ON THE COMMUTATOR OF TWO 2-FORMS IN MINKOWSKI SPACE-TIME *

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Abstract

In the space-time of general relativity, 2-forms play several important roles. But their commutators are generally seen as elements related to their algebraic aspects, rather than to their geometric ones. Here a clear geometric meaning of the commutator of two 2-forms is given. To obtain it, some simple notions on the Minkowski tangent space are needed. The characteristic tetrad and the characteristic planes of two generic planes are introduced. For non generic planes (those that cut each other or coincide), the matrix of their intersections and of the intersections of their orthogonals is considered, and some of its properties analyzed. The orthogonal cut of a plane is defined and the planes that orthogonally cut two given planes are studied. These results allow one to relate very easily the invariant planes of the commutator of two 2-forms to the invariant planes of the 2-forms: the first ones are the single orthogonal cut of the last ones. Some applications are indicated.

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1 Introduction

Local Lorentz transformations (point dependent) are frequently used in general relativity, mainly to adapt observers (time-like directions) or reference systems (tetrads) to the particular data of each problem.

Customarily these transformations are considered as (fields of) second order tensors on the space-time, restricted by the orthogonality conditions [1]. Such a formalism (or representation) has some advantages, but carries also an excessive number of non strict variables [2], which usually masks the *intrinsic elements* of these transformations [3].

In the exponential domain, which suffices for the invoked applications, this difficulty is naturally avoided when one expresses the transformation Lorentz tensor as the exponential of the corresponding Lorentz algebra element, which is a 2-form of the space-time [4, 5]. But the practical and theoretical difficulties become evident in the product of transformations, where the Baker-Campbell-Hausdorff formula is necessarily involved [6].

It follows from the properties of the BCH product that, in order to *predict* which will be the intrinsic elements of the product transformation, we need to know the answer to another simpler problem: which is the relation between the intrinsic elements of two given 2-forms and the intrinsic elements of their commutator? [7]

The article solves this problem.

The linear structure of the space of 2-forms is such that the invariant planes [8] of the sum of two 2-forms depend, *not only* on their invariant planes, *but also* on their eigenvalues. It is striking, and important for us, that this does not happens for their Lie algebra structure as we shall show: *the invariant planes of the commutator of two 2-forms depend exclusively on their invariant planes* [9].

The commutator of the 2-forms thus defines an *internal binary operation* on the space of non space-like planes or, equivalently, on the space of pairs of orthogonal planes. What is the geometric meaning of this operation? [10]

In order to reach the answer (given in the theorem) we shall need to analyze the different reciprocal relations that a pair of planes may have in Minkowski space, a subject of interest in itself [11].

Our result may be of interest, i) for those which are concerned with Lorentz transformations, relative motions and relate questions, the commutator being related locally to the Thomas precession [12], ii) in the study of non linear theories of electromagnetic field, where the commutator appears as the first order (bilinear) 2-form correction to the sum of two electromagnetic fields, iii) in the obtaintion of exact or approximate explicit expressions for the BCH product [6], by obvious reasons.

The paper is organized as follows. The following five sections are devoted to the study of pairs of planes in Minkowski space. Section 2 considers the *generic* case, in which the planes or their orthogonal have no common elements; we show that there always exists a pair of orthogonal planes, the *characteristic planes*, which are *unique* [13]. In Section 3 the matrix $\mathcal{D}(P, Q)$ of the intersec-

tions of P , Q and their orthogonal planes is introduced. This matrix noticeably simplifies the study of the variety of reciprocal relations that two planes may present in Minkowski space. Equivalent definitions for two planes to cut each other orthogonally in euclidean space, give rise to inequivalent ones when the signature of the metric changes; Section 4 presents a good definition of this concept in Minkowski space, as well as the definition of the orthogonal cut of a plane (set of planes that cut orthogonally the given plane). In section 5 we give a characterization for two planes, P and Q , to cut each other orthogonally relating the *diagonal spaces* of the matrix $\mathcal{D}(P, Q)$ (subspaces generated by the elements of its two diagonals). In Section 6, the last one analyzing pairs of planes, we consider the orthogonal cut of a pair of planes and the notion of *single orthogonal cut*, of particular importance in the present context, is introduced.

The last four sections are devoted to relating the preceding results to the commutator of 2-forms. In Section 7, basic notions about 2-forms needed in the paper are reviewed and a lemma concerning a pair of 2-forms is shown. The necessary and sufficient conditions for two 2-forms to commute are given in Section 8 and the relations between the invariant scalars of two 2-forms and of their commutator are derived in Section 9. Finally, Section 10 presents the main result: the invariant spaces of the commutator are the *single orthogonal cut* of the invariant spaces of the factor 2-forms.

2 Characteristic planes

Let (V_4, g) be the Minkowski space with signature -2 . To every plane P , the metric g biunivocally associates its orthogonal $P^\perp$, so that for every non ordered pair of planes (P, Q) we have the *associated set* $\mathcal{A}(P, Q) \equiv \{(P, Q), (P, Q^\perp), (P^\perp, Q), (P^\perp, Q^\perp)\}$. Two planes P and Q *cut* each other if they have a unique common direction, called the *cut direction*. Thus, a pair (P, Q) selects a finite number of privileged directions (the cut ones) whenever the planes of the elements of the associated set $\mathcal{A}(P, Q)$ cut each other. Does (P, Q) select a finite number of privileged directions when no planes of the elements of $\mathcal{A}(P, Q)$ cut each other? We shall see, surprisingly, that the answer is *always* positive.

Denote by a and b two directions of V_4 and by $a + b$ the plane generated by them, and consider the following definition.

Definition 1 *Let P and Q be two planes for which the planes of the elements of $\mathcal{A}(P, Q)$ have no common direction. A tetrad of directions $\{a, b, a', b'\}$ satisfying $P = a + b$, $Q = a' + b'$ and the orthogonality relations $a \perp b \perp a' \perp b' \perp a$ is called a characteristic tetrad of P and Q .*

We have the following result:

Proposition 1 *The characteristic tetrad exists and is unique.*

Proof: Let P be any plane and Q a non null one. First let us remark that the kernel of the orthogonal projection on Q is just $Q^\perp$ so the kernel of the

restriction of this projection to P is $P \cap Q^\perp$; as this intersection is zero in this case, the range of this application is two and every pair of different directions of P will have different orthogonal projections on Q . The pairs of orthogonal directions in P form a one-parameter set; their orthogonal projections in Q are in general non orthogonal. A straightforward calculation shows that the equation imposing that these projections be orthogonal gives the parameter and uniquely determines the pair. It is clear that the tetrad of directions so obtained is characteristic for the planes P and Q and, by construction, unique.

Now let P and Q be a pair of null planes with i and i' their corresponding null direction. It is obvious that the tetrad of directions $\{i, (i+i')^\perp \cap P, i', (i+i')^\perp \cap Q\}$ is characteristic for the planes P and Q ; the uniqueness is a consequence of the following: the characteristic directions in each plane must be orthogonal, so that, as the planes are null, the null directions must belong to the characteristic tetrad.

Let us remark that the planes $a + a'$ and $b + b'$ of the characteristic tetrad of P and Q are orthogonal non null planes and that they do not depend on the pair of planes of the associated set $\mathcal{A}(P, Q)$ used for their obtaintion.

