapes. Starting from equations (3), these can be reformulated by multiplying both sides of these by an arbitrary mode $\phi_m(x)$ and $\varphi_m(x)$, respectively, and integrating with respect to x over the length of the beam L , to obtain

$$\int_0^L EI\phi_n''''(x)\phi_m(x)dx - \bar{m}\omega_n^2 \int_0^L \phi_n(x)\phi_m(x)dx = 0 \quad (8a)$$

$$\int_0^L GJ\varphi_n''(x)\varphi_m(x)dx + \bar{m}r^2\omega_n^2 \int_0^L \varphi_n(x)\varphi_m(x)dx = 0 \quad (8b)$$

Integrating by parts these equations (twice for Eq. (8a) and once for Eq. (8b)) and applying the boundary conditions derived for the problem, gives:

$$\begin{aligned} 0 &= EI \int_0^L \phi_n''(x)\phi_m''(x)dx + GJ \tan(\alpha) [\varphi_n'(L)\phi_m'(L) \\ &\quad - \varphi_n'(0)\phi_m'(0)] - \bar{m}\omega_n^2 \int_0^L \phi_n(x)\phi_m(x)dx \end{aligned} \quad (9a)$$

$$\begin{aligned} 0 &= GJ \left[\tan(\alpha) [\varphi_m'(L)\phi_m'(L) - \varphi_m'(0)\phi_m'(0)] \right. \\ &\quad \left. - \int_0^L \varphi_n'(x)\varphi_m'(x)dx \right] + \bar{m}r^2\omega_n^2 \int_0^L \varphi_n(x)\varphi_m(x)dx \end{aligned} \quad (9b)$$

Interchanging indices n and m in equation (9) and subtracting from its original form yields the following relations for any $n \neq m$:

$$\begin{aligned} GJ \tan(\alpha) [\varphi_n'(L)\phi_m'(L) - \varphi_n'(0)\phi_m'(0) - \varphi_m'(L)\phi_n'(L) \\ + \varphi_m'(0)\phi_n'(0)] - \bar{m}(\omega_n^2 - \omega_m^2) \int_0^L \phi_n(x)\phi_m(x)dx = 0 \end{aligned} \quad (10a)$$

$$\begin{aligned} GJ \tan(\alpha) [\varphi_n'(L)\phi_m'(L) - \varphi_n'(0)\phi_m'(0) - \varphi_m'(L)\phi_n'(L) \\ + \varphi_m'(0)\phi_n'(0)] - \bar{m}r^2(\omega_n^2 - \omega_m^2) \int_0^L \varphi_n(x)\varphi_m(x)dx = 0 \end{aligned} \quad (10b)$$

Next, subtracting equation (10a) from equation (10b) gives:

$$\begin{aligned} (\omega_n^2 - \omega_m^2) \left(\bar{m} \int_0^L \phi_n(x)\phi_m(x)dx \right. \\ \left. + \bar{m}r^2 \int_0^L \varphi_n(x)\varphi_m(x)dx \right) = 0 \end{aligned} \quad (11)$$

Finally, considering $\omega_n \neq \omega_m$, $\int_0^L \phi_n(x)\phi_m(x)dx \geq 0$ and $\int_0^L \varphi_n(x)\varphi_m(x)dx \geq 0$, the orthogonality conditions are:

$$\int_0^L \phi_n(x)\phi_m(x)dx = 0 \quad \text{for } n \neq m \quad (12a)$$

$$\int_0^L \varphi_n(x)\varphi_m(x)dx = 0 \quad \text{for } n \neq m \quad (12b)$$

