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Dissipative flows, global attractors and shape theory [☆]

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ABSTRACT

In this paper we study continuous parametrized families of dissipative flows, which are those flows having a global attractor. The main motivation for this study comes from the observation that, in general, global attractors are not robust, in the sense that small perturbations of the flow can destroy their globality. We give a necessary and sufficient condition for a global attractor to be continued to a global attractor. We also study, using shape theoretical methods and the Conley index, the bifurcation global to non-global.

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Introduction

This paper is devoted to the study of continuous parametrized families of dissipative flows $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$, where $\lambda \in [0, 1]$ and M is a non-compact, locally compact metric space.

Dissipative flows have been introduced by Levinson [18] and they are a very important class of flows since, as it has been proved by Pliss [24], they agree with those flows having a global attractor. We provide a proof of this fact for the sake of completeness (see Proposition 2). Dissipative flows play a crucial role in

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mathematical ecology since they model the limitations of the environment (see for example [3,11,16,34]). An interesting reference regarding dissipative flows is [14].

The main motivation for this paper comes from the observation, pointed out in Example 5, that, in general, global attractors are not robust, in the sense that small perturbations of the flow can destroy their globality.

In section 1 we recall the definition of dissipative flow and we prove, for the sake of completeness, that a flow is dissipative if and only if it possesses a global attractor (Proposition 2). We also see in Example 5 that a small perturbation of a global attractor can produce a non-global attractor. Moreover, we see that in fact, it can have a bounded region of attraction. This fact motivates the definition of coercive family of dissipative flows. In addition, we introduce the concept of uniformly dissipative family of flows and we prove Theorem 9 which establishes that for a family of dissipative flows a global attractor continues to global attractors if and only if the family is uniformly dissipative. We illustrate this result by showing that in the Lorenz equations the global attractor continues to global attractors. We finish this section by proving Theorem 11, in which a topological study of coercive families of dissipative flows in $\mathbb{R}^n$ is made. In particular we see that if a parametrized family of dissipative flows is coercive then, given a continuation K_λ of the global attractor K_0 of φ_0 , an isolated invariant compactum $C_\lambda \subset \mathbb{R}^n \setminus K_\lambda$ with the shape of S^{n-1} which attracts uniformly all the points of the unbounded component of its complement and repels all the points of the bounded one which are not in K_λ is created for all $\lambda > 0$ small. Besides, we see that the diameter of C_λ goes to infinity when λ goes to zero. Moreover, we see that the existence of such a C_λ for all $\lambda > 0$ small is sufficient for the family to be coercive.

In section 2 we study in all its generality the mechanism which produces the bifurcation global to non-global in families of dissipative flows. With this aim we introduce the definition of polar family of dissipative flows and we see (Proposition 14) that a family of dissipative flows is polar if and only if every continuation K_λ of the global attractor K_0 of φ_0 satisfies that K_λ is not a global attractor for all $\lambda > 0$ small. In addition we prove Theorem 15 which establishes that if a parametrized family of dissipative flows is polar then, given a continuation K_λ of the global attractor K_0 of φ_0 , for all $\lambda > 0$ small the maximal invariant compactum $C_\lambda \subset \mathbb{R}^n \setminus K_\lambda$ is non-empty, isolated and has trivial cohomological Conley index.

Through the paper we often consider the Alexandrov compactification $M \cup \{\infty\}$ and we extend the family of flows $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ to a family of flows, which we also denote by φ_λ , on $M \cup \{\infty\}$ by leaving ∞ fixed for all $\lambda \in [0, 1]$.

A form of homotopy theory, namely *shape theory*, which is the most suitable for the study of global topological properties in dynamics, will be occasionally used. Although a deep knowledge of shape theory is not necessary to understand the paper we recommend to the reader Borsuk's monography [2] and the books by Mardešić and Segal [20] and Dydak and Segal [8] for an exhaustive treatment of the subject and [17,25,26,30–33,29,13,12] for a concise introduction and some applications to dynamical systems.

We make use of some notions of algebraic topology. Good references for this material are the books of Hatcher and Spanier [15,35]. We will use the notation H_* and H^* for the singular homology and cohomology with integer coefficients respectively.