Definition 2 *Let P and Q be two planes and $\{a, b, a', b'\}$ their characteristic tetrad; the two orthogonal planes $K \equiv a + a'$ and $K^\perp \equiv b + b'$ will be called the characteristic planes of P and Q .*

3 The matrix $\mathcal{D}(P, Q)$

In order to facilitate the analysis of the cases where the planes of some elements of the associated set $\mathcal{A}(P, Q)$ have non zero intersection, it is convenient to consider the set of 2×2 matrices $\mathcal{D}(P, Q)$

$$\mathcal{D}(P, Q) \equiv \begin{pmatrix} P \cap Q & P \cap Q^\perp \\ P^\perp \cap Q & P^\perp \cap Q^\perp \end{pmatrix} \quad (1)$$

that will be denoted by $\mathcal{D}$:

$$\mathcal{D} \equiv \{\mathcal{D}(P, Q) \mid \forall P, Q \text{ planes of } (V_4, g)\}.$$

We have the following proposition:

Proposition 2 *i) The four components of the matrix $\mathcal{D}(P, Q)$ are mutually orthogonal; ii) The two components of each diagonal of the matrix $\mathcal{D}(P, Q)$ have the same dimension.*

Proof: Point i) is evident from the definition. Point ii) can be shown as follows. If $\dim(P \cap Q) = 1$, as $P \cap Q$ is orthogonal to both $P^\perp$ and $Q^\perp$, they belong to the same hyperplane and consequently $\dim(P^\perp \cap Q^\perp) \neq 0$; this value cannot be 2 because then $P = Q$; thus it is 1. As this argument is invertible, they have the same dimension in this case. A similar argument yields the same

conclusions when the dimension is 2 and, by exclusion, the result is also true for the value 0.

From the definition of $\mathcal{A}(P, Q)$, for non ordered pairs (P, Q) , it follows:

$$\mathcal{A}(P, Q) = \mathcal{A}(Q, P) \quad (2)$$

and

$$\mathcal{A}(P, Q) = \mathcal{A}(P^\perp, Q) = \mathcal{A}(P, Q^\perp) = \mathcal{A}(P^\perp, Q^\perp). \quad (3)$$

Identity (2) leads to consider $\mathcal{D}(P, Q)$ and $\mathcal{D}(Q, P)$ as equivalent. From (2) one has

$$\mathcal{D}(Q, P) = \mathcal{D}^t(P, Q)$$

where “t” stands for the *transposed* matrix, so that we have to consider the set $\mathcal{D}/S_2$ of the equivalent classes of $\mathcal{D}$ by the permutation group $S_2 \equiv \{\text{Identity}, t\}$, finite subgroup of transformations of $\mathcal{D}$ (bijections of $\mathcal{D}$).

Identity (3) leads to consider $\mathcal{D}(P, Q)$, $\mathcal{D}(P, Q^\perp)$, $\mathcal{D}(P^\perp, Q)$ and $\mathcal{D}(P^\perp, Q^\perp)$ as equivalent; denoting by $\mathcal{D}^c(P, Q)$ and $\mathcal{D}^r(P, Q)$, respectively, the matrices obtained from the matrix $\mathcal{D}(P, Q)$ by permutation of the *columns* and of the *rows*, one has:

$$\begin{aligned} \mathcal{D}(P, Q^\perp) &= \mathcal{D}^c(P, Q) \\ \mathcal{D}(P^\perp, Q) &= \mathcal{D}^r(P, Q) \\ \mathcal{D}(P^\perp, Q^\perp) &= \mathcal{D}^{cr}(P, Q). \end{aligned}$$

The fourth order subgroup of transformations of $\mathcal{D}$ $\{\text{Identity}, c, r, cr\}$, generated by “c” and “r” is not cyclic and thus it is isomorphic to the dihedral group D_2 . So we have to consider the set $\mathcal{D}/D_4$ of the equivalent classes of $\mathcal{D}$ by the dihedral eighth order group D_4 generated by D_2 and S_2 .

In this work we are not interested in the particular subspace of each component of the elements of $\mathcal{D}$, but in its dimension and causal character. Matrices $\mathcal{D}(P, Q)$ and $\mathcal{D}(P', Q')$, for planes P, Q, P' and Q' , will be equivalent, if there exists an element $l \in \mathcal{L}$, the Lorentz group, such that $\mathcal{D}(P, Q) = \mathcal{D}(l(P'), l(Q'))$. So we are concerned with quotient space of $\mathcal{D}$ yielded by the subgroup of transformations of $\mathcal{D}$ generated by $D_4 \cup \mathcal{L}$. About this group we the following two results.

Lemma 1 *The subgroups D_4 and $\mathcal{L}$ of transformations of $\mathcal{D}$ commute and have the identity as the sole common element:*

$$\forall d \in D_4, \forall l \in \mathcal{L}, dl = ld \quad \text{and} \quad D_4 \cap \mathcal{L} = \{\text{Identity}\}.$$

Proof: From their definition it is evident that the elements of D_4 and $\mathcal{L}$ commute. The elements of D_4 change the position of the components of each matrix $\mathcal{D}(P, Q)$; as there are matrices whose elements are not all of the same causal character, these transformations cannot belong to $\mathcal{L}$.

It is important to stress here that this result is also valid in euclidean space. Nevertheless, because the notion of orthogonality is different, the proof in this case is not so easy.

Taking into account lemma 1 and the fact that S_2 and D_2 are subgroups of D_4 one has:

Corollary 1 *The subgroups of transformations of $\mathcal{D}$ generated by $S_2 \cup \mathcal{L}$, $D_2 \cup \mathcal{L}$ and $D_4 \cup \mathcal{L}$ are, respectively, $S_2 \times \mathcal{L}$, $D_2 \times \mathcal{L}$ and $D_4 \times \mathcal{L}$, where $\times$ stands for the direct product of groups.*

Making the changes induced by the corresponding group, this result is also valid for the euclidean space.

From proposition 2, the dimensions of the components of the matrices of $\mathcal{D}$ are determined by the dimension of one component of each diagonal. So the elements of $\mathcal{D}/(D_4 \times \mathcal{L})$ will be denoted by m_{ij}^k , $2 \geq i \geq j \geq 0$ and $3 \geq k \geq 0$, where i and j are the dimensions of one component of each diagonal and k is the dimension of the null subspace generated by its components. Abbreviating $m_{ii}^k \equiv m_i^k$ and $m_A \equiv m_A^k$ when there is only one class for subscript A , we have the following proposition.

Proposition 3 *In Minkowski space, the matrix $\mathcal{D}(P, Q)$ associated by (1) to a pair of planes P and Q is, modulo the group $D_4 \times \mathcal{L}$, necessarily of one of the following eight classes:*

$$\begin{aligned}
\left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right] &\equiv m_0, \\
\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right] &\equiv m_{10}^0, \quad \left[\begin{pmatrix} i & 0 \\ 0 & p \end{pmatrix} \right] \equiv m_{10}^2, \\
\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] &\equiv m_1^0, \quad \left[\begin{pmatrix} i & i \\ p & q \end{pmatrix} \right] \equiv m_1^3, \quad \left[\begin{pmatrix} i & i \\ i & i \end{pmatrix} \right] \equiv m_1^1, \\
\left[\begin{pmatrix} R & 0 \\ 0 & R^\perp \end{pmatrix} \right] &\equiv m_{20}, \\
\left[\begin{pmatrix} I & i \\ i & I^\perp \end{pmatrix} \right] &\equiv m_{21},
\end{aligned} \tag{4}$$

where a, b, c, d, p and q are non null directions, i is a null one, and R and I are respectively a non null plane and a null one.