Response to moving loads

We now apply modal superposition for obtaining the response of the skew bridge due to moving loads. Starting with a single moving load P (Fig. 3a), the distributed vertical force and twisting moment for equations (1a), (1b) may be expressed as:

$$p(x, t) = P\delta(x - vt) \quad (13a)$$

$$m_t(x, t) = \frac{P\delta(x - vt)L(\epsilon - \epsilon^2)\cot(\alpha)}{2(1 + K\cot^2(\alpha))} + P\delta(x - vt)e \quad (13b)$$

where $\delta(\bullet)$ is the Dirac delta function, $\epsilon = vt/L$, $K = EI/GJ$ and e the load eccentricity relative to the centreline of the bridge deck section. The first term on the right hand side of Eq. (13b) is due to the skewness of the bridge (Kollbrunner and Basler 1969; Manterola 2006), and the second term is due to the load eccentricity. Using modal superposition and applying the orthogonality relationship, the following uncoupled differential equations in the generalised coordinates are obtained:

$$\ddot{q}_n(t) + 2\zeta\omega_n\dot{q}_n(t) + \omega_n^2q_n(t) = Pa_n + \left[\frac{PL(\epsilon - \epsilon^2)\cot(\alpha)}{2(1 + K\cot^2(\alpha))} + Pe \right] b_n \quad (14)$$

being $a_n = \frac{\phi(vt)}{\bar{m} \int_0^L (\phi_n(x)^2 + r^2\varphi_n(x)^2)dx}$ and $b_n = \frac{\varphi(vt)}{\bar{m} \int_0^L (\phi_n(x)^2 + r^2\varphi_n(x)^2)dx}$ the n th modal participation factors of flexural and torsional modes, respectively.

The uncoupled differential equations (14) may then be solved in time using the piecewise exact method (Clough and Penzien 1993).

Figure 3. Moving loads

For the case of a convoy of moving loads (Fig. 3b), the uncoupled differential equations in the generalised

coordinates for each mode of vibration n are

$$\ddot{q}_n(t) + 2\zeta\omega_n\dot{q}_n(t) + \omega_n^2q_n(t) = \sum_{k=1}^{n_P} \left(P_k a_n + \left[\frac{PL(\epsilon_k - \epsilon_k^2)\cot(\alpha)}{2(1 + K\cot^2(\alpha))} + Pe \right] b_n \right) \left[H\left(t - \frac{d_k}{v}\right) - H\left(t - \frac{d_k}{v} - \frac{L}{v}\right) \right] \quad (15)$$

where n_P is the number of moving loads, d_k is the distance between the first load and the k^{th} load, $H(\bullet)$ is the Heaviside step function, P_k is the magnitude of the k^{th} load, and $\epsilon_k = (vt - d_k)/L$. The solution of Eq. (15) is obtained in similar way as in the case of a moving load. Attention needs to be paid in the determination of the modal loads in the right side of Eq. (15): for loads that have not entered the bridge ($t - d_k/v < 0$) or have left the bridge ($t - d_k/v - L/v > 0$) the value must be set to zero.

Simplified beam model

In this section, a simplified beam model which includes only bending in the vertical xz plane is developed. This model includes an approximation to the bending restraint provided by the skew supports which simplifies the calculations. The torsional moments induced at the supports introduce a negative (hogging) bending moment as shown in Fig. 4a (Kollbrunner and Basler 1969; Manterola 2006). As a result, for the purpose of vertical flexure the simply-supported skew beam behaves like as an elastically-fixed beam, with rotational supports of stiffness constants k_θ^i as shown in Fig. 4b. It is noted that the negative bending moments at the supports change with the position of the load on the bridge. The value of the stiffness of the rotational supports may be determined enforcing the equivalence of the bending moment at the position of the load between the simplified model and the general skew beam with torsion:

$$k_\theta^i \theta_i = M_y = \frac{PL(\epsilon - \epsilon^2)}{2(1 + K\cot^2(\alpha))} \quad \text{for } i = 1, 2 \quad (16)$$

with $\epsilon = a/L$ and where index i refers to the left and right support respectively. Solving the equations for both supports, the stiffness values are obtained:

$$k_\theta^1 = \frac{GJ}{a\cot^2(\alpha)} \quad \text{and} \quad k_\theta^2 = \frac{GJ}{(L - a)\cot^2(\alpha)} \quad (17)$$

This yields different support stiffnesses at both ends, and moreover non-constant as a function of the position a of the load. An approximation is proposed to consider the same stiffness in both supports, by which a constant stiffness of the rotational support is taken for $a = L/2$:

$$k_\theta^1 = k_\theta^2 = k_\theta = \frac{2GJ}{L\cot^2(\alpha)} \quad (18)$$

As a result, considering the above value for the elastic rotational support, this simplified beam model includes only bending in the vertical plane thus allowing simpler 2D beam models to be employed.

