UNIVERSIDAD POLITÉCNICA DE MADRID

**ESCUELA TÉCNICA SUPERIOR DE INGENIERÍA
AGRONÓMICA, ALIMENTARIA Y DE BIOSISTEMAS**

**ESTABILIDAD E INESTABILIDAD EN LA
PROPAGACIÓN NO LINEAL DE HACES
ÓPTICOS DE BESSEL CON MOMENTO
ANGULAR ORBITAL**

TESIS DOCTORAL

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Licenciado en Ciencias Físicas

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**STABILITY AND INSTABILITY IN THE
NONLINEAR PROPAGATION OF OPTICAL
BESSEL BEAMS WITH ORBITAL ANGULAR
MOMENTUM**

Dissertation presented for the Ph.D Degree by

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À Joaquina, que sempre perguntou como iam os meus estudos, ao António, que não teve a oportunidade de fazê-lo, e ao Xavier, que espero que um dia tenha o interesse e a paciência de lê-lo até o final.

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“Nah, you won’t see me in one of those Ph.D. programs, that’s for sure.”

Myself, as a last-year undergraduate.

And still, here I am finishing my Ph.D. dissertation, and actually having a good time writing it down (although I can’t wait to finish it). However, I always thought that getting a Ph.D. wasn’t for me. Not because I didn’t want to keep in touch with Physics (which I deeply love) or didn’t want to deal anymore with the hardships of being a part-time student, full-time worker – but because I know I don’t fit well in strict hierarchical groups, and therefore academic life wasn’t the best of choices under my point of view.

Five years ago I had just finished my MSc., and thought to go back to a standard working routine, while trying every now and then to do (somehow) a bit of Physics. I came back to a full-time job (never completely quitted, bills have to be paid after all, but I did reduce my working hours in order to finish my MSc. and avoid the dreaded Bologna Process), and after a few months working I was lucky enough to come around one of my dearest (and best) professors from my undergraduate years: Professor Isabel Gonzalo Fonrodona. She talked to me about the Ph.D. program in the *Grupo de Sistemas Complejos* at the *Universidad Politécnica de Madrid*, which had already drawn my attention with a few posters I saw around during past years, but which I never really considered as an option. She encouraged me to try it out, saying that it was a cozy group to start doing some serious research and introduced me to the person that would later be my doctoral advisor. With a lot of doubts, I accepted the challenge, as I saw it was a program with a heavy computational component that maybe would be easier to match with my job as a software analyst than, say, a purely theoretical physics Ph.D. Fortunately I was right about the compatibility issue, yet it would still prove itself hard to keep up the level of commitment required.

At the beginning, I was not very sure what research path to take inside the enormously rich field of complex systems, and flirted with a couple of areas before ending up in nonlinear wave dynamics. Here, I would like to mention Professor Ramón Alonso Sanz, who introduced me to the marvelous world of cellular automata and with whom I wrote my first scientific paper, which I really appreciate. I had a lot of fun with the CAs, and I hope someday to come back and study them a bit more in detail. It sure is a fruitful field that allows the researcher a lot of creativity, and that is always something to be grateful for.

Nevertheless, after an uncertain first year, nonlinear waves and their vortices finally caught me up. And also my doctoral advisor, Professor Miguel Ángel Porras. It's hard to express how thankful I am for his generosity in sharing with me both his time and his work the way he did. He didn't teach me all the Physics I know, but he definitely taught me how to persevere in discovering the Physics I don't know (which is a lot more!), and that's priceless. Thank you, Miguel Ángel. And as a bonus, I was also very lucky to have a senior Ph.D. candidate colleague that still to this day shows me the way to follow through when I don't know where the end of the tunnel is, Dr. Carlos Ruiz Jiménez. You helped me a lot more than you can imagine, believe me, both with our work and your words. During the last years I learned much from the two of you, and your patience and kindness were always there, during good and bad moments. I will never forget the endless rehearsal for my presentation at the *XXXVI Bienal de Física* in Santiago de Compostela in 2017, and the adrenaline rush when I finally got it right for the first time. And I truly have the feeling to have collaborated in bringing some new and useful knowledge around, which makes everything worthwhile.

Also, I would like to thank the *Grupo de Sistemas Complejos*, in particular Professor Rosa María Benito, for kindly accepting me in the research group and offering me the opportunity to know first-hand what the world of scientific research is; and Professor Juan Carlos Losada, for always volunteering when help was needed, both for scientific and organizational matters. The research group is fairly large, and it would be out of place to name everybody here, but I'd like to say I'm grateful for the kindness I always received from all of you, and that I'm sorry for the CPU time Matlab[®] ruthlessly ate up at the servers with our endless numerical simulations.

Some special words of gratitude must be dedicated to Marta Antón, who helped me a lot with the front cover during those idle moments at work. I guess my sense of aesthetics needed more inspiration than I expected in first place.

And at last, but surely not least, I would like to thank my family for putting up with me during my long isolation periods necessary to do what I had to do these last years. In fact, I wanted to make a point with that by using quotations from people who are closest to me to introduce each chapter, instead of picking more glamorous ones from historically relevant characters. I would like this choice to be seen not as the result of indolence or arrogance, but as a way of thanking people that, although feeling distant from the “weird things” physicist are usually interested in, they should know that they are still part of it all, at least for me. I hope I will now have more spare time to make up for it correctly. Thank you Marisa, Xavier, Ana and Nico for your support. This particular acknowledgement may be short, but it is from the heart.

Madrid, 2019.

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“Nah, yo no pienso meterme en un programa de doctorado, seguro.”

Yo, en mi último curso de la licenciatura.

Y pese a ello, aquí estoy, terminando de escribir mi tesis, y en realidad pasándomelo muy bien mientras lo hago (aunque con unas ganas tremendas de terminarlo de una vez). Pero siempre he pensado que un doctorado no era lo adecuado para mí. No porque no quisiera seguir haciendo cosas en Física (que es algo que me gusta muchísimo) o porque ya estuviese harto de tener de lidiar con los problemas de trabajar a jornada completa y seguir estudiando e investigando a la vez – sino porque soy consciente de que no encajo bien en grupos con una estructura jerárquica demasiado encorsetada, y por lo tanto la vida académica no sería la decisión más acertada bajo mi punto de vista.

Hace cinco años yo estaba terminando la licenciatura, y pensaba en volver a la rutina laboral de siempre y en los ratos libres encontrar una manera hacer alguna cosilla relacionada con la Física. Volví de hecho a un trabajo a jornada completa (nunca dejé de trabajar del todo, hay que pagar las cuentas de alguna manera al fin y al cabo, pero sí que trabajaba menos horas para poder terminar la convalidación de mi licenciatura y evitar tener de pasar al nuevo grado del Plan Bolonia), y pasados algunos meses tuve la suerte de hablar con una de mis más queridas (y mejores) profesoras de la Complutense: la profesora Isabel Gonzalo Fonrodona. Ella fue quién me habló por primera vez del programa de doctorado del Grupo de Sistemas Complejos de la Universidad Politécnica de Madrid, en el que ya me había fijado yo en años anteriores por algunos pósteres publicitarios, pero que realmente nunca había sido una opción para mí. Ella me animó a probarlo, comentando que era un grupo pequeño, casi “familiar”, interesante para empezar a hacer algo de investigación en serio, y me presentó la persona que más tarde sería mi director de tesis. Con muchas dudas sobre si valdría la pena, acepté el reto al ver que el programa tenía una fuerte componente de computación y simulación que podría ser más fácilmente encajada con mi vida laboral como analista informático que, por ejemplo, un doctorado en el campo de la Física Teórica. Felizmente, yo estaba en lo cierto sobre fijarme en este tema de cómo conciliar trabajo e investigación, y aun así fue un periodo muy difícil para conseguir mantener el nivel de exigencia que el doctorado necesitaba.

Al principio no me era claro cuál sería la mejor área de investigación a seguir dentro de un campo tan rico como el de los sistemas complejos, y tanteé con algunas antes de terminar en la dinámica de ondas no lineal. En este momento me gustaría mencionar al

profesor Ramón Alonso Sanz, quien me introdujo en el fascinante mundo de los autómatas celulares y con quién tuve la oportunidad de escribir mi primer artículo científico, a lo que estoy muy agradecido. Me he divertido mucho con los autómatas, y espero algún día poder volver a ellos y estudiarlos más en detalle. Ciertamente es un área de investigación en el que queda mucho por descubrir y que permite al investigador poner en práctica mucho de su creatividad, algo que siempre es de agradecer.

Pero después de un primer curso algo incierto, las ondas no lineales y sus vórtices me atraparon. Y también mi director de tesis, el profesor Miguel Ángel Porras. Es difícil poner en palabras cuán agradecido estoy por su generosidad al compartir conmigo su tiempo y su trabajo tal y como lo hizo. Él no me ha enseñado todo lo que sé de Física, pero desde luego me ha enseñado cómo perseverar en el descubrimiento de la Física que no sé todavía (¡qué obviamente es mucha más!), y eso no tiene precio. Gracias, Miguel Ángel. Y como bonus, tuve también la suerte de tener un compañero doctorando veterano que aún hoy en día me ayuda a encontrar el camino a seguir en estos menesteres cuando yo no tengo ni idea de por dónde seguir, Dr. Carlos Ruiz Jiménez. Me has ayudado más de lo que te imaginas, créeme, tanto con tus trabajos como con tus palabras. En los últimos años he aprendido mucho de vosotros dos, y vuestra paciencia y amabilidad siempre han estado presentes, en momentos fáciles y difíciles. Jamás se me olvidarán los ensayos sin fin para mi charla en la XXXVI Bienal de Física en Santiago de Compostela en 2017, ni el subidón cuando finalmente me salió bien por primera vez. Y realmente me parece que he podido colaborar para traer a la luz un conocimiento nuevo y útil, lo que hace que todo valga la pena.

Quiero agradecer igualmente al Grupo de Sistemas Complejos, en particular a la profesora Rosa María Benito, por acogerme tan amablemente en el grupo de investigación y ofrecerme esta oportunidad de conocer en primera persona el mundo de la investigación académica; y al profesor Juan Carlos Losada por su disponibilidad total para ayudar, tanto en temas científicos como burocráticos. El grupo de investigación se hace cada vez más grande, y no tendría sentido nombrar a todos aquí, pero me gustaría agradecerlos a todos la colaboración y disculparme por las horas de CPU implacablemente consumidas por Matlab® en nuestras simulaciones numéricas interminables.

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Por último, pero no menos importante por ello, quiero agradecer a mi familia por aguantarme durante los largos periodos de aislamiento necesarios para realizar todo el trabajo que tuve por delante estos últimos años. Por ello, quise también demostrar mi agradecimiento citándoles al inicio de cada capítulo de esta tesis, en vez de usar frases profundas de personajes históricamente relevantes, como es habitual. Espero que esta elección no sea vista como derivada de la pereza o de la arrogancia, sino como una manera de agradecer a aquellas personas que, aunque se sientan distantes de las “cosas raras” que nos interesan a los físicos, deben saber que son partes de todo ello, al menos para mí. Gracias a Marisa, Xavier, Ana y Nico por su apoyo. Este agradecimiento puede que sea corto, pero es sincero.

Madrid, 2019.

List of Abbreviations

Equations

- LSE — Linear Schrödinger Equation.
NLSE — Nonlinear Schrödinger Equation.

Light Beams

- BB — Bessel Beam.
BVB — Bessel Vortex Beam.
UBB — Unbalanced Bessel Beam.

Phenomena

- GVD — Group Velocity Dispersion.
MPA — Multiphoton Absorption.
OAM — Orbital Angular Momentum.
SPM — Self-Phase Modulation.

Abstract

This dissertation aims to elucidate the stability properties of nondiffracting light beams with embedded vortices, and thus transporting angular momentum, namely while propagating in nonlinear self-focusing Kerr media. It comes motivated by the insufficient knowledge available on such a relevant subject, either theoretically or experimentally, as their use in the form of vortex spatial solitons is currently customary in fields like optical tweezers, material processing, long distance communications or transference of angular momentum from light to matter. These vortex light beams are particularly sensitive to the beam collapse instability caused by the Kerr nonlinearity, which lead them to a complete rupture of their initial azimuthally symmetric configuration into different dynamic regimes such as rotating pieces, beam filamentation or seemingly chaotic dynamics. In consequence, the main lines of research on the topic are dedicated to the identification and characterization of stabilizing mechanisms that can be used experimentally, being an area of intense research nowadays.

Nonlinear Bessel vortex beams are the nondiffracting light beams analyzed, always at an intensity sufficiently high so that the relevant nonlinearities to be taken into account are the already mentioned Kerr self-focusing phenomenon and nonlinear absorption, manifested as multiphoton absorption. This dissertation follows suit the work previously done in *“Dinámica no lineal de haces ópticos de Airy y de Bessel en medios Kerr con absorción no lineal”* (2016), doctoral dissertation by Dr. Carlos Ruiz-Jiménez, where nonlinear Bessel vortex beams have been introduced and the properties of their stationary, nondiffracting propagation analyzed thoroughly. In addition, the nonlinear dynamics observed for these beams was explained, allowing the formulation of the laws specifying which nonlinear Bessel vortex beam will act as an attractor for a linear Bessel beam being launched in a nonlinear medium; including the verification of how, in some cases, fully formed nonlinear Bessel vortex beams emerge spontaneously, thus being stable attractors. All the conclusions were verified by means of detailed numerical simulations.

In this work, a step further beyond the study of stationarity is given, and the issue of stability is addressed. It was carried out using a standard technique used in nonlinear wave dynamics, a linearized stability analysis against generic small perturbations, which allowed a meticulous analytical and numerical study on the matter. Nonetheless, nonlinear Bessel vortex beams are weakly localized beams in the transversal plane, and thus not strictly solitons as a standard vortex spatial soliton, which hindered the application of the linearized stability analysis heavily, turning it into a difficult and time-consuming task, as explained later in the text. Yet, relevant results were achieved. In particular, the existence of nonlinear Bessel vortex beams that are fully stable against small perturbations was demonstrated, whenever the nonlinear absorption balances the Kerr self-focusing

nonlinearity. Unstable nonlinear Bessel vortex beams were also better characterized, as the linearized stability analysis gives in this case a criterion to predict the number of rotating pieces that will emerge with the first rupture of the beam. As a final result, the nonlinear absorption is revealed to be the stabilizing mechanism of nonlinear Bessel vortex beams, as its removal from the proposed model causes the stability to disappear, thus also showing that completely transparent media cannot support by themselves the existence of stable nonlinear Bessel vortex beams.

The relevance of the above results lies on the fact that both nonlinearities, self-focusing Kerr and multiphoton absorption, are ubiquitous in most optical materials, leading to the remarkable feat that stable propagation of vortices is possible in almost any transparent material at sufficiently high intensities. Other models to attain stable propagation using specifically tailored nonlinearities and requiring custom-built media and ranges of intensity (such as saturable Kerr nonlinearity, or the cubic-quintic model) are thus rendered unnecessary and evidently difficult to implement. Moreover, the model presented in this dissertation is the first using dissipative nonlinearities (in the form of multiphoton absorption), and having the Kerr nonlinearity as the only conservative nonlinearity, achieving stable vortex solitons for arbitrarily high topological charges (previous works only reported stability for vortex solitons with topological charges up to $s = \pm 1$, as extensively documented in available scientific literature).

The last part of this dissertation is dedicated to analyze the compatibility of the conclusions previously presented with real-case scenarios reported in recent experiments on material processing methods. It is shown that different dynamic regimes observed in cases of axicon-generated (or equivalent devices as spatial light modulators) linear Bessel vortex beams propagating in nonlinear media can be explained under a unified framework supported by the results of the linearized stability analysis. Hence, if the attractor nonlinear Bessel vortex beam is stable, a tubular filamentation propagation regime is observed in the nonlinear media. On the contrary, rotatory or “chaotic” (speckle-like) regimes are observed if the attractor nonlinear Bessel vortex beam is unstable, depending on the extent of the development of the instability in the Bessel zone of the beam.

Resumen

La presente tesis tiene por objetivo dilucidar las propiedades de estabilidad de haces de luz no difractantes con vórtices embebidos, y que por lo tanto transportan momento angular, en particular cuando se propagan en materiales con no linealidad Kerr self-focusing. La motivación detrás de dicho objetivo es el escaso conocimiento disponible sobre la importante cuestión de la estabilidad, tanto desde el punto de vista teórico como del experimental. No obstante, su aplicación bajo la forma de solitones espaciales con vórtices es habitual actualmente en campos como el de las pinzas ópticas, el procesamiento de materiales, las comunicaciones de larga distancia o la transferencia de momento angular de la luz a la materia. Estos haces de luz con vórtices son especialmente sensibles al colapso causado por la no linealidad Kerr, que conduce a la rotura total de su estado inicial con simetría azimutal en diferentes regímenes dinámicos que pueden presentar fragmentos en rotación, filamentación del haz o dinámicas aparentemente caóticas. Como consecuencia natural, las principales líneas de investigación sobre el tema están dedicadas a la identificación y caracterización de los mecanismos estabilizadores que puedan ser utilizados experimentalmente, siendo un área de investigación intensiva hoy en día.

Con respecto a los haces de luz no difractantes analizados, se trata de haces de Bessel no lineales con vórtices, siempre con una intensidad lo suficientemente alta para que sea relevante la consideración en el modelo de la ya mencionada no linealidad Kerr self-focusing y también de la absorción no lineal, manifestada como absorción multifotón. Esta tesis sigue los pasos del trabajo realizado previamente en la tesis doctoral *“Dinámica no lineal de haces ópticos de Airy y de Bessel en medios Kerr con absorción no lineal”* (2016), por el Dr. Carlos Ruiz-Jiménez, donde los haces de Bessel no lineales con vórtices fueron introducidos y sus propiedades de propagación estacionaria y no difractante analizadas en profundidad. Además, la dinámica no lineal de estos haces fue expuesta detalladamente, permitiendo la formulación de las leyes que definen cual es el haz de Bessel no lineal con vórtices que actuará como atractor para un haz de Bessel lineal que se lance en un medio no lineal; incluyéndose la verificación de cómo, en algunos casos, emerge espontáneamente un haz de Bessel no lineal con vórtice completamente formado, indicando en ese caso que es un atractor estable. Todas las conclusiones fueron verificadas mediante minuciosas simulaciones numéricas.

En este trabajo, se da un paso más allá del estudio de la estacionariedad, y el tema de la estabilidad es abordado. Se hizo aplicando una técnica muy usada en la dinámica de ondas no lineal, el análisis de estabilidad linealizado con pequeñas perturbaciones genéricas, lo que ha permitido los estudios analítico y numérico pormenorizados del problema planteado. Sin embargo, dado que los haces de Bessel no lineales con vórtices están débilmente localizados en su plano transversal, y por tanto no son estrictamente solitones

como lo puede ser el solitón espacial con vórtice estándar, la aplicación del análisis de estabilidad linealizado se dificultó enormemente, haciéndola una tarea compleja y larga, como se explicará posteriormente en el texto. Aun así, se han alcanzado resultados muy relevantes. En concreto, se ha demostrado la existencia de haces de Bessel no lineales con vórtices completamente estables frente a pequeñas perturbaciones, siempre que se cumpla que la absorción no lineal compense la no linealidad Kerr self-focusing. La caracterización de haces de Bessel no lineales con vórtices inestables también se ha mejorado, ya que el análisis de estabilidad linealizado da un criterio para la predicción del número de fragmentos en rotación en los que se romperá el haz inestable al perder su simetría inicial. Otro resultado alcanzado ha sido la confirmación de la absorción no lineal como el mecanismo estabilizador de los haces de Bessel no lineales con vórtices, ya que con su retirada del modelo la estabilidad desaparece, lo que implica que medios completamente transparente no pueden soportar por sí mismos la existencia de haces de Bessel no lineales con vórtices que sean estables.

La relevancia de los resultados anteriores se hace patente al considerar que ambas no linealidades, la Kerr self-focusing y la absorción multifotón, son ubicuas en la mayoría de los medios ópticos, lo que lleva a la importante conclusión de que es posible conseguir la propagación estable de vórtices en prácticamente cualquier medio transparente a una intensidad suficientemente alta. Otros modelos propuestos para alcanzar la propagación estable se basan en no linealidades específicas que requieren materiales contruados a medida y rangos de intensidad bien definidos (como el modelo de la no linealidad Kerr saturada, o el modelo cúbico-quíntico), se muestran así innecesarios y de difícil implementación. Además, el modelo presentado en esta tesis es el primero con no linealidades disipativas (bajo la forma de absorción multifotón), y la no linealidad Kerr como la única no linealidad conservativa, que consigue estabilizar solitones con vórtices de cargas topológicas arbitrariamente altas (trabajos anteriores lo hicieron para cargas topológicas $s = \pm 1$ como mucho, algo profusamente documentado en la literatura científica disponible).

El último tramo de esta tesis se dedica a analizar la compatibilidad de las conclusiones presentadas con los resultados obtenidos en escenarios reales informados en experimentos recientes en procesamiento de materiales. Se demuestra cómo los diferentes regímenes dinámicos observados en haces de Bessel lineales con vórtices generados por axicones (o equipos equivalentes, como moduladores espaciales de luz), propagándose en medios no lineales, pueden explicarse de forma unificada a través de los resultados del análisis de estabilidad linealizado realizado en esta tesis. Así, si el haz de Bessel no lineal con vórtice que funciona como atractor es estable, la propagación observada en el medio no lineal tendrá simetría tubular. Caso contrario, es decir, si el haz de Bessel no lineal con vórtice que funciona como atractor es inestable, entonces la propagación observada en el medio no lineal podrá consistir en fragmentos en rotación o en una filamentación aparentemente caótica, dependiendo del grado de desarrollo de la inestabilidad en la zona de Bessel del haz.

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INTRODUCTION

“Uh, what?”

The answer of any non-physicist relative whom I tried to explain what my Ph.D. was about.

The study of complex systems is probably having one of its best historical moments these days. It is not hard to understand, even if not academically, the relevance of the subject when social networks with millions of users are of common use, big data is a commercial hype and data scientists are avidly wanted by all kinds of tech companies. The last efforts are directed very much towards Artificial Intelligence, Computational Modeling and processing vast amounts of data, beginning with those gathered by large-scale research facilities like CERN, with a staggering 30 petabytes per year ([CERN 2018](#)) – actually a small amount of data if commercial sources are included, summing up worldwide around 2500 petabytes every day, according to the last estimates by IBM ([IBM 2013](#)).

Still, the study of complex systems is not new. It can be traced back to at least the 1940’s with the emergence (pun intended) of new conceptual developments in a panoply of different areas like Economics, Biology, Mathematics, Information Theory or Cybernetics. In fact, the mathematical tools necessary to a proper study of “Systems Science”, as it was initially known, were established centuries before by great names like Isaac Newton (Calculus, 18th century) or Henri Poincaré (Algebraic Topology, 19th century). A special mention must be made for the numerous important advances during the 18th century in the establishment of a rigorous basis for hydrodynamics, in particular for vortices and water waves characterization, using advanced calculus techniques by theoreticians like Siméon-Denis Poisson (1781-1840), Claude Louis Marie Henri Navier (1785-1836), Augustin Louis Cauchy (1789-1857), Sir George Gabriel Stokes (1819-1903) or Sir George Biddell Airy (1801-1892).

Nowadays, the study of complex systems has transcended the theoretical realm and it is essentially a multidisciplinary field of study where theory, experimentation, and notably computational simulation all have their place. Moreover, nonlinear dynamics of extended systems, having infinite degrees of freedom, and the nonlinear dynamics of waves in particular, is one of its most fruitful areas, as historically many of its achievements were later applied to explain miscellaneous physical phenomena that unexpectedly shared the same theoretical basis. Pervasive equations such as the nonlinear Schrödinger equation (central in this work) or the Korteweg-de Vries equation successfully describe nonlinear

effects such as solitons in water waves ([Russell 1844](#)), laser beam self-focusing in nonlinear media ([Konno and Suzuki 1979](#)), Bose-Einstein condensates ([Dodd 1996](#); [Tian et al. 2006](#)), Langmuir waves ([Kuo 2014](#)), wave breaking and collapse ([Silberberg 1990](#); [Trillo, Toterogónzora, and Fratalocchi 2014](#)) and ocean turbulence ([Costa et al. 2014](#)) in fields as diverse as optical communications, relativity and hydrodynamics. Solitons, singularities, vortices and other complex structures often emerge from these equations, requiring ever-increasing levels of detail and precision in analytical and numerical models that must be faithful to experimental results done at higher energies or finer scales of space and time measurements. The pursuit of practical applications to wave dynamics phenomena is very active in a variety of fields, as in nanotechnology ([Wang et al. 2016](#)), development of new materials ([Jin et al. 2017](#)), chemistry ([Humble and Cina 2006](#)), medicine ([Jing, Tao, and Clement 2011](#)) or communications ([Sharping et al. 2001](#)), thus helping push forward improvements in theoretical and modeling results.

To the purpose of this dissertation, the research of nonlinear wave dynamics received a huge boost during the 1960s with the invention of laser technology ([Franken et al. 1961](#)). It finally granted a reliable technique to achieve a high intensity, coherent light beam under controlled conditions, allowing direct observation and measurements of several nonlinear effects, some previously predicted by theory, some completely new to experiment. As will be reviewed in the next chapter, a light beam that travels in a medium at a sufficient high intensity modifies it through its polarization property, possibly creating a nonlinear response by the material and generating nonlinear effects to be observed on the light beam itself. These nonlinear interactions that occur between a light beam with high enough intensity and the medium where it travels generate a panoply of different effects, having historically uncovered many previously unknown properties of both light and matter. Its scientific importance can be easily understood if the 2018 Nobel Prize awarded to Arthur Ashkin for his development of microparticle manipulation through the use of radiation pressure (i.e., optical tweezers) ([Ashkin et al. 1986](#); [Ashkin, Dziedzic, and Smith 1982](#); [Ashkin 1970](#)); and shared with Gérard Mourou and Donna Strickland for their work in overcoming the technical issues needed to achieve terawatt-power "tabletop lasers" ([Strickland and Mourou 1985](#); [Strickland and Corkum 1994](#)), is remembered. Among all the nonlinear phenomena that can arise in this manner, this work is centered in a specific type of light beam under a small, fundamental set of nonlinear effects, plus the ubiquitous linear diffraction spreading: Bessel beams propagating in nonlinear self-focusing Kerr media with nonlinear multiphoton absorption.

Bessel beams were first described and observed in 1987 in their linear, zeroth-order (vortexless) version, as reported in ([Durnin, Miceli, and Eberly 1987](#); [Durnin 1987](#)). This breakthrough was immediately under the spotlight thanks to its nondiffracting property, later seen to be shared by few other types of beams like the Airy beam ([Siviloglou and Christodoulides 2007](#)), the Weber beam ([Bandres and Rodríguez-Lara 2013](#); [Zhang et al. 2012](#)) and the Mathieu beam ([Gutiérrez-Vega, Iturbe-Castillo, and Chávez-Cerda 2000](#); [Chávez-Cerda, Gutiérrez-Vega, and New 2001](#); [Yan et al. 2015](#)). In addition to being the first diffraction-free light beam discovered, Bessel beams are also self-healing, allowing them to reconstruct themselves against large perturbations, partial obscuration by obstacles, and so on, just by propagating further in their trajectory. Richer variants of Durnin's Bessel beam were discovered later on, especially high-order Bessel beams, which

carry orbital angular momentum that can be transferred to matter, in addition to the angular momentum associated to the polarization, or spin of photons. High-order Bessel beams can carry orbital angular momentum because they transport an optical vortex in their center. The vortex and the orbital angular momentum are characterized by its topological charge, or number of times that the phase changes by 2π in a closed-path circulation around the vortex center, where the intensity vanishes. Bessel beams can thus be generated either in its fundamental version with null topological charge s , i.e., presenting no vortices; or in its more plentiful, higher order form ($s \neq 0$), presenting an arbitrary topology of vortices. These previously unseen properties caused Bessel beams to receive widespread attention from the scientific community for the last decades, which resulted in numerous theoretical and experimental advances with applications in the creation of laser-processed materials, optical tweezers and different techniques of transference of angular momentum from light to matter, among others (Masalov 1997; Allen, Padgett, and Babiker 1999; Tabosa and Petrov 1999). Experimentally, only approximate versions of the Bessel beam can be created, as it is weakly localized transversally carrying infinite amount of power, and therefore real case scenarios use Gaussian beams focusing on specific configurations of axicon lenses for either fundamental or higher-order (vortex) Bessel beams, which form as a very good approximation to the ideal case.

Bessel beams entered the nonlinear realm with the work of (Porrás et al. 2004). Fundamental nonlinear Bessel beams in Kerr media with nonlinear absorption were then shown to exist propagating stationarily, i.e., without diffraction nor any attenuation, in contrast with solitons, whose strong localization in exponential form cause them to be depleted by the absorptive medium during propagation. Loosely localized Bessel beams, on the contrary, are able to take advantage of its self-healing properties and maintain themselves intact. Shortly after, fundamental nonlinear Bessel beams were demonstrated to spontaneously form from linear Bessel beams when entering a nonlinear medium, and after an initial stage of absorption (Polesana et al. 2006).

This phenomenon was later extended to higher-order nonlinear Bessel beams (Porrás and Ruiz-Jiménez 2014; Jukna et al. 2014; Ruiz-Jiménez 2016). Again, the concept of a *reservoir* of energy and momentum, intrinsic to Bessel beams and detailed in the works already mentioned, was central to understand the stationary propagation with nonlinear absorption, and it will receive special attention in Ch. 2. Another important observation made was that under certain conditions, a high-order linear Bessel vortex beam launched in the nonlinear medium would spontaneously transform into a nonlinear Bessel vortex beam; yet that would not happen for another arbitrary linear Bessel vortex beam with slightly different parameters. In the latter case, instead, different outcomes were observed, where the field amplitude oscillated more or less periodically, quasi-periodically or chaotically, as reported in (Couairon et al. 2013; Porrás, Ruiz-Jiménez, and Losada 2015; Jukna et al. 2014). These findings suggested that there should exist a subset of nonlinear Bessel vortex beams that are stable, acting as attractors for the dynamics of the linear Bessel vortex beam being launched in the nonlinear medium.

It is important at this point to highlight the relevance of the problem of the stability of vortex beams propagating in Kerr self-focusing media, since the Kerr nonlinearity is present in almost all media. As detailed in (Desyatnikov, Torner, and Kivshar 2005), vortex beams are easily led to unstable propagation in Kerr self-focusing media, in par-

ticular vortex beams under azimuthal perturbations. This tendency to instability poses a serious technical problem to their usual applications where precision and control are essential, for instance in material processing at micro-scales or as optical tweezers. A way to circumvent this problem and attain stable propagation is the use of specifically-tailored media that have cubic (focusing) and quintic (defocusing) nonlinearities with opposite signs ([Quiroga-Teixeiro and Michinel 1997](#); [Paz-Alonso and Michinel 2005](#)), but it is a very labor-intensive method.

The current dissertation presents a rigorous linearized stability analysis of nonlinear Bessel vortex beams under general small perturbations. The results of the analysis allow the identification of a simple set of parameters that strictly define nonlinear Bessel vortex beams with completely stable propagation, without the necessity of other media nonlinearities besides the Kerr self-focusing effect and the ubiquitous nonlinear multiphoton absorption that appears at high enough intensities. Here, “complete” or “full” stability is used in the sense that the beam is stable against all types of small perturbations, not only, for instance, radial or azimuthal, as it is usual to research stability only on some specific degree of freedom of the beam. An evident advantage of this discovery, in contrast with other previously presented ways to achieve stable propagation, is that it is solely sustained by common material properties (the Kerr and multiphoton absorption nonlinearities), constituting a tool to foresee stable beam production unmistakably, for its use in any practical applications suitable to nonlinear Bessel vortex beams.

This work builds on the previous research, deepening the understanding of the stability mechanisms of nonlinear Bessel vortex beams propagating in nonlinear media. Although it was known from previous experimental work that nonlinear Bessel vortex beams can be stable in some controlled configurations ([Couairon et al. 2006](#); [Polesana et al. 2008](#); [Polesana et al. 2005](#); [Polesana et al. 2006](#); [Polesana et al. 2007](#)), it was not understood the factors that could stabilize them, and neither there was any criteria to predict a specific parametrization of a nonlinear Bessel vortex beam as stable or unstable. This dissertation solves these problems deepening the study on the stability properties of nonlinear Bessel vortex beams in Kerr media with nonlinear absorption that was introduced first-hand in ([Porrás et al. 2016](#)) and ([Porrás, Ruiz-Jiménez, and Carvalho 2018](#)).

After this introductory chapter, the following work starts out Chapter 1 with the fundamental aspects of theory and practice to bear in mind about light beam propagation in nonlinear media, and relevant effects that usually come up in this scenario. The linear case is briefly seen to introduce the paraxial approximation and the linear Schrödinger equation, which governs propagation of Gaussian and Bessel beams in linear media. It then moves on to the nonlinear case, reviewing elements like the optical Kerr effect, nonlinear multiphoton absorption, orbital angular momentum and spatial solitons; to finally introduce the main problem of this dissertation, the stability of the stationary states of the nonlinear Schrödinger equation. It closes with a case study on stability that will serve as a model for the later chapters.

Chapter 2 offers the main characteristics of the propagation of nonlinear Bessel vortex beams in Kerr media with nonlinear absorption, such as their existence as solutions to the nonlinear Schrödinger equation, the energy reservoir mechanism and their asymptotic behavior, which include studying their underlying dynamics attractors, previously introduced in a thorough research for the fundamental case of Bessel beams and for Airy beams

in (Ruiz-Jiménez 2016). The dynamics attractors revealed themselves as a determinant factor of the propagation results observed experimentally.

Chapter 3 makes a profound study on the stability properties of nonlinear Bessel beams, presenting a linearized stability analysis that gives out the main original results of this work, as reported in (Porras et al. 2016; Carvalho, Ruiz-Jiménez, and Porras 2017), finally providing a predictive criterion for the stability characterization of nonlinear Bessel vortex beams, with new insight of the physics behind the existence of these beams. It corroborates the stability analysis with numerical verifications of the results, and closes with a new validation of the stabilizing role of the nonlinear multiphoton absorption on the nonlinear Bessel vortex beam propagation.

