A NEW CONVERGENT ALGORITHM TO APPROXIMATE POTENTIALS FROM FIXED ANGLE SCATTERING DATA*

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Abstract. We introduce a new iterative method to recover a real compact supported potential of the Schrödinger operator from their fixed angle scattering data. The method combines a fixed point argument with a suitable approximation of the resolvent of the Schrödinger operator by partial sums associated to its Born series. The main interest is that, unlike other iterative methods in the literature, each iteration is explicit (and therefore faster computationally) and a rigorous analytical result on the convergence of the iterations is proved. This result requires potentials with small norm in certain Sobolev spaces. As an application we show some numerical experiments that illustrate this convergence.

Key words. Inverse problem, Helmholtz equation, scattering.

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1. Introduction and statement of results. We consider the scattering problem for the Schrödinger operator $-\Delta + q$ in $\mathbb{R}^d$, $d \geq 2$, where q is a real valued potential with compact support in $B(0, R)$. Here $B(0, R)$ denotes the ball centred at the origin with radius $R > 0$.

Associated to a given wave number $k > 0$ and an incident direction $\theta \in S^{d-1}$ we consider the incident wave $u_i(x) = e^{ik\theta \cdot x}$. Here S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. The outgoing scattering solution $u = u(x, \theta, k)$ with wave number k and direction of propagation θ , is the solution of equation

$$(1.1) \quad (\Delta + k^2) u(x) = q(x)u(x), \quad x \in \mathbb{R}^d,$$

which can be written as $u = u_i + u_s$ with $u_s(x, \theta, k)$ satisfying the outgoing Sommerfeld radiation condition given by

$$(1.2) \quad \partial_r u_s - iku_s = o\left(r^{-(d-1)/2}\right), \quad r = |x| \rightarrow \infty.$$

The function $u_s(x, \theta, k)$ named the scattered wave, is the perturbation of u due to the potential. It is well known that for appropriate q , u_s satisfies the following asymptotic expression as $|x| \rightarrow \infty$,

$$(1.3) \quad u_s(x, \theta, k) = c_d k^{(d-3)/2} e^{ik|x|} |x|^{-(d-1)/2} u_\infty(\theta', \theta, k) + o\left(|x|^{-(d-1)/2}\right),$$

where $\theta' = x/|x|$ is the reflecting angle and

$$(1.4) \quad u_\infty(\theta', \theta, k) = \int_{\mathbb{R}^d} e^{-ik\theta' \cdot y} q(y) u(y, \theta, k) dy.$$

The function u_∞ is called the *scattering amplitude* or *far-field pattern*, and it represents the measurements in the inverse scattering problem. For a successful description of direct and inverse scattering problems we refer the reader to Chapter 5 in [6]. We are interested in recovering the potential $q(x)$ from the knowledge of the scattering amplitudes $u_\infty(\theta', \theta_0, k)$, for fixed incident direction θ_0 and $(\theta', k) \in S^{d-1} \times (0, \infty)$, with $\theta' = x/|x|$. In fact, by fixed incident direction we mean data for both $\theta = \theta_0$ and $\theta = -\theta_0$, since we are considering wave

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numbers $k > 0$. This problem is known as the fixed angle inverse scattering problem and appears naturally in quantum physics. In general, the recovery of information about q from scattering amplitudes is known as inverse scattering problem and it has been studied by several authors. Here we mention some of them that we consider relevant.

Early works on the study of inverse scattering problems can be found in the middle of the last century with the results of Gelfand and Levitan [10], Jost and Kohn [17], and Moses [26]. Later on, Prosser generalized the method in [17] to recover the potential q , based on the Born series (nonlinear approximations), when the Friederichs norm of q is small enough (see [32, 33, 34, 35]). However, the procedures employed in these papers are purely formal, and the smallness condition is difficult to characterize.

In the nineties, Eskin and Ralston (see [7, 8, 9]), and also Stefanov (see [41]) studied the problem of uniqueness. At this time, Päiväranta, Somersalo and Serov introduced new techniques for dealing with the problem of singularities using all the scattering data (see [31, 29, 30]). This problem was also studied by Greenleaf and Uhlmann (see [11]).

In 2001, Ruiz used very precise estimates for the resolvent of the Laplacian to prove that for non-smooth potential the main singularities of the potential (in the scale of Sobolev spaces) are contained in the fixed angle Born approximation, which is a linear approximation of the potential that we define below (see [37]).

More recently, Kilgore, Moskow and Schotland studied the convergence and stability of the Born series and its inverse for several inverse scattering problems (see [24, 19, 20]). They also made numerical studies in [25]. From the numerical point of view, Barceló, Castro and Reyes studied the recovery of a potential from scattering data using a fixed point algorithm which is not justified from the theoretical point of view (see [2]).

All these results are for real potentials, the more general case of complex potentials was treated in [23] by Mochizuki and in [3] by Barceló, Faraco, Ruiz and Vargas.

The aim of this paper is to propose an iterative method for recovering a potential q from fixed angle scattering data. More precisely, we obtain a new convergent algorithm that combines two approaches, the nonlinear approximation described by Prosser in [32]-[35], and the fixed point algorithm of Barceló, Castro and Reyes in [2]. It is worth mentioning that, even if the numerical approximations based on these approaches work fine, there are no rigorous proofs in the literature supporting any of them. The interest of the new algorithm is twofold. On one hand, it is computationally faster than the known iterative methods since they involve, at each iteration, the solution of an implicit integral equation ([32]-[35], [2] or [28] for the fixed energy scattering data), while the algorithm proposed here is explicit at each iteration. This makes the implementation simpler and provides a faster algorithm, particularly for 3-d problems. The second advantage is that we are able to prove the analytical convergence of the iterations rigorously.

To state our main result, we first rewrite the inverse scattering problem in an equivalent integral formulation. Let us define R_k the outgoing resolvent operator of the Laplace operator given, in terms of the Fourier transform, by

$$(1.5) \quad \widehat{R_k(f)}(\xi) = \frac{\widehat{f}(\xi)}{-|\xi|^2 + k^2 + i0}.$$

Then, u_s is the solution of the so-called Lippmann-Schwinger integral equation

$$(1.6) \quad u_s(x, \theta_0, k) = R_k(qe^{ik\theta_0 \cdot (\cdot)})(x) + R_k(qu_s(\cdot, \theta_0, k))(x), \quad x \in \mathbb{R}^d.$$

Moreover, from (1.4) we have that

$$(1.7) \quad u_\infty(\theta, \theta_0, k) = \int_{\mathbb{R}^d} e^{-ik(\theta - \theta_0) \cdot y} q(y) dy + \int_{\mathbb{R}^d} e^{-ik\theta \cdot y} q(y) u_s(y, \theta_0, k) dy.$$

The problem is then to find an approximation of the potential q knowing that it satisfies (1.6) and (1.7) parametrized by the scattering data $u_\infty(\theta, \pm\theta_0, k)$ with $(\theta, k) \in S^{d-1} \times (0, \infty)$.

FIG. 1. Ewald spheres are centered at $-k\theta_0$ with radius k .

If we formally remove the last term in (1.7), the right hand side can be interpreted as a suitable Fourier transform that can be inverted to obtain the so-called Born approximation. More precisely, given θ_0 fixed, we have, up to a zero measure set,

$$(1.8) \quad \mathbb{R}^d = H_{\theta_0} \cup H_{-\theta_0} = \{\xi \in \mathbb{R}^d : \xi \cdot \theta_0 < 0\} \cup \{\xi \in \mathbb{R}^d : \xi \cdot \theta_0 > 0\}.$$

Then, for $\xi \in H_{\pm\theta_0}$, there exists unique $\theta(\xi) \in S^{d-1}$ and $k(\xi) > 0$ such that

$$\xi := k(\xi)(\theta(\xi) \mp \theta_0) \quad (\text{see Figura 1}).$$

Let us write

$$(1.9) \quad \theta_0(\xi) = \begin{cases} \theta_0, & \text{if } \xi \in H_{\theta_0} \\ -\theta_0, & \text{if } \xi \in H_{-\theta_0} \end{cases}$$

Then, the Born approximation for fixed angle scattering data θ_0 of a potential q is defined by

$$(1.10) \quad \widehat{q_{\theta_0}}(\xi) = u_\infty(\theta(\xi), \theta_0(\xi), k(\xi)).$$

Note that this definition requires scattering data for both θ_0 and $-\theta_0$.