The main reference for the elementary concepts of dynamical systems will be [1] but we also recommend [27,22,23]. By the *omega-limit* of a point x we understand the set $\omega(x) = \bigcap_{t>0} \overline{x[t, \infty)}$ while the *negative omega-limit* is the set $\omega^*(x) = \bigcap_{t<0} \overline{x(-\infty, t]}$. An invariant compactum K is *stable* if every neighborhood U of K contains a neighborhood V of K such that $V[0, \infty) \subset U$. Similarly, K is *negatively stable* if every neighborhood U of K contains a neighborhood V of K such that $V(-\infty, 0] \subset U$. The compact invariant set K is said to be *attracting* provided that there exists a neighborhood U of K such that $\emptyset \neq \omega(x) \subset K$ for every $x \in U$ and *repelling* if there exists a neighborhood U of K such that $\emptyset \neq \omega^*(x) \subset K$ for every $x \in U$. An *attractor* (or *asymptotically stable* compactum) is an attracting stable set and a *repeller* is a repelling negatively stable set. We stress the fact that stability (positive or negative) is required in the definition of attractor or repeller. If K is an attractor, its region (or basin) of attraction $\mathcal{A}(K)$ is the set of all points

$x \in M$ such that $\emptyset \neq \omega(x) \subset K$. It is well known that $\mathcal{A}(K)$ is an open invariant set. The region of repulsion of a repeller K , $\mathcal{R}(K)$ is defined in a dual way. If in particular $\mathcal{A}(K)$ is the whole phase space we say that K is a *global attractor*.

Notice that, since we will deal with parametrized families of flows we will use the notation $\omega_\lambda(x)$, $\omega_\lambda^*(x)$, $\mathcal{A}_\lambda(K)$ and $\mathcal{R}_\lambda(K)$ to denote the omega-limit, negative omega-limit, region of attraction and region of repulsion with respect to the flow φ_λ .

There is a nice topological relation between an attractor K and its region of attraction $\mathcal{A}(K)$ expressed in terms of shape theory. It establishes that the inclusion $i : K \hookrightarrow \mathcal{A}(K)$ is a shape equivalence [17, Theorem 3.6]. As a consequence, if K is a global attractor of a flow in an Euclidean space it must have the shape of a point. In particular, it must be connected.

An important class of invariant compacta is the so-called *isolated invariant sets* (see [4,5,9] for details). These are compact invariant sets K which possess an *isolating neighborhood*, i.e. a compact neighborhood N such that K is the maximal invariant set in N .

A special kind of isolating neighborhoods will be useful in the sequel, the so-called *isolating blocks*, which have good topological properties. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and the exit sets, satisfying

1. $\partial N = N^i \cup N^o$;
2. for each $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M - N$ and for each $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M - N$;
3. for each $x \in \partial N - N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \mathring{N}$ and for every $x \in \partial N - N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \mathring{N}$.

These blocks form a neighborhood basis of K in M .

Let K be an isolated invariant set. Its *Conley index* $h(K)$ is defined as the pointed homotopy type of the topological space $(N/N^o, [N^o])$, where N is an isolating block of K . A weak version of the Conley index which will be useful for us is the *cohomological index* defined as $CH^*(K) = H^*(h(K))$. The *homological index* $CH_*(K)$ is defined in an analogous fashion using homology. Notice that in fact, $CH^*(K) \cong H^*(N, N^o)$ and $CH_*(K) \cong H_*(N, N^o)$. Our main references for the Conley index theory are [4,6,28].

In this paper the concept of continuation of isolated invariant sets plays a crucial role. Let M be a locally compact metric space, and let $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ be a parametrized family of flows (parametrized by $\lambda \in [0, 1]$, the unit interval). The family $(K_\lambda)_{\lambda \in J}$, where $J \subset [0, 1]$ is a closed (non-degenerate) subinterval and, for each $\lambda \in J$, K_λ is an isolated invariant set for φ_λ is said to be a *continuation* if for each $\lambda_0 \in J$ and each N_{λ_0} isolating neighborhood for K_{λ_0} , there exists $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$. We say that the family $(K_\lambda)_{\lambda \in J}$ is a continuation of K_{λ_0} for each $\lambda_0 \in J$.

Notice that [28, Lemma 6.1] ensures that if K_{λ_0} is an isolated invariant set for φ_{λ_0} , there always exists a continuation $(K_\lambda)_{\lambda \in J_{\lambda_0}}$ of K_{λ_0} for some closed (non-degenerate) subinterval $J_{\lambda_0} \subset [0, 1]$.

There is a simpler definition of continuation based on [28, Lemma 6.2]. There, it is proved that if $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ is a parametrized family of flows and if N_1 and N_2 are isolating neighborhoods of the same isolated invariant set for φ_{λ_0} , then there exists $\delta > 0$ such that N_1 and N_2 are isolating neighborhoods for φ_λ , for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$, with the property that, for every λ , the isolated invariant subsets in N_1 and N_2 which have N_1 and N_2 as isolating neighborhoods agree.