Chapter 4 applies the theory developed in the previous chapters to real cases of axicon-generated nonlinear Bessel vortex beams, explaining the various dynamic regimes reported on the scientific literature, in particular for some very representative cases observed in (Jukna et al. 2014) and (Xie et al. 2015). It gives a coherent account under the theoretical framework presented in this dissertation for all experimental cases, which now can be interpreted collectively understanding the characteristics of the underlying dynamics attractors and the results of the linearized stability analysis in each case.

A few important appendixes are added at the end to support the reader in some points of the dissertation, mostly with technical details that would be too cumbersome to include in the main chapters. Appendix A details the calculations to obtain the system of coupled differential equations that determines the transversal amplitude and phase profile of nonlinear Bessel vortex beams, and whose solutions will later be subjected to the linearized stability analysis. Appendix C describes the discretization method to numerically solve the eigenvalue problem resulting from the linearized stability analysis. It is the computation and scrutiny of these eigenvalues that will make possible to determine the stability properties of the nonlinear Bessel vortex beams under study. Appendix B gathers some auxiliary calculations from different points of the dissertation that may need some extra steps to satisfy readers who demand more comprehensibility to the mathematical reasoning presented. Appendix D lists the software used to generate the most important numerical results and graphics of the dissertation, so they can be easily reproduced and verified.

Finally, as Supplemental Material of this dissertation a set of videos showing the propagation of several nonlinear Bessel vortex beams is provided. These propagations are used throughout this work to illustrate the behavior of these beams in different situations, but mainly on Ch. 3 in order to analyze their stability properties. Where relevant, information for the browsing of the correspondent video will be indicated for a deeper understanding of the scenario being studied.

Chapter 1

FUNDAMENTALS

My significant other: “Really, another four years at the University? Jeez!”

On 2014, when I was beginning my first year at the Ph.D program.

This chapter features some primary theoretical and experimental topics that will be used throughout this dissertation. Its intention is to set the tone for the discourse and allow a natural presentation of the results. It will also establish the notation to be used in the next chapters, when it comes to equations and physical magnitudes.

Relevant physical phenomena in the propagation of light beams like diffraction, the optical Kerr effect and the nonlinear absorption effects are raised and explained stressing the most relevant points that affect the results obtained. Linear propagation is exemplified with Gaussian and Bessel beams. Bessel beams, or BBs, carrying vortices with orbital angular momentum (or OAM, for short) are also introduced. The nonlinear self-focusing effect brought into the propagation dynamics of light beams by the optical Kerr nonlinearity is shown to be able to balance diffraction to form, ideally, spatial solitons, although this equilibrium is unstable and is swiftly destroyed during propagation. Finally, the important issue of the stability properties of nonlinear propagation of light beams, central to this dissertation, is exemplified (due to its difficulty) by a small case study of instability phenomena that is analytically solved, highlighting the analytical methods used later to derive the results on the determination of the stable or unstable behavior of nonlinear Bessel vortex beams, or BVBs.

The model that rules the nonlinear propagation dynamics of light beams is the nonlinear Schrödinger equation, or NLSE for short. It was introduced in this context for the first time under the paraxial approximation by Talanov ([Litvak and Talanov 1967](#)). This approximation, previously used to research the evolution of radio waves propagating in the ionosphere, was developed by Leontovich and Fock ([Leontovich and Fock 1946](#)), resulting in a nonlinear parabolic partial differential equation. It usually appears as the result of the Maxwell’s equations applied to quasi-plane waves propagating in material media, which allow the use of approximations such as the slow evolution of the wave packet when compared to its internal spatial-temporal oscillations. Consequently, this form of the NLSE is that of a equation for the complex envelope of a wave packet.

The name of the NLSE was given after its evident analogy with the Schrödinger equation used in the field of Quantum Mechanics, with the difference of instead of governing

the evolution of probability wave functions, it governs the evolution of the complex envelope of wave packets of the electromagnetic field. The second order linear differential operator in the NLSE represents the wave diffraction, while the nonlinear terms represent a variety of effects such as the Kerr effect, dissipation, changes on the properties of the medium due to the interaction with the propagation, nonlinear dissipation, or interaction with other existing electromagnetic fields. The direction of the propagation of the light beam is the dimension measuring its evolution, analogous to the time dimension in Quantum Mechanics, and the Laplace operator is thus calculated using the coordinates in the transversal plane during propagation. Due to its central role, linear and nonlinear Schrödinger equations are derived below starting from Maxwell's equations.

1.1 Wave equation

We start with Maxwell's equations for homogeneous dielectric media in the absence of free charges or currents,

$$\begin{cases} \nabla \cdot \mathbf{D} = 0, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \end{cases} \quad (1.1)$$

$\mathbf{E}$ being the electric field and $\mathbf{B}$ the magnetic field. The displacement field $\mathbf{D}$ and the magnetizing field $\mathbf{H}$ are defined by the following so-called *constitutive relations*, to be used when material media are considered (Wangsness 1989)

$$\begin{aligned} \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P}, \\ \mathbf{H} &= \mathbf{B}/\mu_0 - \mathbf{M}, \end{aligned} \quad (1.2)$$

where ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, and $\mathbf{P}$ and $\mathbf{M}$ are, respectively, the induced electric polarization and the induced magnetization. Only nonmagnetic materials shall be considered in this work, making $\mathbf{M} = \mathbf{0}$, and the last of the Eqs. (1.1) turns into

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t}, \quad (1.3)$$

Applying a known identity of differential operators, $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \Delta \mathbf{V}$ (Spiegel 1970), calculating the curl of the third equation in Eqs. (1.1), and using Eq. (1.3) gives

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = \\ &= -\frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t} \right), \end{aligned}$$

where the divergence of the electric field cancels out due to the first of the equations in Eqs. (1.1) since the electric field direction and propagation direction are parallel to

each other, which is valid for the linear media and nonlinear isotropic media (Boyd 2008) considered in this dissertation. Hence,

$$\Delta \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} , \quad (1.4)$$

is the wave equation to be used for the electric field with a source due to the medium polarization.

1.2 Linear Schrödinger equation

At low electric field intensities, material polarization is usually linear, e.g., it is proportional to the existing field. Under these conditions, it can thus be assumed a linear dependence between the polarization and the electric field for any arbitrary time t , and all instants preceding t , that is (Cabrera, López, and Agulló 1992)

$$\mathbf{P} = \mathbf{P}_L(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(\tau) \mathbf{E}(\mathbf{r}, t - \tau) d\tau , \quad (1.5)$$

where $\chi^{(1)}$ is the linear electric susceptibility of the medium, which is in general a tensor, except for the isotropic media considered here, where it is a scalar. Due to causality, the inferior limit of the previous integral is actually $\tau = 0$, as the electric susceptibility does not contribute for $\tau < 0$, i.e., $\chi^{(1)} = 0$ for $\tau < 0$.

From now on, only linear polarization of light shall be considered, which allows to put aside the vectorial representation of the fields being studied; in addition only paraxial waves (that is, highly directional waves) will be considered, as will be seen soon. As a result, the following theory will be scalar in its entirety, not vectorial. In addition, monochromatic waves can always be assumed for the propagation phenomena relevant to this work, with an angular frequency ω . Usual complex notation shall be used, making the electric field considered the real part of the complex electric field $\mathcal{E}(\mathbf{r}) \exp(-i\omega t)$,

$$E(\mathbf{r}, t) = \text{Re}\{\mathcal{E}(\mathbf{r})e^{-i\omega t}\} = \frac{1}{2} [\mathcal{E}(\mathbf{r})e^{-i\omega t} + \mathcal{E}^*(\mathbf{r})e^{i\omega t}] . \quad (1.6)$$

The same applies to the polarization,

$$P_L(\mathbf{r}, t) = \text{Re}\{\mathcal{P}_L(\mathbf{r})e^{-i\omega t}\} = \frac{1}{2} [\mathcal{P}_L(\mathbf{r})e^{-i\omega t} + \mathcal{P}_L^*(\mathbf{r})e^{i\omega t}] . \quad (1.7)$$

The linear susceptibility can also be expressed by a complex quantity using a Fourier transform,

$$\chi_\omega^{(1)} = \int_{-\infty}^{\infty} \chi^{(1)}(\tau) e^{i\omega\tau} d\tau . \quad (1.8)$$

Substituting the previous expressions in Eq. (1.5) and matching the relevant terms, the following important relation is obtained,

$$\mathcal{P}_L(\mathbf{r}) = \epsilon_0 \chi_\omega^{(1)} \mathcal{E}(\mathbf{r}) . \quad (1.9)$$

As a result, the wave equation for the real field in Eq. (1.4) will be valid as long as its complex amplitude satisfies

$$\Delta \mathcal{E} + \frac{\omega^2}{c^2} (1 + \chi_\omega^{(1)}) \mathcal{E} = 0 , \quad (1.10)$$

where the relation $c^2 = 1/\mu\epsilon$ was used. The dielectric constant (in general dependent of the frequency) can also be defined as (Agrawal 2001)

$$\epsilon_r \equiv (1 + \chi_\omega^{(1)}) \equiv \left[n + i \frac{\alpha c}{2\omega} \right]^2 , \quad (1.11)$$

where $n \simeq 1 + (1/2) \text{Re}\{\chi_\omega\}$ is the refractive index and $\alpha = (\omega/nc) \text{Im}\{\chi\}$ is the linear absorption coefficient, the latter not relevant for the transparent media now considered. Hence, the refractive index is $n \simeq \sqrt{\epsilon_r}$. Then, Eq. (1.10) can be rewritten as

$$\Delta \mathcal{E} + \frac{\omega^2 n^2}{c^2} \mathcal{E} = 0 , \quad (1.12)$$

or, using $k = \omega n/c$,

$$\Delta \mathcal{E} + k^2 \mathcal{E} = 0 , \quad (1.13)$$

which is the well-known Helmholtz equation or wave equation for monochromatic waves.

1.2.1 Paraxial approximation

As mentioned before, the light beams considered in this work are monochromatic, linearly polarized, quasi-plane waves, as those generated by a laser. The direction of propagation shall be the z direction, and thus

$$\mathcal{E} = A(x, y, z) e^{ikz} , \quad (1.14)$$

where $A(x, y, z)$ is the complex envelope, which varies slowly in the x, y directions, and particularly in the z direction, when compared to the wave oscillations expressed by the term $\exp(ikz)$. This constitutes the so-called slowly varying envelope approximation, to be used in the substitution of Eq. (1.14) in Eq. (1.13), leading to

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial}{\partial z} \left(\frac{\partial A}{\partial z} + 2ikA \right) = 0 . \quad (1.15)$$

The first term inside the parenthesis can be ignored when compared to the second term, as the slow variation of A with respect to $\exp(ikz)$ implies

$$\left| \frac{\partial A}{\partial z} \right| \ll |kA| , \quad (1.16)$$

leading to the so-called linear Schrödinger equation, or LSE,

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \Delta_\perp A , \quad (1.17)$$

where the transversal Laplace operator is defined as $\Delta_{\perp} \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. The application of the slowly varying envelope approximation to monochromatic light is equivalent to the paraxial approximation, also called Fresnel approximation or small-angle approximation. As will be seen next, the LSE describes the diffraction of a light beam under this approximation. In later chapters, nondiffracting Bessel beams with cylindrical symmetry shall be analyzed, and so $\Delta_{\perp} = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial \varphi^2)$ shall be used in cylindrical coordinates. Bessel beams will be presented as stationary solutions of the LSE (1.17). Note that the role of the time coordinate t for the quantum version of the original LSE, is now played by the z direction, which is the direction of propagation.

Important physical quantities are the intensity and the beam power, because they are more directly measurable than the electric field itself. The intensity, in SI units, is related to the squared modulus of the electric field (or its envelope A) by $I = nc\epsilon_0|A|^2/2$, and the power carried by the light beam is

$$P = \int I dx dy = \frac{nc\epsilon_0}{2} \int |A|^2 dx dy . \quad (1.18)$$

It can readily be shown from the LSE (1.17) that P is independent of z in a transparent medium.

1.2.2 Diffraction. Gaussian beams

Previously, the LSE was obtained using the paraxial approximation, or small-angle approximation. It can be demonstrated that it is a differential version of the Fresnel integral for the diffraction, deduced from the Huygens principle (Born and Wolf 1980).

According to Kirchhoff's scalar theory, the value of a luminous perturbation $\mathcal{E}(x, y, z)$ at a distance z , in the plane (x, y) , of a monochromatic wave traveling through an opening of area S in the plane (x', y') at $z = 0$, can be calculated using the integral equation

$$\mathcal{E}(x, y, z) = \frac{1}{i\lambda} \int_S \mathcal{E}(x', y', 0) \frac{e^{ikr}}{r} \cos \theta dx' dy' , \quad (1.19)$$

where $\lambda = 2\pi/k$ is the wavelength in the medium, θ is the angle formed by the vector that goes from a point (x', y') through the opening to a point (x, y) in the z plane, with $r = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$. In the Fresnel approximation, it can be assumed that $\cos \theta \sim 1$ and $r \sim z$ in the denominator in Eq. (1.19). Also,

$$r \simeq z \left(1 + \frac{1}{2} \frac{(x - x')^2}{z^2} + \frac{1}{2} \frac{(y - y')^2}{z^2} \right) , \quad (1.20)$$

in the phase factor in Eq. (1.19). Taking into account that $\mathcal{E} = A \exp(ikz)$, Eq. (1.19) becomes

$$A(x, y, z) = \frac{1}{i\lambda z} \int_S A(x', y', 0) e^{i \frac{k[(x-x')^2 + (y-y')^2]}{2z}} dx' dy' . \quad (1.21)$$

Equation (1.21) is known as Fresnel diffraction integral, which is the solution in integral form to the LSE (1.17) for an initial condition $A(x, y, 0)$, as can be validated by direct substitution. The expression

$$\frac{1}{z} e^{i \frac{k(x^2 + y^2)}{2z}} \quad (1.22)$$

Figure 1.1: Schematics of the diffraction effects on a Gaussian beam. In red, radial intensity profiles. In green, wavefronts diverging due to diffraction. On the left and on the right, respectively, a Gaussian beam before and after propagating a distance of $2 L_R$.

is the complex envelope under paraxial approximation of a spherical wave emitted at the origin and therefore of radius z at distance z . The Fresnel integral expresses that $A(x, y, z)$ is a superposition of paraxial spherical wavefronts with amplitudes $A(x', y', 0)$, all of them emitted at the different points (x', y') of the light source.

Gaussian beams

The prototypical laser beam that shows the effects of the diffraction during propagation in the simplest way is the Gaussian beam. If the complex amplitude at $z = 0$ is the Gaussian distribution

$$A(x, y, 0) = \exp[-(x^2 + y^2)/w_0^2] = \exp(-r^2/w_0^2) , \quad (1.23)$$

where r is the radial distance in cylindrical coordinates from now on, $r^2 \equiv x^2 + y^2$, and w_0 is the width of the Gaussian distribution, also called the waist width. The solution to the LSE (1.17), or Fresnel integral, at any arbitrary distance z is

$$A(x, y, z) = \frac{-iL_R}{q(z)} e^{i \frac{kr^2}{2q(z)}} , \quad (1.24)$$

$$\text{with } q(z) = z - iL_R , \quad (1.25)$$

where L_R is the diffraction distance or Rayleigh distance, defined as

$$L_R = \frac{kw_0^2}{2} . \quad (1.26)$$

For instance, for a beam with waist $w_0 = 1$ mm propagating in air ($n \simeq 1$), and a wavelength of 800 nm, the diffraction distance is $L_R \simeq 3.93$ m, representing the distance at which the beam width increases by a factor of $\sqrt{2}$.

It is convenient to write

$$\frac{1}{q(z)} \equiv \frac{1}{R(z)} + \frac{2i}{k[w(z)]^2} , \quad (1.27)$$

where

$$R(z) = z \left[1 + \left(\frac{L_R}{z} \right)^2 \right] \quad (1.28)$$

represents the radius of curvature of the wave fronts as a function of propagation distance z , and

$$w(z) = w_0 \sqrt{1 + (z/L_R)^2}, \quad (1.29)$$

is the beam width as a function of z . Thus, the Gaussian beam can be more explicitly written as

$$A(x, y, z) = \frac{1}{\sqrt{1 + (z/L_R)^2}} e^{-i\phi} e^{i \frac{kr^2}{2R(z)}} e^{-\frac{r^2}{w(z)^2}}, \quad (1.30)$$

where $\phi = \arctan(z/L_R)$ is the so-called Gouy's phase, or else

$$A(x, y, z) = \frac{w_0}{w(z)} e^{-i \arctan(z/L_R)} e^{i \frac{kr^2}{2R(z)}} e^{-\frac{r^2}{w(z)^2}}. \quad (1.31)$$

It can be seen that the transversal amplitude distribution always has a Gaussian shape, only with a Gaussian width $w(z)$ that increases with the propagation distance, and since $R(z) > 0$ for $z > 0$, the phase, which was zero (constant) at $z = 0$, acquires a positive parabolic distribution that represents a divergent spherical wavefront in the paraxial approximation, as seen in Eq. (1.22) with z replaced with $R(z)$. Figure 1.1 shows the diffraction of a Gaussian beam, with diverging wavefronts in green and transversal intensity profiles in red, and increasing beam width as the beam propagates. Equations (1.27)-(1.29) show that if $z \rightarrow \infty$ is satisfied, then $w(z) \rightarrow w_0 z/L_R \equiv \theta z$, being $\theta = 2/kw_0$ the divergence angle, indicating that at large distances the beam width increases as a linear function of the distance and proportional to the divergence.

Much in the same fashion, when more complicated functions are used instead of the Ansatz (1.23), for instance a Gaussian distribution multiplied by Hermite polynomials, the so-called high-order Gauss beams (or Hermite-Gauss modes) are obtained. In addition, Gaussian distributions multiplied by Laguerre polynomials lead to Laguerre-Gauss beams or modes.

The Gaussian beam model illustrates the usual diffraction phenomenon observed in most laser beams and affects all types of waves that are transversally localized, i.e., that carries finite amount of power, as defined in Eq. (1.18), because of their strong transversal localization. As mentioned before, this dissertation studies beams that are radically different from the Gaussian beam: the Bessel beams and their nonlinear counterparts, which are nondiffracting beams whose weak localization makes them to ideally carry infinite amounts of power, but finite-power approximations that maintain their nondiffracting property over a finite distance are actually realized in experiments, as explained in the next section.

1.2.3 Nondiffracting Bessel beams

The fundamental Bessel beam was introduced by (Durnin, Miceli, and Eberly 1987) as monochromatic solutions of the fundamental wave equation (without the paraxial approximation) and caused a deep impact in the scientific community because they are

Figure 1.2: Schematics of the diffraction effects on two nondiffracting Bessel beams: above, the fundamental version ($s = 0$); below, the high-order version ($s = 1$). In red, radial intensity profiles suffer no changes with propagation, since both are nondiffracting beams. On the left and on the right, respectively, the Bessel beams before and after propagation.

nondiffracting or diffraction-free. It was actually the second type of nondiffracting beams reported in scientific literature after Airy beams in (Berry and Balazs 1979), in a moment when the interest of the scientific community on the matter was on the rise. Soon higher-order Bessel beams were also introduced, being (Allen et al. 1992) the foundational paper reporting them, followed by others like (Allen, Padgett, and Babiker 1999) or (Volke-Sepulveda et al. 2002). For illustration, examples of fundamental and high order Bessel beams are displayed in Fig. 1.2, with their respective transverse radial intensity profiles.

Fundamental Bessel beams

Bessel beams can be created experimentally using an axicon, a conical lens whose basic workings is shown in Fig. 1.3, where a plane wave propagating on the z direction incides on the left face of the axicon. On the right side, the lens creates plane waves with a fixed angle of inclination θ , called conical angle, and every azimuthal angle φ between 0 and 2π , that is, with a propagation vector with components

$$\begin{aligned} k_x &= k \sin \theta \cos \varphi , \\ k_y &= k \sin \theta \sin \varphi , \\ k_z &= k \cos \theta , \end{aligned} \tag{1.32}$$

Figure 1.3: Basic working of an axicon lens.

which in paraxial approximation $\theta \ll 1$ can be written as

$$\begin{aligned} k_x &= k \theta \cos \varphi, \\ k_y &= k \theta \sin \varphi, \\ k_z &= k (1 - \theta^2/2). \end{aligned} \quad (1.33)$$

Hence, if $\vec{r} = (x, y, z) = (r \cos \alpha, r \sin \alpha, z)$, these plane waves can also be written as $E_\varphi = A_\varphi \exp(ikz - \omega t)$, where

$$A_\varphi = e^{i[k\theta r(\cos \varphi \cos \alpha + \sin \varphi \sin \alpha)]} e^{-i\frac{k\theta^2}{2}z} = e^{i[k\theta r \cos(\varphi - \alpha)]} e^{-i\frac{k\theta^2}{2}z}. \quad (1.34)$$

Taking into account the Bessel integral ([Spiegel 1970](#))

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \varphi} d\varphi, \quad (1.35)$$

it is possible to sum up all the contributions from the plane waves for each value of the azimuthal angle, obtaining $E = A(r, z) \exp[i(kz - \omega t)]$, resulting in a slowly variable complex envelope (for paraxial θ)

$$A(r, z) = \int_0^{2\pi} e^{i[k\theta r \cos(\varphi - \alpha)]} d\varphi e^{-i\frac{k\theta^2}{2}z} = J_0(k\theta r) e^{-i\frac{k\theta^2}{2}z}, \quad (1.36)$$

which constitutes the fundamental linear Bessel beam (or BB). Thus, a BB is formed by a superposition of plane waves with wave vectors uniformly distributed over the surface of a cone of angle of value θ . It is a nondiffracting linear beam, since its intensity $I = |A|^2 = |J_0(k\theta r)|^2$ does not depend on z . The original BB presented in ([Durnin, Miceli, and Eberly 1987](#)) has the form $E = J_0(k \sin \theta r) \exp(ik \cos \theta z - \omega t)$, and thus Eq. (1.36) can be obtaining just by imposing the condition $\theta \ll 1$. In addition, note that the amplitude

of the linear BB decreases in the radial coordinate as $r^{-1/2}$, which implies that the total power carried by the beam is infinite,

$$P \propto 2\pi \int_0^\infty dr r |A(r, z)|^2 = \infty. \quad (1.37)$$

It can be easily verified that the linear BB in Eq. (1.36) is a solution of the LSE (1.17), written in cylindrical coordinates as

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \left(\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \varphi^2} \right). \quad (1.38)$$

Introducing the new parameter

$$\delta \equiv -\frac{k\theta^2}{2}, \quad (1.39)$$

and a constant factor that determines the peak intensity of the beam, the fundamental BB can be expressed as

$$A(r, z) = \sqrt{I_0 J_0} (\sqrt{2k|\delta|} r) e^{i\delta z}. \quad (1.40)$$

Because $\delta < 0$, and considering the conical structure of beam, these BBs have their wave vectors shortened along their direction of propagation, and thus their effective wave vector in this direction is given by

$$k_{\text{eff}} = k + \delta, \quad (1.41)$$

and consequently, a superluminal phase velocity of $v_f = \omega/k_{\text{eff}}$ (Giovannini et al. 2015).

It has been neglected in the previous calculations the fact that the axicon has a finite aperture. Bessel beams created under real experimental conditions by axicons with a radius R . Given the conical geometric disposition presented in Fig. 1.3, the superposition of the axicon-generated waves, and thus the BB, only exists over a finite distance

$$L_B = \frac{R}{\tan \theta} \simeq \frac{R}{\theta}, \quad (1.42)$$

called Bessel zone, which can nonetheless be made as large as desired simply by either increasing R or decreasing θ .

Bessel beams are currently generated and used on a daily basis in laboratories, in applications such as optical tweezers, microparticle trapping and guiding (Sokolovskii et al. 2014), and several other tasks in optical micromanipulation (Arlt et al. 2001).

High order Bessel beams, or Bessel vortex beams: beams carrying orbital angular momentum

There is recently huge interest in light beams carrying optical vortices and the OAM associated with them. A vortex-carrying light beam is a light beam with a phase variation $e^{-is\varphi}$ in the azimuthal direction φ perpendicular to the propagation direction z of the beam. At the beam center, $r = 0$, the phase is undetermined and the electric field strength vanishes, which endows the beam with a ring structure. The integer number s is called the topological charge, and it can take arbitrary positive or negative values, determining

the OAM $s\hbar$ per photon of the beam, in addition to its spin angular momentum $\pm\hbar$ that depends on polarization (0 for linear polarization) (Allen et al. 1992; Yao and Padgett 2011). The topological charge is defined by the circulation of the phase gradient around the vortex (Desyatnikov, Torner, and Kivshar 2005),

$$s = \frac{1}{2\pi} \oint \vec{\nabla}\varphi \cdot d\mathbf{r} , \quad (1.43)$$

where the circulation is calculated over a curve $\mathbf{r}$ containing the singularity. In this work only positive values of s shall be considered, since negative values are completely equivalent, except for the spinning direction of the beam, which is the opposite to that observed in the case of positive sign.

Even if the study of such vortex carrying beams began with Laguerre-Gauss beams (Allen et al. 1992) because they are easily produced from a vortexless, Gaussian-like beam with a fork-type grating or a spiral phase plate (Yao and Padgett 2011), today high-order Bessel beams dominate the scientific interest because of their additional property of being nondiffracting. Recognizing the existence of these new and unlimited degrees of freedom of the OAM of the photons in a light beam has led to a revolution in optics with ramifications in other fields such as astrophysics (Foo, Palacios, and Swartzlander 2005) or medicine (Jeffries et al. 2007), with applications as optical tweezers (He et al. 1995), to laser ablation (Omatsu et al. 2010), classical and quantum information processing (Gibson et al. 2004; Mair et al. 2001), transference of OAM from light to matter, in particular to nanoparticles, Bose-Einstein condensates or atomic interactions (Masalov 1997; Allen, Padgett, and Babiker 1999; Tabosa and Petrov 1999). In the pulsed regime (Shvedov et al. 2010; Yamane, Toda, and Morita 2012), pulsed vortex beams are used to excite high harmonics and extremely short, attosecond pulses carrying also vortices of very high topological charge (Zurch et al. 2012; Hernández-García et al. 2013; Gariépy et al. 2014; Rego et al. 2016).

In paraxial approximation, high order Bessel beams, or BVBs for short, are expressed by

$$A(r, \varphi, z) = \sqrt{I_0} J_s(\sqrt{2k|\delta|r}) e^{i\delta z} e^{is\varphi} \quad s = \pm 1, \pm 2, \dots \quad (1.44)$$

As before, $\delta = -k\theta^2/2$, with θ being the conical angle. Equation (1.44) can be verified to be a solution for the LSE (1.38) by direct substitution. The intensity profile of a BVB of topological charge $s = 1$ is depicted at the bottom of Fig. 1.2. As for the fundamental, vortexless BB, these BVBs are also nondiffracting that carry infinite power. Their wavefronts are of helical shape, given by the expression $kz + \delta z + s\varphi = \text{cte}$, as shown in Fig. 1.4(a). BVBs are experimentally created by axicons much in the same way as their vortexless counterparts, only with the difference that the incident plane wave already has an embedded vortex when enters the axicon. Around the vortex, the phase varies continuously between 0 and 2π a number of times equal to the topological charge s . As demonstrated in (Ruiz-Jiménez 2016), the amplitude A must satisfy $A \propto r^{|s|}$ near the vortex. The experimental construction of the vortex is achieved through the use of holographic devices (Heckenberg et al. 1992) which imprint tailored phase differences using diffraction gratings; another possibility is to use transparent spiral phase plates whose thickness increases with the azimuthal angle (Beijersbergen et al. 1994). Both experimental methods are illustrated in Fig. 1.4(b).

Figure 1.4: (a) Helical wavefronts for different values of the topological charge s . (b) Above, a hologram with a fork at the center, commonly used to generate optical vortices. Below, a phase plate is instead the instrument used to create the optical vortices.

Another way to express linear Bessel beams is by using Hankel functions (Spiegel 1970), as

$$A(r, \varphi, z) = \frac{\sqrt{I_0}}{2} \left[H_s^{(1)}(\sqrt{2|\delta|kr}) + H_s^{(2)}(\sqrt{2|\delta|kr}) \right] e^{i\delta z} e^{is\varphi}, \quad (1.45)$$

where $H_s^{(1,2)}$ are Hankel functions of the first and second kind, respectively. This representation of BB will be further addressed in Ch. 2. For now it is sufficient to know that the first term represents a Hankel beam carrying power in an outward spiral, while the second term represents a Hankel beam carrying an equal amount of power, now in an inward spiral.

1.3 Optical Kerr effect

1.3.1 Nonlinear refractive index

At high intensities, Eq. (1.5) is no longer valid and the value of the polarization is not proportional to the existing electric field. Instead, a term representing the nonlinear response of the medium must be included,

$$\mathbf{P} = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t), \quad (1.46)$$

where the nonlinear term includes contributions of the square (or superior powers) of the electric field; expressing all possible self-interactions of the field at any order. Given that most liquids, gases and amorphous crystals do not give rise to even-order optical nonlinearities (since they present inversion symmetry¹ due to their isotropy); and since the third-order optical nonlinearity is already small regarding the rest of the contributions²

¹Still, in some cases nonlinear effects can break up this symmetry and thus generate even-ordered optical nonlinearities (Moll et al. 2002)

²For solids, $\chi^{(1)}$ has values around the unity, while $\chi^{(3)}$ has values around 10^{-22} (for instance, $\chi^{(3)} = 2, 24 \cdot 10^{-22} \text{m}^2/\text{V}^2$ for fused silica (Boyd 2008)).

and can be overlooked, the following cubic expression is, except for extreme intensities or materials with enhanced nonlinearities, a very good approximation³

$$P_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(3)}(\tau_1, \tau_2, \tau_3) E(\mathbf{r}, t - \tau_1) E(\mathbf{r}, t - \tau_2) E(\mathbf{r}, t - \tau_3) d\tau_1 d\tau_2 d\tau_3, \quad (1.47)$$

where $\chi^{(3)}$ is the third-order nonlinear susceptibility. For the media considered in this work, the dispersion of the nonlinear polarization is usually very small, and in addition irrelevant for quasi-monochromatic light. Hence, the response produced by the medium can be assumed to be instantaneous:

$$\chi^{(3)}(\tau_1, \tau_2, \tau_3) = \chi^{(3)} \delta(\tau_1) \delta(\tau_2) \delta(\tau_3), \quad (1.48)$$

and therefore,

$$P_{\text{NL}}(\mathbf{r}, t) = \epsilon_0 \chi^{(3)} E^3(\mathbf{r}, t). \quad (1.49)$$

Using complex notation, the cube of the electric field is

$$E^3 = \frac{1}{8} [\mathcal{E}^3 e^{-3i\omega t} + cc + 3|\mathcal{E}|^2 \mathcal{E} e^{-i\omega t} + cc], \quad (1.50)$$

where cc denotes the complex conjugate. The first two terms give out terms of frequency 3ω , giving rise to the generation of the third harmonic, which needs particular conditions of phase matching to be effective, and in fact does not have a relevant influence on the nonlinear propagation dynamics (Kolesik et al. 2006). Hence, the cube of the field can be expressed as

$$E^3 = \frac{3}{8} [|\mathcal{E}|^2 \mathcal{E} e^{-i\omega t} + cc]. \quad (1.51)$$

On the other hand, the nonlinear polarization response at the frequency ω can also be expressed, using complex notation, as

$$P_{\text{NL}}(\mathbf{r}, t) = \text{Re}\{\mathcal{P}_{\text{NL}}(\mathbf{r}) e^{-i\omega t}\} = \frac{1}{2} [\mathcal{P}_{\text{NL}}(\mathbf{r}) e^{-i\omega t} + \mathcal{P}_{\text{NL}}^*(\mathbf{r}) e^{i\omega t}], \quad (1.52)$$

and using Eqs. (1.49), (1.51) and (1.52), and matching the corresponding terms with the same exponential dependence, it comes

$$\mathcal{P}_{\text{NL}} = \frac{3}{4} \epsilon_0 \chi^{(3)} |\mathcal{E}|^2 \mathcal{E}. \quad (1.53)$$

As a result, the nonlinear Helmholtz equation (1.13) obtained in a similar way as in the linear case, but including the nonlinear polarization, is

$$\Delta \mathcal{E} + \frac{\omega^2}{c^2} \left(n^2 + \frac{3}{4} \chi^{(3)} |\mathcal{E}|^2 \right) \mathcal{E} = 0. \quad (1.54)$$

Using the intensity $I = n\epsilon_0 c |E|^2/2$ (in W/cm²) which is the most directly accessible magnitude experimentally, and defining the nonlinear refractive index n_2 as

$$n_2 \equiv \frac{3}{4n^2\epsilon_0 c} \chi^{(3)}, \quad (1.55)$$

³Again, for anisotropic media $\chi^{(3)}$ would be a tensor.

Eq. (1.54) becomes

$$\Delta\mathcal{E} + \frac{\omega^2}{c^2}(n^2 + 2nn_2I)\mathcal{E} = 0 . \quad (1.56)$$

This equation indicates that the nonlinearity modified the refractive index of the medium, n_T , given by

$$n_T^2 = n^2 + 2nn_2I , \quad (1.57)$$

and since the nonlinear change of the refractive index is always extremely small compared to the linear index,

$$n_T \simeq n + n_2I . \quad (1.58)$$

The nonlinear refractive index n_2 is measured in cm^2/W . The condition $n_2I \ll n$ is satisfied, even at high intensity values (TW/cm^2) as those considered in this dissertation (n_2 takes values of the order of $10^{-16}\text{cm}^2/\text{W}$ in the media considered, so that $n_2I \sim 10^{-4} \ll n \sim 1$). The variation of the refractive index proportional to the intensity of the field is known as the optical Kerr effect. In this case it will give rise to a cubic nonlinearity in the NLSE, although in some particular cases higher order terms of the refractive index are necessary, $n_T \simeq n + n_2I + n_4I^2$, possibly manifesting as quintic nonlinearities.

Using the quasi-plane wave Ansatz (1.14) in Eq. (1.56), and application of the paraxial approximation in Eq. (1.16), leads, as in the linear case, to (in SI units)

$$\frac{\partial A}{\partial z} = \frac{i}{2k}\Delta_{\perp}A + i\frac{k\epsilon_0 c}{2}n_2|A|^2A , \quad (1.59)$$

which is the NLSE for the slowly varying complex amplitude. In our derivation, we have assumed a monochromatic light beam. In practice, the high intensities needed to trigger nonlinear effects require the use of pulses, for which dispersion in the linear part of the index may be relevant, and would be included in Eq. (1.59) as a second derivative in time. However, as discussed below, the effect of chromatic dispersion is usually negligible for the long pulses used and sufficiently short propagation distances considered.