Proof: It is sufficient to take $P = Q = I$, where I is a null plane and i its null direction for $\mathcal{D}(P, Q) \in m_{21}$; $P = Q = R$ where R is a non null plane for $\mathcal{D}(P, Q) \in m_{20}$; $P = a + b$ and $Q = a + c$ with $\{a, b, c, d\}$ an orthogonal tetrad for $\mathcal{D}(P, Q) \in m_1^0$; $P = i + i'$ and $Q = i + p$ where $\{i, i', p, q\}$ are the four directions of a null tetrad for $\mathcal{D}(P, Q) \in m_1^3$; $P = I$ and $Q = I'$ where I and I' are null planes such that $I \cap I' = i$ for $\mathcal{D}(P, Q) \in m_1^1$; $P = a + c$ and $Q = a + c'$ with $\{a, b, c\}$ and $\{a, b, c'\}$ are two orthogonal different triad for $\mathcal{D}(P, Q) \in m_{10}^0$; $P = i + i'$ and $Q = i + i''$ where $\{i, i', i''\}$ are three independent null directions, p is the space-like direction which is orthogonal to these three and $P^\perp = p + p'$, $Q^\perp = p + p''$, where $\{p, p', p''\}$ are three independent space-like directions orthogonal to i for $\mathcal{D}(P, Q) \in m_{10}^2$. The existence of type m_0

is deduced from proposition 1 as follows. If we take two orthogonal non null planes K and $K^\perp$ and select two non orthogonal directions a and a' of K and two non orthogonal directions b and b' of $K^\perp$, then the planes $P = a + b$ and $Q = a' + b'$ are such that $\mathcal{D}(P, Q) \in m_0$. For $c \equiv K \cap Q^\perp$ and $d \equiv K^\perp \cap Q^\perp$ being one-dimensional subspaces from proposition 2 and, a and c as well as b and d being different by construction, the directions a , b , c and d are independent so that P and $Q^\perp$ do not cut each other.

The orthogonal and dimensional conditions of proposition 2 are necessary but not sufficient to characterize the matrices of $\mathcal{D}$. Those conditions give, in addition to the classes (4), the following classes:

$$\left[\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right] \equiv m_{10}^1, \quad \left[\begin{pmatrix} I & 0 \\ 0 & I^\perp \end{pmatrix} \right] \equiv m_{20}^3,$$

$$\left[\begin{pmatrix} i & i \\ i & p \end{pmatrix} \right] \equiv m_1^2, \quad \left[\begin{pmatrix} p & i \\ i & q \end{pmatrix} \right] \equiv m_1'^3.$$

The reason why these classes are not possible in Minkowski space is easy to see: all of them have a diagonal of null subspaces of null direction i , so that this direction belongs to all the planes and consequently it would appear in the other diagonal.

Proposition 4 *The components of the matrix $\mathcal{D}(P, Q)$ determine the planes P and Q as indicated in Table 1, where $\{a, b, c, d\}$ is a tetrad of orthogonal directions, $\{a, b, c'\}$ and $\{a, b, c''\}$ are different orthogonal triads, $\{i, i', i''\}$ is a triad of null directions and, p and p' are different and non orthogonal directions which are orthogonal to the null directions of each case.*

Proof: Taking into account (1) our statement is obviously true for m_0 , m_1^0 , m_1^1 , m_{20} and m_{21} . For m_{10}^0 , as the cut direction a is not null, one can take $c' \in P$ and $c'' \in Q$ such that $c' \perp a \perp c''$; we also need to impose $c' \not\perp c''$, otherwise $c' \in Q^\perp$ and $c'' \in P^\perp$ so that $\mathcal{D}(P, Q) \in m_1^0$. For m_{10}^2 and m_1^3 the planes P and Q must be either null or time-like planes. If P and Q are null planes then $P \cap Q$ null implies that all the other components of $\mathcal{D}(P, Q)$ are also null directions, so that P and Q are not both null planes. If P is time-like and Q is null then $P \cap Q$ null implies that $P \cap Q^\perp$ is null, so that $\mathcal{D}(P, Q) \in m_1^3$ determines pairs of planes, one null and the other time-like, that cut each other along a null direction. If P and Q are time-like planes then $P \cap Q$ null implies that $P \cap Q^\perp$ is a space-like direction or the zero subspace; from proposition 2 and the fact that $P^\perp$ is space-like, the former case yields a matrix $\mathcal{D}(P, Q)$ with three space-like orthogonal directions which are orthogonal to a null direction; a situation that is impossible in Minkowski space, so that $\mathcal{D}(P, Q) \in m_{10}^2$ determines pairs of time-like planes that cut each other along a null direction.

m_0	m_{10}^0	m_{10}^2	m_1^0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & p \end{pmatrix}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
Undetermined	$\begin{aligned} P &= a + c' \\ Q &= a + c'' \end{aligned}$	$\begin{aligned} P &= i + i' \\ Q &= i + i'' \end{aligned}$	$\begin{aligned} P &= a + b \\ Q &= a + c \end{aligned}$

m_1^3	m_1^1	m_{20}	m_{21}
$\begin{pmatrix} i & i \\ p & q \end{pmatrix}$	$\begin{pmatrix} i & i \\ i & i \end{pmatrix}$	$\begin{pmatrix} R & 0 \\ 0 & R^\perp \end{pmatrix}$	$\begin{pmatrix} I & i \\ i & I^\perp \end{pmatrix}$
$\begin{aligned} P &= i + i' \\ Q &= i + p \end{aligned}$	$\begin{aligned} P &= i + p \\ Q &= i + p' \end{aligned}$	$\begin{aligned} P &= R \\ Q &= R \end{aligned}$	$\begin{aligned} P &= I \\ Q &= I \end{aligned}$

Table 1: Planes determined by $\mathcal{D}(P, Q)$.

4 Orthogonal cut of a plane

In this section we are interested in the set $C_\perp(P)$ of all planes that orthogonally cut the plane P ; we shall call $C_\perp(P)$ the *orthogonal cut* of P . As this notion admits different characterizations, equivalent in the familiar euclidean case but non equivalent for other signatures, we shall provide the following definition.

Definition 3 *In Minkowski space, we shall say that two planes P and Q cut each other orthogonally if they cut each other and there are directions, in P and in Q orthogonal to the cut direction that are orthogonal and different, or if they are null and coincide.*

For an euclidean metric, the first part of this definition resembles the usual one, except for the fact that the condition for the directions in P and Q to be different is redundant: as we know, the cut direction and these two directions generate the three-dimensional space $P + Q$. In Minkowski space this last three-dimensional property is too strong because in some cases it forbids the existence of planes that orthogonally cut a given pair; nevertheless the direct extension of the usual euclidean definition allows those three directions to coincide, giving rise to an excess of multiplicity. Finally, the last part of the definition has been included in order to preserve the universal property that if two planes cut each

other orthogonally, the planes of any element of the associated set $\mathcal{A}(P, Q)$ do as well.

