Takagi-Sugeno Fuzzy Incremental State Model for Optimal Control of a Ball and Beam Nonlinear Model

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Abstract. The optimal control of a ball and beam by an approach based on Takagi-Sugeno incremental state model is proposed. The advantages of incremental state model in comparison with the non incremental one are that the control action cancels steady state errors, the affine terms disappear and incremental state solves the problem of computing the target state, choosing zero as an objective. A generalized version of Takagi-Sugeno identification method is applied. For an optimal control, Linear Quadratic Regulator and optimal state observer are used in each fuzzy rule. Simulation results over the ball and beam nonlinear model show a stable closed loop in the full range, zero steady state error and good transient response.

1 Introduction

The ball and beam system [1] is a classical mechanical system with two degrees of freedom. The beam rotates, driven by a torque at the center of rotation. The ball rolls freely in contact with the beam. In spite of its mechanical simplicity, the ball and beam system is nonlinear and unstable which presents significant challenges from the control point of view.

The ball and beam is a common didactic plant in many control laboratories [2], as it is nonlinear, unstable, which means that it is difficult to control, and can be a benchmark for testing several advanced control techniques [3].

Fuzzy logic has become one of the most popular tools for modelling and control of nonlinear systems, especially the Takagi-Sugeno (T-S) fuzzy model [4], which allows the global identification of a nonlinear system as a set of linearized systems connected by a fuzzy bending.

In [5], an approach developed by the authors is presented to improve the estimation of T-S models. The problem is that the original T-S identification method cannot be applied when the triangular membership functions are overlapped by pairs. This restricts the use of this type of membership functions which have been widely used in the controllers design and are popular in industrial applications. The approach, to search for an exact optimal solution, uses the minimum norm method although it increases complexity and computational cost. Another approach was developed in [6], which can be considered as a generalized version of T-S method. This simple method with not much computational cost is based on weighting of parameters. In [7], this T-S identification method was extended to the multivariable case. These methods are characterized by the high accuracy obtained for modelling nonlinear systems in comparison with the original T-S method [4].

Incremental state model is presented in [8] to model multivariable nonlinear delayed systems expressed by a generalized version of T-S fuzzy model. The advantages of incremental state model compared with non incremental one has been defined. First, it solves the problem of computing the target state, choosing zero incremental state as an objective. Second, the control action in an incremental form is equivalent to introduce an integral action. Third, incremental state model makes the affine terms disappear. A comparative analysis has been developed in [9] between incremental state model and traditional one, the control based on these state models have been developed by a linearized model and applied over a nonlinear model of a wind turbine.

The rest of the work is organized as follows. Section 2 describes ball and beam nonlinear model. The fuzzy T-S model and the fuzzy identification method are described in Sect. 3. In Sect. 4, incremental state model is presented. Optimal state controller and optimal state observer designs are described in Sect. 5. In Sect. 6, the proposed fuzzy optimal controller based on incremental state model is applied to the ball and beam nonlinear model.

2 Ball and Beam Nonlinear Model

In this work, the AMIRA BW500 (Fig. 1) [1] is used as ball and beam model. The ball position p , the objective system output, would be measured by a camera, therefore the discrete sample time is supposed to be large. The beam angle α would be measured by an incremental encoder, thus it is considered as a measurable internal variable. The system input F is a force produced by a DC motor, which causes the beam to rotate around its center.

The nonlinear differential equations of the ball and beam model [1] are:

$$\left(m + \frac{I_b}{r^2}\right) \ddot{p} + (mr^2 + I_b) \frac{1}{r} \ddot{\alpha} - mp\dot{\alpha}^2 = mg \sin(\alpha) \quad (1)$$

$$\begin{aligned} (mp^2 + I_b + I_W) \ddot{\alpha} + (2mpp + bl^2) \dot{\alpha} + Kl^2\alpha + \\ (mr^2 + I_b) \frac{1}{r} \ddot{p} - mgp \cos(\alpha) = Fl \cos(\alpha) \end{aligned} \quad (2)$$

Fig. 1. Ball and beam system.

where p is the ball position, α is the beam angle and F is the drive mechanics force. Table 1 summarizes the model parameters and its values.