The algorithm proposed in [2] used this Born approximation to approximate $u_s(y, \theta_0, k)$ in (1.7) iteratively. More precisely, a sequence of potentials are defined by $q_1 = q_{\theta_0}$ and

$$\widehat{q_{n+1}}(\xi) = u_\infty(\theta(\xi), \theta_0(\xi), k(\xi)) - \int_{\mathbb{R}^d} e^{-ik(\xi)\theta_0 \cdot y} q_n(y) u_s^n(y, \theta_0, k(\xi)) dy,$$

84 where u_s^n solves

$$85 \quad (1.11) \quad u_s^n(x, \theta_0, k) = R_k(q_n e^{ik\theta_0 \cdot (\cdot)})(x) + R_k(q_n u_s^n(\cdot, \theta_0, k))(x), \quad x \in \mathbb{R}^d.$$

86 This requires to solve the Lipmann-Schwinger equation (1.6) for all $k(\xi)$ at each iteration. As described in [2]
87 the numerical version of this algorithm converge in few iterations but each one is expensive computationally,
88 even for 2-d problems.

89 To avoid the solution of equation (1.11) we insert iteratively the Lippmann-Schwinger integral equation
90 (1.6) into (1.4). In this way we obtain

$$\begin{aligned} 91 \quad u_\infty(\theta, \theta_0, k) &= \int_{\mathbb{R}^d} e^{-ik(\theta - \theta_0) \cdot y} q(y) dy \\ 92 \quad (1.12) \quad &+ \sum_{j=1}^m \int_{\mathbb{R}^d} e^{-ik\theta \cdot y} (qR_k)^j (q e^{ik\theta_0 \cdot (\cdot)})(y) dy \\ 93 \quad &+ \int_{\mathbb{R}^d} e^{-ik\theta \cdot y} (qR_k)^m (q u_s(\cdot, \theta_0, k))(y) dy. \end{aligned}$$

94 Taking $m \rightarrow \infty$, we formally get the Born series

$$95 \quad (1.13) \quad u_\infty(\theta, \theta_0, k) = \sum_{j=0}^{\infty} \int_{\mathbb{R}^d} e^{-ik\theta \cdot y} (qR_k)^j (q e^{ik\theta_0 \cdot (\cdot)})(y) dy.$$

96 This is the approach followed by R.T. Prosser. Note that the right hand side of (1.13) is a tensor polynomial
97 where the first term is linear in q , the second one bilinear in q , the third one trilinear in q , and so on. Motivated
98 by this structure, Prosser introduced a classical asymptotic method to recover q based on replacing in (1.13)
99 u_∞ by εu_∞ , writing q as the power series $q = \sum_{n=0}^{\infty} \varepsilon^n q_n$, and then identifying the terms with the same
100 power in ε . This provides a system of equations for q_n that can be solved recursively.

101 Here we follow a different approach. For convenience, we rewrite (1.12) as follows

$$102 \quad (1.14) \quad \widehat{q}(\xi) = \widehat{q_{\theta_0}}(\xi) - \sum_{j=1}^m \widehat{\mathcal{Q}_j(q)}(\xi) - \widehat{q_m^r}(\xi), \quad \xi \in \mathbb{R}^d,$$

103 where

$$104 \quad (1.15) \quad \widehat{\mathcal{Q}_j(q)}(\xi) := \int_{\mathbb{R}^d} e^{-ik(\xi)\theta(\xi) \cdot y} (qR_{k(\xi)})^j (q e^{ik(\xi)\theta_0 \cdot (\cdot)})(y) dy$$

105 and

$$106 \quad (1.16) \quad \widehat{q_m^r}(\xi) := \int_{\mathbb{R}^d} e^{-ik(\xi)\theta(\xi) \cdot y} (qR_{k(\xi)})^m (q u_s(\cdot, \theta_0, k))(y) dy.$$

107 The convergence of this series suggests that the last term in equation (1.14) should be small for large
108 m . Based on this idea we consider the following family of reduced equations for q_m , where we have removed
109 this last term,

$$110 \quad (1.17) \quad \widehat{q_m}(\xi) = \widehat{q_{\theta_0}}(\xi) - \sum_{j=1}^m \widehat{\mathcal{Q}_j(q_m)}(\xi), \quad \xi = k(\xi)(\theta(\xi) \mp \theta_0),$$

111 with $(\theta, k) \in S^{d-1} \times (0, \infty)$. These reduced equations have the advantage that they do not involve u_s , avoiding
112 the solution of the Lipmann-Schwinger equation (1.6). However, for each parameter $(\theta, k) \in S^{d-1} \times (0, \infty)$,

equation (1.17) is still nonlinear in q_m . Moreover, it is not clear if there exists a unique function q_m satisfying (1.17).

Here we propose a fixed point procedure to find approximations of q_m . More precisely we introduce the linear operator $\mathcal{L}_m$ defined by

$$\widehat{\mathcal{L}_m(q)}(\xi) := \widehat{q_{\theta_0}}(\xi) - \sum_{j=1}^m \widehat{\mathcal{Q}_j(q)}(\xi),$$

Then, if q_m is solution of (1.17), it must be also a fixed point of $\mathcal{L}_m$ and we can try the usual iterative method based on powers of $\mathcal{L}_m$ to approximate q_m . This requires in particular that $\mathcal{L}_m(q)$ is of compact support. Therefore, instead of $\mathcal{L}_m$ we consider the modified operator

$$(1.18) \quad \mathcal{T}_m(q) := \phi \mathcal{L}_m(q) = \phi q_{\theta_0} - \phi \sum_{j=1}^m \mathcal{Q}_j(q),$$

where $\phi \in C^\infty$ is a cut-off function with compact support satisfying

$$(1.19) \quad \phi(x) = 1, \text{ if } |x| < R \text{ and } \phi(x) = 0, \text{ if } |x| > 2R.$$

We are now ready to state the main results in this paper. For each $m \in \mathbb{N}$, we consider the sequence $\{q_{m,\ell}\}_{\ell \in \mathbb{N}}$ defined recursively by

$$(1.20) \quad \begin{cases} q_{m,1} = 0, \\ q_{m,\ell+1} = \mathcal{T}_m(q_{m,\ell}), \end{cases} \quad \ell \geq 1.$$

We note that $q_{m,2} = \phi q_{\theta_0}$, which is a good approximation to a potential q with support in $B(0, R)$.

The purpose of this paper is to prove that the sequence of approximations $\{q_{m,\ell}\}_{m,\ell \in \mathbb{N}}$ converges to the potential q in some sense. More precisely, we will prove the following theorem.

THEOREM 1.1. *For $d \geq 2$ and α satisfying*

$$(1.21) \quad 0 < \alpha \leq 1, \quad \text{and} \quad \frac{d}{2} - \frac{d}{d-1} < \alpha < \frac{d}{2},$$

let $q \in W^{\alpha,2}(\mathbb{R}^d)$ be a real valued function with compact support in $B(0, R)$ and such that

$$(1.22) \quad \|q\|_{W^{\alpha,2}} < A,$$

for an appropriate constant $A := A(d, \alpha, R) > 0$ small enough (see Remark 1.2). For each $m \in \mathbb{N}$, let $\{q_{m,\ell}\}_{\ell \in \mathbb{N}}$ be the sequence defined by (1.20). Then, there exists $q_m \in W^{\alpha,2}(\mathbb{R}^d)$ satisfying

$$q_m = \lim_{\ell \rightarrow \infty} q_{m,\ell} \quad \text{in } W^{\alpha,2}(\mathbb{R}^d).$$

Moreover, the sequence $\{q_m\}_{m \in \mathbb{N}}$ satisfies

$$\lim_{m \rightarrow \infty} q_m = q \quad \text{in } W^{\alpha,2}(\mathbb{R}^d).$$

Remark 1.2. Following the proof of Theorem 1.1, one can see that the smallness condition given in (1.22) is

$$\|q\|_{W^{\alpha,2}} < A = \min \left(\frac{1}{C_1}, \frac{1}{C_2}, \frac{1}{2C_1C_3C_4}, \frac{1}{2^{2+\frac{d-1}{2}(\frac{1}{2}-\frac{\alpha}{d})}C_5} \right),$$

where C_1, C_2, C_3, C_4 and C_5 are the constants that appear in (2.4), (2.7), (2.8), (2.9) and (2.22) respectively (see below).

Remark 1.3. Conditions in (1.21) imply that $d < 5$, and therefore Theorem 1.1 is valid for $2 \leq d \leq 4$. Moreover, for $d = 2$, the result holds if $0 < \alpha < 1$, for $d = 3$, if $0 < \alpha \leq 1$, and for $d = 4$, if $2/3 < \alpha \leq 1$.