Figure 4. a) Diagram of bending moment of a skew bridge under a static load, b) Simplified model considering only beam bending in the vertical xz plane

Natural frequencies and mode shapes

The governing equation for the free vibration of the simplified model is similar to Eq. (3a). The solution of this equation is given in (4a). Details for determination of frequencies and associated mode shapes may be found in [Karnovsky and Lebed \(2000\)](#).

Orthogonality conditions

Similar to the analysis in the previous section, equation (3a) may be rewritten, using the boundary conditions of the simplified model as:

$$EI \int_0^L \phi_n''(x) \phi_m''(x) dx + k_{\theta} [\phi_n'(0) \phi_m'(0) + \phi_n'(L) \phi_m'(L)] - \bar{m} \omega_n^2 \int_0^L \phi_n(x) \phi_m(x) dx = 0 \quad (19)$$

Interchanging the indices n and m in Eq. (19) and subtracting the resulting equation from its original form, considering $\omega_n \neq \omega_m$ yields the orthogonality condition:

$$\int_0^L \phi_n(x) \phi_m(x) dx = 0 \quad \text{for } n \neq m \quad (20)$$

Response to moving loads

The dynamic response of the bridge under moving loads is obtained similarly to the model in the previous section “General beam model”, the only difference being that the torsional response is not considered.

Numerical validations

The above models will be applied to a typical box-slab single track bridge, designed for high speed railway, with cross section as shown in Fig. 5. This is a closed-type cross section, which fulfils the assumption made for the beam model of neglecting warping and distortion effects. The geometric and mechanical properties are as follows:

- Span $L = 24$ m, width $B = 8$ m, skew angle $\alpha = 25^\circ$, depth $h = 1.7$ m.
- Elastic modulus $E = 32$ GPa, Poisson ratio $\nu = 0.25$.
- $I = 1.3921 \text{ m}^4$, $J = 2.6741 \text{ m}^4$, $\bar{m} = 9.774 \text{ t/m}$ and $r = 0.5967 \text{ m}$.
- Damping ratio $\zeta = 1\%$.

Figure 5. Cross section of the simply-supported bridge

The results obtained by the two proposed beam models will be compared between themselves and with Finite Element (FE) models of two types: 1) a model with 3D Euler-Bernoulli beam elements (*stick model*, see Fig. 6a), using the program FEAP ([Taylor 2014](#)), and 2) a model with shell elements, using the program Abaqus [Simulia \(2017\)](#) (see Fig. 6b).

The bridge is subjected to the action of high-speed railway traffic, performing dynamic analyses for moving

loads representing each axle of the train. Such moving load models are the basic design methods when dynamic analysis is required (CEN 2003). They are also appropriate for the present case in which the bridge mass amounts to $\bar{m}L \approx 235$ t, compared to the non-suspended mass of 4 t for the at most 2 axles on the bridge (the remaining vehicle mass is largely uncoupled due to primary and secondary suspensions). A convoy of moving loads is applied to the nodal forces along the centreline axis, using time-dependent amplitude functions.

The dynamic response in FE models is obtained using modal analysis in the time domain with a time step $\Delta t = 0.001$ s. For the analysis, the first five modes of the beam model are considered in the calculation, with a common damping ratio ($\zeta_n = 1\%$). The FE models include more modes than those in the analytical beam models, as they have additional degrees of freedom. The FE modes considered and their correspondence with the analytical beam modes are detailed in Table 1. Figure 7 shows the Modal Assurance Criteria (MAC) between the modes of general beam model and FE stick model and between the simplified beam model and FE stick model, in which a very good match is observed.