Therefore, the family $(K_\lambda)_{\lambda \in J}$, with K_λ an isolated invariant set for φ_λ , is a continuation if for every $\lambda_0 \in J$ there are an isolating neighborhood N_{λ_0} for K_{λ_0} and a $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ , for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$.

We will make use of the fact that if $(K_\lambda)_{\lambda \in J}$ is a continuation then, for each $\lambda_1, \lambda_2 \in J$, the Conley indices $h(K_{\lambda_1})$ and $h(K_{\lambda_2})$ agree (see [28, Corollary 6.8]). A consequence of this fact is that if K_{λ_0} is a non-empty attractor and $(K_\lambda)_{\lambda \in J}$ is a continuation of it, then K_λ is non-empty for each $\lambda \in J$.

We are interested in continuations $(K_\lambda)_{\lambda \in J}$, with $0 \in J$, where K_0 is a global attractor. Since K_0 is an attractor, using [32, Theorem 4] it follows that there exists $0 < \lambda_0 \in J$ such that, for $\lambda < \lambda_0$, K_λ is an attractor which has the shape of K_0 . As a consequence, if the phase space is a Euclidean space, then for small values of λ , K_λ has the shape of a point and, in particular, it is connected.

Notice that, since this should not lead to any confusion, sometimes we will only say that K_λ is a continuation of K_{λ_0} without specifying the subinterval $J \subset [0, 1]$ to which the parameters belong.

1. Continuation of global attractors

In this section we study parametrized families of dissipative flows and continuations of global attractors. In particular, we will see that a small perturbation of a global attractor can be a non-global attractor since, in fact, it can have a bounded region of attraction. We will also give a necessary and sufficient condition for the family $(K_\lambda)_{\lambda \in [0, 1]}$ where, for each $\lambda \in [0, 1]$ K_λ is the global attractor of φ_λ , to be a continuation of the global attractor K_0 of φ_0 . Besides all this we give a descriptive characterization of coercive families of dissipative flows.

Definition 1 (*Levinson 1944*). We say that a flow $\varphi : M \times \mathbb{R} \rightarrow M$ is dissipative provided that for each $x \in M$, $\omega(x) \neq \emptyset$ and the closure of the set

$$\Omega(\varphi) = \bigcup_{x \in M} \omega(x)$$

is compact.

Proposition 2 (*Pliss 1966*). A flow $\varphi : M \times \mathbb{R} \rightarrow M$ is dissipative if and only if there exists a global attractor.

Proof. If φ has a global attractor K , then it is clear that $\Omega(\varphi) \subset K$ and hence its closure is compact ensuring the dissipativeness of φ .

Conversely, suppose that φ is dissipative. We extend the flow to the Alexandrov compactification $M \cup \{\infty\}$ leaving ∞ fixed. Let N be a compact neighborhood of $\overline{\Omega(\varphi)}$. Then, $U = (M \cup \{\infty\}) \setminus N$ is a neighborhood of $\{\infty\}$ and, given any point $x \in M \setminus N = U \setminus \{\infty\}$, since $\emptyset \neq \omega(x) \subset N$, there exists a time $t_x > 0$ such that $xt_x \notin U$. Hence, by [28, Lemma 3.1] $\{\infty\}$ is a repeller.

Since $M \cup \{\infty\}$ is compact and $\{\infty\}$ is a repeller, there exists a non-empty attractor $K \subset M$ dual to $\{\infty\}$. It is clear that this attractor K must be a global attractor. $\square$

Remark 3. The dissipativeness of φ is equivalent to $\{\infty\}$ being a repeller.

An easy consequence of these considerations is the following proposition.

Proposition 4. Let φ be a dissipative flow in $\mathbb{R}^n$ and K an attractor of φ . Then K is global if and only if $\mathcal{A}(K) \setminus K$ does not contain bounded orbits.

Hence, if K is an attractor of a dissipative flow, $\mathcal{A}(K)$ being bounded is in sharpest contrast to K being global.