Table 1. Ball and beam parameters

Parameter	Meaning	Value
m	Mass of the ball	0.025 Kg
g	Gravity	9.81 m/s ²
r	Roll radius of the ball	0.0167 m
I_b	Inertia moment of the ball	$3.516 \cdot 10^{-6}$ Kg m ²
I_W	Inertia moment of the beam	0.09 Kg m ²
b	Friction coefficient of the drive mechanics	1.0 Ns/m
K	Stiffness of the drive mechanics	0.001 N/m
l	Radius of force application	0.49 m

Since we consider α as a measurable internal variable, the system is modeled as single-input-single-output (SISO), and the relation between output p and input F is described by two second order differential equations, so the global system is fourth order.

3 Fuzzy Takagi-Sugeno Model and System Identification

3.1 Fuzzy T-S Model

Nonlinear systems can be modelled by T-S model, supposing known a set of measurable nonlinear variables $[z_1(k), z_2(k), \dots, z_m(k)]$ of the system. By choosing $[r_1, r_2, \dots, r_m]$ number of fuzzy sets for these variables, a monovariable fuzzy

system can be defined as follows:

$$S^{(i_1 \dots i_m)}: \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}$$

$$y(k) = a_0^{(i_1 \dots i_m)} + a_1^{(i_1 \dots i_m)} y(k-1) + \dots + a_n^{(i_1 \dots i_m)} y(k-n) \quad (3)$$

$$+ b_1^{(i_1 \dots i_m)} u(k-1) + \dots + b_n^{(i_1 \dots i_m)} u(k-n)$$

In each rule, we can transform the difference equation (3) to state model with affine term as follows:

$$S^{(i_1 \dots i_m)}: \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}$$

$$x(k) \in \mathfrak{R}^n$$

$$x(k+1) = \begin{bmatrix} a_0^{(i_1 \dots i_m)} \cdot a_1^{(i_1 \dots i_m)} \\ a_0^{(i_1 \dots i_m)} \cdot a_2^{(i_1 \dots i_m)} \\ \vdots \\ a_0^{(i_1 \dots i_m)} \cdot a_n^{(i_1 \dots i_m)} \end{bmatrix} + \begin{bmatrix} a_1^{(i_1 \dots i_m)} & 1 & \dots & 0 \\ a_2^{(i_1 \dots i_m)} & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ a_n^{(i_1 \dots i_m)} & 0 & \dots & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1^{(i_1 \dots i_m)} \\ b_2^{(i_1 \dots i_m)} \\ \vdots \\ b_n^{(i_1 \dots i_m)} \end{bmatrix} u(k)$$

$$y(k) = a_0^{(i_1 \dots i_m)} + [1 \ 0 \ \dots \ 0] x(k) \quad (4)$$

In matrix form:

$$S^{(i_1 \dots i_m)}: \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:}$$

$$x(k+1) = a_x^{(i_1 \dots i_m)} + A^{(i_1 \dots i_m)} x(k) + B^{(i_1 \dots i_m)} u(k) \quad (5)$$

$$y(k) = a_y^{(i_1 \dots i_m)} + Cx(k)$$

3.2 Estimation of T-S Model Parameters

The identification method of T-S fuzzy models [4] is based on the estimation of the fuzzy system parameters minimizing a quadratic performance index. The traditional T-S identification method [4] fails if the triangular membership functions are overlapped by pairs, since the T-S matrix is not of full rank and then it is not invertible [5]. Thus, in [6] a generalized T-S identification was proposed, using a parameters weighting method.

The fuzzy estimation of the output becomes:

$$\hat{y} = \sum_{i_1=1}^{r_1} \dots \sum_{i_m=1}^{r_m} \beta^{(i_1 \dots i_m)} (z_{(i_1 \dots i_m)}(k)) \left[a_0^{(i_1 \dots i_m)} + a_1^{(i_1 \dots i_m)} y(k-1) + \dots + a_n^{(i_1 \dots i_m)} y(k-n) + b_1^{(i_1 \dots i_m)} u(k-1) + \dots + b_n^{(i_1 \dots i_m)} u(k-n) \right] \quad (6)$$

where

$$\beta^{(i_1 \dots i_m)} (z_{(i_1 \dots i_m)}(k)) = \frac{\mu_{1i_1}(z_1) \dots \mu_{mi_m}(z_m)}{\sum_{i_1=1}^{r_1} \dots \sum_{i_m=1}^{r_m} (\mu_{1i_1}(z_1) \dots \mu_{mi_m}(z_m))} \quad (7)$$

with $\mu_{ji_j}(z_j)$ being the membership function corresponding to the fuzzy set $M_j^{i_j}$.

