Universidad Politécnica de Madrid

**ESCUELA TÉCNICA SUPERIOR DE INGENIEROS
DE CAMINOS, CANALES Y PUERTOS**

**A TWO-PHASE SPH DEPTH INTEGRATED
MODEL FOR DEBRIS FLOW PROPAGATION
CONSIDERING PORE-WATER PRESSURE
EVOLUTION**

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Dedicated

To my beloved parents

To my dear brother

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ABSTRACT

Landslides are a geological phenomenon which occur frequently in mountainous regions or along rivers, causing major economic damage and a large number of casualties around the world. Development of continuous based models to represent the propagation of these geophysical mass flows, from initiation to deposit in the run-out zones, is not an easy task due to the complexity of the phenomenon. Moreover, many of the interesting fast catastrophic landslides, such as debris flows, involve more than one phase where the coupling between the phases plays a fundamental role. Such multiphase materials can be properly described by using a multiple set of nodes for different phases, and a suitable drag law to take into account the interaction between particles.

The principal objective of this Doctoral Thesis is to develop mathematical and numerical modelling for simulating debris flows in which considering the effect of pore-water pressure dissipation is essential for risk analysis. A debris flow consists of low to high permeable soil in which lateral spreading could be highly affected by excess pore-water pressure. The numerical modeling of these types of debris flow has not been investigated until now.

In this study, the two-phase model proposed by Pastor et al. (2017) is extended and also an improvement in description of pore pressure evolution (Pastor et al., 2015) is presented in order to take into account with more precision changes caused not only by vertical consolidation and height variations, but also by new considerations which include changes of basal surface permeability and porosity variations.

The new mathematical approach is based on the depth integrated mathematical model of Zienkiewicz and Shiomi (1984), and is capable to reproduce the propagation of debris flows with soil permeability ranging from high to low.

These mathematical equations are discretized by using Smooth Particle Hydrodynamic (SPH) technique where a double set of nodes, one to represent the

movement of the solid particles and another to represent the movement of the fluid particles, is defined. As a novelty, this Doctoral Thesis provides a contribution to enhance the two-phase numerical model with adding a 1D finite difference grid to each SPH node that represents a solid particle in order to improve the description of pore-water pressure (SPH-FD model).

The performance and limitations of the model is assessed using a series of benchmark exercises, including (i) dam break problem to show the main features of the model, (ii) flume tests equipped with a (permeable) rack which performed in Trondheim, and (iii) two real cases for which we have had access to their reliable information.

The good results obtained from the validation analysis of the mentioned benchmarks indicated that the proposed model is capable to properly reproduce the propagation velocity, runout distance and deposit thickness of the debris flows, and more importantly to correctly performs the time-space evolution of pore-water pressures during the whole propagation stage from initiation to propagation over an impermeable and permeable bottom boundary, and up to deposition.

RESUMEN

Los flujos de derrubios son un fenómeno geológico que tiene lugar con cierta regularidad en regiones montañosas del planeta causando cada año cuantiosas pérdidas económicas y un elevado número de víctimas en todo el mundo. El desarrollo de modelos continuos que permitan representar la propagación de este tipo de flujos másico geofísicos, desde su desencadenamiento hasta su deposición, no es una tarea sencilla debido a la complejidad del fenómeno, conduciendo a un sistema de ecuaciones en derivadas parciales hiperbólico, dependiente del tiempo y no lineales.

Los flujos de derrubios son uno de los tipos de deslizamiento que causan mayor número de daños, presentando una naturaleza multifísica en el que el comportamiento acoplado esqueleto sólido-fluido intersticial juega un papel fundamental en la fase de propagación.

En esta Tesis Doctoral se presenta un modelo matemático integrado en profundidad para flujos de derrubios basado en las ecuaciones de Biot-Zienkiewicz incluyendo leyes de arrastre para tener en cuenta la interacción entre el esqueleto sólido y el fluido intersticial. Este modelo incorpora la influencia de la disipación de excesos de presión de poros, aspecto crucial para la evaluación de riesgos asociada a esta tipología de flujos gravitatorios.

El modelo matemático de dos fases (esqueleto sólido y fluido intersticial) que se presenta en esta Tesis Doctoral extiende el modelo propuesto por Pastor y colaboradores en 2015 ya que permite una descripción más precisa de la evolución de las presiones intersticiales al incorporar no sólo los cambios de tensión total debidos a la variación de la altura del flujo, sino a cambios en la porosidad.

Este nuevo modelo matemático puede ser empleado en la propagación de flujos de derrubios con distintas permeabilidades permitiendo, permitiendo reproducir la dispersión lateral observada en este tipo de flujos de derrubios la cual depende de la presión intersticial.

El modelo matemático ha sido resuelto numéricamente a través del método Smoothed Particle Hydrodynamics (SPH) con un doble juego de partículas, una para representar el movimiento del esqueleto y otra para representar el movimiento del agua intersticial. Como novedad un esquema de diferencias finitas ha sido incluido a las partículas de sólido para incorporar la evolución de las presiones de poros a lo largo de la fase de propagación del flujo de derrubios. Hasta donde llega el conocimiento del autor, esta es la principal novedad de la presente Tesis Doctoral, ya que es la primera vez en la que un modelo continuo integrado en profundidad desarrollado para la propagación de flujos de derrubios es resuelto numéricamente con la técnica SPH con dos juegos de partículas incluyendo la disipación de los excesos de presión de poros (modelo SPH-DF). De esta forma esta Tesis Doctoral extiende el trabajo de 2017 de Manuel Pastor y colaboradores.

El funcionamiento así como las limitaciones del presente modelo han sido evaluadas a través de una serie de simulaciones de verificación que incluyen 1) problemas de rotura de presa, 2) ensayos a pequeña escala incorporando rejillas permeables en la superficie de deslizamiento y 3) dos casos reales de flujos de derrubios acaecidos en Hong Kong.

Los buenos resultados obtenidos a través de los análisis de validación indican que el modelo matemático propuesto es capaz de reproducir adecuadamente la velocidad de propagación así como la extensión y profundidad de la zona afectada por el deslizamiento. Más aún, el modelo representa adecuadamente la evolución de las presiones intersticiales a lo largo de la fase de propagación considerando una superficie basal tanto permeable como impermeable.

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LIST OF SYMBOLS

a	A non-dimensional parameter in Bingham fluids	$[-]$
b	Vector of body forces	$[m/s^2]$
b_3	Gravity when the axis x_3 is vertical and points upwards ($-g$)	$[m/s^2]$
C_*	Bed sediment concentration by volume for Egashira entrainment rate formula	$[\%]$
c	Cohesion	$[N/m^2]$
c	Sediment concentration of the debris flow by volume for Egashira entrainment rate formula	$[\%]$
$C_{s\alpha}$	Storage coefficient of the solid which relates the variation of saturation with respect to the pressure of fluid phase	$[m^2/N]$
c_v	Coefficient of vertical consolidation	$[m^2/s]$
D_1	Dilatancy which are limited to a single Fourier component	$[-]$
D^e	Elastic constitutive tensor	$[-]$
d	Rate of deformation tensor	$[-]$
d^{vp}	Viscoplastic component	$[-]$
d_v	The increment of volumetric strain	$[-]$
d_{v0}	Extra volumetric component	$[-]$
$\bar{d}$	Averaged rate of deformation tensor	$[-]$
d_{sc}	Deformation caused by vertical consolidation	$[-]$
d_{s2D}	Deformation caused by other two dimensions except vertical	$[-]$
E_m	Oedometric modulus	$[N/m^2]$
E_s	Erosion coefficient	$[m^{-2}]$
e	Voids ratio	$[-]$
e_R	Erosion rate	$[m/s]$
F	A function describing a convex surface in the stress space	$[N/m^2]$
F_0	The value characterizes the stress where no viscoplastic flow occurs.	$[N/m^2]$
g	Magnitude of the gravitational acceleration	$[m/s^2]$

h	Mobilized soil depth or propagation height	[m]
h_{sm}	Smoothing length	[m]
h_p	Height of the plug	[m]
h_w^{rel}	Relative water height (h_{sat}/h)	[%]
h_{sat}	Height of the water table	[m]
I	The second order identity tensor	[-]
K	An empirical factor integrated in Egashira entrainment rate formula	[-]
K_α	Volumetric stiffness of fluid phase	[N/m ²]
K_s	Volumetric stiffness of solid phase	[N/m ²]
k_v	Elastic volumetric stiffness	[N/m ²]
k_α	Permeability tensor describing the flow of the component α which is an immiscible liquid phase	[m ³ .s/kg]
$\bar{k}_\alpha$	Hydraulic conductivity	[m/s]
$\bar{k}_{\alpha 0}$	Initial hydraulic conductivity	[m/s]
k^{intr}	Intrinsic permeability tensor	[m ²]
k_α^{rel}	Relative permeability which depends on S_α	[%]
l	Path length of the erosion zone	[m]
M_i	A fictitious volume for a general fluid or solid node i	[m ³]
m	Material constant	[-]
m_s	Mass of solid particle	[m ³]
m_w	Mass of fluid particle	[m ³]
N	Number of SPH material nodes	[-]
N	Material constant for Perzyna model ($N = 1/m$)	[-]
N_α	Number of immiscible interstitial fluids	[-]
$N_k(x_3)$	Shape function	[-]
n	Porosity or volume fractions for the phases	[-]
n	Manning coefficient	[s/m ^{1/3}]
$\vec{n}$	Unit vector normal to the surface	[-]
n_α	Volume fraction of the voids occupied by fluid phase	[-]

n_a	Volume fraction of the voids occupied by air phase	[-]
n_1, n_2	Material constants for Perzyna model ($N = n_1 + n_2$)	[-]
P_1	Pore-water pressure which are limited to a single Fourier component	[Pa]
$\bar{P}_\alpha$	Averaged pressure acting on solid or fluid phase(α denotes solid or fluid phase)	[Pa]
p	Total pressure, thermodynamic pressure	[Pa]
p_α	Total pore pressure acting on fluid phases (α is an immiscible liquid phase)	[Pa]
p_w	Total pore-water pressure	[Pa]
p_w^{rel}	Relative pore-water pressure	[%]
p_{hyd}	Hydrostatic component of pore-water pressure	[Pa]
p_{hyd}^b	Hydrostatic component of basal pore-water pressure	[Pa]
$\bar{p}$	Average pore pressures of the immiscible fluid phases	[Pa]
p'	Effective confining pressure ($p' = p - p_{hyd} - \Delta p_w$)	[Pa]
$\bar{p}'$	Mean-confining pressure ($\bar{p}' = p - p_{hyd}$)	[Pa]
$p_2(\xi)$	Second order polynomial	[-]
$p_3(\xi)$	Third order polynomial	[-]
p_{atm}	Atmospheric pressure or an alternative unity datum	[Pa]
Q	Mixed volumetric stiffness of water and solid grains	[m ² /N]
R	Interaction forces between phases	[N]
R_α	Drag force of the fluid phase caused by the solid particles	[N]
R_s	Internal force on the solid particles due to relative velocity of all fluid phases	[N]
r_c	Radius of curvature	[m]
S_r	Degree of saturation	[%]
S_α	Degree of saturation of a fluid α contained in the pores of soil	[%]
S_w	Degree of saturation of water	[%]
S_a	Degree of saturation of air	[%]
s	Material strength	[N/m ²]
$\bar{s}$	Average deviatoric stresses of the fluid phases	[N/m ²]

s_α	Deviatoric part of the stress tensor	$[N/m^2]$
V_0	Initial volume entering the erosion zone	$[m^3]$
V_f	Final volume exiting the erosion zone	$[m^3]$
V_T	Terminal velocity	$[m/s]$
$\bar{v}$	Averaged velocity	$[m/s]$
v_α	Velocities of fluid particles in the mixture	$[m/s]$
v_s	Velocity of solid particle in the mixture	$[m/s]$
v_B	Velocity at the surface ($z = h$)	$[m/s]$
$\bar{v}_{sB}$	Basal slip velocity	$[m/s]$
v_p	Velocity of the plug	$[m/s]$
$v_{\max}$	Velocity at the top	$[m/s]$
v_n	Normal velocity	$[m/s]$
W	Smoothing kernel function	$[-]$
z	Ground surface elevation	$[m]$
α	Constitutive coefficient of an oedometric state deformation	$[-]$
α	A momentum correction factor	$[-]$
β_w	Increment of pore pressure	$[-]$
γ	Fluidity parameter	$[s]$
Δp_w	Excess pore-water pressure	$[Pa]$
Δp_w^b	Excess pore-water pressure at the basal surface	$[Pa]$
$\Delta \bar{p}_w^b$	Normalized excess pore-pressure at basal surface	$[-]$
δ	Friction angle between both materials	$[^\circ]$
δ	Dirac delta function	$[-]$
η	Relative plug height	$[-]$
θ	Slope angle or friction angle of the fluidized soil	$[^\circ]$
θ_e	Equilibrium bed slope for Egashira entrainment rate formula	$[^\circ]$
μ	Viscosity coefficient	$[Pa \cdot s]$
μ_B	Bagnold coefficient	$[Pa \cdot s^2]$

μ_{PZ}	Inverse of the fluidity parameter (γ)	$[1/s]$
ν	Poisson's ratio	$[-]$
ξ	Non-dimensional parameter in Bingham fluids	$[-]$
ξ	Turbulence coefficient in frictional fluid model	$[m/s^2]$
ρ	Mixture density	$[kg/m^2]$
$\rho^{(\alpha)}$	Density of fluid phase	$[kg/m^2]$
$\rho^{(s)}$	Density of solid phase	$[kg/m^2]$
ρ_α	Density of the of fluid particle	$[kg/m^2]$
ρ_s	Density of the of solid particle	$[kg/m^2]$
ρ_w	Density of the of water particle	$[kg/m^2]$
$\bar{\rho}'$	Effective density ($\bar{\rho}' = \rho - \rho_w$)	$[kg/m^2]$
σ	Total stress tensor (Cauchy) acting on the mixture	$[N/m^2]$
$\bar{\sigma}$	Depth integrated stress	$[N/m^2]$
$\sigma^{(\alpha)}$	Partial stress of fluid phase	$[N/m^2]$
$\sigma^{(s)}$	Partial stress of solid phase	$[N/m^2]$
σ_α	Stress acting on the pore fluid phase	$[N/m^2]$
σ_s	Stress acting on the solid phase	$[N/m^2]$
σ'	Effective stress tensor	$[N/m^2]$
σ_s^{eff}	Effective component stress	$[N/m^2]$
τ_B	Basal shear stress	$[N/m^2]$
τ_y	Rheological parameter equivalent to yield stress	$[N/m^2]$
ϕ	Internal friction angle of the bed for Egashira entrainment rate formula	$[^\circ]$
ϕ_B	Basal friction angle	$[^\circ]$
$\phi(F)$	An arbitrary function	$[-]$
Ω	Interpolation region	$[m^2]$
Ω_i	2D area associated to a general fluid or solid node i	$[m^2]$
ω	Darcy's velocity	$[m/s]$
ω_j	The volume of a neighboring particle	$[m^3]$

Chapter 1

INTRODUCTION

1.1 Motivation

A landslide is a geological phenomenon that occurs when a large mass of land slides by natural processes or human actions. The most catastrophic landslides correspond to those classified as fast or extremely fast, since they manage to spread long distances in very short times without warning, sweeping away everything in their path and reach areas initially considered safe. The landslides cause considerable damage and loss of a significant number of life around the world every year.

Hazard and risk assessment of landslides, with potentially long run-out, has attracted the attention of numerous researchers in the past decades. The only way to limit risk is to have a better understanding of this phenomenon and its mechanism in order to develop more reliable prediction techniques. These objectives can be achieved nowadays by using advanced simulation tools suitable for foreseeing the landslide propagation path, its velocity and the height of the deposits. In landslide processes, there are three stages that can be model:

i. Triggering is a fundamental aspect of any failure mechanism, which determines the initial conditions for the propagation phase. This stage is represented by constitutive models describing solid behavior up to failure. Concerning triggering mechanisms, two types of failures can be distinguished as slides and flows [1]. Slides present a failure surface and can be described as the movement of a mass of soil, which behave like a rigid body, over a basal surface with a concentration of shear strain on a very narrow zone. Flows are associated with another type of failure mechanism, which is referred to as diffuse, where shear strains distributed along most of the mass and failure takes place over a much larger extension of soil. Chapter 2 is concentrated on diffuse mechanisms and how to relate them to the constitutive properties of the material.

ii. Propagation is the second stage in which liquefaction or fluidization has been taken place, in other words, the soil starts behaving in a fluid like manner. So far, there has not been any satisfactory constitutive model capable to reproduce the behavior of soil mixtures under the full range of strain rates that are generated in a rapid landslide (from its initiation to deposition, passing through its propagation). Because of this fluid-like behavior, rheological models have been used traditionally to describe the propagation of the displacing mass. They are able to reproduce both internal deformations and interaction with ground surface. An important aspect of rheological models is the mathematical models where they will be implemented.

iii. Consolidation is another important aspect of some types of flows where pore pressure dissipation takes place along the normal to the terrain surface and affects the dynamics of the mixture.

The purpose of this study is to present an alternative computational model which can be applied to both the propagation and consolidation mechanisms of fast catastrophic landslides. There are various types of landslide regarding its movements. Here are the most common flows with some basic information about them [2]:

Mudflows are mixtures of fine soil particles and water, and their behavior usually is viscoplastic. They can be modelled with continuum models using a single-phase viscous fluid.

In the case of rock avalanches, the flowing material consists on solid blocks which begin disintegrating into smaller particles as they gain speed. Discrete element model (DEM) is very well suited to modeling of granular flows, such as rock avalanches, with particles which are associated with a different size and shape. However, the model is generally very expensive in terms of computation time. This is the reason why continuum models are frequently applied here, using a single phase granular fluid.

Flow slides consist of mixtures of soil particles and water, with relative displacements between both phases. In this flows, pore pressures might be generated due to the tendency of the solid skeleton to dilate or contract under shear stress affecting the effective stresses. To describe them, mathematical models formulated in terms of velocities and pore pressures (Biot-Zienkiewicz of $u - p_w$ type) provide a good approximation.

Debris flows are a mixture of water and/or air with high concentrations in sediments (rocks, gravel, clay and organic material) which can travel at extremely rapid velocities, and their debris mass moves long distances due to the significant effect of pore-water pressures on the propagation stage. Therefore, a quantitative modelling of both pore pressure time-space evolution and flow propagation is a fundamental issue to assess these hazard. The modeling of debris flows is complicated due to the velocities of both solid and fluid phases, and the stresses acting on them, which are needed to be involved; as both solid and fluid particles can have different velocities.

This study is devoted to debris flows and, accordingly, two-phase modeling will be applied in order to consider both solid and fluid phases, and their mutual interaction. In order to cover different scenarios of debris flows, four case studies will be analyzed in this study to demonstrate the capabilities of the proposed model. The detailed information about these hazard types are provided in Chapter 5. The simulation model, that will be presented in this work, has been developed at Madrid by an expert research team for almost a decade. The model is named GeoFlow_SPH and has previously been applied to theoretical, experimental, and real case histories. It is based on rational models of continuum type which can be applied to a large variety of problems. Rational methods are based on mathematical models, usually expressed by partial differential equations, while empirical methods are used to estimate travel distances rather than to provide quantitative values such as velocity. The advantage of continuum models is its capability to consider coupling of the mechanical and hydraulic behavior in the model, while discrete models are rather suited for the cases where the granularity of the moving mass is important such as debris avalanches.

1.2 Objectives

This doctoral thesis has been carried out with the general objective of developing an effective and accurate methodology for the two-phase numerical modeling, applicable to the real cases of debris flows, in which the effects of pore-water pressures and bed entrainment along the landslide path is considered. The four specific objectives are as follows:

- i. To develop a propagation mathematical model by taking into account the effect of excess pore pressures for particular cases with the low permeability of soils.
- ii. To apply consolidation models in the two-phase modeling of debris flow in order to consider the pore-water pressure evolutions caused by porosity variations and basal surface permeability.
- iii. To implement a frictional rheological model in the numerical simulation which is able to consider the effect of the pore pressure dissipations.
- iv. To improve the representation of the erosion phenomenon of the material over time by implementing new features and conditions into the original program.

1.3 Structure of the work

Based on the objectives mentioned above, the thesis is organized as follows:

Chapter 1, which is the introduction, first briefly describes landslide phenomenon and their basic aspects. Next, the characteristics of different types of landslide are briefly described, and their applicable numerical modeling in particular is presented. The last section deals with the different predictive tool and model which will be applied in this study to simulate the propagation of the rapid landslides.

Chapter 2 presents general framework for constitutive and rheological models. The most suitable expressions for basal friction force and entrainment rate are provided by using different type of rheological and erosion laws. They are used to complete the mathematical equations which will be described in the Chapter 3.

Chapter 3 is dedicated to present different alternative mathematical models which can be used to describe the coupling between solid skeleton and pore fluids. It begins with a basic concepts and theorems of continuum mechanics which will be applied later. Then, the general mathematical model will be presented based on the Biot equations. In the next section, the particularized mathematical model will be derived from Zienkiewicz-Shiomi model and then its governing equations will be integrated along depth in order to provide an excellent combination of accuracy and computational effort. The chapter will end with presenting a propagation-consolidation model where it is assumed that the dissipation of interstitial pressures takes place only along the depth.

Chapter 4 covers the basic concepts of the numerical techniques of discretization used in this study. First, the depth integrated mathematical model, described in Chapter 3, will be discretized by using Smoothed Particle Hydrodynamics Method (SPH). Then, in the cases with a high permeability terrain (rock), the consolidation equation will be discretized by using Finite Difference Method (FDM) in each time step in order to consider sudden pore pressure evolutions. At the end of the chapter, several numerical problems will be described and treated in order to improve the numerical accuracy.

Chapter 5 presents four application cases in order to assess the validity of the proposed model. The first one is dam break problem which is a simple test showing the main features of the two-phase model. The second simulation is a small scale laboratory test with a permeable basal layer (rock), located along the propagation path, aiming to illustrate the performance of the proposed consolidation model and measure propagation heights and basal pore pressures accurately. The last two are real cases, aiming to test the capability of the proposed model to reproduce natural debris-flows. Finally, the obtained results are shown, and their analysis are presented.

Chapter 6 is the conclusive chapter of this PhD thesis, where its contributions and also work recommendations for future investigation are listed.

Chapter 2

CONSTITUTIVE/RHEOLOGICAL MODEL

2.1 Introduction

The purpose of this chapter is to present models able to describe the behavior of fluidized soils with different degrees of approximation. We will begin by describing the triggering mechanisms of liquefaction or fluidization, where a soil with a solid behavior transforms to a fluid-like material, and then we present some of the many available constitutive models which allow describing this phenomenon. From here, we will continue by presenting the most relevant rheological models capable of describing the behavior of fluidized soils. Finally, basal friction forces will be derived from these models in order to determine the terms related to depth integrated stresses of mathematical equations which will be described in Chapter 3.

2.2 Constitutive model

In geotechnical engineering, constitutive equations are a relation between two physical quantities of stress and strain rate. To develop suitable constitutive relations, it is necessary to have experimental results obtained under controlled conditions. It is convenient to use simple stress and strain rate fields to obtain the basic features of the model. At a latter stage, more complex tests can be introduced to improve our understanding.

A great effort [3–7] has been devoted to developing precise constitutive models capable of describing the most relevant aspects of their behavior and as examples we can mention: (i) the elastoplastic models, (ii) viscoplastic, (iii) hypoplastic and (iv) generalized plasticity.

Among these different approaches, the generalized plasticity theory, introduced by Zienkiewicz and Mroz (1985)[8] and elaborated by Zienkiewicz and Pastor [9–11],

provides a framework within which accurate models can be developed to describe softening and liquefaction under monotonic and cyclic loading. The main advantage of the model is to postulate all ingredients without introducing any yield or plastic potential surface.

Viscoplastic (VP) models, which have been applied to describe rate effects on solid materials, were found to provide a suitable and more economic approach than classical plasticity models when computing failure loads and mechanisms [12]. In the case of soils, VP models have been applied both to cohesive [13–15] and frictional materials. They have been found to reproduce slow landslides well [15–17]. The main three classes of VP models are referred to as follows: i. Perzyna viscoplasticity [18,19], ii. Duvaut–Lion [20] and iii. Consistent viscoplasticity [21,22].

Under Perzyna’s approach, the effective stress and the rate of deformation tensor are related by the following expressions:

$$\sigma' = D^e : (d - d^{vp}) \quad (2.1)$$

where,

- σ' is the effective stress and D^e the elastic constitutive tensor.
- d is the rate of deformation tensor which can be related to the the velocity by using kinematic equation as follows:

$$d = \text{grad}_{sym} v \quad (2.2)$$

where v is the velocity field.

- d^{vp} is the viscoplastic component of the rate of deformation tensor which is given by [23],

$$d^{vp} = \gamma m \langle \phi(F) \rangle \quad (2.3)$$

where γ is the fluidity parameter, m a unit norm tensor characterizing the direction of the plastic flow and $\phi(F)$ an arbitrary function. The symbol $\langle \dots \rangle$ represents the Macaulay brackets:

$$\begin{cases} \langle \phi \rangle = \phi & \text{if } \phi \geq 0 \\ \langle \phi \rangle = 0 & \text{otherwise} \end{cases} \quad (2.4)$$

while the function $\phi(F)$ can be chosen to be:

$$\phi(F) = \left(\frac{F - F_0}{F_0} \right)^N \quad (2.5)$$

where N is a model parameter and F a function describing a convex surface in the stress space, which in this case is the Mohr–Coulomb envelope. The value F_0 characterizes the stress where no viscoplastic flow occurs.

So far, some constitutive models have been introduced to describe the triggering mechanism of landslides. Constitutive models can also be used to complete mathematical models, which will be described in chapter 3, by assuming that we can relate deformation to the rate of variation of the effective confining pressure (p'), then the next constitutive equation can be written as follows:

$$-\frac{d^{(s)}p'}{dt} = E_m(d_v - d_{v0}) \quad (2.6)$$

where,

– p' is the effective vertical stress when the state of stress is isotropic and E_m is the oedometric modulus which are given by, respectively:

$$p' = -\frac{1}{3} \text{tr} \sigma' \quad (2.7)$$

$$E_m = \frac{(1 - \nu)}{(1 + \nu)(1 - 2\nu)} E \quad (2.8)$$

where σ' is the effective stress tensor, E the elastic modulus and ν poisson's ratio. An elastic material with zero lateral deformation (oedometric conditions) is another case of interest where E_m is written as:

$$E_m = \frac{k_v}{\alpha} \quad (2.9)$$

where k_v is the elastic volumetric stiffness ratio if the skeleton is elastic and α is a constitutive coefficient of an oedometric state deformation which is obtained as:

$$\alpha = \frac{1 + \nu}{3(1 - \nu)} \quad (2.10)$$

– d_v is the increment of volumetric strain and d_{v0} is an extra volumetric component for the cases that are needed to consider for instance, volume changes caused by breaking of soil grains, thermal effects or dilatancy caused by high rates of shear deformation.

Dilatancy is defined as the volume changes induced by shear deformation and is a fundamental aspect of soil behavior. Obviously, soil dilatancy is not constant when the soil is sheared, as loose soils tend to compact, while dense soils dilate. This fact has been considered to play an important role in triggering of landslides and early stages of propagation [24,25]. There exist different dilatancy laws which can be implemented in the constitutive equations as an extra volumetric component. A simple example is obtained by assuming that dilatancy varies linearly with the distance to the critical state line (CSL)[26]. Roux and Radjai (1998)[27] and Pailha and Pouliquen (2009)[25] proposed laws which describes the evolution towards the dynamic critical state line (DCSL). Alternatively, Pastor et al. (2009)[28] proposed a law in which the

fluidized soil will dilate if its state is below the DCSL and contract when it is over it, which can occur when we decrease the shear strain rate.

In the case of rock avalanche, dilatancy laws have to be modified when particle crushing phenomenon occurred. There exist different fragmentation laws in order to consider particle crushing in the evolution of an avalanche. The interested reader is referred to the article by Longo et al. (2019)[29] in which the rheological models Hatano [30] and Gray [31] are combined with two different fragmentation laws [32,33], a hyperbolic and a fractal based law, to take into account the influence of the rheological and fragmentation laws in the modeling of avalanches.

2.3 Rheological model

So far, some constitutive models have been presented to considered behavior of soil up to failure. These models do not go further because of the lack of capability to reproduce propagation of the of the fluidized material with large deformations. The purpose of this section is to describe rheological models, which are relations between shear stress and rate of shear strain, used to study propagation of the displacing mass.

Fast landslides are complex phenomena which can be described by alternative models with different accuracy. 3D modeling of these phenomena are complex and expensive in terms of calculation time. As an alternative, the depth integrated models, which will be described in details in section 3.4 of Chapter 3, have been introduced, where velocities and stresses are integrated along the vertical axis.

In this study, simple shear infinite landslide models will be used to provide the basal shear stress (τ_B) which is more suitable for implementing in 2D depth integrated models. However, the rheological models can be generalized to the 3D form and can be implemented in 3D numerical modeling such as finite element.

Most of rheological models have been formulated in terms of total stresses for one phase, but they can be generalized including the stresses of the fluid. Therefore, the first refinement consists on considering two phases, a granular skeleton with voids filled with either water or mud. It seems reasonable to consider one-phase in the case

of mudflows, and two-phase in the case of debris flows which are depend on the ability of the fluid phase to percolate through the solid.

The development of rheological models is limited by existing rheometers in applying general stress and rate of deformation fields. However, the stress conditions in the case of fluidized geomaterials are quite simple and most of simplifications used in the analysis are caused by a lack of available data rather than founded in experiments.

2.3.1 General framework

The rheological models have been used to estimate the deformation tensor that appears in the equations of mathematical models. Therefore, the stress will be expressed as a function of the rate of deformation tensor and assumed that the fluid is isotropic. These can be carried out by utilization of representation theorems [34,35] which can be expressed as:

$$\sigma = -pI - \Phi_0 I + \Phi_1 d + \Phi_2 d^2 \quad (2.11)$$

where p is a “thermodynamic” pressure, I the identity tensor, d the rate of deformation tensor, and the term in d^2 provides, in simple-shear flows, normal stresses which depend on shear strain rate, often referred to as dispersive stresses, which agrees with classical Bagnold model [36]. Φ_k , with $k = 0,1,2$, is scalar functions of the invariants of d :

$$\Phi_k = \Phi_k(I_{1d}, I_{2d}, I_{3d}) \quad (2.12)$$

The invariants can be defined as:

$$I_{kd} = \frac{1}{k} \text{tr}(d^k) \quad (2.13)$$

In most of models, it is assumed that the flow is incompressible, and therefore $I_{1d} = 0$. This is consistent with the decomposition into propagation and vertical

consolidation, which will be described in Chapter 3, and also with the fact that soils fail at constant volume. The reader should be aware that this is just a simplification which have been proven to be accurate only under certain assumptions and that is precisely why we have introduced the dilatancy in previous section. Moreover, due to the lack of experimental evidence, we will assume that there is no dependence on the third invariant, I_{3d} . In the case of frictional fluids, where material behavior strongly depends on Lode's angle, this assumption introduces a limitation. In consequence, following the above assumptions, the scalar functions Φ_k are obtained as:

$$\Phi_k = \Phi_k(I_{2d}) \quad (2.14)$$

The stress tensor can be decomposed into hydrostatic and deviatoric components as:

$$\sigma = -\hat{p}I + s \quad (2.15)$$

where $\hat{p}$ is the hydrostatic component and can be obtained as:

$$\hat{p} = -\frac{1}{3}tr(\sigma) \quad (2.16)$$

Taking equation 2.11 into account, $\hat{p}$ can be express by:

$$\hat{p} = -\frac{1}{3}(-3p - 3\Phi_0 + \Phi_2 tr(d^2)) = p + \Phi_0 - \frac{2}{3}\Phi_2 I_{2d} \quad (2.17)$$

In above, we have used the definition of I_{2d} (see equation 2.13):

$$I_{2d} = \frac{1}{2}tr(d^2) \quad (2.18)$$

Finally, the deviatoric component can be easily obtained by substituting in equation 2.15 as:

$$s = \Phi_1 d + \Phi_2 d^2 - \frac{2}{3} \Phi_2 I_{2d} I \quad (2.19)$$

It is important to notice that, as the flow is isochoric (no volume changes), the hydrostatic component of the rate of deformation tensor is zero, and d is a deviatoric tensor. Therefore,

$$d = \text{dev}(d) \quad (2.20)$$

Thus, equation 2.19 can be written as:

$$s = \Phi_1 \text{dev}(d) + \Phi_2 d^2 - \frac{2}{3} \Phi_2 I_{2d} I \quad (2.21)$$

The stress can be written as:

$$\sigma = -pI - \Phi_0 I + \Phi_1 \text{dev}(d) + \Phi_2 d^2 \quad (2.22)$$

Next, a general framework is introduced, which later derives the most common rheological laws use in debris flows modeling. From here, there exist two particular cases of motion, which depend on the mathematical model being used:

- i. Use these laws in simple shear infinite landslide models which will be used in this work.
- ii. Generalize them to 2D flow.