Fig. 1. Phase portrait of the family of equations (1). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Example 5. Consider the family of ordinary differential equations defined on the plane in polar coordinates

$$\begin{cases} \dot{r} = -r^3 \left(\frac{1}{r} - \lambda \right)^2, \\ \dot{\theta} = 1 \end{cases}, \quad \lambda \in [0, 1] \quad (1)$$

The picture on the left in Fig. 1 describes the phase portrait of equation (1) when the parameter $\lambda = 0$. We see that in this case the origin is a globally attracting fixed point and the orbit of any other point spirals towards it. The picture on the right describes the phase portrait of equation (1) when $\lambda > 0$. In this case we see that the origin is still an asymptotically stable fixed point but it is not a global attractor anymore since, for each $\lambda > 0$, the circle centered at the origin and radius $1/\lambda$ is a periodic trajectory which attracts uniformly all the points of the unbounded component of its complement and repels all the points of the bounded one except the origin. Notice that for every $\lambda > 0$ the disk B_λ centered at the origin and radius $1/\lambda$ is a global attractor whose interior agrees with the basin of attraction of $\{0\}$. As a consequence of these remarks, for each $\lambda \in [0, 1]$ the flow φ_λ associated to the corresponding value of λ in (1) is dissipative and, if K_λ is a continuation of the global attractor $\{0\}$ of φ_0 , there exists $\lambda_0 > 0$ such that for each $0 < \lambda < \lambda_0$, $K_\lambda = \{0\}$. Therefore, the global attractor $\{0\}$ cannot be continued to a family of global attractors.

Example 5 is an instance of parametrized family φ_λ of dissipative flows in $\mathbb{R}^2$ with a global attractor K_0 such that every continuation K_λ of K_0 satisfies that there exists $\lambda_0 > 0$ such that $\mathcal{A}_\lambda(K_\lambda)$ is bounded for every $0 < \lambda < \lambda_0$. This motivates the following definition.

Definition 6. A parametrized family of dissipative flows $\varphi : M \times \mathbb{R} \rightarrow M$ is said to be *coercive* if for any continuation K_λ of the global attractor K_0 of φ_0 , there exists $\lambda_0 > 0$ such that $\mathcal{A}_\lambda(K_\lambda)$ is bounded for every $0 < \lambda < \lambda_0$.

Remark 7. Notice that it follows from [28, Lemma 6.2] that if K_0 has a continuation K_λ satisfying that there exists $\lambda_0 > 0$ such that $\mathcal{A}_\lambda(K_\lambda)$ is bounded for every $0 < \lambda < \lambda_0$, then the family φ_λ is coercive.

Before studying coercive families of dissipative flows we will characterize when a global attractor can be continued to a family of global attractors. For this purpose we introduce the following definition.

Definition 8. A parametrized family of flows $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ is said to be uniformly dissipative provided that $\omega_\lambda(x) \neq \emptyset$ for each $x \in M$ and $\lambda \in [0, 1]$ and the closure of

$$\Omega = \bigcup_{\lambda \in [0,1]} \Omega(\varphi_\lambda)$$

is compact.

Theorem 9. Let $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ be a parametrized family of dissipative flows with $\lambda \in [0, 1]$. Suppose that K_λ denotes the global attractor of φ_λ . Then, the family $(K_\lambda)_{\lambda \in [0,1]}$ is a continuation of K_0 if and only if the family $(\varphi_\lambda)_{\lambda \in [0,1]}$ is uniformly dissipative.

Proof. Suppose that the family of global attractors $(K_\lambda)_{\lambda \in [0,1]}$ is a continuation of the global attractor K_0 of φ_0 . Then, for each $\lambda_0 \in [0, 1]$ there exists an isolating neighborhood N_{λ_0} of K_{λ_0} and $\delta > 0$ such that N_{λ_0} is an isolating neighborhood of K_λ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$.

Using this fact and the compactness of $[0, 1]$ it follows that there exist $\lambda_1, \dots, \lambda_k \in [0, 1]$ such that, if we call $K = \bigcup_{\lambda \in [0,1]} K_\lambda$,

$$\overline{K} \subset \bigcup_{i=1}^k N_{\lambda_i}.$$

As a consequence, $\overline{K}$ is compact. Moreover, $\omega_\lambda(x) \neq \emptyset$ for each $\lambda \in [0, 1]$ and $x \in M$ since every flow in the family is dissipative and the closure of Ω is compact being contained in $\overline{K}$. Therefore, the family $(\varphi_\lambda)_{\lambda \in [0,1]}$ is uniformly dissipative.