Remark 1.4. In the proof of Theorem 1.1 (see (2.19) below) we will see that the convergence of the sequence $\{q_m\}_{m \in \mathbb{N}}$ to potential q is faster than $(C_2\|q\|_{W^{\alpha,2}})^m$, where C_2 is the constant that appears in (2.7) (see below).

Notation. For $\alpha \in \mathbb{R}$ we introduce the fractional differentiation operator

$$\Lambda^\alpha = (1 - \Delta)^{\alpha/2} = \mathcal{F}^{-1} \langle \xi \rangle^\alpha \mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

We use the Sobolev spaces

$$W^{\alpha,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^\alpha f \in L^p(\mathbb{R}^d)\}, \quad \alpha \in \mathbb{R}, 1 \leq p \leq \infty,$$

and also their weighted versions

$$W_\delta^{\alpha,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^\alpha f \in L_\delta^p(\mathbb{R}^d)\}, \quad \alpha, \delta \in \mathbb{R}, 1 \leq p \leq \infty,$$

where $L_\delta^p(\mathbb{R}^d) = \{f : \langle x \rangle^\delta f \in L^p(\mathbb{R}^d)\}$.

Throughout this paper C will denote a positive constant that may change from line to line and depend on some parameters such as d, α or R . This dependence will be indicated when relevant. In order to know in detail the required smallness condition on the potential q , in some cases, we will name constants by C_n with n varying in $\mathbb{N}$. Moreover we will write $C_{n_1, \dots, n_m}$ to indicate the product $C_{n_1} \dots C_{n_m}$, $n_1, \dots, n_m \in \mathbb{N}$.

The rest of this paper is organized as follows. The proof of Theorem 1.1 is given in section 2, and also several lemmas needed in the proof. In section 3 we illustrate these results with several numerical experiments.

2. Proofs. This section is devoted to the proof of Theorem 1.1. We split it into two subsections. The first one contains the proof itself and the statement of two results (see Propositions 2.1 and 2.2 below) which are the key points in the proof. The proofs of these propositions are quite technical, and require of several known results, so we postpone them to the second subsection.

2.1. Proof of Theorem 1.1. The first key point in the proof of Theorem 1.1 are some estimates for a family of operators that generalize the operators $\mathcal{Q}_j$ given in (1.15).

For every $j \in \mathbb{N}$, we introduce the following multilinear operator defined via its Fourier transform as follows

$$(2.1) \quad \mathcal{P}_j(\mathbf{f})(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} \widehat{\mathcal{P}_j(\mathbf{f})}(\xi) d\xi = \left(\int_{H_{\theta_0}} + \int_{H_{-\theta_0}} \right) e^{ix \cdot \xi} \widehat{\mathcal{P}_j(\mathbf{f})}(\xi) d\xi,$$

where $\mathbf{f} = (f_1, f_2, \dots, f_{j+1})$, $\theta_0 \in S^{d-1}$ and $H_{\pm\theta_0}$ is given in (1.8).

Taking into account that any $\xi \in H_{\pm\theta_0}$ can be written in a unique way as $\xi = k(\theta \mp \theta_0)$, with $k = k(\xi) > 0$ and $\theta = \theta(\xi) \in S^{d-1}$, we define

$$(2.2) \quad \widehat{\mathcal{P}_j(\mathbf{f})}(\xi) := \int_{\mathbb{R}^d} e^{-ik\theta \cdot y} (f_{j+1} R_k \dots f_2 R_k)(f_1 e^{ik\theta_0 \cdot (\cdot)})(y) dy,$$

Observe that in the particular case $f_i = q$, $i = 1, 2, \dots, j+1$, we have that

$$(2.3) \quad \widehat{\mathcal{P}_j(\mathbf{f})}(\xi) = \widehat{\mathcal{Q}_j(q)}(\xi), \quad \xi \in \mathbb{R}^d.$$

PROPOSITION 2.1. *Let $d \geq 2$ and α satisfying (1.21). For each $j \in \mathbb{N}$ fixed, let $\mathbf{f} = (f_1, f_2, \dots, f_{j+1})$ with $f_\ell \in W^{\alpha,2}(\mathbb{R}^d)$ and compactly supported with support in $B(0, R)$, for $\ell = 1, \dots, j+1$. Then, there exists a constant $C_1 := C_1(d, \alpha, R)$ such that*

$$(2.4) \quad \|\mathcal{P}_j(\mathbf{f})\|_{W^{\alpha,2}} \leq C_1^j \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}.$$

169 The other key point in the proof of Theorem 1.1 concerns with the error term q_m^r defined in (1.16).

170 PROPOSITION 2.2. For $d \geq 2$ and α satisfying

$$171 \quad (2.5) \quad 0 < \alpha \leq 1, \quad \text{and} \quad \frac{d}{2} - 2 < \alpha < \frac{d}{2},$$

172 let $q \in W^{\alpha,2}(\mathbb{R}^d)$ be a real valued function with compact support in $B(0, R)$ and such that

$$173 \quad (2.6) \quad \|q\|_{W^{\alpha,2}} < \frac{1}{2^{2+\frac{d-1}{2}(\frac{1}{2}-\frac{\alpha}{d})} C_5},$$

174 where C_5 is the constant that appears in (2.22). Then, for every $m \in \mathbb{N}$ there exists a constant $C_2 :=$
175 $C_2(d, \alpha, R)$ such that

$$176 \quad (2.7) \quad \|q_m^r\|_{W^{\alpha,2}} \leq C_2^m \|q\|_{W^{\alpha,2}}^{m+1}.$$

177 We also need to control the Born approximation defined in (1.10).

178 COROLLARY 2.3. For $d \geq 2$ and α satisfying (1.21), let $q \in W^{\alpha,2}(\mathbb{R}^d)$ be a real valued function with
179 compact support in $B(0, R)$ satisfying (2.6). Then, there exists a constant $C_3 := C_3(d, \alpha, R)$ such that

$$180 \quad (2.8) \quad \|q_{\theta_0}\|_{W^{\alpha,2}} < C_3 \|q\|_{W^{\alpha,2}}.$$

Proof. From (1.14) with $m = 1$, we have that

$$\|q_{\theta_0}\|_{W^{\alpha,2}} \leq \|q\|_{W^{\alpha,2}} + \|\mathcal{Q}_1(q)\|_{W^{\alpha,2}} + \|q_1^r\|_{W^{\alpha,2}}.$$

181 The result follows from here using (2.3) and (2.4) for $j = 1$, (2.7) for $m = 1$, and (2.6). $\square$

182 Finally, we need the following result concerning the product of functions in Sobolev spaces due to Zolesio
183 (see [44]).

184 LEMMA 2.4. ([12, Theorem 1.4.4.2]) Let $0 \leq \alpha \leq s$ and $s > d/2$, and let ϕ be the cut-off function defined
185 in (1.19), then there exists a constant $C_4 := C_4(d, \alpha, R) > 0$ such that

$$186 \quad (2.9) \quad \|\phi g\|_{W^{\alpha,2}} \leq C \|\phi\|_{W^{s,2}} \|g\|_{W^{\alpha,2}} \leq C_4 \|g\|_{W^{\alpha,2}}, \quad \forall g \in W^{\alpha,2}(\mathbb{R}^d).$$

187 *Proof of Theorem 1.1.* We split the proof in three steps. In the first one we will prove that for each
188 $m \in \mathbb{N}$, the sequence $\{q_{m,\ell}\}_{\ell \in \mathbb{N}}$ is bounded in the space $W^{\alpha,2}(\mathbb{R}^d)$. In the second one we will see that such
189 sequence is a Cauchy sequence in that space, thus it converges to a function q_m . Finally, in the third one we
190 will prove that q is the limit in $W^{\alpha,2}(\mathbb{R}^d)$ of q_m as m goes to infinity.

191 STEP 1. We will prove the boundedness by induction on ℓ . Moreover, for each $m \in \mathbb{N}$, we will prove
192 that

$$193 \quad (2.10) \quad \|q_{m,\ell}\|_{W^{\alpha,2}} \leq 2C_{3,4} \|q\|_{W^{\alpha,2}}, \quad \forall \ell \geq 2,$$

194 whenever α satisfies (1.21) and $q \in W^{\alpha,2}(\mathbb{R}^d)$ is a real valued function with compact support in $B(0, R)$
195 satisfying (2.6) and such that

$$196 \quad (2.11) \quad \|q\|_{W^{\alpha,2}} \leq \frac{1}{2C_{1,3,4}(1+2C_4)}.$$

197 Here $C_{3,4} = C_3 C_4$ with C_3 and C_4 being the constants that appear in (2.8) and (2.9) respectively. Similarly
198 it is defined $C_{1,3,4}$.

