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**MECHANICAL MODELING OF POROELASTIC AND RESIDUALLY
STRESSED HYPERELASTIC MATERIALS AND ITS APPLICATION TO
BIOLOGICAL TISSUES**

DOCTORAL THESIS

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Engineering of Civil-Structures
(M.Sc.)

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CAMINOS, CANALES Y PUERTOS

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EL PRESIDENTE

LOS VOCALES

EL SECRETARIO

To my family

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ABSTRACT

The thesis deals with the modeling of residually stressed hyperelastic as well as poroelastic materials with application in biological tissues. It deals with two different materials, namely, poroelastic and hyperelastic ones. In particular, numerical frameworks able to predict their mechanical response as well as bifurcation and post-bifurcation are provided.

In the research area of poroelasticity, in the context of cancer mechanics, firstly, state of the art of the upscaling process (asymptotic homogenization) to reach the macroscale system of PDEs from microscale setting is given. The macroscale mechanical response can be characterized by means of some coefficients that hold the microscale information. Secondly, the system of PDEs to be solved in microscale cell sub-domains (fluid and solid phases) are explained and appropriate numerical frameworks able to solve them are provided. The macroscale coefficients then are derived from the solution of the latter problems and the dependence of mechanical and hydraulic properties of poroelastic media on the porosity and solid matrix compressibility is studied in detail. The findings are presented by means of a parametric analysis conducted by varying the porosity as well as the Poisson's ratio of the matrix that, for instance, reveal a nontrivial dependency of Biot's modulus on both porosity and compressibility of the matrix.

The numerical three-dimensional results, which are presented in the context of tumor modeling, serve as a robust first step to quantify the macroscale

response of poroelastic materials on the basis of their underlying microstructure. This is done by implementing the macroscale system of equations in a finite element code. The results can be used to relate the mechanical and hydraulic response of the tissue to its microstructural properties, which, in turn, can be useful to distinguish between benign tumor and cancer. Moreover, the macroscale visualization of the displacements serves as a robust tool to quantify the poroelastic properties of tissues such as tumors and brain by exploiting techniques such as poroelastography.

In the research area of hyperelasticity, in the context of aneurysm formation and propagation, first, state of the art of analytical analysis of different bifurcation modes of extended circular cylindrical membranes made of hyperelastic material under inflation is given. The goal is to gain a better understanding of possible bifurcation modes of the model which can guide and benchmark the presented numerical results. Then, different bifurcation modes as well as post-bifurcation of pre-extended thick-walled circular cylinders made of residually stressed hyperelastic material under inflation are provided. The dependence of the instabilities of the models on residual stresses and axial pre-stretch is shown and compared with results when there is no residual stress. The numerical results highlight that bulging bifurcation is expected for large values of the axial stretch while the bending mode happens in the cases with small values of the axial stretch (close to the non-extended configuration). For the latter bifurcation, the structure can support more pressure during post-bifurcation until a one-sided bulge with irregular shape is reached, which is consistent with the formation and propagation of abdominal aortic aneurysms (AAA).

RESUMEN

La tesis presentada trata sobre el modelado de materiales hiperelásticos con tensión residual y el modelado de materiales poroelásticos con aplicación en tejidos biológicos. Este documento trata dos teorías diferentes de los materiales, a saber, la poroelasticidad y la hiperelasticidad. En particular, se proporcionan marcos numéricos capaces de predecir su respuesta mecánica, así como la bifurcación y la post-bifurcación.

En el área de investigación de la poroelasticidad, en el contexto de la mecánica del cáncer, en primer lugar, se proporciona el estado de la técnica del proceso de ampliación de escala (homogeneización asintótica) para obtener el sistema de EDP's de la macroescala a partir de la configuración en la microescala. La respuesta mecánica en la macroescala se puede caracterizar mediante algunos coeficientes que contienen la información de la microescala. En segundo lugar, se explica el sistema de EDP a resolver en subdominios de una celda a nivel de la microescala (fases fluida y sólida) y se proporcionan los marcos numéricos apropiados para resolverlo. Los coeficientes para la macroescala son entonces derivados a partir de la solución de los problemas anteriores y se estudia en detalle la dependencia de las propiedades mecánicas e hidráulicas de los medios poroelásticos en ambos la porosidad y la compresibilidad de la matriz sólida. Los hallazgos se presentan mediante un análisis paramétrico que se realiza variando la porosidad y la relación de Poisson de la matriz que, por ejemplo, revela una depen-

dencia no trivial del módulo de Biot en la porosidad y la compresibilidad de la matriz.

Los resultados numéricos tridimensionales, que se presentan en el contexto del modelado de tumores, sirven como un primer paso para cuantificar la respuesta en la macroescala de los materiales poroelásticos sobre la base de su microestructura subyacente. Esto se hace implementando el sistema de ecuaciones de la macroescala en un código de elementos finitos. Los resultados pueden usarse para relacionar la respuesta mecánica e hidráulica del tejido con sus propiedades microestructurales, que, a su vez, pueden ser útiles para distinguir entre un tumor benigno y un tumor maligno. Además, la visualización de los desplazamientos en la macroescala sirve como una herramienta robusta para cuantificar las propiedades poroelásticas de los tejidos presentes en los tumores y en el cerebro mediante el uso de técnicas como la poroelastografía.

En el área de investigación de la hiperelasticidad, en el contexto de la formación y propagación de aneurismas, en primer lugar, se proporciona el estado del arte del estudio analítico de diferentes modos de bifurcación en membranas cilíndricas circulares elongadas formadas por material hiperelástico sometidas a una presión de inflamamiento. El objetivo es lograr una comprensión de los posibles modos de bifurcación con el fin de guiar y acotar los resultados numéricos presentados. En segundo lugar, se presenta un cilindro con sección circular de pared gruesa pre-elongado de material hiperelástico con tensiones residuales sometido a una presión de inflamamiento y se analizan sus diferentes modos de bifurcación, así como su post-bifurcación. La dependencia de las inestabilidades de los modelos de las tensiones residuales y el estiramiento previo axial es mostrada y se compara con los resultados obtenidos cuando no hay tensión residual. Los resultados numéricos resaltan que se espera una bifurcación por abultamiento para valores grandes del estiramiento axial, mientras que el modo de flexión ocurre en los casos con valores pequeños del estiramiento axial (cerca de la configuración no extendida). Se obtiene que en la anterior bifurcación la estructura puede soportar más presión durante la postbifurcación hasta que se alcanza una protuberancia unilateral con forma irregular que es consistente con la formación y propagación de los aneurismas aórticos abdominales (AAA).

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CHAPTER 1

INTRODUCTION

In this chapter, a descriptive introduction of motivations, methodology, objectives, and contents of the presented thesis is provided. This chapter is organized as follows. In the first section, the motivations behind the mechanical modeling of human tissues, residual stresses, structural instabilities of arteries, and poroelastic tissues are explained. Next, Section 1.2 deals with an introduction to the methods and techniques used in this research work. The principal objectives of the presented thesis are introduced in Section 1.3. Finally, Section 1.4 provides a map of the rest of the thesis.

1.1 Motivation and applications

Modeling of human tissues, such as brain tissue, tumors, cardiovascular system, etc., provides us with the ability to predict the response of the human body subjected to diseases, loads, exercises, remedies, etc. Consequently, a better understanding into the causes and effects of the issues related to the biological organs can be gained which, in turn, gives us a better insight into the risk factors, more efficient and safe choices, etc. Moreover, it can help and speed up the development of modern biomedical and surgical instruments that are well designed according to the target environment which

is nowadays of great importance. In addition, the production of artificial tissues that imitates the properties of the models in nature including human body follows the advances of cutting edge modeling and simulation technologies.

Mechanical Modeling of human tissues requires a complex interplay of strongly coupled multiscale and multiphysics mechanisms together with non-linear elasticity and residual stresses (see, among many others, [12–15, 32, 106–115, 117–123, 132–134] for multiscale/multiphysics and [33, 77–95, 99, 100, 103, 127, 128] for hyperelasticity). Reliable computer modeling (which is the chief motivation and objective of this thesis) has clear and significant advantages over in-vitro and in-vivo experiments as it can be low cost, risk-free, non-invasive, and capable to reveal important physiological mechanisms (see, for example, [3–5, 32, 33, 36, 40, 65, 94]). Despite significant progress in the past few decades ([36, 95, 137, 139]), many challenges remain in the area of instabilities of thin-walled as well as thick-walled cylinders (e.g. arteries) subjected to different boundary conditions (several issues are addressed in previous works such as [36, 79, 81, 83, 84, 86]), determination of poroelastic properties of human tissues (e.g. brain tissues and tumors) and their mechanical response, dependence of macrostructural poroelastic response on different microstructural factors, etc.

1.1.1 Cancer mechanics

Chapters 2, 3, and 4 deal with the analysis of poroelastic materials. The principal motivation behind these works is the study of Cancer mechanics. Poroelasticity is the study of the mechanical behavior of elastic porous media interacting with the percolating fluid in its pores [111]. A poroelastic medium can be studied by means of multiscale analysis since the pore dimension (*microscale*) is considerably smaller than the one of the media (*macroscale*) [76]. Moreover, the microscale exhibits a multiphysics (fluid-solid) problem. This has a wide range of applications including soil and rock mechanics [75], oil and gas recovery, biological tissues [108], artificial constructs [110] etc. The focus of this research work is mainly on the analysis of deformable malignant cell aggregates (i.e. tumors) in the context of cancer mechanics. A poroelastic medium is made of the porous solid structure (the solid matrix) as well as the fluid transporting inside its pores. The interstitial solid matrix acts as a barrier to the flow of fluids such as anti-cancer drugs so, it is of great importance to predict the behavior of poroelastic tissues in order to design effective medicine and take appropriate remedies according to the environment of the tumor. In addition, estimation of the mechanical properties of the brain tissues, tumors, etc. is another important application of this study. This can be done in vivo by means

of Poroelastography [116] [158], Elastography [60], and Viscoelastography [67]. In all these techniques one measures the displacements that are induced in the target tissue from different sources e.g. Valsalva maneuver [98], blood pressure pulses [158], mechanical waves [157], etc. Then, using appropriate constitutive equations they can estimate the effective poroelastic, elastic, or viscoelastic properties of the brain tissue according to the chosen system of equation. Therefore, the estimated poroelastic properties of the tissue are sensitive to the assumptions of the chosen macroscale constitutive equations that are, in turn, affected by assumed microscale geometry as well as mechanical and hydraulic properties.

1.1.2 Aneurysm

Chapters 5 and 6 provide a detailed research work on the bifurcation and post-bifurcation of residually stressed hyperelastic circular cylindrical tubes under axial pre-stretch as well as inflation pressure which is applicable to the context of Aneurysm. Moreover, the mechanical modeling of veins has been studied in several previous experimental and biomechanical works, for instance, in the context of venous tortuosity (see [73] and [53]), which in turn is an important instability associated with various pathologies such as varicose veins.

Residual stresses can exist due to several reasons such as growth and development, remodeling, plastic deformation, etc. In the study of blood vessels, the unloaded tube is commonly taken as the reference state in which the stress is zero everywhere. However, when an unloaded ring of an artery is cut longitudinally, its wall spring opens which is a proof of the residual stress existence. They have an important effect on the mechanical response of bodies such as the cardiovascular system. There are several ways in the literature to include residual stresses in the material such as defining a global virtual pre-stress field, assigning a pre-deformation, and including them in the constitutive equations. The mechanical response of a residually stressed material is different from the same material without them. This means that this type of stresses plays an important role in the deformation of the body. An important application of this analysis is in the context of aneurysms formation and propagation in arterial wall tissue (see, for example [3, 5, 34, 127]).

An aneurysm, which is bulging of the wall of an artery, can be fusiform (when it is uniform and symmetric) or saccular (a small, lop-sided blister on one side of the aorta). In the former case, the symmetric bulging instability mode in a cylinder is exploited while the saccular one can be studied in the post-bifurcation of bending instability [127]. The localized bulging is consistent with the aneurysm rupture. Moreover, the bending bifurcation of the

presented model gives a great understanding of the causes and effects in the context of venous tortuosity and the aneurysm rupture under this condition. In this thesis, the effect of residual stress, as well as axial pre-stretch in the formation and propagation of these pathology, is studied.

1.2 Methodology

In this section, the theories and techniques used in this thesis are briefly explained.

Continuum mechanics is a branch of science that studies the motion of a body subjected to external loads. The material is assumed to be a continuous mass which means that it completely fills the occupied space. The starting point of continuum mechanics is mathematical description of Kinematics including the *reference configuration* (Lagrangian or undeformed body), *deformed configuration* (Eulerian), *rigid motion*, deformation, displacement, velocity etc. This is followed by expression of Balance laws such as the balance of linear momentum, mass, angular momentum, etc. that can be described via Partial Differential Equations PDEs in a moving body. Finally, the material laws in phase space, in general, are used to relate strains and stresses.

There are different types of Constitutive Equations CEs in mechanics according to the material phase (fluid and solids) as well as the material type (non-linear, linear, etc.). Linear elastic CE is an appropriate formulation for materials that are subjected to small strains. In general, this formulation relates the stress tensor to the strain tensor by multiplying the latter by a constant namely elasticity tensor which can be expressed in different ways, for example, providing Young's modulus and Poisson's ratio for the isotropic case. In the case of non-linear elastic materials, the coefficient of the strain is a function of deformation which is appropriate for the materials undergoing large deformation. Hyperelasticity provides a unique constitutive relation between stresses and strain at a certain point that does not depend on the load path. This is described by means of a Strain Energy Density Function (SEDF) such that stresses are obtained from derivatives of this function with respect to the corresponding strains [77] to [88], [90], etc. In the presented thesis we also make use of CEs for incompressible Newtonian fluids. The shear stresses in this type of fluid are linearly proportional to the gradient of velocity.

The analytical analysis is a modeling technique that can be used to provide a solution for a model made of continuous material under different loading condition. This technique examines the relationship between variables for boundary value problems, for instance, in the context of stability theory. Moreover, analytical analysis can provide the instability criteria.

For example, in [93] they use the principles of continuum mechanics and hyperelasticity to provide a balance relationship between load and body motions (namely, axial and azimuthal stretch) for a circular cylindrical tube under axial stretches as well as inner pressure. In addition, they study incremental displacements in order to investigate the possible bifurcation modes.

Bifurcation theory is the analytical study of qualitative changes in the mechanical response of certain structures under certain loads. A bifurcation occurs when a qualitatively large and sudden change is observed in a variable such as displacements due to a small smooth increase in the magnitude of a parameter such as the internal pressure. The Post-Bifurcation denotes the response of material after it bifurcates. The study of different Bifurcation and post-bifurcation modes is of great importance in the investigation of different structural instabilities. The latter, in particular, in the case of cylinders that are subjected to several boundary conditions, have advanced significantly in the last few years (see [35, 93, 127]).

The complex mathematical machinery in analytical modeling is considered as the main difficulty when it comes to complex geometries and boundary conditions of the problems in the real world. This imposes the necessity of considerable simplifications including the analysis in 2-D or 1-D instead of 3-D, neglecting residual stresses, assuming perfect geometry, neglecting the flexural stiffness, etc. Although this type of modeling gives a better understanding of the physical interpretation of the problems, the mentioned issues arise the need for other tools that can analyze and model more detailed problems.

The growth in computing power has allowed the emergence of the "Third Paradigm" of science (after empirical and mathematical analysis), namely computational tools such as FE implementation for detailed simulations of the models in the world. Numerical simulation using FE models is well known as an interdisciplinary science which makes use of multiple branches of mathematics and computer science to describe a physical phenomenon that is difficult to reach its analytical solution. The application of numerical analysis is from economy to physics. Here, we focus on structural analysis. In this framework, in general, a body is discretized into several cells with simple geometries, known as *elements*, in order to be able to solve complex Partial Differential Equations PDEs of a *boundary-value problem* over a complex domain. The starting point for FE implementation is to express *strong form* of PDEs in *variational form*. The former can be utilized in analytical modeling while the latter is the appropriate form for FE implementation. First, we multiply the strong form is multiplied by a *test function* and integrate the resulting equations over the domain. From either Divergence theorem or Integrating by Parts we can reach the *weak form* of the equations which are applicable to an element. Afterward, the problem

has to be discretized to be solved in simple elements such as Hexahedron and Tetrahedron in which we can apply numerical integration methods such as Gauss-Legendre quadrature. The shape functions or interpolation functions interpolate the solutions at the mesh nodes. Moreover, these functions provide the discretized form of spatial derivatives that provide us with the discretized form of the weak formulation. There are important factors in this type of analysis that determine its efficiency and accuracy such as element type, the number of elements, and their distribution over the domain. Element type includes the geometry of the elements, interpolation functions (linear and non-linear), the number of nodes, the number of Gauss points for numerical integration, and several more advanced controls such as Hourglass control for large deformations.

1.2.1 Cancer mechanics

In Chapters 2, 3, and 4 the analysis of poroelastic materials is carried out. As mentioned previously, Poroelasticity deals with the mechanical behavior of elastic porous media in which fluid fills its pores. Multiscale analysis is employed since the pore dimension (*microscale*) is considerably smaller than the one of the media (*macroscale*) [76]. Moreover, the microscale exhibits a multiphysics (fluid-solid interaction) problem. The medium is assumed to have microscale periodicity which means that it can be constructed by repeating one cell [140]. This assumption results in identical displacements on the inlet and outlet of the cell in each principal orthogonal axis. A cubic microscale cell is chosen that consists of a solid matrix and fluid. The solid matrix is a cube from which three cylinders are subtracted. The longitudinal axis of each cylinder coincides with one of the principal cartesian axes and they meet at the center of the cube. The merge of the described cylinders constructs the fluid phase. This implies rotational invariance in both solid and fluid phases. The volume fraction of the fluid phase over the cell is defined as porosity. The solid matrix is assumed of linear elastic type and the fluid is chosen to be incompressible Newtonian. Stress and velocity continuity on their interface is the assumption that allows us to analyze their interaction. The asymptotic homogenization technique serves as an upscaling process to translate the cell problem into a macroscale problem governed by the poroelasticity system of equations. The basis of this technique is the sharp scale separation which is imposed via a very small parameter equal to the dimension of the medium over the one of the cell. Then formally independent spatial variables are chosen to represent the macro and micro scale. This is followed by three main sub-techniques. First, the differential operator in physical scale is transferred into two independent ones on microscale and macroscale. Then, each func-

tion is expressed by its power of series representation to produce the functions of the leading order as well as the first order. The former is achieved by equating the coefficients of the power zero and the latter in the same way for the power one of the scale separation parameter. We name the achieved equations as the principal equations of the upscaling process. Finally, the average operator is applied in microscale so that their results can be utilized in macroscale.

The macroscale equation for the balance of linear momentum is reached by taking the integral average of the sum of the first order equilibrium equations of the solid and fluid phase imposing the no growth limit. Then, by substituting an appropriate Ansatz to the first order equation of the constitutive relationship of the solid phase we reach the macroscale constitutive equation for the stress of poroelastic medium. The mentioned Ansatz relates the first order displacement to the gradient of the zeroth order displacement and macroscale hydrostatic pressure. Next, the first order incompressibility condition of the fluid velocity, y -periodicity, and no growth limit together with some mathematical machinery result in the macroscale mass conservation balance equation for the poroelastic media. Finally, the material law that relates the relative fluid velocity to the gradient of macroscale pore pressure is the Darcy law. Relative fluid velocity is defined as the fluid velocity minus solid velocity. In the described macroscale system of equations, there are four coefficients, namely, Biot's modulus, Biot's coefficient, effective elasticity tensor, and hydraulic conductivity. These retain information of underlying microstructural properties. [11, 107, 112]

The above-mentioned coefficients are to be determined by solving microscale cell problems. From the leading order and first order constitutive equation of fluid phase, leading order no-slip boundary condition, and leading order equilibrium equation of the fluid phase, and by means of appropriate Ansatz, we reach three Stocke's type boundary value problems for the fluid to be solved in the fluid phase of the cell. The assumed Ansatz are the Darcy law together with another one that relates the first order of hydrostatic pressure to the gradient of its leading order. The solution of these problems is 9 velocity components that construct the Hydraulic conductivity second order tensor. In order to determine the remaining coefficients of macroscale system of equations (effective elasticity tensor, Biot's coefficient, and Biot's modulus), we make use of the leading order stress continuity, solid equilibrium equation, and fluid material law together with the first order linear elastic constitutive equation and proper Ansatz to make one set of elastic type boundary value problems (consists of six problems) plus a seventh linear elastic problem. All of the solid problems have Neumann boundary conditions. From the solution (strain components) of the six problems, a fourth rank tensor is constructed that results in effective elasticity tensor

and Biot's coefficient. The Biot modulus is achieved from the solution of the seventh solid problem. Furthermore, If we assume static fluid phase filling the pores, the elastic response of this phase can be achieved which allow us to compute the undrained effective elasticity tensor by substituting the mass conservation law to the constitutive equation for the macroscale stress. Consequently, undrained macroscale properties are also studied.

In this work, the attention is dedicated to a material governed by the equations of poroelasticity. The asymptotic homogenization technique is adopted to investigate the impact of the interstitial fluid volume fraction (*porosity*) and compressibility of the solid matrix on the mentioned coefficients that determine the overall mechanical response of the material. A parametric analysis via finite element numerical simulations is carried out to compute the solution of the relevant pore-scale cell problems in three-dimensions for various values of porosity and Poisson's ratio of the elastic matrix. As a consequence of previously mentioned rotational invariance, it is sufficient to investigate the profile of six independent scalar parameters, namely Young's modulus, shear modulus, Poisson's ratio, Biot's modulus, Biot's coefficient, and hydraulic conductivity, which fully characterize the macroscale mechanical and hydraulic behavior of the medium. In this research work, the mentioned scalar parameters are non-dimensionalized with respect to a characteristic pressure, viscosity, and microscale length. Dimensional Hydraulic Conductivity is obtained by multiplying the non-dimensional value by characteristic square microscale length divided by viscosity. In the case of effective elasticity tensor and Biot's modulus, we multiply them by characteristic pressure. Biot's coefficient and effective Poisson's ratio are non-dimensional.

This work is then followed by the implementation of the macroscale system of PDEs which allows us to model the macroscale poroelastic media and examine its mechanical response.

1.2.2 Aneurysm

In Chapters 5 and 6, the Finite Element Method (FEM), which is an appropriate tool to carry out bifurcation and post-bifurcation analysis, is used to approach this problem. A numerical method that can be used in these cases is the modified Riks method. This method considers the arc length of the load-displacement path as the constraint equation. This is the reason to also mention this method as the arc-length method [141]. In particular, we make use of the well developed FEM code Abaqus/Standard which has this procedure well implemented in its library.

The starting point of this study is the analytical analysis of a hyperelastic long membrane neglecting the flexural stiffness. This is done by making

use of continuum mechanics and the theory of Hyperelasticity. Although the results of this analysis are not expected to be equal to the ones with thickness in our FE model they should show similar and near behavior to the model without residual stresses. Consequently, this analysis is exploited to validate the FE model. Moreover, a bending criterion is presented from the incremental analytical analysis to confirm the validity of the bending bifurcation in the numerical model. The formulation of a strain energy density function for a hyperelastic residually stressed material is also provided in this part. As the latter is not existed in the Abaqus material library, next, we explain the details of its implementation in a User MATerial (UMAT). For this purpose, it is necessary to provide an explicit relationship between stress tensor and deformation followed by the consistent Jacobian Matrix that is proportional to the deformation. One advantage of UMAT is that there is no need for the weak formulation of PDEs. Having implemented the material in 3-D via Fortran programming language, next, a parametric Finite Element model is made by means of writing an input file in Python exploiting the keywords of ABAQUS. The latter allows us to simply change the strength and weight of residual stresses (non-dimensionalized with respect to the geometry), the magnitude of axial pre-stretch, inner pressure (non-dimensionalized with respect to the geometry as well as the material parameter), geometry, material parameter, boundary conditions, mesh setting, etc. The Strength of residual stress is a coefficient that controls the magnitude of its components while the weight for its invariants is dedicated to defining the invariant that is used in the strain energy function to apply it. The presented thesis, in particular, deals with the parametric study of the effects of axial pre-stretch and the strength and weight of residual stresses.