Conversely, suppose that the family $(\varphi_\lambda)_{\lambda \in [0,1]}$ is uniformly dissipative. We see that the family of global attractors $(K_\lambda)_{\lambda \in [0,1]}$ is a continuation of the global attractor K_0 . For this purpose we consider the family of flows extended to the Alexandrov compactification $M \cup \{\infty\}$ leaving ∞ fixed. Let $\lambda_0 \in [0, 1]$, N_{λ_0} an isolating neighborhood of K_{λ_0} and W a compact neighborhood of $N_{\lambda_0} \cup \Omega$. Then, $U = \overline{(M \cup \{\infty\})} \setminus \overline{W}$ is an isolating neighborhood of $\{\infty\}$ for the flow φ_{λ_0} . Since $(K_{\lambda_0}, \{\infty\})$ is an attractor-repeller decomposition, it follows from [28, Lemma 6.4] that there exists $\delta > 0$ such that for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$ the pair (A_λ, R_λ) , where A_λ is the isolated invariant set having N_{λ_0} as isolating neighborhood and R_λ is the maximal invariant set in W for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$, is an attractor-repeller decomposition. Besides, from the choice of W it easily follows that $R_\lambda = \{\infty\}$ and, hence, A_λ must agree with the global attractor K_λ for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$. Thus the result follows. $\square$

Example 10. An important example of a global attractor is provided by the Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

where σ, r and b are three real positive parameters. E.N. Lorenz proved in [19] that for every value of σ, r and b there exists a global attractor of zero volume for the flow associated to these equations. This attractor should not be confused with the famous Lorenz attractor, which is a proper subset of the global attractor.

If we fix σ and b we obtain a family of flows $\varphi_r : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ corresponding to the Lorenz equations for the different values of r . Each of these flows has a global attractor K_r . We shall show that the family φ_r is uniformly dissipative and, hence, for a given r_0 the family of global attractors K_r is a continuation of K_{r_0} .

Sparrow studied in [36] the function

$$V = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$$

which is a Lyapunov function for the flow φ_r . By using this function he was able to prove that K_r lies in a ball B_r centered at 0 and with radius $O(r)$, such that $O(r)$ depends continuously on r . Hence, if we consider an arbitrary r_0 and an interval $[c, d]$ containing r_0 , we have that the set $C = \overline{\cup_{c \leq r \leq d} B_r}$ is compact and that $\emptyset \neq \omega_r(x) \subset C$ for every $x \in \mathbb{R}^3$ and every r with $c \leq r \leq d$. Hence the family of Lorenz flows φ_r is uniformly dissipative for any interval $[c, d]$ of parameters and the corresponding family $(K_r)_{r \in [a, b]}$ of global attractors is a continuation of K_{r_0} .

From now on we will focus on families of flows in $\mathbb{R}^n$. An interesting fact, of topological and dynamical significance, takes place for coercive families. This result provides a graphic characterization of coercive families.

Theorem 11. *Let φ_λ , with $\lambda \in [0, 1]$, be a coercive family of dissipative flows in $\mathbb{R}^n$. We denote by K_0 the global attractor of φ_0 and by K_λ a continuation of K_0 . Then there exists $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ there is an isolated invariant compactum C_λ in $\mathbb{R}^n \setminus K_\lambda$ such that*

- i) C_λ separates $\mathbb{R}^n$ into two components and K_λ lies in the bounded component.
- ii) C_λ has the shape of S^{n-1}
- iii) C_λ attracts uniformly all the points of the unbounded component and repels all the points of the bounded one which are not in K_λ .
- iv) $\text{diam}(C_\lambda) \rightarrow \infty$ when $\lambda \rightarrow 0$, where $\text{diam}(C_\lambda)$ denotes the diameter of C_λ .

Moreover, the existence of such a C_λ for $0 < \lambda < \lambda_0$ is sufficient for the family to be coercive.

Proof. Suppose that φ_λ is a coercive family of flows in $\mathbb{R}^n$. Then, if we extend this family to $\mathbb{R}^n \cup \{\infty\}$ leaving ∞ fixed, there are no connecting orbits between the attractor K_λ and $\{\infty\}$ when $0 < \lambda < \lambda_0$ and, as a consequence, $\mathcal{A}_\lambda(K_\lambda)$ and $\mathcal{R}_\lambda(\{\infty\})$ are disjoint open sets. Besides, $\mathcal{R}_\lambda(\{\infty\})$ is connected by [17, Theorem 3.6] and, since there exists $0 < \lambda_1 \leq 1$ (we may assume that $\lambda_1 = \lambda_0$) such that K_λ is connected, [17, Theorem 3.6] also ensures that $\mathcal{A}_\lambda(K_\lambda)$ is connected for $0 < \lambda_1 \leq 1$. Since $\mathbb{R}^n \cup \{\infty\}$ is connected, then