2.3.1.1 The simple shear flow

Simple shear infinite landslide models are a special case of simple-shear flows where (i) flow is steady, and (ii) all magnitudes are independent on the position along the landslide, which is assumed to have an infinite length. This section is devoted to present the infinite landslide model, and a method to obtain basal friction and depth averaged stresses.

Figure 2-1. Simple shear infinite landslide.

The simple shear flow will be assumed to take place in the x_1x_3 plane (Figure 2-2), with a velocity field of the form:

$$\begin{aligned} v_1 &= v_1(z) \\ v &= (v_1, 0, 0) \end{aligned} \quad (2.23)$$

Figure 2-2. Simple shear flow.

By taking into account that the flow is incompressible, the rate of deformation tensor reads:

$$d = \begin{bmatrix} 0 & 0 & \frac{1}{2} \frac{\partial v_1}{\partial x_3} \\ 0 & 0 & 0 \\ \frac{1}{2} \frac{\partial v_1}{\partial x_3} & 0 & 0 \end{bmatrix} \quad (2.24)$$

and d^2 is given by:

$$d^2 = \begin{bmatrix} \frac{1}{4} \left(\frac{\partial v_1}{\partial x_3} \right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \left(\frac{\partial v_1}{\partial x_3} \right)^2 \end{bmatrix} \quad (2.25)$$

The second invariant I_{2d} (see equation 2.18) is given by:

$$I_{2d} = \frac{1}{2} \text{tr}(d^2) = \frac{1}{4} \left(\frac{\partial v_1}{\partial x_3} \right)^2 \quad (2.26)$$

Collecting the terms and substituting in equation 2.22, the stress tensor is given by:

$$\begin{aligned} \sigma = & -pI - \Phi_0 I + \Phi_1 \begin{bmatrix} 0 & 0 & \frac{1}{2} \frac{\partial v_1}{\partial x_3} \\ 0 & 0 & 0 \\ \frac{1}{2} \frac{\partial v_1}{\partial x_3} & 0 & 0 \end{bmatrix} \\ & + \Phi_2 \begin{bmatrix} \frac{1}{4} \left(\frac{\partial v_1}{\partial x_3} \right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \left(\frac{\partial v_1}{\partial x_3} \right)^2 \end{bmatrix} \end{aligned} \quad (2.27)$$

from where we obtain the components:

$$\begin{aligned} \sigma_{11} = \sigma_{33} &= -p - \Phi_0 + \frac{1}{4} \Phi_2 \left(\frac{\partial v_1}{\partial x_3} \right)^2 \\ \sigma_{22} &= -p - \Phi_0 \\ \sigma_{13} = \sigma_{31} &= \frac{1}{2} \Phi_1 \frac{\partial v_1}{\partial x_3} \end{aligned} \quad (2.28)$$

From above expressions, it can be concluded that both shear and normal stresses depend on the rate of shear strain, $\partial v_1 / \partial x_3$. The latter contribution is often referred to as dispersive stresses. The above expressions of the stress components will be used to generalize the results obtained in simple shear flow rheometers to more general stress conditions.

Other alternative is to neglect Φ_2 , then it is possible to show that Φ_1 does not depend on I_{3d} [37], hence:

$$\begin{aligned}\Phi_1 &= \Phi_1(I_{2d}) \\ \sigma &= pI + \Phi_1(I_{2d})\text{dev}(d)\end{aligned}\tag{2.29}$$

where,

$$p = \bar{p} + \Phi_0\tag{2.30}$$

The next step is to determine the depth integrated stresses and basal shear stresses (τ_B) without having information of the 3D flow structure, which has been lost in the averaging process (depth average model). In this regard, a possible solution is given by the simple shear infinite landslide model in which is assumed that the flow at a given point and time has the same vertical structure with known depth and averaged velocities as a uniform steady state flow moves along a constant slope. The main utilized variables and notation are depicted in Figure 2-3.

Figure 2-3. Infinite landslide model

Since the acceleration along the x-axis is zero, the equilibrium of the column in Figure 2-3 can be expressed as follows:

$$\rho g h \sin \theta = \tau_B \quad (2.31)$$

where ρ is the mixture density which will be described in detail in Chapter 3, g the gravity acceleration, h the depth of flow, θ the slope angle and τ_B the shear stress at the bottom.

By considering the equilibrium of a part of the column extending from the surface to a depth $(h - z)$, we can obtain the shear stress as a function of z as:

$$\tau = \rho g (h - z) \sin \theta \quad (2.32)$$

Therefore, the shear stress varies linearly from zero at the surface to a maximum at the bottom given by equation 2.31. Substituting the basal shear stress given in equation 2.31 in above equation 2.32, then the shear stress will be obtained after elimination of the slope angle as:

$$\tau = \tau_B \left(1 - \frac{z}{h}\right) \quad (2.33)$$

The next step is to analyze the depth averaged stresses and relate it to a depth integrated rate of deformation tensor. The starting point is the general rheological law given in equation 2.22. If the equation is integrated along depth, the averaged magnitudes will not satisfy the averaged equation,

$$\bar{\sigma} \neq -\bar{p}I - \bar{\Phi}_0 I + \bar{\Phi}_1 \text{dev}(\bar{d}) + \bar{\Phi}_2 \bar{d}^2 \quad (2.34)$$

This means that the integrated equation along depth is not fulfilled unless it was a linear equation, which only happens if Φ_0 and Φ_2 are zero and Φ_1 is constant, which

means that dispersive stresses are not considered. In these conditions, the rheological law can be written as:

$$\bar{\sigma} = -\bar{p}I + \bar{\Phi}_1 \text{dev}(\bar{d}) \quad (2.35)$$

Then, the averaged components of the rate of deformation tensor (d_{ij}) is obtained as:

$$\bar{d}_{ij} = \frac{1}{2h} \left(\int_z^{z+h} \frac{\partial v_i}{\partial x_j} dx_3 + \int_z^{z+h} \frac{\partial v_j}{\partial x_i} dx_3 \right) \quad i, j = 1, 2 \quad (2.36)$$

In the case of infinite landslide, a reference system is used where the integral of the previous expression is given by:

$$\frac{1}{h} \int_0^h \frac{\partial v}{\partial x} dz = \frac{1}{h} \left\{ \frac{\partial}{\partial x} \int_0^h v dz - v \frac{\partial h}{\partial x} \Big|_h \right\} \quad (2.37)$$

From where:

$$\frac{1}{h} \int_0^h \frac{\partial v}{\partial x} dz = \frac{\partial \bar{v}}{\partial x} \quad (2.38)$$

The depth of the flow does not depends on x . Thus, the above expression can be generalized as:

$$\frac{1}{h} \int_z^{z+h} \frac{\partial v_i}{\partial x_j} dx_3 = \frac{\partial \bar{v}_i}{\partial x_j} \quad (2.39)$$

$$\frac{1}{h} \int_z^{z+h} \frac{\partial v_j}{\partial x_i} dx_3 = \frac{\partial \bar{v}_j}{\partial x_i}$$

From here, substituting into equation 2.36, we have:

$$\bar{d}_{ij} = \frac{1}{2} \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right) \quad i, j = 1, 2 \quad (2.40)$$

The component $\bar{d}_{33}$ is obtained as follows:

$$\bar{d}_{33} = \frac{1}{h} \int_z^{z+h} \frac{\partial v_3}{\partial x_3} dx_3 = \frac{1}{h} (v_3|_{z+h} - v_3|_z) \approx \frac{1}{h} \frac{dh}{dt} \quad (2.41)$$

Next, the components $\bar{d}_{i3}$ can be calculated using the assumption that the flow is that of an infinite landslide:

$$\bar{d}_{i3} = \frac{1}{2h} \left\{ \int_z^{z+h} \frac{\partial v_i}{\partial x_3} dx_3 + \int_z^{z+h} \frac{\partial v_3}{\partial x_i} dx_3 \right\} \quad i = 1, 2 \quad (2.42)$$

The second term of the previous expression is zero and v_3 does not depend on x_i , finally we arrive at:

$$\bar{d}_{i3} = \frac{1}{2h} \int_z^{z+h} \frac{\partial v_i}{\partial x_3} dx_3 \quad (2.43)$$

which will be evaluated for the particular rheological model being used.

Take into account that the infinite landslide model can be used in order to determine the conditions at which a landslide is triggered, and provide a first estimate of paths and velocities, in addition to provide depth integrated stresses and basal friction.

2.3.1.2 Two-dimensional flow

The second case motion is an isochoric 2D flow, which takes place in the x_1x_3 plane. In order to obtain the component of tensors, the expressions of the last section will be used to generalize their results to more complex conditions (2D). The rate of deformation tensor is now given by:

$$d = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & 0 & \frac{1}{2} \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \\ 0 & 0 & 0 \\ \frac{1}{2} \left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \right) & 0 & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & d_{13} \\ 0 & 0 & 0 \\ d_{31} & 0 & d_{33} \end{bmatrix} \quad (2.44)$$

with $d_{11} + d_{33} = 0$. From here, we obtain:

$$\begin{aligned} d^2 &= \begin{bmatrix} d_{11}^2 + d_{13}^2 & 0 & d_{13}(d_{11} + d_{33}) \\ 0 & 0 & 0 \\ d_{13}(d_{11} + d_{33}) & 0 & d_{33}^2 + d_{13}^2 \end{bmatrix} \\ &= \begin{bmatrix} d_{11}^2 + d_{13}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_{33}^2 + d_{13}^2 \end{bmatrix} \end{aligned} \quad (2.45)$$

In consequence, the components of the stress tensor are given by:

$$\begin{aligned} \sigma &= -pI - \Phi_0 I + \Phi_1 \begin{bmatrix} d_{11} & 0 & d_{13} \\ 0 & 0 & 0 \\ d_{31} & 0 & d_{33} \end{bmatrix} \\ &+ \Phi_2 \begin{bmatrix} d_{11}^2 + d_{13}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_{33}^2 + d_{13}^2 \end{bmatrix} \end{aligned} \quad (2.46)$$

and the components of the tensor (σ) are expressed as:

$$\begin{aligned}
\sigma_{11} &= -p - \Phi_0 + \Phi_1 d_{11} + \Phi_2 (d_{11}^2 + d_{13}^2) \\
\sigma_{33} &= -p - \Phi_0 + \Phi_1 d_{33} + \Phi_2 (d_{33}^2 + d_{13}^2) \\
\sigma_{11} &= \sigma_{33} \\
\sigma_{22} &= -p - \Phi_0 \\
\sigma_{13} &= \sigma_{31} = \Phi_1 d_{13} \\
\sigma_{12} &= \sigma_{23} = 0
\end{aligned} \tag{2.47}$$

Again, it is found that (i) a shear stress depending only on d_{33} , and (ii) dispersive stress terms which depend on $\Phi_2(d_{11}^2 + d_{13}^2)$.

In the next sections, several rheological models will be presented which allow determination of the stress tensor once the rate of deformation tensor is known.

2.3.2 Newtonian fluids

The Newtonian fluid is a simple rheological model characterized by one single constitutive parameter, the fluid viscosity. In the case of a simple shear flow, the shear stress is given by:

$$\sigma_{13} = \mu \frac{\partial v_1}{\partial x_3} \tag{2.48}$$

where we have introduced the viscosity coefficient (μ) with units [Pa-s].

If we combine shear stresses expressions 2.28 and 2.48, the following relation can be obtained:

$$\Phi_1 = 2\mu \tag{2.49}$$

Consequently, the Newtonian fluid model can be written in general conditions as (see equation 2.29):

$$\sigma = -pI + 2\mu \text{dev}(d) \tag{2.50}$$

So far, the general formulation for the stress tensor as a function of the deformation tensor was developed. Then, these expressions will be integrated along the vertical axis in order to derive the expressions of the basal friction and depth average deformation tensor. The starting point is the shear stress equation (2.48) which will be rewritten here as:

$$\tau = \mu \frac{\partial v}{\partial z} \quad (2.51)$$

The last expression can be related to the shear stress equation (2.33) of the infinite landslide model as:

$$\mu \frac{\partial v}{\partial z} = \tau_B \left(1 - \frac{z}{h}\right) \quad (2.52)$$

Integrating the last expression along depth, the resulting velocity is obtained as:

$$v = \int \frac{\tau_B}{\mu} \left(1 - \frac{z}{h}\right) dz = \frac{\tau_B}{\mu} \left(z - \frac{z^2}{2h}\right) + c \quad (2.53)$$

where it is needed to take into account that the velocity is zero at the basal surface, $c = 0$.

The depth integrated velocity, which will be described in detail in section 3.4 of Chapter 3, can be related to the basal shear stress as:

$$\bar{v} = \frac{1}{h} \int_0^h v(z) dz = \frac{h\tau_B}{3\mu} \quad (2.54)$$

and therefore, the basal friction is obtained as:

$$\tau_B = \frac{3\mu\bar{v}}{h} \quad (2.55)$$

On the basis of these results, it can be concluded that an infinite slide in a Newtonian fluid is triggered when the slope is not zero and it does not stop until the slope is zero. Concerning the depth average deformation tensor, The terms $\bar{d}_{i3}$ and $\bar{d}_{33}$ are given by (see equations 2.40 and 2.43):

$$\begin{aligned} \bar{d}_{i3} &= \frac{1}{2h} \int_z^{z+h} \frac{\partial v_i}{\partial x_3} dx_3 = \frac{3\bar{v}}{4h} \quad i = 1, 2 \\ \bar{d}_{33} &= \frac{1}{h} \int_z^{z+h} \frac{\partial v_3}{\partial x_3} dx_3 = \frac{1}{h} \frac{\partial h}{\partial t} \end{aligned} \quad (2.56)$$

Take into account that the governing mathematical equations, which will be given in the Chapter 3, include a basal friction term. Therefore, we focus only on obtaining basal friction expressions for the rest of rheological models from now on. Obtaining expressions of depth average deformation tensor, for all the rheological models, is beyond the scope of this study.

2.3.3 Bagnold fluids

Bagnold (1954)[36] performed tests on a rheometer to study granular flows and proposed a model valid for simple shear conditions which can be used to explained the behavior the flows. It included simple shear and pressure terms, which depended on the shear strain rate, $\partial v/\partial z$. The stresses were given by the following experimental laws:

$$\begin{aligned} \tau &= \mu_B \sin \phi_B \left(\frac{\partial v}{\partial z} \right)^2 \\ \sigma_v = \sigma_h &= p + \mu_B \cos \phi_B \left(\frac{\partial v}{\partial z} \right)^2 \end{aligned} \quad (2.57)$$

where μ_B is a material constant and ϕ_B is the slope angle of a straight line which effective stress state (p, τ) moves along it, as can be seen in Figure 2-4.

Figure 2-4. Stress path in the (p, τ) plane for a Bagnold's fluid.

The Bagnold model is valid to describe the behavior of extremely fast flows of dry material formed by cohesionless particles. It is not recommended to apply it in the case of mixtures of water and soil, such as the flows of debris [38]. It is important to remark that Bagnold model has been obtained for a granular fluid under constant volume conditions, and special care should be taken when applying it to more general situations. Takahashi (1978)[39] also started a research line dedicated to the study of fluidized materials by applying Bagnold's theory, under the assumption that the debris flow's material is made of dispersive particles in a viscous fluid.

From expressions 2.33 and 2.57, after integration with respect to z , and using the boundary condition $v(0) = 0$, we arrive at:

$$v = \frac{2h}{3} \sqrt{\frac{\tau_B}{\mu_B \sin \phi_B}} \left\{ 1 - \left(1 - \frac{z}{h} \right)^{\frac{3}{2}} \right\} \quad (2.58)$$

From here, the following velocity profile can be obtained:

$$v = v_B \left[1 - \left(1 - \frac{z}{h} \right)^{\frac{3}{2}} \right] \quad (2.59)$$

where v_B is the velocity at the surface ($z = h$):

$$v_B = \frac{2h}{3} \sqrt{\frac{\tau_B}{\mu_B \sin \phi_B}} \quad (2.60)$$

The averaged velocity ($\bar{v}$) is obtained by integrating velocity along the vertical axis, and it is given by:

$$\bar{v} = \frac{3}{5} v_B = \frac{2h}{5} \sqrt{\frac{\tau_B}{\mu_B \sin \phi_B}} \quad (2.61)$$

Finally, the basal shear stress can be obtained as:

$$\tau_B = \frac{25}{4} \mu_B \sin \phi_B \frac{\bar{v}^2}{h^2} \quad (2.62)$$

2.3.4 Bingham fluids

So far, the rheological models of Newtonian and Bagnold have been described. These rheological models deform when subjected to shear stress. However, there exist some viscoplastic models that introduce a threshold shear stress that must be overcome so that the flow takes place. There exist different viscoplastic models such as viscoplastic Bingham, Herschel and Bulkley, and Coulomb-viscous [40].

The Bingham rheological model [41] is one of the simplest and the most widely used viscoplastic model, which is able to provide description to phenomena such as formation of plug regions and stoppage of flows. Rheology of hyper-concentrated flows, such as mudflows, lahars, and some cases of debris flows, have been frequently described by models such as Bingham or Herschel-Bulkley [42,43]. Bingham model has been applied in a variety of mathematical models such as the equations relating

averaged velocities. In this study, it will be used in depth integrated models in two dimensions, where bottom friction term is provided.

In the case of cohesive fluids, such as clay and water mixtures, flows exhibit a yield stress and their modeling was the pioneering work of Bingham (1922)[44]. In these models, the flow starts once the shear stresses exceed the threshold, represented by the yield stress (τ_y) then the movement is governed by a linear relationship between shear stress and strain rates, where the constant parameter of proportionality is called the coefficient of viscosity (μ). Rheological law governing the behavior of Bingham fluid is given by:

$$\tau = \tau_y + \mu \frac{\partial v}{\partial z} \quad (2.63)$$

Flow structure of a Bingham fluid, which is depicted in Figure 2-5, consists of two separate parts:

- i. From points S to P, which is called “area of the plug”, the mobilized shear stress is less than the yield stress. Taking into account that the velocity is constant in this region, therefore:

$$\frac{\partial v}{\partial z} = 0 \quad (2.64)$$

- ii. From P to B, which is called “cutting zone”, the mobilized shear stress is greater than the yield stress and varies according to the following expression:

$$\frac{\partial v}{\partial z} = \frac{1}{\mu} (\tau_y - \tau) \quad (2.65)$$

Figure 2-5. Flow structure of a Bingham fluid in an infinite simple shear flow.

Referring to Figure 2-5, Some relations of interest are obtained as follows:

$$h_p = \frac{\tau_y}{\rho g h \sin \theta} = h \frac{\tau_y}{\tau_B} \quad (2.66)$$

$$h_s = h \left(1 - \frac{\tau_y}{\tau_B} \right)$$

Then, the relative plug height (η), which is a relation between the thickness of plug and total thickness, is defined as:

$$\eta = \frac{h_p}{h} = \left(1 - \frac{h_s}{h} \right) = \left(1 - \frac{\tau_y}{\tau_B} \right) \quad (2.67)$$

For Bingham fluid, the following equations can be obtained:

- i. Depth averaged velocity, which relates the averaged velocity to the relative height of the plug η :

$$\bar{v} = \frac{\tau_B h}{6\mu} (1 - \eta)^2 (2 + \eta) \quad (2.68)$$

where η is estimated by using the approximate method, proposed by Pastor et al. (2004)[45], which is based on using a polynomial economization

technique. It consists on obtaining the second order polynomial $p_2(\eta)$ which gives a better approximation to $p_3(\eta)$ in the sense of the uniform distance in the interval $[0,1]$. The second order polynomial is obtained as follows:

$$P_2(\eta) = \frac{3}{2}\eta^2 - \left(\frac{57}{16} + a\right)\eta + \frac{65}{32} \quad (2.69)$$

where the non-dimensional number a can be obtained as:

$$a = \frac{6\mu\bar{v}}{\tau_y h} \quad (2.70)$$

It is important to remark that in order to obtain the flow structure, it is necessary to calculate the root of $p_2(\eta)$, which provide the relative height of the plug. Once this value is known, it will be possible to obtain basal shear stress, momentum correction factor and depth integrated stresses.

ii. Basal Shear Stress, which obtained by considering equations 2.66 and 2.67:

$$\tau_B = \frac{\tau_y}{\eta} \quad (2.71)$$

iii. Momentum correction factor α which is a part of balance of momentum equation which will be presented in Chapter 3:

$$\alpha = \frac{3}{5} \left(\frac{7\eta + 8}{(\eta + 2)^2} \right) - 1 \quad (2.72)$$

iv. Depth averaged stresses:

$$\bar{\sigma} = -\bar{p}I + \left(\frac{\tau_y}{\sqrt{I_{2D}}} + 2\mu \right) \text{dev}(\bar{d}) \quad (2.73)$$

where $\bar{d}$ is the averaged rate of deformation tensor. The averaged components, $\bar{d}_{33}$ and $\bar{d}_{13}$, of the rate of deformation tensor can be written as follows:

$$\begin{aligned}\bar{d}_{33} &= \frac{1}{h} \frac{\partial h}{\partial t} \\ \bar{d}_{13} &= \frac{\tau_B}{4\mu} (1 - \eta)^2\end{aligned}\tag{2.74}$$

The interested reader will find in the articles by Pastor (2004, 2009)[45,46] a detailed explanation about the last expressions.

2.3.5 Generalized viscoplastic model of Chen and Ling

Chen and Ling (1996)[47] proposed the generalized viscoplastic Flow model, which embrace many particular cases of rheological models. In the case of the infinite landslide, the following relations are given as:

$$\begin{aligned}\tau &= s + \mu_1 \left(\frac{\partial v}{\partial z} \right)^{m_1} \\ \sigma_v = \sigma_h &= -p + \frac{1}{3} \mu_2 \left(\frac{\partial v}{\partial z} \right)^{m_2}\end{aligned}\tag{2.75}$$

In above, we have taken compressions as positive. The parameters μ_1 and μ_2 are viscosity-like coefficients and, m_1 and m_2 are material constants, which can be obtained in this particular case as:

$$\begin{aligned}\mu_1 &= \mu_c \sin \phi_c \\ \frac{1}{3} \mu_2 &= \mu_c \cos \phi_c \\ m_1 &= m_2 = m\end{aligned}\tag{2.76}$$

and s characterizes the material strength which depends on the cohesion (c), the effective confining pressure (p') and the effective friction angle (ϕ):

$$s = c \cos \phi + p' \sin \phi \quad (2.77)$$

To better understand these equations and parameters, we have represented in Figure 2-6 the Mohr-Coulomb criterion, the stress conditions at failure and the stress path of the fluidized mixture of a Chen and Ling fluid.

Figure 2-6. Mohr-Coulomb criterion for the 2D incompressible flow of a Chen and Ling fluid.

It is important to notice that the strength term (s) in equation 2.77 depends on the effective confining pressure (p') which is given by:

$$p' = p - p_{hyd} - \Delta p_w \quad (2.78)$$

where the total pressure term (p) is obtained by adding up the contributions from soil skeleton and hydrostatic component of pore-water pressure as:

$$p = g(h - z)\{(1 - n)\rho_s + n\rho_w\} \quad (2.79)$$

where ρ_s and ρ_α are densities of the solid and fluid particles, respectively and n is the porosity. These parameters will be described in detail in Chapter 3.

The pore pressure in excess to the hydrostatic will be written as:

$$p_{hyd} + \Delta p_w = g(h - z)(\rho_w + \beta_w \rho_w) \quad (2.80)$$

where the variable β_w is introduced to characterizes the increment of pore pressure. For liquefaction $\beta_w = \rho'_d / \rho_w$ and the effective confining pressure is reduced to zero, otherwise $0 \leq \beta_w < \rho'_d / \rho_w$.

For convenience, we will introduce the following expressions in order to merge the densities:

$$\begin{aligned} \rho'_s &= \rho_s - \rho_w \\ \rho'_d &= (1 - n)\rho'_s \\ \rho'^*_d &= \rho'_d - \beta_w \rho_w \end{aligned} \quad (2.81)$$

Combine above equations, we arrive at:

$$p' = \rho'^*_d g(h - z) \quad (2.82)$$

Substitute in equation 2.77, then we have:

$$s = c \cos \phi + \rho'^*_d g(h - z) \cos \theta \sin \phi \quad (2.83)$$

The vertical distribution of shear stress and strength of the fluidized soil is depicted in Figure 2-7. The point P at which the shear strength is equal to the shear stress characterizes the plug. All the material above P will flow with a constant velocity of the plug (v_p), while the fluidized soil below P will be sheared. The height of the plug is denoted by h_p which will be zero in the case of a pure frictional fluid (section 2.3.6).

Figure 2-7. Shear strength and shear stress distributions for the infinite landslide with Chen and Ling model.

For Chen and Ling fluid, the following equations can be obtained:

i. Depth averaged velocity:

$$\alpha \eta - (1 - \eta)^{m+1} (m \eta + m + 1)^m = 0 \quad (2.84)$$

which is a polynomial of degree $2m + 1$. Parameter α is a non-dimensional number which is given by:

$$\alpha = \left(\frac{\bar{v} (m + 1) (2m + 1)}{h} \right)^m \left(\frac{\mu_c \sin \phi_c}{s_0} \right) \quad (2.85)$$

One interesting case is that of $m = 2$, which can be viewed as a generalization of Bagnold model. The polynomial is now:

$$P_5(\eta) = (1 - \eta)^3 (2\eta + 3)^2 - \alpha \eta \quad (2.86)$$

$$a = \frac{15}{2} \left(\frac{\mu_c \sin \phi_c}{s_0} \right)^{1/2} \quad (2.87)$$

In order to speed up computations in numerical models, polynomial economization technique can be used to build a second order polynomial $P_2(\eta)$ which is the best approximation of the fifth order polynomial $P_5(\eta)$ in the minimax sense. $P_2(\eta)$ can be obtained as:

$$P_2(\eta) = 10\xi^2 - (a + 19.2188)\xi + 9.1875 \quad (2.88)$$

ii. Momentum correction factor:

$$a = \frac{5}{4} \left\{ \frac{9 - 11\eta}{(3 + 2\eta)^2} \right\} - 1 \quad (2.89)$$

iii. Depth averaged stresses:

$$\begin{aligned} \bar{\sigma} = & -\bar{p}I + \left\{ \frac{s}{\sqrt{I_{2d}}} + 2\mu_c \sin \phi_c (4I_{2d})^{\frac{m_1-1}{2}} \right\} \text{dev}(\bar{d}) \\ & + \left\{ 4\mu_c \cos \phi_c (4I_{2d})^{\frac{m_2-2}{2}} \right\} \bar{d}^2 \end{aligned} \quad (2.90)$$

The interested reader will find in the articles by Pastor (2009) [28,48] a detailed explanation about the above expressions.

Herschel–Bulkey model [49] can be considered a particular case of the Chen's generalized viscoplastic fluid (see equation 2.75 where $\tau_y = s$ and $\varphi_c = 0$ in the Herschel–Bulkey model) and an extension of the Bingham model to account the effects of non-linear viscosity components. It was described by Laigle and Coussot (1997)[43] and is based on the following simple shear law:

$$\begin{aligned} \tau &= \tau_y + K \left(\frac{\partial v}{\partial z} \right)^m \\ \sigma_v &= \sigma_h = -p \end{aligned} \quad (2.91)$$

where m is a material constant. Take into account that Bingham model is recovered by setting $m = 1$. The Herschel–Bulkley fluid assumed to be purely cohesive and its dilatancy considered to be zero. It is recommended to use it for mudflows and muddy debris flows.

2.3.6 The frictional fluid

It is important to note that implementing models, such as the generalized viscoplastic model of Cheng and Ling, in depth integrated codes are not an easy task. Therefore, frictional fluid model can be a good alternative for its simplicity. It is derived from Chen and Ling model where the cohesion and all viscous terms are neglected.

If we further assume that cohesion is zero (frictional fluid), and using equation 2.75, we obtain:

$$\mu_F \left(\frac{\partial v}{\partial x} \right)^m = \tau - s \quad (2.92)$$

From above expression, the following velocity profile is obtained:

$$v = v_B \left[1 - \left(1 - \frac{z}{h} \right)^{\frac{m+1}{m}} \right] \quad (2.93)$$

where v_B is the velocity at the surface:

$$v_B = \left(\frac{\tau_B - s_B}{\mu_F} \right)^{\frac{1}{m}} \frac{m}{m+1} h \quad (2.94)$$

Then, the depth averaged velocity ($\bar{v}$) is given by:

$$\bar{v} = v_B \left[\frac{1+m}{1+2m} \right] \quad (2.95)$$

Finally, the basal shear stress can be obtained as:

$$\tau_B = s_B + \mu_F \left(\frac{1 + 2m}{m} \right)^m \frac{\bar{v}^m}{h^m} \quad (2.96)$$

where s_B is the strength at the basal surface. In case of a frictional law, the basal strength in infinite landslides is given by:

$$s_B = \rho'^* g h \cos \theta \tan \phi \quad (2.97)$$

where the term $\rho'^* g h \cos \theta$ provides the normal stress to the bottom. It is interesting to note that the above expression can be particularized by taking $m = 2$. Then, the basal friction is given by:

$$\tau_B = \rho'^* g h \cos \theta \tan \phi + \mu_F \frac{25}{4} \frac{\bar{v}^2}{h^2} \quad (2.98)$$

It is interesting to note the similarity with Voellmy's law, which has introduced by Voellmy (1955)[49] in the case of snow avalanche. Körner (1976) developed it to model rock avalanches and then, various researchers [50,51] have been successfully using it to model rock avalanches, sliding flows and debris flows. It has the same features of the frictional fluid model, and its basal shear stress is approximated by:

$$\tau_B = (\rho'_d g h - \Delta p_w^b) \frac{\bar{v}_i}{|\bar{v}|} \tan \phi_B + \rho g \frac{|\bar{v}|}{\xi} \bar{v}_i \quad (2.99)$$

where h is the propagation height, ϕ_B the basal friction angle, $\bar{v}$ the depth averaged flow velocity, ξ the turbulence coefficient which takes into account the thickness of the basal layer where most of the tangential deformation in granular materials would take place and Δp_w^b the excess pore-water pressure at the basal surface.

Take into account that whenever the pore-water pressure is higher, the shear stress will be lower at the base of the material.

Manzanal (2016)[50] recommended to include the effect of centripetal accelerations in equation 2.99. In order to implement it, the simplest manner is to obtain the depth averaged velocity ($\bar{v}$) from the balance of momentum equation, which will be described in Chapter 3, assuming a constant vertical acceleration given by $\bar{v}^2/r_c$, where r_c is the main radius of curvature in the direction of the flow. Finally, the term $\bar{v}^2/r_c$ will be added to the parameter g of the equation 2.99.