It is supposed to have a set of input/output system samples and a first affine linear parameters estimation $p^0 = [a_0^0 \ a_1^0 \ \dots \ a_n^0 \ b_1^0 \ \dots \ b_n^0]$, which could be obtained by a classical input/output identification of the data, for example with least squares method. This first approximation can be utilized as reference parameters for all the subsystems. Then, the fuzzy model parameters can be obtained minimizing:

$$J = \sum_{k=1}^s (y(k) - \hat{y}(k))^2 + \gamma^2 \sum_{i_1=1}^{r_1} \dots \sum_{i_m=1}^{r_m} \sum_{j=0}^n \left(p_j^0 - p_j^{(i_1 \dots i_m)} \right)^2 = \quad (8)$$

$$\|Y - XP\|^2 + \gamma^2 \|P_0 - P\|^2 = \left\| \begin{bmatrix} Y \\ \gamma P_0 \end{bmatrix} - \begin{bmatrix} X \\ \gamma I \end{bmatrix} P \right\|^2 = \|Y_a - X_a P\|^2$$

where Y are the output data, X are the input/output fuzzy data, P_0 are the linear estimated parameters repeated as many times as the number of fuzzy rules ($P_0 = [p_0, p_0, \dots, p_0]$), and P are the fuzzy T-S model parameters. The γ factor represents the degree of confidence of the linear estimated parameters, and it must be adjusted. It should be noted that the matrix X_a is of full rank, even if the membership functions are triangular overlapped by pairs, which solves the problem where the traditional T-S identification method fails. Thus, the vector P can be computed as:

$$P = (X_a^t X_a)^{-1} X_a^t Y_a \quad (9)$$

4 Incremental State Model

The control method based on traditional state model, can produce steady state errors in presence of disturbances or modelling errors. This problem can be solved by incremental state model [8,9].

Applying the discrete state model described in (5), at the previous sample ($k-1$):

$$\begin{aligned} x(k) &= a_x + Ax(k-1) + Bu(k-1) \\ y(k-1) &= a_y + Cx(k-1) \end{aligned} \quad (10)$$

Subtracting (10) from (5):

$$\begin{aligned} x(k+1) - x(k) &= A(x(k) - x(k-1)) + B(u(k) - u(k-1)) \\ y(k) - y(k-1) &= C(x(k) - x(k-1)) \end{aligned} \quad (11)$$

where the affine terms are cancelled. Defining the incremental state Δx and the incremental input Δu as follows:

$$\Delta x(k) = x(k) - x(k-1) \quad \Delta u(k) = u(k) - u(k-1) \quad (12)$$

Substituting (12) into (11), it is obtained:

$$\begin{aligned} \Delta x(k+1) &= A\Delta x(k) + B\Delta u(k) \\ y(k) &= y(k-1) + C\Delta x(k) \end{aligned} \quad (13)$$

A new state is introduced to complete the formulation, verifying that:

$$y(k+1) = y(k) + C\Delta x(k+1) = y(k) + C(A\Delta x(k) + B\Delta u(k)) \quad (14)$$

A new expanded incremental state vector $x_a \in \mathfrak{R}^{(1+n)}$ is defined, obtaining the new state model as follows:

$$\begin{aligned} \begin{bmatrix} y(k+1) \\ \Delta x(k+1) \end{bmatrix} &= \begin{bmatrix} 1 & CA \\ 0 & A \end{bmatrix} \begin{bmatrix} y(k) \\ \Delta x(k) \end{bmatrix} + \begin{bmatrix} CB \\ B \end{bmatrix} \Delta u(k) \\ y(k) &= [1 \ 0] \begin{bmatrix} y(k) \\ \Delta x(k) \end{bmatrix} \quad x_a(k) = \begin{bmatrix} y(k) \\ \Delta x(k) \end{bmatrix} \end{aligned} \quad (15)$$