$$C_\lambda = \mathbb{R}^n \setminus (\mathcal{A}_\lambda(K_\lambda) \cup \mathcal{R}_\lambda(\{\infty\}))$$

is non-empty, compact and invariant. It is clear that it is also isolated since every point near C_λ must be either in $\mathcal{A}_\lambda(K_\lambda)$ or in $\mathcal{R}_\lambda(\{\infty\})$. Notice that from the previous discussion it also follows that C_λ decomposes $\mathbb{R}^n$ into two connected components and that K_λ is contained in the bounded one. Besides, it is clear that C_λ must attract every point in $\mathcal{R}_\lambda(\{\infty\})$ (the unbounded component) and repel every point in $\mathcal{A}_\lambda(K_\lambda)$ (the bounded one) which is not in K_λ . Now, we will see that $\mathbb{R}^n \setminus K_\lambda$ has the homotopy type of S^{n-1} and that the inclusion $i_\lambda : C_\lambda \hookrightarrow \mathbb{R}^n \setminus K_\lambda$ is a shape equivalence. To see that choose a closed ball B isolating K_0 . Then, for small values of the parameter B is an isolating neighborhood of K_λ as well, and it is contained in $\mathcal{A}_\lambda(K_\lambda)$ (in fact, we may assume that this happens for $0 < \lambda < \lambda_0$). As a consequence K_λ has a basis of neighborhoods consisting of balls (obtained as images of B by the flow φ_λ), i.e., it is cellular. Then, by [7, Exercise 2, pg. 41] $\mathbb{R}^n \setminus K_\lambda$ is homeomorphic to $\mathbb{R}^n \setminus \{p\}$, where $p \in \mathbb{R}^n$, and hence it has the homotopy type of S^{n-1} .

Let us see that $i_\lambda : C_\lambda \hookrightarrow \mathbb{R}^n \setminus K_\lambda$ is a shape equivalence. To do that notice that, since K_λ is an attractor for $0 < \lambda < \lambda_0$, it possesses a Lyapunov function $\Phi_\lambda : \overline{\mathcal{A}_\lambda(K_\lambda)} \rightarrow [0, \infty]$ which is strictly decreasing on orbits of φ_λ contained in $\mathcal{A}_\lambda(K_\lambda) \setminus K_\lambda$ and such that $\Phi_\lambda|_{K_\lambda} = 0$ and $\Phi_\lambda|_{\partial \mathcal{A}_\lambda(K_\lambda)} = \infty$. Similarly, since $\{\infty\}$ is a repeller for every λ , there exist a Lyapunov function $\Psi_\lambda : \overline{\mathcal{R}_\lambda(\{\infty\})} \rightarrow [0, \infty]$ such that Ψ_λ is strictly increasing on orbits of φ_λ contained in $\mathcal{R}_\lambda(\{\infty\}) \setminus \{\infty\}$, $\Psi_\lambda(\infty) = 0$ and $\Psi_\lambda|_{\partial \mathcal{R}_\lambda(\{\infty\})} = \infty$. Using this fact we define, for each $0 < \lambda < \lambda_0$ a function $F_\lambda : \mathbb{R}^n \cup \{\infty\} \rightarrow [0, \infty]$ by

$$F_\lambda(x) = \begin{cases} \Phi_\lambda(x) & \text{if } x \in \mathcal{A}(K_\lambda) \\ \Psi_\lambda(x) & \text{if } x \in \mathcal{R}(\{\infty\}) \\ 0 & \text{if } x \in K_\lambda \cup \{\infty\} \\ \infty & \text{otherwise.} \end{cases}$$

It is clear that F_λ is continuous for each $0 < \lambda < \lambda_0$ and

$$C_\lambda = F_\lambda^{-1}(\infty).$$

For every $a > 0$ the set $N_a^\lambda = F_\lambda^{-1}[a, \infty]$ is a compact neighborhood of C_λ . On the other hand, for each $x \in \mathbb{R}^n \setminus (K_\lambda \cup \overset{\circ}{N}_a^\lambda)$ there exists a unique time $t_a^\lambda(x) \in \mathbb{R}$ such that $F_\lambda(\varphi_\lambda(x, t_a^\lambda(x))) = a$ and for each $0 < \lambda < \lambda_0$ the function $t_a^\lambda : \mathbb{R}^n \setminus (K_\lambda \cup \overset{\circ}{N}_a^\lambda) \rightarrow \mathbb{R}$ is continuous. Using this fact we can define the retraction $r_a^\lambda : \mathbb{R}^n \setminus K_\lambda \rightarrow N_a^\lambda$ as follows:

$$r_a^\lambda(x) = \begin{cases} \varphi_\lambda(x, t_a^\lambda(x)) & \text{if } x \in \mathbb{R}^n \setminus (K_\lambda \cup N_a^\lambda) \\ x & \text{if } x \in N_a^\lambda. \end{cases}$$