1.3 Objectives

The general objective of the presented thesis is a detailed study of the mechanical response of biological tissues by means of numerical simulation. This is divided into two main categories: 1- Cancer mechanics by means of poroelasticity 2- Aneurysm by means of residually stressed hyperelastic materials.

1.3.1 Cancer mechanics

The main objective of the research work in the context of poroelasticity is to study details of the effects of microscale structural and hydraulic properties on the macroscale mechanical response of the poroelastic media. In order to achieve this goal the following objectives are set

- To set an appropriate macroscale system of PDEs (Partial Differential Equations) for the principal objective of this study in which the information of underlying microscale structural and hydraulic properties is encoded in some parameters (Coefficients of Macroscale System of Equations or CMSEs).
- To define PDE problems to be solved in the cell domain (microscale) in order to reach the mentioned parameters. The latter then is solved via a Finite Element code and the coefficients are derived from their solutions. Furthermore, a thorough dataset of CMSEs for different geometries (porosities) as well as structural properties is provided as a basis for the following.
- To implement the macroscale system of PDEs in an FE code in order to model the poroelastic media which allow us to study the effects of microscale structural and hydraulic properties on the macroscale mechanical response of this type of material.

1.3.2 Aneurysm

In the context of Aneurysm, a residually stressed hyperelastic circular cylindrical thick-walled tube is studied. The principal objective is to provide detailed research work on the effects of residual stresses on the different bifurcation modes and their post-bifurcation. To achieve this goal the following objectives have been set.

- To study analytical analysis of circular cylindrical membrane tubes made of hyperelastic material subjected to axial stretch as well as inflation pressure in order to guide the numerical analyses.
- To implement residually stressed hyperelastic materials in an FE code in order to model a circular cylindrical thick-walled tube made of the mentioned material followed by analyzing it and capturing different bifurcation and post-bifurcation modes.
- To study deeply the effect of residual stresses on the bifurcation modes as well as post-bifurcation of the mentioned model which is followed by the detailed study of the effects of different bifurcation modes on the radial expansion of the bulge in the context of aneurysm rupture.

1.4 Content

Chapter 2 deals with the state of the art of multiscale and multiphysics analytical analysis for poroelastic media. This includes a slight modification

of previous studies in order to achieve a macroscale system of PDEs that is appropriate for the objectives of this thesis. In this chapter stating the microscale material properties and boundary condition, asymptotic homogenization is exploited as the upscaling process to reach a system of PDEs including some coefficients (to be determined from microscale cell problems) with only macroscale variables which describe the response of the porous media.

In Chapter 3, mathematical description of the fluid and solid cell problems is given which is followed by providing an FE framework to solve them. Numerical results include the macroscale coefficients of cell problems with different geometries and material.

The FE implementation of macroscale system of equations is carried out in Chapter 4. This is based on the results of its previous chapter to show the effects of different underlying microscale properties on macroscale response of poroelastic media.

Chapters 5 and 6 deal with the computational analysis of a body with residually stressed hyperelastic materials. In the former chapter, state of the art of analytical solution for the circular cylindrical membrane is given. In Chapter 6, the numerical analysis of long thick-walled circular cylinders made of residually stressed hyperelastic material under different axial pre-stretch as well as inner pressure is carried out to capture their bifurcation and post-bifurcation.

Finally, the last chapter of this thesis is dedicated to the general summary as well as the future works.

CHAPTER 2

MULTISCALE AND MULTIPHYSICS ANALYSIS FOR POROELASTIC MEDIA: STATE OF THE ART

Asymptotic Homogenization for Poroelastic media

2.1 Introduction

Poroelasticity deals with the mechanical behavior of a solid elastic structure interplaying with fluid percolating its pores ([12–15]). It has wide range of applications including soil mechanics [155], (bio) artificial constructs [23, 63, 68], and biological tissues, such as bone [28], organs, healthy and malignant (tumorous) cell aggregates [16].

Poroelastic materials exhibit an intrinsically multiscale structure as the distance between the pores for an approximately uniform pore distribution (*the pore scale*), is typically much smaller than the size of the poroelastic medium (*the macroscale*).

Both the average field techniques or asymptotic homogenization can be exploited as the upscaling process that translates a pore-scale fluid-structure

interaction problem into a macroscale problem governed by the equations of poroelasticity, see, e.g., [31] and [59] for a comparison between these two alternative approaches in the context of fluid and solid mechanics, respectively.

The former approach is focused on the derivation of the macroscopic model as such and relies on suitable relationships between the microscale and macroscopic energy of the system at hand. Models deduced this way can be readily extended to a nonlinear constitutive behavior of the individual phases, however, the coefficients are typically not entirely related to the underlying microstructure (see, e.g. the analytic formulas relating drained and undrained coefficients for interconnected pores reported in [75, 155]), and are usually to be determined also exploiting experimental measurements. There also exist simplified micromechanical approaches that provide the poroelastic coefficients for specific geometries, for example when spherical, ellipsoidal, or "penny-shaped" diluted pores are considered, see, e.g. [25].

The asymptotic homogenization technique (see, e.g., [7, 9, 11, 58, 76, 105]) exploits the sharp length scale separation between a fine and a coarse scale to represent the fields in terms of power series of the ratio between them. The latter approach entails, in general, a higher degree of algebraic complexity and cannot be trivially generalized to nonlinear balance equations, however, it provides a precise prescription of the coefficients of the model. These encode information concerning the microstructure as they are provided in terms of pore-scale averages which involve both the properties of the individual phases and auxiliary variables which are to be computed solving differential problems on the pore-scale geometry. The latter is often assumed to be periodic to allow for actual computations of the coefficients on a small and definite portion of the microstructure.

In [18] the authors derive the standard Biot's system of partial differential equations (PDEs) via asymptotic homogenization and provide closed formulas for the coefficients of the model, as well as their associated pore-scale systems of PDEs. In order to compute the coefficients in practice, we also assume pore-scale periodicity, thus embracing the no-growth limit of [106] (which corresponds to the system of PDEs reported in [18] in non-frequency form) as a starting point.

In this chapter, the state of the art of Asymptotic Homogenization for poroelastic media is provided.

2.2 Upscaling Process

The formal derivation of the equation of poroelasticity obtained accounting explicitly for the porous microstructure has been reported for the first time

Figure 2.1: The microscale cel, on the left, is assumed to be periodic. Macroscale poroe-
lastic medium on the right.

in [18]. The process starts from a fluid-structure interaction problem between a linear elastic matrix and a Newtonian fluid phase at the pore-scale and closes the resulting system of partial differential equations by enforcing continuity of velocities and stresses across the fluid-solid interface. The asymptotic homogenization technique is then exploited to derive the effective governing equations for the medium at the macroscopic scale.

2.2.1 Asymptotic homogenization for poroelastic media

Assuming that the pore scale d and the macroscale L are well separated, microscale and macroscale spatial variations can be decoupled by a parameter as follows

$$\epsilon = \frac{d}{L} \ll 1. \quad (2.1)$$

$$\mathbf{y} = \frac{\mathbf{x}}{\epsilon}. \quad (2.2)$$

Where ϵ is a small scale separation parameter and $\mathbf{x}$ and $\mathbf{y}$ read as the two formally independent macroscale and microscale spatial variables, respectively. A simple poroelastic medium is shown in Figure 2.1.

The problem is then upscaled by following the typical steps of the asymptotic homogenization technique. Every relevant field is represented in power series of ϵ , and each component of the fluid velocity $\mathbf{v}$, pressure p and elastic displacement $\mathbf{u}$, as well as the elastic properties of the matrix, are assumed to be functions of both $\mathbf{x}$ and $\mathbf{y}$.

In physical scale we consider a set $\Omega \in \mathbb{R}^3$, such that $\Omega = \Omega_s \cup \Omega_f$, where Ω_s and Ω_f represent the porous solid and fluid compartment, respectively.

An incompressible Newtonian fluid is assumed so, neglecting body force and inertial effects, the fluid can be described as

$$\nabla \cdot \sigma = 0 \quad (2.3)$$

$$\sigma = -pl + \mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top \right) \quad (2.4)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.5)$$

where σ denotes the fluid stress tensor and $\mathbf{v}$, p , μ are the fluid velocity, pressure, and viscosity respectively.

The solid phase is assumed of the linear elastic type which, neglecting body force and inertia, in Ω_s can be described by

$$\nabla \cdot \tau = 0 \quad (2.6)$$

$$\tau = \mathbb{C} \mathbf{e}. \quad (2.7)$$

where $\mathbf{e}$ is the strain tensor of the solid phase such that

$$\begin{aligned} \mathbb{C} \nabla \mathbf{u} &= \mathbb{C} \mathbf{e}, \\ \mathbf{e} &= \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^\top}{2}. \end{aligned} \quad (2.8)$$

Here, $\mathbf{u}$ is the displacement vector for the solid compartment and $\mathbb{C}$ is the fourth rank elasticity tensor (with components denoted C_{ijkl} for $i, j, k, l = 1, 2, 3$)

Stress continuity as well as no-slip is assumed on the solid-fluid interface $\Gamma := \partial\Omega_s \cap \partial\Omega_f$ and described as

$$\tau \mathbf{n} = \sigma \mathbf{n} \quad \text{on } \Gamma \quad (2.9)$$

$$\dot{\mathbf{u}} = \mathbf{v} \quad \text{on } \Gamma \quad (2.10)$$

The system is non-dimensionalized with respect to appropriate characteristic quantities in terms of length scales and velocity fields. In particular, the characteristic (relative) fluid velocity is typically proportional to d^2 to reflect the fact that the (relative) fluid velocity inside the pores should (approximately) exhibit a parabolic profile, see, e.g. [106, 109, 114] for details concerning non-dimensionalization for porous media flow via asymptotic homogenization. This is indeed formally equivalent to the so-called “ ϵ^2 viscosity scaling”, which is the correct scaling to obtain Darcy’s and poroelasticity equations from the microstructure [18, 105, 106]. This means that the non-dimensionalized form of (2.4) can be written as

$$-pl + \epsilon^2 \mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top \right) = \sigma \quad \text{in } \Omega_f \quad (2.11)$$

According to scale separation, all fields, as well as elasticity tensor, are functions of $\mathbf{y}$ and $\mathbf{x}$. In Asymptotic homogenization, three basic techniques are exploited. First, the differential operators transform into two independent ones for macroscale and microscale as

$$\nabla \rightarrow \nabla_{\mathbf{x}} + \frac{1}{\epsilon} \nabla_{\mathbf{y}}. \quad (2.12)$$

Then, every field replaced by its power of series representation as below

$$\psi_{\epsilon}(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{\infty} \psi^{(l)}(\mathbf{x}, \mathbf{y}) \epsilon^l. \quad (2.13)$$

Finally, the integral average defined as

$$\langle \psi \rangle_k = \frac{1}{|\Omega|} \int_{\Omega_k} \psi(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \quad k = f, s, \quad (2.14)$$

In the following, we apply the mentioned techniques to the basic equations governing solid and fluid phases as well as their interface. From now on, (2.14) should be understood as a cell average, where $|\Omega|$ is replaced by the cell volume and Ω_k by the corresponding k -phase subvolume. Applying the transformation (2.12) the mentioned equations become:

$$\nabla_{\mathbf{y}} \cdot \sigma_{\epsilon} + \epsilon \nabla_{\mathbf{x}} \cdot \sigma_{\epsilon} = 0 \quad \text{in } \Omega_f \quad (2.15)$$

$$p_{\epsilon} \mathbf{I} - \epsilon \mu \left(\nabla_{\mathbf{y}} \mathbf{v}_{\epsilon} + (\nabla_{\mathbf{y}} \mathbf{v}_{\epsilon})^T \right) - \epsilon^2 \mu \left(\nabla_{\mathbf{x}} \mathbf{v}_{\epsilon} + (\nabla_{\mathbf{x}} \mathbf{v}_{\epsilon})^T \right) = \sigma_{\epsilon} \quad \text{in } \Omega_f \quad (2.16)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}_{\epsilon} + \epsilon \nabla_{\mathbf{x}} \cdot \mathbf{v}_{\epsilon} = 0 \quad \text{in } \Omega_f \quad (2.17)$$

$$\tau_{\epsilon} \mathbf{n} = \sigma_{\epsilon} \mathbf{n} \quad \text{on } \Gamma \quad (2.18)$$

$$\dot{\mathbf{u}}_{\epsilon} = \mathbf{v}_{\epsilon} \quad \text{on } \Gamma \quad (2.19)$$

$$\mathbf{u}_{\epsilon}^s = \mathbf{u}_{\epsilon}^f \quad \text{on } \Gamma \quad (2.20)$$

$$\nabla_{\mathbf{y}} \cdot \tau_{\epsilon} + \epsilon \nabla_{\mathbf{x}} \cdot \tau_{\epsilon} = 0 \quad \text{in } \Omega_s \quad (2.21)$$

$$\frac{1}{\epsilon} \mathbb{C} \nabla_{\mathbf{y}} \mathbf{u}_{\epsilon} + \mathbb{C} \nabla_{\mathbf{x}} \mathbf{u}_{\epsilon} = \tau_{\epsilon} \quad \text{in } \Omega_s \quad (2.22)$$

$$(2.23)$$

where the subscripts ϵ indicate the representation in the power series form.

This is then followed by equating the coefficients ϵ^0 and ϵ^1 by means of (2.13). The former case reads

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\sigma}^{(0)} = 0 \quad \text{in } \Omega_f \quad (2.24)$$

$$\sigma^{(0)} = -p^{(0)}\mathbf{I} \quad \text{in } \Omega_f \quad (2.25)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}^{(0)} = 0 \quad \text{in } \Omega_f \quad (2.26)$$

$$\tau^{(0)}\mathbf{n} = \sigma^{(0)}\mathbf{n} \quad \text{on } \Gamma \quad (2.27)$$

$$\dot{\mathbf{u}}^{(0)} = \mathbf{v}^{(0)} \quad \text{on } \Gamma \quad (2.28)$$

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\tau}^{(0)} = 0 \quad \text{in } \Omega_s \quad (2.29)$$

$$\mathbb{C}\nabla_{\mathbf{y}}\mathbf{u}^{(0)} = 0 \quad \text{in } \Omega_s \quad (2.30)$$

whereas equating coefficients of ϵ^1 in equations (2.15-2.22) yields

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\sigma}^{(1)} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}^{(0)} = 0 \quad \text{in } \Omega_f \quad (2.31)$$

$$-p^{(1)}\mathbf{I} + \mu \left(\nabla_{\mathbf{y}}\mathbf{v}^{(0)} + (\nabla_{\mathbf{y}}\mathbf{v}^{(0)})^T \right) = \boldsymbol{\sigma}^{(1)} \quad \text{in } \Omega_f \quad (2.32)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{v}^{(1)} + \nabla_{\mathbf{x}} \cdot \mathbf{v}^{(0)} = 0 \quad \text{in } \Omega_f \quad (2.33)$$

$$\tau^{(1)}\mathbf{n} = \boldsymbol{\sigma}^{(1)}\mathbf{n} \quad \text{on } \Gamma \quad (2.34)$$

$$\dot{\mathbf{u}}^{(1)} = \mathbf{v}^{(1)} \quad \text{on } \Gamma \quad (2.35)$$

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\tau}^{(1)} + \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}^{(0)} = 0 \quad \text{in } \Omega_s \quad (2.36)$$

$$\mathbb{C} \left(\nabla_{\mathbf{y}}\mathbf{u}^{(1)} + \nabla_{\mathbf{x}}\mathbf{u}^{(0)} \right) = \boldsymbol{\tau}^{(0)} \quad \text{in } \Omega_s. \quad (2.37)$$

Equations (2.24-2.25) infer that $p^{(0)}$ does not depend on the microscale $\mathbf{y}$, hence

$$p^{(0)} = p^{(0)}(\mathbf{x}). \quad (2.38)$$

Equation (2.30) implies, together with the elasticity tensor property (2.8), that $\mathbf{u}^{(0)}$ must correspond to a rigid body motion. Since the only $\mathbf{y}$ -periodic solution of this type is $\mathbf{y}$ -constant, we see that:

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x}). \quad (2.39)$$

Hence the leading order solid displacement field is also locally constant.

2.2.2 Macroscale system of equations

In this section we make use of the equations given in the previous section to obtain the governing system of PDEs in the macroscale domain $\Omega \subset \mathbb{R}^3$

2.2.2.1 Macroscale balance law and Cauchy stress

Taking the integral average of the sum of the (2.31) and (2.36) and applying Divergence theorem we have

$$\frac{1}{|\Omega|} \int_{\Gamma} (\sigma^{(1)} \mathbf{n} - \tau^{(1)} \mathbf{n}) dS_y = \langle \nabla_{\mathbf{x}} \cdot \tau^{(0)} \rangle_s - \phi \nabla_{\mathbf{x}} p^{(0)}, \quad (2.40)$$

where the contributions from the periodic boundaries and the knowledge that $p^{(0)}$ is locally constant has been exploited. Given the no-growth limit and normal stress condition on internal boundaries (2.34), (2.40) reduces to:

$$\langle \nabla_{\mathbf{x}} \cdot \tau^{(0)} \rangle_s - \phi \nabla_{\mathbf{x}} p^{(0)} = 0. \quad (2.41)$$

Equation (2.41) can be rewritten in terms of $\langle \tau^{(0)} \rangle_s$ by means of the Reynolds transport theorem and using $\mathbf{y}$ -periodicity, namely:

$$\int_{\Omega_s} \nabla_{\mathbf{x}} \cdot \tau^{(0)} d\mathbf{y} = \nabla_{\mathbf{x}} \cdot \int_{\Omega_s} \tau^{(0)} d\mathbf{y} \quad (2.42)$$

The *porosity* (that is, the interstitial volume fraction) is given by

$$\phi = \frac{|\Omega_f|}{|\Omega_c|}. \quad (2.43)$$

Using (2.41) together with (2.42) we obtain:

$$\nabla_{\mathbf{x}} \cdot \langle \tau^{(0)} \rangle_s - \phi \nabla_{\mathbf{x}} p^{(0)} = 0, \quad (2.44)$$

Defining the tensor τ_E as the effective Cauchy stress tensor of macroscale as

$$\tau_E := \langle \tau^{(0)} \rangle_s - \phi p^{(0)} \mathbf{I}, \quad (2.45)$$

together with (2.44) the balance of linear momentum for macroscale is obtained as

$$\nabla \cdot \tau_E = 0 \quad (2.46)$$

Next, we adopt appropriate Ansatz as below

$$\mathbf{u}^{(1)} = \mathcal{A} \nabla_{\mathbf{x}} \mathbf{u}^{(0)} + \mathbf{a} p^{(0)}. \quad (2.47)$$

Substituting (2.47) into (2.37) we have

$$\tau^{(0)} = (\mathbb{C}\mathbb{M} + \mathbb{C}) \nabla_{\mathbf{x}} \mathbf{u}^{(0)} + \mathbb{C} \mathbf{Q} p^{(0)}, \quad (2.48)$$

where the fourth rank tensor $\mathbb{M}$ and the second rank tensor $\mathbf{Q}$ are defined as follows:

$$\mathbb{M} = \xi_{\mathbf{y}} \mathcal{A}, \quad \mathbf{Q} = \xi_{\mathbf{y}} \mathbf{a}. \quad (2.49)$$

where

$$\xi_{\mathbf{y}}(\bullet) = \frac{1}{2} \left(\nabla_{\mathbf{y}}(\bullet) + \nabla_{\mathbf{y}}(\bullet)^{\top} \right). \quad (2.50)$$

Then exploiting (2.45) we can write the macroscale constitutive law for effective Cauchy stress as

$$\tau_E = \langle \mathbb{C} \mathbb{M} + \mathbb{C} \rangle_s \nabla_{\mathbf{x}} \mathbf{u}^{(0)} + (\langle \mathbb{C} \mathbf{Q} \rangle_s - \phi \mathbf{I}) p^{(0)}. \quad (2.51)$$

2.2.2.2 Macroscale mass conservation and fluid flow

The relative fluid-solid displacement $\mathbf{w}$ defined by:

$$\mathbf{w}(\mathbf{x}, \mathbf{y}) = \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}) - \dot{\mathbf{u}}^{(0)}(\mathbf{x}), \quad (2.52)$$

By taking the average integral of the first order incompressibility condition for the fluid phase (2.33) we have

$$\langle \nabla_{\mathbf{y}} \cdot \mathbf{v}^{(1)} \rangle_f + \langle \nabla_{\mathbf{x}} \cdot \mathbf{v}^{(0)} \rangle_f = 0. \quad (2.53)$$

Applying the divergence theorem in the $\mathbf{y}$ variable, $\mathbf{y}$ -periodicity together with no-growth limit and the Reynolds theorem in $\mathbf{x}$ gives:

$$\frac{1}{|\Omega|} \int_{\Gamma} \dot{\mathbf{u}}^{(1)} \cdot \mathbf{n} \, dS_y + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}^{(0)} \rangle_f = 0 \quad (2.54)$$

Since

$$\int_{\Gamma} \dot{\mathbf{u}}^{(1)} \cdot \mathbf{n} \, dS_y = - \int_{\Omega_s} \nabla_{\mathbf{y}} \cdot \dot{\mathbf{u}}^{(1)} \, d\mathbf{y} = - \int_{\Omega_s} \text{Tr}(\nabla_{\mathbf{y}} \dot{\mathbf{u}}^{(1)}) \, d\mathbf{y}, \quad (2.55)$$

equation (2.54) can be rewritten by means of (2.47) accounting for (2.38, 2.39), as follows:

$$\nabla_{\mathbf{x}} \cdot \langle \mathbf{v}^{(0)} \rangle_f = \langle \text{Tr} \mathbb{M} \rangle_s : \nabla_{\mathbf{x}} \dot{\mathbf{u}}^{(0)} + \langle \text{Tr} \mathbf{Q} \rangle_s \dot{p}^{(0)} \quad (2.56)$$

Moreover, we assume that the governing constitutive equation for fluid phase is Darcy's law expressed as

$$\mathbf{w} = -\mathbf{K} \nabla_{\mathbf{x}} p^{(0)} \quad (2.57)$$

The relative fluid velocity can then also be expressed as

$$\mathbf{w} = \langle \mathbf{v}^{(0)} \rangle_f - \phi \dot{\mathbf{u}}^{(0)}, \quad (2.58)$$

Equations (2.46), (2.51), (2.56), and (2.58) form a closed differential problem for the leading (zeroth) order components, which represents the *homogenized* limit for ϵ approaching zero. In the next chapter, we make use of these equations to compute the coefficients of the macroscale system of equations.