Let us remark that m_0 and m_{20} with m_{21} correspond respectively to the generic and to the degenerated situations and that the other classes correspond to pairs of planes (P, Q) such that the planes of some element of the associated set $\mathcal{A}(P, Q)$ cut each other. In the following result we give the classes that correspond to planes that cut each other orthogonally.

Proposition 5 *In Minkowski space two planes cut each other orthogonally if, and only if, the corresponding matrix class $\mathcal{D}/(D_4 \times \mathcal{L})$ belongs to m_{21} , m_1^0 or m_1^3 .*

Proof: From definition 3, two planes that cut each other orthogonally, cut each other or, if null, they coincide. This eliminates m_0 and m_{20} and shows that, for m_{21} , the pairs (P, Q) and $(P^\perp, Q^\perp)$ with $P = Q = I$ orthogonally cut each other. In this last case the space-like direction p in P and q in $Q^\perp$ are different and orthogonal so that $(P, Q^\perp)$ and $(P^\perp, Q)$ cut each other orthogonally.

Consider now a cut direction a and its orthogonal p in P and q in Q : according to definition 3 both directions p and q have to be different, so that at least one of them is different from the cut direction a , say p . Thus $p \subset P$ is orthogonal to Q , $p \subset Q^\perp$, and consequently $P \cap Q^\perp$ cannot vanish: matrices of $\mathcal{D}$ corresponding to pairs of planes that cut orthogonally each other cannot have vanishing components, i.e. if $\mathcal{D}(P, Q)$ belongs to m_{10}^0 or m_{10}^2 , P and Q do not cut each other orthogonally. That neither (P, Q) verifies this condition when $\mathcal{D}(P, Q)$ belongs to m_1^1 is easy to see: m_1^1 is obtained by two non orthogonal null planes of same null direction, so that it is impossible to find directions in each of them that are orthogonal to the other, except for the null one.

For m_1^0 , if $P \cap Q = a$, $P \cap Q^\perp = b$, $P^\perp \cap Q = c$ and $P^\perp \cap Q^\perp = d$, then it is clear that b and c are the directions in P and Q (*resp.* in $Q^\perp$ and $P^\perp$) that are orthogonal to the cut direction a (*resp.* d), and that a and d are the directions in P and $Q^\perp$ (*resp.* in Q and $P^\perp$) that are orthogonal to the cut direction b (*resp.* c); in both cases b and c , as well as a and d , are orthogonal and different according to definition 3.

Now, if $a = b = i$, $c = p$ and $d = q$, we have the same situation for m_1^3 , showing thus that both, m_1^0 and m_1^3 , together with the above m_{21} are those classes corresponding to pairs of planes that cut each other orthogonally.

As it is implicit in the statement of proposition 5 and explicit in its proof, one has the following property:

Corollary 2 *In Minkowski space (like in the euclidean case) if two planes cut each other orthogonally, all the other elements of their associated set do as well.*

From corollary 2, $M \in C_\perp(P)$ implies that M contains a direction of P and another one of $P^\perp$. When P is non null this is a necessary and sufficient condition for M to belong to $C_\perp(P)$. When P is null this is also true for the non null planes belonging to $C_\perp(P)$; but for null planes definition 3 imposes

$M = P$ or $M = P^\perp$. So, we have:

$$C_\perp(P) = \begin{cases} \{a + b \mid a \subset P, b \subset P^\perp\} & \text{if } P \text{ is non null} \\ \{P, P^\perp, i + x, (i + x)^\perp \mid i = P \cap P^\perp, i \not\subset x\} & \text{if } P \text{ is null} \end{cases} \quad (5)$$

where a , b and x are directions.

5 Diagonal planes

Another characterization of the pairs of planes that cut each other orthogonally may be obtained in the following way. Let us call *diagonal subspaces* the traces of the matrices $\mathcal{D}(P, Q)$ and $\mathcal{D}^c(P, Q)$ as well as its orthogonal. They are thus given by $(P \cap Q) + (P^\perp \cap Q^\perp)$, $(P \cap Q^\perp) + (P^\perp \cap Q)$, $[(P \cap Q) + (P^\perp \cap Q^\perp)]^\perp$, $[(P \cap Q^\perp) + (P^\perp \cap Q)]^\perp$, and may have dimensions going from 0 to 4, according to the class of the matrix $\mathcal{D}(P, Q)$. As follows from point i) of proposition 2 there are at most only *two different* non vanishing diagonal subspaces because of the restriction $\text{tr } \mathcal{D}(P, Q) \subseteq [\text{tr } \mathcal{D}^c(P, Q)]^\perp$ or its orthogonal expression $\text{tr } \mathcal{D}^c(P, Q) \subseteq [\text{tr } \mathcal{D}(P, Q)]^\perp$. Starting from proposition 5, a direct inspection of this restriction for every class gives the following corollary:

Corollary 3 *In Minkowski space, two planes P and Q cut each other orthogonally if, and only if, the diagonal subspaces $\text{tr } \mathcal{D}(P, Q)$ and $\text{tr } \mathcal{D}^c(P, Q)$ are non vanishing and satisfy $\text{tr } \mathcal{D}(P, Q) = [\text{tr } \mathcal{D}^c(P, Q)]^\perp$.*

The diagonal subspaces of dimension two, i.e. the *diagonal planes*, will be of interest in what follows. They will be denoted by Δ and $\Delta^\perp$.

6 Orthogonal cut of two planes

In this section we are interested in the set $C_\perp(P, Q)$ of all planes that orthogonally cut the planes P and Q ; we shall call $C_\perp(P, Q)$ the *orthogonal cut* of P and Q . One has by definition,

$$C_\perp(P, Q) \equiv C_\perp(P) \cap C_\perp(Q). \quad (6)$$

From corollary 2 it follows that $M \in C_\perp(P) \Leftrightarrow P \in C_\perp(M)$, and from corollary 2 $M \in C_\perp(P) \Leftrightarrow M^\perp \in C_\perp(P)$ so that, taking into account (6), one has

$$C_\perp(P, Q) = C_\perp(P, Q^\perp) = C_\perp(P^\perp, Q) = C_\perp(P^\perp, Q^\perp) \quad (7)$$

and

$$M \in C_\perp(P, Q) \Leftrightarrow M^\perp \in C_\perp(P, Q).$$

Because of this last property, we shall say that $C_\perp(P, Q)$ is *unique* for given planes P and Q if it contains only a *single* pair of planes $(M, M^\perp)$.

For planes P, Q such that $\mathcal{D}(P, Q) \in m_{20}$, it follows from (7) and (6) that $C_\perp(P, Q) = C_\perp(P)$, so that $C_\perp(P, Q)$ contains a two-parameter set of planes $(M, M^\perp)$. But this is not the only case of non uniqueness for $C_\perp(P, Q)$, in Minkowski space, as proposition 6 shows.