Yet, it must be noted that the electric field used in this dissertation it is not that defined in the SI system, but it is actually proportional to it with a proportionality constant $\sqrt{n\epsilon_0 c/2}$, and thus measured in $\text{W}^{1/2}/\text{cm}$, so that its squared value is equal to the intensity (irradiance or mean flux value of the Poynting vector, measured in W/cm^2 , $I = n\epsilon c|E|^2/2$ (Griffiths 2008)). Therefore, the previous equation for the complex envelope, measured in $\text{W}^{1/2}/\text{cm}$, is

$$\frac{\partial A}{\partial z} = \frac{i}{2k}\Delta_{\perp}A + i\frac{k}{n}n_2|A|^2A , \quad (1.60)$$

which is the form most commonly used in this dissertation. The beam power in Eq. (1.18) then simply reads

$$P = \int |A|^2 dx dy . \quad (1.61)$$

1.3.2 Spatial self-phase modulation and self-focusing

In qualitative terms, the effect of the Kerr nonlinearity on a light beam is to self-focus it when $n_2 > 0$ and to self-defocus it when $n_2 < 0$. The term “self” is brought here because it is a self-action of the beam through the nonlinear response of the medium. The Kerr nonlinearity may cause the light beam to self-focus ($n_2 > 0$) or self-defocus ($n_2 < 0$) during its propagation. The cubic term in Eq. (1.60) is a nonlinear dispersive term, and therefore affects primarily the phase of the beam. This is the phenomenon of self-phase modulation, or SPM, that makes the phase front to converge (for $n_2 > 0$) or to diverge (for $n_2 < 0$). As a result, also the intensity profile of the beam tends to narrow (for $n_2 > 0$) or to widen (for $n_2 < 0$), which is observed as self-focusing or self-defocusing. An example for the case of a Gaussian beam is provided in Fig. 1.5.

Most materials have $n_2 > 0$ (e.g., glasses, water, air). We shall henceforth focus on self-focusing phenomena. As explained later, the origin of instability observed in many beams is the self-focusing mechanism, and in particular in the nonlinear BVBs considered in this dissertation. In order to understand how SPM acts and causes self-focusing, we consider propagation distances at which diffraction is still negligible (e.g., $z < L_R$ for a Gaussian-like beam). Then the NLSE (1.60) simplifies to

$$\frac{\partial A}{\partial z} = i \frac{k}{n} n_2 |A|^2 A. \quad (1.62)$$

At $z = 0$, for a plane wavefront $A(r, 0) = a(r, 0)$, with $a(r, 0)$ the real amplitude, and $I(r) = a^2(r, 0)$ the intensity profile, the following analytical solution can be explicitly calculated

$$A(r, z) = a(r, 0) e^{i \frac{k}{n} n_2 I(r) z}. \quad (1.63)$$

Figure 1.5: (a) Transversal profiles where self-focusing ($n_2 > 0$) evolution is shown: above, the intensity profiles of the initial wave and the propagated wave; below, the narrowing of the wave during propagation. The wave depicted is a 1+1D Gaussian beam $A(x) = \exp(-x^2/w^2)$, of width $w = 1$ mm, propagating over a distance of $L = 25$ dm ($\lambda = 633$ nm, $n = 1.5$; $n_2 = 2 \times 10^{-13}$ cm²/W). (b) The same wave, now under self-defocusing effects, $n_2 = -2 \times 10^{-13}$ cm²/W.

Figure 1.6: Self-phase modulation and self-focusing for a Gaussian beam. In red, radial intensity profiles at the beginning and the end of propagation. In green, the evolution of the radial profiles of the converging wavefronts due to self-phase modulation. At the left and at the right, numerical simulations of the Gaussian beam under self-focusing effects.

It can be seen that the phase of the light beam acquires a modulation proportional to the intensity profile $I(r)$ during propagation. This is just the phenomenon of self-phase modulation acting. A light beam have typically a maximum of intensity in its center. Thus, to a first order of approximation, the phase modulation consists on an inverted parabola, which is a converging wave front – just the opposite situation originated by diffraction (see e.g., Eqs. (1.22) and (1.16) at $z > 0$ for diverging wavefronts). Eventually, the converging wavefronts cause the beam to narrow, that is, to self-focus, whose accurate description would require the inclusion of the neglected diffraction term in the NLSE (1.62). Figure 1.6 illustrates this phenomenon, the intensity profiles in red color and wavefront evolution in green. The initially collimated beam, or in terms or rays, parallel rays perpendicular to the plane wavefront, bend during propagation, maintaining themselves perpendicular to the wavefronts, what is an evident case of self-focusing or narrowing of the light beam.

From Eq. (1.63), the distance at which SPM and self-focusing start to be relevant is characterized by the SPM distance given by

$$L_{\text{SPM}} \equiv \frac{n}{2kn_2I_0}, \quad (1.64)$$

where I_0 is the peak intensity of the light beam. The SPM distance is typically small, indicating that the Kerr effect is relevant since the beginning of the propagation. As mentioned before, if $n_2I_0 \sim 10^{-4}$, for visible or near infrared light ($k \sim 10^5 \text{ cm}^{-1}$) this distance is of a mere fraction of a centimeter.

Figure 1.7: Intensity profile for the Townes soliton.

It has been previously seen how diffraction increases the width of propagating wave packets. Under the right conditions, diffraction and self-focusing effects can balance each other out, giving rise to what is known as spatial solitons, which are light beams whose transversal intensity profiles remain unchanged during propagation, as the widely known Townes soliton (Chiao, Garmire, and Townes 1964). The transversal intensity profile of the Townes soliton is achieved by a precise equilibrium between the effects of diffraction and self-focusing. Mathematically, the transversal profile of the Townes soliton, shown in Fig. 1.7 is obtained by trying the stationary state $A(r, z) = a(r)e^{i\delta z}$, whose intensity profile $I = |a(r)|^2$ does not depend on the propagation distance z , in the NLSE (1.60) and solving the ordinary differential equation obtained for $a(r)$, as detailed in the next section.

It is also of interest to proceed more qualitatively. The conditions under which diffraction and SPM can cancel each other out can be qualitatively evaluated imposing that the typical lengths of diffraction, $L_R = kw^2/2$, and of SPM, L_{SPM} , are equal, leading to the beam intensity

$$I_S = \frac{n}{k^2 w^2 n_2} . \quad (1.65)$$

Equation (1.65) is a simple, approximate criterion to the formation of a spatial soliton. For a Gaussian profile $I_S \exp(-2r^2/w^2)$, carrying a power of $2\pi \int_0^\infty dr r I(r) = \pi I_S w^2/2$, substituting Eq. (1.65) would give out the necessary power, or critical power, equal to $P_{\text{CRIT}} = \pi n/2k^2 n_2$ (for instance, approximately 10 MW for glass). This is nonetheless only an estimated value, and actual values of the critical power are usually higher and calculated as

$$P_{\text{CRIT}} = \gamma \frac{\pi n}{2k^2 n_2} . \quad (1.66)$$

For example, for a Gaussian beam $\gamma = 3.77$ (Marburger 1975), and for a Townes soliton (Chiao, Garmire, and Townes 1964) $\gamma = 3.72$. Light beams carrying critical power have their diffractive and self-focusing effects balancing each other out (Silberberg 1990), creating a spatial soliton or self-trapped beam (Bjorkholm and Ashkin 1974). If $P < P_{\text{CRIT}}$, diffraction is stronger and the beam constantly widens throughout propagation. If $P > P_{\text{CRIT}}$, the self-focusing effect is stronger; the beam shrinks and its peak intensity grows without bound, until a singularity is formed at a finite propagation distance, known as beam collapse (Sulem and Sulem 1999). Actually, this singularity is an artifact of the mathematical model of the cubic NLSE (1.60). Before the formation of the singularity, the intensity grows to very high values, triggering other linear and nonlinear effects that stop its formation. The specific mechanisms that stop singularity formation depend on the experimental settings, and can be as varied as saturation of the Kerr nonlinearity (negative quintic term, or other models), chromatic dispersion for pulses, nonlinear absorption or medium ionization. In addition, what occurs after stopping the beam collapse is also dependent on the mechanism that stopped it. A self-trapped beam may be formed, or a spatial soliton in stable propagation. If medium ionization occurs, then plasma channels are formed in what is known as filamentation and the plasma in turn modifies the

refractive index creating a self-defocusing effect that may help to stabilize the balance between self-focusing and diffraction. The study of the mechanisms that stop beam collapse leading to self-trapped stable propagation is currently object of intense research.

As mentioned above, the intensities needed for the Kerr effect to be relevant requires the use of pulses made of a spectrum of frequencies rather than a single one. This brings the question of whether chromatic dispersion or dependence of the linear refractive index with frequency are relevant or not. The characteristic length at which dispersive effects begin to be relevant is given by $L_{\text{DISP}} \equiv \Delta t^2 / 2|k''|$, where Δt is the pulse duration and $k'' \equiv \partial^2 k / \partial \omega^2|_{\omega}$, with $k(\omega) = n(\omega)\omega/c$, relates to the group velocity dispersion (or GVD) occurring when different frequencies propagate at different velocities in a medium. For pulses with large duration, around 200 femtoseconds ($1 \text{ fs} = 10^{-15} \text{ s}$), which are those most commonly used, this characteristic length is usually large. For example, in air, at near infrared wavelengths ($\lambda = 2\pi c/\omega = 800 \text{ nm}$), dispersion is $k'' = 0.2 \text{ fs}^2/\text{cm}$, and thus $L_{\text{DISP}} = 1 \text{ km}$, large enough a length to assume a dispersionless propagation if only a few tens of meters of propagation are relevant. Yet, in amorphous glasses, dispersion is usually larger by two orders of magnitude, and for a pulse of the same duration the dispersion is of a few meters. Even in this case, the media used for the study of Bessel beams is much shorter dimensions, making chromatic dispersion negligible.

1.4 Stationary solutions of the nonlinear Schrödinger equation and spatial solitons

The study of the nonlinear dynamics dictacted by the NLSE (1.60) is based on the analysis of its stationary solutions, such as the Townes soliton in Fig. 1.7, and their stability properties. In this and the next sections, a few examples that are simple and relevant will be presented, in order to illustrate the general procedure used later for more complex solutions.

Basic examples of stationary solutions of the NLSE (1.60) have the form

$$A(r, z) = a(r)e^{i\delta z}, \quad (1.67)$$

δ being an arbitrary constant and $a(r)$ the real amplitude, as the intensity does not depend of the propagation distance. Substituting Eq. (1.67) in the NLSE (1.60), results in

$$\frac{1}{2k} \left(\frac{d^2 a}{dr^2} + \frac{1}{r} \frac{da}{dr} \right) + \frac{kn_2}{n} a^3 - \delta a = 0 \quad (1.68)$$

which is an ordinary differential equation to be solved with the initial conditions

$$a(0) = \sqrt{I_0}, \quad (1.69)$$

$$\frac{da(0)}{dr} = 0, \quad (1.70)$$

where I_0 is the intensity at the center of the beam, usually equal to the peak intensity. The second condition is imposed by the fact the beam is azimuthally symmetrical and

therefore cannot have a sharp point (i.e. a cusp) at its center. A stationary solution is physically valid and represents a transversally localized wave if $a(r) \rightarrow 0$ as $r \rightarrow \infty$. As commonly done in nonlinear analysis, and as shall be done in later chapters, the following dimensionless variables are introduced,

$$\tilde{a} = a/\sqrt{I_0}, \quad (1.71)$$

$$\rho = \sqrt{k k_{\text{NL}}} r, \quad (1.72)$$

$$k_{\text{NL}} = k n_2 I_0 / n. \quad (1.73)$$

With the new dimensionless variables, the ordinary differential equation (1.68) becomes

$$\frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} + 2\tilde{a}^3 - 2\alpha \tilde{a} = 0 \quad (1.74)$$

where $\alpha = \delta/k_{\text{NL}}$, and with initial conditions

$$a(0) = 1, \quad (1.75)$$

$$\frac{da(0)}{d\rho} = 0. \quad (1.76)$$

Equation (1.74) must be solved numerically. It can be demonstrated that there exists a localized solution only for a subset of values of α , which are represented in Fig. 1.8 as a partly discrete, partly continuous spectrum of values of α , together with some solutions of $\tilde{a}(\rho)$ (Sulem and Sulem 1999). For $\alpha = 1$, the solution $a = 1$ is not localized, but it is theoretically relevant as it represents a nonlinear plane wave (as a stationary state, its stability is studied in the Sec. 1.5 as a simple illustrative example). For $\alpha_0 = 0.2054517\dots$ the solution is the Townes profile shown in Fig. 1.7, which presents no zeros in its radial profile and is called the fundamental “soliton”. Quotations marks are used because stationary propagation is only the first requirement satisfied by a soliton, being stability the second one, as explained later in Sec. 1.5. For an infinite sequence of positive discrete values of α , satisfying $\alpha_n < \alpha_0$, the so-called high-order spatial “solitons” are solutions that decrease exponentially, with n zeros of the intensity and n concentric rings surrounding the central maximum. On the contrary, for negative values of α localized solutions do also exist, but now they decrease oscillating infinite times as ρ increases, in the same way as Bessel beams. In fact, these solutions for negative values of α are known as nonlinear Bessel beams in media with pure cubic Kerr nonlinearity. Considering now only the discrete part of the spectrum of solutions, in real-valued variables, the localized stationary solutions of spatial “solitons” are expressed as

$$A_n = \sqrt{I_0} \tilde{a}_n(\sqrt{k k_{\text{NL}}} r) e^{i\delta_n z}, \quad (1.77)$$

and the complete electric field as

$$\mathcal{E}_n = \sqrt{I_0} \tilde{a}_n(\sqrt{k k_{\text{NL}}} r) e^{-i[\omega t + (k + \delta_n)z]}, \quad (1.78)$$

where $\tilde{a}_n$ is the numerical solution of Eq. (1.74). In the particular case of the Townes soliton, it can be shown that the power carried by the beam is given by Eq. (1.66), with $\gamma = 3.72$.

Figure 1.8: Discrete and continuous spectra of α . On top, continuous spectrum for negative values of α , discrete spectrum for positive values of α . On the bottom row, three illustrative examples are provided, left to right, of (a) a nonlinear BVB, (b) a higher-order soliton and (c) the Townes “soliton”.

Evidently, the previous Ansatz (1.67) is just one possibility among many. For instance, other localized stationary solutions of the form

$$A(r, \varphi, z) = a(r)e^{i\delta z}e^{is\varphi}, \quad (1.79)$$

can be tested. Equation (1.79) expresses a stationary solution carrying a vortex of topological charge s and OAM. These solutions are known as vortex “solitons”, first reported in (2+1)D in (Kruglov and Vlasov 1985), and since then extensively investigated under very different conditions (Kruglov, Logvin, and Volkov 1992; Desyatnikov, Torner, and Kivshar 2005). The radial amplitude profiles $a(r)$ of vortex “solitons” satisfy, according to the NLSE (1.60),

$$\frac{1}{2k} \left(\frac{d^2 a}{dr^2} + \frac{1}{r} \frac{da}{dr} - \frac{s^2}{r^2} a \right) + \frac{kn_2}{n} a^3 - \delta a = 0 \quad (1.80)$$

or, using the dimensionless normalized variables in Eqs. (1.73),

$$\frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} - \frac{s^2}{\rho^2} \tilde{a} + 2\tilde{a}^3 - 2\alpha\tilde{a} = 0. \quad (1.81)$$

Since Eq. (1.81) has a vortex of topological charge s in its center, it must in turn be solved with the initial condition $\tilde{a} \simeq \rho^{|s|}$. Following the previously described process, again it is

found numerically that there exist localized solutions for a discrete spectrum of positive values of α , called vortex “solitons”, a fundamental one and others of higher order; and weakly localized solutions for a continuous spectrum of negative values of α , which are known as high-order Bessel beam in transparent Kerr media.

1.5 Instability phenomena

Taking into consideration all previously seen on spatial solitons, it should suffice, for instance, to generate a Townes “soliton” $A_0(r, z) = \sqrt{I_0} \tilde{a}_0(\sqrt{k_{\text{NL}}} r) e^{i\delta_0 z}$ (see Fig. 1.7), with a critical power $P_{\text{CRIT}} = \gamma\pi n/2k^2 n_2$, with $\gamma = 3.72$, in order to observe a self-trapped beam experimentally. Unfortunately, things are more complicated because nonlinear waves suffer from instability phenomena. If the NLSE would have only one transversal dimension, say x , the phenomenon of collapse does not exist, and the corresponding stationary solution, known as the sech (hyperbolic secant) soliton would be stable (Agrawal 2001). This is a true soliton that has extensively been used in optical fiber communications. However, instability is a feature easily found for configurations of two or more spatial dimensions, due to a phenomenon already discussed in previous sections as beam collapse (Sulem and Sulem 1999). It affects all previously discussed spatial “solitons”, included the nonlinear BBs in the continuous spectrum.

Instability and beam collapse are intrinsically bound: if a NLSE with certain nonlinear terms presents collapse, as with the cubic Kerr case discussed, all stationary solutions are unstable. On the contrary, adding new nonlinear terms that prevent collapse, then there may exist stable stationary solutions.

More in detail, adding a small perturbation p to a stationary state A_s , thus obtaining $A = A_s + \epsilon p$, with $\epsilon \rightarrow 0$, and substituting it in the NLSE (1.60), which contains only the effects of diffraction and the Kerr nonlinearity, two outcomes are possible. First, if $P_{\text{beam}} < P_{\text{CRIT}}$, diffraction dominates over the Kerr self-focusing and the beam will diffract. Second, if $P_{\text{beam}} > P_{\text{CRIT}}$, self-focusing dominates and the beam collapses at a finite propagation distance, as previously explained in Sec. 1.3.2. The case $P_{\text{beam}} \gtrsim P_{\text{CRIT}}$ is less straightforward, as the beam may diffract or collapse depending on different factors. Therefore, at least with the theoretical model presented until now, it should be impossible to observe stable propagation, or self-trapped beams, experimentally. In fact, no true soliton, i.e. with stable propagation, has been observed under experimental conditions including only diffraction and pure Kerr nonlinearity. Indeed, the Townes beam has never been observed.

Nonetheless, and surprisingly enough, Fig. 1.9 shows such a stable propagation, obtained experimentally, as many more reported (Bjorkholm and Ashkin 1974). The model presented until now is thus flawed, or at least incomplete. The stable propagation is usually achieved in laboratory generating a beam satisfying $P_{\text{beam}} > P_{\text{CRIT}}$, but as the beam collapses with propagation, it reaches a state with a very high (but finite) intensity, the critical collapse being halted by other nonlinearities which increase their magnitude as the intensity grows. These new nonlinearities appear as added terms on the NLSE (1.60), and must be such that the new NLSE must not admit the phenomenon of beam collapse and thus presents stable stationary states as solutions.

Figure 1.9: Photograph of the top view of a soliton propagating in a photorefractive crystal (above); and in a linear media, the soliton showing standard diffraction while propagating (below). In the photorefractive case, the crystal is 5 mm long, and the spatial soliton width is 10 μm . The photograph was published in (Shih et al. 1996).

For example, instead of assuming a nonlinear refractive index as in Eq. (1.58), we take the saturable Kerr medium with refractive index of the form

$$n_{\text{T}} \simeq n + n_2 \frac{I}{1 + (I/I_s)} . \quad (1.82)$$

with usually $I_s \gg I$, I_s being the saturation intensity. Therefore, the nonlinear refractive index will not grow indefinitely with intensity, but it asymptotically tends to saturate at $n = n + n_2 I_s$, avoiding the beam collapse. In this model, the NLSE is easily found to be

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \Delta_{\perp} A + i \frac{kn_2}{n} \frac{|A|^2}{1 + (|A|^2/I_s)} A , \quad (1.83)$$

and the stationary states to be probed are the same as in Eq. (1.67). This leads to the new ordinary differential equation

$$\frac{1}{2k} \left(\frac{d^2 a}{dr^2} + \frac{1}{r} \frac{da}{dr} \right) + \frac{kn_2}{n} \frac{a^3}{1 + a^2/I_s} - \delta a = 0 \quad (1.84)$$

to be solved with the initial conditions

$$a(0) = \sqrt{I_0} , \quad (1.85)$$

$$\frac{da(0)}{dr} = 0 . \quad (1.86)$$

In dimensionless variables, the previous Eqs. (1.84) and (1.86) become

$$\frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} + \frac{2\tilde{a}^3}{1 + (\tilde{a}^2 I_0/I_s)} - 2\alpha \tilde{a} = 0 \quad (1.87)$$

with $\alpha = \delta/k_{\text{NL}}$ and the initial conditions

$$\tilde{a}(0) = 1, \quad (1.88)$$

$$\frac{d\tilde{a}(0)}{dr} = 0. \quad (1.89)$$

Again, as in the case in Fig. 1.8, a continuous spectrum for negative values of α and a discrete spectrum for positive values of α emerge, leading to very similar conclusions. Important to notice the fact that there is no beam collapse in this model, and that there are stable stationary states, in the sense presented in the analysis at the end of this section.

In brief, extra nonlinearities relevant at high intensities and specifically tailored nonlinearities are currently the main lines of research on the search of stable propagation of light beams. This dissertation describes a model that takes the first option, and uses the multiphoton absorption nonlinearity to achieve stable propagation. This choice comes motivated by the fact that MPA is always relevant at high intensities, as the Kerr nonlinearity. It is thus a natural choice, instead of models that need physical media that contain specific nonlinearities, which are more difficult to implement. Next section shall introduce an overview on MPA, necessary to understand how it fits the model presented in this dissertation.

1.5.1 An example of linearized stability analysis

In the following it is introduced an example of the standard way to predict whether a nonlinear wave is stable or unstable by means of the so-called linearized stability analysis of the nonlinear plane wave in cubic Kerr media presented in the previous section (case $\alpha = 1$ in Fig. 1.8), which in real variables reads as

$$A(z) = \sqrt{I_0} e^{ik_{\text{NL}}z}, \quad (1.90)$$

where $k_{\text{NL}} = kn_2 I_0 / n_0$. Although it is possible to study general perturbations in space and time (Malomed et al. 2005; Mihalache 2012), here, for illustration purposes, we shall consider only spatial perturbations in the x direction. Let $\epsilon P \equiv \epsilon [\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z}]$ be a comparatively small ($\epsilon \ll \sqrt{I_0}$) generic spatial modulation in the x direction of the frequency k_x . Applying it to the nonlinear monochromatic plane wave to proceed with a standard linearized stability analysis, the expression obtained is

$$\begin{aligned} A(z) &= \sqrt{I_0} e^{ik_{\text{NL}}z} + \epsilon P e^{ik_{\text{NL}}z} \\ &= \sqrt{I_0} e^{ik_{\text{NL}}z} + \epsilon [\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z}] e^{ik_{\text{NL}}z}, \end{aligned} \quad (1.91)$$

with μ and ν arbitrary constants, and $\kappa = \kappa_R + i\kappa_I$ a complex parameter whose imaginary part can be identified with the gain (or exponential growth rate) of the perturbation, as shown by

$$\begin{aligned} P &= [\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z}] \\ &= [\mu e^{-ik_x x + i(\kappa_R + i\kappa_I)z} + \nu^* e^{ik_x x + i(-\kappa_R + i\kappa_I)z}] \\ &= [\mu e^{-ik_x x + i\kappa_R z} + \nu^* e^{ik_x x - i\kappa_R z}] e^{-\kappa_I z}. \end{aligned} \quad (1.92)$$

Now it is clear the role of the imaginary part κ_I as the gain of the modulation. If $\kappa_I \geq 0$, then the perturbation will exponentially decrease with propagation, meaning that the nonlinear monochromatic plane wave is stable under modulations of the frequency k_x . On the other hand, if $\kappa_I < 0$, then an exponential increase of the modulation will cause the development of the instability at the frequency k_x .

Substituting the previous Ansatz (1.91) into Eq. (1.60) it is possible to analyze the stability or instability of the nonlinear monochromatic plane wave under different frequency modulations. Being this a linearized analysis, it is important to point out that throughout the following calculations all terms of any order greater than linear in ϵ will be neglected. First, some auxiliary calculations:

$$\frac{\partial A}{\partial z} = \left[k_{\text{NL}} \sqrt{I_0} + \epsilon \left[(\kappa + k_{\text{NL}}) \mu e^{-ik_x x} e^{i\kappa z} + (-\kappa^* + k_{\text{NL}}) \nu^* e^{ik_x x} e^{-i\kappa^* z} \right] \right] i e^{ik_{\text{NL}} z}, \quad (1.93)$$

$$\Delta_{\perp} A = \frac{\partial^2 A}{\partial x^2} = -\epsilon k_x^2 \left[\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z} \right] e^{ik_{\text{NL}} z}, \quad (1.94)$$

$$\begin{aligned} |A|^2 A &= \left(\sqrt{I_0} + \epsilon P \right) \left(\sqrt{I_0} + \epsilon P^* \right) \left(\sqrt{I_0} + \epsilon P \right) e^{ik_{\text{NL}} z} \\ &= I_0 \left[\sqrt{I_0} + \epsilon (2P + P^*) \right] e^{ik_{\text{NL}} z} + \mathcal{O}(\epsilon^2) \stackrel{\sim}{\sim} 0 \\ &= I_0 \left[\sqrt{I_0} + \epsilon \left(2 \left[\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z} \right] + \left[\mu^* e^{ik_x x} e^{-i\kappa^* z} + \nu e^{-ik_x x} e^{i\kappa z} \right] \right) \right] e^{ik_{\text{NL}} z} \\ &= I_0 \left[\sqrt{I_0} + \epsilon \left[(2\mu + \nu) e^{-ik_x x} e^{i\kappa z} + (\mu^* + 2\nu^*) e^{ik_x x} e^{-i\kappa^* z} \right] \right] e^{ik_{\text{NL}} z}. \end{aligned} \quad (1.95)$$

Substituting the previous auxiliary results into Eq. (1.60), yields:

$$\begin{aligned} ik_{\text{NL}} \sqrt{I_0} e^{ik_{\text{NL}} z} + \left[i\epsilon \left[(\kappa + k_{\text{NL}}) \mu e^{-ik_x x} e^{i\kappa z} + (-\kappa^* + k_{\text{NL}}) \nu^* e^{ik_x x} e^{-i\kappa^* z} \right] e^{ik_{\text{NL}} z} \right] &= \\ -\frac{i}{2k} \epsilon k_x^2 \left[\mu e^{-ik_x x} e^{i\kappa z} + \nu^* e^{ik_x x} e^{-i\kappa^* z} \right] e^{ik_{\text{NL}} z} + & \\ + i \frac{kn_2}{n_0} I_0 \left[\sqrt{I_0} + \epsilon \left[(2\mu + \nu) e^{-ik_x x} e^{i\kappa z} + (\mu^* + 2\nu^*) e^{ik_x x} e^{-i\kappa^* z} \right] \right] e^{ik_{\text{NL}} z} &\Leftrightarrow \\ (\kappa + k_{\text{NL}}) \mu e^{-ik_x x} e^{i\kappa z} + (-\kappa^* + k_{\text{NL}}) \nu^* e^{ik_x x} e^{-i\kappa^* z} = & \\ \left[-\frac{1}{2k} k_x^2 \mu + k_{\text{NL}} (2\mu + \nu) \right] (e^{-ik_x x} e^{i\kappa z}) + & \\ + \left[-\frac{1}{2k} k_x^2 \nu^* + k_{\text{NL}} (\mu^* + 2\nu^*) \right] (e^{ik_x x} e^{-i\kappa^* z}). & \end{aligned}$$

Separating the terms with $(e^{-ik_x x} e^{i\kappa z})$ and with $(e^{ik_x x} e^{-i\kappa^* z})$, the problem can now be easily expressed as an eigenvalue problem:

$$\begin{cases} (\kappa + k_{\text{NL}}) \mu = -\frac{k_x^2}{2k} \mu + k_{\text{NL}} (2\mu + \nu) \\ (-\kappa^* + k_{\text{NL}}) \nu^* = -\frac{k_x^2}{2k} \nu^* + k_{\text{NL}} (\mu^* + 2\nu^*) \end{cases} \Leftrightarrow$$

$$\begin{pmatrix} k_{\text{NL}} - \frac{k_x^2}{2k} k_{\text{NL}} \\ -k_{\text{NL}} - k_{\text{NL}} + \frac{k_x^2}{2k} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \kappa \begin{pmatrix} \mu \\ \nu \end{pmatrix}. \quad (1.96)$$

In addition to the trivial solution ($\mu = \nu = 0, \forall \kappa$), the eigenvalue problem is solved for $\mu \neq 0, \nu \neq 0$ if

$$\begin{vmatrix} k_{\text{NL}} - \frac{k_x^2}{2k} - \kappa k_{\text{NL}} \\ -k_{\text{NL}} - k_{\text{NL}} + \frac{k_x^2}{2k} - \kappa \end{vmatrix} = 0 \Leftrightarrow \kappa = \pm k_x \sqrt{\frac{k_x^2 - 4k_{\text{NL}}k}{4k^2}}.$$

Now, κ will be imaginary if the frequency $k_x < 2\sqrt{k_{\text{NL}}k}$, and thus the expression that describes the gain of the spatial perturbation is finally

$$|\kappa_I| \equiv k_x \sqrt{\frac{4k_{\text{NL}}k - k_x^2}{4k^2}}. \quad (1.97)$$

Figure 1.10: Non-zero gain values for modulations on k_x . All of them will grow exponentially and lead to instability, but the one with maximum value at $k_x = \sqrt{2k k_{\text{NL}}}$ will impose itself during propagation.

The existence of modulation frequencies in x that grow exponentially implies that the nonlinear monochromatic plane wave is unstable. In practice, it means that small random perturbations will generally excite the unstable modulations, and the modulation with the highest exponential gain will dominate over the others, emerging spontaneously.

Figure 1.10 shows the values of k_x for which a non-zero gain of the growth of the spatial perturbation is obtained, i.e., the instability domain. There is a maximum gain κ_I for a specific value $k_x = \sqrt{2k k_{\text{NL}}}$, which corresponds to the modulation that will exponentially grow and dominate over all other possible modulations also present at the initial random perturbation. Figure 1.11 shows how this predominating modulation evolves with the propagation of the nonlinear plane monochromatic wave,

gradually making the initial profile lose its form until it transforms into a chaotic regime that is actually outside the scope of the previous linearized analysis, since it is already $\epsilon \sim \sqrt{I_0}$.

Figure 1.11: Development of the instability caused by the modulation with the greatest gain, corresponding to $k_x = \sqrt{2k k_{NL}}$. Each figure corresponds to a different step of propagation, at increasing ζ . At $\zeta = 0.05$ and $\zeta = 0.22$, the perturbation is still under a linearized regime where the analysis present is valid; but at superior values of ζ it is already under a chaotic regime of comparatively big perturbations, outside the scope of this study. Note the different scales on the intensity axis.

1.6 Multiphoton absorption

The media considered here are transparent because linear absorption is negligible, since the photon energy is less than the difference between two energy levels of the medium, or less than the energy necessary for ionization. Nonetheless, at high values of the beam intensity it may occur that the summed energy of a number of photons is equal or greater than the energy needed to ionize the medium (multiphoton ionization) (Göppert-Mayer 1931). The medium is thus no longer transparent, and the intensity of the input wave decreases. This multiphoton absorption (or MPA) is in fact one of the mechanisms that can stop the collapse of the beam due to the self-focusing effect seen earlier (LeMesurier 2000), together with the GVD in the case of short pulses, or plasma defocusing due to ionization itself, depending on the experimental settings. On the other hand, defocusing caused by plasma is usually relevant in the self-focusing of Gaussian beams, but are shown to play a minor role with Bessel beams (Jukna et al. 2014; Xie et al. 2015), and therefore we neglect it and consider only the pure effect of the nonlinear absorption.

Usually, the dominant order of the MPA is the minimum number of photons needed for medium ionization. In order to understand the effects of the MPA, consider the case of plane waves. It is known that the intensity of plane waves decrease according to (Sheik-Bahae et al. 1990)

$$\frac{\partial I}{\partial z} = -\beta^{(M)} I^M, \quad (1.98)$$

Figure 1.12: Evolution of the ratio of the nonlinear loss of intensity, I/I_0 , as a function of the normalized coordinate $\zeta = \beta^{(M)} I_0^{M-1} z$.

where M is the number of photons simultaneously absorbed and $\beta^{(M)}$ is the MPA coefficient, also called Keldysh coefficient (Keldysh 1965), which depends on the number of photons, on the incident wavelength and on medium properties like the atomic density and ionization cross sectional area. Equation (1.98) can be analytically solved, giving

$$I = I_0 [1 + \beta^{(M)} I_0^{M-1} (M-1) z]^{-1/(M-1)} . \quad (1.99)$$

where I_0 is the initial intensity. Examples of the behavior of Eq. (1.99) are given in Fig. 1.12. Since $I = A^2$, the equation for the amplitude A is $\partial A / \partial z = -\beta^{(M)} A^{2M-1} / 2$.

Therefore, taking into account the combined effects of diffraction, Kerr nonlinearity and MPA, the NLSE (1.60) is completed with the nonlinear absorption terms as

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \Delta_{\perp} A + i \frac{k}{n} n_2 |A|^2 A - \frac{\beta^{(M)}}{2} |A|^{2M-2} A . \quad (1.100)$$

Due to the fact that MPA effects are generally relevant at much higher intensities than the Kerr effect, usually a saturation in the Kerr nonlinearity, or its cubic-quintic approximation, is also included in Eq. (1.58), $n_T \simeq n + n_2 I + n_4 I^2$, which leads to a quartic amplitude-dependent term, i.e., quintic on the nonlinear equation, expressed as

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \Delta_{\perp} A + i f(|A|^2) A - \frac{\beta^{(M)}}{2} |A|^{2M-2} A , \quad (1.101)$$

where $f(u) \equiv k(n_2 u + n_4 u^2)/n$.