Figure 6. FE models

Table 1. Frequencies of first five modes of vibration of different models (in Hz); numbers in parentheses for the FE models define the corresponding mode numbers

No.	General beam	Simplif. beam	FE Stick	FE Shell
1	6.388	6.386	6.386 (#1)	6.337 (#1)
2	23.404	23.880	23.288 (#2)	21.660 (#4)
3	53.189	53.002	53.001 (#4)	57.161 (#17)
4	94.175	93.761	93.153 (#6)	89.408 (#29)
5	147.23	146.16	146.16 (#8)	145.73 (#52)

The loading corresponds to one of the 10 virtual trains defining the High Speed Load Model (HSLM), in particular

(a) MAC between General beam model and FE-stick model

(b) MAC between Simplified beam model and FE-stick model

Figure 7. Comparison of modal assurance criterion (MAC) between modes of the beam models and FE stick model

the HSLM-A1 (CEN 2003). This train consists of 18 intermediate coaches and power cars at both ends, with a total of 50 axles with loads of 170 kN/axle. The dynamic analyses are carried out for train speeds ranging from 100 km/h to 300 km/h in increments of 2 km/h.

The dynamic response for displacement and acceleration at the train speed of 206 km/h is presented in Fig. 8, showing a very close approximation between the two beam models and the two FE models. Furthermore, for this case a large amplification of the response is obtained, typical of resonance. In effect, for a simply-supported bridge the critical train velocities which may produce resonance can be estimated by the simple relation (CEN 2003):

$$v_i = f_0 \frac{D}{i} \quad \text{with } i = 1, 2, 3, \dots \quad (21)$$

where f_0 is the fundamental frequency and D is the regular distance between load axles (18 m for the HSLM-A1 train). The first three critical velocities are obtained as 413.94 km/h, 206.97 km/h and 137.98 km/h, the second one being very near to the considered train speed of 206 km/h.

The envelope curves of maximum response for the velocity range (100 km/h – 300 km/h) are presented in Fig. 9. These show two response peaks (both for deflection and acceleration), for 206 km/h and 138 km/h, close to the

(a) Deflection at $L/2$ (b) Acceleration at $L/2$ **Figure 8.** Dynamic response for the HSLM-A1 train for $v = 206$ km/h (a) Envelope of maximum deflection at $L/2$ (b) Envelope of maximum acceleration at $L/2$ **Figure 9.** Envelopes of the maximum response for the HSLM-A1 train between speeds of 100 and 300 km/h

predicted critical velocities. Therefore, it can be remarked that the estimation of the train velocities of resonance proposed by (CEN 2003) is still valid for skew bridges. Furthermore, from both Figs. 8 and 9 it can be concluded that the results obtained using the general beam model and the simplified beam model agree well between themselves and with the results from the FE models (stick and shell). It must be remarked that for the FE shell model at non-resonant

speeds, larger differences are obtained with respect to all the other beam models (analytical or FE). These are of minor relevance in the deflection envelope but more significant in the accelerations envelope. This is interpreted as the contribution of vibrations of extra modes which appear only in the 3D shell model due to additional degrees of freedom which do not exist in the beam models, but which do not affect significantly the global vertical deflection.

Finally, the CPU time required for completing each analysis using the beam model was 1.9 s, compared to 130.5 s for the FE stick model (69 times greater) and 207 s for the FE shell model (109 times greater), in a standard PC with Intel Xeon processor of 2.33 GHz and 4 GB of RAM. Considering that dynamic analyses for design a single bridge may involve at least 10 train configurations, more than 100 velocities for each train and several bridge design alternatives, it is easily understood the convenience of the fast analytical beam models proposed.

Parametric studies

Having validated the analytical beam models in the previous section, in the following three parametric studies are performed using the simplified beam model, in order to identify variables which influence significantly the dynamic response. The same structural properties as the skew bridge from the previous section are adopted. In each study, the value of one parameter will be changed, and the dynamic response for the HSLM-A1 train will be shown as a function of the studied parameter. The results presented will be the natural frequencies for the first 2 modes of vibration, as well the maximum displacements and accelerations at mid-span.