CHAPTER 3

COMPUTATIONAL STUDY OF POROELASTIC MEDIA

This chapter deals with a material governed by the equations of poroelasticity, as described in the previous chapter, derived from the equations of asymptotic homogenization technique. The main objective is to investigate the impact of the interstitial fluid volume fraction (*porosity*) and compressibility of the matrix, which is assumed to behave as a compressible linear and isotropic elastic solid, on the coefficients of macroscale system of PDEs that describes the overall mechanical response of the material. First, the mathematical machinery to obtain systems of PDEs in the cell subvolumes (solid matrix and fluid phase) is given and solved numerically.

A parametric analysis via finite element numerical simulations is carried out to compute the solution of the mentioned pore-scale cell problems (which are of Stokes' and linear elastic type) in three-dimensions for various different values of porosity and Poisson's ratio of the elastic matrix. The microscale geometry comprises an interstitial phase made of three interconnected cylinders, which is invariant with respect to permutation of the three orthogonal axes. The mentioned rotational invariance reduces the number of independent scalar parameters which can fully characterize the macroscale mechanical and hydraulic behavior of the medium. As a consequence, it is sufficient to investigate the profile of six independent scalar pa-

rameters, namely Young's modulus, shear modulus, Poisson's ratio, Biot's modulus, Biot's coefficient, and hydraulic conductivity.

Although there are several applicable physical systems to this analysis the analysis of deformable malignant cell aggregates (i.e. tumors) is the main focus of this research work. A tumor mass can be viewed as a multiscale deformable system, which is constructed of several different constituents (such as collagen fibers and proteins forming the extracellular matrix) and cells which collectively form the so-called interstitial matrix ([71, 102]). This matrix plays the role of a fluid (and hence drug) transport barrier which reduces the interstitial fluid volume portion accessible for anti-cancer molecules which is of considerable significance. Moreover, it exhibits deformations which are supposed to play a role in the spatiotemporal distribution of the interstitial fluid volume and pressure which in turn drive drug transport in malignant tissues, as highlighted in the analysis [102].

Although it is well-known that the biological tissues, in particular, porous structure of tumors, affects the observed blood and drug flow, experimental data are typically fitted by using mathematical models (for example of poroelastic and viscoelastic type, see e.g. [101] and [102], respectively) which can be reasonably applicable on the macroscale only. However, the dependence on the underlying microstructure is not explicitly encoded. Furthermore, there only exists a few measurements related to the interstitial volume fraction ([69]), which exhibits a large variability depending on the specific tissue type and tumor region.

Although the nature of tumor (i.e. malignant vs benign) has considerable effects on its the hydraulic and mechanical properties, especially in terms of compressibility [24], a tumor system is often considered as made of intrinsically incompressible phases in order to reduce the number of parameters to be examined, as done for example in [101] among many others works.

These arguments motivate the development of a computational analysis that can highlight the role of porosity and solid matrix compressibility on the mechanical and hydraulic properties of poroelastic materials framed in the context of solid tumors. Nonetheless, the framework as such can also be used to reach target mechanical and/or transport properties of artificial poroelastic constructs with respect to the choice of their parameters.

The rest of this chapter is organized as follows. The poroelastic governing equations at the macroscale together with the definitions of the parameters in terms of the relevant auxiliary variables to be computed by solving pore scale cell problems are provided in Section 3.1. In Section 3.2, the cell problems which are solved to find the auxiliary variables that are needed to compute the macroscale coefficients of the model are presented and discussed. In Section 3.4, the results are presented, which are obtained by

solving the cell problems described in Section 3.2 by finite element analysis.

3.1 Macroscale system of PDEs

In the previous chapter, the governing system of PDEs in the macroscale domain $\Omega \subset \mathbb{R}^3$ are obtained by exploiting the asymptotic homogenization. The coefficients of this system of equations are defined as

$$\mathbf{K} = \langle \mathbf{W} \rangle_f, \quad \tilde{\alpha} = \phi \mathbf{I} - \text{Tr} \langle \mathbb{M} \rangle_s, \quad \tilde{\mathbb{C}} = \langle \mathbb{C} + \mathbb{C}\mathbb{M} \rangle_s, \quad M = -\frac{1}{\langle \text{Tr} \mathbf{Q} \rangle_s}. \quad (3.1)$$

Consequently, the macroscale system of PDEs can be rewritten as

$$\begin{cases} \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}_E = 0, & (3.2) \\ \boldsymbol{\tau}_E =: \tilde{\mathbb{C}} \nabla_{\mathbf{x}} \mathbf{u}^{(0)} - \tilde{\alpha} p^{(0)} & (3.3) \\ \dot{p}^{(0)} = -M [\tilde{\alpha} : \nabla_{\mathbf{x}} \dot{\mathbf{u}}^{(0)} + \nabla_{\mathbf{x}} \cdot \mathbf{w}] & (3.4) \\ \mathbf{w} = -\mathbf{K} \nabla_{\mathbf{x}} p^{(0)}. & (3.5) \end{cases}$$

It worth to mention again that $p^{(0)}$, $\mathbf{u}^{(0)}$, $\dot{\mathbf{u}}^{(0)}$, and $\mathbf{w}$ are the macroscale pressure, solid displacement, solid velocity, and average fluid velocity (relative to the solid displacement). In [106] the authors have reported these equations using similar notation that, in general, they are derived in [18]. Making a suitable notation these equations are shown to coincide with the Biot's equations of poroelasticity.

The effective stress balance equation for a poroelastic medium is (3.2) that is characterized by the constitutive relationship (3.3). The latter consists of two terms which account for the effective response of the linear elastic phase and the interstitial pore pressure. Conservation of mass for a poroelastic medium is represented by Equation (3.4) which relates variations of the interstitial fluid pressure to fluid and solid volume changes. Darcy's law represents suitable last equation of the macroscale system of PDEs (3.5) for the relative fluid velocity $\mathbf{w}$.

To fully characterize the mechanical behavior of the poroelastic media one need to compute the effective elasticity tensor $\tilde{\mathbb{C}}$, the hydraulic conductivity $\mathbf{K}$, Biot's coefficient tensor $\tilde{\alpha}$, and Biot's modulus M . As a consequence of $\mathbf{y}$ -periodicity and according to [18], the above-mentioned coefficients can be expressed in terms of suitable averages over the whole microscale domain, however, suitable regularity assumptions are to be embraced in order to compute the coefficients in practice. The microscale periodicity is the typical and traditional approach of the asymptotic homogenization literature which is shown graphically in Figure 3.1.

Figure 3.1: The color map of displacements showing the periodic solution of one cell problem in Abaqus.

The fourth rank tensor $\mathbb{M}$ and the second rank tensors $\mathbf{Q}$ and $\mathbf{W}$ are to be computed by solving the pore scale periodic cell problems illustrated in the next section.

3.2 Mathematical description of the cell problems

In this section mathematical description of the cell problems to be solved in order to determine the parameters in (3.1) is given. The adopted cell geometry with its different phases are shown in Figure 3.2 where the domain is mentioned again.

3.2.1 The cell problems in the fluid domain

Here, the equations from the upscaling process that are required to achieve the cell problem in the fluid domain are given. The basic equations to reach the PDEs to be solved in Ω_f (in order to determine Hydraulic conductivity) are

Figure 3.2: A cell (Ω_c) and its corresponding subdomains (fluid and solid Ω_f and Ω_s , respectively). Highlighted in red is the interface Γ .

$$\sigma^{(0)} = -p^{(0)}\mathbf{I} \quad \text{in } \Omega_f \quad (3.6)$$

$$\dot{\mathbf{u}}^{(0)} = \mathbf{v}^{(0)} \quad \text{on } \Gamma \quad (3.7)$$

$$\dot{\mathbf{w}}(\mathbf{x}, \mathbf{y}) := \mathbf{v}^{(0)}(\mathbf{x}, \mathbf{y}) - \dot{\mathbf{u}}^{(0)}(\mathbf{x}) \quad (3.8)$$

$$\nabla_{\mathbf{y}} \cdot \sigma^{(1)} + \nabla_{\mathbf{x}} \cdot \sigma^{(0)} = 0 \quad \text{in } \Omega_f \quad (3.9)$$

$$-p^{(1)}\mathbf{I} + \mu \left(\nabla_{\mathbf{y}} \mathbf{v}^{(0)} + (\nabla_{\mathbf{y}} \mathbf{v}^{(0)})^T \right) = \sigma^{(1)} \quad \text{in } \Omega_f \quad (3.10)$$

In order to obtain hydraulic conductivity, the information on the leading order velocity of the fluid $\mathbf{v}^{(0)}$ is needed. Enforcing (3.6), equations (3.7, 3.9, and 3.10) form a Stokes-type boundary value problem for $(\mathbf{v}^{(0)}, p^{(1)})$ which can be rewritten in terms of the relative fluid-solid displacement $\mathbf{w}$ defined by (3.8) as:

$$\mu \nabla_{\mathbf{y}}^2 \dot{\mathbf{w}} - \nabla_{\mathbf{y}} p^{(1)} - \nabla_{\mathbf{x}} p^{(0)} = 0 \quad \text{in } \Omega_f \quad (3.11)$$

$$\nabla_{\mathbf{y}} \cdot \dot{\mathbf{w}} = 0 \quad \text{in } \Omega_f \quad (3.12)$$

$$\dot{\mathbf{w}} = 0 \quad \text{on } \Gamma. \quad (3.13)$$

where the knowledge that $\mathbf{u}^{(0)}$ is constant in microscale has been used. From assumed linearity together with microscale independency of p^0 the following ansatz for the solution $(\dot{\mathbf{w}}, p^{(1)})$ is adopted

$$\dot{\mathbf{w}} = -\mathbf{W}\nabla_{\mathbf{x}}p^{(0)}, \quad (3.14)$$

$$p^{(1)} = -\mathbf{P} \cdot \nabla_{\mathbf{x}}p^{(0)}, \quad (3.15)$$

where $p^{(1)}$ is defined up to an arbitrary $\mathbf{y}$ -constant function and the quantities $(\mathbf{W}, \mathbf{P})$, $(\mathbf{h}, p_h)$ satisfy the following differential *cell problems*, to be solved on the single representative cell

$$\nabla_{\mathbf{y}}^2 \mathbf{W}^T - \nabla_{\mathbf{y}} \mathbf{P} + \mathbf{I} = 0 \quad \text{in } \Omega_f \quad (3.16)$$

$$\nabla_{\mathbf{y}} \cdot \mathbf{W}^T = 0 \quad \text{in } \Omega_f \quad (3.17)$$

$$\mathbf{W} = 0 \quad \text{on } \Gamma. \quad (3.18)$$

Where $\mathbf{P}$ is an auxiliary vector holding microscale information that relates p^1 to $\nabla_{\mathbf{x}}p^{(0)}$, $\mathbf{I}$ is the identity tensor and the (non-dimensional) viscosity of the fluid is set to 1. The second rank tensor $\mathbf{W}$ can be determined by solving the mentioned auxiliary Stokes'-type cell problem on the microscale. The problem is equipped with homogeneous Dirichlet conditions on the interface Γ together with periodicity conditions on $\partial\Omega_f/\Gamma$ and a suitable uniqueness condition for the vector $\mathbf{P}$, e.g.

$$\langle \mathbf{P} \rangle_f = \mathbf{0} \quad (3.19)$$

without loss of generality.

The system of equations 3.16-3.18 can be rewritten componentwise as follows

$$\mu \frac{\partial W_{ji}}{\partial y_k \partial y_k} - \frac{\partial P_i}{\partial y_j} + \delta_{ij} = 0 \quad \text{in } \Omega_f \quad (3.20)$$

$$\frac{\partial W_{ji}}{\partial y_j} = 0 \quad \text{in } \Omega_f \quad (3.21)$$

$$W_{ij} = 0 \quad \text{on } \Gamma, \quad (3.22)$$

i.e. the problem (3.16-3.18) corresponds to three Stokes' problems for $i = 1, 2, 3$. However, since a geometry with rotational invariance with respect to the three orthogonal axes is chosen, the solution of this problem can be obtained by solving, for example, by $i = 1$ only, i.e.

$$\nabla_{\mathbf{y}}^2 \tilde{\mathbf{v}} - \nabla_{\mathbf{y}} \tilde{p} + \mathbf{e}_1 = 0 \quad \text{in } \Omega_f \quad (3.23)$$

$$\nabla_{\mathbf{y}} \cdot \tilde{\mathbf{v}} = 0 \quad \text{in } \Omega_f \quad (3.24)$$

$$\tilde{\mathbf{v}} = 0 \quad \text{on } \Gamma, \quad (3.25)$$

where $\tilde{p} = P_1$. The above problem formally reads as a periodic Stokes' problem for an incompressible fluid driven by a unit body force directed

along e_1 and the solution $\tilde{v}$ is related to the components of the tensor W by means of the following identifications

$$\tilde{v}_1 = W_{11} = W_{22} = W_{33} \quad (3.26)$$

$$\tilde{v}_2 = W_{21} = W_{32} = W_{13} \quad (3.27)$$

$$\tilde{v}_3 = W_{31} = W_{12} = W_{23}, \quad (3.28)$$

In the following, the cell problems to be solved in the solid domain are presented.

3.2.2 The cell problems in the solid domain

The effective elasticity tensor, Biot's coefficient, and Biot's modulus can be determined from the solution of the solid problems. The basic equations from asymptotic homogenization (given also in the previous chapter) to reach the appropriate cell problems are

$$\tau^{(0)} \mathbf{n} = \sigma^{(0)} \mathbf{n} \quad \text{on } \Gamma \quad (3.29)$$

$$\nabla_{\mathbf{y}} \cdot \tau^{(0)} = 0 \quad \text{in } \Omega_s \quad (3.30)$$

$$\mathbb{C} (\nabla_{\mathbf{y}} \mathbf{u}^{(1)} + \nabla_{\mathbf{x}} \mathbf{u}^{(0)}) = \tau^{(0)} \quad \text{in } \Omega_s \quad (3.31)$$

$$\sigma^{(0)} = -p^{(0)} \mathbf{I} \quad \text{in } \Omega_f \quad (3.32)$$

These equations form a linear Neumann-type differential problem for $\mathbf{u}^{(1)}$ which can be written, enforcing microscale independency of $p^{(0)}$ and $\mathbf{u}^{(0)}$ ($p^{(0)} = p^{(0)}(\mathbf{x})$ and $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(\mathbf{x})$), as follows:

$$\nabla_{\mathbf{y}} \cdot (\mathbb{C} \nabla_{\mathbf{y}} \mathbf{u}^{(1)}) = 0 \quad \text{in } \Omega_s, \quad (3.33)$$

$$(\mathbb{C} \nabla_{\mathbf{y}} \mathbf{u}^{(1)} + \mathbb{C} \nabla_{\mathbf{x}} \mathbf{u}^{(0)}) \mathbf{n} = -p^{(0)} \mathbf{n} \quad \text{on } \Gamma. \quad (3.34)$$

Mentioning the appropriate Ansatz as well as the definition of the fourth rank tensor $\mathbb{M}$ and the second rank tensor $\mathbf{Q}$ again

$$\mathbf{u}^{(1)} = \mathcal{A} \nabla_{\mathbf{x}} \mathbf{u}^{(0)} + \mathbf{a} p^{(0)}, \quad (3.35)$$

$$\mathbb{M} = \xi_{\mathbf{y}} \mathcal{A}, \quad \mathbf{Q} = \xi_{\mathbf{y}} \mathbf{a}, \quad (3.36)$$

where

$$\xi_{\mathbf{y}}(\bullet) = \frac{1}{2} \left(\nabla_{\mathbf{y}}(\bullet) + \nabla_{\mathbf{y}}(\bullet)^{\top} \right), \quad (3.37)$$

equations (3.33) and (3.34) can be expanded to the form two system of PDEs as follows

$$\nabla_{\mathbf{y}} \cdot (\mathbb{C}\mathbb{M}) = 0 \quad \text{in } \Omega_s \quad (3.38)$$

$$(\mathbb{C}\mathbb{M})\mathbf{n} + \mathbb{C}\mathbf{n} = 0 \quad \text{on } \Gamma \quad (3.39)$$

$$\langle \mathcal{A} \rangle_s = 0, \quad (3.40)$$

and

$$\nabla_{\mathbf{y}} \cdot (\mathbb{C}\mathbf{Q}) = 0 \quad \text{in } \Omega_s \quad (3.41)$$

$$(\mathbb{C}\mathbf{Q})\mathbf{n} + \mathbf{n} = 0 \quad \text{on } \Gamma. \quad (3.42)$$

where $\mathbf{n}$ is the unit outward vector normal to Γ . The problem is equipped with periodic conditions on $\partial\Omega_s/\Gamma$ and a further condition, for example of the type (3.40), ensures uniqueness of the solution.

The system of equations ((3.38)-(3.40)) is rewritten componentwise as follows

$$\frac{\partial}{\partial y_j} (C_{ijkl} M_{lk\nu\gamma}) = 0 \quad \text{in } \Omega_s \quad (3.43)$$

$$C_{ijkl} M_{lk\nu\gamma} n_j = -C_{ij\nu\gamma} n_j \quad \text{on } \Gamma \quad (3.44)$$

$$\langle A_{ijk} \rangle_s = 0 \quad \forall i, j, k = 1 \dots 3. \quad (3.45)$$

The solution of the problem $M_{lk\nu\gamma}$ can be obtained by solving six elastic-type cell problems by fixing the couple of indices (ν, γ) . This way, $M_{lk\nu\gamma}$ in (3.43) formally represents a strain and for each fixed couple (ν, γ) we have a linear elastic problem equipped with inhomogeneous Neumann interface conditions which can be rewritten componentwise as

$$\nu\gamma = 11: C_{ijkl} M_{lk11} n_j = -C_{ij11} n_j \quad \text{where } -C_{ij11} n_j = f_i \quad (3.46)$$

$$\nu\gamma = 22: C_{ijkl} M_{lk22} n_j = -C_{ij22} n_j \quad \text{where } -C_{ij22} n_j = f_i \quad (3.47)$$

$$\nu\gamma = 33: C_{ijkl} M_{lk33} n_j = -C_{ij33} n_j \quad \text{where } -C_{ij33} n_j = f_i \quad (3.48)$$

$$\nu\gamma = 23: C_{ijkl} M_{lk23} n_j = -C_{ij23} n_j \quad \text{where } -C_{ij23} n_j = -C_{ij32} n_j = f_i \quad (3.49)$$

$$\nu\gamma = 13: C_{ijkl} M_{lk13} n_j = -C_{ij13} n_j \quad \text{where } -C_{ij13} n_j = -C_{ij31} n_j = f_i \quad (3.50)$$

$$\nu\gamma = 12: C_{ijkl} M_{lk12} n_j = -C_{ij12} n_j \quad \text{where } -C_{ij12} n_j = -C_{ij21} n_j = f_i \quad (3.51)$$

Assuming that the elastic matrix is isotropic at the pore scale we have

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (3.52)$$

and substituting the relationship (3.52) into the interface loads ((3.46)-(3.51)) we obtain

$$\nu\gamma = 11: \mathbf{f} = \lambda\mathbf{n} + 2\mu n_1\mathbf{e}_1 \quad (3.53)$$

$$\nu\gamma = 22: \mathbf{f} = \lambda\mathbf{n} + 2\mu n_2\mathbf{e}_2 \quad (3.54)$$

$$\nu\gamma = 33: \mathbf{f} = \lambda\mathbf{n} + 2\mu n_3\mathbf{e}_3 \quad (3.55)$$

$$\nu\gamma = 23: \mathbf{f} = \mu(n_3\mathbf{e}_2 + n_2\mathbf{e}_3) \quad (3.56)$$

$$\nu\gamma = 13: \mathbf{f} = \mu(n_3\mathbf{e}_1 + n_1\mathbf{e}_3) \quad (3.57)$$

$$\nu\gamma = 12: \mathbf{f} = \mu(n_2\mathbf{e}_1 + n_1\mathbf{e}_2) \quad (3.58)$$

Where n_1, n_2, n_3 are the components of the unit vector normal to the interface Γ and $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the standard unit vectors in the Cartesian coordinate system.

As previously, the cell problem (3.41) – (3.42) formally reads as a linear elastic problem equipped with inhomogeneous Neumann interface conditions on Γ and periodic conditions on $\partial\Omega_s \setminus \Gamma$, and an additional condition on $\mathbf{a}$ is required to ensure uniqueness of the solution, e.g.

$$\langle \mathbf{a} \rangle_s = 0. \quad (3.59)$$

The interface load in this problem is directly proportional to the interface normal $\mathbf{n}$.

In the next section, the coefficients of the macroscale system of equations based on the solutions of the mentioned cell problems are presented.

3.3 Coefficients of macroscale system of PDEs

This analysis is performed in the non-dimensional form so that the cell problems for the auxiliary variables $\mathbb{M}$, $\mathbb{Q}$, and $\mathbb{W}$ are to be solved in fluid and solid subsets of a cubic unit cell Ω_c (with $\phi = |\Omega_f|$). A cell consists of the fluid subdomain Ω_f , which is constructed by merging three cross-shaped cylindrical structure representing a fully interconnected porous media, and the elastic matrix host domain $\Omega_s = \Omega_c \setminus \Omega_f$. This is the simplest possible geometry that accounts for three-dimensional flow in a saturated porous media and also ensures that the number of macroscale parameters is kept to a minimum. Figure 3.2 shows the geometry of the cell as well as the different phases depicted above, namely the solid, fluid, and their interface $\Gamma = \partial\Omega_s \cap \partial\Omega_f$. Figure 3.1 shows a typical computational output related to the solution of a periodic, linear-elastic type problem.

Although the equations of poroelasticity are derived (starting from the microstructure) assuming continuity of velocity and stresses, in the resultant macroscopic system of partial differential equations the interface between phases is smoothed out (this is the case when applying asymptotic homogenization to the general system of PDEs describing multiphase problems). The role of the interface between the solid and the fluid phase is encoded in the relevant cell problems, where the latter actually read, formally, as purely (auxiliary) fluid and solid decoupled problems [18, 76].

Next, we introduce the coefficients and the relevant discussion.

3.3.1 Hydraulic conductivity tensor

From a numerical viewpoint, as we have had no access to *Abaqus* fluid dynamics modules, we have solved the problem for a linear, isotropic, and incompressible elastic solid which is equivalent to the incompressible Stokes' problem for the elastic displacement. Solved by the same unit body force, whose solution coincides with the problem (3.23-3.25). The velocity is identified with the resulting displacement and the viscosity (which is set to 1) with the shear modulus. The dimensional hydraulic conductivity K_{dim} can be obtained by multiplying the result by d^2/μ , where μ is an appropriate (dimensional) fluid viscosity.

see also [108] where the same argument is used to compute the hydraulic conductivity tensor associated with the capillary compartment of vascularized tumors. This way the full tensor $\mathbb{W}$ and therefore the hydraulic conductivity $\mathbb{K} = \langle \mathbb{W} \rangle_f$ can be determined by means of (3.1).

3.3.2 Effective elasticity tensor, Biot's coefficient, and Biot's modulus

The fourth rank tensor $\tilde{\mathbb{C}}$ represents the effective elasticity tensor for a drained poroelastic medium (i.e. it fully characterizes the effective mechanical response when setting the interstitial pore pressure $p^{(0)} = 0$, cf. (3.3)) and, according to relationships (3.1), is related to both the elasticity tensor of the matrix $\mathbb{C}$ and the auxiliary tensor $\mathbb{M}$.

The third rank tensor $\mathcal{A}$ and consequently $\mathbb{M}$ and $\tilde{\mathbb{C}}$ can be determined by solving the 6 linear elastic type cell problems derived from Equations (3.38)-(3.40).