According to Eq. (1.99), the length at which MPA effects are relevant can be estimated using

$$L_{\text{MPA}} \equiv \frac{1}{\beta^{(M)} I_0^{M-1}} . \quad (1.102)$$

The intensity at which the effects of nonlinear absorption are relevant can also be calculated, given a medium of length L , simply by manipulating Eq. (1.102), obtaining $I_{\text{MPA}} = (1/\beta L)^{1/M-1}$. A relevant value of the intensity is that at which the effects of nonlinear absorption and diffraction are of similar order of magnitude. Imposing $L_{\text{MPA}} = L_{\text{R}}$, comes

$$I_{\text{MPA-R}} = \left(\frac{2}{\beta^{(M)} k w^2} \right)^{\frac{1}{M-1}}. \quad (1.103)$$

Now is also possible to predict which would be the necessary intensity to have diffraction, self-focusing and MPA effects competing at an arbitrary distance. For example, in air, with $n \simeq 1$ and $n_2 = 3.2 \times 10^{-19} \text{cm}^2/\text{W}$, the usual MPA coefficient at 800 nm is $M = 8$, with $\beta^{(8)} = 1.8 \cdot 10^{-94} \text{cm}^{13}/\text{W}^7$ (Couairon et al. 2013). Supposing on Eq. (1.103) a beam width of 1 millimeter, an intensity of $I_0 \sim 10 \text{ TW}/\text{cm}^2$ is obtained, and thus a competition among the three effects would be observed at a distance of a few meters. In glass, with $n = 1.461$ and $n_2 = 4.1 \times 10^{-16} \text{cm}^2/\text{W}$, the MPA order at 800 nm is $M = 5$, and the coefficient $\beta^{(5)} = 3.5 \times 10^{-50} \text{cm}^7/\text{W}^4$ (Gaizauskas et al. 2009). Using these values, for a beam width of 0.01 mm and an intensity of $I_0 \sim 0.5 \text{ TW}/\text{cm}^2$, the characteristic distances of the three effects coincide around the value 0.5 mm. These values illustrate the usual experimental setting considered in this dissertation.

1.7 Summary

The model to be used in the study of the propagation of Bessel beams in nonlinear media, the NLSE, was presented in this chapter, analyzing separately the main physical mechanisms relevant to the propagation. The first of them, diffraction, causes the light beams to spatially diverge, as illustrated by the case of the Gaussian beam. Another, the spatial self-phase modulation, leads to beam self-focusing when the power carried by the beam has a value which is higher than the critical power P_{CRIT} . The concept of spatial solitons in which linear diffraction is compensated by self-focusing nonlinearities, or stationary solutions of the nonlinear Schrödinger equation, has been introduced, as well as the problem of the instability of all stationary states with the cubic Kerr nonlinearity, and the necessity of searching stabilizing mechanisms aside the Kerr nonlinearity. We finally introduced MPA, which is, as will see, the stabilizing mechanism for nonlinear Bessel vortex beams.

NONLINEAR BESSEL BEAMS IN SELF-FOCUSING KERR MEDIA RESISTANT TO NONLINEAR ABSORPTION

Me: “And so, at high intensities light does these weird things, like breakup into filaments or into pieces going around, unpredictably. Neat science stuff.”

My son: “Like the Divine Beasts’ laser rays¹!”

On my second year at the Ph.D program.

This chapter reviews known properties of nonlinear BVBs, such as their basic features and dynamics in self-focusing Kerr media with nonlinear absorption, in particular their stationarity, as detailed in (Porrás and Ruiz-Jiménez 2014; Ruiz-Jiménez 2016). Beginning with a discussion of how these beams naturally arise from linear Bessel beams when propagating in nonlinear media, they are derived as stationary solutions of the NLSE with Kerr nonlinearities (saturable or not) and MPA. Relevant properties of the nonlinear BVB are then identified and comprehensively described, such as the power loss due to nonlinear absorption, the energy and OAM fluxes and the reservoir mechanism that sustains stationarity of the nonlinear BVB, whose stability will be thoroughly studied in the next chapter as the central part of this dissertation.

2.1 Existence and basic properties

As discussed in the preceding chapter, we consider the propagation of a light beam $E = A \exp[-i(\omega t - kz)]$ of angular frequency ω , linear propagation constant $k = (\omega/c)n$, n and c being the linear refractive index and the speed of light in vacuum, respectively, and of complex envelope A . As concluded in the preceding chapter, nonlinear propagation in self-focusing Kerr media with nonlinear absorption is modeled at sufficiently high intensities

¹Referring to “Breath of the Wild”, one of our favorite games of The Legend of Zelda series to play on the weekends when we had a little spare time together.

(on the order of TW/cm²) by the NLSE

$$\frac{\partial A}{\partial z} = \frac{i}{2k} \Delta_{\perp} A + i f(|A|^2) A - \frac{\beta^{(M)}}{2} |A|^{2M-2} A, \quad (2.1)$$

where $\Delta_{\perp} = \partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_{\varphi}^2$ is again the transversal Laplace operator, (r, φ) are polar coordinates in the transversal plane, the second term accounts for dispersive nonlinearities and the last term accounts for dissipative power losses associated with MPA of coefficient $\beta^{(M)}$, as explained. Note that dispersive nonlinearities are included in a generic function f to stress that the properties of the nonlinear BVB are not dependent on the specific form of the dispersive nonlinearities, or even of its presence at all (Porras, Ruiz-Jiménez, and Carvalho 2018). Nonetheless, ubiquitous cubic nonlinearities (and possibly quintic if the intensity is sufficiently high) shall be used henceforth with $f(u) \equiv k(n_2 u + n_4 u^2)/n$, being n_2 and n_4 nonlinear refractive indexes. The multi-photon number M is determined by the medium and the wavelength (Couairon and Mysyrowicz 2007), ranging between 3 and 8 for a wavelength ranging between 248-800 nm in air, for example (Polesana et al. 2005; Schwarz et al. 2000).

Recall that at sufficiently low intensities, all nonlinear terms in Eq. (2.1) can be neglected, and linear BVBs, or high-order Bessel beams, come up as the solutions $A(r, \varphi, z) \propto J_s(\sqrt{2k|\delta|r})e^{is\varphi}e^{i\delta z}$, where $\delta = -k\theta^2/2 < 0$ is the shortening of the axial component of the wave vector because of the conical structure, $\theta = \sqrt{2|\delta|/k}$ is the cone angle of the Bessel beam, and $s = \pm 1, \pm 2 \dots$ is the topological charge of the vortex at $r = 0$.

As done in Ch. 1 in the search of stationary solutions and the analysis of the dynamics in general, it is convenient to reduce as much as possible the free parameters in Eq. (2.1) by introducing the dimensionless variables

$$\rho \equiv k\theta r = \sqrt{2k|\delta|r}, \quad \zeta \equiv |\delta|z, \quad \tilde{A} \equiv \left(\frac{\beta^{(M)}}{2|\delta|}\right)^{\frac{1}{2M-2}} A, \quad (2.2)$$

rewriting Eq. (2.1) for a given cone angle into

$$\frac{\partial \tilde{A}}{\partial \zeta} = i \Delta_{\perp} \tilde{A} + i \tilde{f}(|\tilde{A}|^2) \tilde{A} - |\tilde{A}|^{2M-2} \tilde{A}, \quad (2.3)$$

where the transversal Laplace operator is now $\Delta_{\perp} \equiv \partial_{\rho}^2 + (1/\rho)\partial_{\rho} + (1/\rho^2)\partial_{\varphi}^2$. The dispersive nonlinearities will be now represented by $\tilde{f}(u) \equiv \alpha_2 u + \alpha_4 u^2$, with

$$\alpha_2 \equiv \frac{kn_2}{n|\delta|} \left(\frac{2|\delta|}{\beta^{(M)}}\right)^{\frac{1}{M-1}}, \quad \alpha_4 \equiv \frac{kn_4}{n|\delta|} \left(\frac{2|\delta|}{\beta^{(M)}}\right)^{\frac{2}{M-1}}, \quad (2.4)$$

given by the medium characteristics for a fixed value of the cone angle and the light wavelength. Henceforth, the values chosen for these parameters correspond to a wide range of common media. For example, $\alpha_2 \simeq 0.76, 1.31, 3.30, 8.31$, and $\alpha_4 \simeq 0$ are respectively obtained in water at 527 nm, with $n = 1.33$, $n_2 = 2.7 \times 10^{-16}$ cm²/W, $n_4 \simeq 0$, $M = 4$ and $\beta^{(4)} = 2.4 \times 10^{-36}$ cm⁵/W³ (Polesana et al. 2007), and with typical cone angles $\theta = 3^\circ, 2^\circ, 1^\circ, 0.5^\circ$. Similar values are obtained for other media at other wavelengths. In

these dimensionless variables, the linear BVB solution to Eq. (2.3) without nonlinear terms is written as

$$\tilde{A}(\rho, \varphi, 0) = b_0 J_s(\rho) e^{is\varphi} e^{-i\zeta}, \quad (2.5)$$

where b_0 determines its amplitude.

Figure 2.1: Illustrating the conversion of a linear BVB with topological charge $s = 1$ and peak amplitude parameter $b_0 = 3$ into a nonlinear BVB with equal cone angle, topological charge and $b_0 = 1.6$ when introduced into a medium with four-photon absorption ($M = 4$) and negligible Kerr nonlinearity ($\alpha_2 = 0, \alpha_4 = 0$), (a-d) shows radial intensity profiles at different propagation distances ζ . The gray lines represent the initial linear BVB and the dashed lines the final nonlinear BVB. (e) Nonlinear power losses per unit length as a function of distance ζ , as calculated using Eq. (2.6).

In the nonlinear regime, generalizations of linear BVBs (i.e. nonlinear BVBs) that are resistant to nonlinear absorption are known to exist in self-focusing Kerr media, both theoretically and experimentally, for at least a decade in the fundamental, vortexless case (Porrás et al. 2004; Polesana et al. 2007). This fact suggested recently the possibility that propagation-invariant, or stationary high-order nonlinear BVBs (i.e., carrying OAM with a central vortex) could also exist, as demonstrated in (Porrás and Ruiz-Jiménez 2014; Ruiz-Jiménez 2016; Jukna et al. 2014).

The existence of nonlinear BVBs is first supported by the following numerical simulations. Using as initial condition a linear BVB, $\tilde{A}(\rho, \varphi, 0) = b_0 J_s(\rho) e^{is\varphi}$, at the entrance plane of the nonlinear medium $\zeta = 0$, to solve the NLSE (2.3), the input linear BVB transforms gradually into a new stationary nonlinear state that suffers no diffraction or attenuation as it propagates, as illustrated in Fig. 2.1.

The dynamics towards the transformation into a stationary state is similar as in Fig. 2.1 regardless the presence or not of the Kerr nonlinearity. Figures 2.1(a) to 2.1(d) represent various radial intensity profiles at increasing propagation distances ζ , for a medium with negligible Kerr nonlinearity. It is important to note that, in such a medium, a Gaussian beam would be completely absorbed. On the contrary, the input linear BVB previously mentioned passes through an initial phase of strong absorption, only to develop finally into a stationary, nondiffracting, nonattenuating beam with a vortex.

Note that this stationary propagation is attained though the beam is constantly subjected to the nonlinear absorption by the media. Additionally, Fig. 2.1(e) reveals that the power losses tend to a positive constant value, which can be calculated with the following expression:

$$N(\infty) = 2\pi \int_0^\infty d\rho \rho |\tilde{A}|^{2M}. \quad (2.6)$$

Equation (2.6) will be further explained in Sec. 2.2 under the context of the intrinsic reservoir of energy and OAM of nonlinear BVBs, being equivalent to Eq. (2.12) when $\rho \rightarrow \infty$.

2.1.1 Stationary solutions to the nonlinear Schrödinger equation with nonlinear absorption

In order to investigate analytically the properties of these stationary states excited by the linear BVBs, the Ansatz now to be tried is

$$A(r, \varphi, z) = a(r) e^{i\phi(r)} e^{is\varphi} e^{i\delta z}, \quad (2.7)$$

which in the dimensionless variables turns into

$$\tilde{A}(\rho, \phi, \zeta) = \tilde{a}(\rho) e^{i\phi(\rho)} e^{is\varphi} e^{i\text{sgn}(\delta)\zeta}, \quad (2.8)$$

where $\phi(\rho)$ and $\tilde{a}(\rho) > 0$, respectively the radial phase and the amplitude profiles, are unknown, and δ is a constant. Note the important difference with respect to the Ansatz for vortex solitons in transparent media in Eq. (1.79) introduced in Sec. 1.4 that the Ansatz in Eq. (2.8) presents a radial variation of the phase $\phi(\rho)$, which, as will be seen,

is necessary for stationarity with nonlinear losses. Equation (2.8) is thus introduced into the NLSE (2.3), obtaining (see App. A for detailed calculations)

$$\frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} - \left(\frac{d\phi}{d\rho} \right)^2 \tilde{a} + \tilde{f}(\tilde{a}^2) \tilde{a} - \text{sgn}(\delta) \tilde{a} - \frac{s^2}{\rho^2} \tilde{a} = 0, \quad (2.9)$$

$$\frac{d^2 \phi}{d\rho^2} + \frac{1}{\rho} \frac{d\phi}{d\rho} + 2 \frac{d\phi}{d\rho} \frac{d\tilde{a}}{d\rho} \frac{1}{\tilde{a}} + \tilde{a}^{2M-2} = 0. \quad (2.10)$$

Now, two boundary conditions must be imposed to obtain physically valid solutions. First, the solution must be localized, as it is to behave similar to a “spatial soliton” : $\tilde{a}(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. It is worth noting that even though we impose this boundary condition to solve the system, the final solution obtained will be only weakly localized (decaying slowly, algebraically), which will have important physical consequences on the system that will be further explained later on.

The second boundary condition comes due to the vanishing amplitude at the vortex center, where nonlinear effects are negligible, and hence the beam behaves as a linear BVB in this zone. Thus, if b_0 is the amplitude of an arbitrary linear BVB, it will be imposed that $a(\rho) \simeq b_0 J_s(\rho)$ close to $\rho = 0$. Using the relation $J_s(\rho) \simeq \rho^{|s|}/(2^{|s|}|s|!)$ for $\rho \rightarrow 0$ (Olver 2010), this boundary condition can be expressed as

$$\tilde{a}(\rho) \simeq \frac{b_0}{2^{|s|}|s|!} \rho^{|s|} \quad \text{for } \rho \rightarrow 0. \quad (2.11)$$

The system can now be solved for physically meaningful solutions (see App. B.1.1). In the case of no dissipation, the system has either a discrete spectrum of solutions for positive delta, and a continuous spectrum for negative delta, as seen in Ch. 1. If it is positive, the solutions correspond to a discrete set of values of b_0 – a family of vortex solitons characterized by transversal profiles that are strongly localized around the vortex. If it is negative, the solutions correspond to a continuous set of values of b_0 from zero to infinity, i.e., it is a family of nonlinear BBs in transparent media, already mentioned in Sec. 1.4. The phase equation (2.10) is commonly omitted when studying fully transparent media (Hamiltonian systems) since it is satisfied with an irrelevant constant phase that can be set to zero (Desyatnikov, Torner, and Kivshar 2005).

On the other hand, if dissipation is considered, the previous set of discrete solutions to a positive sign of δ disappears. The dissipation effectively prevents vortex solitons to exist in nontransparent media. Nonetheless, the continuous spectrum for a negative sign of δ survives even in presence of dissipation, but only to a maximum value $b_{0,\text{max}}$, not infinity as in the non-dissipative case, as no solution to Eqs. (2.9) and (2.10) with the above two boundary conditions exist for $b_0 > b_{0,\text{max}}$. These new set of solutions are the much sought-after family of nonlinear BVBs in Kerr nonlinear media with nonlinear absorption (Jukna et al. 2014; Porras and Ruiz-Jiménez 2014). As mentioned before, they are weakly localized, meaning that they are not strict solitons: the nonlinear BVBs fulfill the transversal plane entirely, decaying slowly as $\rho^{-1/2}$ in amplitude, and therefore carrying infinite power. This feature can be appreciated in Fig. 2.2. This characteristic allows an inward flux of energy and OAM that is constantly transferred to matter and balances out the major nonlinear losses in the central zone, where most of the nonlinear

Figure 2.2: For $\alpha_2 = \alpha_4 = 0$, and $M = 4$, intensity profiles of nonlinear BVBs with $s = 2$ and (a) $b_0 = 1.7$, (b) $b_0 = 1.82$ and (c) $b_0 = 1.896$. For $\alpha_2 = 0.5$, $\alpha_4 = -0.25$ and $M = 4$, intensity profiles of nonlinear BVBs with $s = 2$ and (d) $b_0 = 1.8$, (e) $b_0 = 2.15$ and (f) $b_0 = 2.237$. The dotted curves represent linear BVBs with $s = 2$ and the same b_0 as the nonlinear BVBs.

losses take place. This *intrinsic reservoir mechanism* supports, as shall be seen in this chapter, stationarity, and in Ch. 3, stability.

For now, it is sufficient to compare the solutions to Eqs. (2.9) and (2.10) with the above boundary conditions to a standard linear BVB with the same value of b_0 in each case, equivalent as having the same vortex core in both beams. Figure 2.2 shows intensity profiles, $|\tilde{a}|^2$, as b_0 grows from low values to $b_{0,\max}$, hence with increasing nonlinear absorption and dispersive Kerr nonlinearities: at low values of b_0 , the nonlinear BVB has the behavior of a linear BVB at any radius ρ . As b_0 intensifies up to $b_{0,\max}$, the inner rings are narrower in self-focusing media (respectively, broader in self-defocusing media, a situation not considered here). Consequently, outer rings are likewise radially shifted by the change in the inner rings width, due to stronger nonlinear effects. An additional effect of approaching $b_{0,\max}$ that can be observed on the rings is how they gradually lose their contrast (or depth of modulation). Radial frequency remains the same in every case, and the outer rings always decay in amplitude proportional to $\rho^{-1/2}$, as $\rho \rightarrow \infty$.

Note that no assumption was made while solving the system of equations (2.9) and

(2.10) about the explicit form of the dispersive nonlinearity present, hence the existence of these new beams are independent either from its form or order. Although the specific radial profile and the value of $b_{0,\max}$ will depend on the nonlinear medium chosen (given by the parameters s , α_2 , α_4 and M), there will be always a range of values for the existence of these stationary solutions.

2.2 Stationary propagation: the reservoir mechanism and the transference of energy and orbital angular momentum in non-transparent media

The essential process sustaining stationary propagation of nonlinear BVBs in Kerr media with MPA is the reservoir of energy and OAM intrinsic to these weakly localized solutions. Ideally, it allows the transference of those quantities to matter, effectively balancing out the nonlinear losses that dominate in the central region of these objects.

To better explore how this occurs, Eq. (2.10) can be integrated in the radial variable, giving

$$-F(\rho) \equiv -2\pi\rho \frac{d\phi}{d\rho} \tilde{a}^2 = 2\pi \int_0^\rho d\rho \rho \tilde{a}^{2M} \equiv N(\rho), \quad (2.12)$$

The left side of the equation represents an inward (thus the minus sign) power flux crossing a circumference of radius ρ , coming from large radial distances. This must be equal to the right side of the equation, which represents the nonlinear power losses in a circle with the same radius ρ . Therefore, the energy flux replenishes nonlinear losses occurring at the center of the nonlinear BVB that would cause its destruction. This is possible only in beams transporting infinite power, as originally proposed in Ref. (Porrás et al. 2004), and more comprehensively detailed for the case of nonlinear BVBs in (Porrás and Ruiz-Jiménez 2014).

The following continuity equation can be obtained from the NLSE (2.3) using an arbitrary (maybe not stationary) complex envelope of a light beam $\tilde{A} = \tilde{a}e^{i\Phi}$,

$$\frac{1}{2}\partial_\zeta \tilde{a}^2 + \nabla_\perp \cdot \mathbf{j} = -\tilde{a}^{2M} \quad (2.13)$$

where $\tilde{a}^2$ is the intensity, $\nabla_\perp$ is the transversal nabla operator and the current of intensity is $\mathbf{j} = \tilde{a}^2 \nabla_\perp \Phi$. The intensity pattern will be stationary if the condition

$$-\nabla_\perp \cdot \mathbf{j} = \tilde{a}^{2M} \quad (2.14)$$

is met. This condition is a generalization of previous Eq. (2.10) to beams that do not have circular symmetry. By reason of the divergence theorem (see App. B.1.2) it expresses that the nonlinear power losses in a delimited region of the transversal plane must be equal to the inward power flux that crosses its boundary.

Figure 2.3 depicts a comparative among different stationary beams and a nonlinear BVB. Beam types are displayed in rows, while different attributes are shown in columns:

Figure 2.3: Intensity profiles (first column), phase profiles (second column) and intensity current (third column) of the vortexless ($s = 0$) nonlinear Bessel beam with $M = 4$, $\alpha_2 = \alpha_4 = 0$ and $b_0 = 1.174$ (first row); of the linear BVB with $s = 1$, and $b_0 = 1.666$ (second row); and of the nonlinear BVB with $s = 1$, $M = 4$, $\alpha_2 = \alpha_4 = 0$ and $b_0 = 1.666$ (third row). The normalized Cartesian coordinates in the transversal plane are $\xi = \sqrt{2k|\delta|x}$ and $\eta = \sqrt{2k|\delta|y}$.

intensity profile at the first, phase profile at the second, and intensity current at the third. Fundamental, vortexless nonlinear BVBs (first row) have an intensity current $\mathbf{j} = (\tilde{a}^2 d\phi/d\rho) \mathbf{u}_\rho$ that flows radially inwards with a divergence equal to the nonlinear losses density $\tilde{a}^{2M}$; while linear BVBs and vortex solitons (second row) are on the other extreme, having an azimuthal and solenoidal intensity current $\mathbf{j} = (\tilde{a}^2 s/\rho) \mathbf{u}_\varphi$. The main new feature of the present nonlinear BVBs (last row) is its intensity current with azimuthal and radial inward components, creating a permanent inward spiral current of power transported towards the central zone where nonlinear losses predominate.

OAM has an analogue behavior to the intensity, i.e., presents a stationary distribution in the transversal plane. The axial OAM of a nonlinear BVB is proportional to the topological charge s and to the intensity $\tilde{a}^2$, as its density is given by $L = \tilde{a}^2 \partial\Phi/\partial\varphi = s\tilde{a}^2$

(Desyatnikov, Torner, and Kivshar 2005). It flows permanently inwards with a current of OAM

$$s\mathbf{j} = s\tilde{a}^2 \left(\frac{d\phi}{d\rho} \mathbf{u}_\rho + \frac{s}{\rho} \mathbf{u}_\varphi \right), \quad (2.15)$$

balancing the OAM losses caused by the nonlinear MPA, equal to $s\tilde{a}^{2M}$. The continuity equation in this case is $-\nabla_\perp \cdot (s\mathbf{j}) = s\tilde{a}^{2M}$, whose integration over a circle of radius ρ gives out the following analogue to Eq. (2.12)

$$-2\pi\rho s \frac{d\phi}{d\rho} \tilde{a}^2 = 2\pi \int_0^\rho d\rho \rho s \tilde{a}^{2M}, \quad (2.16)$$

that, as before, states the equality between the OAM losses due to its transmission to matter over a circle of radius ρ and the inward spiral flux through the boundary of that same area.

2.3 Asymptotic behavior of nonlinear Bessel vortex beams

Although nonlinear effects are stronger in the central, more intense zone of nonlinear BVBS, its linear asymptotic behavior in its weak oscillating rings also is crucial for understanding the dynamics of nonlinear BVB formation in nonlinear media. This relevance was first pointed out in (Porrás and Ruiz-Jiménez 2014), and further explored in (Ruiz-Jiménez 2016), introducing the laws that govern the formation of nonlinear BVB of null topological charge. Still, as will be presented in Ch. 4, these criteria can be expanded and adapted to real-world experimental setups where a linear BVB is introduced in a media with nonlinear absorption and spontaneously evolve into a nonlinear BVB, having its propagation dynamics correctly predicted.

First, note that the linear Bessel beam $A = b_0 J_s(\rho) e^{is\varphi} e^{-i\zeta}$ can equivalently expressed as (Olver 2010)

$$\tilde{A} = \frac{1}{2} [b_0 H_s^{(1)}(\rho) + b_0 H_s^{(2)}(\rho)] e^{is\varphi} e^{-i\zeta}. \quad (2.17)$$

where $H_s^{(1)}(\rho)$ and $H_s^{(2)}(\rho)$ are, respectively, the Hänkel functions of the first and second kind, making the linear BVB to be now described as a superposition of two Hänkel beams with the same amplitude. The Hänkel beam $H_s^{(1)}(\rho) e^{is\varphi} e^{-i\zeta}$ transports power spirally outwards and the Hänkel beam $H_s^{(2)}(\rho) e^{is\varphi} e^{-i\zeta}$ does its share spirally inwards, so that the net transport of power in the cross-section is zero (Salo et al. 2000). This balance is lost in media with nonlinear absorption, as the outward and inward Hänkel beams do not have the same amplitude (Porrás et al. 2004), causing the beam to behave as an unbalanced Bessel beam asymptotically at large radii. The complex amplitude is

$$\tilde{A}(\rho, \phi, \zeta) \simeq \frac{1}{2} [b_{\text{out}} H_s^{(1)}(\rho) + b_{\text{in}} H_s^{(2)}(\rho)] e^{is\varphi} e^{-i\zeta}. \quad (2.18)$$

where the sign $\simeq$ means asymptotically at large radii, and b_{in} and b_{out} are generally complex. The inward power flux across a circle or large radius can be evaluated from the

complex amplitude as

$$-F(\rho) = -2\pi\rho\tilde{a}^2 \frac{d\phi}{d\rho} = 2\pi\rho \operatorname{Im} \left[\tilde{A} \left(\frac{\partial \tilde{A}^*}{\partial \rho} \right) \right]. \quad (2.19)$$

From Eq. (2.18) and taking into account the asymptotic expressions at large ρ of Hankel functions (Olver 2010)

$$H_s^{(1,2)}(\rho) \simeq \sqrt{\frac{2}{\pi\rho}} e^{\pm i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})}, \quad (2.20)$$

we obtain (see App. B.1.3) the constant value $-F(\rho) = |b_{\text{in}}|^2 - |b_{\text{out}}|^2$. According to Eq. (2.12), this value must be equal to the total power losses $N(\infty)$ in the cross-section, i.e.,

$$|b_{\text{in}}|^2 - |b_{\text{out}}|^2 = N(\infty) > 0. \quad (2.21)$$

Hence, all nonlinear BVB in a medium with dissipation satisfy $|b_{\text{in}}| > |b_{\text{out}}|$. Equation (2.21) also expresses that the unbalance between the two Hankel components creates the power flux coming from infinity to restore the total power losses $N(\infty)$ occurring mainly in the more intense central rings.

Figure 2.4: (a) For the nonlinear BVB with parameters $M = 4$ (four-photon absorption), $\alpha_2 = \alpha_4 = 0$ (no Kerr nonlinearity), $s = 3$ and $b_0 = 1.9$, its amplitude profile (solid curve), the amplitude profile of its associated asymptotic linear unbalanced Bessel beam (UBB) (dashed curve), and its Hankel inward and outward components (dashed grey curves). (b-c) Values of $|b_{\text{out}}|$ and $|b_{\text{in}}|$ of nonlinear BVBs with $M = 4$, the indicated values of the Kerr nonlinearities, and $s = 0, 1$ and 2 , as functions of their parameter b_0 up to the limit of existence $b_{0,\text{max}}$ in each case (vertical dashed lines).

The calculation of $|b_{\text{in}}|$ and $|b_{\text{out}}|$ for a given nonlinear BVB is of utmost importance, as will be seen in the next section. Using Eqs. (2.18) and (2.20), the radial intensity profile at large radius is of the form

$$2\pi\tilde{a}^2\rho \simeq |b_{\text{out}}|^2 + |b_{\text{in}}|^2 + 2|b_{\text{out}}||b_{\text{in}}|\cos(2\rho + \sigma), \quad (2.22)$$

with $\sigma = -\pi s - \pi/2 + \arg(b_{\text{out}}/b_{\text{in}})$. The right side of Eq. (2.22) represents harmonic oscillations of contrast $C = 2|b_{\text{in}}||b_{\text{out}}|/(|b_{\text{in}}|^2 + |b_{\text{out}}|^2)$ about the central value $R = |b_{\text{in}}|^2 + |b_{\text{out}}|^2$. Since both magnitudes, C and R , can be easily extracted from the intensity radial

profiles of nonlinear BVBs obtained by solving numerically Eqs. (2.9) and (2.10) with a given value of b_0 . Thus, the following expressions give the values of $|b_{\text{out}}|$ and $|b_{\text{in}}|$,

$$|b_{\text{in}}|^2 = \frac{R}{2} \left(1 + \sqrt{1 - C^2}\right), \quad |b_{\text{out}}|^2 = \frac{R}{2} \left(1 - \sqrt{1 - C^2}\right). \quad (2.23)$$

Figure 2.4 illustrates the previous conclusions. Figure 2.4(a) shows the amplitude profiles of a nonlinear BVB, its unbalanced Bessel beam (UBB) fitting the nonlinear BVB at large radius, and its two Hankel components. The interference of the two Hankel components with different amplitudes gives the characteristic oscillations with contrast $C < 1$. Figures 2.4(b-c) show the inward and outward Hankel beam amplitudes as functions of the parameter b_0 that specifies the nonlinear BVB for, respectively, a media with negligible Kerr nonlinearity and another with non-negligible Kerr nonlinearity. It can be shown that b_{in} and b_{out} are real quantities in absence of Kerr nonlinearities and, although the phase of $|b_{\text{in}}|$ and $|b_{\text{out}}|$ is not relevant to our considerations, it can be seen that they satisfy the relation $\arg b_{\text{in}} = -\arg b_{\text{out}}$ with Kerr nonlinearities.

2.3.1 Nonlinear Bessel beams attractors of the nonlinear dynamics

As shown in 2.1, a linear BVB introduced in a nonlinear medium gradually transforms into a nonlinear BVB with the same values of the topological charge and the cone angle. The beam evolves towards an attracting nonlinear BVB, but the parameter b_0 determining its amplitude or intensity was unknown at the time that this “selection problem” was posed for the first time (Polesana et al. 2006). The specific problem refers to determine the parameter b_0 at large propagation distances, here signified by $b_0(\infty)$, which specifies the attracting nonlinear BVB for a given parameter $b_0(0)$, where the value 0 stands for the initial or input configuration of the launched linear BVB. An example can be seen in Fig. 2.1, where a linear BVB with $b_0(0) = 3$ is attracted by the nonlinear BVB with $b_0(\infty) = 1.6$. In fact, this problem existed not only for Bessel beams, but also for Airy beams (Lotti et al. 2011; Polesana et al. 2008; Couairon et al. 2013), and it was solved some years later in (Porras and Ruiz-Jimenez 2014; Ruiz-Jimenez, Nobrega, and Porras 2015; Porras, Ruiz-Jimenez, and Losada 2015; Ruiz-Jimenez 2016).

The solution to the selection problem came with the finding that *the amplitude of the asymptotic inward Hankel component $|b_{\text{in}}|$ is a conserved quantity during the nonlinear wave propagation*, i.e., the inward Hankel component transporting power from the outer rings towards the inner rings where nonlinear absorption takes place, is not altered in the transformation from the linear to the nonlinear BVB. This conclusion is illustrated in Figs. 2.5(c-d), where the values of $|b_0(\infty)|$, $|b_{\text{in}}(\infty)|$ and $b_{\text{out}}(\infty)$ of the spontaneously formed nonlinear BVBs, extracted numerically as explained in the preceding section, are plotted against $b_0(0)$ of the input linear BVBs. It is seen that $|b_{\text{in}}(\infty)| = b_0(0)$. Since $|b_{\text{in}}(0)| = b_{\text{out}}(0) = b_0(0)$ for the input linear BVB entering the nonlinear medium, the amplitude of the asymptotic inward Hankel component is conserved, $|b_{\text{in}}(\infty)| = |b_{\text{in}}(0)|$. Figures 2.5(a) and 2.5(b) show the parameter $b_0(\infty)$ that determines the intensity of the nonlinear attractor, which increases with growing input linear intensities until it matches its maximum value $b_{0,\text{max}}$ supported by the medium in each case ($s = 0$ and $s = 1$). On

the contrary, at low values of input linear intensities $b_0(0)$, $b_0(\infty)$ approaches $b_0(0)$ since nonlinear absorption becomes negligible.

In brief, given M , α_2 and α_4 , the value of $b_0(\infty)$ of the attractor nonlinear BVB can be extracted from Figs. like 2.4(b-c) as those satisfying $|b_{in}(\infty)| = b_0(0)$.

Although the properties presented here are drawn for the ideal case of linear and nonlinear BVBs, in Ch. 4 they will be applied for real-world axicon-generated nonlinear BVBs transporting finite power, and successfully used to predict their dynamics.

Figure 2.5: (a-b) Relation between the amplitude of the input linear BVB, $b_0(0)$, and the amplitude of the attracting nonlinear BVB, $b_0(\infty)$. For input beams with $s = 0$ and $s = 1$, both in a medium and cone angle such that $\alpha_2 = 0.5$ and $\alpha_4 = -0.25$ and $M = 4$. The parameter $b_0(\infty)$ is a growing monotonic function of $b_0(0)$ until it reaches the maximum value supported by the medium. Each point of the curve is obtained from a full numerical simulation of Eq. (2.3) with input linear BVBs with different values of $b_0(0)$. (c-d) For the same media and charges, $b_0(\infty)$, $|b_{in}(\infty)|$ and $|b_{out}(\infty)|$ of the attracting nonlinear BVB versus $b_0(0)$ of the input linear BVB. To be highlighted, the conservation law $|b_{in}(\infty)| = b_0(0)$ holds true in both cases.

2.4 Summary

In this chapter, the main properties and structure of nonlinear BVBs have been studied, as well as its reservoir mechanism, essential to attain its stationary propagation in Kerr media with nonlinear absorption. Contrary to what happens to, for instance, plane or Gaussian waves, nonlinear BVBs are not absorbed in such media, but instead reach a nondiffracting, nonattenuating stationary regime fueled by an inward spiral current of energy and OAM that refills the nonlinear losses occurring mainly in its inner rings. These new stationary solutions are exclusively supported by nonlinear absorption, occurring in all media at sufficiently high intensities, unlike other types of vortex solitons that need specific media with tailored nonlinearities, or a gain term balancing out the losses, to propagate without change.