From (1.20) and (1.18), we have that

$$\|q_{m,2}\|_{W^{\alpha,2}} = \|\phi q_{\theta_0}\|_{W^{\alpha,2}}.$$

Using (2.9) and (2.8) we get

$$\|q_{m,2}\|_{W^{\alpha,2}} \leq C_{3,4}\|q\|_{W^{\alpha,2}} \leq 2C_{3,4}\|q\|_{W^{\alpha,2}},$$

for α and q under the assumptions of Corollary 2.3.

Arguing as before, from (1.20), (1.18), (2.9) and (2.8) we also get

$$\begin{aligned} \|q_{m,\ell+1}\|_{W^{\alpha,2}} &\leq \|\phi q_{\theta_0}\|_{W^{\alpha,2}} + \left\| \phi \sum_{j=1}^m \mathcal{Q}_j(q_{m,\ell}) \right\|_{W^{\alpha,2}} \\ &\leq C_{3,4}\|q\|_{W^{\alpha,2}} + C_4 \sum_{j=1}^m \|\mathcal{Q}_j(q_{m,\ell})\|_{W^{\alpha,2}}. \end{aligned}$$

From here, using identity (2.3), (2.4) and the induction hypothesis (2.10), we obtain

$$\begin{aligned} \|q_{m,\ell+1}\|_{W^{\alpha,2}} &\leq C_{3,4}\|q\|_{W^{\alpha,2}} + C_4 \sum_{j=1}^m C_1^j \|q_{m,\ell}\|_{W^{\alpha,2}}^{j+1} \\ &\leq C_{3,4}\|q\|_{W^{\alpha,2}} + C_4 \sum_{j=1}^m C_1^j (2C_{3,4})^{j+1} \|q\|_{W^{\alpha,2}}^{j+1}. \end{aligned}$$

Estimate (2.10) follows from here since for q satisfying (2.11) we have that

$$2C_4 \sum_{j=1}^m (2C_{1,3,4}\|q\|_{W^{\alpha,2}})^j \leq 1.$$

STEP 2. From (1.20) and (1.18), using (2.9), we have that

$$\|q_{m,\ell+1} - q_{m,n+1}\|_{W^{\alpha,2}} \leq C_4 \left\| \sum_{j=1}^m (\mathcal{Q}_j(q_{m,\ell}) - \mathcal{Q}_j(q_{m,n})) \right\|_{W^{\alpha,2}}.$$

From here, using (2.3), the fact that $\mathcal{P}_j$ is a multilinear operator and the triangular inequality, we get

$$\|q_{m,\ell+1} - q_{m,n+1}\|_{W^{\alpha,2}} \leq C_4 \sum_{j=1}^m \sum_{i=1}^{j+1} \|\mathcal{P}_j(\mathbf{f}_{m,\ell,n,i})\|_{W^{\alpha,2}},$$

where

$$\mathbf{f}_{m,\ell,n,i} = (q_{m,\ell}, \dots, q_{m,\ell}, \underbrace{q_{m,\ell} - q_{m,n}}_{i\text{-position}}, q_{m,n}, \dots, q_{m,n}).$$

Using (2.4) and (2.10) in (2.12) we obtain

$$\begin{aligned} \|q_{m,\ell+1} - q_{m,n+1}\|_{W^{\alpha,2}} &\leq C_4 \sum_{j=1}^m (j+1) (2C_{1,3,4}\|q\|_{W^{\alpha,2}})^j \|q_{m,\ell} - q_{m,n}\|_{W^{\alpha,2}} \\ &\leq B \|q_{m,\ell} - q_{m,n}\|_{W^{\alpha,2}}, \end{aligned}$$

with

$$B = \frac{C_4}{(1 - 2C_{1,3,4} \|q\|_{W^{\alpha,2}})^2}.$$

Observe that $B < 1$ whenever α satisfies (1.21) and $q \in W^{\alpha,2}(\mathbb{R}^d)$ is a real valued function with compact support in $B(0, R)$ satisfying (2.6), (2.11) and

$$(2.14) \quad \|q\|_{W^{\alpha,2}} < \frac{1 + \sqrt{C_4}}{2C_{1,3,4}}.$$

Therefore, for such a q , we have that $\{q_{m,\ell}\}_{\ell \in \mathbb{N}}$ is a Cauchy sequence. As a consequence, there exists $q_m \in W^{\alpha,2}(\mathbb{R}^d)$ such that

$$q_m = \lim_{\ell \rightarrow \infty} q_{m,\ell}, \quad \text{in } W^{\alpha,2}(\mathbb{R}^d).$$

From (2.10), we have that q_m satisfies

$$(2.15) \quad \|q_m\|_{W^{\alpha,2}} \leq 2C_{3,4} \|q\|_{W^{\alpha,2}}.$$

Moreover,

$$(2.16) \quad q_m = \phi q_{\theta_0} - \phi \sum_{j=1}^m \mathcal{Q}_j(q_m),$$

since arguing as we did to get (2.13), we obtain

$$\left\| q_{m,\ell+1} - \left(\phi q_{\theta_0} - \phi \sum_{j=1}^m \mathcal{Q}_j(q_m) \right) \right\|_{W^{\alpha,2}} < \|q_{m,\ell} - q_m\|_{W^{\alpha,2}}.$$

STEP 3. For α satisfying (1.21) and $q \in W^{\alpha,2}(\mathbb{R}^d)$ being a real valued function with compact support in $B(0, R)$ satisfying (2.6), (2.11) and (2.14), from (2.16), (1.14) and (1.19), since $\text{supp}(q) \subset B(0, R)$, we have that

$$\|q_m - q\|_{W^{\alpha,2}} \leq \left\| \phi \sum_{j=1}^m (\mathcal{Q}_j(q_m) - \mathcal{Q}_j(q)) \right\|_{W^{\alpha,2}} + \|\phi q_m^r\|_{W^{\alpha,2}}.$$

From here, arguing as we did to get (2.13), we obtain

$$(2.17) \quad \|q_m - q\|_{W^{\alpha,2}} \leq D \|q_m - q\|_{W^{\alpha,2}} + \|\phi q_m^r\|_{W^{\alpha,2}},$$

where

$$D = C_4 \sum_{j=1}^m C_1^j \sum_{i=1}^{j+1} \|q_m\|_{W^{\alpha,2}}^{i-1} \|q\|_{W^{\alpha,2}}^{j+1-i}.$$

Using (2.15), we have that

$$D \leq C_4 \sum_{j=1}^m (C_1 \|q\|_{W^{\alpha,2}})^j \sum_{i=1}^{j+1} (2C_{3,4})^{i-1},$$

and therefore

$$D \leq \begin{cases} C_4 \sum_{j=1}^m (j+1)(C_1 \|q\|_{W^{\alpha,2}})^j, & \text{if } 2C_{1,3,4} \leq 1, \\ C_4 \sum_{j=1}^m (j+1)(2C_{1,3,4} \|q\|_{W^{\alpha,2}})^j, & \text{if } 2C_{1,3,4} \geq 1. \end{cases}$$

Thus, $D \leq 1/2$ whenever

$$(2.18) \quad \|q\|_{W^{\alpha,2}} < \min \left(\frac{1}{C_1}, \frac{1}{2C_{1,3,4}} \right).$$

Using this in (2.17) we get

$$\|q_m - q\|_{W^{\alpha,2}} \leq 2\|\phi q_m^r\|_{W^{\alpha,2}}.$$

Finally, using (2.9) and (2.7) we have that

$$(2.19) \quad \|q_m - q\|_{W^{\alpha,2}} \leq 2C_4 C_2^m \|q\|_{W^{\alpha,2}}^{m+1}.$$

The result follows from here if q satisfies (2.6), (2.11), (2.14), (2.18), and also $\|q\|_{W^{\alpha,2}} < 1/C_2$. $\square$

2.2. Proofs of the key points. In this subsection we give the proofs of Propositions 2.1 and 2.2. They require of several known estimates for different operators in weighted Sobolev spaces.

We will need to know the behaviour of the outgoing resolvent of the laplacian denoted by R_k (see (1.5)) for k small given in the following lemma.