In matrix notation, the expanded state model becomes:

$$\begin{aligned} x_a(k+1) &= A_a x_a(k) + B_a \Delta u(k) \\ y(k) &= C_a x_a(k) \end{aligned} \quad (16)$$

Using the method described in Eq.(15) in each fuzzy rule defined by T-S state model (5), an incremental T-S state model can be obtained as follows:

$$\begin{aligned} S^{(i_1 \dots i_m)}: & \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:} \\ x_a(k+1) &= A_a^{(i_1 \dots i_m)} x_a(k) + B_a^{(i_1 \dots i_m)} \Delta u(k) \\ y(k) &= C_a x_a(k) \end{aligned} \quad (17)$$

5 Fuzzy Controller and Observer Design Based on Incremental State Model

5.1 Fuzzy Controller for Zero Steady-State Error

Incremental state feedback control can be designed by any method. In order to calculate the coefficients of the state feedback controller, discrete LQR method is chosen, which allows optimal control weighting the dynamic response and the control action.

The goal is to minimize the cost index J:

$$J = \sum_{k=0}^{\infty} \left[(x_r - x_a(k))^t Q (x_r - x_a(k)) + \Delta u(k)^t R \Delta u(k) \right] \quad (18)$$

LQR method is optimal for linear systems, however, in the case of nonlinear systems, it is complex to propose the minimization of any objective function for the global system. In order to solve this problem, it is suggested to minimize the cost of each fuzzy rule instead of the global cost. The solution will be a suboptimal one but with the great advantage of being easy to calculate. With this method, the global stability is not guaranteed, which needs to be analyzed a posteriori, although gaining in return a balance between static and dynamic behavior of the system with admissible control actions.

If the control objective is to approach a steady state output y_r , this is equivalent to achieve the reference expanded state in incremental model [8,9]:

$$x_{ar} = \begin{bmatrix} y_r \\ \Delta x_r \end{bmatrix} = \begin{bmatrix} y_r \\ 0 \end{bmatrix}$$

Then the control action is described as:

$$S^{(i_1 \dots i_m)}: \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:} \\ \Delta u(k) = K(x_{ar} - x_a(k)) = K^{(i_1 \dots i_m)} \begin{bmatrix} y_r - y(k) \\ -\Delta x(k) \end{bmatrix} \quad (19)$$

$$u(k) = u(k-1) + \Delta u(k) \quad (20)$$

It should be noted that the reference expanded state comes naturally without any calculations. Moreover, control action (20) is calculated in incremental form (19), which is equivalent to apply an integral control action. If the feedback system is stable and the steady state is approached, the controlled system has zero steady state error [8,9].

5.2 Fuzzy Observer for Incremental State Model

In the controller algorithm it is supposed that the output $y(k)$ is measurable, but the incremental state $\Delta x(k)$ is not directly accessible, then a state observer is required. The state observer is a parallel dynamic system with a correction term that approximates the estimated state to the real one. The fuzzy incremental state observer [8,9], is formulated as follows:

$$S^{(i_1 \dots i_m)}: \text{ If } z_1(k) \text{ is } M_1^{i_1} \text{ and } \dots \text{ and } z_m(k) \text{ is } M_m^{i_m} \text{ then:} \\ \Delta x_e(k+1) = A^{(i_1 \dots i_m)} \Delta x_e(k) + B^{(i_1 \dots i_m)} \Delta u(k) + \\ H^{(i_1 \dots i_m)} (y(k) - y(k-1) - C^{(i_1 \dots i_m)} \Delta x_e(k)) \quad (21)$$

Thus, the expanded estimated state is obtained as follows:

$$x_{ae}(k) = \begin{bmatrix} y(k) \\ \Delta x_e(k) \end{bmatrix} \quad (22)$$

Matrix $H^{(i_1 \dots i_m)} \in \mathfrak{R}^{(n \times 1)}$ coefficients are obtained by optimal state observer design [8,9] in each fuzzy rule.

The optimal observer solves the problem of calculating a matrix H which minimizes the cost index J_o :

$$J_o(H) = \alpha^T (A - HC) (A - HC)^T \alpha \quad \forall \alpha \in \mathfrak{R}^n \quad (23)$$

It is verified in [8] that for any value of α , it fulfills:

$$H = AC^T(CC^T)^{-1} \quad (24)$$

In the proposed state representation C matrix is defined in observable canonical form, therefore it holds that $CC^T = I$, so that:

$$H = AC^T \quad (25)$$

The separation principle holds between optimal observer and control design.