Effect of skew angle

The skew angle α was varied between 0 and 40 with increments of 2, and for each case all velocities were considered between 100 km/h and 300 km/h with increments of 2 km/h. This amounts to a total of 2121 dynamic analyses in the time domain. Firstly it can be observed from Fig. 10 that the skewness has a significant influence on the first mode of vibration, whereas the frequency of the second mode hardly changes. In particular, the increment of the frequency of the first mode is about 25% with the increase of the skew angle from 0 to 40.

Figure 10. Effect of skewness on the natural frequencies

Figures 11a, 11b show how the dynamic response varies with the skew angle. An important change is also observed for the deflection: in general it decreases with increasing skew angle. A sharp change in slope is observed approximately at the skew angle of 15 (Fig. 11a). It's noted that this sharp change at the same skew angle is also observed in the value of the frequency of the first mode (Fig. 10). From this value of the skew angle, the maximum deflection decreases faster. Furthermore, an increase in the resonant train velocity is also observed when the skew angle increases, due to the growing natural frequency of the first mode. Regarding the maximum acceleration, the skew angle does not have a significant influence: the acceleration hardly increases when the skew angle grows.

(a) Maximum deflection

(b) Maximum acceleration

Figure 11. Effect of skewness on the dynamic response

Effect of torsional to flexural stiffness ratio

For this study, the ratio between the torsional stiffness and flexural stiffness (GJ/EI) is varied in the range from 0.5 to 1.5, with increments of 0.1. The analyses have been carried out for two skew angles, 10 and 30. Considering the same velocity range as before this amounts to a total of 2222 dynamic analyses.

Figure 12 shows the variation of the frequency of the first two modes of vibration, from which it can be observed that the torsional to flexural stiffness ratio has an important

influence on the eigenfrequencies when the skew angle is large (30).

From Fig. 13a, for the case of $\alpha = 10$ the maximum deflection increases slightly as the ratio increases, while the maximum acceleration in Fig. 13b is barely changed. This skew angle is in the range between 0 to 15 in which the skewness has small influence on the dynamic response of the bridge as mentioned in the preceding section and shown in Fig. 11. As a result of this, the torsional stiffness does not have a pronounced influence in the vertical deflection for small skew angles. For larger skew angle ($\alpha = 30$) the torsional stiffness has a noticeable effect on the maximum deflection, as shown in Fig. 14a. The maximum acceleration is hardly affected by the torsional stiffness for both skew angles selected (see Fig. 13b and 14b). This remark is coincident with the observation for the variation of frequency when the skew angle is large.

(a) First mode

(b) Second mode

Figure 12. Effect of torsional to flexural stiffness ratio on the eigenfrequencies

Effect of the span length

In this final parametric study, the influence of the span length on the dynamic response of the simply-supported skew bridge is carried out, ranging between $L = 15$ m to $L = 35$ m with increments of 5 m. For each case different skew angles are considered, from 0 to 40 with increments of 10. Calculations are carried out for the same velocity range as for the previous studies, thus amounting to a total of 2525 dynamic analyses. In order to obtain realistic bridge designs, the cross section of the bridge is changed for each span length, using the simple design criterion of a constant

Figure 13. Effect of torsional to flexural stiffness ratio on the maximum dynamic response for a skew angle $\alpha = 10$

Figure 14. Effect of torsional to flexural stiffness ratio on the maximum dynamic response for a skew angle $\alpha = 30$

ratio between the depth of the cross section and the span length $h/L = 1/14$, common for this type of cross section, maintaining unchanged the other dimensions of the cross section (Table 2). The first natural frequency corresponding

Table 2. Main properties of the bridge for the parametric study for variation of span length L

L (m)	h (m)	EI (GN·m ²)	GJ (GN·m ²)	$\bar{m}$ (t/m)	h/L
15.0	1.07	11.66	9.75	9.092	1/14
20.0	1.42	28.70	22.61	9.616	1/14
25.0	1.78	50.85	36.26	10.101	1/14
30.0	2.14	80.40	53.21	10.603	1/14
35.0	2.50	117.91	71.34	11.116	1/14

to each span length is shown in Fig. 15a for the different skew angles. Additionally, the variation of the first natural frequency between the skew angles of 0 and 40 is presented in Fig. 15b. It can be observed that the variation of frequency for each span length increases with the skew angle α . It is also apparent that this variation decreases with span length in an almost linear fashion.