Proposition 6 *In Minkowski space, the orthogonal cut $C_\perp(P, Q)$ of two planes P, Q , is never empty, and it is unique if, and only if, $\mathcal{D}(P, Q)$ belongs to m_0, m_{10}^0, m_1^0 or m_1^3 .*

Proof:

m_0 : It is clear, from proposition 1, that the characteristic planes of P and Q for this class always exist and are unique. From definition 1, it is obvious that the characteristic planes belong to $C_\perp(P, Q)$. As every pair of orthogonal planes of $C_\perp(P, Q)$ are characteristic planes of P and Q , one has that $C_\perp(P, Q)$ is unique and $C_\perp(P, Q) = \{K, K^\perp\}$, K and $K^\perp$ being the characteristic planes of P and Q .

m_{10}^0 and m_1^0 : If $M \in C_\perp(P, Q)$ is such that $M \not\subset P + Q$, its unique direction in $P + Q$ must be necessarily $P \cap Q$; the direction in M orthogonal to $P \cap Q$ is unique because this last one is non null, thus that direction must be $P^\perp \cap Q^\perp$. Consequently M is the diagonal plane $\Delta = tr \mathcal{D}(P, Q)$. If $M \in C_\perp(P, Q)$ is such that $M \subset P + Q$, one has that $M^\perp \not\subset P + Q$, otherwise $P + Q$ will be null and $P^\perp \cap Q^\perp$ will be the only null direction in this three-dimensional subspace; so that $C_\perp(P, Q) = \{\Delta, \Delta^\perp\}$ with $\Delta = tr \mathcal{D}(P, Q)$.

m_1^3 : From proposition 4, one can take $P = i + i'$ and $Q = i + p$ where $\{i, i', p, q\}$ are the four directions of a null tetrad. From proposition 5 it is clear that Q and $Q^\perp$ belong to $C_\perp(P, Q)$. It is also verified that there are no others: Q (and $Q^\perp$) being null, from (5) all the other planes in $C_\perp(Q)$ are non null and the time-like ones M cut Q along i ; but (5) implies that P and M cannot orthogonally cut each other. In this case $Q^\perp = tr \mathcal{D}(P, Q)$, so we have once more $C_\perp(P, Q) = \{\Delta, \Delta^\perp\}$ with $\Delta = tr \mathcal{D}(P, Q)$.

m_{10}^0 : From proposition 4 one can take two time-like planes P and Q cutting each other along a null direction i . It follows directly from (5) that all null planes containing the common null direction i are in $C_\perp(P, Q)$. That they are the only ones may be seen as follows. Let us take $M \in C_\perp(P, Q)$; if $i \not\subset M$ the cut directions of M with P and Q are different, so that $M \subset P + Q$. But $P^\perp \cap Q^\perp \not\subset P + Q$, so that we can write $M = a + b$ with $a \subset P^\perp$ and $b \subset Q^\perp$; as $a \perp i \perp b$, thus $M^\perp$ is time-like and $i \subset M^\perp$, so, from proposition 5, $M \notin C_\perp(P, Q)$. Let us remark that $C_\perp(P, Q)$ contains the diagonal planes Δ and $\Delta^\perp$ with $\Delta = tr \mathcal{D}(P, Q)$.

m_1^1 : It corresponds to different non orthogonal null planes P and Q with the same null direction. From (5) $C_\perp(P, Q)$ contains all the time-like planes M that cut P and Q along their null common direction as well as their orthogonals and these planes are the only ones in $C_\perp(P, Q)$.

m_{20} and m_{21} : In both cases, one can take $Q = P$, a regular plane for m_{20} and a null one for m_{21} . According to (6) one has $C_\perp(P, Q) = C_\perp(P)$, so that $C_\perp(P, Q)$ is neither empty nor unique.

Types	$\mathcal{D}(P, Q)$	$C_{\perp}(P, Q)$
m_0	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\{K, K^{\perp}\}$
m_{10}^0	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$\{\Delta, \Delta^{\perp}\}$
m_{10}^2	$\begin{pmatrix} i & 0 \\ 0 & p \end{pmatrix}$	$\{\Delta, \Delta^{\perp}\} \cup \{i + p' \mid i \perp p'\}$
m_1^0	$\begin{pmatrix} a & c \\ d & b \end{pmatrix}$	$\{\Delta, \Delta^{\perp}\}$
m_1^3	$\begin{pmatrix} i & i \\ q & p \end{pmatrix}$	$\{\Delta, \Delta^{\perp}\}$
m_1^1	$\begin{pmatrix} i & i \\ i & i \end{pmatrix}$	$\{i + p, (i + p)^{\perp} \mid i \not\perp p\}$
m_{20}	$\begin{pmatrix} R & 0 \\ 0 & R^{\perp} \end{pmatrix}$	$\{a + b \mid a \subset R, b \subset R^{\perp}\}$
m_{21}	$\begin{pmatrix} I & i \\ i & I^{\perp} \end{pmatrix}$	$\{I, I^{\perp}, i + p, (i + p)^{\perp} \mid i \not\perp p\}$

Table 2: The orthogonal cut of a pair of planes.

It is interesting to select from the above results the explicit form of $C_{\perp}(P, Q)$. This is given in the following proposition:

Proposition 7 *In Minkowski space, the relations between the orthogonal cut $C_{\perp}(P, Q)$ and the components of the matrix $\mathcal{D}(P, Q)$ of two planes, according to its class, are given by Table 2, where K and $K^{\perp}$ are the characteristic planes in m_0 , Δ and $\Delta^{\perp}$ are the diagonal planes, a and b are two orthogonal directions in m_{20} and p and p' are space-like directions anywhere.*

When the orthogonal cut is unique, it is given by the characteristic planes or by the diagonal planes according to the class of $\mathcal{D}(P, Q)$. For future applications, it is important to select these particular elements of the orthogonal cut among the other elements shown in Table 2.

Definition 4 In Minkowski space, we shall call single orthogonal cut $S_{\perp}(P, Q)$ the following set:

$$S_{\perp}(P, Q) \equiv C_{\perp}(P, Q) \cap \{\Delta, \Delta^{\perp}, K, K^{\perp}\},$$

where Δ and $\Delta^{\perp}$ are the diagonal planes and K and $K^{\perp}$ the characteristic planes.

Let us remark that for the pairs of planes P and Q where $\mathcal{D}(P, Q)$ belongs to m_1^1 , m_{20} or m_{21} there are neither diagonal planes nor characteristic planes, so that $S_{\perp}(P, Q) = \emptyset$ in these cases.

7 Basic notions about 2-forms

Every 2-form F has two independent *invariant scalars*, namely $\phi_F \equiv (F, *F)$ and $\psi_F \equiv (F, F)$. Here $*$ is the Hodge dual operator [14] and $(\cdot, \cdot)$ stands for the scalar product induced by g on the space of 2-forms: for any pair of 2-forms, F and G , it may be written $(F, G) \equiv -(1/2) \text{tr}(FG)$, where tr is the trace operator and FG is the matrix product of 2-forms [15].

A non vanishing 2-form F is called *null* if $\phi_F^2 + \psi_F^2 = 0$, *regular* if $\phi_F^2 + \psi_F^2 \neq 0$, *simple* if $\phi_F = 0$ and *unitary* if it is simple and $\psi_F^2 = -1$.