As many of the dynamics characteristics of nonlinear BVBs are determined by their asymptotic behavior and their underlying nonlinear attractor, these subjects have also been reviewed here. The problem was reformulated using Hankel functions, and analyzed in terms of the balance of energy entering the system through an inwards Hankel component and the energy lost through the sum of the outward Hankel component and nonlinear effects. Lastly, the procedure to identify the aforementioned dynamics attractor was presented as the solution to the selection problem first reported in ([Polesana et al. 2006](#)), as it will be needed in later chapters to understand different cases to be put under scrutiny.

Chapter 3

ON THE STABILITY PROPERTIES OF NONLINEAR BESSEL VORTEX BEAMS

My mom, or my dad: “But then, will it do any good for you to get a better job after you finish it?”

On my third year at the Ph.D program (and possibly also on the others too).

Once established the existence of propagation-invariant nonlinear BVBs supported by MPA, it is due to investigate their stability properties. As described in Ch. 1 that vortex beams, and nonlinear BVBs in particular, are prone to azimuthal breakup and collapse in Kerr media, triggered by any small perturbation in the amplitude or phase. Determining the existence of stable vortex beams, and the specific conditions leading to the stable propagation, is a subject of deep theoretical and experimental importance since stability means observability in experiments and usefulness for applications, while instability means an interesting dynamics but non-observability in practice and self-destruction of the unstable vortex beam. In fact, from a more general point of view, stability (or self-trapping) of multidimensional spatial solitons propagating in nonlinear media is a central issue in nonlinear optics, with several implications in other areas of investigation (Kivshar and Pelinovsky 2000; Malomed et al. 2005; Desyatnikov, Torner, and Kivshar 2005; Mihalache 2012; Kivshar and Agrawal 2003; Kartashov, Malomed, and Torner 2011).

Some early examples of solitons were already presented in Ch. 1 (e.g. the Townes soliton and its vorticity-carrying counterparts), mainly supported by dispersive nonlinearities such as the ubiquitous cubic self-focusing Kerr nonlinearity. Although stability is easily achieved with one transversal dimension, multidimensional solitons suffer from the critical collapse caused by the cubic self-focusing cubic nonlinearity, which lead, in absence of other collapse-arresting nonlinearities, to the formation of singularities at a finite propagation distance. This tendency worsens if the soliton has intrinsic vorticity, i.e. if it is a vortex beam or vortex ring, as azimuthal perturbations will easily make it split into various (possibly rotating) pieces.

To attain stability for these beams, several models were proposed, mainly Hamiltonian ones. One possibility is to use specific nonlinear terms as self-focusing (positive) cubic

and self-defocusing (negative) quintic nonlinearities, as for instance in (Quiroga-Teixeiro and Michinel 1997) and (Paz-Alonso and Michinel 2005). This configuration relies on the joint effect of diffraction and self-defocusing to balance the collapsing tendency of the self-focusing term, creating stable light beams. Also, complex Ginzburg-Landau equations with cubic-quintic nonlinearities are used to model dissipative solitons and spiral vortices (Malomed, Crasovan, and Mihalache 2002; Aranson and Kramer 2002). On the downside of these models, it is necessary to find materials that have this type of nonlinearities (cubic-quintic, as the mentioned example) as an intrinsic property. Although there are some of them already identified (liquid CS₂, chalcogenide glasses, metal-dielectric nanocomposites), their usefulness cannot be guaranteed at high intensities where nonlinear absorption, not included in the model, begins to play an important role in the phenomenon. Things can be elaborated even further, as cubic-quintic-septimal media have already been proposed (Reyna, Malomed, and de Araújo 2015).

Other possibility to achieve stable propagation, now with a nonconservative model, is the so-called dissipative solitons model (Grelu and Akhmediev 2012; Akhmediev and Ankiewicz 2005; Melo and Douady 1993), which includes terms of gain and losses that balance each other out. These are used to model physical systems like dissipative light bullets and to more technological applications such as semiconductor optical amplifiers, passively mode-locked lasers and hydrodynamic systems. Nonlocal nonlinearities (Kruglov, Logvin, and Volkov 1992; Yakimenko, Zaliznyak, and Kivshar 2005) or spin-orbit coupling in binary systems (Sakaguchi, Li, and Malomed 2014; Zhang et al. 2015) are other models proposed to achieve stable propagation of multidimensional vortex beams. In this case, lead-doped glass is used for the experimental implementation of thermal nonlocal nonlinearities (Rotschild et al. 2005).

Nevertheless, all previous models are very laborious to be made effective in practice, due to the efforts in researching and creating the somewhat exotic media needed for such precisely tailored nonlinearities. The model exposed in this work differs from previous models because it is a nonconservative model, and additionally that it does not need any gain term to balance the losses. As seen in the preceding chapter, stationary nonlinear BVBs exist in self-focusing media at intensities at which nonlinear absorption is significant. The numerical simulations in Sec. 2.2 showing their spontaneous formation from linear BVBs in self-focusing Kerr media with nonlinear absorption, i. e., their attracting nature, suggest that there should exist stable nonlinear BVBs in the presence of self-focusing, and that the stabilizing mechanism is nonlinear absorption. Further, MPA of different orders naturally occurs in ordinary dielectric media like fused silica, water or air, providing a technically accessible means of energy and OAM pumping from the nonlinear BVB to matter (Tabosa and Petrov 1999; Masalov 1997). This pumping accounts for the strong energy depletion in the central zone of the beam, where nonlinear effects dominate, effectively using this natural collapse-arresting mechanism to support both stationarity and stability. Bear in mind that the nonlinear Bessel beams treated here are not strict solitons, as they are weakly localized conical waves, meaning that they can carry (ideally) infinite amounts of power as they propagate. This is the previously described mechanism in Sec. 2.2, in which an inward radial flux of energy and OAM is transported from the outer, asymptotically linear, zones of the beam (a power reservoir) to the central, nonlinear dominated zones, replenishing the depleted power and OAM. In addition to

this mechanism of stationarity, the stabilizing property of nonlinear absorption has been reported previously, for example against radial and temporal perturbations of beams with null vorticity (Polesana et al. 2007; Porras, Ruiz-Jiménez, and Losada 2015) or light bullets (Porras and Redondo 2013), but not against azimuthal perturbations that strongly affect nonlinear vortex beams.

This study shows, for the first time, that complete stability can be achieved for nonlinear BVBs of not only single vorticity, but also multiple vorticities ($|s| > 1$) in Kerr media with nonlinear MPA. In fact, its conclusions were reported in (Porras et al. 2016; Carvalho, Ruiz-Jiménez, and Porras 2017). The self-focusing Kerr nonlinearity is used because it is the ubiquitous nonlinearity in optical media, and the nonlinearity that causes the instability, but the results can be easily generalized to media with other nonlinearities of higher order. As shall be demonstrated in the next sections, the existence of a region of stability of nonlinear BVBs defined by a subset of the parameters that define them is demonstrated through a linear stability analysis, and tested afterwards by direct numerical simulations. The analysis performed in this chapter regards ideal nonlinear BVBs carrying infinite power. As it will be seen in Ch. 4, similar conclusions hold with real-world nonlinear BVBs carrying finite power.

3.1 Linearized stability analysis

As in Sec. 1.5, a linear stability analysis of nonlinear BVBs shall be carried in this and next sections, in order to study how nonlinear BVBs of different vorticities behave against small perturbations during propagation. The strategy followed will be as in Sec. 1.5: starting with the NLSE that governs the propagation, a nonlinear BVB defined by a set of parameters with a comparatively small perturbation will be introduced, and the linearized equations for the perturbation shall lead to an eigenvalue problem. From the spectrum of eigenvalues the stability (or instability) can be predicted for the parameters of the nonlinear BVB. Yet, this case has an important difference from the simple illustrative case discussed in Sec. 1.5, since the eigenvalue problem obtained will have to be solved numerically through a process of discretization.

To start, the NLSE (2.3) shall be conveniently rewritten as

$$\frac{\partial \tilde{A}}{\partial \zeta} = i\Delta_{\perp}\tilde{A} + iF(|\tilde{A}|^2)\tilde{A}, \quad (3.1)$$

with $F(I) = \alpha I + iI^{M-1}$, a compact form more useful for the next calculations. Only positive ($\alpha > 0$) cubic nonlinearity is considered, since it is this nonlinearity that usually brings instability, and nonlinear absorption as a possible stabilizing mechanism. Also, $\Delta_{\perp} = \partial^2/\partial\rho^2 + (1/\rho)(\partial/\partial\rho) + (1/\rho^2)(\partial^2/\partial\varphi^2) \equiv \Delta_{\rho} + (1/\rho^2)(\partial^2/\partial\varphi^2)$. As unperturbed solution to Eq. (3.1) we consider a nonlinear BVB, defined by the medium properties α and M , and specified by its parameters b_0 and s :

$$\tilde{A}_S = \tilde{a}(\rho)e^{i\phi(\rho)}e^{-i\zeta}e^{is\varphi}, \quad (3.2)$$

with $s = 0, \pm 1, \pm 2, \dots$. To this nonlinear BVB a small ($\epsilon \rightarrow 0$) perturbation $P(\rho, \varphi, \zeta)$ is added, giving the following Ansatz to be introduced in Eq. (3.1):

$$\tilde{A} = \tilde{A}_S + \epsilon P(\rho, \varphi, \zeta). \quad (3.3)$$

Therefore,

$$\frac{\partial \tilde{A}}{\partial \zeta} = \frac{\partial \tilde{A}_S}{\partial \zeta} + \epsilon \frac{\partial P}{\partial \zeta} = i\Delta_{\perp} \tilde{A}_S + i\epsilon \Delta_{\perp} P + iF(|\tilde{A}_S + \epsilon P|^2)(\tilde{A}_S + \epsilon P) .$$

Keeping terms only until first order in ϵ , since it is a linearized analysis, it is possible to simplify the last term of the previous equation:

$$\begin{aligned} |\tilde{A}_S + \epsilon P|^2 &= |\tilde{A}_S|^2 + \epsilon P^* \tilde{A}_S + \epsilon P \tilde{A}_S^* + \cancel{\mathcal{O}(\epsilon^2)}^{\sim 0} , \\ F(|\tilde{A}_S + \epsilon P|^2) &\approx F(|\tilde{A}_S|^2 + \epsilon P^* \tilde{A}_S + \epsilon P \tilde{A}_S^*) \\ &= F(|\tilde{A}_S|^2) + \frac{dF}{d|\tilde{A}_S|^2} \epsilon (P^* \tilde{A}_S + P \tilde{A}_S^*) \\ &\equiv F_S + \epsilon F'_S (P^* \tilde{A}_S + P \tilde{A}_S^*) , \end{aligned} \quad (3.4)$$

where

$$F_S \equiv \alpha |\tilde{A}_S|^2 + i |\tilde{A}_S|^{2(M-1)} = \alpha \tilde{a}^2 + i \tilde{a}^{2(M-1)} , \quad (3.5)$$

$$F'_S \equiv \alpha + i(M-1) |\tilde{A}_S|^{2(M-2)} = \alpha + i(M-1) \tilde{a}^{2(M-2)} . \quad (3.6)$$

Substituting and expanding terms:

$$\begin{aligned} \frac{\partial \tilde{A}_S}{\partial \zeta} + \epsilon \frac{\partial P}{\partial \zeta} &= i\Delta_{\perp} \tilde{A}_S + i\epsilon \Delta_{\perp} P + i \left[F_S + \epsilon F'_S (P^* \tilde{A}_S + P \tilde{A}_S^*) \right] (\tilde{A}_S + \epsilon P) \\ &= \underline{i\Delta_{\perp} \tilde{A}_S} + i\epsilon \Delta_{\perp} P + \underline{iF_S \tilde{A}_S} + i\epsilon F_S P + i\epsilon F'_S (P^* \tilde{A}_S + P \tilde{A}_S^*) \tilde{A}_S + \cancel{\mathcal{O}(\epsilon^2)}^{\sim 0} . \end{aligned}$$

Again, terms second-order in ϵ are neglectable. Also, the underlined terms cancel out because the nonlinear BVB $\tilde{A}_S$ satisfies Eq. (3.1), giving the equation for the evolution of the perturbation P

$$\begin{aligned} \frac{\partial P}{\partial \zeta} &= i\Delta_{\perp} P + iF_S P + iF'_S (P^* \tilde{A}_S + P \tilde{A}_S^*) \tilde{A}_S \\ &= i\Delta_{\perp} P + iF_S P + iF'_S P^* \tilde{A}_S^2 + iF'_S P |\tilde{A}_S|^2 \\ &= i\Delta_{\perp} P + i(F_S + F'_S |\tilde{A}_S|^2) P + iF'_S \tilde{A}_S^2 P^* , \end{aligned} \quad (3.7)$$

with $\tilde{A}_S^2 = \tilde{a}^2 e^{2i\phi} e^{-2i\zeta} e^{2is\varphi}$ and $|\tilde{A}_S|^2 = \tilde{a}^2$. Any perturbation $P = P(\rho, \varphi, \zeta)$ can alternatively be expressed using the following expansion in cylindrical harmonics (see App. B.2.1):

$$P(\rho, \varphi, \zeta) = \sum_{m=-\infty}^{+\infty} P_m(\rho, \zeta) e^{im\varphi} . \quad (3.8)$$

The introduction of the parameter m at this point deserves further attention. This so-called *winding number* may assume the values $m = 0, \pm 1, \pm 2, \pm 3, \dots$ and labels the different azimuthal modes of perturbation to the nonlinear BVB, which may be stable or

unstable according to the results of the stability analysis. Calculating the derivative of Eq. (3.8) with respect to ζ , and using Eq. (3.7):

$$\begin{aligned} \frac{\partial P}{\partial \zeta} &= \sum_{m=-\infty}^{+\infty} \frac{\partial P_m}{\partial \zeta} e^{im\varphi} \\ &= i \sum_{m=-\infty}^{+\infty} \left(\Delta_\rho P_m - \frac{m^2}{\rho^2} P_m \right) e^{im\varphi} + i(F_S + F'_S \tilde{a}^2) \sum_{m=-\infty}^{+\infty} P_m e^{im\varphi} \\ &\quad + i(F'_S \tilde{a}^2 e^{2i\phi} e^{-2i\zeta} e^{2is\varphi}) \sum_{m=-\infty}^{+\infty} P_m^* e^{-im\varphi} . \end{aligned} \quad (3.9)$$

Expanding the sums give out a system of infinite equations, of which only those in P_m and P_{2s-m} are coupled and of interest. For easier manipulation of the indexes, m will be redefined to $s+m$, and therefore $2s-m = 2s-(s+m) = s-m$. Hence, the final system of coupled equations to be studied is

$$\begin{aligned} \frac{\partial P_{s+m}}{\partial \zeta} &= i \left(\Delta_\rho P_{s+m} - \frac{(s+m)^2}{\rho^2} P_{s+m} \right) + i(F_S + F'_S \tilde{a}^2) P_{s+m} + i(F'_S \tilde{a}^2 e^{2i\phi} e^{-2i\zeta}) P_{s-m}^* , \\ \frac{\partial P_{s-m}}{\partial \zeta} &= i \left(\Delta_\rho P_{s-m} - \frac{(s-m)^2}{\rho^2} P_{s-m} \right) + i(F_S + F'_S \tilde{a}^2) P_{s-m} + i(F'_S \tilde{a}^2 e^{2i\phi} e^{-2i\zeta}) P_{s+m}^* . \end{aligned} \quad (3.10)$$

It is noteworthy the symmetry of these equations, which makes it necessary to analyze only zero and positive values of m , being those with the opposite sign completely analogous.

To analyze the stability or instability of the azimuthal modes, we choose them in order to have the most generic form possible of the perturbation P (Olver 2010), and thus

$$P_{s+m} = \mu_m(\rho) e^{i\beta_+ \zeta} , \quad P_{s-m} = \nu_m^*(\rho) e^{i\beta_- \zeta} , \quad (3.11)$$

with $\beta_\pm$ are arbitrary complex numbers whose imaginary part account for exponential growth or reduction. Substituting them in Eq. (3.10) and dividing the first equation by $ie^{i\beta_+ \zeta}$ and the second equation by $ie^{i\beta_- \zeta}$, it becomes:

$$\begin{cases} \beta_+ \mu_m = \Delta_\rho \mu_m - \frac{(S+m)^2}{\rho^2} \mu_m + (F_S + F'_S \tilde{a}^2) \mu_m + F'_S \tilde{a}^2 e^{2i\phi} e^{-2i\zeta} \nu_m e^{-i\beta_- \zeta} e^{-i\beta_+ \zeta} \\ \beta_- \nu_m^* = \Delta_\rho \nu_m^* - \frac{(S-m)^2}{\rho^2} \nu_m^* + (F_S + F'_S \tilde{a}^2) \nu_m^* + F'_S \tilde{a}^2 e^{2i\phi} e^{-2i\zeta} \mu_m^* e^{-i\beta_+ \zeta} e^{-i\beta_- \zeta} \end{cases} . \quad (3.12)$$

Since μ and ν do not depend on ζ , the relation

$$-2 - \beta_-^* - \beta_+ = 0 \quad (3.13)$$

is valid, and satisfied if

$$\begin{cases} \beta_+ = -1 + \kappa_m \\ \beta_- = -1 - \kappa_m^* \end{cases} . \quad (3.14)$$

Substituting in Eqs. (3.11), the azimuthal modes are

$$P_{S+m} = \mu_m(\rho) e^{-i\zeta + i\kappa_m \zeta} , \quad P_{S-m} = \nu_m^*(\rho) e^{-i\zeta - i\kappa_m^* \zeta} , \quad (3.15)$$

whose radial structure satisfy

$$\begin{cases} \kappa_m \mu_m = \Delta_\rho \mu_m - \frac{(S+m)^2}{\rho^2} \mu_m + \mu_m + (F_S + F'_S \tilde{a}^2) \mu_m + F'_S \tilde{a}^2 e^{2i\phi} \nu_m \\ \kappa_m \nu_m = -\Delta_\rho \nu_m + \frac{(S-m)^2}{\rho^2} \nu_m - \nu_m - (F_S^* + F'^*_S \tilde{a}^2) \nu_m - F'^*_S \tilde{a}^2 e^{2i\phi} \mu_m \end{cases} . \quad (3.16)$$

Again, as in Sec. 1.5, an eigenvalue problem arises to be solved via the linear stability analysis. The problem can be better visualized in the following, more compact form:

$$\kappa_m \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} = \begin{pmatrix} H_+ & f \\ -f^* & -H_-^* \end{pmatrix} \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} , \quad (3.17)$$

$$\text{with } \begin{cases} H_\pm = \Delta_\rho - \frac{(S \pm m)^2}{\rho^2} + 1 + (F_S + F'_S \tilde{a}^2) \\ f = F'_S \tilde{a}^2 e^{2i\phi} \end{cases} , \quad (3.18)$$

or, substituting

$$F_S + F'_S \tilde{a}^2 = 2\alpha \tilde{a}^2 + iM \tilde{a}^{2(M-1)} , \quad (3.19)$$

$$F'_S \tilde{a}^2 e^{2i\phi} = [\alpha \tilde{a}^2 + i(M-1) \tilde{a}^{2(M-1)}] e^{2i\phi} , \quad (3.20)$$

we finally obtain

$$\kappa_m \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} = \begin{pmatrix} H_+ & f \\ -f^* & -H_-^* \end{pmatrix} \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} , \quad (3.21)$$

$$\text{with } \begin{cases} H_\pm = \Delta_\rho - \frac{(S \pm m)^2}{\rho^2} + 1 + 2\alpha \tilde{a}^2 + iM \tilde{a}^{2(M-1)} \\ f = [\alpha \tilde{a}^2 + i(M-1) \tilde{a}^{2(M-1)}] e^{2i\phi} \end{cases} . \quad (3.22)$$

In the last equations, a and ϕ are the radial amplitude and phase profiles of the nonlinear BVB whose stability is being analyzed, determined by the material properties in the parameters α and M , and b_0 determining its peak amplitude and the topological charge s . Since the profiles a and ϕ of the nonlinear BVB are only known numerically, the above eigenvalue problem can only be solved numerically. The numerical procedure must be repeated, in principle, for all values of m . According to Eq. (3.21), the existence of at least one eigenvalue with negative imaginary part for one value of m implies the existence of an azimuthal eigenmode (μ_m, ν_m) that grows exponentially, therefore leading to the instability of the nonlinear BVB. For the numerical procedure, a discretization method will be applied as presented in App. C and later used to generate the simulation algorithm implemented in Matlab[®]. Of course, a linearized stability analysis only informs us whether a small perturbation can grow exponentially. Once an unstable mode has grown to an amplitude comparable to that of the nonlinear BVB itself, the further development of instability can only be studied by numerical means. For that reason, it is necessary to conduct a numerical resolution of the linearized stability analysis on a nonlinear BVB affected by a random noise term, which fulfills the role of exciting all possible unstable modes therein contained. This method also allows the observation of the development of the unstable modes, if any, as they grow during propagation.

3.2 Numerical resolution and results of the linearized stability analysis

3.2.1 Analysis procedure and preparation

The numerical analysis is carried out with the objective of controlling the effects of truncation in a finite computational box and estimating the eigenvalue spectrum of the problem in Eq. (3.21) with a given nonlinear BVB and extracting conclusions about its stability. A discretized version of the eigenvalue problem is solved numerically by introducing a radial mesh in ρ of N points and a step size h . Such a discretization with N points implies a truncation of the weakly localized BVB at a radius equal to $\rho_{\text{trun}} = Nh$. This procedure results in the continuous system of equations (3.21)-(3.22), with unbounded ρ , being replaced with a discretized version in a finite box, and therefore with a discrete set of eigenvalues. The purpose of this subsection is to analyze the conditions under which the eigenvalues of the discrete system approximate the eigenvalues of the continuous system, i.e., are physically valid. Given the weak localization of nonlinear BVBs, the main source of divergence arises from the truncation of the nonlinear BVB in the finite computational box. Thus, this problem is much more difficult than the usual one where the stability of strongly localized solitons is analyzed (Polesana et al. 2007), where the computational box and number of points needed are comparatively small.

Unfortunately, the relationship between the value of N and the reliability of the numerical analysis is not straightforward. Limiting the size in which the BVB can spread out radially introduces a minimum reliable eigenvalue of the numerical counterpart, as seen from the following analysis.

In the non-truncated, original case, nonlinear terms of Eq. (3.21) tend to zero when $\rho \rightarrow \infty$ for the eigenmodes to be physically valid. It is equivalent to assert that the behavior of the eigenmodes is asymptotically linear in outer zones. Then, at large ρ , Eqs. (3.21)-(3.22) simplifies to

$$\kappa_m \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} \approx \begin{pmatrix} H_+ & f \\ -f^* & -H_-^* \end{pmatrix} \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix}, \quad (3.23)$$

$$\text{with } \begin{cases} H_{\pm} = \Delta_{\rho} - \frac{(S \pm m)^2}{\rho^2} + 1 \\ f = 0 \end{cases}, \quad (3.24)$$

which is equivalent to the uncoupled set of equations

$$\begin{cases} \frac{d^2 \mu}{d\rho^2} + \frac{1}{\rho} \frac{d\mu}{d\rho} - \frac{(S+m)^2}{\rho^2} \mu + (1-\kappa)\mu = 0 \\ \frac{d^2 \nu}{d\rho^2} + \frac{1}{\rho} \frac{d\nu}{d\rho} - \frac{(S-m)^2}{\rho^2} \nu + (1+\kappa)\nu = 0 \end{cases}. \quad (3.25)$$

Note that the subindex m is omitted for conciseness. This set can be solved using the changes

$$z_1 = \sqrt{1-\kappa}\rho, \quad z_2 = \sqrt{1+\kappa}\rho, \quad (3.26)$$

which turns Eq. (3.25) into

$$\begin{cases} \frac{d^2\mu}{dz_1^2} + \frac{1}{z_1} \frac{d\mu}{dz_1} - \frac{(S+m)^2}{z_1^2} \mu + \mu = 0 \\ \frac{d^2\nu}{dz_2^2} + \frac{1}{z_2} \frac{d\nu}{dz_2} - \frac{(S-m)^2}{z_2^2} \nu + \nu = 0 \end{cases} . \quad (3.27)$$

The general solution to Eqs. (3.27) can be expressed in terms of Bessel functions as

$$\mu = A_1 H_{S+m}^{(1)}(z_1) + B_1 H_{S+m}^{(2)}(z_1) , \quad (3.28)$$

$$\nu = A_2 H_{|S-m|}^{(1)}(z_2) + B_2 H_{|S-m|}^{(2)}(z_2) . \quad (3.29)$$

In our numerical approximation to the eigenvalue problem in Eqs. (3.27), the most difficult problem is to ascertain whether small eigenvalues are physically valid or an artifact of the discretization in a finite box. For small values of $|\kappa|$, Eqs. (3.26) can be approximately written as

$$z_1 \simeq \left(1 - \frac{1}{2}\kappa\right) \rho = \left(1 - \frac{1}{2}\kappa_R - \frac{1}{2}i\kappa_I\right) \rho , \quad (3.30)$$

$$z_2 \simeq \left(1 + \frac{1}{2}\kappa\right) \rho = \left(1 + \frac{1}{2}\kappa_R + \frac{1}{2}i\kappa_I\right) \rho . \quad (3.31)$$

Leaving aside irrelevant constants to this analysis (see asymptotic expressions in Eq. (2.20)), for small gains κ_I Eqs. (3.28)-(3.29) become

$$\begin{cases} \mu \propto \frac{A_1}{z_1^{1/2}} e^{i(1-\frac{1}{2}\kappa_R)\rho} e^{(-\frac{1}{2}\kappa_I)\rho} + \frac{B_1}{z_1^{1/2}} e^{-i(1-\frac{1}{2}\kappa_R)\rho} e^{(\frac{1}{2}\kappa_I)\rho} \\ \nu \propto \frac{A_2}{z_2^{1/2}} e^{i(1+\frac{1}{2}\kappa_R)\rho} e^{(\frac{1}{2}\kappa_I)\rho} + \frac{B_2}{z_2^{1/2}} e^{-i(1+\frac{1}{2}\kappa_R)\rho} e^{(-\frac{1}{2}\kappa_I)\rho} \end{cases} . \quad (3.32)$$

Coefficients A_2 and B_1 must be zero for μ and ν to tend to zero as $e^{(-\frac{1}{2}\kappa_I)\rho}$. Hence

$$\begin{cases} \mu \propto \frac{A_1}{z_1^{1/2}} e^{i(1-\frac{1}{2}\kappa_R)\rho} e^{(-\frac{1}{2}\kappa_I)\rho} \\ \nu \propto \frac{B_2}{z_2^{1/2}} e^{-i(1+\frac{1}{2}\kappa_R)\rho} e^{(-\frac{1}{2}\kappa_I)\rho} \end{cases} . \quad (3.33)$$

The system of equations (3.33) describes the approximate, asymptotic behavior of eigenmodes with small eigenvalue. Hence, to adequately reproduce in the discretized model (limited by a finite box of size Nh) those modes presenting small negative imaginary parts κ_I must satisfy $|\kappa_I|Nh/2 \gg 1$, that is, the minimum reliable $|\kappa_I|$ in the discrete model is

$$|\kappa_I| \gg \frac{2}{Nh} . \quad (3.34)$$

Condition (3.34) limits the reliability of the eigenvalues obtained from the numerical analysis carried out through a specifically designed discretization algorithm. When calculating

the eigenvalue spectrum of the problem (3.21), only eigenmodes that meet the condition (3.34) are considered physically valid, i. e., reproduce eigenmodes of the original continuous system. The algorithm constructed to discretize the eigenvalue problem (3.21) is thoroughly detailed in the App. C, while the software implementation is included in App. D.

To begin with the numerical stability analysis, it is necessary to explicitly define the nonlinear BVB to be inspected by first selecting the α parameter, which determines the Kerr nonlinearity of the medium being traveled with respect to the nonlinear MPA [see Eqs. (2.4)]. Nonlinear MPA is in turn fixed by the parameter M . Two additional parameters exclusively related to the nonlinear BVB are necessary, the amplitude parameter b_0 , which settles the peak amplitude of the beam, and the topological charge s . The amplitude a and the phase ϕ of the nonlinear BVB is then sampled at N points separated by a sampling distance (or step) h and introduced in the discretized version of Eq. (3.21). The discretization procedure over the radial variable in N points generates a non-symmetric, non-Hermitian $2N \times 2N$ matrix to be solved for a solution set of $2N$ complex eigenvalues whose imaginary parts determine the gain of the modes in the nonlinear BVB.

Special attention was given when determining the adequate value of the step h for the numerical resolution of Eq. (3.21). Different values of h were tested, and for a given eigenvalue and a fixed value of the product Nh , no relevant differences were obtained for small values of h , if the nonlinear BVB was correctly sampled. The value of h was therefore fixed in all numerical resolutions, while the value of N was gradually increased, improving precision of the results progressively.

For each value of the winding number m identifying the mode being analyzed, the numerical resolution was executed, beginning with a small sample of $N = 1000$ points. The number of points was then doubled each time until the maximum computational capacity available were reached, going through samplings of $N = 2000$, $N = 4000$, $N = 8000$ and $N = 16000$ points. Notably, it was necessary to reach such high values for N to verify the nature of the eigenmodes with particularly small growth rates $|\kappa_I|$, for which it was required to develop special routines to calculate eigenvalues of sparse matrixes up to 32000×32000 (see App. D). A special mention must be made at this point about the computational requirements needed for the smallest growth rates analyzed. The need for a mesh of $N = 16000$ points arose mainly when it was necessary to accurately determinate if a specific eigenvalue was not indeed an artifact of the discretization method, in particular for those eigenvalues in a limit situation between the stability and instability of a specific nonlinear BVB. The correct computation of these eigenvalues meant to handle amounts of data of the order of the gigabytes in-memory, and computational times on the order of a day for only one eigenvalue – evidently a heavy toll for every computer available to us at the time. Moreover, these computations had to be rerun for every case of interest for this research. In every case, only valid eigenvalues according to the condition (3.34) were considered. Observing the variation of valid eigenvalues with increasing N , those satisfying Eq. (3.21) that would rapidly stabilize their imaginary parts (or remain almost unchanged) were considered physically relevant; on the other hand, those with imaginary parts rapidly decreasing to 0 were considered an artifact of the discretization process, bound to disappear as the limit $N \rightarrow \infty$ was approached.

Note that this numerical procedure must be repeated to each value of m to be studied

for a given nonlinear BVB, univocally determined by its parameters M , α and b_0 . Consequently, this is an extremely long, time-consuming task, whose results are detailed in the next subsection, where the trend of the calculated gains with increasing N allow to conclude if the mode (identified by the value of m) is stable or not: given a fixed set of parameters M , α , b_0 and s , stable modes have their gain progressively reduced to zero with increasing N . This occurs in clear opposition to what happens in the case of unstable modes, where they stabilize at a nonzero value. If there exists at least one eigenvalue for an arbitrary mode m with negative imaginary part κ_I , the mode will exponentially grow with a gain $|\kappa_I|$ and cause the instability of the nonlinear BVB considered. Complete stability is demonstrated only in the case of the non-existence of eigenvalues with negative imaginary part κ_I , when extrapolating the numeric results to the limit $N \rightarrow \infty$.

3.2.2 Results of the linearized stability analysis: existence of completely stable nonlinear Bessel vortex beams

Again, let us remind that “complete” or “full” stability is used in this dissertation in the sense that the nonlinear BVB is stable against small perturbations in all of its dynamic variables, instead of only in a specific one, as could be, for example, its radial or azimuthal variable, as it is sometimes researched in scientific literature. The results of the simulations described in the previous section are presented next. In all cases, the MPA parameter was set to $M = 4$, corresponding to a common value for a wide range of materials used in experimental settings. The results presented do not critically depend on this choice. The stability properties of nonlinear BVBs are studied for three different values of their topological charge, $s = 1$, $s = 2$ and $s = 3$. To analyze the influence of the intensity parameter b_0 of the nonlinear BVB on their stability character, for each s two different values of b_0 were selected, $b_0 = 1.2$ and $b_0 = 1.6$. The effect of the variation of the ratio between the Kerr nonlinearity and the nonlinear absorption on the nonlinear BVB (expressed by the parameter α) was inspected through a detailed scanning of a significant interval of values ranging from $\alpha = 0$ (meaning that nonlinear absorption is strong, while Kerr nonlinearity is negligible) to $\alpha = 6$ (strong Kerr nonlinearity, negligible nonlinear absorption). Hence, instability is expected to increase as α increases.

Fig. 3.1 gathers the results obtained for nonlinear BVBs with $s = 1$, with case $b_0 = 1.2$ in the first column and case $b_0 = 1.6$ in the second. The horizontal axis shows α values, giving the relation between the Kerr nonlinearity and the nonlinear absorption affecting each propagating nonlinear BVB. The vertical axis shows the exponential growth rate (i.e., the gain, $|\kappa_I|$) of the most unstable mode for each winding number m indicated. Values of m ranging from $m = 1$ to $m = 8$ were explored in each setting. Figs. 3.2 and 3.3 are analogue cases for $s = 2$ and $s = 3$, respectively. Stability is demonstrated to exist if, given a nonlinear BVB univocally defined by some fixed values of s , b_0 and α , no mode m can be found with positive exponential growth rate.

Figure 3.1: For a topological charge $s = 1$, and two different values of the intensity parameter ($b_0 = 1.2$ at first column, $b_0 = 1.6$ at the second), this figure shows dimensionless growth rates for winding numbers $m = 1, \dots, 8$. First row compares gain curves for all m with $N = 2000$, while the second row presents a relevant growth rate (here, $m = 1$), with a step size of $h = 0.2$. The last row explicitly details the convergence to zero of the growth rates with increasing N (circles for $N = 1000$, squares for $N = 2000$, upwards triangles for $N = 4000$, downwards triangles for $N = 8000$ and rhombuses for $N = 16000$).

Figure 3.2: For a topological charge $s = 2$, and two different values of the intensity parameter ($b_0 = 1.2$ at first column, $b_0 = 1.6$ at the second), this figure shows dimensionless growth rates for winding numbers $m = 1, \dots, 8$. First row compares gain curves for all m with $N = 2000$, while the second row presents a relevant growth rate (here, $m = 2$), with a step size of $h = 0.2$. The last row explicitly details the convergence to zero of the growth rates with increasing N (circles for $N = 1000$, squares for $N = 2000$, upwards triangles for $N = 4000$, downwards triangles for $N = 8000$ and rhombuses for $N = 16000$).