As the input parameters of the mentioned problems are the matrix's Young's modulus E and Poisson's ratio ν , the standard relationships between the latter and the Lamé constants are reminded below

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}; \quad \mu = \frac{E}{2(1+\nu)}. \quad (3.60)$$

For an isotropic linear elastic problem, Voigt notation can be used to reduce the 81 components of the fourth rank auxiliary tensor $\mathbb{M}$ and the effective elasticity tensor $\tilde{\mathbb{C}}$ to 36 and express them in the form of $C_{\alpha\beta}$ and $M_{\beta\kappa}$, where α, κ , and $\beta = 1, 2, 3, \dots, 6$. As the solid matrix is assumed to be isotropic and from the invariance properties of the geometry, both $\mathbb{M}$ and $\tilde{\mathbb{C}}$ are expected to have cubic symmetry. An example of the resulting cubic symmetric tensors (for porosity $\phi = 0.286$, Poisson's ratio $\nu=0.35$ and non-dimensional $E = 13.5$) is reported below.

$$\mathbb{M} = \begin{pmatrix} -0.35 & -0.094 & -0.094 & 0 & 0 & 0 \\ -0.094 & -0.35 & -0.094 & 0 & 0 & 0 \\ -0.094 & -0.094 & -0.35 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.167 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.167 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.167 \end{pmatrix} \quad (3.61)$$

$$\tilde{\mathbb{C}} = \begin{pmatrix} 8.59 & 3.23 & 3.23 & 0 & 0 & 0 \\ 3.23 & 8.59 & 3.23 & 0 & 0 & 0 \\ 3.23 & 3.23 & 8.59 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.97 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.97 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.97 \end{pmatrix} \quad (3.62)$$

In fact, the analysis of the elastic moduli in terms of the effective Young's modulus, Poisson's ratio and shear modulus E_p , ν_p , and μ_p are conducted,

respectively. They are related to the independent components $\tilde{C}_{11}$, $\tilde{C}_{12}$, $\tilde{C}_{44}$ via the following relationships, see, e.g. [107]

$$E_p = \frac{\tilde{C}_{11}(\tilde{C}_{11} + \tilde{C}_{12}) - 2\tilde{C}_{12}^2}{\tilde{C}_{11} + \tilde{C}_{12}} \quad (3.63)$$

$$\nu_p = \frac{\tilde{C}_{12}}{\tilde{C}_{11} + \tilde{C}_{12}} \quad (3.64)$$

$$\mu_p = \tilde{C}_{44}. \quad (3.65)$$

Finally, the Biot's tensor given by relationship (3.1) is actually diagonal as a consequence of cubic symmetry of $\mathbb{M}$. We have

$$\text{Tr } \mathbb{M} = M_{ijkl}\delta_{lk} = (M_{1111} + 2M_{1122})\delta_{ij}, \quad (3.66)$$

so that the Biot's tensor reads

$$\tilde{\alpha} = \alpha \mathbf{I}, \quad (3.67)$$

with scalar Biot's coefficients given by (cf. (3.1)):

$$\alpha = \phi - \langle M_{11} + 2M_{12} \rangle_s, \quad (3.68)$$

where the Voigt notation has been used.

The Biot modulus is defined, according to (3.1), as

$$M = -\frac{1}{\langle \text{Tr } \mathbf{Q} \rangle_s}, \quad (3.69)$$

The scalar quantity $\text{Tr } \mathbf{Q} = \nabla \cdot \mathbf{a}$ actually represents the solid volume variation of an elastic solid (characterized by the elastic response of the matrix given by the stiffness tensor $\mathbb{C}$) subjected to such a load on the interface Γ and periodic boundary condition on $\partial\Omega_s/\Gamma$.

In the next Section, the results that are based on the numerical solutions of the cell problems are presented. Numerical simulations are performed using the commercial software Abaqus, see also [45, 149] for other examples of poroelastic finite element modeling using this software.

3.4 Results

The results of effective hydraulic conductivity, poroelastic parameters (Biot's modulus and Biot's coefficient), elastic moduli (Young's and shear modulus) and Poisson's ratio are presented. The cell problems presented in the previous section are solved via finite elements. In the Appendix, the sensitivity analysis is carried out in order to verify the accuracy of the results.

The analysis is performed in non-dimensional form, therefore, the assumed cell is a cubic unit cell Ω_c with its subdomains i.e. Ω_f and Ω_s for fluid and solid phase, respectively. The dimensional counterpart of any parameter, having the physical dimension of stress ([Pa]), such as Young's modulus, shear modulus, and Biot's modulus, can be obtained by multiplying the result by a characteristic value (for example a representative pressure as done in [106]). The hydraulic conductivity ($[m^2/(\text{Pa} \cdot s)]$) can be obtained by multiplying the non-dimensional result by d^2/μ , where d is the reference microscale length and μ the dimensional viscosity of the microscale fluid. Biot's coefficient and Poisson's ratio are non-dimensional. The elastic moduli are computed for the drained case (i.e. by using directly the effective stress tensor $\tilde{\mathbb{C}}$) as well as for the undrained case. Considering the elastic response for a static fluid phase filling the pores one can write

$$p^{(0)} = -M[\tilde{\alpha} : \nabla_{\mathbf{x}} \mathbf{u}^{(0)} + \nabla_{\mathbf{x}} \cdot \langle \mathbf{W} \rangle_f], \quad (3.70)$$

where $\mathbf{u}^{(0)}$ and $\langle \mathbf{W} \rangle_f$ are the solid and fluid displacements.

By using (3.70), (3.3) can be written in the form of

$$\tau_E := (\tilde{\mathbb{C}} + M\tilde{\alpha}\tilde{\alpha})\nabla_{\mathbf{x}} \mathbf{u}^{(0)} + (M\tilde{\alpha})\nabla_{\mathbf{x}} \cdot \langle \mathbf{W} \rangle_f. \quad (3.71)$$

The fourth rank tensor

$$\tilde{\mathbb{C}} + M\tilde{\alpha}\tilde{\alpha} \quad (3.72)$$

indicates the undrained elasticity tensor of the porous medium. As in this case $\tilde{\alpha}$ is diagonal, Biot's modulus and coefficient in 3.72 only affects the effective Young's modulus and Poisson's ratio, as in the case of a macroscopically isotropic poroelastic medium described in [18]. The input parameters are the (non-dimensional) Young's modulus and Poisson's ratio of the matrix E and ν , respectively, and the porosity ϕ . As the focus is on the role of porosity and solid matrix compressibility on the effective hydraulic and mechanical properties of poroelastic materials, Young's modulus of the matrix is fixed to a reference value, which is set to 13.5. The latter non-dimensional value corresponds to a dimensional value of 202.5 mmHg, i.e. $\approx 2 \cdot 10^{-3}$ Pa (taken from the Lamé constants reported in [16] for poroelastic tumors), when a dimensional reference (interstitial) pressure of 15 mmHg (see, e.g. [62]) is adopted.

The parametric analysis is performed in terms of the porosity and Poisson's ratio of the matrix. In particular, the radius of the three interconnected cylinders is varied from 0.1 to 0.4 with steps of 0.01, which corresponds to a porosity range $\phi \in [0.083 \ 0.784]$. The profile of every elastic and poroelastic parameter is also evaluated with respect to four different Poisson's ratios, namely, 0.35, 0.4, 0.45, and 0.49, to study both the role of tumor compressibility recently reported in [145], and also account for the incompressible

Figure 3.3: Normalized and Non-dimensionalized hydraulic conductivity versus the solid volume fraction.

case (approximated by setting $\nu = 0.49$). First, the resulting hydraulic conductivity profile against porosity is provided.

3.4.1 Hydraulic conductivity and poroelastic parameters

The hydraulic conductivity tensor $\mathbf{K}$ is computed for several values of the cylinders' radius in $(0.1, 0.4)$ with steps of 0.01, which means the results correspond to the porosity $\phi \in [0.083, 0.784]$.

As a result of the previously explained rotational invariance, a diagonal conductivity tensor is expected, i.e.

$$\mathbf{K} = \langle W_{11} \rangle_f \mathbf{I}, \quad (3.73)$$

which is indeed consistent with the provided numerical results and also those shown in [108].

The profile of hydraulic conductivity $\langle W_{11} \rangle_f$ against the volume fraction of the elastic matrix $\phi_s = 1 - \phi$ is shown in Figure 3.3. A nonlinear drop of $\langle W_{11} \rangle_f$ with negative curvature at decreasing porosity is observed, which is consistent with the parabolic-like profile obtained as a solution to the problem (3.23-3.25). The values shown are normalized with respect to the maximum value of hydraulic conductivity $1.41 \cdot 10^{-2}$, obtained for a cylinder radius equal to 0.4 (i.e. $\phi \approx 0.7838$), which means that the minimum value (for a cylinder radius equal to 0.1, i.e. $\phi \approx 0.08$) is $3.8 \cdot 10^{-5}$.

The dimensional hydraulic conductivity K_{dim} can be obtained by multiplying the dimensionless one by d^2/μ , where d is the reference microscale length and μ the interstitial fluid viscosity. Considering the plasma viscosity $\mu = 4 \cdot 10^{-3} \text{ Pa} \cdot \text{s}$ and a typical $d \approx \mu\text{m} = 10^{-6} \text{ m}$ a physiologically reasonable range is obtained (see, e.g., [8]), that yields

$$10^{-14} \text{ m}^2/(\text{Pa} \cdot \text{s}) \lesssim K_{\text{dim}} = (d^2/\mu)\langle W_{11} \rangle_f \lesssim 10^{-12} \text{ m}^2/(\text{Pa} \cdot \text{s})$$

which corresponds to $10^{-8} - 10^{-6} (\text{cm})^2/(\text{mmHg} \cdot \text{s})$. Tumor hydraulic conductivity values reported in the experimental literature typically fall within this range, see for example [101] and [102] among many others. Next, the profile of the poroelastic parameters against porosity and Poisson's ratio of the matrix is provided.

For each value of porosity and Poisson's ratio, the second and fourth rank tensors $\mathbf{Q}$ and $\mathbf{M}$ are computed numerically. Then, the scalar Biot's coefficient α and (non-dimensional) Biot's modulus M are obtained via equations (3.68) and (3.69).

The profile of Biot's coefficient versus porosity for the four different Poisson's ratios is shown in Figure 3.4. The Biot's coefficient exhibits a monotonically increasing profile which tends to its horizontal asymptote $\alpha = 1$ for all Poisson's ratios as the porosity approaches 1. This coefficient physically represents the ratio of macroscale solid to fluid volume changes (at constant pressure), and $\alpha = 1$ indeed represents the classical poroelastic upper bound for the Biot's coefficient (see, e.g. [37]) when both phases are intrinsically incompressible which means that $\alpha = 1$ for any porosity value when $\nu = 0.5$). Similarly at increasing the incompressible fluid volume fraction and consequently decreasing the compressible elastic volume fraction and at increasing Poisson's ratio Biot's coefficient approaches to 1.

Biot's modulus represents the macroscale inverse of the fluid volume variation due to variation of the macroscale pore pressure, in other words, for a fixed pressure variation, the lower the M , the higher the variation of the fluid volume is. This coefficient is computed by solving equations (3.41)-(3.42) and then exploiting relationship (3.69).

An interesting and not completely intuitive behavior is observed in the profile of Biot's modulus versus porosity. As expected, $\text{Tr } \mathbf{Q} = \nabla \cdot \mathbf{a}$ decreases monotonically towards zero as the Poisson's ratio of the matrix approaches the incompressible limit ($\nu = 0.5$). consequently, as intrinsic incompressibility of the fluid phase is considered in the upscaling process, M approaches infinity for every value the porosity, which is consistent with classical poroelasticity. As the problem is solved in the elastic domain Ω_s only, according to our numerical results, $\text{Tr } \mathbf{Q} = \nabla \cdot \mathbf{a}$ increases at increasing porosity and therefore, at fixed Poisson's ratio, volume variations increase. As Biot's modulus is related to the average variation of interstitial

Figure 3.4: Computed Biot's coefficient versus porosity for four different solid matrix compressibility.

fluid volume in response to variations of the interstitial pressure, the provided results can be exploited to relate compressibility and fluid (and hence drug) transport in compressible tumors. Starting from low values of pore volume fraction, the Biot modulus reduces non-linearly with positive curvature up to the minimum point then, it soars and approaches infinity for large values of porosity. Technically, there are two factors determining the overall behavior of Biot's modulus, namely the divergence of the seventh solid cell problem's solution and the volume fraction of the solid phase. The former increases by reducing the solid's volume while it decreases sharply increasing the Poisson's ratio of the solid phase. On the other hand, This coefficient is computed for the poroelastic medium that means the incompressible fluid inside is taken into account. It is also averaged over the solid phase so, the volume fraction of solid has an effect on the result. The overall profile of Biot's modulus is a consequence of the interplay between the mentioned factors. Noteworthy is that having an incompressible solid matrix the decrease prior to the definite minimum becomes negligible and so the minimum itself. The Biot's modulus versus porosity for the four considered values of the Poisson's ratio is shown in figures 3.5a, 3.5b, 3.5c, and 3.5d.

Furthermore, a parametric analysis by studying the Biot's modulus profile versus the Poisson's ratio of the matrix is performed, since the observed behavior suggests that there is a non-trivial interplay between compressibility of the matrix and porosity. The same range $\nu \in (0.35, 0.49)$ is considered with more refined steps of 0.01. Different values of the porosity ϕ are also considered, namely 0.0828, 0.1737, 0.2864, 0.4122, 0.5427, 0.6694,

(a) $\nu = 0.35$ (b) $\nu = 0.40$ (c) $\nu = 0.45$ (d) $\nu = 0.49$

Figure 3.5: Non-dimensionalized Biot's modulus against porosity for four different solid matrix compressibility.

Figure 3.6: Biot's modulus Vs. Poisson's ratio of the elastic matrix for different porosities

and 0.7838. The profile of this coefficient against the Poisson's ratio of the solid matrix demonstrate intersections which are the result of the above-mentioned interplay. The results are shown in Figure 3.6.

By increasing Poisson's ratio for every porosity value, Biot's modulus increases. Due to the dependency on the porosity, the curves representing the profile of Biot's modulus versus the Poisson's ratio of the matrix can actually cross each other, in other words, the same values of M at fixed Poisson's ratio can be obtained for different porosities. This behavior is not observed for very high values of the porosity, as in this case the Biot's modulus is almost monotonically increasing at increasing porosity, see for instance Figure 3.5d. The results concerning the effective elastic moduli are shown in the following.

3.4.2 The effective elasticity tensor

From Equation (3.1) the drained effective elasticity tensor $\tilde{\mathbb{C}}$ is obtained. The undrained one is computed by Equation (3.72) by adding the contribution of the poroelastic parameters M and α . The profile of drained and undrained Young's moduli and shear modulus E_p and E_{ps} , respectively, versus porosity is shown in Figures 3.7a and 3.7b. The shear modulus μ_p against porosity is shown in Figure 3.8.

All of them exhibit a decreasing profile at increasing porosity. The drained Young's modulus is slightly lower than the undrained one. This behavior is expected because the effect of the fluid inside the pores on the latter modulus is negligible as its strength is much lower than the solid phase which

Figure 3.7: Normalized non-dimensionalized effective Young's modulus against porosity for drained and undrained cases.

Figure 3.8: Normalized and non-dimensionalized effective shear modulus against porosity.

Figure 3.9: Drained effective Poisson's ratio against porosity for different solid matrix compressibility

implies that the drained and undrained values are almost equal. Thus, the porosity plays a major role in determining the stiffness of poroelastic materials, while the results indicate that the dependency of the elastic moduli on the Poisson's ratio of the solid matrix is non-significant in the investigated range. The effective Poisson's ratio, however, strongly depends on both the Poisson's ratio of the matrix and the porosity, as shown below.

3.4.3 The effective Poisson's ratio

The effective drained and undrained Poisson's ratios ν_p and ν_{ps} are shown in Figures 3.9 and 3.10, respectively. The effect of solid matrix Poisson's ratio on the effective Poisson's ratio is considerable, particularly, for the cases with low porosity. The value of this coefficient (drained one) decreases sharply by increasing the porosity or solid matrix compressibility, however, considering the fluid inside the pores it increases and approaches to incompressibility for larger porosity. The reason is that the fluid in the upscaling process is assumed to be incompressible so, by increasing the fluid volume fraction the overall behavior of the cell approaches to the one of fluid. Moreover, the increase in saturated effective Poisson's ratio Vs. porosity is non-linear with positive curvature. This is the expected behavior, as ν_{ps} measures the macroscale compressibility of the material as though the pores were filled by an incompressible static fluid.

Figure 3.10: Undrained effective Poisson's ratio against porosity for different solid matrix compressibility

Appendix - Mesh sensitivity analysis

In order to improve the efficiency of the numerical simulations in terms of accuracy and computational cost, the sensitivity analysis for mesh setting is performed. Since a parametric analysis by varying the pore radius is carried out, the parameters that characterize the mesh setting should be geometry-dependent. This issue can be addressed by means of a mesh sensitivity analysis finding appropriate relationships between the mentioned parameters and the pore radius. In particular, the mesh around the interface should be finer than other parts far away from it as the loads are applied in this region. The target mesh density can be assigned by using markers ("seeds") that are placed along the edges of the specific region under consideration. It is necessary to divide the whole geometry into several simpler partitions. Therefore, the desired mesh dimension in every direction can be reached by assigning the density of seeds on the edges of partitions. A partitioned geometry including two groups of edges is shown in Figure 3.11 which is used to control the dimension of the mesh in both radial and circumferential directions. The density of seeds on each edge can be assigned by the parameter "seed number" that shows the number of markers on the chosen edge. The element type is linear hexahedron, type *C3D8R* elements (8-node linear brick, reduced integration) in the solid phase. In order to generate a geometry-dependent mesh, we relate the seed number in both

Figure 3.11: Edges of the geometry divided into two groups to control the mesh size. The *Circumferential* in red and the *Radial* in black. The size of meshes can be controlled by changing the number of seeds of the mentioned groups.

directions to the radius of the pores R , namely

$$\text{Radial edges seed number} = (0.5 - R) * X_1 \quad (3.74)$$

$$\text{circumferential edges seed number} = R * 10 * X_2. \quad (3.75)$$

A parametric analysis in terms of X_1 and X_2 is performed in order to minimize the numerical errors, as well as the computational cost. Equations 3.74 and 3.75 show the relationship between seed number and mentioned parameters. The seed number is equal for both the inner and outer circumferential edges which result in a mesh density which is uniformly increasing towards the interface (see figure 3.12). Since the auxiliary forth rank tensor $\mathbb{M}$ exhibits cubic symmetry then for instance

$$M_{11} = M_{22}. \quad (3.76)$$

The above equation is chosen to check the numerical error as the quantity $M_{11} - M_{22}$ should be approaching zero. Figure 3.13 shows the results of the sensitivity analysis for the solid phase at $R = 0.1$.

In order to obtain an acceptable computational cost (*total cpu time* = 20 ± 0.5 seconds) together with acceptable accuracy of the results, the values $X_1 = 110$ and $X_2 = 7$ are chosen for the solid phase. The results of the parametric analysis conducted for the pore radius $R = 0.2$ and $X_2 = 7$ are also shown in figure 3.14.

Figure 3.12: The mesh setting for two cases with $R = 0.1$ and $R = 0.4$.

Figure 3.13: Mesh sensitivity analysis for the case with $R = 0.1$: $\frac{M_{11}-M_{22}}{M_{11}}$ vs X_1 for the three cases $X_2 = 6, 7$, and 8

Figure 3.14: Mesh sensitivity analysis for the case with $R = 0.2$: $\frac{M_{11} - M_{22}}{M_{11}}$ vs X_1

An alternative approach to generate mesh in ABAQUS is called "*global seed size*". Here, one chooses an approximate size for seeds (more precisely approximate seed density) for whole geometry. This approach together with 10-node quadratic tetrahedron element (*C3D10*) is used to generate mesh for fluid phase problem. In this case, the only required relationship to relate the pore radius with the mesh density is the following

$$global\ seed\ size = R * X_3 \quad (3.77)$$

where X_3 is another tuning parameter and unlike the seed number in the solid phase, by reducing the value of the global seed size, the mesh density increases. Figure 3.15 shows the parametric analysis of Hydraulic Conductivity W_{11} against $0.5 - X_3$ to determine an efficient value for the tuning parameter X_3

Here, for the fluid phase we choose $X_3 = 0.2$. For this value, the error reduces to less than 1%

Figure 3.15: The results of mesh sensitivity analysis for the fluid phase for pore radius=0.2

CHAPTER 4

NUMERICAL SIMULATION OF THE MACROSCALE RESPONSE OF POROELASTIC MEDIA

**Finite element implementation of fully-coupled anisotropic
poroelasticity**

4.1 Introduction

The numerical implementation of the three-dimensional anisotropic Biot's poroelasticity with coefficients derived from the microstructure, as prescribed by the asymptotic homogenization technique [18] is presented. The system of partial differential equations (PDEs) is discretized by finite elements exploiting a formal analogy with the fully coupled thermal displacement systems of PDEs implemented in the commercial software ABAQUS. The robustness of our computational framework is confirmed by comparison with well-known simplified examples when an analytic solution can be found, such as the Therzaghi's problem (i.e. the one arising when considering both the solid and the fluid phases as intrinsically incompressible at the microscale) for simplified geometries. Then three-dimensional numerical simulations are performed to test the importance of the underlying microstructure encoded in the coefficients of the model, where the latter are computed via three-dimensional microstructural cell problems by following the approach described in [32]. In particular, the results are presented by performing a parametric analysis by varying the solid matrix' compressibility and investigate how the resulting macroscale poroelastic coefficients affect the macroscale solution in terms of the effective solid displacement and pressure. Finally, we apply our results to investigate the poromechanics of relevant biological tissues.

In this chapter, the implementation of fully-coupled quasi-static anisotropic macroscale system of equations for poroelastic media with coefficients derived from the microstructure, as prescribed by the asymptotic homogenization technique is carried out. This is done by exploiting the analogy between the latter case and the fully-coupled thermal-displacement system of equations that is well implemented in a commercial software ABAQUS. This is followed by the validation of the presented framework by making comparison with two analytical works (One for Transient analysis and the other one for Steady-state).

In particular, the focus is on the effect of the compressibility of the microscale solid matrix on the macroscale mechanical response of the poroelastic tissues which in turn, among all the effects, causes anisotropy of the macroscale system of PDEs. It is shown that there are several important factors for the mechanical properties estimation of the poroelastic tissues such as the loading condition (immediate or gradual application), time, anisotropy, displacement distribution profile, porosity, etc. Moreover, measuring the displacements, relative fluid velocity and their divergence give us the best estimation of the material properties. Several different time spans, microscale material properties, and different analyses types (Transient and Steady-state) are analyzed. The results reveal the fact that the presented

numerical framework is very useful in the prediction of the mechanical response and estimation poroelastic properties from in vivo experiments.

It is well known that the poroelastic tissues such as human brain inside the unopened skull are compressible [98]. Moreover, asymptotic homogenization shows that even with an isotropic microstructure we may have anisotropic effective mechanical properties. The mechanical properties of the brain tissues, tumors, etc. can be estimated in vivo by means of Poroelastography [116] [158], Elastography [60], and Viscoelastography [67]. In all these techniques one measures the displacements that are induced in the target tissue from different sources e.g. Valsalva maneuver [98], blood pressure pulses [158], mechanical waves [157], etc. Then, using appropriate constitutive equations they can estimate the effective poroelastic, elastic, or viscoelastic properties of the brain tissue according to the chosen system of equation. So, the estimated poroelastic properties of the tissue are sensitive to the assumptions of the chosen constitutive equations e.g. in [116] a linear isotropic one is chosen.