In the case of a pure frictional mass, the vertical distributions of shear stress and strength are depicted in Figure 2-8, are given by:

$$\begin{aligned}\tau &= \rho g(h-z) \sin \theta \\ s &= p' \tan \phi\end{aligned}\tag{2.100}$$

Figure 2-8. Shear strength and shear stress distributions for the infinite landslide in the case of pure frictional fluids (without cohesion).

It is important to note that it is not possible to get the velocity distribution, the correction factor α in depth integrated models and, the distribution of the shear stress along the vertical without having additional data. Concerning the basal shear stress, it is usually approximated as:

$$\tau_B = -p' \tan \phi \frac{\bar{v}_i}{|\bar{v}|} \quad (2.101)$$

Pastor et al. (2009) [28] developed the basal shear stress equation based on this approach for the cases in which the fluidized soil flows over a basal surface made of a different material. The basal shear stress is given by:

$$\tau_B = -(\rho'_d - \beta_w \rho_w)gh \tan \phi_B \frac{\bar{v}_i}{|\bar{v}|} \quad (2.102)$$

where the basal friction ϕ_B is

$$\phi_B = \min(\delta, \theta) \quad (2.103)$$

The parameter δ is the friction angle between both materials, and θ the friction angle of the fluidized soil.

Chezy–Manning equation can also be used in the case of channel flows as an alternative frictional fluid model. Its basal friction is given by:

$$\tau_B = -\rho \frac{gn^2 |\bar{v}|}{h^{7/3}} \bar{v}_i \quad (2.104)$$

where n is the Manning number.

2.3.7 Perzyna viscoplastic constitutive model

So far, the approaches for triggering are usually based on a suitable rate dependent constitutive law relating the increment of effective stress and strain, while the approaches for propagation are based on rheological laws relating the stress and rate of deformation. Under this approach, constitutive laws have to switch to a rheological ones at a certain moment and this makes the whole process difficult [51]. In this section, a rheological model, based on the Perzyna viscoplastic constitutive model, will be described which could be the link between triggering and propagation models. Considering the simple shear rheological law given in equation 2.75, the rate of shear strain can be written as:

$$\frac{\partial v}{\partial z} = \frac{1}{\mu^{\frac{1}{m}}} (\tau - s)^{\frac{1}{m}} \quad (2.105)$$

which it has an interesting similitude to the viscoplastic fluid rheological models.

The next step is to compare Perzyna's viscoplastic models to the rheological laws described in section 2.3. If we obtain similar results, the model is able to reproduce both the triggering and propagation of the landslide in a simpler manner. Pastor (2015)[52,53] explored the applicability of 3D infinite landslide models by implementing Perzyna viscoplastic laws to describe fast landslides. In these studies, three different viscoplasticity models of Von Mises type, a Mohr–Coulomb and Cam Clay model were chosen. It was found that Perzyna-Von Mises models provided the same results obtained with Bingham models, while Cam Clay and Mohr–Coulomb types predicted velocity profiles closer to linear.

Next, a Perzyna model based on simple shear infinite landslide models will be presented. Considering the arbitrary function in equation 2.5, if $\varphi(F)$ is chosen as the rate of shear strain, $F = \tau$ and $F_0 = s$, we will have:

$$\frac{\partial v}{\partial z} = \gamma \left(\frac{\tau - s}{s} \right)^N \quad (2.106)$$

where the shear stress and strength (see equations 2.32) have already been defined previously. The above equation will coincide with equation 2.75 when the elastic part of the viscoplastic strain rate is negligible, $N = 1/m$ and $\gamma = 1/\mu^{1/m}$.

Substituting shear stress (τ) and strength equation (2.100) in above expression, we obtain:

$$\frac{\partial v}{\partial z} = \gamma \left(\frac{\tan \theta - \tan \varphi}{\tan \varphi} \right)^N \quad (2.107)$$

Integrating the above equation along depth, then depth integrated velocity can be obtained as:

$$\bar{v} = \frac{1}{2} \gamma c t^N h \quad (2.108)$$

where,

$$c t = \frac{\tan \theta - \tan \varphi}{\tan \varphi} \quad (2.109)$$

and consequently,

$$\frac{\partial v}{\partial z} = \gamma c t^N = \frac{2\bar{v}}{h} \quad (2.110)$$

In order to obtain basal shear stress, the shear stress and strength equations 2.33, and above expression are substituted in equation 2.106, we arrive at:

$$\tau_B = s_B \left[1 + \left(\frac{2\mu_{PZ}\bar{v}}{h} \right)^{\frac{1}{N}} \right] \quad (2.111)$$

where we have introduced μ_{PZ} which is the inverse of the fluidity parameter (γ).

Pastor (2015)[53] proposed an alternative way to define the rate of basal shear stress as:

$$\frac{\partial v}{\partial z} = \gamma \frac{(\tau - s)^N}{P_{atm}^{n_1} s^{n_2}} \quad (2.112)$$

where $N = n_1 + n_2$, and p_{atm} is the atmospheric pressure or an alternative unity datum.

The velocity profiles will depend now on n_1 and after some algebra it is given by:

$$v = v_{\max} \left[1 - \left(1 - \frac{z}{h} \right)^{n_1+1} \right] \quad (2.113)$$

where $v_{\max}$ is the velocity at the top.

Considering equations 2.110 and 2.112, the shear stress τ is given by:

$$\tau = s \left[1 + \left(\frac{\bar{v} 2 \mu}{h} \right)^{\frac{1}{N}} P_{atm}^{\frac{n_1}{N}} s^{\frac{-n_1}{N}} \right] \quad (2.114)$$

The presented alternative rheological model derives from the continuum mechanics rather than empirical obtained from previous rheological laws, and can provide suitable laws for both fluid and solid behaviors, as demonstrated by Pastor et al. (2010, 2015)[53,54] and Blanc (2012, 2013)[55–57].

So far, many alternative rheological models were described. These laws have been implemented in a GeoFlow_SPH depth integrated code, and will be applied to

simulate debris flows to consider the influence of the rheology on the material properties. The selection of a particular rheological model is not a trivial task. In fact, different models can provide similar results. Rheological parameters are often obtained from the back-analysis of case histories, and thus simple rheological models, with a limited number of parameters, are recommended to use. In this study, the cases represent debris flows, where basal pore pressure plays an important role. For these reasons, frictional rheological model will be used as it considers the effect of pore pressure at the basal surface.

It is important to note that if the effect of entrainment is considered in modeling, it is recommended to use Perzyna viscoplastic model because its velocity profile is linear, as shown in Figure 2-9, and therefore the model is more adaptable to erosion laws, which will be described in the next section.

Figure 2-9. Velocity profile obtained using a Perzyna's Mohr-Coulomb model.

2.4 Bed entrainment

Bed entrainment, also called basal erosion, is a process which causes an increase in landslide volume due to the inclusion of soil, debris and trees uprooted from the ground surface during the flow propagation [58]. Entrainment of material, located along the landslide path, is outlined as a key factor in controlling the dynamics of landslides with a flow-like character as observed in many catastrophic landslides. Therefore, it is necessary to understand its dynamics and consider the entrainment effects on runout analysis results.

The key parameter in evaluating of entrainment process is called entrainment rate (e_R) which is defined as the derivative of the ground surface elevation (z) with respect to the propagation time of flow:

$$e_R = -\frac{\partial z}{\partial t} \quad (2.115)$$

which is generally agreed that the entrainment is positive if the elevation of ground surface diminishes [59].

It is important to note that the entrainment rate depends on several variables, including the flow structure, the slope angle, the sizes and densities of the solid particles, the saturation degree and effective stresses at the base of the flow. Moreover, in order to consider properly the influence of bed entrainment in modeling, various parameters such as velocity and height of the flow, basal pore-water pressure and the rheology of the flow if the flowing mass and the entrained materials are different, are needed to be modified at each time step.

Based on these aspects, various empirical and process-based laws governing the landslide growth rate (E_r) have been proposed in the literature for different types of flow. Pirulli and Pastor (2012)[60] and also Cascini et al. (2014)[61] have provided a comprehensive review of the models. Here, the laws which already implemented in the GeoFlow_SPH code will be described.

i. Hungr erosion law, which is the formulation of Hungr et al. (2005)[62], is based on algorithm that the total volume of debris increases in accordance with a specified rate. Therefore, the erosion rate (e_R) is defined as:

$$e_R = E_s \cdot h \cdot \bar{v} \quad (2.116)$$

which indicates a direct proportionality between the entrainment rate and the product of depth average velocity ($\bar{v}$) and mobilized soil depth (h). The landslide growth rate (E_s) can be obtained directly from the initial volume entering the erosion zone (V_0), final volume exiting the erosion zone (V_f) and the path length of the erosion zone (l) as:

$$E_s = \frac{\ln\left(\frac{V_f}{V_0}\right)}{l} \quad (2.117)$$

ii. Egashira (1993)[63] proposed an entrainment rate formula which is based on flume tests, as well as numerical and dimensional analyses. Referring to the Figure 2-10, the entrainment rate formula can be written as:

$$e_R = K C_* \bar{v} \tan(\theta - \theta_e) \quad (2.118)$$

with

$$\tan\theta_e = \frac{(\rho_s - \rho_w)c}{(\rho_s - \rho_w)c + \rho} \tan\phi \quad (2.119)$$

where K is a empirical factor, C_* the bed sediment concentration by volume, $\bar{v}$ depth average velocity, θ the bed slope, θ_e the equilibrium bed slope, ρ_s the mass density of the sediment particle, ρ_w the mass density of the water, c the sediment concentration of the debris flow by volume, and ϕ internal friction angle of the bed approximated by the basal friction angle $\tan\phi_B$.

Figure 2-10. Definition sketch of the Egashira erosion rate.

iii. Blanc (2008)[64] applied the Egashira erosion law in depth integrated equation in order to consider entrainment effects in the modeling of debris flows. This study concluded that the Egashira erosion law is able to predict more accurately some characteristics of debris flows, such as deposition pattern and flow velocity. He also proposed an erosion law composed of the parameters, such as the channel slope, found important by assessing the Hungr and Egashira erosion laws. The formulation is written as:

$$e_R = h \cdot \bar{v} \cdot K (\tan \theta)^{2.5} \quad (2.120)$$

where h is the propagation height, $\bar{v}$ the depth average velocity of flow, θ the slope angle, K an empirical parameter to be calibrated, and the exponent equal to 2.5 is purely empirical.

The proposed erosion laws describe the rate at which the basal surface is eroded, and therefore can be implemented very easily in depth integrated models.

Chapter 3

MATHEMATICAL MODEL

3.1 Introduction

A mathematical model is a set of equations in which describes the most important aspects of a physical phenomenon. Geotechnical phenomenon has also been described by a general mathematical model based on the Biot equations (section 3.2).

In landslide propagation, the fluidized geomaterials are mixtures of solid particles and pore fluids, in which coupling between them plays a paramount role in the behavior of geomaterials [65–68]. Therefore, the mathematical model describing the problem has to implement this effect. There are different possibilities to describe the coupling between skeleton and pore fluids. In this study, the most general framework is chosen which is given by the mixture theory [69–71]. In this theoretical framework, porosities, densities and partial stresses are defined for all constituents, and balance of mass and momentum equations is established for all species and the mixture [11,72].

A few realistic simplifying assumptions will be considered to obtain the approximate solutions which are acceptable for many engineering purposes. The fluidized soil will be assumed to be consist of a solid skeleton and a fluid phase (two-phase), in which the fluid fills completely the voids (fully saturated: $S_r = 1$) (see Figure 3-1). We will denote them by the sub-indexes s for solid and w for fluid. In most of cases, the fluid phase is inviscid and it is either water, or a mixture of water with very fine soil particles such as clay, which can be considered as a fluid. In this study, the behavior of soil will be analyzed at macroscopic scale since presenting soils in a variable geometry at microscopic scale can complicate the results.

Figure 3-1. Two phase soil sketch.

The solid skeleton is made of solid particles with density (ρ_s) having a porous structure which is characterized by its porosity (n , volume percent of voids in the mixture) and a voids ratio (e , volume of voids per unit volume of solid fraction). Both are related by:

$$n = \frac{e}{1 + e} \quad (3.1)$$

In the most general cases, there are N_α fluid phases which are immiscible. For each fluid it is possible to introduce n_α as the volume of fraction of the voids occupied by phase α .

$$n = \sum_{\alpha=1}^{N_\alpha} n_\alpha \quad (3.2)$$

In the case of non-saturated soils, assuming there are only two phases, fluid and air, in voids and then we have:

$$n = \sum_{\alpha=1}^{N_\alpha} n_\alpha + n_a \quad (3.3)$$

where we have introduced n_α and n_a which correspond to the volume fraction of the voids occupied by fluid phases and air phase, respectively.

The fraction of the total volume of the mixture occupied by the solid and water particles are:

$$\begin{aligned} n_s &= 1 - n \\ n_w &= n \end{aligned} \tag{3.4}$$

In the general case of a mixture of solid particles and fluid phases is possible to define degree of saturation (S_α) which is expressed as the ratio of a fluid α contained in the pores of soil. It can be written as:

$$S_\alpha = \frac{n_\alpha}{n} \tag{3.5}$$

In this study, it is useful to introduce the degree of saturation as the fraction of voids occupied by water. It is usually denoted as S_r , but in the framework of mixtures theory, we will prefer to use S_w and S_a for water and air, respectively. The most frequently saturation state of soils are:

$$\begin{aligned} S_w &= 0, S_a = 1 && \text{dry soil} \\ S_w &= 1, S_a = 0 && \text{saturated soil} \\ S_w &\neq 1, S_a \neq 0, S_w + S_a = 1 && \text{unsaturated soil} \end{aligned} \tag{3.6}$$

Density (ρ) is introduced as the relationship between the mass and volume of the mixture. The density of the solid phase ($\rho^{(s)}$) and fluid phase ($\rho^{(\alpha)}$) are given by:

$$\begin{aligned} \rho^{(s)} &= (1 - n)\rho_s \\ \rho^{(\alpha)} &= \sum_{\alpha=1}^{N_\alpha} (nS_\alpha\rho_\alpha) \end{aligned} \tag{3.7}$$

where ρ_s and ρ_α are densities of the solid and fluid particles, respectively and N_α the number of immiscible interstitial fluids.

Total stress tensor (Cauchy) acting on the mixture is denoted by σ and assuming that it can be decomposed of stresses for each phase:

$$\sigma = \sigma^{(s)} + \sum_{\alpha=1}^{N_{\alpha}} \sigma^{(\alpha)} \quad (3.8)$$

where $\sigma^{(s)}$ and $\sigma^{(\alpha)}$ are the partial stresses of solid and fluid phases, respectively and, α denotes an immiscible liquid phase. They provide a complementary explanation to the magnitudes used in geotechnical engineering. The partial stresses of solid and fluid phases will be defined as:

$$\begin{aligned} \sigma^{(s)} &= (1 - n)\sigma_s \\ \sum_{\alpha=1}^{N_{\alpha}} \sigma^{(\alpha)} &= \sum_{\alpha=1}^{N_{\alpha}} (nS_{\alpha}\sigma_{\alpha}) \end{aligned} \quad (3.9)$$

where σ_s and σ_{α} are the stresses acting on the solid and pore fluid phases respectively.

It is essential to determine the derivative of stresses here, as it will be widely used later. The derivative of the stress acting on fluid phase is:

$$\sum_{\alpha=1}^{N_{\alpha}} \text{div} \sigma^{(\alpha)} = \sum_{\alpha=1}^{N_{\alpha}} \text{div}(nS_{\alpha}\sigma_{\alpha}) = \sum_{\alpha=1}^{N_{\alpha}} nS_{\alpha} \text{div} \sigma_{\alpha} + \sum_{\alpha=1}^{N_{\alpha}} \sigma_{\alpha} \text{grad}(nS_{\alpha}) \quad (3.10)$$

As already mentioned in the section 2.3.1 of the previous chapter, the partial stress of fluid phase can be decomposed into hydrostatic and deviatoric components as follows:

$$\sum_{\alpha=1}^{N_{\alpha}} \sigma^{(\alpha)} = - \sum_{\alpha=1}^{N_{\alpha}} nS_{\alpha} p_{\alpha} I + \sum_{\alpha=1}^{N_{\alpha}} nS_{\alpha} s_{\alpha} \quad (3.11)$$

where p_α is the total pore pressure acting on fluid phases, I the second order identity tensor and s_α the deviatoric part of the stress tensor:

$$s_\alpha = \text{dev}(\sigma) = \sigma - \frac{1}{3} \text{tr}(\sigma) I \quad (3.12)$$

which is neglectable in the case of an inviscid pore fluid. Then, its derivative is given by:

$$\sum_{\alpha=1}^{N_\alpha} \text{div} \sigma^{(\alpha)} = \sum_{\alpha=1}^{N_\alpha} n S_\alpha (\text{div} s_\alpha - \text{grad} p_\alpha) + \sum_{\alpha=1}^{N_\alpha} (s_\alpha - p_\alpha) \text{div}(n S_\alpha) \quad (3.13)$$

The stresses and its derivative acting on the fluid particle respectively are given by:

$$\sigma_\alpha = - \sum_{\alpha=1}^{N_\alpha} (S_\alpha p_\alpha I) + \sum_{\alpha=1}^{N_\alpha} (S_\alpha s_\alpha) \quad (3.14)$$

$$\text{div} \sigma_\alpha = - \sum_{\alpha=1}^{N_\alpha} S_\alpha \text{grad} p_\alpha + \sum_{\alpha=1}^{N_\alpha} S_\alpha \text{div} s_\alpha \quad (3.15)$$

By considering equations 3.8 and 3.11, we obtain the Cauchy stress and its derivative as:

$$\sigma = \sigma^{(s)} - \sum_{\alpha=1}^{N_\alpha} n S_\alpha p_\alpha I + \sum_{\alpha=1}^{N_\alpha} n S_\alpha s_\alpha \quad (3.16)$$

$$\text{div} \sigma = \text{div} \sigma^{(s)} + \sum_{\alpha=1}^{N_\alpha} n S_\alpha (\text{div} s_\alpha - \text{grad} p_\alpha) + \sum_{\alpha=1}^{N_\alpha} (s_\alpha - p_\alpha) \text{div}(n S_\alpha) \quad (3.17)$$

The average pore pressures ($\bar{p}$) and deviatoric stresses ($\bar{s}$) of the fluid phases will be introduced as:

$$\begin{aligned}\bar{p} &= \sum_{\alpha=1}^{N_{\alpha}} (S_{\alpha} p_{\alpha}) \\ \bar{s} &= \sum_{\alpha=1}^{N_{\alpha}} (S_{\alpha} s_{\alpha})\end{aligned}\tag{3.18}$$

Therefore, taking into account equations 3.8, 3.9, 3.14, and 3.18, the total stress tensor can be written as:

$$\sigma = (1 - n)\sigma_s - n\bar{p}I + n\bar{s}\tag{3.19}$$

According to the concept of effective stress in Soil Mechanics, introduced by Terzaghi and Rendulic [73,74], deformations in the solid phase are due to effective stress tensor (σ'), and therefore the mechanical behavior of soil–water mixtures depends on them. Accordingly, the stress tensor is decomposed into an effective stress tensor acting on soil skeleton and average pore pressures:

$$\sigma = \sigma' - \bar{p}I\tag{3.20}$$

where $nS_{\alpha} \approx 1$ in order to consider effective area coefficient in hydrostatic pressure term. If we define the effective component stress as:

$$\sigma_s^{eff} = \sigma_s + \bar{p}I\tag{3.21}$$

We are now in position to convert the concept of effective stress to more general expression for mixtures of solid particles and pore fluids.

Therefore, the following relation is obtained:

$$\sigma' = (1 - n)\sigma_s^{eff} = (1 - n)(\sigma_s + \bar{p}I)\tag{3.22}$$

Both effective stresses are related by:

$$\sigma' = \sum_{\alpha=1}^{N_{\alpha}} (1 - nS_{\alpha}) \sigma_s^{eff} = (1 - nS_{\alpha}) \{\sigma_s + p_{\alpha} I\} \quad (3.23)$$

By considering equation 3.16, 3.18 and 3.20, the effective stress is related to the solid partial stress by:

$$\sigma' = \sigma^{(s)} + \sum_{\alpha=1}^{N_{\alpha}} (1 - nS_{\alpha}) p_{\alpha} I \quad (3.24)$$

and its derivative is given by:

$$\text{div} \sigma' = \text{div} \sigma^{(s)} + \sum_{\alpha=1}^{N_{\alpha}} (1 - nS_{\alpha}) \text{grad} p_{\alpha} - \sum_{\alpha=1}^{N_{\alpha}} p_{\alpha} I \text{grad} n \quad (3.25)$$

By substituting solid partial stress equation (3.9) in effective stress equation (3.24), we could use the relation between σ_s and the effective stress tensor:

$$\sigma' = \sum_{\alpha=1}^{N_{\alpha}} (1 - nS_{\alpha}) (\sigma_s + p_{\alpha} I) \quad (3.26)$$

So far, porosities, densities and partial stresses is defined for all constituents, and in the next session balance of mass and momentum equations is established for all species and for the mixture.

3.2 General model

This model is applicable to the most general geotechnical systems with capability of representing both static and dynamic behavior of soils. The dynamics of the soil, as a mixture in flow-like landslides, are governed by the sets of equations, which include balance of mass and balance of linear momentum. Then, these equations have to be

complemented by suitable constitutive relations and kinematic equations. It is important here to recall the concept of material derivative and Darcy's velocity, as they will be of wide use from now on.

The material derivative describes the time derivative, rate of change of some physical quantity for a material element, and in the case of a scalar field $\varphi(x, t)$, it is defined as:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + v^T \text{grad}\varphi \quad (3.27)$$

where $d\varphi/dt$ describes the time rate of change of some physical quantity. The first and second terms on the right hand side represent the change in time of a fixed point and the spatial rate of change of a point, respectively.

One important feature of debris flows is that the velocities of both phases can be different. Accordingly, the mathematical models use following derivatives, for the solid ($d^{(s)}/dt$) and fluid ($d^{(\alpha)}/dt$) phases, in order to describe the balance of mass and momentum for both phases:

$$\begin{aligned} \frac{d^{(s)}\varphi}{dt} &= \frac{\partial\varphi}{\partial t} + v_s^T \cdot \text{grad}\varphi \\ \frac{d^{(\alpha)}\varphi}{dt} &= \frac{\partial\varphi}{\partial t} + v_\alpha^T \cdot \text{grad}\varphi \end{aligned} \quad (3.28)$$

where v_s and v_α are the velocities of solid and fluid particles in the mixture.

There is a relation between material derivatives following the fluid and solid particles, which can be obtained by subtracting the definitions of them (equation 3.28) in the following way:

$$\frac{d^{(s)}\varphi}{dt} - \frac{d^{(w)}\varphi}{dt} = (v_\alpha - v_w)^T \text{grad}\varphi \quad (3.29)$$

When a mixture of solid and fluid particles is moving, all the phases will have different velocities in the most general conditions. In classical Soil Mechanics, the analysis is performed in a lagrangian framework for the solid skeleton, which moves with its velocity, where relative velocities are small. Movement of the fluid can be considered as the motion of pore water relative to soil skeleton which is given by the Darcy's velocity (ω_α) and expressed mathematically as follows:

$$\omega_\alpha = nS_\alpha(v_\alpha - v_s) \quad (3.30)$$

Taking into account equation 3.29 and introducing the Darcy's velocity of the fluid phase, it is possible to arrive at the following equation:

$$\frac{d^{(\alpha)}\varphi}{dt} - \frac{d^{(s)}\varphi}{dt} = \frac{\omega_\alpha^T}{nS_\alpha} \cdot \text{grad}\varphi \quad (3.31)$$

3.2.1 Balance of mass

The balance of mass equation is based on the conservation of mass which states that the quantity of mass is conserved over time for any closed system. It can be formulated mathematically in the fields of fluid and continuum mechanics by considering a representative volume fixed in space, using the differential eulerian form of the continuity equation. In this study, the equation is obtained by using the notation given in Figure 3-2:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho v) = 0 \quad (3.32)$$

Figure 3-2. notation and reference system.

where the time derivative is the accumulation (or loss) of mixture density (ρ) and the divergence term represents the difference between inputs and outputs of a flow. The above balance of mass equation can be still rewritten in a eulerian form as:

$$\frac{\partial \rho}{\partial t} + v \cdot \text{grad} \rho + \rho \text{div} v = 0 \quad (3.33)$$

Taking into account the definition of material derivative (see equation 3.27), lagrangian expression of balance of mass equation can be written as follows:

$$\frac{d\rho}{dt} + \rho \text{div} v = 0 \quad (3.34)$$

These density evolutions are calculated throughout the simulation by solving the balance of mass equation in a numerical model. Taking into account that the mass of each particle remains constant ($dm/dt = 0$) and only their associated density changes where the flow is considered weakly-compressible. In the case of incompressible materials ($\text{div} v = 0$), the above equation particularizes to:

$$\frac{d\rho}{dt} = 0 \quad (3.35)$$

In the most general condition, when a mixture of solid and fluid particles moves, each phase has a different speed. Therefore, balance of mass in each mixture component is generated separately. The balance of mass equation corresponding to the fluid phase are given by:

$$\frac{d^{(\alpha)}\rho^{(\alpha)}}{dt} + \rho^{(\alpha)}\text{div}v_{\alpha} = 0 \quad (3.36)$$

If we consider the density of fluid phase in equation 3.7, the balance of mass equation for the fluid is expressed as:

$$\frac{d^{(\alpha)}\sum_{\alpha=1}^{N_{\alpha}}(nS_{\alpha}\rho_{\alpha})}{dt} + \sum_{\alpha=1}^{N_{\alpha}}(nS_{\alpha}\rho_{\alpha})\text{div}v_{\alpha} = 0 \quad (3.37)$$

For simplicity, we consider only one fluid phase and the above equation can be rewritten as:

$$\frac{n}{S_{\alpha}\rho_{\alpha}}\frac{d^{(\alpha)}(S_{\alpha}\rho_{\alpha})}{dt} + \frac{d^{(\alpha)}n}{dt} + n\text{div}v_{\alpha} = 0 \quad (3.38)$$

By expanding the first term, we arrive at the following formula:

$$\frac{n}{\rho_{\alpha}}\frac{d^{(\alpha)}\rho_{\alpha}}{dt} + \frac{n}{S_{\alpha}}\frac{d^{(\alpha)}S_{\alpha}}{dt} + \frac{d^{(\alpha)}n}{dt} + n\text{div}v_{\alpha} = 0 \quad (3.39)$$

If we consider an elastic behavior for both of the solid and fluid grains, and assume that volumetric deformations induced only by the pore pressure (p_{α}), the constitutive equations of the solid and fluid phases can be expressed in terms of the volumetric stiffness K_s and K_{α} for solid and fluid phases respectively, as:

$$\begin{aligned}\frac{1}{\rho_s} \frac{d^{(s)}\rho_s}{dt} &= \frac{1}{K_s} \frac{d^{(s)}p_\alpha}{dt} \\ \frac{1}{\rho_\alpha} \frac{d^{(\alpha)}\rho_\alpha}{dt} &= \frac{1}{K_\alpha} \frac{d^{(\alpha)}p_\alpha}{dt}\end{aligned}\tag{3.40}$$

and after substituting the above expressions in the equation 3.39, finally, we obtain the balance of mass equation of the fluid phase as:

$$\frac{1}{K_\alpha} \frac{d^{(\alpha)}p_\alpha}{dt} + \frac{n}{S_\alpha} \frac{d^{(\alpha)}S_\alpha}{dt} + \frac{d^{(\alpha)}n}{dt} + n \operatorname{div} v_\alpha = 0\tag{3.41}$$

We introduce the specific storage coefficient of the solid ($C_{s\alpha}$), which relates the variation of saturation with respect to the pressure of fluid phase as follows:

$$C_{s\alpha} = n \frac{d^{(\alpha)}S_\alpha}{d^{(\alpha)}p_\alpha}\tag{3.42}$$

and after substituting in equation 3.41, finally, we obtain the balance of mass equation of the fluid phase as:

$$\left\{ \frac{n}{K_\alpha} + \frac{C_{s\alpha}}{S_\alpha} \right\} \frac{d^{(\alpha)}p_\alpha}{dt} + \frac{d^{(\alpha)}n}{dt} + n \operatorname{div} v_\alpha = 0\tag{3.43}$$

For the balance of mass of solid particles, following a similar procedure, it is possible to arrive at:

$$\frac{d^{(s)}\rho^{(s)}}{dt} + \rho^{(s)} \operatorname{div} v_s = 0\tag{3.44}$$

The above equation can be written in term of solid particle density based on equation 3.7 as follows:

$$\frac{d^{(s)}}{dt} [(1-n)\rho_s] + [(1-n)\rho_s] \text{div} v_s = 0 \quad (3.45)$$

By expanding the first term, we rewrite the above equation as:

$$\frac{1-n}{\rho_s} \frac{d^{(s)}\rho_s}{dt} - \frac{d^{(s)}n}{dt} + (1-n) \text{div} v_s = 0 \quad (3.46)$$

We will relate the rate of volumetric strain to the material derivative of the averaged pressure:

$$\frac{1}{\rho_s} \frac{d^{(\alpha)}\rho_s}{dt} = \frac{1}{K_s} \frac{d^{(\alpha)}\bar{p}}{dt} \quad (3.47)$$

Therefore, the equation 3.46 can be written as:

$$\frac{(1-n)}{K_s} \frac{d^{(s)}\bar{p}}{dt} - \frac{d^{(s)}n}{dt} + (1-n) \text{div} v_s = 0 \quad (3.48)$$

where average interstitial pressure $\bar{p}$, is given by equation 3.18. Next, we will derive the averaged pressure with respect to time:

$$\frac{d^{(s)}\bar{p}}{dt} = \frac{d^{(s)}}{dt} \left(\sum_{\alpha=1}^{N_\alpha} (S_\alpha p_\alpha) \right) = \sum_{\alpha=1}^{N_\alpha} \left(S_\alpha \frac{d^{(s)}p_\alpha}{dt} + p_\alpha \frac{d^{(s)}S_\alpha}{dt} \right) \quad (3.49)$$

Substituting the above expression in the balance of mass equation 3.48, we have:

$$\frac{(1-n)}{K_s} \left(\sum_{\alpha=1}^{N_\alpha} \left(S_\alpha \frac{d^{(s)}p_\alpha}{dt} + p_\alpha \frac{d^{(s)}S_\alpha}{dt} \right) \right) - \frac{d^{(s)}n}{dt} + (1-n) \text{div} v_s = 0 \quad (3.50)$$

Taking into account the following expression:

$$C_{ss} = n \frac{d^{(s)}S_\alpha}{d^{(s)}p_\alpha} \quad (3.51)$$

and substitute it in equation 3.50, we obtain:

$$\frac{1-n}{K_s} \left(\sum_{\alpha=1}^{N_\alpha} \frac{d^{(s)}p_\alpha}{dt} \left(S_\alpha + p_\alpha \frac{C_{ss}}{n} \right) \right) - \frac{d^{(s)}n}{dt} + (1-n)\text{div}v_s = 0 \quad (3.52)$$

These balance of mass equations are written for each solid and fluid phases of the mixture, and also can be written for the mixture itself. To this purpose, sometimes it is convenient to combine the balance of mass equations of fluid phase with solid particles. Thus, adding both equations 3.43 and 3.52, we have:

$$\begin{aligned} \left\{ \frac{n}{K_\alpha} + \frac{C_{s\alpha}}{S_\alpha} \right\} \frac{d^{(\alpha)}p_\alpha}{dt} + \frac{1-n}{K_s} \left(\sum_{\alpha=1}^{N_\alpha} \frac{d^{(s)}p_\alpha}{dt} \left(S_\alpha + p_\alpha \frac{C_{ss}}{n} \right) \right) + \frac{d^{(\alpha)}n}{dt} \\ - \frac{d^{(s)}n}{dt} + n\text{div}v_\alpha + (1-n)\text{div}v_s = 0 \end{aligned} \quad (3.53)$$

Taking into account the definition of material derivative, given in equation 3.29, the porosity derivatives in the left hand side $(d^{(w)}n/dt - d^{(s)}n/dt)$ can be identified with:

$$\frac{d^{(\alpha)}n}{dt} - \frac{d^{(s)}(n)}{dt} = (v_\alpha - v_s)^T \text{grad}n \quad (3.54)$$

In this way, the porosity derivatives are eliminated, and the resulting equation is cast in terms of the Darcy's velocity (ω_α) , defined in equation 3.30, as:

$$\frac{d^{(w)}n}{dt} - \frac{d^{(s)}(n)}{dt} = \frac{\omega_\alpha}{S_\alpha} \text{grad}n \quad (3.55)$$

After substituting above equation in the balance of mass equation 3.53 for mixture, arriving at:

$$\left\{ \frac{n}{K_\alpha} + \frac{C_{s\alpha}}{S_\alpha} \right\} \frac{d^{(\alpha)}p_\alpha}{dt} + \frac{1-n}{K_s} \left(\sum_{\alpha=1}^{N_\alpha} \frac{d^{(s)}p_\alpha}{dt} \left(S_\alpha + p_\alpha \frac{C_{ss}}{n} \right) \right) + \text{div} \left(\frac{\omega_\alpha}{S_\alpha} \right) + \text{div}v_s = 0 \quad (3.56)$$

If solid grains are considered much stiffer than pore fluid, then the first term in equations 3.52 and the first term 3.56 can be neglected, arriving finally at:

$$-\frac{d^{(s)}n}{dt} + (1-n)\text{div}(v_s) = 0 \quad (3.57)$$

$$\left\{ \frac{n}{K_\alpha} + \frac{C_{s\alpha}}{S_\alpha} \right\} \frac{d^{(\alpha)}p_\alpha}{dt} + \text{div} \left(\frac{\omega_\alpha}{S_\alpha} \right) + \text{div}v_s = 0 \quad (3.58)$$

which are the balance of mass equations for the solid phase and the mixture, respectively.