6 Results

The ball and beam model is defined in continuous time, but the controller has been developed in discrete time, therefore, a sampler and zero order holding device have been added to the model, with the proposed sampling time $T=0.05$ s. All system variables are supposed to be ideally measured.

For a generalized T-S identification of the system, the following weighting factor $\gamma = 10^{-11}$ and the following membership functions defined in Fig. 2 has been chosen, obtaining an identification error of $1.6169 \cdot 10^{-11}$.

Fig. 2. Membership functions of the fuzzy sets.

With the generalized T-S identification method and Eq. (15), a fuzzy T-S incremental state model has been obtained:

$$S^{(1,1,1,1)}: \text{ If } p(k) \text{ is } M_1^1 \text{ and } \dot{p}(k) \text{ is } M_2^1 \text{ and } \alpha(k) \text{ is } M_3^1 \text{ and } \dot{\alpha}(k) \text{ is } M_4^1 \text{ then:}$$

$$\begin{bmatrix} y(k+1) \\ \Delta x(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 3.9181 & 1 & 0 & 0 \\ 0 & 3.9181 & 1 & 0 & 0 \\ 0 & -5.7542 & 0 & 1 & 0 \\ 0 & 3.7543 & 0 & 0 & 1 \\ 0 & -0.9181 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(k) \\ \Delta x(k) \end{bmatrix} + 10^{-3} \begin{bmatrix} -0.0502 \\ -0.0502 \\ 0.1562 \\ 0.1262 \\ -0.0571 \end{bmatrix} \Delta u(k)$$

$$y(k) = [1 \ 0 \ 0 \ 0 \ 0] \begin{bmatrix} y(k) \\ \Delta x(k) \end{bmatrix}$$

Thus, the controller matrix K is designed in each rule by discrete LQR algorithm, using the incremental state matrices A_a and B_a , and the positive

definite weighting matrices $Q = I$ and $R = 1$. The observer matrix is defined by the optimal state observer method [8, 9], using Eq. (25). Obtaining:

$S^{(1,1,1,1)}$: If $p(k)$ is M_1^1 and $\dot{p}(k)$ is M_2^1 and $\alpha(k)$ is M_3^1 and $\dot{\alpha}(k)$ is M_4^1 then:

$$K^{(1,1,1,1)} = 10^3 \begin{bmatrix} 0.0008 & 4.0174 & 3.2390 & 2.5627 & 1.9811 \end{bmatrix}$$

$$H^{(1,1,1,1)} = \begin{bmatrix} 3.9181 & -5.7542 & 3.7543 & -0.9181 \end{bmatrix}^t$$

With this fuzzy controller and observer design method, the global stability cannot be theoretically proved, so it has to be analyzed a posteriori with the simulation results.

Figure 3, shows the ball position and the observation error of the ball position, when the controlled system is subjected to changes in the ball position reference. Moreover, Fig. 4 shows the beam angle and the input force.

Fig. 3. Ball and beam variables: (left) Ball position. (right) Observation error of ball position.

Fig. 4. Ball and beam variables: (left) Beam angle. (right) Input force.

In Fig. 3 (left) it can be seen that the system present stable and good transient response and zero steady state error. In Fig. 3 (right) it is shown that the observation error is of small range and tends to zero. In Fig. 4 it can be seen that, the system variables present smooth and stable transient responses. Thus, the controlled ball and beam model has a stable response in the full range of the system, presents a good transient response and zero steady state error.

7 Conclusion

In this work, we have shown the obtained results of T-S incremental state model and optimal state controller and observer, applied in a ball and beam nonlinear model. The advantages of incremental state model over traditional one comes in a natural way and they are that the affine terms disappear, the problem of computing the target state is solved and the control action cancels steady state errors. The optimal controller and observer has been designed in each fuzzy rule, so a suboptimal solution have been found, but easy of calculate and compute. The results show that the ball and beam controlled nonlinear model has a stable behavior, good transient response and zero steady state error on the full range of the ball and beam system.

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