The results of the parametric study of the maximum response to the traffic loads is presented in Fig. 16,

showing the results corresponding to the main peak (second resonant velocity) for each span length. A direct comparison of all results such as presented in the previous studies (Figs. 11, 13, 14) is not meaningful, as the characteristics of bridges are different for each span length. The ratios of maximum dynamic deflection to static deflection (dynamic amplification factor, DAF) are presented in Fig. 16a), showing that the DAF decreases with the increasing span length. It is also observed that the range of variation of the DAF does not change significantly for different skew angles as the span length increases. However, for maximum accelerations (Fig. 16b) a reduction of the range of variation of magnitude can be observed for different skew angles. It can then be remarked that the span length reduces the skewness effect on the dynamic response of the bridge in term of the maximum acceleration.

Conclusions

In this paper analytical beam models for determining the dynamic response of simply-supported skew bridges under moving loads have been proposed. These include a general model with bending and torsion and a simplified model

Figure 15. Influence of the span length on the natural frequency of the simply-supported skew bridge

Figure 16. Maximum dynamic responses at mid-span

including only bending in the vertical plane. Both models are based on analytical determination of eigenfrequencies and eigenmodes taking into account the boundary conditions and orthogonality relationships. The dynamic response is obtained in the time domain following a piecewise exact integration. The resulting models are robust, accurate, simple to define and computationally efficient. They have been successfully validated comparing to FE beam and shell models with numerical modal analysis. The application to parametric analyses of representative railway bridges under moving loads yield the following conclusions.

- The analytical beam models proposed are suitable for simple and quick dynamic analyses, allowing for parametric calculations for massive amounts of cases.
- The criterion for estimation of critical (resonant) train velocities in (CEN 2003) is basically valid also for simply-supported skew bridges.
- The degree of skewness of the bridge plays an important role in the dynamic behaviour of the bridge in terms of deflections, which decrease with the skew angle. However the maximum accelerations are hardly affected by the skewness.
- There is a critical skew angle from which the effect of the skewness is more noticeable; for the cross section used in this parametric study it was determined as $\alpha = 15$.
- The torsional stiffness has a significant influence on the vibration of the bridge in terms of deflections when the skew angle is larger than the critical value. However maximum accelerations are hardly affected.
- For larger span lengths the skewness effect on the dynamic behaviour is reduced, both in terms of changes in natural frequencies and of maximum accelerations.

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Appendix 1: Characteristic Matrix

The characteristic matrix $\mathbf{A}$ in Eq. (7) can be obtained as:

$\mathbf{A} =$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ a_4 & a_5 & a_6 & a_7 & 0 & 0 \\ -a_1 & 0 & -a_1 & 0 & 0 & \cos(\alpha) \\ -a_1 a_5 & a_1 a_4 & -a_1 a_7 & -a_1 a_6 & a_8 \cos(\alpha) & a_9 \cos(\alpha) \\ 0 & -a_2 & 0 & a_2 & a_3 & 0 \\ -a_2 a_4 & -a_2 a_5 & a_2 a_6 & a_2 a_7 & a_3 a_9 & -a_3 \sin(\lambda L) \end{bmatrix}$$

with $a_1 = \beta \sin(\alpha)$, $a_2 = EI\beta^2 \cos(\alpha)$, $a_3 = GJ\lambda \sin(\alpha)$, $a_4 = \sin(\beta L)$, $a_5 = \cos(\beta L)$, $a_6 = \sinh(\beta L)$, $a_7 = \cosh(\beta L)$, $a_8 = \sin(\lambda L)$ and $a_9 = \cos(\lambda L)$.