The other alternative is to come back to equation 3.38, we can see that we have the first term of $(n/S_\alpha \rho_\alpha)(d^{(\alpha)}S_\alpha \rho_\alpha/dt)$ where a similar development can be done by using material derivative equation (3.29) to transform the derivative following the fluid phase into the derivative following the solid particles:

$$\frac{n}{S_\alpha \rho_\alpha} \frac{d^{(\alpha)}}{dt} (S_\alpha \rho_\alpha) = \frac{n}{S_\alpha \rho_\alpha} \left\{ \frac{d^{(s)}}{dt} (S_\alpha \rho_\alpha) + \frac{\omega_\alpha}{n S_\alpha} \text{grad}(S_\alpha \rho_\alpha) \right\} \quad (3.59)$$

Replacing the derivative of the solid phase in equation 3.38 and adding to the balance of mass equation (3.46) of solid phase, we have:

$$\begin{aligned} \frac{n}{S_\alpha \rho_\alpha} \left\{ \frac{d^{(s)}}{dt} (S_\alpha \rho_\alpha) + \frac{\omega_\alpha}{n S_\alpha} \text{grad}(S_\alpha \rho_\alpha) \right\} + \frac{(1-n)}{\rho_s} \frac{d^{(s)} \rho_s}{dt} + \frac{d^{(\alpha)} n}{dt} \\ - \frac{d^{(s)} n}{dt} + n \text{div} v_\alpha + (1-n) \text{div} v_s = 0 \end{aligned} \quad (3.60)$$

For the balance of mass of the solid and multi-phase fluid particles, following a similar procedure, as the one described in the previous section, it is possible to arrive at:

$$\begin{aligned} \left\{ \frac{n}{K_\alpha} + \frac{C_{s\alpha}}{S_\alpha} \right\} \frac{d^{(s)} p_\alpha}{dt} + \frac{1-n}{K_s} \left[\sum_{\alpha=1}^{N_\alpha} \frac{d^{(s)} p_\alpha}{dt} \left(S_\alpha + p_\alpha \frac{C_{ss}}{n} \right) \right] + \frac{1}{S_\alpha} \text{div} \left(\frac{\omega_\alpha}{S_\alpha} \right) \\ + \text{div} v_s = 0 \end{aligned} \quad (3.61)$$

The interested reader will find in the text by Zienkiewicz et al. [11], and Lewis and Schrefler [75] a detailed explanation.

3.2.2 Balance of momentum

The second set of equations of mathematical models is balance of linear momentum which provides the amount of movement that should be constant over time, and is given by:

$$\rho \frac{dv}{dt} = \text{div} \sigma + \rho b \quad (3.62)$$

where b is vector of volume forces; and σ is total stress tensor.

Taking into account equations 3.8 and 3.10, the balance of momentum equations of the fluid and solid phases, respectively, are:

$$\rho^{(\alpha)} \frac{d^{(\alpha)} v_\alpha}{dt} = \sum_{\alpha=1}^{N_\alpha} n S_\alpha \text{div} \sigma_\alpha + \rho^{(\alpha)} b + R^{(\alpha)} \quad (3.63)$$

$$\rho^{(s)} \frac{d^{(s)} v_s}{dt} = \text{div} \sigma^{(s)} + \sum_{\alpha=1}^{N_\alpha} \sigma_\alpha \text{grad}(nS_\alpha) + \rho^{(s)} b + R^{(s)} \quad (3.64)$$

where $R^{(s)}$ and $R^{(w)}$ are the internal force on the solid and fluid phases, respectively.

This form is more frequently found in Solid Mechanics, while in Fluid Dynamics the stress tensor is decomposed into a hydrostatic and a deviatoric component as already mentioned in the section 2.3.1 of previous chapter.

Taking into account equations 2.15, 3.8 and 3.15, the balance of momentum equation of solid and fluid phases are:

$$\rho^{(\alpha)} \frac{d^{(\alpha)} v_\alpha}{dt} = - \sum_{\alpha=1}^{N_\alpha} nS_\alpha \text{grad} p_\alpha + \sum_{\alpha=1}^{N_\alpha} nS_\alpha \text{div} s_\alpha + \rho^{(\alpha)} b + R^{(\alpha)} \quad (3.65)$$

$$\rho^{(s)} \frac{d^{(s)} v_s}{dt} = \text{div} \sigma^{(s)} + \sum_{\alpha=1}^{N_\alpha} (s_\alpha - p_\alpha I) \text{grad}(nS_\alpha) + \rho^{(s)} b + R^{(s)} \quad (3.66)$$

Taking into account equation 2.15 and 3.10, the balance momentum equation (3.63) of fluid phases can be written alternatively in terms of either the partial or the total stress as:

$$\rho^{(\alpha)} \frac{d^{(\alpha)} v_\alpha}{dt} = \begin{cases} \sum_{\alpha=1}^{N_\alpha} \text{div} \sigma^{(\alpha)} - \sum_{\alpha=1}^{N_\alpha} \sigma_\alpha \text{grad}(nS_\alpha) \\ \sum_{\alpha=1}^{N_\alpha} \text{div} \sigma^{(\alpha)} + \sum_{\alpha=1}^{N_\alpha} (p_\alpha I - s_\alpha) \text{grad}(nS_\alpha) \end{cases} + \rho^{(\alpha)} b + R^{(\alpha)} \quad (3.67)$$

Taking into account equation 3.17 and 3.24, the balance of momentum equation (3.64) of solid phase can be written alternatively in terms of either the effective or the total stress as::

$$\rho^{(s)} \frac{d^{(s)} v_s}{dt} = \begin{cases} \operatorname{div} \sigma' - \sum_{\alpha=1}^{N_\alpha} (1 - nS_\alpha) \operatorname{grad} p_\alpha + \sum_{\alpha=1}^{N_\alpha} s_\alpha \operatorname{grad} (nS_\alpha) \\ \operatorname{div} \sigma + \sum_{\alpha=1}^{N_\alpha} nS_\alpha \operatorname{grad} p_\alpha - \sum_{\alpha=1}^{N_\alpha} nS_\alpha \operatorname{div} s_\alpha \\ + \rho^{(s)} b + R^{(s)} \end{cases} \quad (3.68)$$

Sometimes is preferable to express the motion of fluid phases relative to the solid. For obtaining this purpose, we will express the material derivative (see section 3.2) following the solid particles, using the relation:

$$\frac{d^{(\alpha)} v_\alpha}{dt} = \frac{d^{(\alpha)}}{dt} \left\{ v_s + \frac{\omega_\alpha}{nS_\alpha} \right\} \quad (3.69)$$

where we have used the Darcy's velocity equation (3.30). It will be further expanded as:

$$\frac{d^{(\alpha)} v_\alpha}{dt} = \frac{d^{(s)}}{dt} \left\{ v_s + \frac{\omega_\alpha}{nS_\alpha} \right\} + \frac{\omega_\alpha}{nS_\alpha} \operatorname{grad} \left\{ v_s + \frac{\omega_\alpha}{nS_\alpha} \right\} \quad (3.70)$$

where we have used the definition of derivative material (see equation 3.31). From here, we can rearrange the terms to get:

$$\frac{d^{(\alpha)} v_\alpha}{dt} = \frac{d^{(s)} v_s}{dt} + \left\{ \frac{d^{(s)}}{dt} \left(\frac{\omega_\alpha}{nS_\alpha} \right) + \frac{\omega_\alpha}{nS_\alpha} \operatorname{grad} v_s + \frac{\omega_\alpha}{nS_\alpha} \operatorname{grad} \left(\frac{\omega_\alpha}{nS_\alpha} \right) \right\} \quad (3.71)$$

From where we obtain the balance of momentum for fluid phases by taking into account the equation 3.63:

$$\begin{aligned}
& \rho^{(\alpha)} \frac{d^{(s)} v_s}{dt} + \rho^{(\alpha)} \left\{ \frac{d^{(s)}}{dt} \left(\frac{\omega_\alpha}{nS_\alpha} \right) + \frac{\omega_\alpha}{nS_\alpha} \text{grad} v_s + \frac{\omega_\alpha}{nS_\alpha} \text{grad} \left(\frac{\omega_\alpha}{nS_\alpha} \right) \right\} \\
& = \sum_{\alpha=1}^{N_\alpha} nS_\alpha \text{div} \sigma_\alpha + \rho^{(\alpha)} b + R^{(\alpha)}
\end{aligned} \tag{3.72}$$

To obtain the balance of linear momentum for the mixture, we will add above equation, which was obtained for fluid phases, to equation 3.64, which was obtained for fluid phases. Next, by taking into account the equality 3.10, we arrive at the resulting equation of the balance of linear momentum of the mixture as:

$$\rho \frac{d^{(s)} v_s}{dt} + c.t = \rho b + \text{div} \sigma + R \tag{3.73}$$

where the correcting term $(c.t)$ are given by:

$$c.t = \sum_{\alpha=1}^{N_\alpha} nS_\alpha \rho_\alpha \left\{ \frac{d^{(s)}}{dt} \left(\frac{\omega_\alpha}{nS_\alpha} \right) + \frac{\omega_\alpha}{nS_\alpha} \text{grad} v_s + \frac{\omega_\alpha}{nS_\alpha} \text{grad} \left(\frac{\omega_\alpha}{nS_\alpha} \right) \right\} \tag{3.74}$$

3.3 Particular cases

The purpose of this section is to show how different approximations, frequently used in Geomechanics and Geology, can be obtained as particular cases of the theory presented above. We will start by recalling the general equations or Biot equations which can be used in the most general case. Much effort has been devoted in the past and considerable progress has been achieved to understand coupling between soil skeleton and pore water since the early work of Biot [76], who proposed the first mathematical model describing the coupling for linear elastic materials.

For the sake of simplicity, we assume that the flow material is a fully saturated ($S_r = 1$) two-phase fluidized soil. It consists of a solid skeleton and a fluid phase which is either water, or a mixture of water with very fine soil particles such as clay. We will denote them by the sub-indexes s and w . We will further assume that the pore fluid has a

negligible internal viscosity. Therefore, the fluid phase is inviscid ($s_\alpha = 0$). The general model consists of the following equations:

i. The balance of mass equations for fluid phase (see equation 3.43):

$$\frac{n}{K_w} \frac{d^{(w)}p_w}{dt} + \frac{d^{(w)}n}{dt} + n \operatorname{div} v_w = 0 \quad (3.75)$$

and for solid phase (see equation 3.48):

$$\frac{(1-n)}{K_s} \frac{d^{(s)}p_w}{dt} - \frac{d^{(s)}n}{dt} + (1-n) \operatorname{div} v_s = 0 \quad (3.76)$$

ii. The balance of momentum equations for fluid phase (see equations 3.63, 3.65 and 3.67):

$$n\rho_w \frac{d^{(w)}v_w}{dt} = \left\{ \begin{array}{c} n \operatorname{div} \sigma_w \\ -n \operatorname{grad} p_w + n \operatorname{div} s_w \\ \operatorname{div} \sigma^{(w)} - \sigma_w \operatorname{grad} n \\ \operatorname{div} \sigma^{(w)} + (p_w - s_w) \operatorname{grad} n \end{array} \right\} + n\rho_w b - R \quad (3.77)$$

and for solid phase (see equations 3.64, 3.66 and 3.68):

$$(1-n)\rho_s \frac{d^{(s)}v_s}{dt} = \left\{ \begin{array}{c} \operatorname{div} \sigma^{(s)} + \sigma_w \operatorname{grad} n \\ \operatorname{div} \sigma^{(s)} + (s_w - p_w I) \operatorname{grad} n \\ \operatorname{div} \sigma' - (1-n) \operatorname{grad} p_w + s_w \operatorname{grad} n \\ \operatorname{div} \sigma + n \operatorname{grad} p_w - n \operatorname{div} s_w \end{array} \right\} + (1-n)\rho_s b + R \quad (3.78)$$

where the density of the solid and fluid phases has been introduced according to equations 3.7. Concerning the stresses, they can be obtained by complementing these equations with suitable constitutive or rheological laws. Internal force (R) has been

introduced in section 3.2.2, which account for the interaction between the solid and fluid phases.

$$R = R^{(s)} = (1 - n)R_s = -R^{(\alpha)} = -\sum_{\alpha=1}^{N_\alpha} nS_\alpha R_\alpha \quad (3.79)$$

where R_s is the internal force on the solid particles due to relative velocity of all fluid phases, while R_α is the force of the fluid phase caused by the solid particles. In two-phase modeling, the magnitude of the internal force ascertains whether the solid and fluid phases separate from each other or remain mixed. The following equation can also be written:

$$(1 - n)R_s + \sum_{\alpha=1}^{N_\alpha} nS_\alpha R_\alpha = 0 \quad (3.80)$$

For cases that solid grains are much stiffer than pore fluid, arriving to:

i. Balance of mass for solid phase (see equation 3.57):

$$-\frac{d^{(s)}n}{dt} + (1 - n)\text{div}v_s = 0 \quad (3.81)$$

ii. Balance of mass for mixture (see equation 3.58):

$$\frac{n}{k_w} \frac{d^{(w)}p_w}{dt} + \text{div}\omega + \text{div}v_s = 0 \quad (3.82)$$

iii. Balance of momentum for solid and fluid phases are the same. (see equations 3.77 and 3.78)

General model equations are composed of various variables, and in the case of saturated two-phase debris flow, they are made according to the following six variables:

- Volume fractions for the phases (n)
- Stress (σ) and pore pressures (p_w)
- Velocities of the solid phase (v_s) and interstitial fluids (v_w)

The general model, described by equations 3.75 and 3.78, can be simplified in cases where velocities of both phases are similar ($v_s = v_w$) by assuming that

$$\frac{d^{(w)}}{dt} \approx \frac{d^{(s)}}{dt} = \frac{d}{dt} \quad (3.83)$$

which implies that the gradient term of equation 3.31 is small. Moreover, if porosity is assumed to be constant, the resulting model is written in terms of the following unknowns:

- Stress (σ) and pore pressures (p_w)
- Velocities of the mixture (v) and Darcy's velocity (ω)

For some special cases, it is possible to make some additional approximations or interesting simplifications to the general model due to their geometry and flow conditions. These allow developing less computationally expensive models. This study has been devoted to the cases of debris flow, where both phases and their mutual interaction have to be modeled. In the next part, different mathematical models of debris flows, which have been developed in the past, will be described.

First, the work of Biot [76,77] was extended to non-linear materials and large deformation problems by Zienkiewicz and co-workers [11,78–80] and it is known for the first successful comparison of mathematical and physical models [79]. Then, Zienkiewicz and Shiomi [78] proposed a general formulation which can be applied to debris flows. Its equations are presented in terms of the velocities of solid skeleton (v_s) (or the displacements), the velocities of the pore water relative to the skeleton (ω) and the averaged pore pressure (p_w). It is also referred to as $v - p_w - \omega$ model in the case of saturated soils [11,79]. The case of non-saturated soils with air at atmospheric pressure

was proposed by Zienkiewicz et al (1990)[80] which the averaged pore pressure was defined in following equation as:

$$\bar{p} = S_a p_a + S_\alpha p_\alpha \quad (3.84)$$

If the air considered to be at atmospheric pressure, the averaged pressure is:

$$\bar{p} = S_\alpha p_\alpha \quad (3.85)$$

The model presented in preceding section can be used when the movement of all fluids and solid phase is relevant, which is the case of debris flows or mixtures of mobilized rock fragments and water where the permeability is high. Indeed, it is more convenient to think in terms of a mixture than a solid skeleton with interstitial fluids filling its voids. In some occasions, the phenomenon can be described by the movement of a solid skeleton, and the movement of the pore fluids relative to it, being the case of most geotechnical problems.

The $v - p_w - \omega$ model for non-saturated soils with air at atmospheric pressures consists of following three equations:

- i. Balance of mass equation for fluid phases (see equation 3.61).
- ii. Balance of linear momentum equation for fluid phases (see equation 3.72).
- iii. Balance of linear momentum equation for mixture (see equation 3.73).

A particular case of great interest is the $v - p_w - \omega$ model for saturated soils, where the model reduces to following expressions:

- i. Balance of mass for mixture (see equation 3.61):

$$\frac{1}{Q} \frac{d^{(s)} p_w}{dt} + \text{div} \omega + \text{div} v_s = 0 \quad (3.86)$$

where Q is introduced which is the joint volumetric stiffness of water and solid grains, given by the following:

$$\frac{1}{Q} = \frac{n}{K_w} + \frac{1-n}{K_s} \quad (3.87)$$

ii. Balance of momentum for mixture:

$$\rho \frac{dv_s}{dt} + n\rho_w \frac{d}{dt} \left(\frac{\omega}{n} \right) = \text{div} \sigma' - \text{grad} p_w + \rho b \quad (3.88)$$

In order to obtain above equation, first, add up the balance of momentum equations 3.63 and 3.64 for fluid and solid phases, respectively. Then, substitute the equations 3.24, 3.30 and 3.83 into the result.

iii. Balance of momentum for fluid (see equation 3.65):

$$n\rho_w \frac{d^{(w)}v_w}{dt} = -n\text{grad} p_w + n\rho_w b - R \quad (3.89)$$

The resulting model can be casted in terms of v_s , p_w and ω , hence its name. Based on the above formulations, the Biot–Zienkiewicz ($v - p_w$) equations can be derived by assuming that the velocities and accelerations of the fluid phases in the pores are small with respect to solid skeleton in some cases and accordingly, it is possible to eliminate the Darcy's velocity from the model [81]. Therefore, we can derive models formulated in terms of velocities of soil skeleton (v_s) and pressures of the pore fluids (p_w), which is more convenient for fluidized geomaterials.

We will consider first the balance of mass equation (3.86) for mixture and the linear momentum equation (3.89) for fluid. Then, Darcy law can be used to describe the interaction between pore water and soil skeleton, therefore, the interaction term can be expressed as:

$$R = C_d \omega \quad (3.90)$$

where $C_d = n^2 k_w^{-1}$, and k_w is a permeability tensor.

From the balance of momentum equation (3.89) for fluid, the Darcy's relative velocity (ω) is obtained as:

$$\omega = -\frac{n}{\bar{C}_d} \text{grad} p_w + \frac{n}{\bar{C}_d} \rho_w b - \frac{n}{\bar{C}_d} \rho_w \frac{d^{(w)} v_w}{dt} \quad (3.91)$$

Substitute above equation into balance of mass equation (3.86) for mixture, where we have $\text{div} \omega$, we arrive at:

$$\begin{aligned} \frac{1}{Q} \frac{d^{(s)} p_w}{dt} &= \text{div} \left(\frac{n}{\bar{C}_d} \text{grad} p_w \right) - \text{div} v_s - \text{div} \left(\rho_w b \frac{n}{\bar{C}_d} \right) \\ &+ \text{div} \left(\rho_w \frac{n}{\bar{C}_d} \frac{d^{(w)} v_w}{dt} \right) \end{aligned} \quad (3.92)$$

It is interesting to note that the divergence of the body forces, in the third term, is zero except in some special cases such as centrifuge testing machines, where the centripetal acceleration depends on the position. The forth term can be considered as the volumetric component of the rate of deformation tensor describing the extra volumetric deformation (d_{v0}). Finally, the $v - p_w$ model for saturated soils consists of following two equations:

- i. Balance of momentum equation for mixture, where the two terms on LHS of equation 3.88 are combined by using the equation 3.30:

$$\rho \frac{dv}{dt} = \text{div} \sigma' - \text{grad} p_w + \rho b \quad (3.93)$$

- ii. Balance of mass and momentum for mixture (see equation 3.92):

$$\frac{1}{Q} \frac{d^{(s)}p_w}{dt} = \text{div}(k_w \text{grad} p_w) - \text{div} v_s \quad (3.94)$$

where the last two terms of equation 3.92 are frequently omitted in the analysis, because the divergence of the body forces is zero and the space derivatives of accelerations are assumed to be small.

These Zienkiewicz-Shiomi ($v - p_w - w$) and Biot-Zienkiewicz ($v - p_w$) models consist on three and two equations of balance respectively, plus a suitable constitutive equation relating the rate of deformation to the rate of effective stress tensors (see equation 2.6) and a kinematical relation relating displacements or velocities to strain or rate of deformation tensors (see equation 2.2). The resulting model can be casted in terms of v_s , v_w and p_w , plus effective stress tensor (p') and rate of deformation (d_s).

In this study, we will mainly focus on the two-phase model which can be applied for debris flows, where the difference of velocities between solid grains and fluid is important, and both phases with their mutual interaction have to be modeled. The origin of multi-phase modeling can be tracked back within the area of physics of granular media, where coupling between solid particles and pore fluids appeared to be important in industrial processes such as fluidized beds. Regarding this, we can mention the work done by Bowen [71], Anderson and Jackson [82] where mixture of several interacting phases has been considered. Regarding applications to debris flows, a more general approach for two-phase modeling is required. Such models have been proposed by Pitman and Le [83] and Pudasaini [84]. The approach we will follow here is proposed by Pastor, et al. [2] which is based on the mathematical model proposed by Zienkiewicz and Shiomi [78] and is similar to those of Pitman [83] and Pudasaini [84]. The governing equations will be described in the next section.

3.4 Depth averaged model

So far, we have considered three-dimensional continuum models which can describe flow of geomaterials in both triggering and propagation phases. 3D formulations of mathematical models describing fast landslides are generally very expensive in terms of calculation time. These models can be further simplified and reduced to a 2D formulation through a depth integration approximation. Since many flow-like landslides have small average depths, or propagation height, in comparison with their length or width, the given equations can be integrated along the vertical axis. The resulting 2D depth integrated model provides a good relation between accuracy and time-consuming, which justify the suitability to model in most of fast landslides.

Depth integrated models have been used in coastal and hydraulics engineering since the work of Barré de Saint-Venant [85] and applied to landslide propagation since the work of Savage and Hutter [58]. Examples of case studies have been provided in recent years, and it is worth mentioning the modelling of debris flows performed by Cascini [61,86] and debris avalanches by Cuomo [59,87]. Advantages of this technique include: (1) A lower number of unknowns required to be solved due to eliminating the vertical variable, and (2) free surface is not needed to be track with special techniques such as level set.

We will use the reference system given in Figure 3-3 with axes $\{x_1, x_2, x_3\}$, where we have depicted some magnitudes of interest which will be used in this section. Z denotes the ground surface elevation, and h the flow depth, defined as the increment of height of the moving soil per unit time.

Figure 3-3. Reference system, coordinates and notation used in the analysis.

Depth averaged models are obtained by integrating the balance of mass and momentum equations along depth (x_3), and using Leibniz's rule:

$$\int_a^b \frac{\partial}{\partial s} F(r, s) dr = \frac{\partial}{\partial s} \int_a^b F(r, s) dr - F(b, s) \frac{\partial b}{\partial s} + F(a, s) \frac{\partial a}{\partial s} \quad (3.95)$$

where a and b are differentiable functions of s , $F(r, s)$ and $\partial F(r, s)/\partial s$ continuous in r and in s .

Take into account that an overbar over a magnitude (θ) denotes a depth integrated value. For instance:

$$\bar{\theta} = \frac{1}{h} \int_z^{z+h} \theta dx_3 \quad (3.96)$$

In the previous sections, an lagrangian description of the balance of mass and momentum equations has been made. In depth integrated equations, there is a significant difficulty of obtaining directly a lagrangian form because the vertical integration is not performed in a material volume. Sometimes, it has been found convenient to refer to an equivalent 2D continuum having the velocities of their material points as depth integrated velocities. This cannot be considered as a lagrangian formulation, because the moving points have no exact connection with material particles. The new description form is called Quasi-Lagrangian formulation which we derive depth integrated equations base on it. The quasi-material derivative will be introduced as:

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \bar{v}_j \frac{\partial}{\partial t} \quad j = 1, 2 \quad (3.97)$$

We will show here how the general depth integrated model can be obtained, and we start by converting the **balance of mass** equations (3.75-3.76) into the eulerian form as:

$$\frac{\partial n_\alpha}{\partial t} + \text{div}(n_\alpha v_{\alpha i}) = 0 \quad (3.98)$$

where α indicates the types of phases, s for solid and w for water. The pore pressure terms are neglected in the balance of mass equation under the assumptions that the mixture is weakly-compressible and density is constant.

It is assumed that the permeability is very large, therefore the pore pressure terms are neglected.

Integrate the first term of the above equation along x_3 , results in :

$$\int_z^{z+h} \frac{\partial n_\alpha}{\partial t} dx_3 = \frac{\partial}{\partial t} \int_z^{z+h} n_\alpha dx_3 - n_\alpha \frac{\partial(Z+h)}{\partial t} \Big|_{z+h} + n_\alpha \frac{\partial Z}{\partial t} \Big|_z \quad (3.99)$$

It is then possible to get a depth integrated average value of the flow velocity during propagation (see equation 3.96), so that the average value is no more dependent on the vertical direction (x_3), as:

$$\frac{\partial}{\partial t} \int_z^{z+h} n_\alpha dx_3 = \frac{\partial(\bar{n}_\alpha h)}{\partial t} \quad (3.100)$$

Considering the divergence term of equation 3.98, the derivatives with respect to x_1 and x_2 are as follows:

$$\begin{aligned} & \int_z^{z+h} \frac{\partial(n_\alpha v_{\alpha i})}{\partial x_i} dx_3 \\ &= \frac{\partial}{\partial x_i} \int_z^{z+h} n_\alpha v_{\alpha i} dx_3 - n_\alpha v_{\alpha i} \frac{\partial(Z+h)}{\partial x_i} \Big|_{z+h} \\ &+ n_\alpha v_{\alpha i} \frac{\partial Z}{\partial x_i} \Big|_z, \quad i = 1, 2, j = 1, 2 \end{aligned} \quad (3.101)$$

Then, the term including the derivative with respect to x_3 is as follows:

$$\int_z^{z+h} \frac{\partial(n_\alpha v_{\alpha i})}{\partial x_3} dx_3 = n_\alpha v_3|_{z+h} - n_\alpha v_3|_z, i = 3, j = 1, 2 \quad (3.102)$$

Adding up both contributions, we obtain:

$$\begin{aligned} \int_z^{z+h} \frac{\partial(n_\alpha v_{\alpha i})}{\partial x_i} dx_3 \\ = \frac{\partial}{\partial x_i} \int_z^{z+h} n_\alpha v_{\alpha i} dx_3 - \left(n_\alpha v_{\alpha i} \frac{\partial(Z+h)}{\partial x_i} - n_\alpha v_3 \right)_{z+h} \\ + \left(n_\alpha v_{\alpha i} \frac{\partial Z}{\partial x_i} - n_\alpha v_3 \right)_z, i = 1-3, j = 1, 2 \end{aligned} \quad (3.103)$$

The first term of the above equation can be written as:

$$\begin{aligned} \frac{\partial}{\partial x_i} \int_z^{z+h} n_\alpha v_{\alpha i} dx_3 &= \frac{\partial}{\partial x_i} \int_z^{z+h} (\bar{n}_\alpha + n_\alpha^*)(\bar{v}_{\alpha i} + v_{\alpha i}^*) dx_3 \\ &= \frac{\partial}{\partial x_j} \int_z^{z+h} (\bar{n}_\alpha \bar{v}_{\alpha i}) dx_3 + \frac{\partial}{\partial x_j} \int_z^{z+h} (n_\alpha^* v_{\alpha i}^*) dx_3 \\ &\quad + \frac{\partial}{\partial x_j} \int_z^{z+h} (\bar{n}_\alpha v_{\alpha i}^*) dx_3 + \frac{\partial}{\partial x_j} \int_z^{z+h} (n_\alpha^* \bar{v}_{\alpha i}) dx_3 \end{aligned} \quad (3.104)$$

where $v_\alpha(x_1, x_2, x_3) = \bar{v}_\alpha(x_1, x_2) + v_\alpha^*(x_1, x_2, x_3)$ and $n_\alpha(x_1, x_2, x_3) = \bar{n}_\alpha(x_1, x_2) + n_\alpha^*(x_1, x_2, x_3)$ are precisely the Reynolds decomposition. $\bar{v}_\alpha(x_1, x_2)$ and $\bar{n}_\alpha(x_1, x_2)$ represent the depth integrated average velocity and porosity field, while $v_\alpha^*(x_1, x_2, x_3)$ and $n_\alpha^*(x_1, x_2, x_3)$ is the velocity and porosity fluctuation field respectively.

The first two terms on the right hand side of equation 3.104 can be written as:

$$\frac{\partial}{\partial x_i} \int_z^{z+h} (\bar{n}_\alpha \bar{v}_{\alpha i}) dx_3 = \frac{\partial(\bar{n}_\alpha \bar{v}_{\alpha i} h)}{\partial x_i} \quad (3.105)$$

$$\frac{\partial}{\partial x_i} \int_z^{z+h} (n_\alpha^* v_{\alpha i}^*) dx_3 = \frac{\partial(\alpha \bar{n}_\alpha \bar{v}_{\alpha i} h)}{\partial x_i} \quad (3.106)$$

where the coefficient α is a momentum correction factor which can be obtained from:

$$\alpha = \frac{1}{h} \frac{\int_z^{z+h} n_\alpha^* v_{\alpha i}^* dx_3}{\bar{n}_\alpha \bar{v}_{\alpha i}} \quad (3.107)$$

where α will depend on the rheological behavior which describes the fluid. For a Newtonian fluid is generally taken as 0.2 and for a Bingham fluid it varies linearly with height (see equation 2.72); while in frictional fluid, which is the type of rheological that will be used for our case studies, is not taken into account.