LEMMA 2.5. ([22, Chapter VI], [21, lemma 21.4]). *Let $\alpha \in [0, 2]$, $k \in [0, b]$, $b > 0$ and $\delta > 1$. Then*

$$(2.20) \quad \|R_k f\|_{W_{-\delta}^{\alpha,2}} \leq C_{b,\delta} \|f\|_{L_{\delta}^2}.$$

For $\theta_0 \in S^{d-1}$ and $k > 0$ fixed, we introduce the following operator involving the outgoing resolvent of the laplacian,

$$(2.21) \quad R_{k,\theta_0} f(x) = e^{-ik\theta_0 \cdot x} R_k(f(\cdot) e^{ik\theta_0 \cdot (\cdot)})(x).$$

The following result can be obtained by interpolation of several estimates due to Agmon (see [1]), Kenig, Ruiz and Sogge (see [18]), Ruiz and Vega (see [39]), and Barceló, Ruiz and Vega (see [4]). For details we refer the reader to [37].

LEMMA 2.6. ([37, Lemma 3.4]). *Let $\alpha \geq 0$, r and t such that $0 \leq \frac{1}{t} - \frac{1}{2} \leq \frac{1}{d+1}$ and $0 \leq \frac{1}{2} - \frac{1}{r} \leq \frac{1}{d+1}$, then there exist $\delta > 1$ and a constant $C_5 := C_5(\delta)$ such that*

$$(2.22) \quad \|R_{k,\theta_0} f\|_{W_{-\delta}^{\alpha,r}} \leq C_5 k^{-1 + \frac{d-1}{2}(\frac{1}{t} - \frac{1}{r})} \|f\|_{W_{\delta}^{\alpha,t}}.$$

We also introduce the restriction operator given by

$$(2.23) \quad S_{k,\theta_0} f(\theta) = \int_{\mathbb{R}^d} e^{-ik(\theta - \theta_0) \cdot y} f(y) dy, \quad \theta \in S^{d-1}.$$

The following lemma is a consequence of Theorem 3(c) in [4].

LEMMA 2.7. *If $\delta > 1/2$ then, there exists $C > 0$ such that*

$$(2.24) \quad \|S_{k,\theta_0} f\|_{L^2(S^{d-1})} \leq C k^{-\frac{d-1}{2}} \|f\|_{L_{\delta}^2}.$$

Using the previous lemma and the Stein-Tomas restriction theorem(see [42]) we can get the following result which generalizes the previous one.

LEMMA 2.8. ([37, Lemma 3.7]). *Let $\alpha \geq 0$ and t satisfying $0 \leq \frac{1}{t} - \frac{1}{2} \leq \frac{1}{d+1}$; then there exists $\delta(t) > 0$ and $C := C_t > 0$ such that for $w(\theta) = |\theta - \theta_0|^\alpha$, we have*

$$(2.25) \quad \|w S_{k,\theta_0} f\|_{L^2(S^{d-1})} \leq C k^{\frac{d-1}{2}(\frac{1}{t} - \frac{3}{2}) - \alpha} \|f\|_{W_{\delta(t)}^{\alpha,t}}.$$

Now we state the following result concerning the product of functions in weighted Sobolev spaces due to Zolesio (see [44]). For a more general version and details we refer the reader to Theorem 1.4.4.2 (pp. 28) of [12]. See also Proposition D.3 (pp. 182) of [36] for functions compactly supported.

LEMMA 2.9. *Let f be compactly supported and $\delta \in \mathbb{R}$.*

(i) *For α satisfying*

$$(2.26) \quad 0 \leq \alpha \quad \text{and} \quad \alpha > \frac{d}{2} - 2,$$

or

$$(2.27) \quad 0 < \alpha \quad \text{and} \quad \alpha \geq \frac{d}{2} - 2,$$

we have that there exists a constant $C := C(\text{supp}(f))$ such that

$$(2.28) \quad \|fg\|_{L^2_\delta} \leq C\|f\|_{W^{\alpha,2}}\|g\|_{W^{2,2}_{-\delta}}, \quad \forall g \in W^{2,2}_{-\delta}(\mathbb{R}^d).$$

(ii) *For α, t and r satisfying*

$$1 \leq t < \min(2, r) \quad \text{and} \quad 0 \leq \frac{1}{2} + \frac{1}{r} - \frac{1}{t} \leq \frac{\alpha}{d},$$

or

$$1 \leq t \leq \min(2, r) \quad \text{and} \quad 0 \leq \frac{1}{2} + \frac{1}{r} - \frac{1}{t} < \frac{\alpha}{d},$$

we have that there exists a constant $C := C(\text{supp}(f))$ such that

$$(2.29) \quad \|fg\|_{W^{\alpha,t}_\delta} \leq C\|f\|_{W^{\alpha,2}}\|g\|_{W^{\alpha,r}_{-\delta}}, \quad \forall g \in W^{\alpha,r}_{-\delta}(\mathbb{R}^d).$$

(iii) *For α, β and p satisfying*

$$0 \leq \beta \leq \min(\alpha, 2), \quad 1 \leq p < 2 \quad \text{and} \quad 0 \leq d \left(1 - \frac{1}{p}\right) \leq \alpha + 2 - \beta,$$

we have that there exists a constant $C := C(\text{supp}(f))$ such that

$$(2.30) \quad \|fg\|_{W^{\beta,p}} \leq C\|f\|_{W^{\alpha,2}}\|g\|_{W^{2,2}_{-\delta}}, \quad \forall g \in W^{2,2}_{-\delta}(\mathbb{R}^d).$$

Proof of Proposition 2.1. From (1.8), for a fixed $\theta_0 \in S^{d-1}$ we have that

$$\|\mathcal{P}_j(\mathbf{f})\|_{W^{\alpha,2}}^2 = \left(\int_{H_{\theta_0}} + \int_{H_{-\theta_0}} \right) \langle \xi \rangle^{2\alpha} \left| \widehat{\mathcal{P}_j(\mathbf{f})}(\xi) \right|^2 d\xi = I + II.$$

We will prove the estimate for term I , the other one is obtained in a similar way.

Observe that in H_{θ_0} we can make the following change of variables

$$(2.31) \quad \xi = k(\theta - \theta_0), \quad d\xi = k^{d-1}|\theta - \theta_0|^2 d\sigma(\theta) dk.$$

Then,

$$\begin{aligned} I &\leq C \int_0^1 k^{d-1} \int_{S^{d-1}} \left| \widehat{\mathcal{P}_j(\mathbf{f})}(k(\theta - \theta_0)) \right|^2 d\sigma(\theta) dk \\ &\quad + C \int_1^\infty k^{d-1+2\alpha} \int_{S^{d-1}} |\theta - \theta_0|^{2\alpha} \left| \widehat{\mathcal{P}_j(\mathbf{f})}(k(\theta - \theta_0)) \right|^2 d\sigma(\theta) dk \\ &= C(I_1 + I_2), \end{aligned} \quad (2.32)$$

whenever $0 \leq \alpha \leq 1$.

In order to control I_1 , we rewrite $\mathcal{P}_j$ in terms of the resolvent operator R_k and the restriction operator S_{k,θ_0} defined in (2.23), as follows:

$$\widehat{\mathcal{P}_j(\mathbf{f})}(k(\theta - \theta_0)) = S_{k,\theta_0} \left(e^{-ik\theta_0 \cdot (\cdot)} f_{j+1} R_k \cdots f_2 R_k \left(f_1 e^{ik\theta_0 \cdot (\cdot)} \right) \right) (\theta).$$

Using this identity we have that

$$I_1 = \int_0^1 k^{d-1} \left\| S_{k,\theta_0} \left(e^{-ik\theta_0 \cdot (\cdot)} f_{j+1} R_k \cdots f_2 R_k \left(f_1 e^{ik\theta_0 \cdot (\cdot)} \right) \right) \right\|_{L^2(S^{d-1})}^2 dk.$$

We can bound the $L^2(S^{d-1})$ -norm that appears in the last identity using (2.24), (2.28) and (2.20) as the following diagram illustrates

$$L_\delta^2 \cdots \xrightarrow{j\text{-times}} L_\delta^2 \xrightarrow{R_k} W_{-\delta}^{2,2} \xrightarrow{\text{Zolesio}} L_\delta^2 \xrightarrow{S_{k,\theta_0}} L^2(S^{d-1}).$$

Therefore, if $\delta > 1$ and α satisfies (2.26) or (2.27) we get

$$\begin{aligned} I_1 &\leq C^{2j} \|f_{j+1}\|_{W^{\alpha,2}}^2 \|f_j\|_{W^{\alpha,2}}^2 \cdots \|f_2\|_{W^{\alpha,2}}^2 \|f_1\|_{L_\delta^2}^2 \\ &\leq C^{2j} \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}^2. \end{aligned} \quad (2.33)$$