It is straightforward to see that r_a^λ is a deformation retraction from $\mathbb{R}^n \setminus K_\lambda$ onto N_a^λ . Besides, if $a' > a$ it is also easy to see that $r_{a'}^\lambda|_{N_a^\lambda} : N_a^\lambda \rightarrow N_{a'}^\lambda$ is a deformation retraction as well. Since the family $\{N_a^\lambda\}_{a>0}$ is a basis of neighborhoods of C_λ it follows that the inclusion $i_\lambda : C_\lambda \hookrightarrow \mathbb{R}^n \setminus K_\lambda$ is a shape equivalence.

Moreover, for each $0 < \lambda < \lambda_0$, the invariant compactum C_λ contains the compact set $\partial\mathcal{A}_\lambda(K_\lambda)$ and, as a consequence, $\text{diam}(C_\lambda) \geq \text{diam}(\partial\mathcal{A}_\lambda(K_\lambda)) = \text{diam}(\overline{\mathcal{A}_\lambda(K_\lambda)})$. Besides, since K_0 is a global attractor it follows that $\text{diam}(\overline{\mathcal{A}_\lambda(K_\lambda)})$ must go to infinity when λ goes to zero. Therefore, $\text{diam}(C_\lambda) \rightarrow \infty$ when $\lambda \rightarrow 0$.

Conversely, assume now that there exists such a C_λ . Since C_λ is an isolated invariant compactum which decomposes $\mathbb{R}^n$ into two components with K_λ lying in the bounded one it follows that $\mathcal{A}_\lambda(K_\lambda)$ is also contained in the bounded component. $\square$

2. The general case

In view of the previous results, it is interesting to study in all its generality the mechanism which produces the bifurcation global to non-global in families of dissipative flows. With this aim we introduce the following definition.

Definition 12. Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, with $\lambda \in [0, 1]$, be a parametrized family of dissipative flows. The family is said to be *polar* if it has arbitrarily large bounded trajectories. More precisely: for every $L > 0$ (arbitrarily large) there is a $\lambda_0 > 0$ such that for every $0 < \lambda < \lambda_0$ there is a bounded trajectory γ_λ of φ_λ and a $t_\lambda < 0$ such that $\|\gamma_\lambda(t)\| > L$ for every t with $-\infty < t < t_\lambda$.

Remark 13. Obviously, if K_λ is a continuation of the global attractor K_0 of φ_0 , for L sufficiently large, γ_λ lies in $\mathbb{R}^n \setminus K_\lambda$.

The following proposition makes it clear that polarity is a key notion regarding the transition global to non-global.

Proposition 14. Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, with $\lambda \in [0, 1]$, be a parametrized family of dissipative flows. Then the family is polar if and only if for every continuation K_λ of the global attractor K_0 of φ_0 there exists λ_0 such that K_λ is not a global attractor for every $0 < \lambda < \lambda_0$.

Proof. If φ_λ is polar then, by Remark 13, for every λ with $0 < \lambda < \lambda_0$ there is a bounded trajectory γ_λ of φ_λ contained in $\mathbb{R}^n \setminus K_\lambda$ and, thus, K_λ is not a global attractor. Conversely, if K_λ is not a global attractor for every λ with $0 < \lambda < \lambda_0$, then there exists at least a bounded orbit γ_λ in $\mathcal{A}_\lambda(K_\lambda) \setminus K_\lambda$. The negative omega-limit of γ_λ must be contained in $\partial\mathcal{A}_\lambda(K_\lambda)$, which is a closed set all whose points are arbitrarily close to ∞ when $\lambda \rightarrow 0$. This implies that the orbits γ_λ satisfy the requirements of the definition of polarity when λ_0 is sufficiently small. $\square$

The following result describes the overall picture of polar families of dissipative flows.