Since,

$$\begin{aligned} \int_z^{z+h} \bar{n}_\alpha v_{\alpha i}^* dx_3 &= \bar{n}_\alpha \int_z^{z+h} v_{\alpha i}^* dx_3 = 0 \\ \int_z^{z+h} n_\alpha^* \bar{v}_{\alpha i} dx_3 &= \bar{v}_{\alpha i} \int_z^{z+h} n_\alpha^* dx_3 = 0 \end{aligned} \quad (3.108)$$

Then, the integral results on:

$$\frac{\partial}{\partial x_i} \int_z^{z+h} n_\alpha v_{\alpha i} dx_3 = \frac{\partial(h \bar{n}_\alpha \bar{v}_{\alpha i})}{\partial x_i} \quad (3.109)$$

The second term on the right hand side of the equation 3.99 can be related to the material velocity of the particles at the surface, because their velocity is given by:

$$v_{\alpha 3} = \frac{d(Z + h)}{dt} \quad (3.110)$$

Expanding the material derivative:

$$\frac{d(Z + h)}{dt} = \frac{\partial(Z + h)}{\partial t} + v_{\alpha i} \frac{\partial(Z + h)}{\partial x_i} = v_{\alpha 3} \quad (3.111)$$

and rearranging, we obtain:

$$-\frac{\partial(Z + h)}{\partial t} \Big|_{Z+h} - v_{\alpha i} \frac{\partial(Z + h)}{\partial x_i} \Big|_{Z+h} + v_{\alpha 3} \Big|_{Z+h} = 0 \quad (3.112)$$

Then, the summation of the second terms on the right hand side of equations 3.99 and 3.103 become zero. The summation of the third terms can be written as:

$$n_{\alpha} \frac{\partial Z}{\partial t} \Big|_Z + n_{\alpha} v_{\alpha i} \frac{\partial Z}{\partial x_i} \Big|_Z - n_{\alpha} v_{\alpha 3} \Big|_Z, i = 1, 2 \quad (3.113)$$

In general, the basal surface will not change but in some occasions, erosion may occur. This phenomenon is taken into account in the model by introducing an erosion rate (e_R) which already defined in the equation 2.115. In the section 2.4 of previous chapter, several erosion laws has been described such as the one proposed by Hungr [62], which relates erosion rate to the depth averaged velocity of the flowing material.

Consider that scalar product of two perpendicular vectors is equal to zero, therefore, the last two term of the equation 3.113 can be eliminated as:

$$n_\alpha v_{\alpha i} \frac{\partial Z}{\partial x_i} - n_\alpha v_{\alpha 3} = v_\alpha \left[\frac{\partial Z}{\partial x_1}, \frac{\partial Z}{\partial x_2}, 1 \right]^T = 0 \quad (3.114)$$

After integrating along depth and applying the depth integrate model to the balance of mass equation (3.98), the following depth integrated expression is obtained in a more compact eulerian form as:

$$\frac{\partial(h\bar{n}_\alpha)}{\partial t} + \frac{\partial}{\partial x_i}(h\bar{n}_\alpha \bar{v}_{\alpha i}) - \bar{n}_\alpha e_R = 0 \quad (3.115)$$

Since $h_\alpha = h\bar{n}_\alpha$, the balance of mass equation for both phases can be written as:

$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial}{\partial x_{\alpha i}}(h_\alpha \bar{v}_{\alpha i}) = \bar{n}_\alpha e_R \quad (3.116)$$

The above equation can be expanded as:

$$\frac{\partial h_\alpha}{\partial t} + \bar{v}_{\alpha i} \frac{\partial h_\alpha}{\partial x_{\alpha i}} + h_\alpha \frac{\partial \bar{v}_{\alpha i}}{\partial x_{\alpha i}} = \bar{n}_\alpha e_R \quad (3.117)$$

Finally, a quasi-lagrangian formulation of the balance of mass equation will be derived, by considering the quasi-material derivative (see equation 3.97) as:

$$\frac{\bar{d}^{(\alpha)} h_\alpha}{dt} + h_\alpha \text{div} \bar{v}_{\alpha i} = \bar{n}_\alpha e_R \quad (3.118)$$

After we obtain the quasi-lagrangian form of the depth integrated equation, it can be written for solid, fluid and mixture, respectively, as follows:

$$\frac{\bar{d}^{(w)}(\bar{n}h)}{dt} + (\bar{n}h) \text{div} \bar{v}_{wi} = \bar{n} e_R \quad (3.119)$$

$$\frac{\bar{d}^{(s)}((1-\bar{n})h)}{dt} + ((1-\bar{n})h) \text{div} \bar{v}_{si} = (1-\bar{n}) e_R \quad (3.120)$$

$$\begin{aligned} \frac{\bar{d}^{(s)}((1-\bar{n})h)}{dt} + \frac{\bar{d}^{(w)}(\bar{n}h)}{dt} + ((1-\bar{n})h)\text{div}\bar{v}_{si} + (\bar{n}h)\text{div}\bar{v}_{wi} \\ = (1-\bar{n})e_R + \bar{n}e_R \end{aligned} \quad (3.121)$$

The general two-phase depth integrated equations described above can be simplified if we assume the gradient term of equation 3.31 is small and the acceleration of the fluid relative to the solid can be neglected (like in the $u - p_w$ model), results in:

$$\frac{\bar{d}h}{dt} + h\text{div}\bar{v}_i = e_R \quad (3.122)$$

where $\bar{v}$ is depth averaged velocity along x_3 ,

$$\bar{v} = \frac{1}{h} \int_0^h v dx_3 \quad (3.123)$$

Next, we will integrate the **balance of linear momentum** equations (3.77 and 3.78) along depth. In what follows, coordinate directions will be added to sub-index s and w , which referred in the previous section to the solid and fluid phases.

$$n\rho_{w_ij} \frac{d^{(w)}v_{w_ij}}{dt} = n\text{div}\sigma_{w_ij} + n\rho_{w_ij}b - R, i, j = 1 - 3 \quad (3.124)$$

$$(1-n)\rho_{s_ij} \frac{d^{(s)}v_{s_ij}}{dt} = \text{div}\sigma_{ij}^{(s)} + \sigma_{w_ij}\text{grad}n + (1-n)\rho_{s_ij}b + R, i, j = 1 - 3 \quad (3.125)$$

We will use balance of mass for fluid given in equation 3.75 and assume p_w for fluid is constant, we arrive at:

$$\rho_w \frac{d^{(w)}n}{dt} + n\rho_w\text{div}v_w = 0 \quad (3.126)$$

where the terms are multiplied by the density of water (ρ_w).

Adding above equation to the derivative of the balance of momentum on the left hand side of equation 3.124 which can be developed as:

$$n\rho_w \frac{dv_i}{dt} + \rho_w v_i \frac{dn}{dt} + (n\rho_w v_i) \text{div}(v_j) = \rho_w \frac{\partial(nv_i)}{\partial t} + \rho_w \frac{\partial(nv_i v_j)}{\partial x_j} \quad (3.127)$$

We will integrate the results to the components $i = 1, 2$. Applying the Leibnitz rule to the previous expression, it is possible to obtain the first integral as:

$$\begin{aligned} \rho_w \int_z^{z+h} \frac{\partial(nv_i)}{\partial t} dx_3 \\ = \rho_w \frac{\partial}{\partial t} \int_z^{z+h} nv_i dx_3 - \rho_w nv_i \frac{\partial(Z+h)}{\partial t} \Big|_{Z+h} \\ + \rho_w nv_i \frac{\partial Z}{\partial t} \Big|_Z, i = 1, 2 \end{aligned} \quad (3.128)$$

The second integral can be treated as follow:

$$\begin{aligned} \rho_w \int_z^{z+h} \frac{\partial(nv_i v_j)}{\partial x_i} dx_3 \\ = \rho_w \frac{\partial}{\partial x_i} \int_z^{z+h} nv_i v_j dx_3 - \rho_w nv_i v_j \frac{\partial(Z+h)}{\partial x_i} \Big|_{Z+h} \\ + \rho_w nv_i v_j \frac{\partial Z}{\partial x_i} \Big|_Z \quad i = 1, 2, j = 1, 2 \end{aligned} \quad (3.129)$$

$$\begin{aligned} \rho_w \int_z^{z+h} \frac{\partial(nv_i v_3)}{\partial x_3} dx_3 = \rho_w nv_i v_3 \Big|_{Z+h} - \rho_w nv_i v_3 \Big|_Z \\ i = 3, j = 1, 2 \end{aligned} \quad (3.130)$$

From here, and using the definition of depth averaged velocity (see equation 3.123), we obtain:

$$\rho_w \frac{\partial}{\partial t} \int_z^{z+h} n v_i dx_3 = \rho_w \frac{\partial(\bar{n} \bar{v}_i h)}{\partial t} \quad (3.131)$$

Next, we will deal with the integral involving products of velocities $(v_i v_j)$. The velocity can be decomposed as: $v_i(x_1, x_2, x_3) = \bar{v}_i(x_1, x_2) + v_i^*(x_1, x_2, x_3)$, and the integral results on:

$$\begin{aligned} \int_z^{z+h} v_i v_j dx_3 &= \int_z^{z+h} (\bar{v}_i + v_i^*)(\bar{v}_j + v_j^*) dx_3 \\ &= \int_z^{z+h} \bar{v}_i \bar{v}_j dx_3 + \int_z^{z+h} v_i^* v_j^* dx_3 \\ &= h \bar{v}_i \bar{v}_j + \int_z^{z+h} v_i^* v_j^* dx_3 \end{aligned} \quad (3.132)$$

where the other two terms are already eliminated:

$$\begin{aligned} \int_z^{z+h} v_i^* \bar{v}_j dx_3 &= \bar{v}_j \int_z^{z+h} v_i^* dx_3 = 0 \\ \int_z^{z+h} \bar{v}_i v_j^* dx_3 &= \bar{v}_i \int_z^{z+h} v_j^* dx_3 = 0 \end{aligned} \quad (3.133)$$

A simplifying assumption can be made by approximating a term and using a correction factor (α) (see equation 3.107). we can assume that the velocity profile along depth does not change over time (steady state flow condition). The integral can be expressed as:

$$\frac{1}{h} \int_z^{z+h} v_i^* v_j^* dx_3 = \alpha \bar{v}_i \bar{v}_j \quad (3.134)$$

Then, the left hand side (LHS) is given by:

$$\text{LHS} = \frac{\partial}{\partial t}(h\bar{v}_i) + (1 + \alpha) \frac{\partial}{\partial x_j}(h\bar{v}_i\bar{v}_j) \quad (3.135)$$

The correction coefficient is often neglected. In order to simplify the result, before collecting them up, we will group them as follows:

$$\begin{aligned} & -\rho_w n v_i \frac{\partial(Z+h)}{\partial t} \Big|_{Z+h} - \rho_w n v_i v_j \frac{\partial(Z+h)}{\partial x_i} \Big|_{Z+h} + \rho_w n v_i v_3 \Big|_{Z+h} \\ & \rho_w n v_i \frac{\partial Z}{\partial t} \Big|_Z + \rho_w n v_i v_j \frac{\partial Z}{\partial x_i} \Big|_Z - \rho_w n v_i v_3 \Big|_Z \end{aligned} \quad (3.136)$$

The first group, on the surface, can be assumed to be zero (see equation 3.112). In the second group, on the terrain, if we assume the basal velocity is zero ($v_i|_Z = 0$), then

$$\rho_w n v_i \frac{\partial Z}{\partial t} \Big|_Z = 0 \quad (3.137)$$

Concerning the other two terms on the surface, it is the product of the normal vector to the bottom $[\partial Z/\partial x_1, \partial Z/\partial x_2, 1]^T$ and the velocity. The result is the normal velocity (v_n), which is zero (same as equation 3.114) if there is no infiltration through the bottom:

$$\rho_w n v_i v_j \frac{\partial Z}{\partial x_i} - \rho_w n v_i v_3 = \rho_w n v_i v_j \left[\frac{\partial Z}{\partial x_1}, \frac{\partial Z}{\partial x_2}, 1 \right]^T = 0 \quad (3.138)$$

Therefore, the LHS of the balance of linear momentum equation (3.124) for fluid reduces to:

$$\text{LHS} = n \rho_w \frac{d^{(w)} v_i}{dt} = \rho_w \frac{\partial}{\partial t}(\bar{n}\bar{v}_i h) + \rho_w \frac{\partial}{\partial x_i}(h \bar{n}\bar{v}_i \bar{v}_j) \quad (3.139)$$

The result will be expanded as:

$$\begin{aligned}
& \rho_w \frac{\partial(\bar{n}\bar{v}_i h)}{\partial t} + \rho_w \frac{\partial}{\partial x_i} (h\bar{n}\bar{v}_i \bar{v}_j) \\
&= \rho_w h \bar{n} \frac{\partial \bar{v}_i}{\partial t} + \rho_w \bar{v}_i \frac{\partial h \bar{n}}{\partial t} + \rho_w h \bar{n} \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_i} \\
&+ \rho_w \bar{v}_i \frac{\partial h \bar{n} \bar{v}_j}{\partial x_i} \\
&= \rho_w h \bar{n} \frac{\partial \bar{v}_i}{\partial t} + \rho_w h \bar{n} \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_i} \\
&+ \rho_w \bar{v}_i \left(\frac{\partial(h\bar{n})}{\partial t} + \frac{\partial}{\partial x_i} (h\bar{n}\bar{v}_i) \right)
\end{aligned} \tag{3.140}$$

Finally, the balance of momentum on the LHS for fluid, by considering equation 3.119, can be computed as:

$$n\rho_w \frac{d^{(w)}v_i}{dt} = \rho_w h \bar{n} \frac{\partial \bar{v}_i}{\partial t} + \rho_w h \bar{n} \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_i} + \rho_w \bar{v}_i \bar{n} e_R \tag{3.141}$$

Next, we will use the balance of mass equation (3.81) for solid:

$$\rho_s v_i \frac{dn}{dt} - ((1-n)\rho_s v_i) \text{div} v_j = 0 \tag{3.142}$$

where the terms are multiplied by density and velocity of solid, $\rho_s v_i$.

Add the above equation to the derivative of the balance of momentum equation (3.125) on the left hand side, we arrive at:

$$\begin{aligned}
L.H.S &= (1-n)\rho_s \frac{d^{(s)}v_s}{dt} \\
&= \rho_s (1-n) \rho_s \frac{dv_i}{dt} + \rho_s v_i \frac{dn}{dt} \\
&- ((1-n)\rho_s v_i) \text{div}(v_j)
\end{aligned} \tag{3.143}$$

Following the same steps as given in the last part, the balance of momentum on the LHS for solid, by assuming incompressibility ($\text{div}(v) = 0$) for soil, can be computed as:

$$\begin{aligned}
(1-n)\rho_s \frac{d^{(s)}v_s}{dt} \\
= \rho_s h(1-\bar{n}) \frac{\partial \bar{v}_i}{\partial t} + \rho_s h(1-\bar{n}) \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_i} \\
+ \rho_s \bar{v}_i (1-\bar{n}) e_R
\end{aligned} \tag{3.144}$$

So far, we have performed the integration along depth of the left hand side terms. The right hand side consists of several terms which are divergence of stress tensor, gradient of porosity, body force and interaction force (see equations 3.124 and 3.125).

The divergence of the stress tensor acting on fluid is calculated, by using the Leibnitz rule, as follows:

$$\begin{aligned}
\bar{n} \int_z^{z+h} \frac{\partial \sigma_{w_{ij}}}{\partial x_j} dx_3 = \bar{n} \frac{\partial}{\partial x_j} \int_z^{z+h} \sigma_{w_{ij}} dx_3 - \\
\sigma_{w_{ij}} \bar{n} \frac{\partial (z+h)}{\partial x_j} \Big|_{z+h} + \sigma_{w_{ij}} \bar{n} \frac{\partial z}{\partial x_j} \Big|_z, j = 1,2, i = 1,2
\end{aligned} \tag{3.145}$$

$$\bar{n} \int_z^{z+h} \frac{\partial \sigma_{w_{i3}}}{\partial x_3} dx_3 = \sigma_{w_{i3}} \bar{n} \Big|_{z+h} - \sigma_{w_{i3}} \bar{n} \Big|_z, j = 3, i = 1,2$$

where $\bar{n}$ is depth averaged porosity along depth, x_3 :

$$\bar{n} = \frac{1}{h} \int_z^{z+h} n dx_3 \tag{3.146}$$

The divergence of the partial stress tensor for solid is calculated as follows:

$$\int_z^{z+h} \frac{\partial \sigma_{ij}^{(s)}}{\partial x_j} dx_3 = \frac{\partial}{\partial x_j} \int_z^{z+h} \sigma_{ij}^{(s)} dx_3 - \sigma_{ij}^{(s)} \frac{\partial(Z+h)}{\partial x_j} \Big|_{z+h} + \sigma_{ij}^{(s)} \frac{\partial Z}{\partial x_j} \Big|_z$$

$$, j = 1, 2, i = 1, 2 \quad (3.147)$$

$$\int_z^{z+h} \frac{\partial \sigma_{i3}^{(s)}}{\partial x_3} dx_3 = \sigma_{i3}^{(s)} \Big|_{z+h} - \sigma_{i3}^{(s)} \Big|_z \quad , j = 3, i = 1, 2$$

If depth integrated stress tensor is defined as:

$$\bar{\sigma}_{ij} = \frac{1}{h} \int_z^{z+h} \sigma_{ij} dx_3 \quad , i = j = 1, 2 \quad (3.148)$$

Then, the terms of divergence of stress tensor can be written as:

$$\bar{n} \frac{\partial}{\partial x_j} \int_z^{z+h} \sigma_{w_{ij}} dx_3 = \bar{n} \text{div}(h \bar{\sigma}_{w_{ij}})$$

$$\frac{\partial}{\partial x_j} \int_z^{z+h} \sigma_{ij}^{(s)} dx_3 = \text{div}(h \bar{\sigma}_{ij}^{(s)}) \quad (3.149)$$

The surface forces at the top are usually assumed to be zero. However, in some cases such as submarine landslides, they should be taken into account.

$$\bar{n} \sigma_{w_{ij}} \left(-\frac{\partial(Z+h)}{\partial x_1} - \frac{\partial(Z+h)}{\partial x_2} + 1 \right) = 0 \quad , j = 1, 2, 3, i = 1, 2$$

$$\sigma_{ij}^{(s)} \left(-\frac{\partial(Z+h)}{\partial x_1} - \frac{\partial(Z+h)}{\partial x_2} + 1 \right) = 0 \quad , j = 1, 2, 3, i = 1, 2 \quad (3.150)$$

The forces in the bottom depend on the rheological model used (see section 2.3 of previous chapter), thus:

$$-\bar{n} \sigma_{w_{i3}} \Big|_z = \tau_{B_i}^{(w)} \quad , i = 1, 2$$

$$-\sigma_{i3}^{(s)} \Big|_z = \tau_{B_i}^{(s)} \quad , i = 1, 2 \quad (3.151)$$

where $\tau_B^{(s)}$ and $\tau_B^{(w)}$ are introduced as the shear basal stresses of the solid and fluid phase, respectively. It is important to note that in cases where solid–pore fluid interactions are important, the rheological model should describe separately the behavior of solid grains and fluid phase, as done in geotechnical engineering [11].

The average depth of the gradient term can be written as:

$$\sigma_{w_{ij}} \text{grad} n_{ij} = \frac{1}{x_j} \int_z^{z+h} \sigma_{w_{ij}} \text{grad} n_{ij} dx_3 \quad (3.152)$$

Take into account the depth average of stress tensor equation (3.48) and assume that porosity is zero at the surface, the term is obtained as:

$$\sigma_{w_{ij}} \text{grad} n_{ij} = h \bar{\sigma}_{w_{ij}} \text{grad} \bar{n} \quad (3.153)$$

which can be expanded as:

$$h \bar{\sigma}_{w_{ij}} \text{grad} \bar{n} = \text{div}(h \bar{\sigma}_{w_{ij}} \bar{n}) - h \bar{\sigma}_{ij} \text{grad} \bar{n} \quad (3.154)$$

The stress tensor acting on fluid phase will be decomposed into hydrostatic and deviatoric components (see equation 3.11) as follows:

$$\begin{aligned} \text{div}(h \bar{\sigma}_{w_{ij}} \bar{n}) &= -\text{grad}(h \bar{p}_w \bar{n}) + \text{div}(h \bar{s}_w \bar{n}) \\ h \bar{\sigma}_{w_{ij}} \text{grad} \bar{n} &= h(\bar{p}_w I - \bar{s}_w) \text{grad} \bar{n} \end{aligned} \quad (3.155)$$

Concerning the body and interaction force, their integration along depth is immediate:

$$b_i h = \int_z^{z+h} b_i dx_3 \quad (3.156)$$

$$\bar{R}h = \int_z^{z+h} R dx_3 \quad (3.157)$$

Collecting all the terms after integrating along depth (local x_3 axis), we obtain the depth integrated balance of linear momentum equation for fluid and solid phase, respectively in Eulerian form:

$$\begin{aligned} \rho_w h \bar{n} \frac{\partial \bar{v}_{w-i}}{\partial t} + \rho_w h \bar{n} \bar{v}_{w-j} \frac{\partial \bar{v}_{w-i}}{\partial x_i} + \rho_w \bar{v}_{w-i} \bar{n} e_R \\ = -\text{grad}(h \bar{p}_w \bar{n}) + h \bar{p}_w \text{grad} \bar{n} + \text{div}(h \bar{s}_w \bar{n}) \\ - h \bar{s}_w \text{grad} \bar{n} + \tau_B^{(w)} + \bar{n} \rho_w b_i h + \bar{R}h \end{aligned} \quad (3.158)$$

$$\begin{aligned} \rho_s h (1 - \bar{n}) \frac{\partial \bar{v}_{s-i}}{\partial t} + \rho_s h (1 - \bar{n}) \bar{v}_{s-j} \frac{\partial \bar{v}_{s-i}}{\partial x_i} + \rho_s \bar{v}_{s-i} (1 - \bar{n}) e_R \\ = \text{div}(h \bar{\sigma}_{ij}^{(s)}) + (h \bar{s}_w - h \bar{p}_w I) \text{grad} \bar{n} + \tau_B^{(s)} \\ + (1 - \bar{n}) \rho_s b_i h + \bar{R}h \end{aligned} \quad (3.159)$$

To derive a quasi-lagrangian formulation of the depth integrated equation, the quasi-material derivative (see equation 3.97) can be used to transform the eulerian form of the above balance of momentum equation into:

$$\begin{aligned} \rho_w h_w \frac{d^{(w)} \bar{v}_w}{dt} = -\text{grad}(h \bar{p}_w \bar{n}) + h \bar{p}_w \text{grad} \bar{n} + \text{div}(h \bar{s}_w \bar{n}) \\ - h \bar{s}_w \text{grad} \bar{n} + \tau_B^{(w)} + \bar{n} \rho_w b h - \bar{R}h - \rho_w \bar{v}_w \bar{n} e_R \end{aligned} \quad (3.160)$$

$$\begin{aligned} \rho_s h_s \frac{d^{(s)} \bar{v}_s}{dt} = \text{div}(h \bar{\sigma}^{(s)}) + (h \bar{s}_w - h \bar{p}_w I) \text{grad} \bar{n} + \tau_B^{(s)} \\ + (1 - \bar{n}) \rho_s b h + \bar{R}h - \rho_s \bar{v}_s (1 - \bar{n}) e_R \end{aligned} \quad (3.161)$$

Consideration of the development of solid volume fraction (n) could improve the model by taking into the account the effect of dilatancy on the flow.

The equations are reduced from 3D to a 2D formulation through a depth integration approximation, which is suitable for flow-like landslides because of a low ratio of the soil

thickness to the landslide length. The presented quasi-3D depth integrated model is both accurate and less time-consuming than a fully 3D model [59,61,86,87] .

3.5 Special case: Hydrostatic and excess pore-water pressure

In previous sections, we have excluded the divergence of the averaged stresses acting on fluid as a source term on the right hand side of balance of moment equations and replaced it with a pressure term which has a significant effect on a flux and should be consider as a part of the LHS. A classical case is that of an inviscid fluid ($\bar{s}_w = 0$), where:

$$\sigma_{w_{ij}} = -p_w I_{ij} \quad (3.162)$$

Note that b_3 is gravity when the axis x_3 is vertical and points upwards ($-g$).

In one special case of interest, the stresses in the solid phase can be considered as hydrostatic ($\sigma_{s_{ij}} = \sigma_{hyd}$), the pore fluid is inviscid and its total pore-water pressure (p_w) being composed of a hydrostatic part (p_{hyd}), which varying linearly from zero at the surface to ρgh at the bottom, and an excess pore-water pressure which will be denoted as Δp_w ,

$$p_w = p_{hyd} + \Delta p_w \quad (3.163)$$

Integrating the total pore-water pressure along its depth, we arrive at:

$$\begin{aligned} \bar{p}_w h &= \int_z^{z+h} (p_{hyd} + \Delta p_w) dx_3 = \int_z^{z+h} (\rho_w b_3 (x_3 - h) + \Delta p_w) dx_3 \\ &= -\frac{1}{2} \rho_w b_3 h^2 + \Delta \bar{p}_w h \end{aligned} \quad (3.164)$$

By considering equations 3.162 and 3.163, the partial stress of fluid phase (see equation 3.11) and its depth integration, respectively, are given by:

$$\begin{aligned}\sigma_{ij}^{(w)} &= -np_w I = n\rho_w b_3(h - x_3)I - n\Delta p_w I \\ \bar{\sigma}_{ij}^{(w)} h &= \frac{1}{2} \bar{n} \rho_w b_3 h^2 I - \bar{n} \Delta \bar{p}_w h I\end{aligned}\tag{3.165}$$

Then, the depth integrated equation for solid partial stress is obtained as:

$$\bar{\sigma}_{ij}^{(s)} h = \frac{1}{2} (1 - \bar{n}) \rho_s b_3 h^2 I + \bar{n} \Delta \bar{p}_w h I\tag{3.166}$$

After substituting the expressions for pore-water pressure, fluid and solid partial stress, that appear in equations 3.164-3.166, in the balance of momentum equations 3.160 and 3.161, the depth integrated equation for the fluid and solid phases are given by:

$$\begin{aligned}\rho_w h_w \frac{\bar{d}^{(w)} \bar{v}_w}{dt} &= \text{grad} \left(\frac{1}{2} \rho_w b_3 h h_w - \bar{n} \Delta \bar{p}_w h \right) \\ &\quad - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_w + \tau_B^{(w)} + \rho_w b h_w \\ &\quad + \bar{R} h - \rho_w \bar{v}_w \bar{n}_w e_R \\ \rho_s h_s \frac{\bar{d}^{(s)} \bar{v}_s}{dt} &= \text{grad} \left(\frac{1}{2} \rho_s b_3 h h_s + \bar{n} \Delta \bar{p}_w h \right) \\ &\quad - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_s + \tau_B^{(s)} + \rho_s b h_s + \bar{R} h \\ &\quad - \rho_s \bar{v}_s \bar{n}_s e_R\end{aligned}\tag{3.167}$$

Take into account that if we do not consider the excess pore-water pressures (Δp_w), it is equivalent to assume that permeability is very large [2]. In low permeable soil, the excess pore-water pressure maintains through the entire solid skeleton of the debris-flow and increases its mobility. Balance of momentum for both phases can be written in a more compact form as:

$$\begin{aligned}\rho_\alpha h_\alpha \frac{\bar{d}^\alpha \bar{v}_\alpha}{dt} &= \rho_\alpha \text{grad} \bar{P}_\alpha - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_\alpha + \tau_B^{(\alpha)} \\ &\quad + \rho_\alpha b h_\alpha + \bar{R} h - \rho_\alpha \bar{v}_\alpha \bar{n}_\alpha e_R\end{aligned}\tag{3.168}$$

where α denotes the phase (s or w), $\bar{n}_\alpha$ being the volume fractions of solid and fluid phases ($\bar{n}_s = 1 - \bar{n}$ and $\bar{n}_w = \bar{n}$), and $h_\alpha = h\bar{n}_\alpha$.

The gradient of pore pressure ($\text{grad}P_\alpha$) refers to the diffusion term where we have introduced the averaged pressure ($\bar{P}_\alpha$) acting on solid or fluid phase as:

$$\begin{aligned}\bar{P}_s &= \frac{1}{2}b_3hh_s + \frac{\Delta\bar{p}_wh\bar{n}}{\rho_s} && \text{For Solid} \\ \bar{P}_w &= \frac{1}{2}b_3hh_w - \frac{\Delta\bar{p}_wh\bar{n}}{\rho_w} && \text{For Fluid}\end{aligned}\tag{3.169}$$

The mathematical model presented in current section are completed by using various rheological and empirical laws. Therefore, we need to provide the following items: (1) the basal friction laws, (2) the Interaction force laws and (3) the Erosion laws.

The **bottom friction term** ($\tau_B^{(\alpha)}$) can be provided by using different types of the rheological model which a few of them were described in the section 2.3 of previous chapter such as (i) the Newtonian fluid which is a simple rheological model characterized by one single constitutive parameter (see equation 2.55), (ii) the Bingham fluid which is able to exhibit a yield stress in the case of cohesive fluid and describes two paramount phenomena such as stoppage of the flow and formation of plugs (see section 2.3.4), and (iii) the frictional fluid which is a simple model where the cohesion and all viscous terms are disregarded. Moreover, the model is able to consider the effect of pore pressure at the basal surface (see equation 2.99). These laws have already been implemented in a GeoFlow_SPH code. In this study, the numerical simulations were performed using the frictional rheological model with considering the dissipation of basal pore pressures.

The **interaction term** (R) can be provided by following laws as (see also equation 3.80) :

- i. Darcy's law: It is a simple Interaction law in which the internal force is given by:

$$R = n^2 k_\alpha^{-1} (v_\alpha - v_s) \quad (3.170)$$

where k_α is the permeability tensor describing the flow of the component α and its unit is $L^3 T M^{-1}$. The presented permeability tensor is given by:

$$k_\alpha = \frac{k^{intr} k_\alpha^{rel}}{\mu_\alpha} \quad (3.171)$$

where k^{intr} is the intrinsic permeability tensor, with dimensions L^2 , k_α^{rel} is the relative permeability which depends on S_α , and μ_α is the dynamic viscosity of the fluid.

In geotechnical analysis, the permeability tensor is usually defined in a slightly different way as:

$$\bar{k}_\alpha = k_\alpha g \rho_\alpha \quad (3.172)$$

where $\bar{k}_\alpha$ is hydraulic conductivity and its unit is LT^{-1} .