In order to control I_2 we need to take more advantage of oscillations, so we write $\mathcal{P}_j$ in terms of the operator $R_{k,\theta}$ introduced in (2.21) and the restriction operator S_{k,θ_0} defined in (2.23), as follows:

$$\widehat{\mathcal{P}_j(\mathbf{f})}(k(\theta - \theta_0)) = S_{k,\theta_0} (f_{j+1} R_{k,\theta_0} \cdots f_2 R_{k,\theta_0} f_1) (\theta).$$

Using this identity, and writing $w = |\theta - \theta_0|^\alpha$, we have that

$$I_2 = \int_1^\infty k^{d-1+2\alpha} \|w S_{k,\theta_0} (f_{j+1} R_{k,\theta_0} \cdots f_2 R_{k,\theta_0} f_1)\|_{L^2(S^{d-1})}^2 dk,$$

In this case, we can bound the $L^2(S^{d-1})$ -norm that appears in the last identity using (2.25), (2.29) and (2.22) as the following diagram illustrates

$$W_\delta^{\alpha,2} \cdots \xrightarrow{(j-1)\text{-times}} W_\delta^{\alpha,t_j} \xrightarrow{R_{k,\theta_0}} W_{-\delta}^{\alpha,r_j} \xrightarrow{\text{Zolesio}} W_\delta^{\alpha,t_{j+1}} \xrightarrow{S_{k,\theta_0}} L^2(S^{d-1}),$$

whenever there exist r_ℓ and $t_{\ell+1}$ satisfying for $\ell = 1, 2, \dots, j$

$$0 \leq \frac{1}{t_{\ell+1}} - \frac{1}{2} \leq \frac{1}{d+1} \quad \text{and} \quad 0 \leq \frac{1}{2} - \frac{1}{r_\ell} \leq \frac{1}{d+1}, \quad (2.34)$$

$$t_{\ell+1} < \min(2, r_\ell) \quad \text{and} \quad 0 \leq \frac{1}{2} + \frac{1}{r_\ell} - \frac{1}{t_{\ell+1}} \leq \frac{\alpha}{d}. \quad (2.35)$$

Therefore, writing $t_1 = 2$, we have

$$\begin{aligned} I_2 &\leq C^{2j} \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}^2 \int_1^\infty k^{(d-1)\left(\frac{1}{t_{j+1}} - \frac{1}{2}\right) + \sum_{\ell=1}^j \left(-2 + (d-1)\left(\frac{1}{t_\ell} - \frac{1}{r_\ell}\right)\right)} dk \\ &= C^{2j} \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}^2 \int_1^\infty \frac{dk}{k^{(d-1)\sum_{\ell=1}^j \left(\frac{1}{r_\ell} - \frac{1}{t_{\ell+1}}\right) + 2j}}. \end{aligned}$$

Since $t_{\ell+1}$ and r_ℓ have to satisfy (2.35), the best choice to get convergence of the previous integral is

$$(2.36) \quad \frac{1}{r_\ell} - \frac{1}{t_{\ell+1}} = \frac{\alpha}{d} - \frac{1}{2}.$$

With this choice we get

$$(2.37) \quad \begin{aligned} I_2 &\leq C^{2j} \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}^2 \int_1^\infty \frac{dk}{k^{(d-1)(\frac{\alpha}{d}-\frac{1}{2})j+2j}} \\ &\leq C^{2j} \prod_{\ell=1}^{j+1} \|f_\ell\|_{W^{\alpha,2}}^2, \end{aligned}$$

for any $j \in \mathbb{N}$ fixed, whenever

$$(2.38) \quad \alpha > d \left(\frac{1}{2} - \frac{1}{d-1} \right),$$

and there exist r_ℓ and $t_{\ell+1}$ satisfying (2.34), (2.35) and (2.36), for $\ell = 1, 2, \dots, j$. Such r_ℓ and $t_{\ell+1}$ exist if

$$(2.39) \quad 0 \leq \alpha < d/2.$$

Finally, the result follows from (2.32), (2.33) and (2.37) if $\alpha \leq 1$ and satisfies (2.38), (2.39) and, (2.26) or (2.27), that is, if α satisfies (1.21). $\square$

In order to prove Proposition 2.2, we need the existence and uniqueness of solution of the direct scattering problem in $W_{-\delta}^{2,2}(\mathbb{R}^d)$ for any fixed wave number $k > 0$. This is equivalent to prove that there exists a unique u_s , satisfying (1.6). For completeness we present here that result. The following lemma will be needed (see Theorem 6.5 in [38] or [22]).

LEMMA 2.10. *Let $k > 0$. There exists $\delta > 1$ such that*

$$(2.40) \quad \|R_k f\|_{W_{-\delta}^{2,2}} \leq c_k \|f\|_{L_\delta^2}.$$

THEOREM 2.11. *For α satisfying $\alpha > 0$ and $\alpha > \frac{d}{2} - 2$, let $q \in W^{\alpha,2}(\mathbb{R}^d)$ be real valued and compactly supported. For any given $k > 0$ and $\theta_0 \in S^{d-1}$, there exists a unique solution $u_s = u_s(x, \theta_0, k)$ of equation (1.6) such that $u_s \in W_{-\delta}^{2,2}(\mathbb{R}^d)$ for certain $\delta > 1$.*

Proof. We begin by proving the uniqueness. In order to do that, we have to prove that if u is a solution in $W_{-\delta}^{2,2}(\mathbb{R}^d)$ of $u(x) = R_k(qu)(x)$ with $x \in \mathbb{R}^d$, then $u(x) = 0$ for all $x \in \mathbb{R}^d$.

We have that, in a weak sense

$$(2.41) \quad (\Delta + k^2)u(x) = q(x)u(x), \quad x \in \mathbb{R}^d,$$

and u satisfies the outgoing Sommerfeld condition.

Consider $M > 0$ large enough such that $\text{supp } q \subset B_M = B(0, M)$. Multiplying (2.41) by $\bar{u}$ and integrating by parts we get

$$(2.42) \quad - \int_{B_M} |\nabla u|^2 + \int_{S_M} \partial_r u \bar{u} + k^2 \int_{B_M} |u|^2 = \int_{B_M} q|u|^2,$$

where $S_M = \partial B_M$.

From here, since q is real we have that

$$(2.42) \quad \Im \int_{S_M} \partial_r u \bar{u} = 0.$$

Writing the outgoing Sommerfeld condition given in (1.2) in the following equivalent form

$$\lim_{M \rightarrow \infty} \int_{S_M} |\partial_r u - iku|^2 = 0,$$

and using (2.42) we get

$$\lim_{M \rightarrow \infty} \int_{S_M} |u|^2 = 0.$$

From here, since u is a radiating solution of equation $(\Delta + k^2)u = 0$ in the exterior domain $\mathbb{R}^d/B_M$, we conclude using Rellich's lemma (see Lemma 2.11 in page 32 of [5]) that $u = 0$ out of B_M , and by a unique continuation argument (see [16]), $u(x) = 0$ for all $x \in \mathbb{R}^d$.

To prove existence of solution we will use Fredholm alternative. We introduce the operator T_k defined by $T_k u = R_k(qu)$. From (2.40), since $\text{supp } q \subset B_M$, we have that there exists $\delta > 1$ such that

$$(2.43) \quad \|T_k u\|_{W_{-\delta}^{2,2}} \leq C_{k,M} \|qu\|_{L^2(B_M)}.$$

On the other hand, from Sobolev embedding theorem we have that

$$(2.44) \quad W^{\beta,p}(B_M) \xhookrightarrow{\text{compact}} L^2(B_M),$$

whenever

$$\beta \geq 0, \quad p > 1, \quad \beta p < d, \quad 2 < \frac{dp}{d - \beta p}.$$

Using (2.43), (2.44) and (2.30), we have that $T_k : W_{-\delta}^{2,2} \rightarrow W_{-\delta}^{2,2}$ is a linear compact operator whenever $\alpha > 0$ and $\alpha > \frac{d}{2} - 2$. Thus, Fredholm alternative ensures the existence of solution of $(I - T_k)u = R_k(qe^{ik\theta_0 \cdot (\cdot)})$, since we have proved that any solution of $(I - T_k)u = 0$ in $\mathbb{R}^d$ satisfying the outgoing Sommerfeld radiation conditions must be the trivial solution, and $R_k(qe^{ik\theta_0 \cdot (\cdot)}) \in W_{-\delta}^{2,2}(\mathbb{R}^d)$. $\square$

We will also need to know the behaviour of the scattered solution $u_s(x, \theta_0, k)$ when k tends to zero (zero energy case). The following lemma, which can be found in [37], collects several results given in [13, 14, 15, 27].