Figure 3.3: For a topological charge $s = 3$, and two different values of the intensity parameter ($b_0 = 1.2$ at first column, $b_0 = 1.6$ at the second), this figure shows dimensionless growth rates for winding numbers $m = 1, \dots, 8$. First row compares gain curves for all m with $N = 2000$, while the second row presents a relevant growth rate (here, $m = 3$), with a step size of $h = 0.2$. The last row explicitly details the convergence to zero of the growth rates with increasing N (circles for $N = 1000$, squares for $N = 2000$, upwards triangles for $N = 4000$, downwards triangles for $N = 8000$ and rhombuses for $N = 16000$).

The first row of each figure [3.1(a), 3.2(a), 3.3(a)] shows the final curves obtained with the extrapolation of the results to the limit $N \rightarrow \infty$. As explained previously in 3.2.1, the extrapolation procedure is necessary to establish the physical validity of any small eigenvalues calculated during the numerical analysis. Thus, any small eigenvalues were recalculated at increasing precision using larger meshes with $N = 4000$, $N = 8000$ and $N = 16000$ points (second row of each figure, 3.1(b), 3.2(b), 3.3(b)). If the result of such recalculation was fast stabilization of the imaginary part κ_I of the eigenvalue, then it was assumed to have positive gain; on the contrary, if $\kappa_I \rightarrow 0$ then it was considered a spurious result of the discretization process and actually equal to zero in the limit $N \rightarrow \infty$. This process is illustrated on the third row of each figure (3.1(c), 3.2(c), 3.3(c)), where the resulting curves at increasing precision are compared for some relevant, particular cases of small eigenvalues whose gain transitions from positive to zero at a fixed α value. If the region below this fixed α is considered, the extrapolation process evidences how the gain rapidly decreases to zero for that eigenvalue; otherwise a constant value for the gain is attained in the region over the fixed α .

The main conclusion stemming from Figs. 3.1(a), 3.2(a) and 3.3(a) is that there exists, always, and independently of the values of s or b_0 chosen, a set of nonlinear BVBs with small α values (thus, with a dominant nonlinear absorption in comparison with the Kerr nonlinearity) that do not present any eigenmode with exponential growth, and these nonlinear BVB are then *completely stable against small perturbations*.

Figures 3.2(b)-(c) and 3.3(b)-(c) compile another important first-hand contribution of this work, demonstrating the existence of truly stable nonlinear BVBs with topological charge $s > 1$ in the case that the only dispersive nonlinearity is the cubic Kerr nonlinearity. Past attempts to achieve these results were unsuccessful, in the case of Kerr-only nonlinearity, such as those in Refs. (Dalfovo and Stringari 1996; Dodd 1996; Adhikari 2001; Saito and Ueda 2002; Alexander and Bergé 2002; Carr and Clark 2006; Mihalache et al. 2006; Malomed et al. 2007; Sakaguchi, Li, and Malomed 2014; Zhang et al. 2015). These results are expected to be valid also at $s > 3$, as no particular hypothesis was made on the previous analysis about the topological charge of the nonlinear BVBs.

In Figs. 3.2(b), 3.2(b) and 3.3(b), the gain curve that cuts the α axis at its smallest value actually marks the limit between the stability and the instability regions. At greater values of α , the growth rates quickly converge to positive values with increasing N , implying genuine instability of the nonlinear BVB. At smaller values of α , the growth rates tend to zero with increasing N , denoting stability of the nonlinear BVB. This is also an important contributions of this work, the finding of an effective criterion to determine if a particular nonlinear BVB in a Kerr medium with nonlinear MPA, uniquely determined by fixed values of M , α and its intensity parameter b_0 , will develop into a stable or an unstable regime with propagation. Moreover, in the unstable case, it additionally predicts that the nonlinear BVB will breakup, expectedly, in a number of fragments equal to the mode m with the highest gain.

3.3 Numerical verification

The results of the numerical analysis and results presented in the Sec. 3.2.2 were verified through direct simulations of Eq. (3.1). A perturbation consisting of random noise was added to the nonlinear BVB being studied, in order to excite any existing unstable mode that could manifest during the propagation, revealing an unstable nonlinear BVB. On the other hand, truly stable nonlinear BVBs are expected to have the random noise absorbed, resulting in a stationary, robust propagation. In all cases, propagations were carried out to the maximum of the available computational capacity (up to propagation distance $\zeta = 400$), to ensure that any possible instability can fully develop. These numerical simulations allow, first, to verify the predictions of the linearized stability analysis in the so-called small perturbation regime, i. e., when the unstable growing modes are small compared to the nonlinear BVB itself; and second, to study the dynamics of the further development of the instability in the so-called large perturbation regime when perturbation and nonlinear BVB become comparable and, indeed, there is no distinction or separability between them.

In the following, the transversal intensity ($|\tilde{A}|^2$) maps resulting from the numerical simulations are presented and analyzed for a set of relevant nonlinear BVBs. The color scale used for the intensity maps is given in Fig. 3.4 on the right. The Cartesian coordinates in the following figures are normalized as done for the radial coordinate, hence being $\xi = k\theta x$, $\eta = k\theta y$ in the transversal plane. In all figures, the peak intensity values are normalized to unity, since the only relevant phenomenon to be observed is the instability character, and the absolute values of the intensities unimportant. An example of the input transversal intensity profile of the nonlinear BVBs used in these simulations is shown in Fig. 3.5. On the left the input nonlinear BVB is depicted without the random noise and, on the right, the same nonlinear BVB with the random noise added. The level of noise is randomly determined at each point of the transversal plane, being a complex number whose real and imaginary parts vary between 0% and 5% of the input intensity at that point.

Figure 3.4: Color scale used at the intensity maps.

Figure 3.6 shows transversal intensity profiles at the maximum propagation distance $\zeta = 400$ for two nonlinear BVBs with added noise launched into the nonlinear medium. In (a), the nonlinear BVB is determined by $s = 1$, $b_0 = 1.2$, $\alpha = 1$; while in (b) the parameters of the nonlinear BVB are $s = 2$, $b_0 = 1.2$, $\alpha = 1$. According to the linearized stability analysis [see Figs. 3.1(a) and 3.2(a)], these nonlinear BVBs are predicted to be stable against small perturbations. Indeed, the intensity profiles in Fig. 3.6 are identical to their input intensity profiles at $\zeta = 0$, with the noise being eliminated. These numerical simulations of the propagation thus confirm the predictions of the linearized stability analysis, since a stationary and robust (stable) propagation is clearly observed as a result. As Supplemental Material, the full propagation of case 3.6(a) is included as the file Video_3-6a.avi, while case 3.6(b) is included as the file Video_3-6b.avi.

Figure 3.5: An example of the transversal intensity profile $|\tilde{A}|^2$ of the input ($\zeta = 0$) nonlinear BVB whose stability is to be numerically tested is shown, (a) without the random noise used to excite the unstable modes, and (b) with the random noise added (ranging between 0% and 5% of the input intensity at each point of the transversal plane). The nonlinear BVB in this example is determined by $s = 1$, $b_0 = 1.2$ and $\alpha = 1$.

Figure 3.6: Transversal intensity profiles at the maximum propagation distance $\zeta = 400$ of the indicated nonlinear BVBs. These intensity profiles are identical to the input nonlinear BVBs. As expected from the linearized stability analysis, no unstable modes develop in neither case, as both are stable as predicted in Figs. 3.1(a) and 3.2(a).

The input beams in Fig. 3.5 and the stable propagation cases in Fig. 3.6 provide a convenient baseline to assess the following unstable cases presented next, from Figs. 3.7 to 3.11, except 3.9 (which contains stable cases). Varying the different parameters in each case (the topological charge s , the intensity parameter b_0 and the Kerr intensity α), the stability of other nonlinear BVBs is analyzed observing the transverse intensity maps at propagation distances of interest. As will be seen, the modulation m with the largest growth rate always determines the number of pieces the unstable nonlinear BVBs break

up into, coinciding with the predictions of the linearized stability analysis made in Sec. 3.2.2, Figs. 3.1(a), 3.2(a) and 3.3(a). In addition, some general trends can be identified on the simulations. For example, lower values of s , b_0 or α delay the development of instability, while higher values of these parameters accelerate the development. Likewise, higher values of α usually make the small perturbation regime linger less propagation steps than it would at lower values of α , making any large perturbation regime come up earlier in the propagation. This is to say that unstable behavior is stronger as α increases.

For $s = 1$, Fig. 3.7 displays transversal intensity profiles of initial nonlinear BVBs plus added noise at selected propagation distances. In (a), the nonlinear BVB with parameters $b_0 = 1.2$ and $\alpha = 2.6$ on the left column; in (b), the nonlinear BVB with parameters $b_0 = 1.6$ and $\alpha = 2.6$ on the right column. Case (a) shows a split up of the nonlinear BVB into one fragment, in good agreement with the calculated winding number with the highest gain, $m = 1$ [see Fig. 3.1(a)]. Further propagation of the nonlinear BVBs shows a secondary breakup into two rotatory pieces, and finally into three rotatory fragments. No further ruptures are observed until the final propagation step. Case (b) presents a situation known as *mode competition*: since the nonlinear BVB has two unstable modes of nearly equal gain, $m = 2$ and $m = 4$ [see Fig. 3.1(b)], it can be seen in Fig. 3.7(b) how the competition between them leads to a rupture in two fragments in the first ring and four fragments in the outer rings. The last figure illustrate the development of the instability predicted in the small perturbation regime when it further grows into the large perturbation regime, leading to a quite complex dynamics with further splitting into more rotatory fragments, and also into nonrotatory fragments that randomly appear and disappear. In all cases, the fragments predicted by the linearized stability analysis are rotatory. Rotation is a manifestation of the OAM carried by the nonlinear BVB carrying a vortex, which however can only be seen once the cylindrical symmetry of the nonlinear BVB is lost because of the instability. Full propagations of the nonlinear BVBs shown in Figs. 3.7(a) and 3.7(b) are included, respectively, as the video files Video_3-7a.avi and Video_3-7b.avi in the Supplemental Material.

In Fig. 3.8, the propagation of two nonlinear BVBs of topological charge $s = 2$ are analyzed. Case (a) is the nonlinear BVB with $b_0 = 1.2$ and $\alpha = 1.6$, its small perturbation regime consists in breaking into two rotating pieces. This result agrees with the results obtained from the linearized stability analysis and presented in Fig. 3.2(a). The two rotating pieces persist almost until the end of the simulation ($\zeta \gtrsim 390$), when it begins to show signs of imbalance. Next, case (b) is the nonlinear BVB with $b_0 = 1.6$ and $\alpha = 3$, a configuration whose results of the linearized stability analysis in Fig. 3.2(a) indicate the greatest gain for the mode $m = 3$. Indeed, the nonlinear BVB breaks into three rotating splinters early in the propagation; afterwards the three rotating pieces enter into a speckle-like or chaotic regime of nonrotating filaments appearing and disappearing randomly. Comparison between (a) and (b) indicates that the higher the nonlinear BVB intensity, the stronger the instability, which also agrees with the fact that the gain of the mode $m = 2$ in (a) is higher than the gain of the mode $m = 3$ in (b). Again, full propagations of the nonlinear BVBs shown in Figs. 3.8(a) and 3.8(b) are included, respectively, as the video files Video_3-8a.avi and Video_3-8b.avi in the Supplemental Material.

Figure 3.7: Numerical verification of stability results: nonlinear BVBs with $s = 1$. (a) For the nonlinear BVB with $b_0 = 1.2$ and $\alpha = 2.6$, the prediction of the linearized stability analysis [Fig. 3.1(a)] of a rupture in one fragment (maximum gain of mode with $m = 1$) is fulfilled. At longer distances, secondary breakups in two and three fragments are observed. (b) The nonlinear BVB with $b_0 = 1.6$ and $\alpha = 2.6$ is predicted by Fig. 3.1(a) to develop instability with a mode competition between $m = 2$ and $m = 4$. In effect, the inner ring breaks up in two fragments and the second one in four fragments, enduring until a chaotic dynamics with spurious fragments appearing and disappearing at random dominates.

Figure 3.8: Numerical verification of stability results: nonlinear BVBs with $s = 2$. Figure 3.2(a) predicts a rupture in two fragments for (a), with maximum gain of mode with $m = 2$; and three fragments for (b), with maximum gain of mode with $m = 3$. Indeed, the nonlinear BVB (a), with parameters $b_0 = 1.2$ and $\alpha = 1.6$, breaks up into two rotating pieces. At longer propagation distances, the inner rings begin to take an asymmetrical configuration and the two rotating pieces begin to lose their shape. In (b), the nonlinear BVB with parameters $b_0 = 1.6$ and $\alpha = 3$ is also in good agreement, first breaking up into three splinters, and then exhibiting a rather complex dynamics of the central vortices.

3.3.1 A further novelty: stability of nonlinear Bessel vortex beams of topological charge 3

As stated above, no stable vortex solitons with $s > 1$ in media with cubic-only dispersive nonlinearity have been described. Figures 3.9 and 3.10 gather the results obtained for nonlinear BVBs with $s = 3$ and added noise. In particular, Fig. 3.9 highlights the existence of stable vortices with $s = 3$. In (a), a nonlinear BVB with parameters $b_0 = 1.2$ and $\alpha = 0.8$ is revealed to remain stable up to the end of the propagation; the nonlinear BVB (b) with $b_0 = 1.6$ and $\alpha = 0.8$ is also seen to have the same stability character. This is in agreement with the linearized stability analysis predictions [see Fig. 3.3(a)], and the confirmation of this remarkable fact is given in Fig. 3.9, where both beams are seen to propagate in a robust way up to the maximum propagation distance allowed by our numerical facility.

Figure 3.9: Numerical verification of stability results: nonlinear BVBs with $s = 3$. Two nonlinear BVBs with, (a) $b_0 = 1.2$ and $\alpha = 0.8$, and (b) $b_0 = 1.6$ and $\alpha = 0.8$, are shown to be stable against small perturbations, as predicted by the linearized stability analysis.

On the contrary, Fig. 3.10 shows two examples of unstable cases. For the nonlinear BVB (a), with parameters $b_0 = 1.2$ and $\alpha = 3$; and for the nonlinear BVB (b), with parameters $b_0 = 1.6$ and $\alpha = 2$, the linearized stability analysis predicts a rupture in 3 pieces, as the mode with greatest gain is $m = 3$ [see Fig. 3.3(a)]. The prediction is fulfilled in both cases around the same propagation step. The similar propagation distances for the observable development of the instability is understood from the fact that the larger Kerr coefficient in (a) than in (b) is compensated by the smaller intensity in (a) than in (b), leading to a similar trend towards instability. Case (b) is particularly interesting, as it reaches a stationary rotating state that does not decay nor evolves into a chaotic dynamical regime at least until the computational limit of the verification. It was the only observed unstable case to develop such a stationary behavior. Case (a) also shows a curious behavior, seemingly with a periodic oscillation between three and four rotating, flashing pieces without any chaotic behavior emerging until the final propagation distance, $\zeta = 400$.

Figure 3.10: Numerical verification of stability results: nonlinear BVBs with $s = 3$. (a) The nonlinear BVB with $b_0 = 1.2$ and $\alpha = 3$ first breaks up in 3 rotating pieces, later changing to a 4-piece scheme, and then alternating between the two until the end of the propagation. No signs of chaotic dynamics were observed, suggesting a possibly periodic behavior. (b) The nonlinear BVB with $b_0 = 1.6$ and $\alpha = 2$ evolves differently, breaking up in 3 rotating pieces emerging as a seemingly stationary rotational state that dominates the entire propagation, with no further ruptures. Cases (a) and (b) comply with the predictions of Fig. 3.3(a), both with mode $m = 3$ having maximum gain.

As Supplemental Material, the full propagation of case 3.9(a) is included as the file Video_3-9a.avi, while case 3.9(b) is included as the file Video_3-9b.avi. In addition, complete propagations of the nonlinear BVBs shown in Figs. 3.10(a) and 3.10(b) are included, respectively, as the video files Video_3-10a.avi and Video_3-10b.avi.

It is also noteworthy to observe how unstable nonlinear BVBs transition from the small perturbation regime to the large perturbation regime. Previously known unstable vortex solitons in other models, as with tailored nonlinearities (Kivshar and Pelinovsky 2000; Malomed et al. 2005; Desyatnikov, Torner, and Kivshar 2005; Mihalache 2012), have the their large perturbation regime determined by the rupture occurring in their small perturbation regime, with no further dynamics being observed. On the contrary, for nonlinear BVBs the number of circulating fragments in the small perturbation regime does not determine how the instability will further develop. This feature could already be seen in Figs. 3.7, 3.8 and 3.10 above, but a couple more of interesting cases are presented below.

Figure 3.11 depicts in the left column the first rupture suffered by different nonlinear BVBs as predicted by the linearized stability analysis in a small perturbation regime, and in the right column the secondary rupture, under a large perturbation regime where the instability is already growing to be comparable to the amplitude of the nonlinear BVB itself.

In case 3.11(a), a beam with $s = 1$, $b_0 = 1.6$ and $\alpha = 4$ shows a primary breakup into two fragments in the main ring as predicted by Fig. 3.1(a), with $m = 2$ being the mode with the greatest gain. Continuing with the propagation, the two-fragment setup degenerates into one flashing fragment that also rotates around the center, after which a seemingly chaotic regime develops into rotating chips that appear and disappear randomly throughout the propagation.

On the second row, Fig. 3.11(b) shows a nonlinear BVB with parameters $s = 2$, $b_0 = 1.2$ and $\alpha = 2$, which should break into two pieces as forecasted by the results of stability analysis in Fig. 3.2(a). For this specific configuration, the two modes with greatest gains are $m = 2$ and $m = 3$, in decreasing order. Indeed, the primary breakup into two pieces occurs as expected, and apparently has no influence over the subsequent breakup where the two initial pieces change to three circulating pieces, coinciding with the mode with the second greatest gain.

Finally, for 3.11(c) the nonlinear BVB with $s = 3$, $b_0 = 1.6$ and $\alpha = 3.5$, the mode with the greatest gain is $m = 3$ as shown in Fig. 3.3(a). In conformity with the calculated values, Fig. 3.11(c) shows how the main ring of the nonlinear BVB first breaks up into three pieces. Shortly after, the secondary breakup occurs into four pieces, also coinciding with the mode with the second greatest gain for this configuration, $m = 4$.

As Supplemental Material, the full propagation of case 3.11(a) is included as the file Video_3-11a.avi, case 3.11(b) is included as the file Video_3-11b.avi, and case 3.11(c) is included as the file Video_3-11c.avi.

Figure 3.11: Some fascinating cases of the change occurring between the primary and secondary ruptures for nonlinear BVBs with (a) $s = 1$, $b_0 = 1.6$, $\alpha = 4$ and greatest gain on mode $m = 2$; (b) $s = 2$, $b_0 = 1.2$, $\alpha = 2$ and greatest gain also on mode $m = 2$; and (c) $s = 3$, $b_0 = 1.6$, $\alpha = 3.5$ and greatest gain on mode $m = 3$.

3.4 Verification of the stabilizing effect of the nonlinear absorption

The linearized stability analysis provides another opportunity to confirm one of the most important finding of this dissertation: the fact that the mechanism or effect that renders the nonlinear BVBs stable in presence of self-focusing Kerr nonlinearity is MPA, i. e., a dissipative mechanism that renders the dynamical system non-Hamiltonian can also stabilize it. Since it is now possible to determine with certainty if a particular nonlinear BVB is stable or not, we can repeat the simulations carried out in Sec. 3.2.2 removing the effect of the nonlinear absorption, and analyze the new results. In other words, a particular configuration of a nonlinear BVB, known to be stable in a Kerr medium with nonlinear absorption, is now studied in a fully transparent medium.

This can be done neglecting the dissipative (last) term of Eq. (2.3) and repeating the linearized stability analysis described at the end of Sec. 3.2.1. Results are then compared to those presented also in the same section, as shown in Figs. 3.12, 3.13 and 3.14. There is no room for doubt about the crucial stabilizing effect of the nonlinear MPA: without it there is always at least one unstable mode m with positive gain for all values of α , and no stable nonlinear BVB exists anymore in a transparent medium with self-focusing Kerr nonlinearity. Although only the most representative modes m are displayed below, the result applies with generality, and the region of stable nonlinear BVBs ($\alpha \lesssim 1$ in the nonlinearly absorbing medium) completely disappears for the nonlinear BVBs propagating in transparent media.

These results also clarify the observations of previous works reporting stabilization effects on nonlinear BVBs through the increase of the cone angle [which is equivalent to decreasing α , due to the definition in Eqs. (2.4)]. Refs. (Jukna et al. 2014; Xie et al. 2015) hypothesize that at higher cone angles stabilization is achieved thanks to nonlinear wave mixing or suppressed growth of the unstable modulations in a qualitatively manner. Nonetheless, Figs. 3.12, 3.13 and 3.14 demonstrate quantitatively that stabilization of a nonlinear BVB can never be achieved only by increasing the cone angle (decreasing α). The observed stable propagation of nonlinear BVB in these numerical and experimental works is due either to the effect of nonlinear absorption present in the experiment or to the fact that the growth rate for smaller values of α is too small and therefore the instability does not develop in the finite length of the material sample.

Figure 3.12: Growth rates of nonlinear BVBs with $s = 1$ and (a) $b_0 = 1.2$, (b) $b_0 = 1.6$, of unstable modes with representative winding numbers m in Kerr media with nonlinear absorption $M = 4$ (solid curves), and in a transparent media, i. e., neglecting the last term in Eq. (2.3) (dashed curves). The stability region for $\alpha \lesssim 1$ disappears when nonlinear absorption is neglected, confirming that the stabilizing mechanism of the nonlinear BVBs is nonlinear absorption.

Figure 3.13: Growth rates of nonlinear BVBs with $s = 2$ and (a) $b_0 = 1.2$, (b) $b_0 = 1.6$, of unstable modes with representative winding numbers m in Kerr media with nonlinear absorption $M = 4$ (solid curves), and in a transparent media, i. e., neglecting the last term in Eq. (2.3) (dashed curves). The stability region for $\alpha \lesssim 1$ disappears when nonlinear absorption is neglected, confirming that the stabilizing mechanism of the nonlinear BVBs is nonlinear absorption.

Figure 3.14: Growth rates of nonlinear BVBs with $s = 3$ and (a) $b_0 = 1.2$, (b) $b_0 = 1.6$, of unstable modes with representative winding numbers m in Kerr media with nonlinear absorption $M = 4$ (solid curves), and in a transparent media, i. e., neglecting the last term in Eq. (2.3) (dashed curves). The stability region for $\alpha \lesssim 1$ disappears when nonlinear absorption is neglected, confirming that the stabilizing mechanism of the nonlinear BVBs is nonlinear absorption.

3.5 Summary

The stability properties of nonlinear BVBs in Kerr media with nonlinear absorption, present in the form of MPA, were investigated in this chapter. In particular, and as reported in (Porrás et al. 2016; Carvalho, Ruiz-Jiménez, and Porrás 2017), a linearized stability analysis of nonlinear BVBs propagating according to the nonlinear Schrödinger equation was performed. The procedure revealed a new parameter, the winding number m , that identified the modulations introduced in the initial stationary state, a nonlinear BVB. The resulting eigenvalue problem obtained was numerically solved through a discretization process, thus making it necessary to study the problem of truncation in the new discretized system, in order to ensure that the solutions were physically valid and equivalent to the original, continuous system. The discernment between valid small eigenvalues and those which were only an artifact of the discretization process was the most difficult step, solved with the deduction of condition (3.34) to decide on the reliability of small eigenvalues.

The numerical analysis carried out was a time-consuming task, involving recalculation of the eigenvalues for all relevant winding values m at ever greater precision, given by the number of mesh points N , and the gain curves obtained for each mode disclosed the existence of a well-defined stability zone for the nonlinear BVBs, allowing the prediction of the regime into which its propagation would lead, a stable or unstable regimes. Several particular cases are discussed, and the stable or unstable regime conjectured in each is verified numerically through direct simulations of the NLSE, always with satisfactory results.

Thus, the first main result achieved was the existence of a region of stability for small values of the parameter α , which is independent of the value of the intensity peak parameter b_0 and the value of the topological charge s . This result applies to a wide variety of physical media, since the Kerr nonlinearity and the MPA introduced in this model needed no specific tailoring, and both are ubiquitous at high enough intensities. This result also defines, for the first time, a predictive criterion on the stability regime into which a nonlinear BVB will develop with propagation in a Kerr media with nonlinear absorption.

The second important result was the confirmation of stable nonlinear BVB with arbitrary topological charge. Though the highest value of s analyzed was $s = 3$, no special assumption had to be made in the model for the topological charge. Hence, the results seemingly do not depend on the values of the topological charge and can most probably be extrapolated for any value of s .

Furthermore, a confirmation of the stabilizing role of the nonlinear absorption is carried out with a comparison between the gain curves obtained in a fully transparent medium and in a Kerr medium with MPA, endorsing the conclusion that MPA is the physical mechanism that impedes destabilization of the nonlinear BVB.

MANIFESTATIONS OF STABILITY AND INSTABILITY IN FINITE-POWER NONLINEAR BESSEL VORTEX BEAMS GENERATED BY AXICONS

My son: “But how can it be that you do such cool things at school?” (While pointing at a full-colored transversal intensity map from one of our papers).

On my fifth year at the PhD program.

In the previous chapter, it was shown that stable propagation may be achieved for nonlinear BVBs of any vorticity, solely sustained by the energy and OAM transferred to the medium through the mechanism of MPA, i.e., by nonlinear absorption. Moreover, the stability conditions for such propagation were detailed using a standard linearized stability analysis and the predictions made over the stability or instability were verified through direct numerical simulations of the NLSE (3.1).

As explained in Ch. 1, however, linear and nonlinear BBs, carrying vortices or not, are weakly localized so that they carry infinite power, or a power reservoir. Approximations to BBs carrying finite power, and hence physically realizable BBs, are generated by different optical systems, e.g., by illuminating an axicon, or conical lens, with a standard Gaussian beam in which a punctual vortex is nested in its center. As explained in Ch. 2, the BB then transforms spontaneously into a finite-energy nonlinear BVB if a nonlinear medium is placed after the axicon.

In recent experiments (Polesana et al. 2008; Jukna et al. 2014; Xie et al. 2015; Porras, Ruiz-Jiménez, and Losada 2015) such a setup has been implemented in practice with high-power laser systems, so that self-focusing Kerr nonlinearity and MPA in the medium are important. The absorbed energy ionizes the medium creating plasma, and the propagation of the light beam with the generated plasma is usually referred to as light filamentation. The novelty of the set up in Refs. (Jukna et al. 2014; Xie et al. 2015) is that with the nonlinear BVB filamentation takes place in the main ring of the nonlinear BVB

and therefore filamentation has a tubular geometry. However, depending on the input power, the tubular filamentation may break its cylindrical symmetry leading to a number of helical or rotatory filaments, or to speckle-like, disordered filaments appearing and disappearing randomly.

In this chapter, we explain the different observed regimes of filamentation induced by finite-power nonlinear BVBs generated by axicons as manifestations of the stability properties of ideal, infinite-power nonlinear BVBs studied in Ch. 3 that act as stable or unstable attractors of the propagation dynamics, as also detailed in (Porrás et al. 2016; Carvalho, Ruiz-Jiménez, and Porrás 2017).

4.1 Usual regimes observed in axicon-generated nonlinear Bessel vortex beams

In most experimental configurations, as in Refs. (Jukna et al. 2014; Xie et al. 2015), nonlinear Bessel beams are generated by making a broad and collimated Gaussian beam to pass through a spiral phase plate (a transparent dielectric with an azimuthally varying thickness such that an azimuthal phase variation $2\pi s$ is introduced in the Gaussian beam) and then making the beam to cross an axicon, or equivalent device (e. g., a programmable spatial light modulator can impose the same conical phase variation as the axicon).

The optical field immediately after the axicon can be modeled by

$$A_G(r, \varphi, 0) = \sqrt{I_G} e^{-r^2/w_0^2} e^{-ikr\theta} e^{is\varphi}, \quad (4.1)$$

where $\sqrt{I_G}$ is the amplitude of the Gaussian beam, w_0 its width, and $\exp(-ikr\theta)$ is the conical phase (a phase varying linearly with radial distance r) imprinted by the axicon. In the free space beyond the axicon, a linear BVB gradually builds up starting from low intensities in the so-called Bessel zone depicted in gray color in Fig. 4.1, whose axial length is w_0/θ . The Bessel zone is where the conical plane waves formed by the axicon interfere to build the linear BVB. The focus of the axicon is at middle of the Bessel zone, at $w_0/2\theta$, and is the location where the best finite-power approximation to the linear BVB $A_B = \sqrt{I_B} J_s(k\theta\rho) \exp(is\varphi)$ of the cone angle θ is formed. In the second half of the Bessel zone, the finite-power linear BVB decays and spreads gradually as the conical plane waves forming it cease to interfere. The intensity of the formed BVB is related to that of the input Gaussian beam by Eq. (4.3). With the dimensionless variables introduced in Eqs. (2.2), the field in Eq. (4.1) can be rewritten as

$$\tilde{A}_G(\rho, \varphi, 0) = b_G e^{-\rho^2/\rho_0^2} e^{-i\rho} e^{is\varphi}, \quad (4.2)$$

where the Bessel distance is $\rho_0/2$ and the focus of the axicon is situated at $\rho_0/4$. If the medium following the axicon were linear, a linear BVB carrying a finite amount of power would form at the axicon focus with the intensity parameter

$$b_B^2 = \pi\rho_0 e^{-1/2} b_G^2 \quad (4.3)$$

and an (ideal) amplitude $\tilde{A}_B \simeq b_B J_s(\rho) \exp(is\varphi)$.

Figure 4.1: The above figure shows a typical experimental configuration to achieve the generation of nonlinear Bessel beams using axicons. The grey zone indicates the Bessel zone, the green zone the nonlinear medium and the beam profile marks the focus of the axicon. Two types of configuration are shown: on the top, an abrupt input in the nonlinear media; on the bottom, a soft input.

The top drawing in Fig. 4.1 illustrates the most straightforward way to introduce a linear BVB with finite power in a nonlinear medium. This configuration is usually referred to as *abrupt input* because the BVB enters into the nonlinear medium when it is already formed at the focus of the axicon and its intensity I_B is maximum. This input BVB transforms gradually within the second half of the Bessel zone into the nonlinear BVB characterized by the same cone angle, same topological charge, and amplitude parameter such that the amplitude of the inward Hankel beam component is preserved, i.e., into the nonlinear BVB with $|b_{\text{in}}| = b_B$, as explained in Ch. 2. However, as demonstrated in Ref. (Polesana et al. 2008), this abrupt input configuration allows the formation of uncontrollable temporal instabilities that lead to a complicated temporal dynamics.

The bottom drawing in Fig. 4.1 illustrates the so-called *soft input*, in which the medium is placed immediately after the axicon, preventing the formation of a linear BVB as in the free space propagation of the abrupt input and immediately causing the formation of a nonlinear BVB at the axicon focus, with a lower intensity. With this configuration, Kerr nonlinearity and MPA effects are negligible at the entrance of the medium. Indeed, this configuration prevents the development of spurious temporal instabilities, as demonstrated experimentally in (Polesana et al. 2008).

With the soft input configuration, different regimes of nonlinear propagation were observed for different experimental values of the energy of the input Gaussian beam. These can be grouped mainly under three categories, the first of them being a quasi-stationary regime within the Bessel zone in the nonlinear medium, where a Bessel-like beam builds up, remains fundamentally unchanged in its amplitude profile and decays at the end of the Bessel zone. The second of them gathers those beams that break up their input azimuthal

symmetry into any number of possibly rotating pieces, maintaining this rotation within the Bessel zone, and decaying afterwards. The third and last of the observed regimes in the nonlinear propagation in the Bessel zone groups those input beams that lose their initial cylindrical symmetry and break up into non-rotating splinters that appear and disappear in a seemingly chaotic dynamics. These three regimes induce, respectively, and as stated above, tubular, helical (or rotating), and disordered (or speckle-like) filamentation regimes. For practical applications as, for instance, material processing, tubular filamentation is desired. Therefore, understanding the origin of these three regimes is a problem of practical relevance.

The quasi-stationary and rotary regimes of nonlinear propagation after the axicon were reported by means of numerical simulations in Ref. (Jukna et al. 2014); and the quasi-stationary, rotary and speckle-like (chaotic) regimes experimentally in Ref. (Xie et al. 2015). The quasi-stationary regime was attributed to the formation of a supposedly stable nonlinear BVB. The unstable regimes are given more dubious explanations: Ref. (Jukna et al. 2014) relates the formation of a rotary regime either with an unstable nonlinear BVB or to the nonexistence of a stationary state in the specific experimental configuration. On the other hand, Ref. (Xie et al. 2015) associates both rotary and speckle-like regime with the effect of increasing “appearance of wave-mixing and modulation instability, which generates novel spatial frequencies in the beam”. None of these conjectures was demonstrated in these studies, in particular the supposedly stationary nonlinear BVB in the quasi-stationary regime was not identified or specified in terms of the parameters that univocally define them. Hence, its stability or instability character cannot be unmistakably elucidated.

4.2 An interpretation of the observed dynamic regimes

This section will see how the conclusions obtained in previous sections could apply to the dynamic regimes observed experimentally, establishing a common physical interpretation for all cases. In few words, our hypothesis is that in all three observed dynamic regimes after the axicon, a finite-power version of a nonlinear BVB acts as an attractor of the dynamics. We completely identify it by specifying the parameters that define a nonlinear BVB. In case that it is stable according to our linearized stability analysis in Ch. 3, the quasi-stationary regime inducing tubular filamentation is observed. On the contrary, in case that the attracting nonlinear BVB is unstable according to the linearized stability analysis, the rotary or speckle-like regimes are observed, depending on the strength of the instability and its further development into the large perturbation regime within the Bessel zone.