By considering Darcy's velocity equation (3.30), we can express internal force equation (3.170) as:

$$R = \bar{c}_d \omega \quad (3.173)$$

where $\bar{c}_d = C_d/n$, $C_d = n k_\alpha^{-1}$ and ω Darcy's velocity.

ii. Anderson and Jackson law [82]: It is second alternative law which was used by Pitman and Le [83]. It can be used when the relative velocity is larger and is given by:

$$R = \frac{n(1-n)}{V_T n^m} (\rho_s - \rho_\alpha) g (v_\alpha - v_s) \quad (3.174)$$

where V_T is the terminal velocity, g the acceleration of gravity and m a constant.

iii. Kozeny-Carman law [88–90]: It is the third alternative law which can be used to calculate the drag force of fluid-like flow, passing through densely packed grains. It is given by,

$$R = \frac{(1 - n^2)}{n} \frac{1}{\bar{k}_{\alpha 0}} (v_\alpha - v_s) \quad (3.175)$$

where $\bar{k}_{\alpha 0}$ is the initial hydraulic conductivity. The hydraulic conductivity is obtained as follows:

$$\bar{k}_\alpha = \bar{k}_{\alpha 0} \frac{n^3}{(1 - n^2)} \quad (3.176)$$

The effect of **entrainment** is taken into account in the source terms by its implementation in the balance equations of linear momentum and mass. The GeoFlow_SPH model contains various empirical laws, which were described in section 2.4, such as: (i) Hungr erosion law [91] which is based on algorithm that the total volume of debris increases in accordance with a specified rate (see equation 2.116), (ii) Egashira erosion law (1993)[63] which is based on flume tests (see equation 2.118) and (iii) Blanc erosion law [64,92] which allows the entrainment rate to be calculated depending on the local slope angle (see equation 2.120). It is important to consider the difference between depth averaged and basal velocities in modeling of the entrainment. In order to obtain basal slip velocity ($\bar{v}_{sB}$), we multiply depth averaged velocity by a correction factor (α), which depends on the rheological models and properties of the material in the basal layer, as follows:

$$\begin{aligned} \bar{v}_{wB} &= \alpha \bar{v}_w \\ \bar{v}_{sB} &= \alpha \bar{v}_s \end{aligned} \quad (3.177)$$

Moreover, the propagation height (h) can be related to the total height (h_{TOT}) by the correction factor, as depicted in the Figure 3-4 and following equation:

Figure 3-4. Velocity and slip velocity profile in frictional-viscous model.

$$h = (1 - \alpha)h_{TOT} \quad (3.178)$$

By considering the equation 3.177, the balance of momentum for both phases can be rewritten as:

$$\begin{aligned} \rho_w h_w \frac{d^{(s)} \bar{v}_w}{dt} = & \text{grad} \left(\frac{1}{2} \rho_w b_3 h h_w - \bar{n} \Delta \bar{p}_w h \right) \\ & - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_w + \tau_B^{(w)} + \rho_w b h_w \\ & + \bar{R} h - \rho_w (\bar{v}_w - \bar{v}_{wB}) \bar{n}_w e_R \end{aligned} \quad (3.179)$$

$$\begin{aligned} \rho_s h_s \frac{d^{(s)} \bar{v}_s}{dt} = & \text{grad} \left(\frac{1}{2} \rho_s b_3 h h_s + \bar{n} \Delta \bar{p}_w h \right) \\ & - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_s + \tau_B^{(s)} + \rho_s b h_s + \bar{R} h \\ & - \rho_s (\bar{v}_s - \bar{v}_{sB}) \bar{n}_s e_R \end{aligned}$$

3.6 Propagation-Consolidation model

Geomaterials are mixtures of solids with voids which are filled by fluids such as air and water. Pore pressures of fluids filling the voids play a paramount role in the behavior of a soil structure [65–68]. Their variations can induce failure for instance initiation of landslide due to an increase of pore-water pressure after a heavy rainfall event. The tendency of the soil to compact results on the generation of high pore pressures and a decrease of the effective confining pressure which causes in some cases liquefaction. As the fluidized material propagates, phenomena of dissipation of pore pressures and dilatancy takes place. Therefore, mathematical models have to include a proper description of time-space variations of pore-water pressures during propagation and consolidation processes. The rheological models have to include mechanisms responsible for pore-water pressures evolutions as well. Modelling of these phenomena is complex because of the many difficulties encounter in mathematical, rheological and numerical modelling [93,94].

The time-space evolution of the pore-water pressures during the propagation stage of a flow-like landslide has been much investigated in the last decades considering that the interstitial fluid affects both the run-out distances and the kinematic characteristics of the propagating masses. Most of models developed so far assume that velocities of solid skeleton and pore fluid are similar ($v - p_w$ approximation). First approximation extending the vertical consolidation mechanism has been modelled by Hutchinson [95], who proposed a simple sliding-consolidation mechanism for a block to consider coupling of pore water and air with the solid grains. The dilatancy or contraction of the soil skeleton changes the pore pressures and accordingly, the effective stresses. In consequence, basal friction and mobility of the soil mass will be much affected. The effect of pore pressure has been described by Major and Iverson [96], who provided experimental data describing the evolution of basal pore-water pressure in debris flows [24].

In order to describe the behavior of rapid landslides in the form of flow, the propagation-consolidation model has been proposed by Pastor et al., (2004) [45]. In this study, It was assumed that the velocity field can be decomposed as $v = v_0 + v_1$ where v_0 and v_1 are the velocity field corresponding to propagation and consolidation respectively, and the pore pressure field was assumed to be in the form of $p_w = p_{w0} +$

p_{w1} where p_{w0} is a hydrostatic field varying linearly from zero at the surface to ρgh at the bottom. From these assumptions, the following well known 1D consolidation equation, which is found in geotechnical engineering text books, can be derived:

$$\frac{dp_{w1}}{dt} = c_v \frac{\partial^2 p_{w1}}{\partial x_3^2} + E_m d_{v0} \quad (3.180)$$

where E_m is introduced as the oedometric modulus and c_v the coefficient of vertical consolidation with dimensions $L^2 T^{-1}$.

By following the procedures described in section 3.4, a depth integrated equation for the vertical consolidation equation can be obtained as:

$$\frac{\partial}{\partial t}(P_1 h) + \frac{\partial}{\partial x_j}(\bar{v}_j P_1 h) = -\frac{\pi^2}{4h} c_v P_1 - \frac{\pi}{2} C.T. + E_m D_1 h \quad (3.181)$$

where P_1 and D_1 are pore-water pressure and dilatancy, respectively, which are limited to a single Fourier component.

Next, the above vertical consolidation equation will be transformed as follows:

$$\begin{aligned} h \left\{ \frac{\partial P_1}{\partial t} + \bar{v}_j h \frac{\partial P_1}{\partial x_j} \right\} + P_1 \left\{ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x_j}(\bar{v}_j h) \right\} \\ = -\frac{\pi^2}{4h} c_v P_1 - \frac{\pi}{2} C.T. + E_m D_1 h \end{aligned} \quad (3.182)$$

By applying the quasi-material derivative (see equation 3.97) and considering the balance of mass equation (3.122), finally, we obtain:

$$\frac{\bar{d}P_1}{dt} + \frac{1}{h} P_1 e_r = -\frac{\pi^2}{4h} c_v P_1 - \frac{\pi}{2} C.T. + E_m D_1 h \quad (3.183)$$

which is the quasi-lagrangian form of the vertically integrated 1D consolidation equation.

Pastor et al. (2015)[53] extended this approach in order to take into account with more precision the evolutions caused by vertical consolidation, total stresses resulting from height variations, and basal surface permeability. In this section, the consolidation equation describing the pore pressure evolution along the depth will be described in detail.

First, by taking into account the equation 2.78, the expression describing pore pressure changes inside the flow can be written as:

$$\frac{d^{(s)}p'}{dt} = \frac{d^{(s)}\bar{p}'}{dt} - \frac{d^{(s)}p_w}{dt} \quad (3.184)$$

where $\bar{p}'$ is the mean-confining pressure and obtained as follows:

$$\bar{p}' = p - p_{hyd} \quad (3.185)$$

Next, we will recall constitutive equation (2.6) which relate deformation to the rate of variation of the effective confining pressure as follows:

$$-\frac{d^{(s)}p'}{dt} = \frac{k_v}{\alpha}(d_v - d_{v0}) \quad (3.186)$$

where,

– d_v is the increment of volumetric strain. In this model, the deformation caused by vertical consolidation (d_{vc}) will be obtained as:

$$d_{vc} = \frac{\partial v_{s3}}{\partial x_3} \quad (3.187)$$

The deformation caused by other two dimensions (d_{s2D}) can be obtained by considering balance of mass equation (3.81) for solid phase as:

$$d_{v2D} = -\text{div}\bar{v}_{v2D} = -\frac{1}{1-n} \frac{d^{(s)}n}{dt} \quad (3.188)$$

– d_{v0} is the extra volumetric deformations to include dilatancy in the constitutive equations as described in section 2.2. Dilatancy can be well-described by using 3D constitutive models which have a high computational cost. For this reason, researchers have preferred to use simpler model such as depth integrated model, although the implementation of dilatancy rules in the model presents the difficulty of introducing a dilatancy angle which should not be constant, as it depends on the stress state and material history. In the proposed model, deformation of soil skeleton is described by a suitable constitutive equation, and pore pressure variations along depth are considered as well, therefore it is not needed to add an extra term to consider dilatancy.

In the case of landslide propagation, the main responsible of changes in the total pressure (p) will be the height variations. We can relate these parameters by taking into account the increment of stress ($\sigma_3 = \rho'_d b_3 h$). Then, following expression can be obtained:

$$\frac{d\sigma_3}{dt} = \rho'_d b_3 \frac{dh}{dt} \quad (3.189)$$

where we have neglected the term $\bar{v}^2/R$ which accounts for centrifugal accelerations.

Next, we will have to relate increments of stress (σ_3) and pressures. When an increment of stress is applied to a soil element, its mean-confining stress ($\bar{p}'$) will change.

The relation between both increments can be obtained by using a suitable constitutive equation, describing its behavior, as follows:

$$\frac{d^{(s)}\bar{p}'}{dt} = -\frac{d\sigma_3}{dt} \quad (3.190)$$

From here, we arrive at:

$$\frac{d^{(s)}p_w}{dt} = -\rho'_d b_3 \frac{dh}{dt} + \frac{k_v}{\alpha} \left(\frac{\partial v_{s3}}{\partial x_3} - \frac{1}{1-n} \frac{d^{(s)}(n)}{dt} \right) \quad (3.191)$$

which is the equation describing the evolution of pore pressure along x_3 .

The next step is to obtain the acceleration caused by vertical consolidation, $\partial v_{s3}/\partial x_3$. The starting point is the balance of mass equation (3.86) for saturated soil obtained in the section 3.3.

$$\frac{\partial v_{s3}}{\partial x_3} = -\frac{\partial \omega_3}{\partial x_3} \quad (3.192)$$

where we have neglected the first term, $(1/Q)(d^{(s)}p_w/dt) = 0$.

Substituting the Darcy's interaction equation (3.173) in the balance of linear momentum equation (3.77) for fluid phase, we arrive at:

$$\omega_3 = -\frac{n}{\bar{c}_d} \frac{\partial p_{w0}}{\partial x_3} + \frac{n}{\bar{c}_d} \rho_w b_3 - \frac{n}{\bar{c}_d} \rho_w \frac{d^{(w)}v_{w3}}{dt} \quad (3.193)$$

Merging above equations of balance of mass and linear momentum, we will have:

$$\frac{\partial v_{s3}}{\partial x_3} = \frac{\partial}{\partial x_3} \left(\frac{n}{\bar{c}_d} \frac{\partial p_{w0}}{\partial x_3} \right) - \frac{\partial}{\partial x_3} \left(\frac{n}{\bar{c}_d} \rho_w b_3 \right) + \frac{\partial}{\partial x_3} \left(\frac{n}{\bar{c}_d} \rho_w \frac{d^{(w)}v_{w3}}{dt} \right) \quad (3.194)$$

Substituting the above equation in the original equation 3.191, the result is:

$$\frac{d^{(s)}p_w}{dt} = -\rho'_d b_3 \frac{dh}{dt} + c_v \frac{\partial^2 p_{w0}}{\partial x_3^2} - E_m \frac{1}{(1-n)} \frac{d^{(s)}n}{dt} \quad (3.195)$$

where we have neglected the second and third terms on RHS of the equation 3.194, because body forces will not depend in general on space coordinates (except in centrifuge tests), and the space derivatives of accelerations are assumed to be small, respectively. The first term on RHS illustrates that the pore-water pressures inside a flowing mass are influenced by the propagation height and its velocity. The second term refers to the diffusion term where parameter c_v is a consolidation coefficient which is obtained by using geotechnical parameters, including permeability, $k_w = n^2/C_d$, and elastic volumetric stiffness, k_v/α , as:

$$c_v = k_w \frac{k_v}{\alpha} \quad (3.196)$$

It is important to note that a high value of consolidation coefficient allows a rapid dissipation of excess pore-water pressure through the consolidation process.

In the third term, which was not considered in Pastor et al. (2015)[52] formulation, the pore-water pressures are influenced by the porosity variations and the content of fine material in the soil [97]. Parameter E_m is implemented as the oedometric modulus which was given in equation 2.9.

Chapter 4

NUMERICAL METHODS

4.1 Introduction

This chapter is devoted to present numerical models which will be used to discretize the equations of depth integrated and pore pressure evolution in propagation and consolidation stage, respectively. In many applications, partial differential equations (PDEs) are impossible to solve analytically. Therefore, efforts have been made to obtain approximate solutions by using numerical methods. In order to obtain approximate solutions, it is necessary to discretize the model in time and space, and then transform these PDEs into ordinary differential equations (ODEs) which are a suitable form for a particle based simulation.

Because of rapid development of numerical techniques based on meshing, either well-structured type such as the finite difference method (FDM) or unstructured such as the finite element method (FEM) [98], and meshless such as smooth particle hydrodynamics (SPH); and also increase in the power of computers, numerical models have become a fundamental and powerful tool to obtain approximations to physical problems. In geotechnical studies, they are able to describe the fluidization behavior of a solid mass in a fast catastrophic landslide and estimate parameters that cannot easily be measured in the field such as pore-water pressures.

During the past decades, grid-based numerical methods have been powerful tools, and solved a large number of major scientific and engineering problems. However, in the fields of hydraulics and fluid mechanics, large displacements are usually involved in many occasions and using meshes, with lagrangian formulation, can lead to numerical difficulties and distortions. In order to avoid these numerical problems, the eulerian formulations have been used in most fluid dynamics cases, although the free surface has to be tracked, with a high computational cost, by using special techniques such as Volume of Fluid (VOF) or Level Set [99]. Due to these

reasons, the meshless methods such as the SPH become a very interesting alternative method which do not rely on meshes for discretization but on moving points or nodes, where functions with their derivatives are approximated. By using this method, the grid can be completely or partially eliminated which is a great advantage with respect to accuracy and efficiency for analysis of large deformation. Regarding mesh-free numerical models, there exist today some interesting approaches, other than SPH, which could be used such as the Material Point Model (MPM) and discrete element methods (DEM), which in terms of similarities, all three have Lagrangian-based reference frames for the material description. MPM is more expensive in terms of computational time compare to other methods, as it makes use of mesh as well as particle data, and DEM is superior for smaller scale simulations. The author has explored that SPH method is best suited to represent the proposed model and accurately described the failure mechanisms of landslides.

In previous chapter, the lagrangian and quasi-lagrangian formulations of the mathematical model, describing the propagation and consolidation of debris flows, have been presented. The equations of the proposed model will be solved by using two different numerical methods, either the FDM which is more suitable for the lagrangian formulations, or the SPH method for quasi-lagrangian formulations.

SPH allows using a set of ODEs by solving the problem that are basically in the form of PDEs, in order to obtain the movement of the flow along time and space. Then, the field variables such as density, velocity and their derivatives are linked to the particles. SPH method first invented to solve astrophysical problems in three-dimensional space [100,101]. Since the collective movement of star particles is similar to the movement of a liquid or gas flow, SPH is well suited to solve the governing equations of the classical hydrodynamics and many other areas. Good reviews can be found in the texts of Liu and Liu (2004) or Li and Liu (2003) [102,103]. Problems found in Solid Mechanics, such as propagation of catastrophic landslides, have also been solved using the SPH technique [104–106], because of its relatively strong ability to incorporate complicated physical effects and model large strain phenomenon. Soils are a special case of solids where their skeleton is filled with pore fluids. Despite the complexity of this case, SPH method is able to applied to problems involving soil–

water interaction and failure mechanism [56,57,107,108]. Concerning depth integrated models for landslide propagation, it is worth mentioning, for finite volumes or elements, the work of Pudasaini (2012)[84], Pastor et al (2002)[109] for eulerian type and Pastor et al (2009a)[46], McDougall, S. and Hungr (2004)[104] and Rodríguez Paz and Bonet (2005)[106] for lagrangian type (SPH).

The following sections describe the main features of SPH method to understand its fundamentals idea, and how it applies to the proposed mathematical model, described in the previous chapter. Finally, several numerical techniques are introduced to improve the numerical accuracy and we will treat some special problems such as boundary conditions.

4.2 SPH: Fundamentals

SPH is represented as a set of particles having individual material properties. It is based on approximating functions and differential operators, such as gradient or divergence, by integral approximations defined in terms of a smoothed kernel function. In this numerical method, an interpolation process calculates the relevant properties on each particle over neighboring particles, without having to define any connectivity element among them.

The SPH procedure is started by approximating a given function $f(x)$ at a position vector x in space which can be expressed as:

$$f(x) = \int_{\Omega} f(x') \delta(x - x') dx' \quad (4.1)$$

where $\delta(x - x')$ is the Dirac delta function defined as:

$$\delta(x - x') = \begin{cases} \infty & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases} \quad (4.2)$$

with the additional requirement of unity as:

$$\int_{\Omega} \delta(x) dx = 1 \quad (4.3)$$

This definition is sufficient for many applications in science and engineering, although not completely rigorous. It is not possible to use a delta function in our computations since it is infinitesimally narrow which means that the interpolation region (Ω) would not overlap with other particle interpolation points. In fact, the Dirac delta distribution is called singular distribution or generalized function, which is a kind of linear functionals which can be obtained as the limit of a sequence of regular distributions. The interested reader will find in the book by A.I.Saichev and W.A.Woyczynski (1997)[110] an excellent introduction to distributions. Take into account that the value of a function can be approximated by using regular distributions. In order to use the integral representation of a function in a continuum model, the Dirac delta function is replaced by a smoothing kernel function, $W(x - x', h_{sm})$, and the equation 4.1 is rewritten as:

$$f(x) = \int f(x') W(x - x', h_{sm}) dx' \quad (4.4)$$

The smoothing kernel function, or weighting function, determines the pattern to interpolate for a function approximation and defines the dimension of the support domain. The kernel W depends on two quantities: (i) The interpolation distance (distance between particles = $x - x'$) (ii) The smoothing length (h_{sm}) defining the influence area of the smoothing function W . Therefore, the performance of the SPH model is closely linked to the type of the weighting functions and their properties. The smoothing kernel must fulfil following conditions in order to give consistent and accurate results:

- i. The unity condition or normalization condition: The integration of the smoothing function produces the unity. This condition can be interpreted as the ability of the approximation to reproduce a constant or degree of the zero polynomial (zero order consistency).

$$\int W(x - x', h_{sm}) dx' = 1 \quad (4.5)$$

ii. The Delta function property: This condition can be fulfilled when the smoothing kernel function tends to a delta function as the smoothing length tends to zero.

$$\lim_{h_{sm} \rightarrow 0} W(x - x', h_{sm}) = \delta(x - x'), \quad h_{sm} \rightarrow 0 \quad (4.6)$$

iii. Compact support: In this condition, the integration over the entire problem domain is localized over the support domain of the smoothing function. Therefore, the integration domain (Ω) usually coincide with the support domain. The compact support of a kernel is as follows:

$$W(x - x', h_{sm}) dx' = 0 \text{ if } |x - x'| > kh_{sm} \quad (4.7)$$

where k is a scalar factor which defines the effective area of the smoothing function and is usually taken as 2, however it depends on the kernel.

iv. Positivity: The smoothing kernel function should be positive for any point at x' within the support domain of the particle at point x . It is important to ensure a physically meaningful representation of some physical phenomena.

$$W(x - x', h_{sm}) > 0 \quad (4.8)$$

v. Decay: The kernel $W(x - x', h_{sm})$ is a monotonically decreasing function of ξ which is given by:

$$\xi = \frac{|x - x'|}{h_{sm}} \quad (4.9)$$

which means that interaction force and its influence on the concerned particle decrease as distance between particles increases.

- vi. Symmetric property: The kernel is usually chosen to be an even function, and particles should have the same influence from the same distance but different position.

$$W_{ij} = W(x_i - x_j, h_{sm}) = W(x_j - x_i, h_{sm}) = W_{ji} \quad (4.10)$$

- vii. Smoothness: The smoothing function should be sufficiently smooth. It aims to get a better approximation.

Taking into account the properties of monotonically and even function condition, it can be concluded that the kernel approximation is second order accurate,

$$\langle \phi(x) \rangle = \phi(x) + O(h^2) \quad (4.11)$$

Any function which satisfies the properties mentioned above can be used as a smoothing kernel function. In the framework of SPH formulations, the proper choice of the smoothing kernel is necessary to be considered. Within GeoFlow_SPH, the user is able to choose from one of the following functions to define the kernel:

- i. The Gaussian kernel proposed by Gingold and Monaghan (1977)[101]: The major advantage is its sufficiently smoothness even for high orders of derivatives. It results are accurate and very stable, especially for disordered particles. However, it introduces an unnecessary number of particles in the calculation, because the kernel tends to zero when the support domain tends to infinity. The kernel proposed is defined as:

$$W(R, h) = \begin{cases} \frac{1}{\pi^{\frac{N_{dim}}{2}} h^{N_{dim}}} \exp(-R^2) & R \leq 3 \\ 0 & R > 3 \end{cases} \quad (4.12)$$

where R is the relative distance between two particles at points x and x' .

ii. The cubic spline introduced by Monaghan and Gingold (1983)[111,112]: It is similar to the Gaussian, but has a very compact support with a less amount of computation. However, its limitations are the function, which is not smooth enough as it is a piecewise function, and the second derivative, which is a polyline. The kernel is defined as:

$$W(R, h) = \alpha_d \begin{cases} \frac{2}{3} - R^2 + \frac{1}{2} R^3 & 0 \leq R < 1 \\ \frac{1}{6} (2 - R)^3 & 1 \leq R < 2 \\ 0 & R \geq 2 \end{cases} \quad (4.13)$$

where α_d is equal to $2/3\pi h$ in 1D, $10/7\pi h^2$ in 2D and $1/\pi h^2$ in 3D. Cubic spline kernel is used more frequently in this study and Figure 4-1 depicts the kernel and its first derivative.

Figure 4-1. The Cubic spline kernel and its first derivative.

iii. The quartic spline kernel: The accuracy of smoother kernels can be improved by increasing the size of the compact support region and by using higher order polynomials functions which correspondingly increases the cost of computation. To achieve this goal, the quartic spline kernel has been proposed [113] to provide an excellent balance between accuracy and computational cost which defined as:

$$W(R, h) = \frac{C}{h^{N_{dim}}} \begin{cases} (3 - R)^5 - 6(2 - R)^5 + 15(1 - R)^5 & 0 \leq R < 1 \\ (3 - R)^5 - 6(2 - R)^5 & 1 \leq R \leq 2 \\ (3 - R)^5 & 2 \leq R \leq 3 \\ 0 & R > 3 \end{cases} \quad (4.14)$$

where C is a constant and equal to $1/120$ in 1D, $7/478\pi$ in 2D and $3/359\pi$ in 3D.

4.3 The discrete SPH interpolation procedure

The integral approximation of function 4.4, presented in the previous section, can be applied in a continuum level, although it is not a practical implementation approach. Another alternative is to construct discrete approximations and store the information in a series of points or nodes. The SPH method introduces the concept of particles so that information including field variables and their derivatives is linked to a set of nodes $\{x_k\}$ with $K = 1..N$. Then, the kernel approximation can be converted to discretize forms of summation over all the particles in the support domain (see Figure 4-2). The level of approximation will depend on nodes locations and how they are spaced, a technique called particle approximation. As already mentioned above, equation 4.4 is an integral approximation of a function and since the information concerning the function is only available at a set of nodes, evaluation of the integral can be performed using a numerical integration technique as:

$$\langle f(x_i) \rangle_h \cong \sum_{j=1}^N f(x_j) W(x_i - x_j, h_{sm}) \omega_j \quad (4.15)$$

where the integral interpolation procedure is approximated by replacing the integral with a summation over all the particles within the region of compact support of kernel function. The sub index h have been used to denote the discrete approximation. The infinitesimal volume (dx') of the continuous integral representations is replaced by the volume of a neighboring particle (ω_j). This concept of "volume" associated with the node is very important, because it allows the approximation to be more general than those which are based on the "mass" of particles SPH formulations. Subscripts i and j in the above equation denote particles i and j as shown in Figure 4-2 which illustrates the performed numerical integration procedure.

Figure 4-2. Numerical integration in the particle support.

Take into account that the kernel function has local support and the summation extends only to the set of points (Nh) which fulfill the following condition:

$$f_i = \langle f(x_i) \rangle_h = \sum_{j=1}^{Nh} f(x_j) W(x_j - x_i, h_{sm}) \omega_j \quad (4.16)$$

where f_i is introduced in order to simplify notation, and x_j with $j = 1 \dots Nh$ is the set of nodes fulfilling the relation $|x_j - x_i| < kh_{sm}$.

Consider that the volume of the particle ω_j is related to the mass and density of the particle associated to node j . Thus, it can be written as:

$$\omega_j = \frac{m_j}{\rho_j} \quad (4.17)$$

The particle approximation for a function at particle i (see equation 4.16), which is the most commonly used expression among SPH form, can be finally rewritten as the averaged summation over a set of particles, each associated with a mass elements, as follows:

$$f_i \cong \sum_{j=1}^{Nh} f(x_j) W_{ij} \frac{m_j}{\rho_j} \quad (4.18)$$

4.4 Integral approximation of SPH gradients and derivatives

The derivative of function $f(x)$ in SPH can be written in the integral representation as:

$$\langle f'(x_i) \rangle = \int f'(x_j) W_{ij} dx_j \quad (4.19)$$

The idea is that as smoothing length decreases and the kernel approaches the Dirac delta function, the proposed representation will also approach the value of the derivative.

Integrated by parts can be applied in one dimensional problem for this expression as:

$$\langle f'(x_i) \rangle = f(x_j) W_{ij} \Big|_{-\infty}^{+\infty} - \int f(x') W'_{ij} dx' \quad (4.20)$$

Taking into account that the kernel has compact support (see equation 4.7) and is defined as being zero at its limits, we arrive at:

$$\langle f'(x_i) \rangle = - \int f(x_j) W'_{ij} dx' \quad (4.21)$$

Classical differential operators of continuum mechanics can be approximated in the same way. First, the gradient of function $f(x)$ in equation 4.4 will be defined as:

$$\langle \text{grad} f(x_i) \rangle = \int [\text{grad} f(x_j)] W_{ij} dx_j \quad (4.22)$$

Manipulating the equation by using integration by parts, we obtain:

$$\langle \text{grad} f(x_i) \rangle = [f(x_j) W_{ij}] - \int f(x') \text{grad} W_{ij} dx_j \quad (4.23)$$

The Gauss theorem can be applied to the equation in order to evaluate surface integral (s) of the domain (Ω).

$$\langle \text{grad} f(x_i) \rangle = \int_s f(x_j) W_{ij} \vec{n} ds - \int_{\Omega} f(x') \text{grad} W_{ij} d\Omega \quad (4.24)$$

where $\vec{n}$ is the unit vector normal to the surface.

Take into account the fact that the kernel has compact support and is defined as being zero at its limits ($\partial\Omega$). Therefore, the surface integral on the RHS of above equation is zero. The result is:

$$\langle \text{grad} f(x_i) \rangle = - \int_{\Omega} f(x') \text{grad} W_{ij} d\Omega \quad (4.25)$$

In SPH formulations, the kernels are written as functions $W_{ij}(\xi_{ij})$, where depends on $\xi_{ij} = |x_j - x_i|/h$. The gradient of the kernel ($\text{grad} W_{ij}$) with respect to x_j can be obtained as:

$$\text{grad}_j W_{ij} = \frac{1}{h} \frac{dW_{ij}}{d\xi_{ij}} \left(\frac{x_j - x_i}{r} \right) = \frac{1}{h} W'_{ij} \left(\frac{x_j - x_i}{r} \right) \quad (4.26)$$

where W'_{ij} denotes its derivative with respect to ξ_{ij} , and $r = |x_j - x_i|$.

Finally, the gradient of a scalar function $f(x_i)$ can be obtained by differentiating the kernel in equation 4.25 as follows:

$$\begin{aligned} \langle \text{grad} f(x_i) \rangle &= - \int_{\Omega} f(x_j) \frac{1}{h} \text{grad}_j W_{ij} d\Omega \\ &= - \int_{\Omega} f(x_j) \frac{1}{h} W'_{ij} \left(\frac{x_j - x_i}{r} \right) d\Omega \end{aligned} \quad (4.27)$$

If we consider the fact that the smoothing kernel is symmetric, then its gradient should be asymmetric as:

$$W(x_j - x_i, h) = W(x_j + x_i, h) \rightarrow \text{grad}_j W_{ij} = -\text{grad}_i W_{ij} \quad (4.28)$$

Therefore, we arrive at:

$$\langle \text{grad} f(x_i) \rangle = \int_{\Omega} f(x_j) \frac{1}{h} \text{grad}_i W_{ij} d\Omega \quad (4.29)$$

Then, the discrete form is obtained immediately following the procedure described in the preceding section as:

$$\text{grad} f_i = \sum_{j=1}^{Nh} f_j \text{grad} W_{ij} \frac{m_j}{\rho_j} \quad (4.30)$$

where the notation $\text{grad} f_i (= \langle \text{grad} f(x_i) \rangle)$ is introduced, and the approximation $f_j (= f(x_j))$ is often used in practice.

The discrete forms of the divergence of a first order tensor (vector function) $v(x_i)$ and a second order tensor $\sigma(x_i)$ are obtained in the same way. Their discretized operators are as follows:

$$\begin{aligned}\text{div}v_i &= \sum_{j=1}^{Nh} v_j \text{grad}W_{ij} \frac{m_j}{\rho_j} \\ \text{div}\sigma_i &= \sum_{j=1}^{Nh} \sigma_j \text{grad}W_{ij} \frac{m_j}{\rho_j}\end{aligned}\tag{4.31}$$

So far, we have shown in equations 4.30-4.31 the discretization of the different functions in basic form. One interesting aspect of SPH is the existence of several alternative discretized forms for the differential operators. These alternatives are called "symmetrized forms", which are preferred to use in practice to improve the computations accuracy. Three symmetrized forms, for each operator, are listed as follow:

i. Gradient of a scalar function:

$$\text{grad}f_i = \frac{1}{\rho_i} \sum_{j=1}^N m_j (f_j - f_i) \text{grad}W_{ij} \quad 1^{st} \text{ form} \tag{4.32}$$

$$\text{grad}f_i = \rho_i \sum_{j=1}^N m_j \left(\frac{f_j}{\rho_j^2} + \frac{f_i}{\rho_i^2} \right) \text{grad}W_{ij} \quad 2^{nd} \text{ form} \tag{4.33}$$

$$\text{grad}f_i = \sum_{j=1}^N \frac{m_j}{\rho_j} (f_i + f_j) \text{grad}W_{ij} \quad 3^{rd} \text{ form} \tag{4.34}$$

ii. Gradient of a vector field:

$$\text{grad}v_i = \frac{1}{\rho_i} \sum_{j=1}^N m_j (v_j - v_i) \otimes \text{grad}W_{ij} \quad 1^{st} \text{ form} \tag{4.35}$$

$$\text{grad}v_i = \rho_i \sum_{j=1}^N m_j \left(\frac{v_j}{\rho_j^2} + \frac{v_i}{\rho_i^2} \right) \otimes \text{grad}W_{ij} \quad 2^{nd} \text{ form} \quad (4.36)$$

$$\text{grad}v_i = \sum_{j=1}^N \frac{m_j}{\rho_j} (v_j - v_i) \otimes \text{grad}W_{ij} \quad 3^{rd} \text{ form} \quad (4.37)$$

iii. Divergence of a vector field:

$$\text{div}v_i = -\frac{1}{\rho_i} \sum_{j=1}^N m_j (v_j - v_i) \nabla W_{ij} \quad 1^{st} \text{ form} \quad (4.38)$$

$$\text{div}v_i = \rho_i \sum_{j=1}^N m_j \left(\frac{v_j}{\rho_j^2} + \frac{v_i}{\rho_i^2} \right) \nabla W_{ij} \quad 2^{nd} \text{ form} \quad (4.39)$$

$$\text{div}v_i = -\sum_{j=1}^N \frac{m_j}{\rho_j} (v_j - v_i) \nabla W_{ij} \quad 3^{rd} \text{ form} \quad (4.40)$$

iv. Divergence of a tensor field:

$$\text{div}\sigma_i = -\frac{1}{\rho_i} \sum_{j=1}^N m_j (\sigma_j - \sigma_i) \nabla W_{ij} \quad 1^{st} \text{ form} \quad (4.41)$$

$$\text{div}\sigma_i = \rho_i \sum_{j=1}^N m_j \left(\frac{\sigma_j}{\rho_j^2} + \frac{\sigma_i}{\rho_i^2} \right) \nabla W_{ij} \quad 2^{nd} \text{ form} \quad (4.42)$$

$$\text{div}\sigma_i = -\sum_{j=1}^N \frac{m_j}{\rho_j} (\sigma_j - \sigma_i) \nabla W_{ij} \quad 3^{rd} \text{ form} \quad (4.43)$$

4.5 Propagation model

In previous sections, the fundamental concepts of SPH numerical method has been described. The next step is to discretize the depth integrated equations derived from the proposed mathematical model by using the SPH method, properties of

integrated approaches and symmetrized forms for the differential operators discussed in the previous section.