The *principal directions* of F are the common eigendirections of F and $*F$, which are necessarily null. Let $\pm\alpha$ and $\pm i\tilde{\alpha}$ be the eigenvalues of a regular 2-form verifying $\alpha > 0$ and $\phi_F = -2\alpha\tilde{\alpha}$, $\psi = -\alpha^2 + \tilde{\alpha}^2$. Then F admits a unique decomposition of the form $F = \alpha U - \tilde{\alpha} * U$ where U is unitary. The pair $\{U, *U\}$ is called the *geometry* of F . For a null F , it is the pair $\{F, *F\}$ itself which is called the *geometry* of F .

It is *always* possible to write univocally any 2-form F in terms of its geometry $\{U, *U\}$ as $F = \alpha U - \tilde{\alpha} * U$, where the *weights* α and $\tilde{\alpha}$ take the value $\alpha = 1$ and $\tilde{\alpha} = 0$ when U is null. This decomposition is useful when (as in the present case) only algebraic calculations are involved.

An *inner direction* x of a simple 2-form F is a direction such that any of its elements $\hat{x}$ verifies $\hat{x} \wedge F = 0$, where $\wedge$ stands for the exterior product. The plane of inner directions, will be noted $\pi(F)$. The *invariant planes* of a 2-form F are the planes of inner directions of its geometry: $\{\pi(U), \pi(*U)\}$. It is clear that $\pi(*U) = \pi(U)^{\perp}$.

To a pair of 2-forms, F and G , there are six corresponding invariant scalars, namely ϕ_F , ψ_F , ϕ_G , ψ_G and the two *mixed invariant scalars* $\rho_{FG} \equiv (F, *G)$ and $\sigma_{FG} \equiv (F, G)$. The relation between the mixed invariant scalars and the invariant planes is as follows.

Lemma 2 Let F and G be two simple 2-forms. Their invariant planes $\pi(F)$ and $\pi(G)$ have common directions if, and only if, the mixed invariant ρ_{FG} vanishes.

It is to be noted that this lemma also asserts that $\pi(F)$ and $\pi(*G)$ have common directions iff $\sigma_{FG} = 0$, because of the identity $\sigma_{FG} = -\rho_{F*G}$.

Proof of lemma 2: Let $\hat{x}$ be any non vanishing element of a direction x . Then:
 $\exists x \mid x \subset \pi(F) \cap \pi(G) \Leftrightarrow \exists \hat{x} \mid \hat{x} \wedge F = \hat{x} \wedge G = 0 \Leftrightarrow F \wedge G = 0 \Leftrightarrow (F \wedge G, \eta) = 0$
 $\Leftrightarrow (F, *G) = 0 \Leftrightarrow \rho_{FG} = 0$.

Let us remark that this proof is *independent* of the signature of the metric.

8 Commuting 2-forms

Let us recall that 2-forms build the Lie algebra of Lorentz group with the commutator as the Lie bracket

$$[F, G] = FG - GF.$$

In order to see when this commutator vanishes, let us consider the univocal decompositions $F = \alpha U - \tilde{\alpha} * U$ and $G = \beta V - \tilde{\beta} * V$ of F and G in terms of their geometries $\{U, *U\}$ and $\{V, *V\}$. Taking into account the identities

$$[F, *G] = [*F, G] = *[F, G], \quad (8)$$

one obtains:

$$[F, G] = (\alpha\beta - \tilde{\alpha}\tilde{\beta})[U, V] - (\alpha\tilde{\beta} + \tilde{\alpha}\beta) * [U, V], \quad (9)$$

which shows that $[U, V] = 0 \Rightarrow [F, G] = 0$. Conversely, if $[F, G] = 0$ one has either $[U, V] = 0$ or $(\alpha\beta - \tilde{\alpha}\tilde{\beta})^2 + (\alpha\tilde{\beta} + \tilde{\alpha}\beta)^2 = 0$, because real 2-forms are linearly independent of their duals. But the first member of this last equation may be written as $(\alpha^2 + \tilde{\alpha}^2)(\beta^2 + \tilde{\beta}^2)$ which cannot vanish for non vanishing 2-forms; thus $[F, G] = 0 \Rightarrow [U, V] = 0$. Let us see when this last equation holds. If U is regular, the minimal polynomials restricted to its invariant subspaces, i.e. $\{\pi(U), \pi(*U)\}$, are prime, so that according to a classical result [16] $[U, V] = 0$ implies $U = \pm V$ or, in other words, U and V have the same principal directions which, by proposition 4, is the case whenever $\mathcal{D}(\pi(U), \pi(V)) \in m_{20}$. Consequently, if U is null, $[U, V] = 0$ implies V null. Let $\hat{i}$ be a null vector in the principal direction i of U , so $i(\hat{i})U = 0$ and $i(\hat{i}) * U = 0$; putting $\hat{i}' \equiv i(\hat{i})V$, from $[U, V] = 0$ and (8) it follows that $i(\hat{i}')U = 0$ and $i(\hat{i}') * U = 0$, so that $\hat{i}' = \lambda \hat{i}$; that is, $i(\hat{i})V = \lambda \hat{i}$; V and $\hat{i}$ being null, this last equation implies $\lambda = 0$ and consequently i is the principal direction of V . As two null planes of the same null direction correspond, by proposition 4, to the matrices $\mathcal{D}(\pi(U), \pi(V))$ belonging to m_1^1 or to m_{21} , we have the following result.

Proposition 8 *Let F and G be two 2-forms and $\{U, *U\}$ and $\{V, *V\}$ respectively their geometries. The following conditions are equivalent:*

- i)* $[F, G] = 0$
- ii)* $[U, V] = 0$
- iii)* F and G have the same principal directions.
- iv)* $[\mathcal{D}(\pi(U), \pi(V))] \in \{m_1^1, m_{21}, m_{20}\}$.

It is to be noted that, in the null case, to have the same principal direction is a weaker condition than to have the same geometry.

9 Invariant scalars of the commutator

Let us evaluate the invariant scalars of the commutator. We have:

$$([F, G], [F, G]) = -\frac{1}{2} \text{tr}(FG - GF)^2 = \text{tr}(F^2G^2 - FGF G),$$

but, taking into account the identity $FG - *G * F = -\sigma_{FG}g$, we obtain:

$$F^2G^2 + GF^2G = -\psi_F G^2 - \rho_{FH} * FG - \sigma_{FG} FG, \quad FGF G = \frac{1}{4} \phi_F \phi_G g - \sigma_{FG} FG.$$

As $\text{tr}[A, B] = 0$ for any second order tensors A and B , one has from these expressions

$$\text{tr } F^2G^2 = \psi_F \psi_G + \rho_{FG}^2 + \sigma_{FG}^2, \quad \text{tr } FGF G = \phi_F \phi_G + 2\sigma_{FG}^2$$

that yields $\psi_{[F, G]}$. A similar process for $\phi_{[F, G]}$ allows one to state the following result.

Lemma 3 *The invariant scalars $\phi_{[F, G]}$ and $\psi_{[F, G]}$ of the commutator of the 2-forms F and G are related to the invariant scalars of the pair F, G by*

$$\begin{aligned} \phi_{[F, G]} &= \phi_F \psi_G + \psi_F \phi_G - 2\rho_{FG} \sigma_{FG} \\ \psi_{[F, G]} &= -\phi_F \phi_G + \psi_F \psi_G + \rho_{FG}^2 - \sigma_{FG}^2. \end{aligned}$$

Consider two simple 2-forms F and G . From the above lemma one has $\phi_{[F, G]} = -2\rho_{FG} \sigma_{FG}$ so that $[F, G]$ is not simple if, and only if, $\rho_{FG} \sigma_{FG} \neq 0$. But, from lemma 2, this condition is satisfied if, and only if, all the components of the matrix $\mathcal{D}(\pi(F), \pi(G))$ vanish. Thus we have the following proposition.