In (Porras, Ruiz-Jiménez, and Losada 2015; Ruiz-Jiménez 2016), it is demonstrated for the fundamental ($s = 0$) nonlinear BVB that the conservation law of the conical angle and the amplitude of the inward Hänkel component which determines the transformation of a fundamental linear Bessel beam into a nonlinear Bessel beam are also valid for finite-power experimental settings, either under abrupt or soft inputs configurations. For soft inputs, the nonlinear Bessel beam acting as an attractor is that which satisfies $|b_{\text{in}}| = b_{\text{B}}$, being b_{B} the amplitude of the linear beam that would form in free space propagation, and

with the same conical angle as the linear beam that would form in free space propagation. In this case, the nonlinear attractor Bessel beam forms itself gradually along the Bessel zone, decaying at the end of it due to the depletion of its energy supply. Yet, this criterion must now be adapted to higher-order ($s > 0$) BVBs, asserting that the nonlinear BVB that acts as an attractor in the Bessel zone is the one with the same cone angle, same vorticity s and with an amplitude $|b_{\text{in}}|$ equal to that of the linear Bessel beam that would emerge at the axicon focus in linear propagation. The stability properties of this underlying attractor would determine the dynamics in the Bessel zone, i.e., a stable attractor would cause the beam to smoothly develop into a stable regime about the center of the Bessel zone, where a finite-power version of the attracting stable nonlinear BVB is formed, and decay afterwards. On the contrary, an unstable underlying nonlinear BVB would cause the exponential growth of unstable modes intrinsic to the input beam that would finally end in an unstable regime in the Bessel zone.

Our numerical simulations of the nonlinear propagation according to the NLSE (3.1) starting with the initial condition in Eq. (4.2) created by the spiral phase plate and the axicon, corroborate the above hypothesis and allow to explain in a unified basis the different dynamic regimes of filamentation observed after the axicon, as detailed below.

Three relevant beam configurations were chosen. In all cases, the width of the Gaussian beam is $\rho_0 = 400$ and the topological charge imprinted by the experimental setup is $s = 1$. Their amplitudes are $b_G = 0.0402$ (left), 0.0341 (center) and 0.030 (right), the reason for the choice of these values to be justified soon enough. According to Eq. (4.3), the amplitudes of the linear Bessel beams that would form at the center of the Bessel zone if in free space propagation are, respectively, $b_B = 1.11, 0.94$ and 0.829 .

Instead, if a nonlinear medium is located immediately after the axicon (thus, a soft input), the nonlinear BVBs formed satisfy the condition $|b_{\text{in}}| = b_B$. In all three cases exposed before, the nonlinear medium chosen to be located after the axicon has a MPA factor $M = 4$, and the conical angle imprinted by the axicon and the Kerr nonlinearity are such that the parameter α has the values $\alpha = 1$ (left), $\alpha = 3$ (center), and $\alpha = 6$ (right).

To calculate in each case the nonlinear attractor BVB, an asymptotic analysis must be carried out, as explained in Sec. 2.3, in order to obtain a graph analogous to that in Fig. 2.4(c), relevant to the beam being analyzed – in the cases above, the nonlinear BVBs with $\alpha = \alpha_2 = 1, 3, 6$ and $\alpha_4 = 0$, and the other parameters as previously specified. These graphs allow to conclude that the peak intensity parameter b_0 determining the amplitude of the nonlinear attractor BVB is the same for the three cases, $b_0 = 1.2$. In fact, the values of b_G previously chosen are such that the three BVBs have an attractor with the same value of b_0 , for the sake of simplicity when comparing the test cases.

In brief, the three nonlinear attractor BVBs will be defined by the parameters $M = 4$, $s = 1$, $b_0 = 1.2$ (in all cases) and $\alpha = 1$ (left), $\alpha = 3$ (center) and $\alpha = 6$ (right).

Figure 4.2: Dynamics in the Bessel zone for three Gaussian beams carrying a vortex of order $s = 1$ and width $\rho_0 = 400$, propagating in a nonlinear medium with four-photon absorption. Their amplitudes are $b_G = 0.0402$ (left), 0.0341 (center) and 0.030 (right) so that, respectively, their equivalent linear Bessel amplitudes are $b_B = 1.11, 0.94$ and 0.829 . Also, the three beams have a nonlinear attractor BVB with amplitude parameter $b_0 = 1.2$, satisfying $|b_{\text{in}}| = b_B$ while conserving other relevant parameters (M , $s = 1$ and α in each case), as explained in the text. The cone angles and the Kerr nonlinearity of the medium are chosen such that $\alpha = 1$ (left), $\alpha = 3$ (center) and $\alpha = 6$ (right). Figures (a-f) present normalized intensity profiles at the propagation distances ζ indicated. (g-i) Individual peak intensities as a function of the propagation distance in the central part of the Bessel zone. Fig. 4.3 correctly predicts the behavior illustrated in all three cases.

The stability properties of the three nonlinear BVBs were successfully studied in Ch. 3, and Fig. 3.1(a) is again reproduced on the right in Fig. 4.3 because of its relevance to the understanding of the filamentation dynamics in each case. For easier reference, each nonlinear BVB is explicitly indicated with a red dot.

The nonlinear BVB that serves as an attractor in the first case ($\alpha = 1$) is clearly inside the stability region. A stable dynamic regime after the axicon should then be the expected outcome, featuring a quasi-stationary propagation in which the stable nonlinear BVB builds up until the center of the Bessel zone and decays in the second half. This prediction is confirmed by the result of the numerical simulation of the NLSE (3.1) with the initial condition in Eq. (4.2) created by the phase plate and the axicon illuminated by the Gaussian beam (with $\rho = 400$ and $b_G = 0.0402$) presented in Fig. 4.2(a-b). It can be seen that once the nonlinear BVB has been formed in the central region of the Bessel zone, i.e., about its focus, it propagates almost unchanged, resulting in a tubular filamentation regime.

Comparing this result with that reported in Refs. (Jukna et al. 2014) and (Xie et al. 2015), the quasi-stationary regime reported is completely analogous to the previous $\alpha = 1$ case. The decay in the second half of the Bessel zone is due to the depletion of the finite-power reservoir in the input Gaussian beam. Our proof of stability ensures that the power available in the input Gaussian beam is now the only constraint to achieve tubular-like propagation. If a specific nonlinear BVB is known to be stable, then by incrementing the power supply and enlarging the Bessel zone, e. g., enlarging the width ρ_0 of the input Gaussian beam, the tubular regime can be made to remain over arbitrarily long distances. Indeed, it can be seen in Fig. 4.2(g) how the peak intensity remains almost constant in the center of the Bessel zone.

On the contrary, the attracting nonlinear BVBs for cases $\alpha = 3$ and $\alpha = 6$ lie outside the stability region, that is, they are unstable attractors. According to Fig. 4.3, the nonlinear BVB with $\alpha = 3$ must initially break in one rotating splinter in the small perturbation regime, and according to Fig. 3.7(a) the single rotating splinter should experience a secondary breaking into two rotating splinters as a further development of instability in the large perturbation regime. Indeed in Figs. 4.2(c) and (d) we observe such a behavior in the Bessel zone after the axicon, which confirms that it is the nonlinear BVB with $s = 1$, $b_0 = 1.2$ and $\alpha = 3$ that acts as an attractor and thus its properties of

Figure 4.3: For $b_0 = 1.2$, $s = 1$, dimensionless growth rates that serve as a reference to predict the dynamic regimes of Fig. 4.2. Each value of α to be referred to is marked with a red dot. Note that $\alpha = 1$ is inside the stability zone, indicating a stable dynamic regime for the underlying BVB; $\alpha = 3$ is unstable with maximum gain on the $m = 1$ mode and $\alpha = 6$ is unstable with maximum gain on the $m = 2$ curve (slopes are almost constant).

stability determine the dynamics after the axicon. At last, for the nonlinear BVB with $\alpha = 6$, the first rupture must be into two rotating fragments, in good agreement with Fig. 4.2(e). This rotatory state soon evolves into a chaotic dynamic with random rotating fragments appearing and disappearing in the Bessel zone after the axicon, as seen in Fig. 4.2(f), and coinciding with the final behavior of the simulation in Fig. 3.7(a).

For the unstable cases, the similarity of the nonlinear BVBs with $\alpha = 3$ and $\alpha = 6$ occurs now, respectively, with the rotary regime and the speckle-like regime, reported in Refs. (Jukna et al. 2014) and (Xie et al. 2015). Both of them have their first rupture to agree with the results of the linear stability analysis gathered in Fig. 4.3, and then evolve to more complex dynamic regimes as shown in Figs. 4.2(d) and 4.2(f), always within the Bessel zone. In the former (rotary regime), the Bessel zone is dominated by rotating, symmetric fragments resulting from the manifestation of a less intense instability than the one present, and Fig. 4.2(h) shows how the peak intensity oscillates periodically. In the latter (speckle-like regime), the Bessel zone is dominated by completely asymmetric splinters that resemble the initial tubular structure no more, and Fig. 4.2(i) depicts how the peak intensity oscillates erratically, suggesting a filamentated beam with a disordered propagation.

The hypothesis made at the beginning of this section on the validity of the conservation laws for the prediction of how fundamental linear Bessel beams transform into nonlinear BVBs when propagating in Kerr self-focusing media are thus demonstrated to be also valid in the case of high-order nonlinear BVBs, as proven by the successful analysis of the experimental observations just presented.

For a more illustrative example, one of the specific cases reported in (Xie et al. 2015) was analyzed more thoroughly. Using the values given for the material constants, cone angles and three pulse energies, it was possible to determine the correspondent attracting nonlinear BVBs as being those with parameters $M = 5$, $s = 3$, $\alpha = 14.89$, and $b_0 = 0.368$, 0.822 and 1.644 . Performing the stability analysis gave, respectively, the following largest dimensionless growth rates for each case: 0.026 , 0.245 and 0.784 (or 2.48 cm^{-1} , 23.4 cm^{-1} and 74.9 cm^{-1} in physical units, after multiplying the dimensionless values by $|\delta|$). Taking into account that the Bessel zone has an approximate length of $\sim 0.072 \text{ cm}$ in this case, with the associated characteristic lengths of the instability development equal to 0.402 cm , 0.043 cm and 0.013 cm , the interpretation of the reported results is in line with the previous conclusions. The first case displays an instability still incipient as it does not have time to develop in the Bessel zone and be fully appreciated. The second case shows the small perturbation regime of the nonlinear BVB unstable mode with an azimuthal rupture into rotating filaments. Finally, a full-blown instability of the nonlinear BVB is depicted in the third case as the observation of random filaments propagating.

4.3 Summary

This chapter has seen the application of the theoretical model developed in Chs. 2 and 3 to experimental configurations of finite-power nonlinear BVBs generated by axicons. In particular, different experiments reported a variety of dynamic regimes that form in

the Bessel zone under a soft input configuration, such as a tubular geometry, a rotatory filamentation or a speckle-like, disordered filamentation regime.

Assuming the validity of the conservation laws devised for the fundamental nonlinear BVBs ($s = 0$) presented in (Porrás, Ruiz-Jiménez, and Losada 2015; Ruiz-Jiménez 2016), its conclusions were extrapolated successfully for the case of higher-order nonlinear BVBs ($s > 0$). Having identified the Gaussian beams used as input into the axicon-based experimental setups, it was possible to identify the nonlinear attractor BVBs in each case, and thus apply the results of the linearized stability analysis conducted in Ch. 3, relating the observed results with the expected ones. Indeed, the tubular geometry can be associated with a stable nonlinear attractor BVB, while the rotatory and speckle-like geometries are associated with unstable nonlinear attractor BVBs. The different results obtained in the Bessel zone can be explained in terms of the relation between the length of the Bessel zone and the characteristic length of the development of the instability, as reported in (Porrás et al. 2016; Carvalho, Ruiz-Jiménez, and Porrás 2017).

If the characteristic length of the development of the instability is greater than the length of the Bessel zone, then it will not have time to develop and no symmetry breakup will be observed, perhaps even misleading into the conclusion that the beam being propagated is stable if no stability analysis was carried out. As a second possibility, if the characteristic length of the development of the instability is equal (or very similar) to the length of the Bessel zone, then the first rupture due to the instability will be observed, leading to a rotatory (or helical) filamentation regime. The last possibility appears if the characteristic length of the development of the instability is significantly smaller than the length of the Bessel zone, then the instability may have time to achieve a seemingly chaotic behavior (in case it does not result in a stationary rotatory regime) and a speckle-like filamentation will be observed in the Bessel zone.

Hence, the validity of the linearized stability analysis as a prediction model is confirmed for axicon-generated real nonlinear BVBs, confirming the conjecture that there exists an underlying attractor governing the propagation, also for high-order Bessel vortex beams. This attractor must satisfy the conservation of the topological charge, of the cone angle, and must have an amplitude b_0 such that $|b_{\text{in}}| = b_{\text{B}}$.

CONCLUSIONS

This chapter presents the conclusions reached in this dissertation. These are gathered in the six points below, as follows:

- It has been demonstrated the existence of nonlinear Bessel vortex beams propagating in media with self-focusing Kerr nonlinearity that are completely stable against small perturbations; their existence is therefore true for most common optical media used in practical applications. Unlike other vortex solitons reported in scientific literature, the nonlinear Bessel vortex beams described in this work do not need specifically tailored nonlinearities to exist (such as saturated Kerr nonlinearity, among others), nor optical materials expressly built to sustain them. The only requirement needed is nonlinear absorption, which always exist in the form of multiphoton absorption at sufficiently high beam intensities.
- The method used to demonstrate the previously stated stability, a linearized stability analysis, also allowed the determination of a region of stability in terms of the parameters defining a nonlinear Bessel vortex beam, giving for the first time a criterion to predict the stability character of a given nonlinear Bessel vortex beam. Stability is found to exist at sufficiently small values of the Kerr nonlinearity, or sufficiently high values of the multiphoton absorption.
- The results of the linearized stability analysis were verified using direct numerical simulations of the propagation of the nonlinear Bessel vortex beams with a small added random noise. The numerical simulations were thus performed in order to allow any possible outcome during the propagation of the nonlinear BVBs being studied. The agreement with the predictions of the linearized stability analysis was complete. When stable, the noise fade and the nonlinear Bessel vortex beams propagate without any changes. When unstable, the nonlinear Bessel vortex beams break up in a number of fragments predicted by the linearized stability analysis.
- It has also been observed that, contrary to what happens with standard solitons, the first rupture, occurring under a small perturbation regime, does not determine further stages of the dynamics (already outside the small perturbation regime), where more complex phenomena are observed as a variety of dynamic regimes: further breakups into rotating, flashing fragments, which may be in stationary rotation or in a “chaotic” behavior.
- Stable nonlinear Bessel vortex beams were directly confirmed for values of the topological charge $s < 4$, and they are expected to exist at any arbitrary value of s ,

as no premise was made on it to the accomplishment of the results. The existence of stable nonlinear Bessel vortex beams is thus expected to be independent of the value of the topological charge s . This is the first model reporting this feature using the cubic Kerr nonlinearity as the only conservative (dispersive) nonlinearity.

- It was confirmed that the physical mechanism sustaining the stability is indeed the nonlinear absorption, since its elimination causes the stability to disappear, turning all nonlinear Bessel vortex beams into unstable beams.
- The conclusions obtained on the stability properties of nonlinear Bessel vortex beams were successfully applied to explain the observations of real experimental settings using finite-power, high intensity nonlinear Bessel vortex beams to produce beam filamentation. In the usual configuration for these laboratory setups, consisting of a Gaussian beam illuminating a spiral phase plate and an axicon, followed by the nonlinear medium, it was possible to explain all three dynamical regimes observed (stable, rotatory and chaotic) as manifestations of the stability properties of the nonlinear Bessel vortex beam acting as an attractor of the dynamics of the input beam.
- The hypothesis that the conservation law of the conical angle and the amplitude of the inward Hankel component determining the transformation of a fundamental linear Bessel beam into a nonlinear Bessel beam was proven to be valid also for high-order Bessel vortex beams through the analysis of the observations of real experimental settings mentioned in the previous conclusion.

Future work

After verifying the existence of nonlinear Bessel vortex beams which are stable against small perturbations, their stability can be tested further against stronger perturbations. Some advances have already been made and reported in ([García-Riquelme, Ramos, and Porras 2018](#)), which shows how robust can be a stable nonlinear Bessel vortex beams against such a large perturbation as displacing the vortex from its natural spot at the center of the beam. This behavior seems due to the conservation of the total topological charge during the propagation, which also allows engineering vortex couplers and decouplers as desired.

Another new line of research produced during the last two years was the case of stationary and stable propagation with controllable dissipation patterns, the so-called dissipatons, reported in ([Porras, Ruiz-Jiménez, and Carvalho 2017](#)). These new luminous objects, called dissipatons, arise as a natural generalization of the present work when not only one linear Bessel vortex beam is launched in the medium, but the superposition of several.

Finally, those nonlinear Bessel vortex beams with $s = 3$ analyzed in Sec. 3.3 showed a seemingly stationary, periodic and rotatory behavior until the end of the propagation (see Fig. 3.7), a feature not seen at lower values of the topological charge. This seemingly new

type of rotatory, but stationary in rotating frame, type of beam surely deserves further exploration to understand the mechanisms that support them.

APPENDIXES

Appendix A

Obtaining the ODE System for Nonlinear Stationary Solutions

This appendix presents the detailed deduction of the system of coupled ordinary differential equations for the amplitude, Eq. (2.9), and the phase, Eq. (2.10), to be solved in order to obtain the nonlinear stationary solutions to the NLSE (2.3). Introducing Eq. (2.8) into Eq. (2.3), it becomes

$$\begin{aligned}
\frac{\partial \tilde{A}}{\partial \zeta} &= \frac{\partial [\tilde{a}(\rho) e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta}]}{\partial \zeta} = \tilde{a}(\rho) e^{i\phi(\rho)} e^{is\varphi} i \operatorname{sgn}(\delta) e^{i \operatorname{sgn}(\delta) \zeta}, \\
\Delta_{\perp} \tilde{A} &= \partial_{\rho}^2 \tilde{A} + \left(\frac{1}{\rho} \right) \partial_{\rho} \tilde{A} + \left(\frac{1}{\rho^2} \right) \partial_{\varphi}^2 \tilde{A}, \\
\Delta_{\perp} \tilde{A} &= \frac{\partial}{\partial \rho} \left[e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left(e^{i\phi(\rho)} \frac{d\tilde{a}(\rho)}{d\rho} + i\tilde{a}(\rho) e^{i\phi(\rho)} \frac{d\phi(\rho)}{d\rho} \right) \right] + \\
&\quad \frac{1}{\rho} \left[e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left(e^{i\phi(\rho)} \frac{d\tilde{a}(\rho)}{d\rho} + i\tilde{a}(\rho) e^{i\phi(\rho)} \frac{d\phi(\rho)}{d\rho} \right) \right] + \\
&\quad \frac{1}{\rho^2} [i^2 s^2 \tilde{a}(\rho) e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta}] = \\
&\quad = (\dots) = \\
&= e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left[2i \frac{d\phi(\rho)}{d\rho} \frac{d\tilde{a}(\rho)}{d\rho} + \frac{d^2 \tilde{a}(\rho)}{d\rho^2} + i\tilde{a}(\rho) \frac{d^2 \phi(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}(\rho)}{d\rho} \right] + \\
&\quad e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left[-\tilde{a}(\rho) \left(\frac{d\phi(\rho)}{d\rho} \right)^2 - \frac{1}{\rho} i\tilde{a}(\rho) \frac{d\phi(\rho)}{d\rho} - \frac{1}{\rho^2} s^2 \tilde{a}(\rho) \right], \\
\tilde{f}(|\tilde{A}|^2) \tilde{A} &= \tilde{f}(\tilde{a}^2) \tilde{a} e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta}, \\
|\tilde{A}|^{2M-2} \tilde{A} &= \tilde{a}(\rho)^{2M-1} e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta}.
\end{aligned}$$

Substituting all previously calculated terms, it yields

$$\frac{\partial \tilde{A}}{\partial \zeta} = i\Delta_{\perp} \tilde{A} + i\tilde{f}(|\tilde{A}|^2) \tilde{A} - |\tilde{A}|^{2M-2} \tilde{A} \Leftrightarrow$$

$$\begin{aligned}
& \Leftrightarrow \tilde{a}(\rho) e^{i\phi(\rho)} e^{is\varphi} i \operatorname{sgn}(\delta) e^{i \operatorname{sgn}(\delta) \zeta} = \\
& i e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left[2i \frac{d\phi(\rho)}{d\rho} \frac{d\tilde{a}(\rho)}{d\rho} + \frac{d^2 \tilde{a}(\rho)}{d\rho^2} + i \tilde{a}(\rho) \frac{d^2 \phi(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}(\rho)}{d\rho} \right] + \\
& i e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \left[-\tilde{a}(\rho) \left(\frac{d\phi(\rho)}{d\rho} \right)^2 - \frac{1}{\rho} i \tilde{a}(\rho) \frac{d\phi(\rho)}{d\rho} - \frac{s^2}{\rho^2} \tilde{a}(\rho) \right] + \\
& i \tilde{f}(\tilde{a}^2) \tilde{a} e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} - \\
& \tilde{a}(\rho)^{2M-1} e^{i\phi(\rho)} e^{is\varphi} e^{i \operatorname{sgn}(\delta) \zeta} \\
& \Leftrightarrow \tilde{a}(\rho) i \operatorname{sgn}(\delta) = \\
& i \left[2i \frac{d\phi(\rho)}{d\rho} \frac{d\tilde{a}(\rho)}{d\rho} + \frac{d^2 \tilde{a}(\rho)}{d\rho^2} + i \tilde{a}(\rho) \frac{d^2 \phi(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}(\rho)}{d\rho} \right] + \\
& i \left[-\tilde{a}(\rho) \left(\frac{d\phi(\rho)}{d\rho} \right)^2 - \frac{1}{\rho} i \tilde{a}(\rho) \frac{d\phi(\rho)}{d\rho} - \frac{s^2}{\rho^2} \tilde{a}(\rho) \right] + \\
& i \tilde{f}(\tilde{a}^2) \tilde{a} - \\
& \tilde{a}(\rho)^{2M-1} \\
& \Leftrightarrow \tilde{a}(\rho) i \operatorname{sgn}(\delta) = \\
& i \left[\frac{d^2 \tilde{a}(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}(\rho)}{d\rho} - \tilde{a}(\rho) \left(\frac{d\phi(\rho)}{d\rho} \right)^2 - \frac{s^2}{\rho^2} \tilde{a}(\rho) + \tilde{f}(\tilde{a}^2) \tilde{a} \right] - \\
& \left[2 \frac{d\phi(\rho)}{d\rho} \frac{d\tilde{a}(\rho)}{d\rho} + \tilde{a}(\rho) \frac{d^2 \phi(\rho)}{d\rho^2} + \frac{1}{\rho} \tilde{a}(\rho) \frac{d\phi(\rho)}{d\rho} + \tilde{a}(\rho)^{2M-1} \right].
\end{aligned}$$

Separating real and imaginary parts will give out two coupled ordinary differential equations for the amplitude [Eq. (2.9)] and the phase [Eq. (2.10)], as follows:

$$\begin{aligned}
& \frac{d^2 \phi(\rho)}{d\rho^2} + \frac{2}{\tilde{a}(\rho)} \frac{d\phi(\rho)}{d\rho} \frac{d\tilde{a}(\rho)}{d\rho} + \frac{1}{\rho} \frac{d\phi(\rho)}{d\rho} + \tilde{a}(\rho)^{2M-2} = 0, \\
& \frac{d^2 \tilde{a}(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}(\rho)}{d\rho} - \left(\frac{d\phi(\rho)}{d\rho} \right)^2 \tilde{a}(\rho) + \tilde{f}(\tilde{a}^2) \tilde{a} - \operatorname{sgn}(\delta) \tilde{a}(\rho) - \frac{s^2}{\rho^2} \tilde{a}(\rho) = 0,
\end{aligned}$$

or, reorganizing the equations in their definitive form:

$$\begin{aligned}
& \frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} - \left(\frac{d\phi}{d\rho} \right)^2 \tilde{a} + \tilde{f}(\tilde{a}^2) \tilde{a} - \operatorname{sgn}(\delta) \tilde{a} - \frac{s^2}{\rho^2} \tilde{a} = 0, \\
& \frac{d^2 \phi}{d\rho^2} + \frac{1}{\rho} \frac{d\phi}{d\rho} + 2 \frac{d\phi}{d\rho} \frac{d\tilde{a}}{d\rho} \frac{1}{\tilde{a}} + \tilde{a}^{2M-2} = 0.
\end{aligned}$$

Appendix B

Some Mundane Algebra

Next, ancillary calculations that may be of interest to throw light on some minor analytical reasoning are presented under different sections of this dissertation.

B.1 Entries for Chapter 2

B.1.1 Solution to the coupled ordinary differential system of equations with boundary conditions

This entry details the steps to solve the coupled system of ordinary differential equations represented by Eqs. (2.9) and (2.10), reproduced below for easier reference:

$$\frac{d^2 \tilde{a}}{d\rho^2} + \frac{1}{\rho} \frac{d\tilde{a}}{d\rho} - \left(\frac{d\phi}{d\rho} \right)^2 \tilde{a} + \tilde{f}(\tilde{a}^2) \tilde{a} - \text{sgn}(\delta) \tilde{a} - \frac{s^2}{\rho^2} \tilde{a} = 0 , \quad (\text{B.1})$$

$$\frac{d^2 \phi}{d\rho^2} + \frac{1}{\rho} \frac{d\phi}{d\rho} + 2 \frac{d\phi}{d\rho} \frac{d\tilde{a}}{d\rho} \frac{1}{\tilde{a}} + \tilde{a}^{2M-2} = 0 . \quad (\text{B.2})$$

As discussed in Sec. 2.1.1, searching for physically significant solutions implies imposing the following boundary conditions, (B.3) and (B.4) for Eq. eqrefAMPcopy; and additionally conditions (B.5) and (B.6) for Eq. (B.2):

$$\tilde{a}(\rho) \rightarrow 0 \quad \text{for} \quad \rho \rightarrow \infty , \quad (\text{B.3})$$

$$\tilde{a}(\rho) \simeq b_0 J_s(\rho) = \frac{b_0}{2^{|s|} |s|!} \rho^{|s|} \equiv c(\rho) \rho^{|s|} \quad \text{for} \quad \rho \rightarrow 0 , \quad (\text{B.4})$$

$$\phi(0) = 0 , \quad (\text{B.5})$$

$$\left. \frac{d\phi}{d\rho} \right|_{\rho=0} = 0 . \quad (\text{B.6})$$

Using the variable change in Eq. (B.4), the problem can be reformulated as

$$\frac{d^2 c}{d\rho^2} + \frac{2s+1}{\rho} \frac{dc}{d\rho} - \left(\frac{d\phi}{d\rho} \right)^2 c + \tilde{f}(\rho^{2s} c^2) - \text{sgn}(\delta) c = 0 , \quad (\text{B.7})$$

$$\frac{d^2 \phi}{d\rho^2} + \frac{2s+1}{\rho} \frac{d\phi}{d\rho} + \frac{2}{c} \frac{d\phi}{d\rho} \frac{dc}{d\rho} + \rho^{s(2M-2)} c^{2M-2} = 0 , \quad (\text{B.8})$$

with the new set of boundary conditions

$$c(\rho)\rho^s \rightarrow 0 \quad \text{for } \rho \rightarrow \infty , \quad (\text{B.9})$$

$$c(\rho) = \frac{b_0}{2^{|s|}|s|!} \quad \text{for } \rho \rightarrow 0 , \quad (\text{B.10})$$

$$\left. \frac{dc}{d\rho} \right|_{\rho=0} = 0 , \quad (\text{B.11})$$

$$\phi(0) = 0 , \quad (\text{B.12})$$

$$\left. \frac{d\phi}{d\rho} \right|_{\rho=0} = 0 . \quad (\text{B.13})$$

This system must be solved numerically, and in this case it was done through the *ode45* function from the Matlab[®] computing environment, as indicated in Sec. D.2. In order to do so, a last step rewriting the system as a set of ordinary differential equations is needed, defining

$$Y_1 \equiv c \quad , \quad Y_2 \equiv \frac{dc}{d\rho} \quad , \quad Y_3 \equiv \phi \quad , \quad Y_4 \equiv \frac{d\phi}{d\rho} \quad , \quad (\text{B.14})$$

which finally gives

$$\frac{dY_1}{d\rho} = Y_2 , \quad (\text{B.15})$$

$$\frac{dY_2}{d\rho} = -\frac{2s+1}{\rho}Y_2 + Y_4^2 Y_1 + \text{sgn}(\delta)Y_1 - \tilde{f}(\rho^{2s} Y_1^2) , \quad (\text{B.16})$$

$$\frac{dY_3}{d\rho} = Y_4 , \quad (\text{B.17})$$

$$\frac{dY_4}{d\rho} = -\frac{2s+1}{\rho}Y_4 - 2\frac{Y_4 Y_2}{Y_1} - \rho^{s(2M-2)} Y_1^{2M-2} , \quad (\text{B.18})$$

with the respective boundary conditions

$$Y_1(0) = \frac{b_0}{2^{|s|}|s|!} , \quad (\text{B.19})$$

$$Y_2(0) = 0 , \quad (\text{B.20})$$

$$Y_3(0) = 0 , \quad (\text{B.21})$$

$$Y_4(0) = 0 . \quad (\text{B.22})$$

to be numerically solved through the specifically developed software detailed in App. D.

Circumventing the indetermination at $\rho = 0$

A small technical problem must be treated when solving the coupled system made up by Eqs. (B.7) and (B.8). Both equations have an indetermination at $\rho = 0$ that can be resolved using a second-order Taylor series at that point:

$$c(\rho) \simeq K + \left. \frac{dc}{d\rho} \right|_{\rho=0} \rho + \frac{1}{2} \left. \frac{d^2c}{d\rho^2} \right|_{\rho=0} \rho^2 \quad \text{for } \rho \rightarrow 0 . \quad (\text{B.23})$$

Since the first derivative of $c(\rho)$ is zero in virtue of Eq. (B.11), Eq. (B.23) can be in turn derived to obtain the relation

$$\frac{1}{\rho} \frac{dc}{d\rho} = \frac{d^2c}{d\rho^2} \quad \text{for } \rho \rightarrow 0. \quad (\text{B.24})$$

Now, when implementing the numerical resolution of the coupled system (B.7) and (B.8), which will be evaluated at steps of size h (a standard input parameter for many solvers, as the *ode45* function used in this case), a different system, already without the indetermination problem and with the same boundary conditions, must be used to calculate the solution in $[0, h]$:

$$(2s+2) \frac{d^2c}{d\rho^2} - \left(\frac{d\phi}{d\rho} \right)^2 c + \tilde{f}(\rho^{2s}c^3) - \text{sgn}(\delta)c = 0, \quad (\text{B.25})$$

$$(2s+2) \frac{d^2\phi}{d\rho^2} + \frac{2}{c} \frac{d\phi}{d\rho} \frac{dc}{d\rho} + \rho^{s(2M-2)} c^{2M-2} = 0. \quad (\text{B.26})$$

The output of the system (B.25) and (B.26) may then be used as the initial values for the original coupled system (B.7) and (B.8), thus avoiding the indetermination completely. The software implementation of the procedure just described is also included in the Matlab[®] code presented in Sec. D.2.

B.1.2 Analysis of the balance between inward and outward power fluxes from Equation (2.14) and the divergence theorem

From the continuity equation (2.13) obtained in Sec. 2.2, the following condition necessary to a stationary intensity pattern

$$-\nabla_{\perp} \cdot \mathbf{j} = \tilde{a}^{2M}, \quad (\text{B.27})$$

was derived, with $\nabla_{\perp} = \partial_r + (1/r)\partial_{\phi}$ the transversal gradient and $\mathbf{j}$ the transversal intensity current. It is equivalent to the differential equation on the angular variable (A.1) from the coupled system calculated at Eqs. (A) [Eq. (2.10) in the main text], and it gives a hint on the inner workings of the refilling mechanism that supports the stationarity of the nonlinear BVBs studied at this work.

Recalling one of the most fundamental theorems in Calculus, the divergence theorem (Abramowitz and Stegun 1972), it states in its two-dimensional version that the flux of a vectorial field F through a closed boundary C is equal to the surface integral of the divergence of the field F over the surface R inside that boundary. An analogous formulation can be made for a superior number of dimensions and tensorial fields, the divergence theorem being a special case of the Stokes' theorem. That is,

$$\int_C F \cdot \hat{n} \, ds = \iint_R \nabla \cdot F \, dA, \quad (\text{B.28})$$

with $\hat{n}$ a unitary vector perpendicular to the boundary C . Applying it to Eq. (B.27), comes

$$-\nabla_{\perp} \cdot \mathbf{j} = - \int_C \mathbf{j} \cdot \hat{n} \, ds = - \int_C (\tilde{a}^2 \nabla_{\perp} \Phi) \cdot \hat{n} \, ds.$$

Supposing a beam with a radially symmetric intensity profile $\tilde{a}^2$, phase $\Phi = \phi(\rho) + s\varphi - \zeta$, and a transversal intensity current $\mathbf{j} = \tilde{a}^2 [(d\phi/d\rho)\mathbf{u}_\rho + (s/\rho)\mathbf{u}_\varphi]$, and taking C as a circumference of radius ρ around the center of the beam, comes

$$\begin{aligned} - \int_C (\tilde{a}^2 \nabla_\perp \Phi) \cdot \hat{\mathbf{n}} \, ds &= - \int_C \tilde{a}^2 \left[\frac{d\phi}{d\rho} \mathbf{u}_\rho + \frac{s}{\rho} \mathbf{u}_\varphi \right] \cdot \hat{\mathbf{n}} \, ds = \\ &= - \int_0^{2\pi} \tilde{a}^2 \frac{d\phi}{d\rho} \rho \, d\varphi = -2\pi \tilde{a}^2 \frac{d\phi}{d\rho} \rho \end{aligned}$$

The last equation obtained is the left side of Eq. (2.12), and as explained at that point, represents an inward power flux coming from large radii and crossing the closed boundary C , here a circumference of radius ρ . It is thus ensured by the divergence theorem that this inward power flux through C must be equal to

$$\begin{aligned} - \int_C \mathbf{j} \cdot \hat{\mathbf{n}} \, ds &= - \iint_R \nabla \cdot \mathbf{j} \, dA = \\ &= - \int_0^{2\pi} \int_0^\rho (-\tilde{a}^{2M}) \rho \, d\rho \, d\varphi = 2\pi \int_0^\rho \tilde{a}^{2M} \rho \, d\rho, \end{aligned}$$

which is now the right side of Eq. (2.12), expressing the nonlinear losses inside the surface R , here a circle with radius ρ and, by the divergence theorem, equal to the surface integral of the divergence of the $\mathbf{j}$ over R .