For example, in [116] and [158] the effective Poisson's ratio of the brain tissue was determined by means of Poroelastography, assuming isotropic effective properties and porosity equal to 0.2, in most of the cases equal to $\nu_p = 0.4$. On the other hand, from table 4.4 we can see that having almost incompressible solid and fluid phases in microscale an almost isotropic effective elasticity tensor with $\nu_p = 0.4$ (for drained media) is achieved.

The remainder of this work is organized as follows. In Chapter 4.2 the governing equations of the poroelastic media together with its Finite element formulation is presented. Chapter 4.3 deals with the details of Finite Element Implementation of the macroscale PDEs. Finally, in Chapter 4.4 the benchmarks, as well as some practical scenarios of the interest, is given.

4.2 Governing equations

4.2.1 Balance laws

The porous media problem is solved for small strains, such that the small strain measure is defined as usual

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left[\nabla_{\mathbf{x}} \mathbf{u}^{(0)} + [\nabla_{\mathbf{x}} \mathbf{u}^{(0)}]^{\mathbf{t}} \right] \quad (4.1)$$

where $\mathbf{u}^{(0)}$ denotes the solid displacement. The essential balance equations at the macroscale are the balance of linear momentum and the continuity equation, respectively the conservation of mass. Their local forms allow

representation as

$$\mathbf{0} = \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}, \quad (4.2)$$

$$\dot{p}^{(0)} = -M [\tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle_{rf}], \quad (4.3)$$

wherein the quasi-static case is assumed and additional body forces are neglected. Equation (4.3) provides us with a continuity statement that relates the change in the volume of fluid and solid to the time derivative of pore pressure. Moreover, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $p^{(0)}$ describes the present pore pressure, $\langle \mathbf{v} \rangle_{rf}$ represents the relative pore fluid velocity, M and α are the Biot modulus and Biot coefficient, respectively. The notation $\dot{\bullet}$ is used to define the time derivative of the quantity $\bullet$. In order to close the overall system of equations, the following constitutive relations shall be introduced for the stresses as well as for the relative pore fluid velocity, to be specific

$$\boldsymbol{\sigma} = \tilde{\mathbb{C}} : \boldsymbol{\varepsilon} - \tilde{\boldsymbol{\alpha}} p^{(0)}, \quad (4.4)$$

$$\langle \mathbf{v} \rangle_{rf} = -\langle \mathbf{W} \rangle_f \cdot \nabla_{\mathbf{x}} p^{(0)}, \quad (4.5)$$

where $\tilde{\mathbb{C}}$ denotes the effective elasticity tensor and where $\langle \mathbf{W} \rangle_f$ symbolises the hydraulic conductivity.

The above system refers to the anisotropic quasi-static Biot's equations by using the notation that can be found in [106], where the authors obtain a more general form of the above-mentioned equations from the microstructure.

4.2.2 Finite element formulation

In this section, the weak form is derived by considering the contributions from both mechanical equilibrium in the absence of mechanical body forces for a porous medium and the continuity statement for the liquid phase in the porous medium. Afterward, the problem has to be discretized. This is the basis for the Finite Element implementation discussed in section 4.3.

The weak form is obtained by multiplying (4.2) with an arbitrary test function (virtual displacement) $\delta \mathbf{u}$ that is admissible regarding the continuity and the Dirichlet boundary conditions. The result is then integrated with respect to the volume of the underlying body $\mathcal{B}$

$$\mathbf{0} = \int_{\mathcal{B}} [\nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma}] \cdot \delta \mathbf{u} \, dv. \quad (4.6)$$

Applying the divergence theorem and using Cauchy's theorem $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$, where $\mathbf{t}$ represents surface tractions per unit area and where $\mathbf{n}$ denotes the outward normal unit vector, results in

$$0 = \int_{\partial\mathcal{B}} \mathbf{t} \cdot \delta \mathbf{u} \, da - \int_{\mathcal{B}} \boldsymbol{\sigma} : \nabla_{\mathbf{x}} \delta \mathbf{u} \, dv. \quad (4.7)$$

In analogy, the same procedure is used for (4.3) with an admissible, continuous test function δp , resulting after integration over the domain $\mathcal{B}$ in

$$0 = \int_{\mathcal{B}} \left[\dot{p}^{(0)} + M \left[\nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle_{rf} + \tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} \right] \right] \delta p \, dv. \quad (4.8)$$

Afterwards, the integral is splitted into three parts and divided by the Biot modulus M

$$0 = \int_{\mathcal{B}} \frac{1}{M} \dot{p}^{(0)} \delta p \, dv + \int_{\mathcal{B}} \left[\nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \rangle_{rf} \right] \delta p \, dv + \int_{\mathcal{B}} \tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} \delta p \, dv. \quad (4.9)$$

Finally, integration by parts and the divergence theorem can be applied such that

$$\begin{aligned} 0 = \int_{\mathcal{B}} \frac{1}{M} \dot{p}^{(0)} \delta p \, dv &+ \int_{\partial\mathcal{B}} \langle \mathbf{v} \rangle_{rf} \cdot \mathbf{n} \delta p \, da \\ &- \int_{\mathcal{B}} \langle \mathbf{v} \rangle_{rf} \cdot \nabla_{\mathbf{x}} \delta p \, dv + \int_{\mathcal{B}} \tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} \delta p \, dv. \end{aligned} \quad (4.10)$$

As a next step, the discretisation of the problem in space and time has to be performed. Therefore, the body $\mathcal{B}$ is decomposed into n_{el} small elements such that the spacial discretisation of the weak forms is represented by

$$\sum_{e=1}^{n_{\text{el}}} \left[\int_{\partial\mathcal{B}^e} \mathbf{t} \cdot \delta \mathbf{u}^e \, da - \int_{\mathcal{B}^e} \boldsymbol{\sigma} : \nabla_{\mathbf{x}} \delta \mathbf{u}^e \, dv \right] = 0, \quad (4.11)$$

$$\begin{aligned} \sum_{e=1}^{n_{\text{el}}} \left[\int_{\mathcal{B}^e} \frac{1}{M} \dot{p}^{(0)} \delta p^e \, dv + \int_{\partial\mathcal{B}^e} \langle \mathbf{v} \rangle_{rf} \cdot \mathbf{n} \delta p^e \, da \right. \\ \left. - \int_{\mathcal{B}^e} \langle \mathbf{v} \rangle_{rf} \cdot \nabla_{\mathbf{x}} \delta p^e \, dv + \int_{\mathcal{B}^e} \tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} \delta p^e \, dv \right] = 0. \end{aligned} \quad (4.12)$$

In addition, the test function are discretised elementwise for each element node n_{en} in the following manner

$$\delta \mathbf{u}^e \approx \sum_{A=1}^{n_{\text{en}}^u} N_A^u \delta \mathbf{u}^{eA}, \quad \delta p^e \approx \sum_{C=1}^{n_{\text{en}}^p} N_C^p \delta p^{eC}, \quad (4.13)$$

where N denotes the corresponding set of shape functions. The contributions $\delta \mathbf{u}^{eA}$ and δp^{eC} represent the respective values of the test function at the nodes. With these reformulations at hand and by additionally defining the assembly operator $\mathbf{A}$, the discretised weak form can finally be given by

$$\mathbf{A}_{\substack{A=1 \\ e=1}}^{n_{\text{el}}^u} \left[\int_{\partial \mathcal{B}^e} \mathbf{t} N_A^u \, da - \int_{\mathcal{B}^e} \boldsymbol{\sigma} \cdot \nabla_{\mathbf{x}} N_A^u \, dv \right] = \mathbf{0}, \quad (4.14)$$

$$\mathbf{A}_{\substack{C=1 \\ e=1}}^{n_{\text{el}}^p} \left[\int_{\mathcal{B}^e} \frac{1}{M} \dot{p}^{(0)} N_C^p \, dv + \int_{\partial \mathcal{B}^e} \langle \mathbf{v} \rangle_{rf} \cdot \mathbf{n} N_C^p \, da \right. \\ \left. - \int_{\mathcal{B}^e} \langle \mathbf{v} \rangle_{rf} \cdot \nabla_{\mathbf{x}} N_C^p \, dv + \int_{\mathcal{B}^e} \tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}} N_C^p \, dv \right] = \mathbf{0}. \quad (4.15)$$

The same approximation is applied analogously for the physical fields of $\mathbf{u}^e$ and p^e according to (4.13). For the time discretisation of the pressure $\dot{p}^{(0)}$ and strain $\dot{\boldsymbol{\varepsilon}}$ an implicate Backward Euler integration scheme is adopted. For the respective variable $\bullet$ the definition $\dot{\bullet} = [\bullet^{n+1} - \bullet^n] / \Delta t$ is used, where $n + 1$ denotes to the current time step ${}^{n+1}t$ and $\Delta t = {}^{n+1}t - {}^nt$.

4.3 Finite Element implementation

In this section, we discuss the implementation of the macroscale system of equations into the commercial finite element program Abaqus by considering the contribution from both – the mechanical equilibrium statement for a porous medium and the continuity statement for the liquid phase in the porous medium. To solve this problem, a coupled pore fluid diffusion/stress analysis is usually used. Abaqus provides a built-in feature for such a problem, however, it is assumed that both solid and fluid phases are incompressible which in turn means that Biot's modulus is approaching infinity. Therefore this ansatz is not appropriate for the poroelastic problem at hand. In the following section, the analogy to the heat equation shall be examined. With the help of the subroutines UMAT and UMATHT the porous media material model can be realized. This allows the implementation of the framework at hand without writing a user element subroutine UEL in Abaqus.

4.3.1 Analogy to heat equation

The strong form of the energy balance can be expressed as

$$\rho c \dot{\theta} = -\nabla_{\mathbf{x}} \cdot \mathbf{q} + r \quad (4.16)$$

where ρ is the mass density of the material, c represents the specific heat capacity, θ defines the temperature, $\mathbf{q}$ denotes the heat flux vector and where r

symbolises heat sources. The heat sources in a thermo-mechanical problem can be additively decomposed into the sum of external heat supply r_E and the volumetric heat generation per unit time caused by mechanical working of the material r_M defined as

$$r = r_E + r_M. \quad (4.17)$$

The heat conduction is represented by the Fourier law

$$\mathbf{q} = -\boldsymbol{\kappa} \cdot \nabla_x \theta \quad (4.18)$$

with the thermal conductivity tensor $\boldsymbol{\kappa}$.

The weak form of (4.16) is derived by analogy with the procedure in section 4.2.2. Multiplying (4.16) with an admissible test function $\delta\theta$ and integrating over the domain results in

$$0 = \int_{\mathcal{B}} \rho c \dot{\theta} \delta\theta \, dv + \int_{\mathcal{B}} [\nabla_x \cdot \mathbf{q}] \delta\theta \, dv - \int_{\mathcal{B}} r \delta\theta \, dv. \quad (4.19)$$

Applying integration by parts and the divergence theorem, the aforementioned equation can be reformulated to

$$0 = \int_{\mathcal{B}} \rho c \dot{\theta} \delta\theta \, dv + \int_{\partial\mathcal{B}} \mathbf{q} \cdot \mathbf{n} \delta\theta \, da - \int_{\mathcal{B}} \mathbf{q} \cdot \nabla_x \delta\theta \, dv - \int_{\mathcal{B}} r \delta\theta \, dv. \quad (4.20)$$

The similarity of (4.10) and (4.20) is already obvious. As a final step, the aforementioned equation is discretised in space and time in the same manner as in section 4.2.2. By dividing the body $\mathcal{B}$ into a finite number of elements n_{el} and discretising the test functions for each element node n_{en} , i.e.

$$\delta\theta^e \approx \sum_{C=1}^{n_{\text{en}}^\theta} N_C^\theta \delta\theta^{eC} \quad (4.21)$$

equation (4.20) can be finally assembled as

$$\mathbf{A} \left[\sum_{C=1}^{n_{\text{el}}} \sum_{e=1}^{n_{\text{en}}^\theta} \left[\int_{\mathcal{B}^e} \rho c \dot{\theta} N_C^\theta \, dv + \int_{\partial\mathcal{B}^e} \mathbf{q} \cdot \mathbf{n} N_C^\theta \, da - \int_{\mathcal{B}^e} \mathbf{q} \cdot \nabla_x N_C^\theta \, dv - \int_{\mathcal{B}^e} r N_C^\theta \, dv \right] \right] = \mathbf{0}. \quad (4.22)$$

Comparing the latter equation with (4.14), it can be summarized that the heat equation as well as the continuity equation are both partial differential equations in a similar form. With this property at hand the implementation into Abaqus is possible in a straightforward manner as defined in the next section.

4.3.2 Implementation in Abaqus

Due to the similarity of the continuity equation and the heat equation, it is possible to use the coupled thermal-displacement analysis within the commercial FEM software Abaqus. The mechanical and thermal material behavior with the respective subroutines UMAT and UMATHT can be adapted by the user. This way, the heat equation is 'manipulated', such that the temperature degree of freedom is equal to the pore pressure and the heat flux corresponds to the relative pore fluid velocity. Then, Abaqus can be used to implement the discretised macroscale system of equations as defined in (4.14) and (4.15). Contrary to the formulation of a user element in Abaqus via the UEL subroutine, the time for development and for testing the model is lower when using a coupled temperature-displacement analysis. This also enables to use all standard features within Abaqus. The heat equation with the subroutine UMATHT has been manipulated due to these advantages before, see for example [104] for a finite deformation gradient-enhanced damage model. To gain a better understanding of the implementation into Abaqus, all return values for both subroutines shall be analyzed in the following paragraphs.

For a thermo-mechanically coupled user material, the general return quantities STRESS, STATEV, DDSDDE and the thermo-mechanical quantities RPL, DRPLDT, DRPLDE and DDSDDT must be defined within the UMAT user subroutine for the mechanical material model.

Within the subroutine, not only the strain tensor ϵ is provided as a input variable, but also the temperature, here respectively the pore pressure, as the heat transfer equation is 'abused' to solve the partial differential equation of the pore pressure (4.3). Thus it is possible to update the stresses σ according to the constitutive equation (4.4) within the routine UMAT, where STRESS is used as the return variable in Abaqus. In addition, volumetric heat generation caused by mechanical working r_M is incorporated into RPL. For a thermo-mechanically coupled problem, the dissipation and the Gough-Joule effect would be expected here. In view of (4.15), the only contribution dependent on the displacement field, i.e.

$$r_M = -\tilde{\alpha} : \dot{\epsilon} = -\frac{1}{\Delta t} \tilde{\alpha} : [{}^{n+1}\epsilon - {}^n\epsilon] , \quad (4.23)$$

is incorporated into the mechanical model. Furthermore, the respective derivatives, respectively contributions to the Jacobian within the Newton-Raphson-based solution scheme, are necessary as summarised in table 4.1.

In addition, the thermal material model can be adapted within the user subroutine UMATHT for heat transfer, where the quantities U, DUDT, DUDG for the internal energy and FLUX, DFDT, DFDG for the heat flux have to be specified. The temperature degree of freedom θ is substituted by the pore

Table 4.1: Specification of all return values when implementing the user subroutine UMAT for the porous media model compared to a fully coupled thermo-mechanical model.

UMAT variable	thermo-mechanical model	porous media model
STATEV	$\mathcal{V}$	-
STRESS	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma} = \tilde{\mathbb{C}} : \boldsymbol{\varepsilon} - \tilde{\boldsymbol{\alpha}} p^{(0)}$
DDSDDE	$\frac{d\boldsymbol{\sigma}}{d\boldsymbol{\varepsilon}}$	$\tilde{\mathbb{C}}$
DDSDDT	$\frac{d\boldsymbol{\sigma}}{d\theta}$	$-\tilde{\boldsymbol{\alpha}}$
RPL	r_M	$-\tilde{\boldsymbol{\alpha}} : \dot{\boldsymbol{\varepsilon}}$
DRPLDT	$\frac{dr_M}{d\theta}$	0
DRPLDE	$\frac{dr_M}{d\boldsymbol{\varepsilon}}$	$-\frac{\tilde{\boldsymbol{\alpha}}}{\Delta t}$

Table 4.2: Specification of all return values when implementing the user subroutine UMATHT for a classic rigid heat conductor and the porous media model.

UMATHT variable	standard heat transfer	porous media model
U	$U = c\theta$	$\frac{1}{M} p^{(0)}$
DUDT	$\frac{dU}{d\theta} = c$	$\frac{1}{M}$
DUDG	$\frac{dU}{d\nabla_{\mathbf{x}}\theta} = \mathbf{0}$	$\mathbf{0}$
FLUX	$\mathbf{q} := -\boldsymbol{\kappa} \cdot \nabla_{\mathbf{x}}\theta$	$\langle \mathbf{v} \rangle_{rf} = -\langle \mathbf{W} \rangle_f \cdot \nabla_{\mathbf{x}} p^{(0)}$
DFDT	$\frac{d\mathbf{q}}{d\theta} = \mathbf{0}$	$\mathbf{0}$
DFDG	$\frac{d\mathbf{q}}{d\nabla_{\mathbf{x}}\theta} = -\boldsymbol{\kappa}$	$-\langle \mathbf{W} \rangle_f$

pressure $p^{(0)}$ as mentioned before. For the internal energy U , the expression $U = p^{(0)}/M$ is valid when comparing the respective equations (4.15) and (4.22). The heat flux $\mathbf{q}$ is replaced with the relative pore fluid velocity $\langle \mathbf{v} \rangle_{rf}$, where the constitutive relation (4.5) is used for the return variable FLUX. The respective derivatives are shown in table 4.2.

4.4 Numerical examples

In this section, first, the presented numerical framework is validated by comparing its results of some simple 1-D problems with their exact analytical solution. Then, more complex examples with their specific scenarios of interest are provided.

4.4.1 Terzaghi's 1D problem

In order to validate the presented computational framework (the implementation of the system of equations) a similar approach as in [46] is adopted. A saturated column of sand under a compressive pressure at the top of it is solved and compared with its analytical solution of Terzaghi as below

$$P_0 = \frac{\tilde{\alpha} M P_p}{2\mu + \lambda + \tilde{\alpha}^2 M} \quad (4.24)$$

$$u_0 = \frac{(L - z) P_p}{2\mu + \lambda} \quad (4.25)$$

$$P^0(z, t) = \frac{4P_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left(\frac{-t(2n+1)^2 \pi^2 c}{4L^2}\right) \sin\left(\frac{(2n+1)\pi z}{2L}\right) \quad (4.26)$$

$$u^0(z, t) = \frac{P_0}{\lambda + 2\mu} \left[(L - z) - \frac{8L}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left(\frac{-t(2n+1)^2 \pi^2 c}{4L^2}\right) \cos\left(\frac{(2n+1)\pi z}{2L}\right) \right] + u_0 \quad (4.27)$$

where M , c , $P_0 = 4662.4$, $u_0 = 6.6720e - 04$, and $L = 15$ are the biot modulus, the consolidation coefficient, instant interstitial pore pressure, instant settlement, and the height of the undrained column of sand, respectively. At time $t = 5e - 05$, in FEM analysis, $P_0 = 4662.4$ and $u_0 = 6.6720e - 04$ that show this is a sufficiently small time increment for numerical analysis in this framework. The mechanical and hydraulic properties are presented in Table 4.3

In this example, drainage is only allowed from the top of the column where the load is assigned, zero displacements are imposed in all directions on the bottom and free axial displacements for the sides so that the consolidation is allowed. As expected, the dimensions and the mesh setting of the section does not have any effect on the results so, we have just one element in the section. There are 100 elements with quadratic interpolation function for the displacement and linear ones for the pore pressure in the height of the column as shown in Figure 4.1.

Hydraulic and mechanical properties		
Hydraulic conductivity	K	10^{-5}
Lamé constants	$\lambda = \mu$	$4 * 10^7$
Biot coefficient	$\tilde{\alpha}$	1.
Biot modulus	M	$1.05 * 10^8$
Consolidation coefficient	c	559.5

Table 4.3: The values that are exploited in both analytical and numerical analysis

Figure 4.1: The geometry and mesh setting of the column of sand with the typical color map of the pore pressure

In a consolidation analysis, the time increment is of great importance as it is a time-dependent process. When the number of time increments approaches to infinity the numerical and analytical results converge to the same value. From sensitivity analysis, 1000 increments is an efficient number with good accuracy.

The results provided in Figure 4.2 and Figure 4.3 show a very good agreement between the numerical and analytical results which can validate the presented numerical framework.

4.4.2 Brain tissue

One important application of asymptotic homogenization is hydraulic and mechanical properties estimation such as hydraulic conductivity and biot modulus of biological tissues for which the mechanical tests are costly. Brain tissues (both gray and white) are known as poroelastic media. In order to predict their mechanical response such as interstitial fluid velocity and pressure, displacements, strain, etc., the presented numerical framework is used. This problem has been investigated with assumptions like isotropy, incompressibility, homogeneity, etc. in several previous works such as [10] and [156]. In this section, the goal is to treat the latter in our framework. First, a simple one that is solved previously in 1-D [10] is carried out in or-

(a) Pore pressure over height. $t=1$ sec

(b) Settlement over height. $t=1$ sec

Figure 4.2: Numerical and analytical results of Terzaghi example after one second, demonstrating very good agreement between them

(a) Pore pressure over height. $t=0.1$, $t=0.01$, and $t=0.005$ sec

(b) Settlement over height. $t=0.1$, $t=0.01$, and $t=0.005$ sec

Figure 4.3: Numerical and analytical results of Terzaghi example after one second, demonstrating very good agreement between them

der to validate the presented framework for this problem. Then, eliminating the mentioned simplifications more complex examples are performed.

4.4.2.1 Constant pressure source in an isotropic, homogeneous, infinite medium with incompressible phases

A sphere of radius $R = 2\text{cm}$ from which a very small cavity of radius $a = 0.03\text{cm}$ is cored, as a chemotherapeutic agent, is modelled. Imagine that the infusion is from a constant pressure source $P_0 = 6664 \text{ dynes/cm}^2$ and the material is isotropic with $\mu_p = 2E + 04 \text{ dynes/cm}^2$ and $\lambda_p = 9 * E + 05 \text{ dynes/cm}^2$ as its effective shear modulus and other lamé constant.

As it is reported in several experimental, analytical, and numerical works such as [32] and [126] there is a considerable difference between microstructural and macrostructural (effective) poroelastic properties (or elastic properties of a drained porous medium in some studies [126]). Having the effective material properties, the Biot modulus ($M = 2.17E + 06 \text{ dynes/cm}^2$) was calculated by means of a numerical framework based on asymptotic homogenization that is presented in the previous chapter. First, assuming a porosity of $\phi = 0.2$, underlying microstructural elastic properties of the solid matrix is estimated. $E = 96700$ and $\nu = 0.49$ are Young's modulus and Poisson's ratio, respectively.

There is no contact stress or other types of loading except for the mentioned constant infusion pressure source. In [10] the hydraulic permeability is chosen $K = 5 * 10^{-9} \frac{\text{cm}^4}{\text{dynes*sec}}$ which in its basic equations is divided by the porosity while, in Equation (4.5) we assume it (the porosity) to be embedded in the hydraulic conductivity. Neglecting the gravity, assuming a density equal to 1, and dividing it by the porosity we choose the hydraulic conductivity $\langle \mathbf{W} \rangle_f = 25 * 10^{-9} \frac{\text{cm}^4}{\text{dynes*sec}}$ so that, it gives identical results as the analytical one presented in [10].

$$p^0(r) = \frac{aP_0}{r} \quad (4.28)$$

$$\mathbf{v}_r(r) = \frac{KP_0a}{r^2\phi} \quad (4.29)$$

where r indicates the position in the radial direction and, when it is a subscript, it denotes the radial direction. The results are in the steady state therefore, the velocity of the solid skeleton is zero then, the absolute fluid velocity, $\mathbf{v}$, is equal to the relative one $\mathbf{v}_{rf}$.