Proposition 9 *The commutator $[F, G]$ of simple 2-forms is not simple if, and only if, the planes $\pi(F)$ and $\pi(G)$ are such that $\mathcal{D}(\pi(F), \pi(G)) \in m_0$.*

10 Geometry of the commutator

Let $F = \alpha U - \tilde{\alpha} * U$ and $G = \beta V - \tilde{\beta} * V$ be two arbitrary 2-forms. The *linear structure* of the space of 2-forms is such that the geometry $\{Z, *Z\}$ of the sum $H \equiv F + G$, $H = \delta Z - \tilde{\delta} * Z$, is *not*, in general, a function of the sole geometries $\{U, *U\}$ and $\{V, *V\}$ of F and G . The geometry $\{Z, *Z\}$ *also* depends on the weights $\alpha, \tilde{\alpha}, \beta$ and $\tilde{\beta}$ of F and G . At first glance, it would seem that this dependence would remain for the geometry of the commutator in the Lie structure, but this is not true: if $\{W, *W\}$ is the geometry of the non vanishing 2-form $[U, V] = \gamma W - \tilde{\gamma} * W$, putting $\kappa = \alpha\beta - \tilde{\alpha}\tilde{\beta}$ and $\tilde{\kappa} = \alpha\tilde{\beta} + \tilde{\alpha}\beta$, equation (9) may be written

$$[F, G] = (\kappa\gamma - \tilde{\kappa}\tilde{\gamma})W - (\kappa\tilde{\gamma} + \tilde{\kappa}\gamma) * W$$

which, from the uniqueness of the decomposition, allows one to state the following result.

Proposition 10 *The geometry $\{W, *W\}$ of the commutator $[F, G]$ of 2-forms depends only on the geometries $\{U, *U\}$ and $\{V, *V\}$ of F and G , but not on their weights $(\alpha, \tilde{\alpha})$ and $(\beta, \tilde{\beta})$.*

We shall now see the nature of this dependence. Let us observe that, for any 2-forms F and G , we have:

$$\rho_{F[F,G]} = \sigma_{F[F,G]} = 0. \quad (10)$$

This follows directly from the definition of the mixed invariant scalars and the elementary properties $[F, *F] = 0$ and $\text{tr}([A, B]) = 0$ for any 2-form F and any second order tensors A and B . Let us first assume that $[U, V]$ does not vanish and is *not* simple, so that it is of the form $[U, V] = \gamma W - \tilde{\gamma} * W$, with W non null; contracting this expression respectively with U and $*U$, the left hand members vanish due to (10), and we obtain the system $0 = \gamma \sigma_{UW} - \tilde{\gamma} \rho_{UW}$, $0 = \tilde{\gamma} \sigma_{UW} + \gamma \rho_{UW}$ which, because $\gamma^2 + \tilde{\gamma}^2 \neq 0$, gives necessarily $\rho_{UW} = \sigma_{UW} = 0$. This insures by lemma 2 that the matrix $\mathcal{D}(\pi(U), \pi(W))$ has no vanishing components. As W is not null, by propositions 3 and 4, $[\mathcal{D}(\pi(U), \pi(W))] \in \{m_1^0, m_1^3\}$, so that, by proposition 5, $\pi(W) \in C_\perp(\pi(U))$. A similar argument shows that $\pi(W) \in C_\perp(\pi(V))$.

Assume now that $[U, V]$ does not vanish and *is* simple, $[U, V] = \gamma W$. According to (10), one has $\rho_{UW} = \sigma_{UW} = \rho_{VW} = \sigma_{VW} = 0$, so that, by lemma 2, the matrices $\mathcal{D}(\pi(U), \pi(W))$ and $\mathcal{D}(\pi(V), \pi(W))$ have no vanishing components. If both matrices belong to m_1^1 , by proposition 4, $(\pi(U), \pi(W))$ and $(\pi(V), \pi(W))$ are pairs of null planes with same null direction, and then $(\pi(U), \pi(V))$ are null having same principal direction; proposition 8 yields $[U, V] = 0$, an inconsistent result. If one of these matrices, say the first one, belongs to m_1^1 , one has, by proposition 4 $U = \hat{i} \wedge \hat{p}$ and $W = \hat{i} \wedge \hat{p}'$ with $\hat{i}$ a null direction and, $\hat{p}$ and $\hat{p}'$ different non orthogonal space-like directions orthogonal to $\hat{i}$. $\pi(W)$ being null and cutting orthogonally the non space-like plane $\pi(V)$, $\pi(V)$ must contain the direction $\hat{i}$ by (5); as $[U, V]$ does not vanish, V is time-like by proposition 8 and $V = \hat{i} \wedge \hat{i}'$ with $\hat{i}'$ another null direction. Now a straightforward calculation gives $[U, V] = -U$, which is also inconsistent. Consequently, by proposition 3, the only possible classes for both matrices are m_1^0 , m_1^3 or m_{21} . We have the following result.

Lemma 4 *The invariant planes of the commutator of two 2-forms orthogonally cut the invariant planes of both 2-forms.*

Consider the matrix $\mathcal{D}(\pi(U), \pi(V))$ corresponding to the invariant planes of the 2-forms F and G of non vanishing commutator $[F, G] \neq 0$. This last condition insures by proposition 8 that this matrix does not belong to m_1^1 , m_{21} or m_{20} . For the other classes, the orthogonal cut $C_\perp(\pi(U), \pi(V))$ is unique except for m_{10}^2 (proposition 7), so that, from lemma 4, in the cases of uniqueness, $\pi(W) \in S_\perp(\pi(U), \pi(V))$ for the geometry W of the commutator. Class m_{10}^2 occurs (proposition 4) when $\pi(U)$ and $\pi(V)$ are time-like and cut each other along a null direction; so, there are three different null directions $\hat{i}$, $\hat{i}'$ and $\hat{i}''$

such that $U = \hat{i} \wedge \hat{i}'$ and $V = \hat{i} \wedge \hat{i}''$. A straightforward calculation yields $[U, V] = -U + V$ and consequently $i(\hat{i})[U, V] = -\hat{i} + \hat{i} = 0$ and $i(\hat{p})[U, V] = 0$, p being the direction $\pi(*U) \cap \pi(*V)$. Thus $\pi([U, V]) = i + q$ for a space-like direction q with $i \perp q \perp p$, and consequently $\pi([U, V])$ is a diagonal plane so that $\pi([U, V]) \in S_{\perp}(\pi(U), \pi(V))$. We have proved the following result.