We will recall here for convenience the governing equations which are needed to be solved, expressed in quasi_lagrangian form:

- The balance of mass equation for both phases:

$$\frac{\bar{d}^{(\alpha)} h_\alpha}{dt} + h_\alpha \text{div} \bar{v}_{\alpha i} = \bar{n}_\alpha e_R \quad (3.118)$$

- The linear momentum equation for both phases:

$$\begin{aligned} \rho_\alpha h_\alpha \frac{\bar{d}^\alpha \bar{v}_\alpha}{dt} = & \rho_\alpha \text{grad} \bar{P}_\alpha - \left(\frac{1}{2} \rho_w b_3 h^2 - \Delta \bar{p}_w h \right) \text{grad} \bar{n}_\alpha + \tau_B^{(\alpha)} \\ & + \rho_\alpha b h_\alpha + \bar{R} h - \rho_\alpha \bar{v}_\alpha \bar{n}_\alpha e_R \end{aligned} \quad (3.168)$$

where α denotes the phase s or w . Take into account that the equations must be generalized to deal with two-phase flow involving solid and water as depicted in Figure 4-3. Following the procedure outlined in previous sections, we will introduce two sets of nodes $\{x_{\alpha k}\}$ with $k = 1..N_\alpha$, and then define the nodal variables.

Figure 4-3. SPH interactions for two-phase: 1) soil –soil (I - J) and 2) soil-water (I - K).

If the 2D area associated to a general fluid or solid node i is Ω_i , we will introduce for convenience, a fictitious volume M_i with dimensions L^3 moving with this node:

$$M_i = \Omega_i h_i \quad (4.44)$$

It is important to note that M_i has no physical meaning, as when node i moves, the material contained in a column of base Ω_i has entered or left it as the column moves with an averaged velocity which is not the same for all particles in it.

The quasi_lagrangian form of the balance of mass was given in equation 3.118, where its divergence term is evaluated as (see equation 4.31):

$$\text{div} \bar{v}_i = \sum_{j=1}^{Nh} \frac{m_j}{h_j} \bar{v}_j \text{grad} W_{ij} \quad (4.45)$$

In this way, it is possible to write the evolution of flow height in a basic discrete form as:

$$\frac{\bar{d}h_i}{dt} + h_i \sum_{j=1}^{Nh} \frac{m_j}{\rho_j} \bar{v}_j \text{grad} W_{ij} = \bar{n}_i e_R \quad (4.46)$$

As already mentioned in the previous section, there are several alternative symmetrized forms to discretize $\text{div} \bar{v}_i$, therefore it is possible to write the discretized balance of mass equation in the following forms:

$$\begin{aligned} \frac{\bar{d}h_i}{dt} - \frac{h_i}{\rho_i} \sum_{j=1}^{Nh} m_j \bar{v}_{ij} \text{grad} W_{ij} &= \bar{n}_i e_R & 1^{st} \text{ form} \\ \frac{\bar{d}h_i}{dt} + h_i \rho_i \sum_{j=1}^{Nh} m_j \left(\frac{\bar{v}_j}{\rho_j^2} - \frac{\bar{v}_i}{\rho_i^2} \right) \text{grad} W_{ij} &= \bar{n}_i e_R & 2^{nd} \text{ form} \\ \frac{\bar{d}h_i}{dt} - h_i \sum_{j=1}^{Nh} \frac{m_j}{\rho_j} \bar{v}_{ij} \text{grad} W_{ij} &= \bar{n}_i e_R & 3^{rd} \text{ form} \end{aligned} \quad (4.47)$$

where we have introduced $\bar{v}_{ij} = \bar{v}_i - \bar{v}_j$

Alternatively, the height can be obtained once the position of the nodes is known. In this case, we choose the function f_i in the equation 4.18 to represent the height which obtains as:

$$h_i = h(x_i) = \sum_j h_i \Omega_i W_{ij} = \sum_j m_j W_{ij} \quad (4.48)$$

Next, the balance of linear momentum equation 3.168 will be discretized. Depending on the symmetrized form chosen (see section 4.3) to discretize the gradient of the pressure and porosity, we obtain the following discretized forms of the balance of momentum equation:

$$\begin{aligned} \frac{d\bar{v}_i}{dt} = & - \sum_{j=1}^{Nh} m_j \left(\frac{\bar{P}_j}{h_i h_j} \right) \text{grad} W_{ij} \\ & - \left(\frac{1}{2} \frac{\rho_w}{\rho_\alpha} b_3 h_i^2 - \frac{\Delta \bar{p}_w h_i}{\rho_\alpha} \right) \sum_{j=1}^N m_j \left(\frac{\bar{n}_j}{h_i h_j} \right) \text{grad} W_{ij} \\ & + \frac{1}{\rho_\alpha h_i} \tau_B + b_i + \frac{1}{\rho_\alpha} \bar{R}_\alpha - \frac{1}{h_i} \bar{v}_i \bar{n}_i e_R \quad \text{Basic Form} \\ \frac{d\bar{v}_i}{dt} = & - \sum_{j=1}^{Nh} m_j \left(\frac{\bar{P}_j - \bar{P}_i}{h_i^2} \right) \text{grad} W_{ij} \\ & - \left(\frac{1}{2} \frac{\rho_w}{\rho_\alpha} b_3 h_i^2 - \frac{\Delta \bar{p}_w h_i}{\rho_\alpha} \right) \sum_{j=1}^N m_j \left(\frac{\bar{n}_j - \bar{n}_i}{h_i^2} \right) \text{grad} W_{ij} \\ & + \frac{1}{\rho_\alpha h_i} \tau_B + b_i + \frac{1}{\rho_\alpha} \bar{R}_\alpha - \frac{1}{h_i} \bar{v}_i \bar{n}_i e_R \quad 1^{st} \text{ form} \\ \frac{d\bar{v}_i}{dt} = & - \sum_{j=1}^{Nh} m_j \left(\frac{\bar{P}_i}{h_i^2} + \frac{\bar{P}_j}{h_j^2} \right) \text{grad} W_{ij} \\ & - \left(\frac{1}{2} \frac{\rho_w}{\rho_\alpha} b_3 h_i^2 - \frac{\Delta \bar{p}_w h_i}{\rho_\alpha} \right) \sum_{j=1}^N m_j \left(\frac{\bar{n}_i}{h_i^2} + \frac{\bar{n}_j}{h_j^2} \right) \text{grad} W_{ij} \\ & + \frac{1}{\rho_\alpha h_i} \tau_B + b_i + \frac{1}{\rho_\alpha} \bar{R}_\alpha - \frac{1}{h_i} \bar{v}_i \bar{n}_i e_R \quad 2^{nd} \text{ form} \end{aligned} \quad (4.49)$$

Most of the depth integrated models available in the literature do not consider basal pore pressure, although they play a paramount role. In above equations, there is a term describing excess pore-water pressure ($\Delta\bar{p}_w$) at node i , which takes into account pore pressure dissipation during the propagation phase, and it has to be computed at each node and time step.

It is needed to add one more equation,

$$\frac{dm_i}{dt} = \frac{e_R}{h_i} m_i \quad (4.50)$$

which means the mass of each particle does not change in the standard formulation of proposed model unless we have the phenomenon of soil erosion.

4.6 Consolidation models

4.6.1 Finite Difference Method

The numerical model, that will be described in this section, was proposed by Pastor (2015)[52], who combined a Finite Difference Method (FDM) for the 1D equation of vertical consolidation and depth integrated Smooth Particles Hydrodynamics (SPH) model for the propagation analysis (SPH-FD model). The model is also discussed and used by Cascini et al. (2016)[86] to simulate a flume test (one-phase modeling) equipped with a permeable rack (screen) at the end of the channel. The FD scheme chosen is explicit and is centered in space and forward in time (FTCS). One of the advantages of incorporating a set of finite difference meshes at each SPH node (see Figure 4-4) is the ability to simulate cases where basal pore pressures go to zero as a consequence of the landslide crossing a terrain with very high permeability, or a particular control work [52].

Figure 4-4. A 1D finite difference mesh at each SPH node for modeling pore pressure dissipation along depth.

The presented model differs from other depth integrated models with coupled pore pressures had been proposed in the past because in those models pore pressures were described in the soil mass. Here the analysis is not restricted to an assumed shape function fulfilling boundary conditions, but a full approximation of pore pressures inside the landslide is developed. In this framework, the spatial discretization of the pore pressure evolution will be simply made by a set of 1D finite difference meshes, while the time discretization will be carried out with an updated lagrangian approach where the reference configuration will be at time t as depicted in Figure 4-5. In order to solve this equation, we will consider the landslide mass decomposed into differential elements of volume having, at a given time (t), a height (h), and a cross section.

Figure 4-5. Initial and deformed configuration of a column of the landslide mass [52].

For the differential volume element sketched in Figure 4-5, the increment of effective stress, given in equation 3.189, can be written in derivative form as follows:

$$\frac{d\sigma_3}{dt} = \rho g \left(\frac{dh}{dt} - \frac{x_3}{h} \frac{dh}{dt} \right) = \rho g \frac{dh}{dt} \left(1 - \frac{x_3}{h} \right) \quad (4.51)$$

which has been obtained using:

$$x_3 = X_3 \frac{h}{H} \quad (4.52)$$

Finally, we will rewrite the equation 3.195, which describes the evolution of pore pressure along depth, as:

$$\frac{d^{(s)}p_w}{dt} = -\rho' b_3 \frac{dh}{dt} \left(1 - \frac{x_3}{h}\right) + c_v \frac{\partial^2 p_{w0}}{\partial x_3^2} - E_m \frac{1}{(1-n)} \frac{d^{(s)}(n)}{dt} \quad (4.53)$$

which is more suitable for the formulation of FDM. The consolidation equation given in equation 4.53 is discretized by making two changes:

- Height variation: the pore pressure increases by an amount of the first term on the RHS as the soil column varies its height. This allows us to take into account the effects of large deformations in the x_3 axis on the consolidation model.
- Porosity variation which is considered in the third term.

It is important to note that two time scales exist in the equation, (i) a propagation time related to the rate of variation of h and (ii) a consolidation time. The solution depends on the ratio between both time scales.

4.6.2 Quarter cosines shape function

In the second alternative of the consolidation model, the vertical distribution of pore-water pressure is approximated using a quarter cosines shape function, with a zero value at the surface and zero gradient at the basal surface (Pastor et al., 2009)[46].

To solve the second term of the equation 4.53, it is possible to assume that the excess pore-water pressure can be approximated by the following Fourier series:

$$\Delta p_w(x_1, x_2, x_3, t) = \sum_{k=1}^{N_k} \Delta p_{wk}(x_1, x_2, t) N_k(x_3) \quad (4.54)$$

where $N_k(x_3)$ are shape functions which will be used to approximate the pore pressure variations along x_3 .

Among different alternatives, harmonic functions have been chosen because it satisfies the boundary conditions. Assuming the hypothesis that the pore pressure is zero on the free surface and the bottom is impermeable, the harmonic function is:

$$N_k(x_3) = \cos \left\{ \frac{2k-1}{2h} \pi (x_3 + z) \right\} \quad (4.55)$$

where $k = 1 \dots N_k$. For $k = 1$, equation 4.55 transforms into:

$$N_1(x_3) = \cos \left\{ \frac{\pi}{2} \frac{(x_3 + z)}{h} \right\} \quad (4.56)$$

If the analysis is limited to only one of the Fourier components, the expression of the pore pressure becomes:

$$\Delta p_w(x_1, x_2, x_3, t) = \Delta p_{w1}(x_1, x_2, t) \cos \left\{ \frac{\pi}{2} \frac{(x_3 + z)}{h} \right\} \quad (4.57)$$

At basal $x_3 = 0$, we arrive at:

$$\Delta p_w = \Delta p_w^b \cos \left\{ \frac{\pi}{2} \frac{z}{h} \right\} \quad (4.58)$$

where Δp_w^b is the excess pore-water pressure at basal surface. The above equation can be written as:

$$\frac{\partial p_w}{\partial t} = \frac{\partial p_w^b}{\partial t} \cos\left(\frac{\pi z}{2h}\right) \quad (4.59)$$

At basal, we also have following equation:

$$\frac{\partial p_w}{\partial t} = c_v \frac{\partial^2 p_{w0}}{\partial z^2} \quad (4.60)$$

Considering the Fourier series given in equation 4.57, the term on the RHS of the above equation is written as:

$$c_v \frac{\partial^2 p_{w0}}{\partial z^2} = -\frac{c_v \pi^2}{4h^2} \cos\left(\frac{\pi z}{2h}\right) \Delta p_{w0}^b \quad (4.61)$$

Combining the last three equations, the second term of equation 4.53 can be written as:

$$\frac{\partial p_w^b}{\partial t} = -\frac{c_v \pi^2}{4h^2} \Delta p_{w0}^b \quad (4.62)$$

Considering that (i) at basal $x_3 = 0$, thus the first term (RHS) of equation 4.53 is eliminated, (ii) the quarter cosines shape function is used to solve the second term although it has some limitation to a single Fourier component, and (iii) there exists no porosity in basal (the third term is eliminated), the time-evolution of excess pore-water pressure is given by:

$$\frac{d^{(s)} p_w}{dt} = -\rho' b_3 \frac{dh}{dt} - \beta \Delta p_{w0}^b \quad (4.63)$$

where for simplification, we introduce β as:

$$\beta = \frac{c_v \pi^2}{4h^2} \quad (4.64)$$

4.7 Governing equations

So far, we have discretized the equations of balance of mass (4.46), balance of momentum (4.49) and pore pressure dissipation (4.53). The resulting equations are transformed from their partial differential form to a form suitable for particle based simulation.

The dynamics of the soil at **propagation stage** are governed by the following equations

i. Height re-initialization (see equation 4.48):

$$h_{ai} = \sum_j m_{ai} W_{ij} \quad (4.65)$$

ii. Balance of momentum equation (see equation 4.49):

$$\begin{aligned} \frac{d\bar{v}_{ai}}{dt} = & - \sum_{j=1}^{Nh} m_{aj} \left(\frac{\bar{P}_{ai}}{h_{ai}^2} + \frac{\bar{P}_{aj}}{h_{aj}^2} \right) \text{grad} W_{ij} \\ & - \left(\frac{1}{2} \frac{\rho_w}{\rho_\alpha} b_3 h_{ai}^2 - \frac{\Delta \bar{p}_w h_{ai}}{\rho_\alpha} \right) \sum_{j=1}^{Nh} m_{aj} \left(\frac{\bar{n}_{ai}}{h_{ai}^2} \right. \\ & \left. + \frac{\bar{n}_{aj}}{h_{aj}^2} \right) \text{grad} W_{ij} + \frac{1}{\rho_\alpha h_{ai}} \tau_B^{(\alpha)} + b_i + \frac{1}{\rho_\alpha} \bar{R}_\alpha \\ & - \frac{1}{h_{ai}} \bar{v}_{ai} \bar{n}_{ai} e_R \quad 2^{nd} \text{ Form} \end{aligned} \quad (4.66)$$

which are the standard formulations used to solve many geotechnical problems.

It is possible to use either the eulerian or the quasi-lagrangian approximations with SPH models, but the latter presents clear advantages over the former.

Concerning **consolidation model**, two alternative methods are applied to describe the evolution of pore pressure:

i. Finite Difference scheme (SPH-FD model):

$$\frac{d^{(s)}p_w}{dt} = -\rho' b_3 \frac{dh}{dt} \left(1 - \frac{x_3}{h}\right) + c_v \frac{\partial^2 p_{w0}}{\partial x_3^2} - E_m \frac{1}{(1-n)} \frac{d^{(s)}(n)}{dt} \quad (4.53)$$

ii. Quarter cosines shape function:

$$\frac{d^{(s)}p_w}{dt} = -\rho' b_3 \frac{dh}{dt} - \beta \Delta p_{w0}^b \quad (4.63)$$

4.8 Numerical techniques

In previous sections, the SPH numerical method has been completely described. Next, various numerical techniques, which are implemented in GeoFlow_SPH code, will be introduced. They are implemented in order to increase the accuracy of analysis as well as reducing the computational time and complexity.

4.8.1 Time integration scheme

The resulting equations are ODEs which is often computed using a step-by-step time integration scheme. They are used to calculate the system of equations which are dependent on time. In this study, the resulting equations are discretized in time with a suitable algorithm such as 4th order Runge Kutta method in the SPH and FTCS in the FDM which has already been implemented in GeoFlow_SPH. The both numerical schemes are explicit because of its simplicity and speed of computation.

The 4th order Runge Kutta algorithm gives an approximation with greater accuracy in the calculation of the ODE. A detailed description of this method can be found in Hirsch (1988)[114]. The general form of the Runge Kutta algorithm can be obtained as follows:

First we consider a first order differential equation given by:

$$\varphi_n = \varphi(t_n) \text{ with } \varphi(t_0) = \varphi_0 \quad (4.67)$$

where φ is an unknown function of time t which has to be approximated.

The increment based on the slope at the present value n is written as:

$$k_1 = f(\varphi(t_n), t_n) \quad (4.68)$$

The function is evaluated by using k_1 as:

$$\varphi^{(1)}\left(t_n + \frac{\Delta t}{2}\right) = \varphi(t_n) + \frac{\Delta t}{2} k_1 \quad (4.69)$$

The increment based on the slope at the midpoint, between n and $n + 1$, is written as:

$$k_2 = f\left(\varphi^{(1)}\left(t_n + \frac{\Delta t}{2}\right) + \frac{\Delta t}{2} k_1, t_n + \frac{\Delta t}{2}\right) \quad (4.70)$$

The function is evaluated by using k_2 as:

$$\varphi^{(2)}\left(t_n + \frac{\Delta t}{2}\right) = \varphi(t_n) + \frac{\Delta t}{2} k_2 \quad (4.71)$$

Another estimate of the slope at the midpoint can be written as:

$$k_3 = f\left(\varphi^{(2)}\left(t_n + \frac{\Delta t}{2}\right) + \frac{\Delta t}{2} k_2, t_n + \frac{\Delta t}{2}\right) \quad (4.72)$$

The function is evaluated by using k_3 as:

$$\varphi^{(3)}(t_n + \Delta t) = \varphi(t_n) + \Delta t k_3 \quad (4.73)$$

The increment based on the slope at $n + 1$ is written as:

$$k_4 = f(\varphi^{(3)}(t_n + \Delta t) + \Delta t k_3, t_n + \Delta t) \quad (4.74)$$

The final estimate of first order differential equation is reached as:

$$\varphi(t_n + \Delta t_n) = \varphi(t_n) + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (4.75)$$

4.8.2 Time stepping

Time step is a very important factor for calculating new physical quantities of each particle which move according to the updated values at each time step. Therefore, a certain quantity of time step is needed in order to assure the stability of the method and get reasonable results. GeoFlow_SPH includes two choices to obtain the parameter:

- i. A constant value for the time increment can be used.
- ii. Adaptive time stepping can be performed. It is calculated under certain conditions which in this study, Courant Friedrichs-Levy (CFL) condition is implemented. To meet the condition, there are numerous proposals in the literature. In this study, the following one is used:

$$\Delta t_{SPH} = \min \left(\min \left(\frac{h_{sm_i}}{\sum |v_i| + 3\sqrt{c_i}} \right); \left(\frac{1}{2} \right) \min \left(\frac{h_{sm_{ij}}}{\sum |v_{ij}|} \right) \right) \quad (4.76)$$

where, speed is calculated as $c_i = \sqrt{gh_i}$, g is the acceleration of gravity, h the height of the particle, h_{sm} the smoothing length, v_i the speed of the particle i , and v_{ij} the relative speed between the particles i and j .

4.8.3 Smoothing length

The smoothing length (h_{sm}) is a distance based function which defines the extent of the kernel and determines the set of neighboring particles. It has a direct influence on the calculation efficiency and the accuracy of the results. It can be defined either as a constant value or adapted during the simulation. An adapted smoothing length improves the quality of the results since it guarantees that a minimum number of neighboring particles enter the computation. Therefore, it makes the resolution more homogeneous, otherwise the resolution is better in high density regions and lower in low density.

In the case of variable smoothing length, the GeoFlow_SPH code presents two algorithms:

- i. Smoothing length varies according to Monaghan (1989, 1992)[115,116] in which the estimation is made only as a function of the density of the particles.

$$h_{sm} = k \left(\frac{m}{\rho} \right)^{\frac{1}{ndim}} \quad (4.77)$$

where k is an integer and usually is equal to 2.

- ii. Smoothing length varies according to formulation proposed by Benz (1990)[117] which takes the time derivative of the smoothing function in terms of the continuity equation. The smoothing length of particle i at time step $n + 1$ is written as:

$$h_i^{n+1} = h_i + \frac{dh_i}{dt} \Delta t \quad (4.78)$$

where the derivative of the smoothing function with respect to time described by using the concept of kernels and particle approximation, as:

$$\frac{dh_i^n}{dt} = -\frac{h_i^n}{\rho_i^n} \sum_{j=1}^N m_j (v_i^n - v_j^n) \Delta^n W_{ij} \times \left(\frac{1}{ndim} \right) \quad (4.79)$$

4.8.4 Terrain

As shown in Figure 4-6, two different meshes will be used in the proposed model, one to describe the terrain topography while the other is a SPH mesh of nodes. In the case of landslides, the terrain information is given on a digital terrain model (DTM), which consists on a series of values (x_k, y_k, Z_k) at the nodes of a structured grid. The main advantage of presented numerical technique is that the grid pattern of terrain is completely independent on the number of particles (SPH nodes) used to discretize the flowing mass. The number of SPH nodes usually range from hundreds to few thousands, while a structured grid consists of millions of points, from one to few meters spaced in a regular grid of squares (Figure 3-2). In order to represent the changes of terrain surface in the model, as it greatly influences the results, the GeoFlow_SPH use a classical nodal recovery technique on a finite element mesh in which the gradient of the terrain height (gradZ) and its second derivatives, which is the radius of curvature along the tangent to the node path, have to be obtained.

Figure 4-6. SPH mesh and the terrain topography in modelling of a debris flow in Hong Kong.

4.8.5 Computational strategies (Neighbor Search)

In SPH method, a list with finite number of particles, which are in the compact support domain with radius of kh_{sm} , is established at each time step. The procedure to find the particles is called Nearest Neighboring Particle Searching (NNPS). The program allows you to choose between two search algorithms which are the most common in SPH applications: (i) general algorithm (all-pair search) and (ii) temporal-mesh algorithm (linked-list).

All-pair search is a very simple algorithm in which the distances between all the particles, in the domain, are calculated and then the results compare with the smoothing length of the particle constituting the pair. The algorithm considers only those particles that their distance is smaller than smoothing length as neighbor particles. This method is easy to implement but the computation time can be too long.

Linked-list algorithm is based on an auxiliary structured grid covering the problem domain. Spacing is taken as the minimum smoothing length. In each cell a certain number of particles enter, depending on where they are located. Thus, for a particle located inside a cell, the interactions with the other particles are restricted only to the cell it belongs and its neighbors (see Figure 4-3). This temporary grid is valid only for a given time step because the linked-list, corresponding to both solid and fluid particles, is updated every time step. It works very well in cases that their smoothing length is constant or its evolution is low, otherwise it is not easy to reach an optimized spacing of the temporal mesh. The detail of this procedure has been described by Monaghan (1985)[9]. In this study, the linked-list algorithm has been performed.

4.9 Special problems

In any numerical method, there are drawbacks which have to be considered to avoid loss of accuracy and unphysical oscillations. Among the disadvantages and difficulties presented by the SPH method, we can mention (i) the boundary deficiency, (ii) the tensile instability and (iii) the boundary conditioned. Next, these difficulties and the methods to overcome them will be discussed.

4.9.1 Boundary deficiency

The integration near boundaries is carried out over insufficient number of particles in the summation process due to the absence of neighboring particles beyond the boundary, consequently the simulation exhibits large oscillations and displays smaller density values close to boundaries. The problem was detected by Morris (1996)[113], and since then, several approaches have been proposed to overcome this drawback, such as those of Randles and Libersky (2000)[118], Chen et al. (1999)[119], and Liu and Liu (2003)[102]. In GeoFlow_SPH, the normalization is applied to nodes close to boundaries of wall type (Chen et al. [119]) which is one of the most straight forward and computationally least expensive technique. Considering the height re-initialization equation (4.48), the height can be normalized by performing a filter over the height of the particles and then re-assign a height to each particle, which allows improving the approximation close to the boundary nodes:

$$h_i = \frac{\sum_j m_i W_{ij}}{\sum_j \left(\frac{m_i}{h_j}\right) W_{ij}} \quad (4.80)$$

4.9.2 Stability

The second problem of SPH is called tensile instability that arises in dynamics problems with material strength. This instability results in an unphysical clumping of particles and complete collapse of the computation [120]. Swegle et al (1995)[91] described the problem in two dimensional arrangements of particles and studied the stability, which was found to depend on the properties of the kernel. It is important to notice that there are several alternative methods to overcome the tensile instability:

- Use of special kernel functions (Morris, 1996)[113]: The SPH gives good results for the gradients of the first order but in higher-order derivatives, the tensile instability may be appeared. One solution can be to use a special kernel function.
- Introducing artificial stabilizing forces (Monaghan, 2000[121]; Gray et al, 2001[122]): Monaghan suggested that the SPH tensile instability can be prevented by applying an artificial force to control it. Generally, this force is

called artificial stress and consists on repulsion when neighboring particles get closer.

- Adding a new set of stress points between the SPH nodes: Dyka et al. [123,124], and Randles and Libersky [118,124] proposed a new approach to deal with this problem by introducing a new set of stress points other than the SPH nodes. The stress is calculated on the stress points using the information of the SPH nodes, and the linear momentum is calculated on the SPH nodes from the stress points.

- The last approach proposed by Blanc and Pastor [55,56] which will be used in this study to overcome the SPH tensile instability. Its algorithm which is an extension of the Taylor Galerkin method proposed by Peraire et al. [125] and Donea et al. [126], and developed by Mabssout and Pastor [127] to solid dynamic problems. This method requires a SPH grid similar to the one proposed by Randles and Libersky [118,128]. An initial staggered node arrangement will be taken into consideration at first, which consist on a double set of SPH nodes. The nodes of the first set are called material SPH nodes, while the nodes of the second set are called auxiliary SPH nodes. Thus, each variable of the model is defined at both material and auxiliary SPH nodes. However, the material is only represented by the SPH nodes. In consequence, the total mass of the material is

$$M_{total} = \sum_{i=1}^N m_i \quad (4.81)$$

where m_i is the mass of each SPH material node and N the number of SPH material nodes.

4.9.3 Boundary conditions

Boundary conditions require special treatment in SPH. One of them is introduced by Monaghan in 1994 [129], where a set of virtual particles are defined in order to apply a repulsive force on particles approaching the boundary, which prevents boundary penetration by them. Libersky et al (1993)[130], and Randles and Libersky (1996)[128] introduced a second type of virtual particles, which are located symmetrically respect to the real particle approaching the boundary. These methods have been improved by Bonet et al (2004)[131] for cases where rigid boundaries are present. In the case of shallow water waves, boundary conditions of absorbing and prescribed incoming waves are important, and have been studied by Lastiwka et al (2005)[132] and Vacondio et al (2012)[133].

In GeoFlow_SPH, the governing equations are subject to kinematic and tension boundary conditions for both solid and fluid phases at the upper surface, sliding-atmosphere interface, and the base, on which the fluidized mass moves. On the surface, the tension boundary conditions of both phases are considered to be zero. At the base, it depends on rheological laws applied in the analysis. It is also assumed by default that there is no exit or entry of material through the flow. Therefore, it is not needed to define kinematic boundary conditions for both phases, unless the role of entrainment or ejection of material are needed to be considered along the landslide path. Concerning the ejection of material, Lin and Pastor (2019)[134] developed a new type of boundary conditions describing the injection process from a channel into the domain which is based on Riemann invariants.

These boundary conditions, at the upper surface and the base, are already implemented in the proposed governing equations but for some cases, different approach is applied to treat boundary conditions on borders. In real cases, it is not needed to consider any boundary conditions on borders because their computational domain is sufficiently large. Concerning applications with finite domains, such as dam break problem or a small scale laboratory test, wall boundary condition will be implemented. The method we have used consists of (1) defining the wall by a set of wall particles, (2) make zero the normal velocity to the wall if directed against it, and (3) normalize height of interacting nodes (see equation 4.70).

In order to include boundary condition in the pore-water pressure system of equation, one alternative is to use simple shape functions fulfilling boundary conditions at the surface and the basal surface which was described in detail in the section 04.6.2. This approach presents the limitation of not being able to model changes of boundary conditions at the bottom. For instance, when a landslide runs over a permeable basal layer or a rack, pore pressure becomes zero there, while in the body of the landslide is not zero. If a single shape function is used, once the basal value is set to zero, the pressure becomes zero in the whole depth. In order to overcome this drawback, 1D vertical FDM model, describing the evolution of pore pressure (section 4.6.1), developed to be able to define a hydraulic boundary condition at the bottom of the propagating mass.

Chapter 5

EXAMPLES AND APPLICATIONS

5.1 Introduction

In previous chapters, we have presented (i) different rheological models such as frictional fluid model which is able to provide basal friction by considering the effect of basal pore pressure, (ii) a two-phase depth integrated mathematical model based on the Zienkiewicz and Shiomi (1984) equations which is completed by using different approaches for the pore pressure evolution, and (iii) the SPH numerical technique which is incorporated by a set of finite differences meshes at each SPH node to model pore pressure evolution.

The purposes of this chapter are to (i) illustrate the performance and limitations of the proposed two-phase model, and (ii) obtain important flow characteristics include the runout distance, velocity, deposition heights, erosion rate and pore pressure evolution in order to assess the validity of the proposed model and the potential of future damage of debris flows.

We will present four case studies in this section to cover different types of debris flow. First, a simple dam break problem will be analyzed. The second simulation reproduces a flume test, with a permeable rack located along the path, in order to evaluate SPH-FD model by comparing their results. Finally, we will model and analyze the results of two real cases include (i) Yu Tung Road debris flow and (ii) the Johnsons Landing debris avalanche which happened in Hong Kong and Kootenay, respectively. In the first real case, the bed entrainment was a key aspect that effected the dynamics of the moving mass, therefore the role of both parameters, entrainment and pore-water pressure, can be assessed. In the second real case, the initial thickness deposit trim-line and the distribution of deposit volume were provided which make it an appropriate case to evaluate the proposed model.

We start by explaining the way in which the following simulated models are represented. By taking into account that the proposed mathematical equations are integrated in depth, the particles cannot be defined along the depth. In this model, the flow mass was divided into a finite number of columns represented by particles, that keeps the volume constant throughout the path unless we have the phenomenon of soil erosion. Therefore, two heights corresponding to the solid (h_s) and fluid (h_w) phases will appear in each model, by considering the assumption that the fluidized soil is fully saturated, and the sum of these partial heights is the total height (h) of the mixture. Therefore, the phases are simulated in different layers which interact with each other by using the permeability, porosity, and interaction force. Take into account that what we will see in the next figures is the evolution of the height of both phases with respect to the total height, not the relative position in vertical of one phase with respect to the other, as depicted in Figure 5-1.

Figure 5-1. Scheme for obtaining partial heights of both phases (provided by Yagüe, 2018)[135].

5.2 Case 1: Breaking of a dam

One-dimensional dam break exercises provide interesting information and some insight on the propagation of two-phase debris flows in simple and controlled situations.