LEMMA 2.12. ([37, Proposition 2.1]) *Let $q \in L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ for some $r > d/2$, be real valued and compactly supported, and let $u_s(x, \theta_0, k)$ be the solution of (1.6). For $\delta > 1$ and , we have that*

1. *If $d \geq 3$,*

$$(2.45) \quad \|u_s\|_{W_{-\delta}^{2,2}} = O(k^{-1}), \quad k \rightarrow 0.$$

2. *If $d = 2$,*

$$(2.46) \quad \|u_s\|_{W_{-\delta}^{2,2}} = O((k \log k)^{-1}), \quad k \rightarrow 0.$$

Proof of Proposition 2.2. Arguing as in the proof of Proposition 2.1, we split the norm to control into two pieces I and II , and we will just bound I , since II can be bounded in a similar way. In this case, making the change of variables given in (2.31), we write

$$\begin{aligned} I &= \int_{H_{\theta_0}} \langle \xi \rangle^{2\alpha} |\widehat{q_m^r}(\xi)|^2 d\xi \\ &\leq C \int_0^{\frac{1}{2}} k^{d-1} \int_{S^{d-1}} |\widehat{q_m^r}(k(\theta - \theta_0))|^2 d\sigma(\theta) dk \\ &\quad + C \int_{\frac{1}{2}}^\infty k^{d-1+2\alpha} \int_{S^{d-1}} |\theta - \theta_0|^{2\alpha} |\widehat{q_m^r}(k(\theta - \theta_0))|^2 d\sigma(\theta) dk \\ &= C(I_1 + I_2), \end{aligned}$$

377 whenever $0 \leq \alpha \leq 1$.

To estimate I_1 , from the definition of q_m^r given in (1.16), using Cauchy-Schwarz inequality, since q is compactly supported, we get

$$|\widehat{q_m^r}(k(\theta - \theta_0))| \leq \|(qR_k)^m(qu_s)\|_{L^1} \leq C \|(qR_k)^m(qu_s)\|_{L_\delta^2}.$$

378 We can bound the L_δ^2 -norm that appears in the last inequality using (2.28) and (2.20) as the following
379 diagram illustrates

$$380 \quad W_{-\delta}^{2,2} \xrightarrow{\text{Zolesio}} L_\delta^2 \dots \xrightarrow{(m-1)\text{-times}} L_\delta^2 \xrightarrow{R_k} W_{-\delta}^{2,2} \xrightarrow{\text{Zolesio}} L_\delta^2.$$

Therefore, if $\delta > 1$ and α satisfies (2.26) or (2.27) we get

$$\|(qR_k)^m(qu_s)\|_{L_\delta^2} \leq C^m \|q\|_{W_{\alpha,2}^{m+1}}^{m+1} \|u_s\|_{W_{-\delta}^{2,2}},$$

thus,

$$I_1 \leq C^{2m} \|q\|_{W_{\alpha,2}^{2(m+1)}}^2 \int_0^{\frac{1}{2}} k^{d-1} \|u_s(\cdot, \theta_0, k)\|_{W_{-\delta}^{2,2}}^2 dk.$$

381 Since $q \in W^{\alpha,2}(\mathbb{R}^d)$, Sobolev embedding theorem guarantees that $q \in L^2(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ with $r \in \left(\frac{d}{2}, \frac{2d}{d-2\alpha}\right]$
382 whenever $\alpha \geq 0$ and $\frac{d}{2} - 2 \leq \alpha \leq \frac{d}{2}$, so we can use estimates (2.45) and (2.46) in the previous inequality to
383 obtain

$$384 \quad (2.48) \quad I_1 \leq C^{2m} \|q\|_{W_{\alpha,2}^{2(m+1)}}^2.$$

385 On the other hand, denoting $w = |\theta - \theta_0|^\alpha$, we can write

$$386 \quad (2.49) \quad I_2 = \int_{\frac{1}{2}}^\infty k^{d-1+2\alpha} \|w \widehat{q_m^r}(k(\theta - \theta_0))\|_{L^2(S^{d-1})}^2 dk.$$

387 Arguing as in the proof of Proposition 2.1, to control I_2 we write q_m^r in terms of the operators $R_{k,\theta}$ and S_{k,θ_0}
388 introduced in (2.21) and (2.23) respectively,

$$389 \quad \widehat{q_m^r}(k(\theta - \theta_0)) = S_{k,\theta_0} \left((qR_{k,\theta_0})^m \left(e^{-ik\theta_0 \cdot (\cdot)} qu_s \right) \right) (\theta).$$

390 From here, using (2.25), (2.29) and (2.22) as the following diagram illustrates

$$391 \quad W_\delta^{\alpha,t_1} \dots \xrightarrow{(m-1)\text{times}} W_\delta^{\alpha,t_m} \xrightarrow{R_{k,\theta_0}} W_{-\delta}^{\alpha,r_m} \xrightarrow{\text{Zolesio}} W_\delta^{\alpha,t_{m+1}} \xrightarrow{S_{k,\theta_0}} L^2(S^{d-1}),$$

392 where $0 \leq \frac{1}{t_1} - \frac{1}{2} \leq \frac{1}{d+1}$ and, r_ℓ and $t_{\ell+1}$ satisfy (2.34) and (2.35) for $\ell = 1, 2 \dots m$, we get

$$393 \quad (2.50) \quad \|w \widehat{q_m^r}(k(\theta - \theta_0))\|_{L^2(S^{d-1})}^2 \leq C^{2m} k^a \|q\|_{W_{\alpha,2}^{2m}}^{2m} \left\| e^{-ik\theta_0 \cdot (\cdot)} qu_s \right\|_{W_\delta^{\alpha,t_1}}^2,$$

394 with

$$395 \quad a = (d-1) \left(\frac{1}{t_{m+1}} - \frac{3}{2} \right) - 2\alpha + \sum_{\ell=1}^m \left(-2 + (d-1) \left(\frac{1}{t_\ell} - \frac{1}{r_\ell} \right) \right).$$

396 We can control the norm on the right hand side of (2.50) multiplying (1.6) by $q(x)e^{-ik\theta_0 \cdot x}$, using the operator
397 R_{k,θ_0} , the triangular inequality and estimate (2.29) to write

$$398 \quad (2.51) \quad \left\| e^{-ik\theta_0 \cdot (\cdot)} qu_s \right\|_{W_\delta^{\alpha,t_1}} \leq C \|q\|_{W_{\alpha,2}} \left(\|R_{k,\theta_0} q\|_{W_{-\delta}^{\alpha,r_0}} + \|R_{k,\theta_0} (e^{-ik\theta_0 \cdot (\cdot)} qu_s)\|_{W_{-\delta}^{\alpha,r_0}} \right),$$

whenever

$$\alpha \geq 0, \quad t_1 < \min(2, r_0) \quad \text{and} \quad 0 \leq \frac{1}{2} + \frac{1}{r_0} - \frac{1}{t_1} \leq \frac{\alpha}{d}.$$

Using (2.22) we get

$$\left\| R_{k, \theta_0}(e^{-ik\theta_0(\cdot)} qu_s) \right\|_{W_{-\delta}^{\alpha, r_0}} \leq k^b C \left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}},$$

with

$$b = -1 + \frac{(d-1)}{2} \left(\frac{1}{t_1} - \frac{1}{r_0} \right),$$

whenever

$$(2.52) \quad \alpha \geq 0, \quad 0 \leq \frac{1}{t_1} - \frac{1}{2} \leq \frac{1}{d+1} \quad \text{and} \quad 0 \leq \frac{1}{2} - \frac{1}{r_0} \leq \frac{1}{d+1},$$

From here, using (2.6), since $b < 0$, then for $k \geq 1/2$, we have that

$$(2.53) \quad \begin{aligned} \|q\|_{W^{\alpha, 2}} \left\| R_{k, \theta_0}(e^{-ik\theta_0(\cdot)} qu_s) \right\|_{W_{-\delta}^{\alpha, r_0}} &\leq \frac{A}{2^b} C \left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}} \\ &< \frac{1}{2} \left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}} \end{aligned}$$

whenever $AC \leq 2^{b-1}$.