Figure 4.4a and 4.4b show a good agreement between the results of the numerical and analytical solution of the problem at hand.

The small difference between the analytical and numerical results at the end of the dimensionless radius in figure 4.4a is because of the imposed free

seepage boundary condition at the outer surface of the sphere which means zero pore pressure while, as expected from equation 4.28, the analytical result approaches to zero when the radius tends to infinity. Moreover, Figure 4.5 demonstrates a similar profile to the profile of pore pressure presented in 4.4a. This is due to the fact that in this analysis the only source that can cause volumetric strain is pore pressure which, in turn, validates the coupling of the solid-fluid formulation.

The implementation of the macroscale system of equations for both transient and steady-state analysis is now validated by these examples. In following, four anisotropic examples with compressible solid matrix together with their applications are presented.

4.4.2.2 Anisotropy and compressibility

It is well known that the poroelastic tissues such as human brain inside the unopened skull are compressible [98]. Moreover, asymptotic homogenization shows that even with an isotropic microstructure we may have anisotropic effective mechanical properties [32]. The mechanical properties of the brain tissues, tumors, etc. can be estimated in vivo by means of Poroelastography [116] [158], Elastography [60], and Viscoelastography [67]. In all these techniques one measures the displacements that are induced in the target tissue from different sources e.g. Valsalva maneuver [98], blood pressure pulses [158], mechanical waves [157], etc. Then, using appropriate constitutive equations they can estimate the effective poroelastic, elastic, or viscoelastic properties of the brain tissue according to the chosen system of equation. So, the estimated poroelastic properties of the tissue are sensitive to the assumptions of the chosen constitutive equations e.g. in [116] a linear isotropic one is chosen.

It is shown in a previous study [32] that if we change microscale Poisson's ratio of the solid phase, the effective Young's modulus does not suffer considerable changes. We choose a reasonable effective Young's modulus for biological applications such as brain tissue $E_p = 16000 \text{ Pa}$ [10, 116, 125, 144, 158] which, exploiting the framework presented in the previous chapter, results in $E \approx 25700 \text{ Pa}$ for microscale solid skeleton. Then, we choose four different Poisson's ratio for microscale solid skeleton ($\nu = 0.49$, $\nu = 0.45$, $\nu = 0.4$, and $\nu = 0.35$) so, assuming the porosity $\phi = 0.2$, we have four different effective poroelastic properties as in table 4.4

First, the Steady-state analyses in which the cavity pressure is applied gradually is performed.

One effect of the cubic anisotropy is heterogeneous displacement distribution in the azimuthal direction (see figures 4.6a and 4.6b). Figures 4.7a

(a) Interstitial pore pressure Vs. non-dimentinalized radius in the steady state showing the agreement between the analytical results and the presented numerical framework.

(b) Radial interstitial fluid velocity Vs. non-dimentinalized radius in the steady state showing the agreement between the analytical results and the presented numerical framework.

Figure 4.4: The profile of fluid velocity as well as porepressure

Figure 4.5: ϵ_v Vs. non-dimensionalized radius in the steady state showing its agreement with the pore pressure shown in Figure 4.4a.

Case number -->	#	1	2	3	4	-
Poroelastic property	Symbol	$\nu = 0.49$	$\nu = 0.45$	$\nu = 0.40$	$\nu = 0.35$	Unit
Hydraulic conductivity	K	$5e-10$	$5e-10$	$5e-10$	$5e-10$	$\frac{cm^2}{Pa.s}$
Effective Young's modulus	E_p	16000	16000	16000	16000	Pa
Effective Poisson's ratio	ν_p	0.394	0.3664	0.332	0.3	-
Effective Shear modulus	μ_p	6172	6351	6577	6855	Pa
Biot coefficient	α	0.94	0.76	0.63	0.543	-
Biot modulus	M	$4.59e+05$	$1.2e+05$	$7.9e+04$	$6.6e+04$	Pa

Table 4.4: The effective poroelastic properties derived from the underlying microstructural properties

and 4.7b show the radial displacement and relative fluid velocity which are to be compared with the results of the transient analysis.

Next, the Transient-analysis is carried out. From figure 4.9 it can be concluded that the compressibility of the interstitial solid matrix does not have considerable effects on the profile of pore pressure. Fluid drainage and percolation are also very important factors in the determination of the material properties such that, immediately after application of the pressure the tissue behaves as incompressible [60] which is shown in Figure 4.10 when the time approaches zero. This behavior can be seen also in the undrained Poisson's ratio of poroelastic material in [32] (i.e. when considering the elastic response for a static fluid phase filling the pores). However, time passing, due to fluid drainage and percolation the compressible behavior of the material is observed. The mentioned factors lead us to a parametric analysis with respect to Poisson's ratio and its effect on the macroscale anisotropy.

In order to see the injection development in time, transient analysis is performed. $\Delta t = 5e-02$ for this analysis is chosen from a sensitivity analysis. We can rewrite Equation (4.3) as

$$\nabla_x \cdot \langle \mathbf{v} \rangle_{rf} = -\left(\frac{\dot{p}^{(0)}}{M} + \tilde{\alpha} : \nabla_x \dot{\mathbf{u}}^{(0)}\right) \quad (4.30)$$

Having very small time increments we can integrate the latter equation in time

$$\Delta \nabla_x \cdot \langle \mathbf{u} \rangle_{rf} = -\left(\frac{\Delta p^{(0)}}{M} + \tilde{\alpha} : \Delta \nabla_x \mathbf{u}^{(0)}\right) \quad (4.31)$$

where $\langle \mathbf{u} \rangle_{rf}$ is the fluid displacement and $\Delta \nabla_x \cdot \langle \mathbf{u} \rangle_{rf} = \epsilon_v^f$ indicates the change in the volume of the fluid in one increment. We can also write $\nabla_x \cdot \mathbf{u}^{(0)} = \epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ is the volume change of the solid. In the case of a diagonal homogeneous $\tilde{\alpha}$ we can write

$$\Delta \tilde{\alpha} : \nabla_x \mathbf{u}^{(0)} = \Delta \tilde{\alpha}_{11} \nabla_x \cdot \mathbf{u}^{(0)} = \tilde{\alpha}_{11} \Delta \epsilon_v \quad (4.32)$$

then,

$$\Delta \nabla_x \cdot \langle \mathbf{u} \rangle_{rf} = -\left(\frac{\Delta p^{(0)}}{M} + \tilde{\alpha}_{11} \Delta \epsilon_v\right) \quad (4.33)$$

As previously mentioned and from Figure 4.9 one can see that there is no considerable difference in pore pressure between the mentioned cases in Table 4.4 so, only the macroscale coefficients M and $\tilde{\alpha}_{11}$ as well as ϵ_v determine the volume of the injected fluid shown in Figure 4.10.

Figure 4.10 reveals the fact that decreasing Poisson's ratio (increasing compressibility), the volumetric strain decreases. The reason is that as shown in [32] by increasing the compressibility of the microscale solid skeleton, Biot's coefficient $\tilde{\alpha}$ that is the solid-fluid coupling term decreases

(a) Color map of displacements (magnitude) demonstrating heterogeneous distribution in the azimuthal direction due to cubic anisotropy

(b) Radial displacement Vs. Dimensionless radius for $\phi = 0$ and $\phi = 45$. This is taken out from 4.6a.

Figure 4.6: Heterogeneous distribution of displacement in the azimuthal direction

(a) Radial displacement Vs. non-dimentinalized radius from the Steady-state.

(b) Radial interstitial fluid velocity Vs. non-dimentinalized radius in the Steady-state

Figure 4.7: The profile of radial fluid velocity and displacement

Figure 4.8: ϵ_v Vs. non-dimentinalized radius from the Steady state analysis.

Figure 4.9: Mean pore pressure over time for 4 cases in a Transient analysis.

Figure 4.10: Mean solid volumetric strain $\epsilon_v = \nabla \cdot \mathbf{u} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ and Mean fluid volumetric strain $\epsilon_v^f = \nabla \cdot \mathbf{u}^f = \epsilon_{11}^f + \epsilon_{22}^f + \epsilon_{33}^f$ over time for 4 cases taken from Transient analysis.

which impose a decrease in the mechanical response of the solid part. This fact is also shown in Figures 4.11a and Figure 4.11b where the displacements in Case 4 is less than in Case 1. Another point from Figure 4.10 is that the more compressible the cases are, the wider is the differences between the change in the volume of the solid and the fluid which is because of the less Biot's modulus for the more compressible solid matrix. From a physiological point of view, the difference in the fluid volumetric strain and the one of the solid can be concluded to represent the volume of fluid stored in the poroelastic medium due to the volume change of interstitial solid matrix (because of its compressibility) which, in turn, differs the porosity of the medium.

The difference between Figure 4.11b and Figure 4.7a is because of the analysis type which is Transient and Steady-state, respectively. In the former analysis we apply all the pressure instantly at the beginning of the first time increment while, in the latter, in each time increment, we apply a portion of the total cavity fluid pressure which is equal to the portion of the total time. This reveals the fact that the loading condition also plays an important role.

Time plays a very important role in determining the poroelastic properties by means of Poroelastography. Figure 4.12 show the displacement of Case 1 at different times. This is followed by Figure 4.13 for the same argument on the mean value of solid volumetric strain which shows a similar profile as the pore pressure given in Figure 4.14. These plots together with the other ones in this section can give a great insight into the mechanical response of a poroelastic material which in turn can lead to better accuracy of poroelastic properties determination from in vivo experiments.

(a) $t=50$ sec (b) $t=1000$ sec

Figure 4.11: Displacement distribution at $\phi = 0$ along the dimensionless radius at times $t=5, 10, 50, 100, 500$, and 1000 sec.

Figure 4.12: Displacement distribution at $\phi = 0$ along the dimensionless radius at times $t=5, 10, 50, 100, 500$, and 1000 sec. for case1

Figure 4.13: Volumetric strain $\epsilon_v = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ along the dimensionless radius at times $t=5, 10, 50, 100, 500$, and 1000 sec. for case1

Figure 4.14: Pore Pressure distribution along the dimensionless radius at times $t=5, 10, 50, 100, 500$, and 1000 sec.

CHAPTER 5

ANALYTICAL ANALYSIS OF EXTENDED CIRCULAR CYLINDRICAL MEMBRANE UNDER INFLATION: STATE OF THE ART

**Structural instabilities of pre-extended thin-walled circular cylindrical
tubes with application to aneurysms**

5.1 Introduction

The study of structural instabilities of the models subjected to significant mechanical stresses undergoing large deformation can be addressed by means of boundary value problems in the context of stability theory. Although analytical solutions require complex mathematical approaches they provide a better understanding of the problems. These approaches are exploited to provide us with prior knowledge of the mechanical response of the model at hand and validate the numerical results.

One important application of this work is the study of the effects of different factors on Aneurysms. The attention here is dedicated to the analytical solution at hand for extended thin-walled circular cylindrical tubes under inflation. The analysis of circular cylindrical cylinders has attracted many authors focusing on different aspects of this problem including various geometries, materials, and boundary conditions with many applications such as arteries [93] [51]. The mentioned parameters have important influences on the mechanical behavior of arteries and hence may affect aneurysm formation and propagation. The bulging of the wall of the aorta is called aortic aneurysm. This can take place anywhere in the vascular tree which is defined as a fusiform one when it is uniform and symmetric. The localized bulging, from an analytical point of view, is well developed in [52] and, for infinitely long tubes, in [93]. Fu and co-workers have achieved a bifurcation criterion for localized bulging [52].

In Section 5.2, a circular cylindrical tube subjected to a uniform internal pressure combined with a uniform axial loading is mathematically modeled. From the bifurcation analysis of the deformed circular cylindrical geometry, three modes are possible: an axisymmetric bulging; a prismatic mode; and a bending mode which is a composite mode bifurcation.

5.2 Basic equations

In this section, basic equations for the mathematical analysis of the described problem is given. The starting point is the equations for the geometry and deformation, followed by the material model, balance equations and bifurcation of a thin-walled tube due to a combination of axial loading and internal pressure.

5.2.1 Geometry and deformation

The geometry of a circular cylindrical membrane of diameter D , and length L in cylindrical coordinates, (R, Θ, Z) , and in the reference configuration

$\mathcal{B}_r$ is defined by

$$R = \frac{D}{2} = B, \quad 0 \leq \Theta \leq 2\pi, \quad -L/2 \leq Z \leq L/2, \quad (5.1)$$

Each material point in the undeformed configuration $\mathcal{B}_r$ has a position vector that can be written

$$\mathbf{X} = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z. \quad (5.2)$$

$\mathbf{E}_R$ and $\mathbf{E}_Z$ denote unit vectors in their subscript directions.

The cylinder is assumed to remain a circular tube after it is inflated and extended. p denotes the inner pressure and N the axial load. The equations that describe the cylinder in deformed configuration $\mathbf{x} \in \mathcal{B}$ are

$$\mathbf{x} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z, \quad r = \frac{d}{2} = b, \quad 0 \leq \theta \leq 2\pi, \quad -\ell/2 \leq z \leq \ell/2 \quad (5.3)$$

(r, θ, z) and ℓ indicate cylindrical coordinates and the length in the deformed configuration. The displacement is $\mathbf{u} = \mathbf{x} - \mathbf{X}$ and the general deformation gradient in cylindrical coordinate is

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} = \begin{bmatrix} 1 + \frac{\partial u_r}{\partial R} & \frac{1}{R} \frac{\partial u_r}{\partial \Theta} - \frac{u_\theta}{R} & \frac{\partial u_r}{\partial Z} \\ \frac{\partial u_\theta}{\partial R} & 1 + \frac{1}{R} \frac{\partial u_\theta}{\partial \Theta} + \frac{u_r}{R} & \frac{\partial u_\theta}{\partial Z} \\ \frac{\partial u_z}{\partial R} & \frac{1}{R} \frac{\partial u_z}{\partial \Theta} & 1 + \frac{\partial u_z}{\partial Z} \end{bmatrix}. \quad (5.4)$$

Due to the incompressibility condition the deformation gradient tensor is in the form $\text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$, in which, $\lambda_r \lambda_\theta \lambda_z = 1$. Consequently, we can write

$$\lambda_r = \lambda_\theta^{-1} \lambda_z^{-1} \quad (5.5)$$

where $\lambda_\theta = r/R > 0$ is the azimuthal principal stretch and λ_z is the axial stretch so that $\ell = \lambda_z L$.

5.2.2 Balance law

Cauchy stress $\boldsymbol{\sigma}$ is the stress with respect to the deformed area while the nominal stress $\mathbf{S}$ is defined as force divided by reference area. Throughout this paper, we assume that there are no body forces present. So, the

equilibrium equation to be satisfied is

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \quad \operatorname{Div} \mathbf{S} = \mathbf{0}, \quad (5.6)$$

where div and Div are the divergence operators with respect to deformed and reference configuration, respectively. Cauchy and Nominal stresses for incompressible materials are connected by

$$\boldsymbol{\sigma} = \mathbf{F} \mathbf{S} \quad (5.7)$$

$\boldsymbol{\sigma}$ is symmetric if there is no intrinsic couple stresses hence

$$(\mathbf{F} \mathbf{S})^T = \mathbf{F} \mathbf{S} \quad (5.8)$$

5.2.3 Hyperelastic material model

In this section the invariant analysis of an incompressible residually stressed hyperelastic body with a constitutive equation given by a strain energy function W is introduced. For an incompressible isotropic material, the independent invariants are

$$I_1 = \operatorname{tr} \mathbf{C}, \quad I_2 = \operatorname{tr} (\mathbf{C}^{-1}) \quad (5.9)$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the right Cauchy-Green deformation tensor, $\mathbf{F}$ is the deformation gradient, and tr denotes the trace of a second-order tensor.

Using a hyperelastic material, a strain energy function is needed to obtain a stress-strain relationship. Here the upturn of the stress-strain curve is not considered and relatively small strain increments are used. Therefore, Neo-Hookean strain energy function can provide an adequate simple first approximation to the behavior of rubber-like solids in the form [103] [136] [141]

$$W = C_{10}(\bar{I}_1 - 3) + \frac{1}{D_1}(J^{el} - 1)^2$$

Where C_{10} is the material constant that controls the shear behavior and can be determined from uniaxial, biaxial and planar tests, D_1 controls bulk compressibility, and $\bar{I}_1$ is the first deviatoric strain invariant (what causes distortion of the body) defined as $\bar{I}_1 = \bar{\lambda}_r^2 + \bar{\lambda}_\theta^2 + \bar{\lambda}_z^2$ where $\bar{\lambda}_i = J^{-\frac{1}{3}} \lambda_i$ (J is total volume ratio) [141]. In an incompressible material volume ratio is equal one, so, $\bar{\lambda}_i = \lambda_i$. Finally, strain energy function can be written as:

$$W = C_{10}(I_1 - 3) \quad (5.10)$$

5.2.4 Bifurcation of thin-walled tubes

For the analysis of bifurcation of circular cylindrical membranes, the energy minimization can be exploited to obtain the equilibrium equations for the

models with no residual stress. Incremental displacements with respect to the deformed configuration should be considered to investigate the possible bifurcation modes of the tube. the general form is

$$\delta \mathbf{u} = \delta u_r(\theta, z) \mathbf{e}_r + \delta u_\theta(\theta, z) \mathbf{e}_\theta + \delta u_z(\theta, z) \mathbf{e}_z. \quad (5.11)$$

In the case of extended circular cylinders, there are three possible bifurcation modes. A prismatic mode where the incremental displacement is assumed to be independent of z ($\delta u_z = 0$) without any loss of generality but the longitudinal dimension of the cylinder is lost. The second bifurcation is a bulging mode where the incremental displacements are assumed to be independent of θ ($\delta u_\theta = 0$). Finally, the composite mode, where all the incremental displacements are considered.

These equations are for a cylindrical membrane (i.e. $\sigma_{rr} = 0$ and no residual stress). In this thesis results for a membrane are exploited and in the next chapter, they are compared with the numerical results for a thick-walled tube to guide and validate them.

For this particular problem, one can write

$$\sigma_{\theta\theta} = \lambda_\theta \frac{\partial \hat{W}}{\partial \lambda_\theta} \quad \text{and} \quad \sigma_{zz} = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z}, \quad (5.12)$$

The equilibrium equations for a membrane give the inflating pressure in the form

$$p = \frac{H \hat{W}_{\lambda_\theta}}{R \lambda_\theta \lambda_z}, \quad (5.13)$$

where $\hat{W}_{\lambda_\theta}$ denotes $\frac{\partial \hat{W}}{\partial \lambda_\theta}$. Balance equation (5.13) can be seen graphically in 5.3.

In this section, we mention the state of the art of different bifurcation criteria. The finite element method is a helpful tool to deal with this problem. In what follows more information about the prismatic, bulging, and bending modes are provided which is helpful to validate and gain a better understanding of the numerical analysis provided in the next chapter.

5.2.4.1 Prismatic mode

Prismatic bifurcation can be considered as a consequence of the azimuthal surface tension. Since in thin-walled cylinder the amount of azimuthal stress generated by inflation is not enough and also the use of rubber-like material, this mode may not control the inflation problem [55]. However, because of the azimuthal component of residual stress, this mode should be considered as a possible bifurcation mode. In this bifurcation mode the section of the cylinder would not remain circular

In this section formulation of Prismatic mode is given. We consider incremental displacements of the form

$$\delta \mathbf{u} = \delta u_r(\theta) \mathbf{e}_r + \delta u_\theta(\theta) \mathbf{e}_\theta \quad (5.14)$$

After complex mathematical machinery (for more details see [127]) the equilibrium equations can be written as

$$\hat{W}_{\lambda_\theta \lambda_\theta} \frac{\partial \delta \lambda_\theta}{\partial \theta} = 0 \quad (5.15)$$

which is satisfied when

$$\hat{W}_{\lambda_\theta \lambda_\theta} = 0. \quad (5.16)$$

A more important result associated with the condition can be obtained by extracting the critical values of the pressure from 5.13, which is

$$p = \frac{\hat{W}_{\lambda_\theta \lambda_\theta} H}{R \lambda_z} \quad (5.17)$$

The bifurcation associated with 5.16 occurs at values of pressure greater than the pressure values given by 5.17.

5.2.4.2 Bulging mode

Bulging mode is a bifurcation mode of an inflated tube in which it remains a circular one. To obtain the mathematical formulation, an axisymmetric displacement field ($\delta u_\theta = 0$) of the form

$$\delta \mathbf{u} = \delta u_r(z) \mathbf{e}_r + \delta u_z(z) \mathbf{e}_z \quad (5.18)$$

needs to be considered.

Equilibrium in axial and radial directions of an infinitesimal volume element gives the differential equations

$$\frac{\partial}{\partial z} (\delta(\sigma_{zz} h r)) d\theta dz - p r d\theta dz \frac{\partial \delta u_r}{\partial z} = 0 \quad (5.19)$$

$$\delta(p r) d\theta dz - \delta(\sigma_{\theta\theta} h) d\theta dz + \frac{\partial}{\partial z} (\sigma_{zz} h r d\theta \frac{\partial \delta u_r}{\partial z}) dz = 0 \quad (5.20)$$

According to [127] after complex mathematical machinery, the bulging bifurcation criterion without axial extension is given by

$$f(\hat{W}, \lambda_\theta, \lambda_z) = 0 \quad (5.21)$$

Respectively. Where

Figure 5.1: Bulging instability in a long tube

$$f(\hat{W}, \lambda_\theta, \lambda_z) = \lambda_z^2 \hat{W}_{\lambda_z \lambda_z} (\lambda_\theta^2 \hat{W}_{\lambda_\theta \lambda_\theta} - \lambda_\theta \hat{W}_{\lambda_\theta}) - (\lambda_\theta \lambda_z \hat{W}_{\lambda_\theta \lambda_z} - \lambda_\theta \hat{W}_{\lambda_\theta})^2. \quad (5.22)$$

It follows that applying axial extension with inflation pressure bulging bifurcation happens prior to the pressure turning point given by

$$\hat{W}_{\lambda_z \lambda_z} = 0 \quad (5.23)$$

Among the instability modes of stretched pressurized cylinders bulging is likely to be the first that appears.

5.2.4.3 Bending mode: combination of prismatic mode and bulging mode

In the context of venous tortuosity blood vessels, such as retinal arteries and veins, they often become tortuous along their paths. The possible causes could be hypertension, diabetes, and retinopathy (see [56] and [146]). It has been shown that reducing axial extension in arteries can lead to the development of tortuous shape [61]. These are factors that modify the artery as a perfect circular cylindrical geometry. In addition, there are arteries that undergo significant bending, such as the superficial femoral artery. Specialized stent designs are needed in these arteries due to the large bending deformations [47]. When the artery shows significant bending saccular aneurysms seem to appear [47]. It is important to study different factors associated with the bending of an artery. A great insight will be given in the next chapter with the bending instability and it is important to describe here its mathematical formulation.

Bending instability of a perfect circular cylindrical membrane under internal pressure and axial loading is considered. The combination of prismatic and bulging mode may be interpreted in terms of bending.

From figure 5.2 and neglecting the bending stiffness according to the steps taken in [127] the equations for bending are mentioned.