The initial conditions of two-phase dam break problem over a horizontal plane are presented in Figure 5-2 where the initial heights are $h_s = 4\text{ m}$ and $h_w = 6\text{ m}$. The 1D channel, with a length of 90 meters, contains a dam situated at a distance of 15 meters.

Figure 5-2. Initial configuration of the dam break problem with two-phase on a horizontal surface.

It is important to note that in these simulations, it has been assumed that the granular soil has a basal frictional angle (ϕ) of 1 and the turbulence coefficient (ξ) of 100 m/s^2 . Concerning internal force (R), we use Anderson interaction law (see equation 3.174) by assuming that the terminal velocity (V_T) to be equal to 0.01 m/s and 0.75 m/s , correspond to the cases having low and high permeable soil respectively, and $m = 1$.

The solutions are depicted in Figure 5-3 and Figure 5-4, corresponds to the cases having low and high permeable soil, respectively, where we provide the profiles of total height and soil fraction at times 0, 0.5, 1, 1.5 and 2s.

Figure 5-3. Profiles of h_s and $h_s + h_w$ for low permeable soil at times 0, 0.5, 1, 1.5 and 2s.

Figure 5-4. Profiles of h_s and $h_s + h_w$ for high permeable soil at times 0, 0.5, 1, 1.5 and 2s.

In Figure 5-4, we can see how the water is abandoning the debris mass. The reason is that in the high permeable soil, the forces of interaction between phases are not able to mobilize the solid as much as in the low permeable soil (see Figure 5-3), therefore the solid grains stop while the water leaves the skeleton.

We provide for all the dam break problems, with different permeability, the front propagations of the solid and fluid phases, as can be seen in Figure 5-5. Then, the pore-water pressure evolution is considered in these one-phase and two-phase cases (see Figure 5-6) to explore its effect on the front evolution.

Figure 5-5. Comparison of the front propagations for the one-phase and two-phase cases.

Figure 5-6. Comparison of the front propagations by considering pore-water pressure evolution.

The fronts of these dam break cases are then examined to study the influences of soil permeability and pore-water pressure evolution. The comparisons of flow front propagation show clear trends, and it can be observed that:

- In Figure 5-5, the double line is the result of one-phase modeling with the negligible effects of interstitial fluid which represents granular flows, and it presents the smallest runout (28 m). One-phase modeling can be used for limited cases where the soil can be considered as a single phase material such as (i)

granular flows where the permeability is high enough so that the consolidation time is much smaller than the time of propagation and the material behaves as drained and (ii) flow of slurries with a high water content, where the time of dissipation is much higher than the propagation time, and flow behavior can be assumed to be undrained.

- The solid phases of the both low permeable and high permeable soil (brown line and brown dash, respectively) have longer travel distances (31m and 36m, respectively) in comparison with the one-phase case, due to considering the interaction forces in the two-phase analyzes, otherwise they will coincide with one another.

- In two-phase case with low permeable soil, where interaction forces between phases play an important role, the front progress curves of both solid and fluid phases (brown line and blue dash dot, respectively) coincide with one another. On the other hand, it can be observed in the case with high permeability how the solid phase stops (brown dash) while the water (blue round dot) continues flowing.

- In the Figure 5-6, the role of pore-water pressure was fully taken into account in the simulated results. A notable change of front evolution demonstrates the obvious effect of the pore-water pressure on the case with low permeability. On the other hand, the pore-water pressure does not present any considerable influence on the case of high permeable soil. In the latter scenario, the excess pore-water pressure has not been generated due to the high permeability of the soil and high velocity of the flow, and the consolidation process did not take place during propagation stage.

- The double dash line is the result of one-phase modeling with a high consolidation coefficient which coincides with the result of solid phase (brown dash) in high permeable soil. However, the one-phase modeling is not able to reproduce the behavior of the fluid phase (blue round dot) due to assumption that the acceleration of the fluid phase is very small. Therefore, it is not a sufficient model for cases of high permeable soil, but it can be applied to landslides where the relative movement of water and solid is negligible. On the other hand, two-phase modeling is able to consider solid and fluid phases separately and take into account difference of velocities in both phases.

- For the case of low permeable soil, the results of one-phase (double line) and two-phase (brown line for solid phase and blue dash dot for fluid phase) modeling are very close to each other, as depicted in Figure 5-6. The little difference is due to considering the effect of excess pore-water pressure in propagation stage in two-phase modeling.

A comparison between the performances of two different models has been carried out. The advantages of using a two-phase model respect to the one-phase model are as follows:

- i. In two-phase modeling, the distribution of the porosity can be obtained and analyzed due to capability of the model to consider the velocities of two phases.
- ii. The interactions between the fluid and solid phases in simulations are considered.
- iii. In one-phase model, the results are more sensitive to parameters of frictional angle and pore-water ratio. On the other hand, employing more input parameter in two-phase model allows better matching between the simulation results and measurements [136].

5.3 Case 2: Flume test

5.3.1 Introduction

In the second simulation, the capacity of the proposed model was evaluated to reproduce a debris flow on a laboratory scale inclined channel, by comparing the numerical results with the measurements obtained from experiments and assessing the performance of the time-space evolution of pore-water pressure. The small scale laboratory test performed by Yifru at Trondheim (2019)[137] to illustrate the effect of the permeable debris flow rack and its working mechanism.

5.3.2 Description of the flume tests

The flume test was performed in a 10m long channel and has two parts: a main channel and a deposition area. The former is 6m long and 0.3m wide, with a 17° inclination, and the latter is 4m long and 2.2m wide, with a 2° inclination. Figure 5-7

provides a configuration of the flume test and the location of the screen (a permeable rack).

Figure 5-7. Schematic representation of the flume model. (Provided by Yifru)

A vertical cylinder was used to release a fully saturated debris material with a total volume of 25L and solid concentration of 60%. Therefore, the densities of soil particles, interstitial fluid and mixture have been taken with the values of 2750 kg/m^3 , 1000 kg/m^3 and 2050 kg/m^3 respectively.

In this modeling, three flume test with different conditions were conducted. In the first test, the moving mass was able to flow freely in the channel and the deposition area without any obstacle. In the second test, a solid plate with the same size and shape of the rack was placed in the same position as where the rack was intended to be placed in order to evaluate the topographical change due to the application of the rack. In the last test, the rack was placed in the same location as the solid plate.

The run-out distance, and maximum flow height on the rack and deposition areas were recorded by using four flow-height sensors and three video cameras which were installed in different locations of the channel, as depicted in the Figure 5-7. The deposition shapes and the run-out distances of the flume tests are presented in the Figure 5-8. The run-out distance of flume test without any obstacle, with a solid plate and with a rack are 2.86m, 2.08m, and 1.61m, respectively, from the beginning of the deposition area.

Figure 5-8. The deposition shapes and the run-out distances of the tests (a) without obstacle (b), with solid plate and (c) with rack. (Photos taken by Yifru)

There are two distinguishing parts of the debris deposit: (i) The first is the debris mass accumulation which took place over the solid plate and the rack. The debris mass accumulated with a uniform thickness of 2cm over the former. On the other hand, accumulation height on the latter varied between 5cm and 6cm , as depicted in Figure 5-9. (ii) The second is the debris mass that accumulated with maximum height of 2cm in the deposition area for all the three tests.

Figure 5-9. Accumulated debris over: (a) the solid plate and (b) the rack. (Photos taken by Yifru)

5.3.3 Numerical modeling

The one-dimensional numerical simulation has been performed through two-phase SPH model described in Chapter 4. In this case, the consolidation equation will be discretized by using FDM in each time step (SPH-FD model) in order to improve the quality of the predictions. The advantage of using a set of finite difference meshes is the ability to consider sudden pore pressure evolutions and height variations as a consequence of the debris flows crossing a rack with high permeability. The single shape function, described in the section 4.6.2, presents some limitation of the ability to model the cases with a permeable terrain. If the latter method is used, once the basal pore pressure is set to zero at the permeable rack, the whole pressure along the vertical axis becomes zero, while it should not be zero in the body of the landslide.

The SPH-FD model was discussed and used by Cascini et al. (2016)[86] to simulate a flume test, conducted by Gonda (2009)[138], equipped with a permeable rack at the end of the channel. In the above mentioned models, the permeable rack was placed in the deposition area. Therefore, the moving mass was retained on the rack. However, Yifru believes that the entire moving mass might not be deposited on rack and he found it interesting to evaluate the behavior of the materials after they passes over the rack.

The flume test was simulated by using a special boundary condition in which the total pore pressures are considered to be zero at the basal surface where the flow mass is passing over a permeable rack. In Figure 5-10, the vertical profiles of total pore-water pressure on the run-out channel and the rack have been plotted. The numerical simulation of the pore-water pressure is based on the following assumptions: (i) The distribution of total pore-water pressure (p_w) is a sum of the hydrostatic pressure (p_{hyd}) and the excess pore-water pressure (Δp_w). (ii) When the SPH nodes cross a permeable rack, the excess pore-water pressure (Δp_w^b) instantly becomes equal, but opposite in sign, to the hydrostatic pressure (p_{hyd}^b) at the basal surface.

Figure 5-10. The total pore-water pressure distribution of debris-flow on an impermeable terrain (run-out channel) and a permeable rack.

The solid and fluid phases are modelled with frictional rheological laws based on the Voellmy model (see equation 2.99) and Chezy-Manning formula (see equation 2.104), respectively. The frictional rheological law is used in order to implement the effect of pore pressure at the basal surface. It is important to note that the friction between the solid phase and the rack increases as the excess pore-water pressure decreases on the surface of rack.

A back-calculation of the flume test with a rack was carried out to determine the governing parameters in such a way that the results coincide well with the observed run-out time, run-out distance and deposit thickness. The most suitable values, that replicated the event of the selected test, are given in Table 5-1. The same input data was used for all the flume tests.

Table 5-1. Geotechnical parameters of the flume tests.

Parameter	Value
Manning roughness coefficient (n)	0.025
Basal friction angle ($\tan \varphi_B$)	0.6
Turbulence coefficient (ξ)	1000 m/s^2
Permeability tensor (k_w)	$15 \times 10^{-7} \text{ m/s}$
Volumetric stiffness (k_v)	10^6 N/m^2

5.3.4 Numerical results

The simulation results obtained by the GeoFlow_SPH program are presented in this section. Concerning the total pore-water pressure (p_w^b) at basal surface, the results are reported in Figure 5-11 at two different time steps.

Figure 5-11. Total pore-water pressure a) at starting condition and b) at time 1.8s of the flume test.

By assuming that the flowing material was fully liquefied at triggering stage and considering equation 3.163, the maximum total pore-water pressure at starting conditions can be obtained as follows:

$$p_w = 2\rho_w gh = 2 \times 1000 \frac{\text{kg}}{\text{m}^3} \times 10 \frac{\text{N}}{\text{kg}} \times 0.2\text{m} = 4\text{kPa} \quad (5.1)$$

Which implies that the excess pore-water pressure is equal to the hydrostatic pore pressure. During propagation stage, the total pore-water pressure is composed of a hydrostatic part and an excess pore-water pressure. Except when the moving mass is propagating over a rack which in this case, the total pore-water pressure at basal surface becomes zero. The location of the rack is shown in Figure 5-11b.

In the present study, the two-phase modeling has been developed by adding a 1D finite difference grid to each SPH node to improve the description of the evolution-space pore-water pressure. Therefore, the GeoFlow_SPH program is able to present how the relative pore-water pressure changes along the flow depth. In Figure 5-12, we provide the distribution of pore-water pressure at different time steps.

Figure 5-12. The distribution of relative pore-water pressure a) at the start of the simulation, b) before the flow hits the rack (a zoomed portion of the plot), c) over the rack (a zoomed portion of the plot) and d) after the flow passes the rack.

At the start, the relative pore-water pressure (p_w^{rel}) is equal to 1 indicating that the flowing material was completely liquefied. The pore-water pressure persisted during propagation until the debris-flow reached to the rack. During the propagation over the rack, the total pore-water pressure at the basal surface becomes zero and the excess pore-water pressure instantly becomes equal, but opposite in sign, to the hydrostatic pressure, as follows (see Figure 5-10c):

$$\Delta p_w^b = -p_{hyd}^b \quad (5.2)$$

The normalized excess pore pressure ($\Delta \bar{p}_w^b$) at basal surface can be obtained as:

$$\Delta \bar{p}_w^b = \frac{\Delta p_w^b}{p'} = -\frac{\rho_w g h_{sat}}{\bar{\rho}' g h} \quad (5.3)$$

where relative water height (h_w^{rel}), which is the ratio of the height of the water table (h_{sat}) to the soil thickness (h) is considered to be 1, and $\bar{\rho}'$ is the effective density ($\bar{\rho}' = \rho - \rho_w$). Then, $\Delta \bar{p}_w^b$ can be calculated as (see Figure 5-12d):

$$\Delta \bar{p}_w^b = -\frac{\rho_w}{\rho - \rho_w} = -\frac{1000}{2050 - 1000} = -0.95 \quad (5.4)$$

A back analysis was carried out in order to estimate the run-out distance, and the deposition heights on the rack and the deposition area of the flume test, as depicted in Figure 5-13a. The simulation results of the other two flume tests, without obstacle and with solid plate, are given in Figure 5-13b and Figure 5-13c, respectively, which have been simulated by using the parameters that replicate the first test.

Figure 5-13. The final deposition height and run-out distance of the flume test a) with a rack, b) with a solid plate and c) without obstacle. Note that the height has been amplified by a factor of 5.

Results are satisfactorily reproducing the the runout distance and final deposition height over the rack and the deposition area for the first test. The back-calculated parameters of the first test are used for the simulating the other test. The estimated runout distances are found to be longer than what is observed in the laboratory test. However, the depth averaged flow height is seen to be satisfactorily reproduced for all the cases. The arrival time of the flow front fits with the laboratory result when using the rack however in the other two test, the simulation is looked to be faster than what is observed in the laboratory.

The results demonstrated that the proposed model is capable to consider the effect of a terrain with high permeability and describe properly the behavior of a debris flow propagating over a rack. However, the following restrictions apply to the proposed model and flume test:

- i. The main drawback lies in considering the treatment of the flume tests as one-dimensional which simplifies some features.
- ii. The presence of the cylindrical container was not included in the numerical model. Therefore, fully liquefied debris, with a trapezoidal shaped, instantaneously released in the simulation.
- iii. Because of the 1D numerical modeling, the interaction within the flow and the boundary wall could not be considered.
- iv. The model is able to consider the pore-water pressure dissipation in the shearing zone. However, the loss of water through the rack during flow propagation was not included into the model.
- v. There are some drawbacks to simulate debris flows at the small scale flume tests. Discussion about these drawbacks are beyond the scope of this work, however the interested reader will find in the articles by Iverson (2010)[139] and Cascini (2016)[86] an interesting description.

5.4 Case 3: Yu Tung road debris flow

5.4.1 Introduction

The next case was selected from the benchmarking exercises, which proposed by the second Joint Technical Committee on Natural Slopes and Landslides (JTC1) Workshop. The aim of proposing these exercises was to evaluate the latest methodology for simulation of landslide runout analysis and the corresponding development trend.

5.4.2 Description of the real case

On 7 June 2008, an intense rainstorm triggered a debris flow on the natural hillside in Lantau Island, Hong Kong (Figure 5-14). Over $3000m^3$ of debris ran down the hillside and travelled $510m$ at an apparent travel angle of 17° to the Yu Tung Road which resulted in significant entrainment and deposition along the debris path and serious road blockage. A rich documentation with data and details were provided by the Geotechnical Engineering Office of Hong Kong (GEO Report No. 271)[140].

Figure 5-14. Aerial View of the debris flow event after the landslide incident.

5.4.3 Numerical modeling

The developed model was applied to simulate the dynamics of the Yu Tung road debris flow which is an appropriate benchmark case to examine the applicability of the proposed model. For this debris flow case, it is required to consider a relatively high fluid content to obtain acceptable results. One-phase continuum-based approaches may experience difficulties in capturing intense solid-fluid interactions which is a key aspect of debris flows. Therefore, the event was simulated by using the two-phase model which has implemented in the GeoFlow_SPH code. The numerical analysis of the debris flow propagation and consolidation was performed through the SPH model, described in Chapter 4.

Debris flows are known to be highly saturated in which the fluid phase can significantly increase the mobility of the flow. Therefore, the porosity or void fraction is one of the most important parameters used to characterize two-phase flow. In this model, the void fraction is equal to 0.28 which implies the presence of interstitial fluid in the mixture.

5.4.4 Numerical results

Field mapping revealed that significant entrainment of loose materials and erosion of the side slopes had occurred. It is estimated that the active volume increased to about $3400m^3$ which implies that the bed entrainment was a key aspect in effecting the dynamics of the moving mass. Therefore, the entrainment law of Hungr (1995)[91]

(see section 2.4) was used which is based on algorithm that the total volume of debris increases in accordance with a specified rate. The entrainment rate or coefficient is adopted to be 0.001 m^{-1} which have been back-calculated based on the super-elevation data. As you can see in Figure 5-15, the GeoFlow_SPH code is able to consider the change in the topographic profile due to soil erosion by updating the topography continuously with the eroded material. The amount of deposited material reaching Yu Tung Road was directly calculated by the model and it was very close to the estimated maximum active volume.

Figure 5-15. Final erosion depths at time 40s.

As already noted, if the effect of entrainment is considered in modeling, such as Yu Tung Road Case in this study, it is recommended to use Perzyna viscoplastic model (see section 2.3.7) because its velocity profile is linear, as shown in Figure 2-9, which makes the model more adaptable to erosion laws. Therefore, the solid phase is modelled as a granular flow with a frictional rheological law based on the Perzyna visco-plastic constitutive model while the fluid phase is treated as a Newtonian fluid. The analysis was performed with a fixed viscosity coefficient ($\mu_{PZ} = 10^{-4}\text{ Pa.s}$), and the parameters $n_1 = 1$ and $N = 1$.

In this model, the vertical distribution of pore-water pressure is approximated using a quarter cosines shape function (see section 4.6.2), with a zero value at the surface and zero gradient at the basal surface. The relative pore-water pressure (p_w^{rel}) was taken as 1 indicating that the flowing material was completely liquefied. The consolidation coefficient (c_v) is obtained by using back-calculated parameters include terminal velocity, $V_T = 10^{-4}m/s$, and stiffness of the mixture, $k_v = 8 \times 10^7 N/m^2$. Take into account that the high value of c_v can cause rapid dissipation of excess pore pressure through the consolidation process. In Figure 5-16, we present a results sequence of the pore pressure evolution at different time steps.

Figure 5-16. Results sequence of pore pressure evolution at 0, 15 and 30 seconds.

The Tung Road debris flow was simulated based on the provided initial thickness and topography, which is a regular 325x88 grid with a 2 meter step. We provide in Figure 5-17, a topographical map showing the landslide path and positions of the flowing mass produced by GeoFlow_SPH model from initial to final deposition.

Figure 5-17. Results sequence of the debris flow simulation at different positions.

In GeoFlow_SPH code, run-out distance is measured using the straight line distance from source point to evaluation point. As can be seen in Figure 5-17, the simulated landslide debris reaches the final deposition (Yu Tung road) and the obtained runout distance is equal to 450m at final step. Since the final deposit profile was not provided, it was not possible to compare the measured and computed deposit profiles.

Debris flow velocities were estimated at various locations along the flow path by using super-elevation data and a video record which captured the whole debris flow process. Figure 5-18 shows the measured frontal velocities (shown in red dots) along chainage.

Figure 5-18. The measured front velocities of the Yu Tung Debris Flow by labeling them as A, B, C, D and E. (The images provided by Second JTC1 Workshop)

The numerical results of the debris flow frontal velocities are reported in Figure 5-19. The velocity of the debris was computed to be about 12m/s at time of 4s and a distance of 100m from the source location. Then, the debris flow travelled at a higher speed with an average velocity of about 17m/s at propagation time of 13s and a distance of 350m . Finally, it slows down to 11m/s at time of 23s .

Figure 5-19. The numerical simulation with the spatial velocity evolution for different propagation times given by GeoFlow_SPH model.

The SPH-FD model is capable to obtain reasonable results and properly reproduce the propagation velocity and runout distance, and more importantly to correctly performs the time-space evolution of pore-water pressures and bed entrainment during the whole propagation stage from initiation to propagation over an impermeable bottom boundary, and up to deposition.¹

¹ The results and an abridged version of this section has been presented for Second JTC1 and 29th Alert Workshops [144,145].

5.5 Case 4: Johnsons Scott debris avalanche

5.5.1 Introduction

The Johnsons Landing landslide is another real case exercise which was proposed by the Second JTC1 Workshop in order to benchmark the capabilities of recent computational models. Therefore, it was modeled based on the information provided by Hong Kong Geotechnical Office, which included topography, initial thickness deposit trim-line and the distribution of deposit volume.

5.5.2 Description of the real case

It occurred approximately two km northeast of the small community of Johnsons Landing, located on Kootenay Lake, on the morning of July 12th 2012. Figure 5-20 provides a general view of the avalanche and its location.

Figure 5-20. Aerial view of the the Johnsons Landing debris avalanche. (Copyright © Province of British Columbia)

As shown in Figure 5-20, the debris avalanche was triggered in the upper channel where the unstable materials, include soil and rock, flowed into the Gar Creek channel. Then, a small portion of the material flow avulsed from the mid channel at a sharp bend (It is labelled as ‘avulsion point’) and travel along the drainage line until it reached to the mouth of the gully (labelled as ‘Gar Creek Fan’). A video of this secondary event is available online (<https://www.youtube.com/watch?v=n1cCs-S5EKc>). Much of the debris avalanche flowed out of the channel, ran down to the Johnsons Landing bench and spread out over the terrace surface (The location of the

bench is shown on Figure 5-20), causing the loss of four fatalities and the damage of several homes and a public road.

Nicol et al. (2013)[141] has conducted an in-depth investigation on the day of the debris avalanche. It is estimated that about $320,000 \text{ m}^3$ of unstable material travelled down the channel, at the flow velocities of between 25-35 m/s based on super-elevation, It was estimated that $140,000 \text{ m}^3$ of material deposited in the upper channel, $55,000 \text{ m}^3$ deposited in the mid channel, and $169,000 \text{ m}^3$ deposited on the bench. Therefore, the deposit volume increased to about $364,000 \text{ m}^3$.

5.5.3 Numerical modeling

Nicol et al. (2013)[141], Marinelli et al. (2015)[142] and Aaron (2017)[143] conducted a numerical modeling, with the computer-based models DanW and Dan3D, in order to back-analyze the dynamics of the event, and estimate its potential run-out distances and deposit thickness.

The key challenge is to reproduce the process of the debris avalanche flowing along the channel until it reached to the mid channel where the debris avalanche flowed out and ran down to the Johnsons Landing bench (see Figure 5-20). In order to reproduce the observed deposit distribution, the mentioned authors had two options: (i) applying a two-rheology model and (ii) assuming that a channel obstruction existed due to an accumulation of timber and located downstream of the 70° bend. Otherwise, significant amount of volume was predicted to deposit in the channel downstream of the bend which does not agree with the observed results. The two-rheology model can be used to consider a low basal resistance to flow on the channel and a high flow resistance, which represent the presence of vegetation, on the bench. As an alternative, a channel obstruction can be assumed to be present in order to reproduce the large deposition volume on the bench. In present study, the second alternative is implemented in the numerical analysis.

The numerical analysis of Johnsons Landing debris avalanche is performed through the GeoFlow_SPH model, which is a depth integrated two-phase model proposed by Pastor et al. (2017)[2] and extended in this study.

In this model, the vertical distribution of pore-water pressure is approximated using a quarter cosines shape function (see section 4.6.2). The relative pore-water pressure (p_w^{rel}) was taken as 1 indicating that the flowing material was completely liquefied. The consolidation coefficient (c_v) is obtained by using back-calculated parameters including terminal velocity, $V_T = 10^{-3}m/s$, and stiffness of the mixture $k_v = 4 \times 10^7 N/m^2$.

We provide in Figure 5-21 a topographical map showing the landslide path and positions of the sliding mass from initial to final deposition, produced by GeoFlow_SPH model. The Johnson's Landing debris avalanche has been studied with the frictional rheological model. Concerning the two rheological parameters, the basal frictional angle is taken as 30° , and the turbulence coefficient (ξ) of $1000 m/s^2$ has been obtained based on previous studies.

Figure 5-21. Results sequence of the debris flow simulation at different positions at times a) 0s, b) 20s, c) 40s and d) 60s.

Figure 5-22a shows the observed deposit, based on the information provided by Hong Kong Geotechnical Office, and Figure 5-22b shows the comparison of deposit shapes obtained from simulation and observed results. As you can see from the figure,

the deposit shape of the debris avalanche was reasonably predicted in all the zones. The deposited material could well spread in the upper channel, the mid-channel and the bench, but it did not cover the whole trimlines. It may require further research in order to obtain the exact mechanism of the flow spreading.

Figure 5-22. a) Johnsons Landing debris avalanche's deposit trimline and the distribution based on provided information b) the well-matched deposit shapes obtained from simulation and observed results.

The numerical results, obtained with GeoFlow_SPH model, are reported in Figure 5-23 concerning deposit thickness. As shown in Figure 5-20, the deposition areas can be divided into three zones: (i) **the upper channel** with an approximated area of $55 \times 900 \text{ m}$, a thickness between 1 m and 7 m , an average thickness of 3 m , and an estimated volume of $148,500 \text{ m}^3$, (ii) **the mid channel** with an approximated area of $70 \times 150 \text{ m}$, a thickness between 6 m and 13 m , an average thickness of 8 m , and an estimated volume of $84,000 \text{ m}^3$, and (iii) **the bench** with an approximated area of $150 \times 300 \text{ m}$, a thickness between 2 m and 6 m , an average thickness of 3 m , and an estimated volume of $135,000 \text{ m}^3$. The computed (GeoFlow_SPH) and estimated deposition volume [141] are almost matches. However, more amount of material was deposited on the mid channel in the simulation, and not enough in the bench. These differences might be solved by using two-rheology model in order to consider different rheological parameter for different zones.

Figure 5-23. Final deposit thickness computed by GeoFlow_SPH program by considering a channel obstruction at the beginning of the sharp bend.

The average velocity was obtained by Nicol et al. (2013)[141] based on super-elevation data and estimated to be between 25 – 35 m/s as the flow travelled down on to the bench. The numerical results are shown in Figure 5-24, concerning the front and average velocities.

Figure 5-24. The computed front and average velocities of debris avalanche for Johnsons Landing.

Chapter 6

CONCLUSION

6.1 General aspects

In this Doctoral Thesis, a new two-phase SPH model has been developed with the aim of achieving a precise and effective methodology to replicate the complex behavior of debris flows by including many essential physical phenomena such as variation of pore-water pressures and bed entrainment. The presented work can be divided into five areas:

- Chapter 1 briefly described the general aspects of rapid landslides such as debris flows in which considering the time-space evolution of pore-water pressure is a fundamental issue.
- In Chapter 2, different rheological laws have been studied which allow us to describe the behavior of the fluidized geomaterials. Then, the depth integrated rheological relations have been presented in order to complete the governing equations developed in Chapter 3. In the last part, various empirical erosion laws have been presented in order to assessing the role of bed entrainment during the propagation stage. These empirical relations are implementable in depth integrated governing equations.
- In Chapter 3, the two-phase mathematical model, proposed by Pastor (2017), has been extended to take into account the excess pore pressures in debris flows, particularly the ones that consist of low-medium permeable soils. In the proposed model, the phenomenon of consolidation and dissipation of pore pressure have been considered in analysis by adding the one-dimensional consolidation equation, with height and porosity dependent source, into the system of equations.

– In Chapter 4, two numerical methods have been presented to transform the partial differential equations, presented in Chapter 3, into ordinary differential equations. The balance of mass and momentum equations have been discretized with SPH, while the consolidation equation has been discretized using a set of finite difference meshes associated with each SPH node that represents a solid particle.

– Chapter 5 has been dedicated to illustrate the performance of the proposed two-phase SPH model so that it can be practically applied to assess the potential of future damage of landslides. Moreover, the proposed model has been used to investigate the role played by excess pore-water pressure during the propagation of debris flows focusing on four different case studies include dam break problems, flume test and two real cases:

i. For dam break problems, different scenarios have been considered, and the front propagation along time for each of the scenarios has been provided. The results indicated that run-out distance are strongly affected by the time-space evolution of the pore-water pressures in the dam break problems with low permeable soil.

ii. The proposed model was also applied to simulate the flume tests performed by Yifru at Trondheim (2019) in a 10m long channel, equipped with a permeable basal screen (rack) located at middle of the channel to illustrate the effect of the permeable rack and its working mechanism. The results of the numerical analyses show that the presented SPH-FD model is capable to correctly measured the deposition heights and run-out distances of the flume test with a permeable rack, and more importantly properly reproduce the time-space evolution of the pore-water pressures during the propagation stage over both permeable and impermeable bottom boundaries.

iii. In the third stage, we studied two real cases which occurred in Hong Kong (Yu Tung Road, 2008) and Canada (Johnsons Scott, 2012), respectively. The cases were selected based on reliable information includes the provided topography, initial thickness of the landslide, the distribution of deposit volume, final run-out and estimated velocities. The geotechnical parameters were estimated by taking into account these field evidences. The results indicated that the GeoFlow_SPH

model is a suitable tool for the analysis and forecasting the propagation and deposition in the run out zones of debris flows.

6.2 Final conclusions

This PhD thesis has contributed to improve the formulation of the two-phase modeling in order to overcome some limitations identified in previous models. Firstly, the framework of balance of momentum equation has been improved by implementing additional terms such as excess pore-water pressure. Secondly, propagation-consolidation model has been used to evaluate the pore-water pressure along the depth of the propagating mass. As a result, the modifications and improvements of the two-phase modeling allowed to consider several dominant physical aspects of debris flows such as the generation of excess pore-water pressure which increases the mobility of the flow and reduces the basal shear stress.

These mathematical equations have been discretized in space by the Smooth Particle Hydrodynamic (SPH) technique where a double set of particles, one to represent the movement of the solid skeleton and another to represent the movement of the pore-water, has been considered. Then, a 1D finite difference grid has been attached to each solid particle to account the pore-water pressure evolution during propagation. This is the main novelty of the present Doctoral Thesis, as it is the first time that a depth-integrated model for debris flows is numerically solved with two sets of particles with the SPH technique including pore-water pressure dissipation (SPH-FD Model).

Finally, the performance and limitations of the model has been assessed using a series of benchmark exercises, including dam break problems, flume tests and real cases of debris flows. The numerical results show the capability of the proposed two-phase model to properly reproduce the dynamics behavior of debris flows by taking into account bed entrainment along the landslide path and variation of pore-water pressures. It provides a very good combination of computation cost and accuracy, and can be used to predict the potential hazard due to debris flows.

6.3 Future investigation

In order to improve the proposed computational method, the following future research lines can be proposed:

- The proposed two-phase model can only be used to investigate the propagation of landslides. However, Initiation is a fundamental aspect of any failure mechanism, which determines the initial conditions for the propagation phase. So far, there is no single model that able to reproduce all phenomena existing in different stages and facilitate the modeling by transferring the results from the initiation stage to the propagation stage.
- An important question concerns the accuracy of the depth integrated approach. Although it has been able to efficiently reproduce many of the rapid landslides, 3D models are still more accurate in cases which the vertical column of the flow is not homogeneous.
- In order to reduce the computational cost, many different techniques have been used such as depth integrated model and SPH numerical method which reduce the time of calculation by separating the mesh of moving particles from the mesh of topography. However, the computational effort can still be so high in simulation of large domains with millions of particles. One solution to this problem is to implement parallel algorithms on Graphic processing units (GPUs) to improve the performance and reduce the execution times.
- In order to increase user engagement, it is suggested to (i) provide a user-friendly interface, (ii) incorporate the simulation model in a geographic information system (GIS) in order to manage geo-referenced data and facilitate the use of the results, and (iii) maintain simplicity of the program by not increasing the number of parameters.

Chapter 7

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