Theorem 15. *If $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, with $\lambda \in [0, 1]$, is a polar family of dissipative flows then there exists $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ the maximal invariant compactum lying in $\mathbb{R}^n \setminus K_\lambda$ for the flow φ_λ , which we denote by C_λ , is non-empty, isolated and its cohomological Conley index is trivial in every dimension. Moreover, the family φ_λ is coercive if and only if C_λ has the shape of S^{n-1} .*

Proof. Let K_λ be a continuation of the global attractor of φ_0 . Since φ_λ is dissipative there exists a global attractor A_λ of φ_λ and, since φ_λ is polar, there exists $\lambda_0 > 0$ such that K_λ is a connected attractor and $K_\lambda \subsetneq A_\lambda$ for $0 < \lambda < \lambda_0$.

Since K_λ is an attractor of φ_λ , it must be also an attractor of $\varphi_\lambda|_{A_\lambda}$. Let C_λ be the complementary repeller of K_λ for $\varphi_\lambda|_{A_\lambda}$. It is clear that C_λ is a non-empty and invariant compactum. Moreover, if x is a point in $\mathbb{R}^n \setminus K_\lambda$ not belonging to C_λ then the trajectory of x either approaches ∞ for negative times (if $x \in \mathbb{R}^n \setminus A_\lambda$) or approaches K_λ for positive times (if $x \in A_\lambda$). Both situations prevent x from lying in a compact invariant set contained in $\mathbb{R}^n \setminus K_\lambda$ and, thus, C_λ is the maximal invariant compactum lying in $\mathbb{R}^n \setminus K_\lambda$. The same argument proves that C_λ is isolated.

Now we will see that the cohomological Conley index $CH^*(C_\lambda) = 0$. Consider the family of flows extended to $\mathbb{R}^n \cup \{\infty\}$ leaving ∞ fixed. Let N be a positively invariant isolating block of K_λ such that $N \cap C_\lambda = \emptyset$ and let N_∞ be a negatively invariant isolating block of $\{\infty\}$ with $N_\infty \cap C_\lambda = \emptyset$. Then, the compactum $\hat{N} = \mathbb{R}^n \setminus (\hat{N} \cup \hat{N}_\infty)$ is an isolating block of C_λ whose exit set $\hat{N}^o$ agrees with ∂N and its entrance set $\hat{N}^i$ agrees with ∂N_∞ . Moreover, it is easy to see that $\hat{N}/\partial N$ and $\hat{N}/\partial N_\infty$ are connected. This implies that $CH^0(C_\lambda) = 0$ and that $CH_0^-(C_\lambda) = 0$, where $CH_*^-(C_\lambda)$ is the homological Conley index for the reverse flow. Then, by the time duality of the Conley index [21] we also have that $CH^n(C_\lambda) = 0$.

We study now the Morse-Conley sequence of cohomology of the attractor-repeller decomposition (K_λ, C_λ) of A_λ [10]

$$\cdots \xrightarrow{\delta} CH^k(C_\lambda) \rightarrow CH^k(A_\lambda) \rightarrow CH^k(K_\lambda) \xrightarrow{\delta} \cdots$$

Since A_λ is a global attractor it follows from [32, Lemma 2] that $CH^k(A_\lambda) = 0$ for every $k > 0$ and $CH^0(A_\lambda) = \mathbb{Z}$. On the other hand, since K_λ is a continuation of A_0 and the Conley index is preserved by continuation we also have that $CH^k(K_\lambda) = 0$ for every $k > 0$ and that $CH^0(K_\lambda) = \mathbb{Z}$. We deduce from this that $CH^k(C_\lambda) = 0$ as well for $k > 1$. Since we already know that $CH^0(C_\lambda) = CH^n(C_\lambda) = 0$ it remains only to prove that $CH^1(C_\lambda)$ is also trivial.

Consider the initial segment of the Morse-Conley sequence

$$0 \rightarrow CH^0(C_\lambda) \rightarrow CH^0(A_\lambda) \rightarrow CH^0(K_\lambda) \rightarrow CH^1(C_\lambda) \rightarrow CH^1(A_\lambda) \rightarrow \cdots$$

Then, the connectedness of A_λ and K_λ ensure that the homomorphism

$$\mathbb{Z} \cong CH^0(A_\lambda) \rightarrow CH^0(K_\lambda) \cong \mathbb{Z}$$

is an isomorphism. Therefore, since $CH^1(A_\lambda) = 0$, it follows that $CH^1(C_\lambda) = 0$.

The last part of the theorem follows easily from Theorem 11. $\square$

Remark 16. If we only assume that the family φ_λ is not uniformly dissipative then we have an analogous proposition but replacing “there exists λ_0 such that for every $0 < \lambda < \lambda_0$ ” by “for every λ_0 there exists $0 < \lambda < \lambda_0$ ”.

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