It would seem realistic to expect that an increase in λ_z requires an increase in N (axial load). It is shown by [27] that $N = 0$ when $\lambda_\theta \lambda_z^2 = 1$, in which case inflation accompanies axial shortening in the absence of axial constraint, that is, $\lambda_\theta > 1$ implies $\lambda_z < 1$. It follows that for $\lambda_z = 1$ to be maintained during inflation, we must have $N > 0$, in other words, if we fix both ends of the cylinder, inflation would cause axial load [55]. Bifurcation associated with the bending mode is shown in [127]. The cylinder axis is subjected to virtual radial displacement δu_r^0 which in this case is because of residual stress. The balance of moment can be written as

$$M = N\delta u_r^0 - p\pi r^2\delta u_r^0 \quad (5.24)$$

where M is the value of resultant momentum. The left term of the right hand side of the equation is related to moment caused by axial load and the right term of right hand side of the equation is related to the moment caused by inner pressure. Stable equilibrium is associated to $M = 0$ ($N - p\pi r^2 = 0$ or $\delta u_r^0 = 0$). Hence, using 5.24, $N = 2\pi r h \sigma_{zz}$ and $p = \sigma_{\theta\theta} \frac{h}{r}$, this particular condition can be written

$$\sigma_{\theta\theta} - 2\sigma_{zz} = 0 \quad (5.25)$$

The unstable configurations is given by the criterion $\sigma_{\theta\theta} - 2\sigma_{zz} > 0$ as in [55].

This criterion is used for thick-walled tubes to guide the numerical analysis. There is no analytical solution for bending bifurcation under those circumstances.

Figure 5.2: Balance of forces for the bending mode. [127]

Figure 5.3: Analytical plot of Normalized inflation pressure vs λ_θ

CHAPTER 6

COMPUTATIONAL ANALYSIS OF BIFURCATION AND POST-BIFURCATION OF EXTENDED RESIDUALLY-STRESSED THICK-WALLED CIRCULAR CYLINDRICAL TUBES UNDER INFLATION

**Bifurcation and post-bifurcation of pre-extended residually stressed
thick-walled circular cylindrical tube with application to aneurysms**

6.1 Introduction

The study of different instability modes, in the context of large deformation stability, of thick-walled cylinders have advanced considerably in the last few years e.g. in [35] - [128]. Aneurysms formation and propagation, in other words, bifurcation and post-bifurcation, in the arterial wall tissue is an important scenario of interest for this study (see for example [35]- [3] and [5]- [52], and references therein). Some significant factors related to this pathology, among others, are geometry, nonhomogeneous material, growth, anisotropy, age, etc. Mechanical stresses play important role in regulating vascular cell biology, vascular function, and pathological development of vascular diseases (see [54], [70] and [72]). Therefore, mechanical modeling serves as the first step in order to gain a better understanding of the problem although the evolution and formation of an aneurysm is based on biochemical processes. The focus of this study is on this aspect considering the effects of residual stress.

Analytical solutions of circular hyperelastic membranes have given a better understanding of the problem at hand. However, in the case of bifurcation of thick-walled circular cylinders, they require complex mathematical machinery. Consequently, it is important to provide alternatives to these difficult formulations. Here the problem is solved by means of finite element method which is an appropriate tool to approach more complex problems. Modified Riks method is a numerical method that considers the arc length of the load-displacement path as the constraint equation which can be used in these cases. Finite element code Abaqus/Standard has been used in this study.

Residual stresses can exist due to several reasons such as growth and development, remodeling, plastic deformation, etc. In the study of blood vessels, the unloaded tube is commonly taken as the reference state in which the stress is zero everywhere. However, when an unloaded ring of an artery is cut longitudinally, its wall spring opens which is a proof of the residual stress existence. They have an important effect on the mechanical response of bodies such as the cardiovascular system. There are several ways in the literature to include residual stresses in the material such as defining a global virtual pre-stress field, assigning a pre-deformation, and including them in the constitutive equations. The mechanical response of a residually stressed material is different from the same material without them. This means that this type of stresses plays an important role in the deformation of the body. To consider the latter effect, we define the Strain Energy Function as a function of the residual stresses $W(\mathbf{C}, \boldsymbol{\sigma}_0)$ [128] [95]

An aneurysm, which is bulging of the wall of an artery, can be fusiform (when it is uniform and symmetric) or saccular (a small, lop-sided blister on

one side of the aorta). In the former case, the symmetric bulging instability mode in a cylinder is exploited while the saccular one can be studied in the post-bifurcation of bending instability [127]. The localized bulging is consistent with the aneurysm rupture. Moreover, the bending bifurcation of the presented model gives a great understanding of the causes and effects in the context of venous tortuosity and the aneurysm rupture under this condition. In this chapter, the effect of residual stress, as well as axial pre-stretch in the formation and propagation of these pathologies, is studied. This is an important application of the presented analysis is in the context of aneurysms formation and propagation in arterial wall tissue (see for example [127]-[3] and [5]-[34]). Fu and co-workers have made the most important contributions obtaining a bifurcation condition for localized bulging from an analytical view point [52]. The case with infinitely long tubes or tubes whose length is much greater than its radius are studied in [93]. Numerical simulations are carried out by Merodio and co-workers in a series of papers with the goal also to promote this (nonlinear) bifurcation analysis in engineering applications. However, these papers dealing with bulging did not include residual stresses which are the main focus of this chapter. The same residual stress formulation exists in [95] and [99] and in the context of helical buckling bifurcation analysis due to torsion in [128].

This chapter is organized as follows. In Section 6.2, the required basic equations to describe thick-walled circular cylinders from a mathematical viewpoint is given which includes the geometry, equilibrium equation including residual stress, the material law for a residually stressed hyperelastic case, and different bifurcations from the deformed circular cylindrical geometry of the tube. Three modes of bifurcation are possible: a prismatic mode in which the tube becomes noncircular with uniform cross-section; a bulging mode, which is axisymmetric; and a composite mode also known as bending bifurcation. In Section 6.3, the numerical framework to analyze the bifurcation and post-bifurcation of the problem at hand is provided. Numerical results are presented in Section 6.4. The captured Bifurcation modes are Bending and Bulging. The former happens in the cases with small axial pre-stretches while the latter is expected by imposing sufficiently large values of the axial pre-stretch. The plots of bulging bifurcation, as well as Post-bifurcation, are also studied. One observation is that the more axial pre-stretch we impose, the less is the pressure associated with the bulging bifurcation. Moreover, positive strength of the residual stress (that imposes tensile stresses in the outer surface and compressive ones in the inner surface of the cylinder's wall) causes a decrease in the pressure at the bulging instability while the negative one has the inverse effect. This is followed by the effects of the weight parameter for the residual stress invariants demonstrating when it is zero the residual stresses have more effects on the model

behavior. Post-bifurcation analysis of bulging shows a catastrophic instability in which the model can not support more pressure. The bulge is observed to expand radially in a uniform and symmetric shape which is consistent with the fusiform aneurysm rupture.

6.2 Basic equations

In this section, the mathematical description of the problem at hand is given. This includes the geometry, material model, equilibrium equations and bifurcation of a thick-walled tube subjected to combined axial loading and internal pressure. Detailed equations of the finite deformation boundary-value problem for the residually-stressed elastic body at hand can be found in [95].

6.2.1 Geometry

The geometry of a circular cylindrical tube of diameter D , thickness H , and length L which, in terms of cylindrical polar coordinates, (R, Θ, Z) , is defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad -L/2 \leq Z \leq L/2, \quad (6.1)$$

in the reference configuration $\mathcal{B}_r$ (A and B denote the internal and external radii). The position vector of a material point is given by

$$\mathbf{X} = R\mathbf{E}_R(\Theta) + Z\mathbf{E}_Z \quad (6.2)$$

where $\mathbf{E}_R$, $\mathbf{E}_\Theta$ and $\mathbf{E}_Z$ are unit vectors in the indicated directions.

After inflation and extension such that the cylinder remains a circular tube in the deformed configuration $\mathbf{x} \in \mathcal{B}$, the geometrical properties and displacement of the cylinder is described by

$$\mathbf{x} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z, \quad a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad -\ell/2 \leq z \leq \ell/2 \quad (6.3)$$

where (r, θ, z) , are cylindrical polar coordinates in this configuration. The deformed length of the cylinder is ℓ . Referred to cylindrical coordinates, the associated deformation gradient tensor $\mathbf{F}$ for an incompressible material has components $\text{diag}(\lambda_r, \lambda_\theta, \lambda_z)$, where $\lambda_\theta = r/R > 0$ is the azimuthal principal stretch and λ_z is the axial stretch, so that $\ell = \lambda_z L$. Because of the assumed incompressibility, the principal radial stretch is $\lambda_r = \lambda_\theta^{-1} \lambda_z^{-1}$. The displacement is $\mathbf{u} = \mathbf{x} - \mathbf{X}$.

6.2.2 Equilibrium equations including residual stress

The equilibrium equation of the Cauchy stress σ and the nominal stress S is the same as for thin-walled cylinders.

$$\text{div} \sigma = 0, \quad \text{Div} S = 0, \quad (6.4)$$

Residual stresses exist with no boundary surface traction and body force. If the reference configuration $\mathcal{B}_r$ contains residual stress σ_0 then $S = \sigma = \sigma_0$ which means that there is no difference between Cauchy and Nominal stresses as the deformation is measured from $\mathcal{B}_r$.

No intrinsic couple stresses are also assumed so, $\sigma_0^T = \sigma_0$. Consequently the rotational balance equations as well as the equilibrium equation are satisfied in the reference configuration

$$\text{div} \sigma_0 = 0. \quad (6.5)$$

The divergence of a tensor function (here, the residual stress) can be written as

$$\text{div} \sigma_0 = \begin{pmatrix} \frac{\partial \sigma_{0RR}}{\partial R} + \frac{\sigma_{0RR}}{R} + \frac{1}{R} \frac{\partial \sigma_{0\Theta R}}{\partial \Theta} + \frac{\partial \sigma_{0ZR}}{\partial Z} - \frac{\sigma_{0\Theta\Theta}}{R} \\ \frac{1}{R} \frac{\partial \sigma_{0\Theta\Theta}}{\partial \Theta} + \frac{\partial \sigma_{0R\Theta}}{\partial R} + 2 \frac{\sigma_{0R\Theta}}{R} + \frac{\partial \sigma_{0Z\Theta}}{\partial Z} \\ \frac{\partial \sigma_{0ZZ}}{\partial Z} + \frac{\partial \sigma_{0RZ}}{\partial R} + \frac{\sigma_{0RZ}}{R} + \frac{1}{R} \frac{\partial \sigma_{0\Theta Z}}{\partial \Theta} \end{pmatrix}. \quad (6.6)$$

The boundary condition which is no surface tractions for σ_0 can be written as

$$\sigma_0 n = 0 \quad \text{on} \quad \partial \mathcal{B}_r. \quad (6.7)$$

Since the trivial uniform solution for (6.7) is zero stress everywhere then, residual stresses are necessarily non-uniform and geometry dependent. This means inhomogeneous elastic response of the material [57]. For the geometry at hand a planar residual stress with components σ_{0RR} , $\sigma_{0\Theta\Theta}$ and σ_{0ZZ} i.e. there is no residual shear stress which is compatible with (6.7). Here it is also assumed that the Z component of the equilibrium equation is zero everywhere that satisfies the corresponding boundary conditions $\sigma_{0ZZ} = 0$ on the ends of the (finite length) tube. The remaining components, σ_{0RR} and $\sigma_{0\Theta\Theta}$, can then be homogeneous in azimuthal direction which means that they only depend on R . The non-trivial component of the equilibrium equation (6.5) is

$$\frac{d\sigma_{0RR}}{dR} + \frac{1}{R}(\sigma_{0RR} - \sigma_{0\Theta\Theta}) = 0, \quad (6.8)$$

with the boundary conditions from (6.7):

$$\sigma_{0RR} = 0 \quad \text{on} \quad R = A, B. \quad (6.9)$$

Note that if σ_{0RR} is known then $\sigma_{0\Theta\Theta}$ can be obtained by (6.8) as

$$d(R\sigma_{0RR})/dR. \quad (6.10)$$

6.2.3 Residually stressed hyperelastic material model

The cylinder is made of an incompressible hyperelastic material with a constitutive equation given by a strain energy function W . For an isotropic material, the independent invariants are, in the case of an incompressible material,

$$I_1 = \text{tr} \mathbf{C}, \quad I_2 = \text{tr} (\mathbf{C}^{-1}) \quad (6.11)$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the right Cauchy-Green deformation tensor, $\mathbf{F}$ is the deformation gradient, and tr denotes the trace of a second-order tensor. When the material is residually stressed, then it is also inhomogeneous.

Including residual stress in the mentioned material two more invariant are needed

$$I_5 := \text{tr}(\boldsymbol{\sigma}_0 \mathbf{C}), \quad (6.12)$$

$$I_6 := \text{tr}(\boldsymbol{\sigma}_0 \mathbf{C}^2). \quad (6.13)$$

Its response is considered to be elastic relative to $\mathcal{B}_r$, with dependence on $\mathbf{X}$ through $\boldsymbol{\sigma}_0(\mathbf{X})$, and we account for this by including $\boldsymbol{\sigma}_0$ in the argument of W . One can write

$$W = W(\mathbf{F}, \boldsymbol{\sigma}_0) \quad (6.14)$$

In the deformed configuration $\mathcal{B}$ in which W depends on $\mathbf{F}$ only via $\mathbf{C} = \mathbf{F}^T \mathbf{F}$. Then, (6.14) is automatically objective as $\boldsymbol{\sigma}_0$ is unaffected by rotations.

For an incompressible residually stressed material, W depends on nine invariants of $\mathbf{C}$, $\boldsymbol{\sigma}_0$ and their combination. The particular strain energy function for residually stressed incompressible hyperelastic we consider here is

$$W := \frac{\mu}{2}(I_1 - 3) + \frac{f}{2}(I_5 - \text{tr} \boldsymbol{\sigma}_0) + \frac{1-f}{4}(I_6 - \text{tr} \boldsymbol{\sigma}_0), \quad (6.15)$$

where $\mu = 2C_{10}$ is a material constant. The value $f \in \{0, 1\}$. If $f = 0$, then (6.15) does not include the invariant I_5 while if $f = 1$ it follows that (6.15) does not include the invariant I_6 . If there is no residual stress ($\boldsymbol{\sigma}_0 = 0$) then the material model is just the well known neo-Hookean one. Here, the initial stresses that we impose in the strain energy function are (see [95])

$$\sigma_{0RR} = \bar{\beta}(R - A)(R - B) \quad \text{and} \quad (6.16)$$

$$\sigma_{0\Theta\Theta} = \bar{\beta}[3R^2 - 2(A + B)R + AB], \quad (6.17)$$

where $\bar{\beta}$ is the strength of the residual stress with units of stress per square length. It can be non-dimensional taking $\bar{\beta} = \mu\beta/(2BT)$. This is an axisymmetric stress field with $\sigma_{0zz} = 0$.

Figure 6.1: Residual stress components for the described cylinder with $\beta = 0.5$

As pointed out in [1], we take these expressions for the residual stresses to be consistent with the ones of the opening angle method. The results are non-dimensionalized so, there is no need to specify values for the material constants. For values of the residual stress, we refer to [2].

W now can be expressed as a function of the mentioned invariants. Thus, it can be taken as $W = W(I_1, I_5, I_6)$, and use the notation $W_i = \partial W / \partial I_i$, $i = 1, 5, 6$ to write the Cauchy stress tensor as

$$\begin{aligned} \boldsymbol{\sigma} &= 2W_1\mathbf{B} + 2W_5\mathbf{F}\boldsymbol{\sigma}_0\mathbf{F}^T + 2W_6(\mathbf{F}\boldsymbol{\sigma}_0\mathbf{F}^T\mathbf{B} + \mathbf{B}\mathbf{F}\boldsymbol{\sigma}_0\mathbf{F}^T) \\ &- q\mathbf{I}, \end{aligned} \quad (6.18)$$

The notation $\mathbf{F}\boldsymbol{\sigma}_0\mathbf{F}^T$ for the Eulerian tensor is the transformation of residual stress $\boldsymbol{\sigma}_0$ from $\mathcal{B}_r$ to $\mathcal{B}$. q is a Lagrange multiplier that is necessary due to the incompressibility constraint. Finally, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green tensor. Equation 6.18 can be implemented in a UMAT in the same form and there is no need to variational formulation.

6.2.4 Bifurcation of thick-walled circular cylinders

In this section, the incremental displacements for each bifurcation mode with respect to the deformed configuration are given to investigate the possible bifurcation modes of the model. the general form is

$$\delta\mathbf{u} = \delta u_r(r, \theta, z)\mathbf{e}_r + \delta u_\theta(r, \theta, z)\mathbf{e}_\theta + \delta u_z(r, \theta, z)\mathbf{e}_z. \quad (6.19)$$

In the case of extended thick-walled circular cylinders, there are three possible bifurcation modes. A prismatic mode where the incremental displacement is assumed to be independent of z ($\delta u_z = 0$) without any loss of generality but the longitudinal dimension of the cylinder is lost.

$$\delta\mathbf{u} = \delta u_r(r, \theta)\mathbf{e}_r + \delta u_\theta(r, \theta)\mathbf{e}_\theta \quad (6.20)$$

The second bifurcation is a bulging mode where the incremental displacements are assumed to be independent of θ ($\delta u_\theta = 0$).

$$\delta\mathbf{u} = \delta u_r(r, z)\mathbf{e}_r + \delta u_z(r, z)\mathbf{e}_z \quad (6.21)$$

Finally, the composite mode, where all the incremental displacements are considered.

6.3 Finite element analysis

In this section, we provide information about the Finite Element analysis of the problem at hand which is a well-established procedure. Here, the numerical technique developed in the commercial code Abaqus [141].

6.3.1 Geometry

A cylinder with $L/D = 20$ and $H/D = 0.05$ is considered with the length of 200mm , diameter of 10mm and thickness of 0.5mm . Three-dimensional hybrid linear solid elements (C3D8RH in the Abaqus notation) is used. In general, as a purely hydrostatic pressure can be added without changing the displacements, an incompressible material behavior cannot be obtained in terms of the displacement history only. This the reason to use the hybrid formulation available in Abaqus solver. The pressure in a hybrid formulation is treated as an independent interpolated basic solution variable (degree of freedom) which is coupled to the displacement solution through a Lagrange multiplier, the constitutive theory, and the compatibility condition. In addition, this formulation is needed in order to avoid the problem of strain locking. In the thickness of the tube, five elements are taken in order to capture the residual stress field. In the azimuthal direction, 80 and 32 elements are taken to capture bending and bulging instabilities, respectively. In the axial direction, according to the different axial pre-stretches, different numbers of elements varying from 200 to 800 are considered. Figure 6.2 demonstrates a detailed view of the mesh.

6.3.2 Material model

Abaqus subroutine User MATerial (UMAT) is exploited to implement the material for incompressible hybrid elements. The Cauchy stress tensor σ must be provided. In particular, for a fully incompressible user material, the deviatoric stress must be provided [141]. The incompressibility condition is then imposed by the finite element code using a Lagrange multiplier.

In the following, the mathematical formulation required for material implementation is briefly given. For more details, see [128].

The Cauchy stress tensor is given by

$$\bar{\sigma} = \frac{2}{J} \mathbf{F} \frac{\partial \bar{W}}{\partial \bar{\mathbf{C}}} \mathbf{F}^T, \quad (6.22)$$

where $\bar{W}(\bar{\mathbf{C}}, \sigma_0)$ is another notation of the strain energy function (6.15) and $\bar{\mathbf{C}} := I_3^{-1/3} \mathbf{C}$ with $I_3 = \det \mathbf{C} = J^2$. Moreover,

$$\frac{\partial \bar{W}}{\partial \bar{\mathbf{C}}} = \frac{\partial \bar{W}}{\partial I_i} \frac{\partial I_i}{\partial \bar{\mathbf{C}}}, \quad (6.23)$$

which can be computed in this case as

$$\frac{\partial \bar{W}}{\partial \bar{\mathbf{C}}} = I_3^{-1/3} \left(c\mathbf{1} + \frac{f\sigma_0}{2} - \frac{cI_1\mathbf{C}^{-1}}{3} - \frac{fI_5\mathbf{C}^{-1}}{6} \right) + (1-f)I_3^{-2/3} \left(\frac{1}{4} \frac{\partial I_6}{\partial \bar{\mathbf{C}}} - \frac{I_6\mathbf{C}^{-1}}{6} \right). \quad (6.24)$$

The consistent Jacobian $\mathcal{C}^J := \frac{1}{J} \frac{\partial(J\boldsymbol{\sigma})^{\nabla J}}{\partial \mathbf{D}}$ can be computed as

$$\mathcal{C}_{ijkl}^J = \mathcal{C}_{ijkl}^c + \frac{1}{2}(\delta_{ik}\sigma_{lj} + \delta_{il}\sigma_{kj} + \delta_{jl}\sigma_{ik} + \delta_{jk}\sigma_{il}), \quad (6.25)$$

where

$$\mathcal{C}_{ijkl}^c = \frac{4}{J} F_{i\alpha} F_{j\beta} F_{k\gamma} F_{l\delta} \left. \frac{\partial^2 \bar{W}}{\partial \mathbf{C}^2} \right|_{\alpha\beta\gamma\delta}. \quad (6.26)$$

and

$$\begin{aligned} \frac{\partial^2 \bar{W}}{\partial \mathbf{C}^2} = & I_3^{-1/3} \left[-\frac{c}{3}(\mathbf{1} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{1}) - \frac{f}{6}(\boldsymbol{\sigma}_0 \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \boldsymbol{\sigma}_0) + \right. \\ & \left. \left(\frac{cI_1}{9}l + \frac{fI_5}{18} \right) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - \left(\frac{cI_1}{3} + \frac{fI_5}{6} \right) \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \right] + \\ & (1-f)I_3^{-2/3} \left[\frac{I_6}{9} \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} - \frac{1}{6} \left(\mathbf{C}^{-1} \otimes \frac{\partial I_6}{\partial \mathbf{C}} + \frac{\partial I_6}{\partial \mathbf{C}} \otimes \mathbf{C}^{-1} \right) - I_6 \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} + \frac{1}{4} \frac{\partial^2 I_6}{\partial \mathbf{C}^2} \right], \end{aligned} \quad (6.27)$$

which completes all the necessary details for material implementation.

6.3.3 Details of the analysis

Each simulation is carried out via three basic steps. In the first step, residual stresses are generated. The color map of Von Mises residual stress at the end of the first step is given in Figure 6.3. The radial and hoop residual stresses along the thickness direction are shown in Figure 6.4. Axial pre-stretch is imposed by imposing relative axial displacements at one end of the tube in the second step. Azimuthal displacements at both ends of the tube are prevented during the first and second steps. Finally, the internal pressure is applied in a modified Riks analysis ([29]). The magnitude of the inner pressure is $p = 10 * \mu * H/D$. There are two different boundary conditions in this step with respect to displacement restrictions according to two different types of analyses. In one type, hoop displacements are prevented to capture just bulging while, to capture all the possible instability modes in the other type, circumferential and axial displacements are restricted just at the ends of the tube. Any other movement is allowed along the tube.

Two concepts are introduced by the modified Riks method: the load proportional factor (LPF) and the arc length, for details, see [29]. The arc length is a quantity describing the evolution of the structure that combines displacements and loads. The load proportional factor indicates the applied load. This method is not very useful when the load proportional factor is monotonically increasing.