Theorem *The invariant planes of the commutator of two 2-forms are the single orthogonal cut of the invariant planes of both 2-forms.*

11 Acknowledgement

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References

- [1] Just as ordinary rotations, which are usually considered as 3×3 matrices in our euclidean space.
- [2] In general, the dimension ρ of the (pseudo-)rotation group of a n -dimensional space being $\rho = n(n-1)/2$, and the dimension of the space of second rank tensors on it being n^2 , the number ν of non strict variables in the formalism is $\nu = n^2 - n(n-1)/2 = n(n+1)/2$. Thus, for ordinary rotations $\rho = 3$ but $\nu = 6$ and for Lorentz transformations $\rho = 6$ but $\nu = 10$.
- [3] The intrinsic elements of a tensor are the elements in which the tensor may be *biunivocally* and *covariantly* decomposed. Thus, the intrinsic elements of an ordinary 3-dimensional rotation in the exponential domain are one scalar (determining biunivocally its rotation angle) and one unit vector (determining biunovocally its rotation axis); no other algebraically independent quantity may be extracted from the matrix in a covariant way. The corresponding intrinsic elements of a Lorentz matrix in Minkowski space-time are two scalars and one non space-like plane; in the regular case, this plane is time-like (the boost plane), one of the scalars is a hyperbolic angle (relative velocity of the boost) and the other a trigonometric angle (spatial rotation orthogonal to the boost). Of course, in both cases, the variables needed to describe these intrinsic elements are ρ (see [2]), a number favourably smaller than the n^2 variables involved in the matrix formalism.
- [4] In contrast to the n^2 components of a rotation matrix, the $n(n-1)/2$ components of a 2-form are all *strict* because they are biunivocally related to their intrinsic elements. For an ordinary rotation, 1 angle plus 2 parameters to fix a unit vector, for a Lorentz transformation 2 angles plus 4 parameters to fix a 2-plane; specifically, denoting by α, β the angles ($\alpha > 0$),

by u the unit 3-dimensional vector, by U the unit 4-dimensional 2-form (2-plane) and by $*$ the Hodge dual operator on the corresponding space, any 3-dimensional 2-form may be biunivocally written as $f = \alpha * u$ and any 4-dimensional 2-form as $F = \alpha U - \beta * U$ (see Section 7).

- [5] The expression for the exponential of a 2-form on the space-time has been given in generic and covariant form by B. COLL, F. SAN JOSÉ, *Gen. Rel. Grav.*, **22**, pag. 811 (1990).
- [6] If L and M are two (fields of) Lorentz transformations generated by the 2-forms F and G respectively, $L \equiv \exp F$ and $M \equiv \exp G$, the product transformation $N \equiv LM$ is of the form $N \equiv \exp H$, H being given by BCH formula; this defines the so called BCH product by the expression $F \bullet G = H$. For the original papers see: J.E. CAMPBELL: *Proc. London. Math. Soc.*, **28**, pag. 381 (1897); **29**, pag. 14 (1898). H.F. BAKER: *Proc. London. Math. Soc.*, **34**, pag. 347 (1902); **2**, pag. 293 (1904); **3**, pag. 24 (1905). F. HAUSDORFF: *Ber. Verhandl. Saechs. Akad. Wiss. Leipzig, Math. Naturw. Kl.*, **58**, pag. 19 (1906).
- [7] In the case of ordinary rotations, the answer is very simple. Let f , g and h be three 3-dimensional 2-forms of rotation axes determined by the unit vectors u , v and w and rotation angles α , β and γ respectively: $f = \alpha * u$, $g = \beta * v$, $h = \gamma * w$. If h is the commutator of f and g , $[f, g] \equiv fg - gf$, then $[f, g] = \alpha\beta * (v \times u)$ ($\times$ is the usual cross-product in euclidean 3-dimensional space) that is to say: *the rotation axis of the commutator of two 2-forms is the orthogonal direction to the plane defined by their rotation axes.*
- [8] *Invariants planes* means here 2-dimensional eigen-spaces; every eigen-space is, of course, an intrinsic element, but the set of all of them is not an algebraically independent set.
- [9] This is an unexpected generalization of the 3-dimensional result (see [7]) that the rotation axis of the commutator of two 2-forms depends exclusively on their rotation axes (and not on their rotation angles).
- [10] The corresponding internal binary operation on straight lines, defined by the commutator of 2-forms in ordinary space, follows from the expression $[f, g] = \alpha\beta * (v \times u)$ of [7]: to the straight lines αu ($\forall \alpha$) and βv ($\forall \beta$) the commutator associates univocally the straight line $\gamma * (v \times u)$ ($\forall \gamma$). Its geometric meaning is very clear: it is the straight line orthogonal to the given ones. Nevertheless, the geometric meaning for the analogous operation in Minkowski space is not so trivial. The main result of this paper states properly these geometric meanings.
- [11] Geometric generalized photons, point particles with spin, straight strings, define at every point of space-time a plane; in the scattering or shock of two of them, the description of the reciprocal relations of the corresponding planes is of interest because it gives a direct, simple and invariant

characterization of the different possible relative positions of the two objects considered. A complete analysis of these reciprocal relations will be given elsewhere; here we present only the part of the subject that concerns the commutator.

- [12] L. M. THOMAS: *Nature*, **117**, n° 2945, pag. 514 (april 1926).
- [13] This is a surprise. Customarily, we are used to seeing lorentzian geometry to split the general statements of euclidean geometry in several cases. However, this result goes in the opposite direction: the unicity of the characteristic planes in Minkowski space fails for euclidean 4-dimensional space. See H. P. MANNIG: "*Geometry of Four Dimensions*", Dover Pub. Inc, N.Y. (1956), pag. 119. As a consequence, the geometric interpretation that we shall found in this paper is not signature independent, a feature that may have contributed for the subject to pass unnoticed.
- [14] Our convention is $(*F)_{\alpha\beta} \equiv \frac{1}{2}\eta_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ where $\eta_{\alpha\beta\gamma\delta}$ is the volumen element induced by the Minkowski metric.
- [15] In local coordinates $(FG)_{\alpha\beta} = F_{\alpha\rho}G^{\rho}_{\beta}$.
- [16] F.R. GANTMACHER: "*Théorie de Matrices*" Tome I; Dunod, Paris (1966), pag. 227.

ANSWER TO THE REFEREE'S REPORT OF THE
FIRST VERSION

We believe that we have taken into account all the referee's comments and criticisms ¹. Specifically:

- The main part of the introduction has been modified, with the intention of making the terminology clear and explaining the motivations and possible applications of the results.
- The word “commutator” appears everywhere in place of “bracket” (point 1 of the report).
- The notions of intrinsic elements and invariant planes have been clearly defined in references [3] and [8] respectively (point 2 of the report).
- In the first line of Section 7 it was written “Another characterization of the orthogonal cut of two planes ...” where it had to be written “Another characterization of the pairs of planes that cut each other orthogonally ...” (point 3 of the report).
- Our convention for the Hodge dual operator has been explicitly written in reference [14] (point 4 of the report).
- The product FG has been defined in local coordinates in reference [15] (anecdote: our three-lines unconvenable phrase for FG was due to the exigence, by the referee of another paper, of a definition without indices. We agree with the present referee). The corresponding expression for the commutator is now a direct consequence of its definition $[F, G] = FG - GF$ given in Section 8 (point 5 of the report).

We believe that the new introduction may reveal to more people the interest of the subject.

¹Perhaps except point 6 of his report concerning english language. Be sure we expend many time and effort on this subject. If the number of errors or inadecuate expressions is small, we would be very glad if the referee accept to correct them.