B.1.3 Computation of the relation among b_{in} , b_{out} and $N(\infty)$

Equation (2.21) summarizes the deep relation existing in nonlinear Bessel vortex beams between Hankel inward and outward components and the nonlinear losses occurring at the center of the beam. It makes it easy to understand how the inward component offsets these losses, contributing to the beam stationarity and, ultimately, to the beam hypothetical stability (in addition to other parameters). To deduce this relation, it is useful to remember the following equality:

$$N(\infty) = 2\pi \int_0^\infty d\rho \rho |\tilde{A}|^{2M} = -F(\rho) = 2\pi \rho \operatorname{Im} \left[\tilde{A} \left(\frac{\partial \tilde{A}^*}{\partial \rho} \right) \right]. \quad (\text{B.29})$$

Taking into account the asymptotic expressions in Eq. (2.18)

$$\tilde{A}(\rho, \phi, \zeta) \simeq \frac{1}{2} [b_{\text{out}} H_s^{(1)}(\rho) + b_{\text{in}} H_s^{(2)}(\rho)] e^{is\varphi} e^{-i\zeta} \quad (\text{B.30})$$

and in Eq. (2.20)

$$H_s^{(1,2)}(\rho) \simeq \sqrt{\frac{2}{\pi\rho}} e^{\pm i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})}, \quad (\text{B.31})$$

comes

$$\begin{aligned} \tilde{A}^*(\rho, \phi, \zeta) &\simeq \frac{1}{2} [b_{\text{out}}^* H_s^{(1)*}(\rho) + b_{\text{in}}^* H_s^{(2)*}(\rho)] e^{-is\varphi} e^{i\zeta} = \\ &= \frac{1}{2} [b_{\text{out}}^* H_s^{(2)}(\rho) + b_{\text{in}}^* H_s^{(1)}(\rho)] e^{-is\varphi} e^{i\zeta}. \end{aligned} \quad (\text{B.32})$$

Then, knowing that the Hänkel functions satisfy

$$\begin{aligned} H_s^{(1)} H_s^{(2)} &= \frac{2}{\pi \rho} \quad ; \quad H_s^{(1)^2} = \frac{2}{\pi \rho} e^{2i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})} \quad ; \quad H_s^{(2)^2} = \frac{2}{\pi \rho} e^{-2i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})} \quad ; \\ \frac{\partial H_s^{(1)}}{\partial \rho} &= i H_s^{(1)} \quad ; \quad \frac{\partial H_s^{(2)}}{\partial \rho} = -i H_s^{(2)} \quad ; \end{aligned} \quad (\text{B.33})$$

from Eq. (B.32) it is possible to calculate

$$\begin{aligned} \tilde{A} \left(\frac{\partial \tilde{A}^*}{\partial \rho} \right) &= \frac{1}{4} [b_{\text{out}} H_s^{(1)} + b_{\text{in}} H_s^{(2)}] [-i b_{\text{out}}^* H_s^{(2)} + i b_{\text{in}}^* H_s^{(1)}] = \\ &= \frac{1}{4} \left[-i |b_{\text{out}}|^2 H_s^{(1)} H_s^{(2)} + i b_{\text{out}} b_{\text{in}}^* H_s^{(1)^2} - i b_{\text{in}} b_{\text{out}}^* H_s^{(2)^2} + i |b_{\text{in}}|^2 H_s^{(1)} H_s^{(2)} \right] = \\ &= \frac{i}{2\pi \rho} \left[|b_{\text{in}}|^2 - |b_{\text{out}}|^2 + b_{\text{out}} b_{\text{in}}^* e^{2i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})} - i b_{\text{in}} b_{\text{out}}^* e^{-2i(\rho - \frac{\pi}{2}s - \frac{\pi}{4})} \right]. \end{aligned} \quad (\text{B.34})$$

Taking the imaginary part of the last equation and using the relation $\cos(\beta) = \cos(-\beta)$,

$$\begin{aligned} \frac{1}{2\pi \rho} \left[|b_{\text{in}}|^2 - |b_{\text{out}}|^2 + \cos(\arg b_{\text{out}}) \cos(\arg b_{\text{in}}) \cos \left(2i \left(\rho - \frac{\pi}{2}s - \frac{\pi}{4} \right) \right) - \right. \\ \left. - \cos(-\arg b_{\text{out}}) \cos(-\arg b_{\text{in}}) \cos \left(-2i \left(\rho - \frac{\pi}{2}s - \frac{\pi}{4} \right) \right) \right] = \\ = \frac{1}{2\pi \rho} [|b_{\text{in}}|^2 - |b_{\text{out}}|^2]. \end{aligned} \quad (\text{B.35})$$

Finally, the desired relation is obtained, since

$$\begin{aligned} N(\infty) &= 2\pi \rho \operatorname{Im} \left[\tilde{A} \left(\frac{\partial \tilde{A}^*}{\partial \rho} \right) \right] = \\ &= 2\pi \rho \frac{1}{2\pi \rho} [|b_{\text{in}}|^2 - |b_{\text{out}}|^2] = |b_{\text{in}}|^2 - |b_{\text{out}}|^2 \end{aligned} \quad (\text{B.36})$$

is satisfied.

B.2 Entries for Chapter 3

B.2.1 Expressing the perturbation P in terms of cylindrical harmonics

During the linearized stability analysis presented in Sec. 3.1, a perturbation $P(\rho, \varphi, \zeta)$ is added to the stationary solution (3.2) of the NLSE (3.1) in order to evaluate the stability properties of that solution. It is known that Bessel functions satisfy Eq. (3.1), and so it is a good guess to use the same class of functions to express the perturbation $P(\rho, \varphi, \zeta)$, for the purpose of introducing a parameter m , the *winding number*. This parameter allows the identification of a single mode (stable or not), intrinsic to the nonlinear BVB, and

arrive at the eigenvalue problem that can be numerically solved to determine the stability character of the beam.

Cylindrical harmonics form a complete set of linearly independent solutions to the Laplace's equation $\nabla^2 F = 0$. They are composed of the product of three terms, each one dependent on one of the usual cylindrical coordinates. The term dependent on the radius ρ is expressed in terms of Bessel functions, making cylindrical harmonics a good candidate to build an expansion of the perturbation P , as they constitute an algebraic basis to our solution. ([Abramowitz and Stegun 1972](#))

A common formulation of a cylindrical harmonic F would be

$$F_m(\rho, \varphi, \zeta) = R_n(\rho) \Phi_n(\varphi) Z(\zeta) , \quad (\text{B.37})$$

where each function on the right side of the equation can be explicitly calculated from the application of separation of variables to Laplace's equation, originating the well-known results

$$Z(\zeta) = e^{\pm i|k|z} \quad , \quad k \text{ a complex constant} , \quad (\text{B.38})$$

$$\Phi_n(\varphi) = e^{\pm in\varphi} \quad , \quad n \text{ a non-negative integer} , \quad (\text{B.39})$$

and the most relevant in this case,

$$R_n(\rho) = J_n(k\rho) \quad , \quad k \text{ real constant, } J_n \text{ the ordinary Bessel function, or,} \quad (\text{B.40})$$

$$R_n(\rho) = I_n(|k|\rho) \quad , \quad k \text{ imaginary constant, } I_n \text{ the modified Bessel function.} \quad (\text{B.41})$$

Respectively, the ordinary and the modified Bessel functions $Y_n(k\rho)$ and $K_n(|k|\rho)$ also serve as solutions for real and imaginary values of the constant k . Now, for the spanning of the perturbation $P(\rho, \varphi, \zeta)$ in Eq. (3.8), functions $R_n(\rho)$ and $Z(\zeta)$ are kept in a single term, and the phase $\Phi_n(\varphi)$ is represented separately for convenience, and a series expansion in terms of cylindrical harmonics is used, finally resulting in

$$P(\rho, \varphi, \zeta) = \sum_{m=-\infty}^{+\infty} P_m(\rho, \zeta) e^{im\varphi} . \quad (\text{B.42})$$

Appendix C

Discretization Method

This appendix will lay out the discretization process of the eigenvalue problem obtained through the linear stability analysis carried out in Sec. 3.1. The following eigenvalue problem was then calculated:

$$\kappa_m \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix} = \begin{pmatrix} H_+ & f \\ -f^* & -H_-^* \end{pmatrix} \begin{pmatrix} \mu_m \\ \nu_m \end{pmatrix}, \text{ with } \begin{cases} H_{\pm} = \Delta_{\rho} - \frac{(S \pm m)^2}{\rho^2} + 1 + 2\alpha a^2 + iMa^{2(M-1)} \\ f = [\alpha a^2 + i(M-1)a^{2(M-1)}] e^{2i\phi} \end{cases}$$

with $\Delta_{\rho} \equiv d^2/d\rho^2 + (1/\rho)(d/d\rho)$.

A grid of $i = 1, \dots, N$ points shall be considered, together with the boundary conditions specified in Sec. 2.1, which materialize here under the form

$$\mu_m \underset{\rho \rightarrow 0}{\sim} \rho^{S+m}, \quad \nu_m \underset{\rho \rightarrow 0}{\sim} \rho^{|S-m|}. \quad (\text{C.1})$$

The discretization process will follow a standard finite element method, for the general expressions to be applied are

$$\left(\frac{du}{d\rho} \right)_i = \frac{u_{i+1} - u_{i-1}}{2h}, \quad \left(\frac{d^2u}{d\rho^2} \right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad (\text{C.2})$$

with $i \in]1, \dots, N[$, $u \equiv \mu, \nu$ and h is the interval between any two adjacent points i of the grid. For $i = 1, N$, the previous expressions must be adapted imposing $\mu_m = \nu_m = 0$ for $i = 0, N+1$ in order to provide existence and continuity for the derivative in those points, becoming

$$i = 1 : \left(\frac{du}{d\rho} \right)_1 = \frac{u_2 - u_0}{2h} = \frac{u_2}{2h}, \quad (\text{C.3})$$

$$\left(\frac{d^2u}{d\rho^2} \right)_1 = \frac{u_2 - 2u_1 + u_0}{h^2} = \frac{u_2 - 2u_1}{h^2}, \quad (\text{C.4})$$

$$i = N : \left(\frac{du}{d\rho} \right)_N = \frac{u_{N+1} - u_{N-1}}{2h} = -\frac{u_{N-1}}{2h}, \quad (\text{C.5})$$

$$\left(\frac{d^2u}{d\rho^2}\right)_N = \frac{u_{N+1} - 2u_N + u_{N-1}}{h^2} = \frac{-2u_N + u_{N-1}}{h^2}. \quad (\text{C.6})$$

Now, for each i , the previously calculated expressions must be applied to the eigenvalue problem, giving:

- $i \in]1, \dots, N[$

$$\begin{aligned} H_+\mu_i + f\nu_i &= \frac{\mu_{i+1} - 2\mu_i + \mu_{i-1}}{h^2} + \frac{1}{\rho_i} \frac{\mu_{i+1} - \mu_{i-1}}{2h} - \frac{(S+m)^2}{\rho_i^2} \mu_i + \\ &\quad \mu_i + 2\alpha_i a_i^2 \mu_i + iM a_i^{2(M-1)} \mu_i + \\ &\quad \left[\alpha_i a_i^2 + i(M-1)a_i^{2(M-1)} \right] e^{2i\phi_i} \nu_i, \\ -f^*\mu_i - H_-\nu_i &= - \left[\alpha_i a_i^2 - i(M-1)a_i^{2(M-1)} \right] e^{-2i\phi_i} \mu_i - \\ &\quad \frac{\nu_{i+1} - 2\nu_i + \nu_{i-1}}{h^2} - \frac{1}{\rho_i} \frac{\nu_{i+1} - \nu_{i-1}}{2h} + \frac{(S-m)^2}{\rho_i^2} \nu_i - \\ &\quad \nu_i - 2\alpha_i a_i^2 \nu_i - iM a_i^{2(M-1)} \nu_i \end{aligned}$$

- $i = 1$

$$\begin{aligned} H_+\mu_1 + f\nu_1 &= \frac{\mu_2 - 2\mu_1}{h^2} + \frac{1}{\rho_1} \frac{\mu_2}{2h} - \frac{(S+m)^2}{\rho_1^2} \mu_1 + \\ &\quad \mu_1 + 2\alpha_1 a_1^2 \mu_1 + iM a_1^{2(M-1)} \mu_1 + \\ &\quad \left[\alpha_1 a_1^2 + i(M-1)a_1^{2(M-1)} \right] e^{2i\phi_1} \nu_1, \\ -f^*\mu_1 - H_-\nu_1 &= - \left[\alpha_1 a_1^2 - i(M-1)a_1^{2(M-1)} \right] e^{-2i\phi_1} \mu_1 - \\ &\quad \frac{\nu_2 - 2\nu_1}{h^2} - \frac{1}{\rho_1} \frac{\nu_2}{2h} + \frac{(S-m)^2}{\rho_1^2} \nu_1 - \\ &\quad \nu_1 - 2\alpha_1 a_1^2 \nu_1 - iM a_1^{2(M-1)} \nu_1 \end{aligned}$$

- $i = N$

$$\begin{aligned} H_+\mu_N + f\nu_N &= \frac{-2\mu_N + \mu_{N-1}}{h^2} + \frac{1}{\rho_N} \frac{-\mu_{N-1}}{2h} - \frac{(S+m)^2}{\rho_N^2} \mu_N + \\ &\quad \mu_N + 2\alpha_N a_N^2 \mu_N + iM a_N^{2(M-1)} \mu_N + \\ &\quad \left[\alpha_N a_N^2 + i(M-1)a_N^{2(M-1)} \right] e^{2i\phi_N} \nu_N, \end{aligned}$$

$$\begin{aligned}
-f^* \mu_N - H_-^* \nu_N = & - \left[\alpha_N a_N^2 - i(M-1) a_N^{2(M-1)} \right] e^{-2i\phi_N} \mu_N - \\
& \frac{-2\nu_N + \nu_{N-1}}{h^2} - \frac{1}{\rho_N} \frac{-\nu_{N-1}}{2h} + \frac{(S-m)^2}{\rho_N^2} \nu_N - \\
& \nu_N - 2\alpha_N a_N^2 \nu_N - iM a_N^{2(M-1)} \nu_N
\end{aligned}$$

with $\rho_i = hi$, $a_i \equiv a(\rho_i)$ and $\phi_i \equiv \phi(\rho_i)$.

An algorithm to numerically solve the initial eigenvalue problem can be implemented from the previous expressions, applying them to a $2N \times 2N$ matrix. Blocks A and D are band matrixes, while blocks B and C are diagonal matrixes:

$$\left(\begin{array}{c|c} A_{k,j} & B_{k,j} \\ \hline C_{k,j} & D_{k,j} \end{array} \right),$$

with

$$\begin{aligned}
A &\equiv \begin{pmatrix} A_{1,1} & A_{1,2} & 0 & 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & A_{2,3} & 0 & 0 & \dots & 0 \\ 0 & A_{3,2} & A_{3,3} & A_{3,4} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & A_{N,N-1} & A_{N,N} \end{pmatrix}, \\
B &\equiv \begin{pmatrix} B_{1,N+1} & 0 & 0 & 0 & \dots & 0 \\ 0 & B_{2,N+2} & 0 & 0 & \dots & 0 \\ 0 & 0 & B_{3,N+3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & B_{N,2N} \end{pmatrix}, \\
C &\equiv \begin{pmatrix} C_{N+1,1} & 0 & 0 & 0 & \dots & 0 \\ 0 & C_{N+2,2} & 0 & 0 & \dots & 0 \\ 0 & 0 & C_{N+3,3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & C_{N,N} \end{pmatrix}, \\
D &\equiv \begin{pmatrix} D_{N+1,N+1} & D_{N+1,N+2} & 0 & 0 & 0 & \dots & 0 \\ D_{N+2,N+1} & D_{N+2,N+2} & D_{N+2,N+3} & 0 & 0 & \dots & 0 \\ 0 & D_{N+3,N+2} & D_{N+3,N+3} & D_{N+3,N+4} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & D_{2N,2N-1} & D_{2N,2N} \end{pmatrix},
\end{aligned}$$

where each point (k, j) of the previous matrix will be evaluated as follows:

1. If $k, j \in (i = 1, \dots, N)$, i.e., it is on the block A of the matrix, then it must be evaluated as

	$(i, i - 1)$	(i, i)	$(i, i + 1)$
$i = 1$	0	$-\frac{2}{h^2} - \frac{(S + m)^2}{\rho_1^2} + 1 + 2\alpha a_1^2 + i\alpha a_1^{2(M-1)} M$	$\frac{1}{h^2} + \frac{1}{\rho_1} \frac{1}{2h}$
$1 < i < N$	$\frac{1}{h^2} - \frac{1}{\rho_i} \frac{1}{2h}$	$-\frac{2}{h^2} - \frac{(S + m)^2}{\rho_i^2} + 1 + 2\alpha a_i^2 + i\alpha a_i^{2(M-1)} M$	$\frac{1}{h^2} + \frac{1}{\rho_i} \frac{1}{2h}$
$i = N$	$\frac{1}{h^2} - \frac{1}{\rho_N} \frac{1}{2h}$	$-\frac{2}{h^2} - \frac{(S + m)^2}{\rho_N^2} + 1 + 2\alpha a_N^2 + i\alpha a_N^{2(M-1)} M$	0

2. If $k \in (i = 1, \dots, N)$ and $j \in (i = N + 1, \dots, 2N)$, i.e., it is on the block B of the matrix, then it must be evaluated as

$$B_{i,i} = \left[\alpha a_i^2 + i(M - 1)a_i^{2(M-1)} \right] e^{2i\phi_i}$$

3. If $k \in (i = N + 1, \dots, 2N)$ and $j \in (i = 1, \dots, N)$, i.e., it is on the block C of the matrix, then it must be evaluated as

$$C_{i,i} = - \left[\alpha a_i^2 - i(M - 1)a_i^{2(M-1)} \right] e^{-2i\phi_i}$$

4. If $k, j \in (i = N + 1, \dots, 2N)$, i.e., it is on the block D of the matrix, then it must be evaluated as

	$(i, i - 1)$	(i, i)	$(i, i + 1)$
$i = N + 1$	0	$\frac{2}{h^2} + \frac{(S - m)^2}{\rho_{N+1}^2} + 1 + 2\alpha a_{N+1}^2 + i\alpha a_{N+1}^{2(M-1)} M$	$-\frac{1}{h^2} - \frac{1}{\rho_{N+1}} \frac{1}{2h}$
$N + 1 < i < 2N$	$-\frac{1}{h^2} - \frac{1}{\rho_i} \frac{1}{2h}$	$\frac{2}{h^2} - \frac{(S - m)^2}{\rho_i^2} - 1 - 2\alpha a_i^2 + i\alpha a_i^{2(M-1)} M$	$-\frac{1}{h^2} - \frac{1}{\rho_i} \frac{1}{2h}$
$i = 2N$	$-\frac{1}{h^2} + \frac{1}{\rho_{2N}} \frac{1}{2h}$	$\frac{2}{h^2} + \frac{(S + m)^2}{\rho_{2N}^2} - 1 - 2\alpha a_{2N}^2 + i\alpha a_{2N}^{2(M-1)} M$	0

The software with the Matlab[®] implementation of the discretization algorithm is presented in Sec. D. It was used to produce all the numerical simulations presented in this dissertation, as well as those in (Porrás et al. 2016).

Appendix D

Software Used

This appendix gathers all Matlab[®] code used to produce the results of the linearized stability analysis, numerical simulations and most relevant figures included in this dissertation.

D.1 Program for eigenvalues computation

This program calls a routine that computes the eigenvalues for a specific set of input parameters defining a nonlinear BVB, each of them possibly assuming a range of values. Exit data is saved on files, which was later manually analyzed to produce the gain curves collected in Figs. [3.1](#), [3.2](#) and [3.3](#).

Source code:

```
clear all;
format shortG;
%
% INPUT PARAMETERS
%
M=4; % M: MPA order
s=1; % s: topological charge
b0=1.2; % b0: peak intensity parameter
step=0.2; % step: discretization step
vec_mesh = [2000]; % Vector with values of N
vec_alpha = [0:0.1:6]; % Vector with values of alpha
% parameter
vec_m = [0:1:10]; % Vector with values of the
% winding mode m
beta = 1; % 1: absorptive medium ,
% 0: transparent medium
%
% PROGRAM START
%
vec_final_tam = length(vec_m) * length(vec_mesh) * ...
length(vec_alpha);
```

```

vec_final = zeros(vec_final_tam,4);
linea = 0;
%
nom_fich_sal = ['Resultados (M=',num2str(M),',s=',num2str(s), ...
               ',b0=',num2str(b0),',m=',num2str(vec_m(1)), ' a ', ...
               num2str(vec_m(end)),',alpha=',num2str(vec_alpha(1)), ...
               ' a ',num2str(vec_alpha(end)),',mesh=', ...
               num2str(vec_mesh(1)),').txt'];
FID_salida = fopen(nom_fich_sal,'wt');
%
for i=1:length(vec_mesh)

    mesh=vec_mesh(i);

    for k=1:length(vec_m)

        m=vec_m(k);

        for j=1:length(vec_alpha)

            alpha = vec_alpha(j);
            alpha_char = num2str(vec_alpha(j));

            mensaje=strcat(['Se procesará mesh=',num2str(mesh),', m=', ...
                           num2str(m),', alpha=',alpha_char,'].']);
            disp(mensaje)
            linea = linea + 1;
            vec_final(linea,1)=vec_mesh(i);
            vec_final(linea,2)=vec_m(k);
            vec_final(linea,3)=vec_alpha(j);

            min_eigenenv = ...
                fun_diagonalization_new_param(b0,M,s,m,alpha,mesh,step,beta);

            vec_final(linea,4) = min_eigenenv;

            disp('          Mesh          m          Alpha          DiagNuevo')
            disp(vec_final)

            fprintf(FID_salida,'%5.0f    %1.0f    %3.1f    %10.9e\n', ...
                    vec_final(linea,:))

        end
    end
end
end
end

```

```
ST = fclose(FID_salida);
```

D.2 Routine for eigenvalues computation

This routine contains the software implementation of the discretization method explained in App. C. It is called from the program previously presented in D.1, with a specific set of parameters that univocally identifies a nonlinear BVB and the way it is to be sampled in the discretization process.

Source code:

```
function min_eigenv = ...
    fun_diagonalization_new_param(b0,M,s,m,alpha,mesh,step,beta)

M1=2*(M-1);
c_0 = b0/((2^s)*factorial(s));
h = step;
A = (2*s) + 2;
rho_paso1 = [0 h];
rho_resto = [h:h:mesh*h];
B = (2*s)+1;
%
if s == m
    factor = 1.5
else
    factor = 2
end
%
funcion_paso1 = @(rho_paso1,Y) [Y(2);(-Y(1)- ...
    alpha*(rho_paso1^(2*s))*(Y(1)^2+Y(3)^2)*Y(1)+ ...
    beta*(rho_paso1^(s*M1))*Y(3)*(Y(1)^2+ ...
    Y(3)^2)^(M-1))/A;Y(4);(-Y(3)- ...
    alpha*(rho_paso1^(2*s))*(Y(1)^2+Y(3)^2)*Y(3)- ...
    beta*(rho_paso1^(s*M1))*Y(1)*(Y(1)^2+ ...
    Y(3)^2)^(M-1))/A];
funcion_resto = @(rho_resto,Z) [Z(2);(-B*Z(2)/rho_resto)-Z(1)- ...
    alpha*(rho_resto^(2*s))*(Z(1)^2+Z(3)^2)*Z(1)+ ...
    beta*(rho_resto^(s*M1))*Z(3)*(Z(1)^2+ ...
    Z(3)^2)^(M-1);Z(4);(-B*Z(4)/rho_resto)-Z(3)- ...
    alpha*rho_resto^(2*s)*(Z(1)^2+Z(3)^2)*Z(3)- ...
    beta*(rho_resto^(s*M1))*Z(1)*(Z(1)^2+Z(3)^2)^(M-1)];
cond_init_paso1 = [c_0 0 0 0];
%
% Computation of the first step:
```

```

%
options = odeset('reltol', 1e-20,'abstol',1e-25);
[rho_paso1_out,Y_out] = ode45(funcion_paso1,rho_paso1, ...
                             cond_init_paso1,options);

%
%   Computation of the next steps:
%
[rho_resto_out,Z_out] = ode45(funcion_resto,rho_resto, ...
                             Y_out(length(rho_paso1_out),:),options);

%
x = rho_resto_out;
c_out = Z_out;
a = (x.^s).*sqrt((c_out(:,1).^2+c_out(:,3).^2));
denom=(c_out(:,1).^2+c_out(:,3).^2);
phas = ((c_out(:,1).^2-c_out(:,3).^2)./denom) + ...
        ((1i*2*c_out(:,1).*c_out(:,3))./denom);
phasc = ((c_out(:,1).^2-c_out(:,3).^2)./denom) - ...
        ((1i*2*c_out(:,1).*c_out(:,3))./denom);

mesh2=2*mesh; % First two lines must be the same

z=sparse(mesh2,mesh2); % Sparse matrix, all zeros

for i=1:mesh % Elements different from zero
    if i==1
        z(1,1)=-2/h^2-(s+m)^2/x(1)^2+1+2*alpha*a(1)^2+1i*beta*M*a(1)^M1;
        z(1,2)=1/h^2+1/(x(1)*2*h);
        z(mesh+1,mesh+1)=factor/h^2+abs(s-m)^2/x(1)^2-1- ...
            2*alpha*a(1)^2+1i*beta*M*a(1)^M1;
        z(mesh+1,mesh+2)=-1/h^2-1/(x(1)*2*h);
    else
        z(i,i)=-2/h^2-(s+m)^2/x(i)^2+1+2*alpha*a(i)^2+ ...
            1i*beta*M*a(i)^M1;
        z(i,i-1)=1/h^2-1/(x(i)*2*h);
        z(i+mesh,i+mesh)=2/h^2+abs(s-m)^2/x(i)^2-1- ...
            2*alpha*a(i)^2+1i*beta*M*a(i)^M1;
        z(i+mesh,i-1+mesh)=-1/h^2+1/(x(i)*2*h);
        if i<mesh
            z(i,i+1)=1/h^2+1/(x(i)*2*h);
            z(i+mesh,i+1+mesh)=-1/h^2-1/(x(i)*2*h);
        end
    end
end
z(i,i+mesh)=(alpha*a(i)^2+1i*beta*(M-1)*a(i)^M1)*phas(i);
z(i+mesh,i)=(-alpha*a(i)^2+1i*beta*(M-1)*a(i)^M1)*phasc(i);
end

```



```

kappa=eigs(z,mesh2);          % Finds eigenvalues of mesh2
loc=imag(kappa)<0;
relevant=kappa(loc);
result=[real(relevant),imag(relevant)];
result=sortrows(result,-2);
tam_result = size(result);
tam_result = tam_result(1,1);
if tam_result == 0
    min_eigenvalue = 9999;
else
    min_eigenvalue = result(tam_result,2);
end

end

```

D.3 Program for the numerical verification

This program was used to numerically simulate the propagation of a specific nonlinear BVB up to propagation distance $\zeta = 400$, and produce the transverse radial intensity profiles presented in Figs. 3.5 to 3.10. It generates a file with intensity values for each step, as well as a maximum intensities file and auxiliary files with radial, x and y coordinates for plotting the output maps.

Source code:

```

close all
clear all

step=0.01;
alpha=0.8;          % Kerr coefficient
beta=1;             % Nonlinear absorption
M=4;                % Order of absorption

                    % beam properties
b0=1.2;
s=3;

%%%%
%%%%
%%%% Computation of the initial condition
%%%%
%%%%

mesh=500000;

```

```

M1=2*(M-1);

c_0 = b0/((2^s)*factorial(s));
h = step;
A = (2*s) + 2;
rho_paso1 = [0 h];
rho_resto = [h:h:mesh*h];
B = (2*s)+1;

funcion_paso1 = @(rho_paso1,Y) [Y(2);...
    (-Y(1)-alpha*(rho_paso1^(2*s))*(Y(1)^2+Y(3)^2)*Y(1)+ ...
    beta*(rho_paso1^(s*M1))*Y(3)*(Y(1)^2+Y(3)^2)^(M-1))/A;...
    Y(4);...
    (-Y(3)-alpha*(rho_paso1^(2*s))*(Y(1)^2+Y(3)^2)*Y(3)- ...
    beta*(rho_paso1^(s*M1))*Y(1)*(Y(1)^2+Y(3)^2)^(M-1))/A];
funcion_resto = @(rho_resto,Z) [Z(2);...
    (-B*Z(2)/rho_resto)-Z(1)-alpha*(rho_resto^(2*s))*(Z(1)^2+ ...
    Z(3)^2)*Z(1)+ ...
    beta*(rho_resto^(s*M1))*Z(3)*(Z(1)^2+Z(3)^2)^(M-1);...
    Z(4);...
    (-B*Z(4)/rho_resto)-Z(3)-alpha*(rho_resto^(2*s))*(Z(1)^2+ ...
    Z(3)^2)*Z(3)- ...
    beta*(rho_resto^(s*M1))*Z(1)*(Z(1)^2+Z(3)^2)^(M-1)];
cond_init_paso1 = [c_0 0 0 0];

%
%   Computation of the first step:
%
options = odeset('Stats','on','reltol', 1e-15,'abstol',1e-20)
[rho_paso1_out,Y_out]=ode45(funcion_paso1,rho_paso1, ...
    cond_init_paso1,options);

%
%   Computation of the following steps:
%
[rho_resto_out,Z_out] = ode45(funcion_resto,rho_resto, ...
    Y_out(length(rho_paso1_out),:),options);

rho_todo = [rho_paso1_out(1:end-1);rho_resto_out];
c_out = [Y_out(1:end-1,:);Z_out];
a = (rho_todo.^s).*sqrt((c_out(:,1).^2+c_out(:,3).^2));
denom=(c_out(:,1).^2+c_out(:,3).^2);
phas=(c_out(:,1) + 1i .* c_out(:,3)) ./ ...
    sqrt((c_out(:,1).^2+c_out(:,3).^2));

```

```

%%
%%
%% Propagation
%%
%%
% d_z A=i gamma (d_xx A+d_yy A)+i alpha |A|^2 A-beta |A|^{2M-2}A

                                % medium properties
gamma=1;                       % Diffraction
beta=1;                        % MPA

z_max=400;                      % Propagation distance
nz=40001;
hz=z_max/(nz-1);
v_z=[0:hz:z_max];

Ngrids=401;
grids=1:(nz-1)/(Ngrids-1):nz;
grids=round(grids);
zgrids=v_z(grids);

x_max=3.2*z_max;                % to avoid border effects in the center
nx=2^13;                       % #points FFT in X
hx=2*x_max/nx;
x=[-x_max:hx:x_max-hx];

p=nx/2;
w1=(2*pi/(nx*hx))*[0:(p-1)];
w2=(2*pi/(nx*hx))*[-p:-1];
k_x=[w1 w2];

y_max=x_max;
ny=nx;                          % #points FFT in Y
hy=2*y_max/ny;
y=[-y_max:hy:y_max-hy];

p=ny/2;
w1=(2*pi/(ny*hy))*[0:(p-1)];
w2=(2*pi/(ny*hy))*[-p:-1];
k_y=[w1 w2];

[KX,KY]=meshgrid(k_x,k_y);
[X,Y]=meshgrid(x,y);

XX=X(3*nx/8:5*nx/8,3*ny/8:5*ny/8);

```

```

YY=Y(3*nx/8:5*nx/8,3*ny/8:5*ny/8);

ampinterp=interp1(rho_todo,a,sqrt(X.^2 + Y.^2));
phasinterp=interp1(rho_todo,phas,sqrt(X.^2 + Y.^2));

% Initial condition - nonlinear BVB with random noise:
u=ampinterp.*phasinterp.*exp(1i*s*atan2(Y,X))...
    .*(1+0.05*(-1+2*rand(size(ampinterp))- ...
        1i+2*1i*rand(size(ampinterp))));

u0=u;

D=exp(hz*(1i*gamma*(-KX.^2-KY.^2))/2);

r_max= 3*z_max;
loc=(X.^2 + Y.^2)>= r_max^2;

mat_rho=sqrt(X(3*nx/8:5*nx/8,3*ny/8:5*ny/8).^2 + ...
    Y(3*nx/8:5*nx/8,3*ny/8:5*ny/8).^2);

tam_II = size(XX); tam_II = tam_II(1,2);
save(['s',num2str(s),'_b0',num2str(b0),'_alpha',num2str(alpha), ...
    '_II',num2str(tam_II),'_XX.dat'],'XX','-ascii');
save(['s',num2str(s),'_b0',num2str(b0),'_alpha',num2str(alpha), ...
    '_II',num2str(tam_II),'_YY.dat'],'YY','-ascii');
save(['s',num2str(s),'_b0',num2str(b0),'_alpha',num2str(alpha), ...
    '_II',num2str(tam_II),'_rho.dat'],'mat_rho','-ascii');

for z=v_z

u(loc)=u0(loc).*exp(-1i*z);    % no nonlinear effects
                                % far from center

if z == floor(z)
    I=u(3*nx/8:5*nx/8,3*ny/8:5*ny/8);

    II=abs(I).^2;

    save(['s',num2str(s),'_b0',num2str(b0),'_alpha', ...
        num2str(alpha),'_z',num2str(z),'_II',num2str(tam_II), ...
        '.dat'],'II','-ascii')

    mensaje=strcat('Inicio de FTT del paso:',num2str(z));
    disp(mensaje);
end

```

```
u=ifft2(D.*fft2(u));
u=u.*exp((1i*alpha*((abs(u)).^2)-beta*(abs(u)).^(2*M-2))*hz);
u=ifft2(D.*fft2(u));
I=abs(u).^2;
[Imax,ind] = max(I(:));
save(['MaxIntensity_s ',num2str(s),'_b0',num2str(b0), ...
      '_alpha',num2str(alpha),'.dat'],'Imax','-ascii','-append')

end

return
```


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