A sample curve of the load proportional factor against the arc length is shown in Figure 6.6. The maximum point is an indicator that the system ex-

Figure 6.2: One example of the mesh setting of the tube.

Figure 6.3: The typical color map of the von Mises stress shwing the residual stress distribution in its thikness.

Figure 6.4: Components of residual stress (non-dimensionalized with respect to the material constant $\frac{E}{2}$) in the normalized thickness of the model from Numerical analysis and the analytical one. A good agreement between the numerical and analytical results can be observed.

hibits an instability. LPF is proportional to the applied load. It is impossible to apply more loads than the maximum LPF under static equilibrium. Furthermore, a sudden and catastrophic rearrangement could be achieved by the structure for loads close to this point. An imperfection is introduced in the geometry of the problem to capture bifurcation which causes a smoother behavior of the structure as loading approaches the instability point.

One can follow the inflation of two points which are located in the middle of the thickness of the cylinder at two different cross sections to capture bulging. One point should be outside the bulge and the other point inside the bulge. Bulging bifurcation is identified when the hoop stretches of these two points differ. In this case, the bifurcation point is taken when the deviation of the hoop stretches of these two points is more than a small fraction of the hoop stretch average of both points (for instance, 1%). Post-bifurcation can be followed in this framework. There are two criteria that help to capture the onset of the bending mode. First, in analogy with the membrane analytical analysis in which unstable configurations is given as

Figure 6.5: A bent cylinder the mode $m = 1$

Figure 6.6: Load Proportionality Factor against arc length. The maximum is associated with an instability

$\sigma_{\theta\theta} - 2\sigma_{zz} > 0$. In a thick-walled tube, the values of stresses considered in that condition are average stresses at the cross sections of the tube. Second, the displacement analysis of the model also helps to establish the onset of bending. The numerical results in the next section show that the onset of bending is associated with the mode $m = 1$ which is shown in 6.5 (see [93] for details).

6.4 Numerical results

In this section, the numerical results of the described problem are provided in the non-dimensional form e.g. the stress is non-dimensionalized with respect to the material constant $\frac{\mu}{2}$. A small imperfection at the center of the tube with a maximum deviation of $0.001B$ with respect to the reference configuration is introduced in order to avoid a strict numerical bifurcation (see for further details [4]). 5 different cases in terms of residual stresses are considered: four with different residual stresses from combinations of the values of $f \in \{0, 1\}$ and $\beta \in \{-0.5, 0.5\}$ and one with Neo-Hookean material. A positive value of β imposes tensile azimuthal residual stress component in the outer wall of the tube and a negative one has the inverse effect.

The numerical results highlight that the bifurcation modes are either bulging or bending. In addition, the onset of bending is associated with the mode $m=1$. In order to distinguish between two bifurcation modes the values of $\sigma_{\theta\theta} - 2\sigma_{zz}$ (average stresses of the tube middle cross-section) versus arc length for two cases are provided in Figure 6.7. One case with $\lambda_z = 1$, $\beta = 0.5$ and $f = 0$ and the other with $\lambda_z = 1.4$, $\beta = 0.5$, and $f = 0$. The first case obey $\sigma_{\theta\theta} - 2\sigma_{zz} > 0$ (solid line) indicating that bending may give the onset of bifurcation although this is not enough to establish the bifurcation scenario. On the other hand, the second case obey $\sigma_{\theta\theta} - 2\sigma_{zz} < 0$ (dashed line) which can be interpreted as the bending mode is not expected.

In order to establish the bifurcation scenario values of LPF vs arc length for the first case are shown in Figure 6.8, dashed line. As it is explained previously the analysis can be restricted to capture just bulging and the same case is considered under those circumstances. The corresponding values of LPF vs arc length for this simulation are also shown in Figure 6.8, solid line. A simple observation of the two curves given in Figure 6.8 shows that the two curves at a specific point separate. This point is taken as the bending bifurcation and, therefore, the onset of bifurcation is not bulging. The maximum point of the dashed line is associated with the formation of a bulge (after bending), which will be explained later in detail.

In order to illustrate more details of the bifurcation of cases $\lambda_z = 1$, $\beta = 0.5$ and $f = 0$ and $\lambda_z = 1.4$, $\beta = 0.5$, and $f = 0$ the focus is now on the stresses prior to the onset of bifurcation in each case. Figure 6.9 shows values of azimuthal and axial average stresses on each cross-section along the tube length. For the first case prior to the bending bifurcation. Figure 6.10 is the same plot for the second case prior to the bulging bifurcation. These two Figures are representative of other simulations for other values of β and f .

Figure 6.7: Check of bending criterion $\sigma_{\theta\theta} - 2\sigma_{zz}$ for two models. One with bending bifurcation ($\lambda_z = 1$, $\beta = 0.5$, $f = 0$) and another for bulging bifurcation ($\lambda_z = 1.4$, $\beta = 0.5$, $f = 0$). The stresses are averaged over the cross section of the middle of the tube.

Figure 6.8: Plot of Load Proportionality Factor against arc length for a model with $\lambda_z = 1$, $\beta = 0.5$ and $f = 0$ with different bifurcation modes. The model shown by the solid line is restricted to just bulging mode (bending is not allowed) while, all the bifurcations are allowed in the other one (dashed line).

Figure 6.9: Average of azimuthal and axial components of Cauchy stress on each cross-section Vs. the position in the length of the tube for $\beta = 0.5$, $f = 0$ and $\lambda_z = 1$ at the bending bifurcation.

Figure 6.10: Average of azimuthal and axial components of Cauchy stresses on each cross-section Vs. the position in the length of the tube for $\beta = 0.5$, $f = 0$ and $\lambda_z = 1.4$ at the bulging bifurcation.

Figure 6.11: Normalized inflation pressure against λ_z at the bulging mode for the five models with different residual stresses that are restricted to capture just bulging mode.

Figure 6.12: Normalized inflation pressure against λ_z at the onset of bifurcation for the five models with different residual stresses that are free to capture all the bifurcation modes.

To investigate the effect of residual stress on the bifurcation of the model at hand, first, consider simulations that only capture bulging for the five models at hand. Figure 6.11 shows the results of these analyses, which gives values of pressure vs axial stretch at the onset of bulging bifurcation. On the other hand, corresponding results for simulations that are not restricted to capture just bulging are shown in Figure 6.12. Values for the case with no residual stress, $\beta = 0$ are the same in both cases. The considerable difference between Figure 6.12 and Figure 6.11 for the other cases reveal the effect of residual stress on the initiation of bifurcation. A comparison between the curves for the cases with residual stress in Figure 6.12 and Figure 6.11 highlights that the onset of bifurcation for sufficiently small values of λ_z is not bulging. From Figure 6.12 it can be concluded that the domain of bending bifurcation obeys approximately $1.2 > \lambda_z \geq 1$. Numerical simulations show that the bending mode is $m = 1$. For values of $\lambda_z > 1.2$, both figures are identical, which indicates that bifurcation is associated with bulging. On the other hand, different residual stresses cause no considerable quantitative differences at the onset of bending as it is shown in Figure 6.12 where the curves of all these four cases are very close to each other. However, in the case of bulging bifurcation, for a particular value of λ_z , there are considerable quantitative differences among these four cases. In particular, the values of pressure at bulging bifurcation for the models with tensile residual hoop stress in the outer wall of the tube (positive value of β) are lower than the one of the cases with compressive residual hoop stress in the outer wall of the tube (negative value of β). It follows that the quantitative effect of residual stress for the models with the invariant I_5 ($f = 0$) is more than the cases with the invariant I_6 ($f = 1$).

The investigation of the post-bifurcation can give a better insight into the application of this work to aneurysm formation and propagation. First, the post-bifurcation of bulging is studied. In all cases, the propagation of bulge is radially (localized) which is consistent with an aneurysm rupture as it cannot sustain more pressure. Figure 6.13 shows the radial expansion of bulging. In this figure values of pressure vs hoop stretch when $\lambda_z = 1.2$ for different material models are shown and compared by the membrane analytical solution given by (5.13) prior to bifurcation (solid line). Moreover, in the numerical case, this plot is extracted for two nodes: one in the bulge (the blue dashed line) another outside of the bulge (the black dashed line). The decrease of azimuthal stretch (when bifurcation occurs close to maximum points in each curve) of the latter case demonstrates a localized bulge. Furthermore, the pressure decreases showing a catastrophic instability which would be associated with aneurysm rupture.

In the case of bending bifurcation, the instability is ductile meaning that the structure in the post-bifurcation can support higher pressures. The load-

Figure 6.13: The plot of Normalized pressure Vs. azimuthal stretch from analytical analysis of a membrane and numerical analysis of the case $\lambda_z = 1.2$, $\beta = 0$, $f = 0$ (Neo-Hookean material). They show similar response although, as expected, the values of the pressure at a fixed azimuthal stretch of the analytical one is less than the one of the thick-walled cylinder. This is due to neglecting the thickness in a membrane. Moreover, in the numerical case, this plot is extracted for two nodes: one in the bulge (the blue dashed line) another outside of the bulge (the black dashed line). The decrease of the azimuthal stretch of the latter case demonstrates a localized bulge.

ing continues until a bulge appears on one side of the tube showing an irregular shape which is consistent with the development of abdominal aortic aneurysm (AAA). It is also shown in Figure 6.14 and Figure 6.8 that in this case, the values of pressure at the formation of the bulge are lower than the ones associated with the simulation in which bending is not allowed and the onset of bifurcation is bulging. Figure 6.14, for the case $\beta = 0.5$ and $f = 0$, shows three curves for small values of λ_z . The dashed-dot line is for the model restricted to capture just axisymmetric bulging mode. the solid line is for the same material but free to capture all the bifurcation modes and the dashed line is for the one-sided bulge happening in the post-bifurcation of bending. This reveals the effect of the bending bifurcation on aneurysm rupture of tortuous arteries. Moreover, it is consistent with the development of abdominal aortic aneurysm (AAA) and for sufficiently small values of λ_z delayed aneurysm formation is possible if bending is not permitted.

In the end, several graphical representations of the initiation and development of different bifurcation modes are provided. Figure 6.15 shows the

Figure 6.14: Three different instabilities of models with the same material $\beta = 0.5$ and $f = 0$ against different axial pre-stretches. The dashed-dot line is for the model restricted to capture just axisymmetric bulging mode, the solid line is for the same material but free to capture all the bifurcation modes and the dashed line is for the one-sided bulge happening in the post-bifurcation of bending. This reveals the effect of the bending bifurcation on aneurysm rupture of tortuous arteries.

Figure 6.15: A graphic representation of the initiation and development of the bulging. Different configurations of a bulging instability, the beginning is shown on the left hand-side from where the development is demonstrated up to the final configuration on the right handside for the Neo-Hookean material with $\lambda_z = 1$.

Figure 6.16: A graphic representation of the initiation and development of the bending bifurcation as well as the one-sided bulge at the end of it. Different configurations of the bending instability, the beginning is shown on the left handside from where the development is demonstrated up to the final configuration on the right handside for the case with $\beta = 0.5$, $f = 0$ with $\lambda_z = 1$.

bulging onset and propagation. The material of the model in this Figure is without residual stress (Neo-Hookean one). Finally, for completeness, the graphic representation of the initiation and development of the bending bifurcation as well as the one-sided bulge for the model with $\beta = 0.5$ and $f = 0$ (the same case as in Figure 6.14) for different axial pre-stretches, namely, $\lambda_z = 1$, $\lambda_z = 1.04$, $\lambda_z = 1.12$, and $\lambda_z = 1.18$ are provided. The general conclusion of these figures is that the more axial pre-stretch, the less curvature is. Figures 6.16 to 6.19 demonstrate the transition from the onset of bending to the onset of bulging as well as the formation of the one-sided bulge. Moreover, for sufficiently large values of λ_z the tube does not deflect and an axisymmetric bulge develops.

Figure 6.17: A graphic representation of the initiation and development of the bending bifurcation as well as the one-sided bulge at the end of it. Different configurations of the bending instability, the beginning is shown on the left handside from where the development is demonstrated up to the final configuration on the right handside for the case with $\beta = 0.5$, $f = 0$ with $\lambda_z = 1.04$.

Figure 6.18: A graphic representation of the initiation and development of the bending bifurcation as well as the one-sided bulge at the end of it. Different configurations of the bending instability, the beginning is shown on the left handside from where the development is demonstrated up to the final configuration on the right handside for the case with $\beta = 0.5$, $f = 0$ with $\lambda_z = 1.12$.

Figure 6.19: A graphic representation of the initiation and development of the bending bifurcation as well as the one-sided bulge at the end of it. Different configurations of the bending instability, the beginning is shown on the left handside from where the development is demonstrated up to the final configuration on the right handside for the case with $\beta = 0.5$, $f = 0$ with $\lambda_z = 1.18$. The last one shown on the right is a magnification of the one-sided bulge, which is propagating radially. Furthermore, for sufficiently large values of λ_z the tube does not deflect and an axisymmetric bulge develops.

CHAPTER 7

CONCLUSION AND FUTURE WORKS

In this thesis, in general, mechanical modeling of poroelastic materials, as well as residually stressed hyperelastic materials, is carried out. Numerical frameworks for this objective are developed. This work is divided into two categories according to their principal application, namely, cancer mechanics and Aneurysm.

7.0.1 Cancer mechanics

7.0.1.1 *Multiscale*

The starting point of this work is a review and slight modification of multiscale analysis for poroelastic media. The chosen upscaling process is Asymptotic homogenization which provides an analytical approach to achieve the macroscale system of PDEs from microscale (or physical scale) in which the microscale information is encoded in some coefficients. This technique also provides PDEs for cell problems to be solved in the solid and fluid phases of the cell domain [18, 106]. By solving these problems the coefficients of macroscale system of equations can be determined. A three-dimensional analysis to determine the role of porosity and solid ma-

trix compressibility on the mechanical and hydraulic properties of poroelastic materials is performed. Numerical analyses by means of finite elements, making use of material properties that are typical of biological and cancerous tissues, are carried out. A robust and accurate computational framework which can be used to obtain the three-dimensional numerical solutions of the cell problems is provided and exploited to determine the effective Young's and shear moduli, Poisson's ratio, hydraulic conductivity, Biot's modulus, and Biot's coefficient. Although a simple cross-shaped (cylindrical) interstitial phase is chosen this computational framework can be readily generalized to account for highly complicated pore-scale structures.

The results provided in this research work show the interplay between compressibility and porosity and their combined effect on the mechanical and hydraulic properties of the poroelastic media.

First, we consider non-dimensionalized Hydraulic conductivity that demonstrates a non-linear decrease with a negative curvature by reducing the porosity. This means that the fluid transport in poroelastic media becomes slower for the smaller radius of the pores. Effective shear modulus is not subjected to considerable change for the cases with various compressibilities, especially, when the porosity is high. However, a considerable linear decrease can be seen by increasing the porosity. As expected, the effect of the fluid inside the pores on the latter modulus is negligible as its strength is much lower than the solid phase which implies that the drained and undrained values are almost equal. Similar behavior is observed in the profile of the effective Young's modulus. On the other hand, the effect of solid matrix Poisson's ratio on the effective one is considerable, particularly, for the cases with low porosity. The value of this coefficient (drained one) decreases sharply by increasing the porosity or solid matrix compressibility, however, considering the fluid inside the pores it increases and approaches to incompressibility for larger porosity. The reason is that the fluid in the upscaling process is assumed to be incompressible so, by increasing the fluid volume fraction the overall behavior of the cell approaches to the one of fluid. Moreover, the increase in saturated effective Poisson's ratio Vs. porosity is non-linear with positive curvature. Similar behavior with more curvature is observed in the profile of Biot's coefficient Vs. fluid volume fraction in which Biot's coefficient approaches to 1 for large porosities. One of the novel and important results of this study is the variation of Biot's modulus against different porosities that shows an absolute minimum for compressible solid matrix. As Biot's modulus is related to the average variation of interstitial fluid volume in response to variations of the interstitial pressure, the provided results can be exploited to relate compressibility and fluid (and hence drug) transport in compressible tumors. Starting from low values of

pore volume fraction, the Biot modulus reduces non-linearly with positive curvature up to the minimum point then, it soars and approaches infinity for large values of porosity. Technically, there are two factors determining the overall behavior of Biot's modulus, namely the divergence of the seventh solid cell problem's solution and the volume fraction of the solid phase. The former increases by reducing the solid's volume while it decreases sharply increasing the Poisson's ratio of the solid phase. On the other hand, this coefficient is computed for the poroelastic medium that means the incompressible fluid inside is taken into account. It is also averaged over the solid phase so, the volume fraction of solid has an effect on the result. The overall profile of the Biot's modulus is a consequence of the interplay between the mentioned factors. Noteworthy is that having an incompressible solid matrix the decrease prior to the definite minimum becomes negligible and so the minimum itself. The profile of this coefficient against the Poisson's ratio of the solid matrix demonstrate intersections which are the result of the above-mentioned interplay. Furthermore, the results can pave the way for biomimetic applications concerning the optimal design of (biological) artificial constructs with respect to the average interstitial volume. Poroelastic artificial constructs such as scaffolds and implants are actually encountered in a number of biomedical applications, such as bone in-growth and eye implants, see, e.g., [68] and [23, 63]

Some simplifying assumptions such as the geometry (and periodicity of the microstructure), linear elastic type solid matrix, and also neglected vascularization and growth of the tumor mass is considered. However, the presented framework can be readily generalized to more complicated (tortuous) structures, as well as anisotropy of the elastic matrix, depending on the tissue type at hand.

Finally, a sensitivity analysis of the mesh setting is carried out to show the reliability of the results provided in this research work.

The next natural step is the implementation of the macroscale poroelastic governing equations to capture the influence of poroelastic deformations of the solid tumor (possibly accounting for its heterogeneous microstructure informed by appropriate medical images) on the spatio-temporal profile of the fluid interstitial pressure. Predictions from such a model could provide meaningful insights into the optimization of drug injection conditions [101]. In fact, our predictions can also provide a basis in the context of poroelastic vascularized tumors [113] (by accounting for more sophisticated geometries in order to account for the blood vessels network), where the intra-grain porosity (i.e. the porosity of the solid matrix itself) is explicitly taken into account, see also [6]. Solutions arising from the coupling between this model and multiscale drug transport in solid tumors, as addressed for exam-

ple in [74, 109, 140], will provide predictions that could be used to design improved anti-cancer therapies for deformable tumors.

7.0.1.2 *Macroscale*

A numerical framework for the implementation of fully-coupled quasi-static anisotropic macroscale system of equations for poroelastic media derived from Asymptotic homogenization is presented and validated by means of two analytical works. This is done by exploiting the analogy between the latter and the fully-coupled thermal-displacement system of equations that is well implemented in a commercial software ABAQUS. The importance of considering underlying microstructure solid skeleton of the poroelastic medium, which is one important advantage of this framework, is shown via studying four different cases with different microscale mechanical properties that result in anisotropic and compressible macroscale effective elasticity tensor as well as different macroscale coefficients. The application of this study can be seen in the estimation of poroelastic properties of the human tissues (healthy and cancerous) using Poroelastography. It is shown that there are several different important factors in the latter such as the loading condition (immediate or gradual application), time, anisotropy, displacement distribution profile, porosity, etc. Moreover, measuring the displacements, relative fluid velocity and their divergence give us the best estimation of the material properties. Several different time spans, microscale material properties, and different analyses types (Transient and Steady-state) are analyzed.

One very important advantage of the presented framework is that it is derived from an analytical homogenization tool (Asymptotic homogenization) which means one can directly introduce the macroscale coefficients as a function of microscale properties. Here, the application of the latter in Poroelastography is shown. Moreover, the mentioned advantage has some very important applications such as non-linearity and the growth of poroelastic tissue that can be interesting future studies. This work represents the first step to build up a computational framework for multiscale computational analysis of growing poroelastic media such as tumors [106]. The macroscale coefficients which hold the information of the underlying microscale properties are subjected to changes due to the growth condition (e.g. a change in the porosity). Having a robust dataset from the presented study the coefficients can be modified at the beginning of each increment in order to consider the effects of the growth.

7.0.2 Aneurysm

A numerical framework is provided in order to capture bifurcation and post-bifurcation of residually stressed thick-walled tubes subjected to axial pre-stretch combined with inflation pressure. An analytical approach from previous studies (for example in [127]) to solve a circular cylindrical membrane is mentioned which gives a better understanding of the numerical analyses. the latter which is based on the finite element method is an alternative tool to solve thick-walled residually stressed models without the complexity of analytical analyses.

A numerical framework to implement residual stresses that are in equilibrium in the reference configuration and absence of external loads has been developed by means of a commercial finite element code. It has been shown that residual stresses have considerable qualitative effects on the results compared to the model with no residual stress. It seems of great importance to consider residual stresses to model biological tissues such as arteries and their associated cardiovascular diseases such as aneurysms. It is expected that other residual stress fields and boundary conditions may give rise to other modes of bifurcation. The presented framework here can be easily modified include other conditions.

Several factors affect the critical pressure and axial stretch associated with mechanical buckling. Here, the dependence of bifurcation behavior of tubes under the loading at hand on residual stresses has been addressed. The captured Bifurcation modes are Bending and Bulging. The former happens in the cases with small axial pre-stretches while the latter is expected imposing sufficiently large values of the axial pre-stretch. The plots of bulging bifurcation as well as Post-bifurcation are also studied. One observation is that the more axial pre-stretch we impose, the less is the pressure associated with the bulging bifurcation. Moreover, positive strength of the residual stress (that imposes tensile stresses in the outer surface and compressive ones in the inner surface of the cylinder's wall) causes a decrease in the pressure at the bulging instability while the negative one has the inverse effect. This is followed by the effects of the weight parameter for the residual stress invariants demonstrating when it is zero the residual stresses have more effects on the model behavior. Post-bifurcation analysis of bulging shows a catastrophic instability in which the model can not support more pressure. The bulge is observed to expand radially in a uniform and symmetric shape which is consistent with the fusiform aneurysm rupture. Furthermore, the study of stresses at the bulging bifurcation points out the stress concentration in the area that the bulge is happening. In the Bending cases, the graphs demonstrate that although the residual stresses play an important role in triggering the bending instability they do not have

a considerable effect on the pressure associated with this bifurcation. It is also practiced that by imposing azimuthal restrictions to the displacements on the outer/inner surface of the cylinder we can limit the bifurcation to be just bulging, in other words, we can avoid bending. This is used to illustrate the considerable effect of bending bifurcation. By increasing the axial pre-stretch this bifurcation happens at the higher pressures up to a transition area in which the bending happens at pressures very near to the bulging. In this area, we have observed a change in the shape of the bulge but not considerable effect on the associated pressure which is consistent with saccular aneurysm of the aorta. The study of post-bifurcation of this instability reveals the fact that the structure demonstrates a ductile instability which means that in subsequent motion after the onset of bifurcation it can support higher pressures. The loading continues until a one-sided bulge is obtained which is consistent with the saccular aneurysm. Furthermore, the maximum inflation pressure associated with this case is less than one of the same case when bending is avoided (using azimuthal displacement restrictions). This shows the possibility of delayed aneurysm formation if bending is not permitted. In addition, the study of axial and azimuthal stresses along the length of the cylinder shows very good agreement with the criterion provided by analytical analysis.

The residual stress field in this work has just azimuthal and radial components. Moreover, it is homogeneous in azimuthal and axial directions. Therefore the development of a three-dimensional residual stress field in which there exist axial, as well as shear components, is to be considered as future work. Moreover, the effects of design parameters of the model at hand such as the geometry and boundary condition are of interest for a next step based on what has been done in this study.

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