Using (2.53) in (2.51) we obtain

$$\left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}} \leq 2C \|q\|_{W^{\alpha, 2}} \|R_{k, \theta_0} q\|_{W_{-\delta}^{\alpha, r_0}}.$$

From here, since $0 \leq \frac{1}{2} - \frac{1}{r_0} \leq \frac{1}{d+1}$ (see (2.52)) we can use (2.22), to get

$$\left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}} \leq C k^{-1 + \frac{(d-1)}{2} \left(\frac{1}{2} - \frac{1}{r_0} \right)} \|q\|_{W^{\alpha, 2}} \|q\|_{W_{\delta}^{\alpha, 2}}.$$

And since q has compact support, we can use (2.29) to obtain

$$\left\| e^{-ik\theta_0(\cdot)} qu_s \right\|_{W_{\delta}^{\alpha, t_1}} \leq C k^{-1 + \frac{(d-1)}{2} \left(\frac{1}{2} - \frac{1}{r_0} \right)} \|q\|_{W^{\alpha, 2}}^2.$$

Using this inequality and (2.50) in (2.49) we get

$$I_2 \leq C^{2m} \|q\|_{W^{\alpha, 2}}^{2(m+2)} \int_{\frac{1}{2}}^{\infty} k^{-2(m+1) - (d-1) \sum_{\ell=0}^m \left(\frac{1}{r_{\ell}} - \frac{1}{t_{\ell+1}} \right)} dk,$$

for any r_{ℓ} and $t_{\ell+1}$ satisfy (2.34) and (2.35) for $\ell = 0, 1 \dots m$.

If we choose r_{ℓ} and $t_{\ell+1}$ satisfying (2.36), the last integral is convergent if and only if

$$2(m+1) + (d-1)(m+1) \left(\frac{\alpha}{d} - \frac{1}{2} \right) > 1,$$

or equivalently

$$\alpha > \frac{d}{2} - \frac{(2m+1)d}{(m+1)(d-1)}.$$

Therefore, for any $m \in \mathbb{N}$, we have that

$$(2.54) \quad I_2 \leq C^{2m} \|q\|_{W^{\alpha,2}}^{2(m+1)}$$

whenever

$$0 \leq \alpha < d/2 \quad \text{and} \quad \alpha > \frac{d}{2} - \frac{2d}{d-1}.$$

Finally, the result follows from (2.47), (2.48) and (2.54) for α satisfying (2.5). $\square$

3. Numerical experiments. In this section we show two numerical experiments in dimension 2 that illustrate the efficiency of the proposed algorithm. We follow the discretization in [2], based on a trigonometric collocation method introduced by G. Vainikko in [43]. Note that the numerical version of the sequence (1.20) only requires the numerical approximation of some integral equations involving powers of the resolvent R_k (see the right hand side of (1.15)-(1.16)) and the inverse Fourier transform. Both ingredients are described in [2] and therefore the adaptation to the new algorithm defined here is straightforward. We refer to [2] for implementation details.

We have considered two different examples corresponding to a piecewise constant potential (Example 1) and a smooth one (Example 2). Note that the piecewise constant potential in Example 1 does not satisfy the regularity condition in Theorem 1.1, but nevertheless the algorithm provides good results. This is somehow a numerical evidence that the regularity hypotheses are probably not sharp in Theorem 1.1.

To compute the scattering data, i.e. the far field pattern, we have implemented the numerical algorithm described in [43], with a mesh twice finer than the mesh used to solve the inverse problem. In this way we try to simulate real data to recover the potential. Given a potential q , we first approximate the solution u_s of the Lippman-Schwinger equation (1.6) by trigonometric collocation, and then u_∞ in (1.7) by quadrature. A convergence result for this numerical approximation can be found in [43] and [40].

In the experiments below we have considered a computational domain $[-2.1, 2.1] \times [-2.1, 2.1]$ with N grid points in each variable uniformly distributed. The potential is supported in the region $[-1, 1] \times [-1, 1]$.

3.1. Example 1. In the first example the potential is a piecewise constant function, given by

$$(3.1) \quad q(x_1, x_2) = \begin{cases} 1.2, & \text{if } |x_1| + |x_2| < 0.3, \\ 1, & \text{if } 0.7 < |(x_1, x_2)| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

The error of the approximation is computed using the discrete L^2 -norm of the difference between the numerical approximation of q and the projection of the continuous functional in the mesh.

In Figure 2 we show the behaviour of the error for the discrete approximations of $q_{m,\ell}$ in log-scale when considering a mesh with 32×32 points. Each line corresponds to $q_{m,\ell}$ for fixed m and represents the error (Y-axis) with respect to ℓ (X-axis). Thus, the first line (in blue) illustrates the error of $q_{1,\ell}$. We see that it decreases as ℓ grows and it attains the lower error for $\ell = 3$.

The second line (in red) corresponds to $q_{2,\ell}$. It becomes stable for $\ell = 4$. The other lines correspond to $q_{3,\ell}$ and $q_{4,\ell}$. We see that they exhibit the same behaviour as $q_{2,\ell}$. Thus, we deduce that for $m = 2$ and $\ell = 3$ we almost attain the minimal error.

In Figure 3 we illustrate the same as in Figure 2 but this time with a mesh grid containing 128×128 points. The behaviour is almost the same but the error becomes smaller.

In Figure 4 we show a section at $x_2 = 0$, of the graph of the potential given in Example 1 and its approximation $q_{4,7}$ considering a mesh with $N = 128$.

Finally, in Figure 5 we illustrate the behaviour of the error as we consider finer meshes. To this end, we take $q_{4,7}$ as the best approximation for each mesh grid, since we have seen that larger values of m and ℓ do not improve the error significantly. Then, we compare the error as the mesh becomes finer. This illustrates the convergence of the approximations as N goes to infinity.

FIG. 2. Error in log-scale for Example 1 with a mesh of 32×32 points. FIG. 3. Error in log-scale for Example 1 with a mesh of 128×128 points.

3.2. Example 2. Here we consider a smooth potential given by

$$q(x_1, x_2) = \max(0, e^{-5|x_1-0.5|^2}) + 1.5e^{-4|(x_1, x_2)-(-1, 0.8)/2|^2} + 2e^{-7|(x_1, x_2)-0.4(-1, -1)|^2-0.4}.$$

The numerical results for this example are completely similar to those of the previous one, but the errors are significantly smaller. This is due to the regularity of the potential q that we recover. We illustrate this in Figure 6, which is the analogous to Figure 5. Note that in this case we obtain errors of $10^{-4.4}$ instead of $10^{-1.8}$ of the previous example.

One of the hypothesis in Theorem 1.1 is that the size of the potential in the $W^{\alpha, 2}$ norm, with $\alpha \in (0, 1)$, must be small (see Remark 1.2 above). A precise estimate of this size is not easy since it involves non-explicit

FIG. 4. Section at $x_2 = 0$ of the graph of q in Example 1 and its approximation $q_{4,7}$. FIG. 5. Error of the approximation $q_{4,7}$ in Example 1 for different meshes.

467 constants that depend on certain bounds of the resolvent of the Hemholtz equation. In this second example
 468 we can give a numerical estimate of this norm for this particular potential, that we compute approximating

$$\|q\|_{W^{\alpha,2}} = \left(\int_{\mathbb{R}^2} (1 + |\xi|^2)^\alpha |\widehat{q}(\xi)|^2 d\xi \right)^{1/2}$$

with the discrete Fourier Transform. We obtain,

$$1.12 \sim \|q\|_{L^2} = \|q\|_{W^{0,2}} \leq \|q\|_{W^{\alpha,2}} \leq \|q\|_{W^{1,2}} \sim 2.73, \quad 0 < \alpha < 1.$$

469 Therefore, the norm of the potential is not particularly small in this example.

FIG. 6. Error of the approximation $q_{4,7}$ in Example 2 for different meshes.

We have tried the algorithm with potentials of the form $\lambda q(x)$ with $\lambda > 1$ to understand how the algorithm works for potentials with larger norm. As λ grows, the algorithm requires larger values of ℓ and m to attain a stable error (as in Figure 3 for the Example 1) and the convergence rate is smaller, as N grows. On the other hand, the larger value of λ for which the algorithm converges is around $\lambda = 2$.

Finally, it is worth mentioning that the numerical approximations obtained with the algorithm introduced here are similar to those obtained by the iterative one in [2], at least in the experiments that we have considered. The main point is that the computational cost is reduced drastically for large N since, as we said in the introduction, we do not have to solve a Lipmann-Schwinger equation for each point in the mesh, and for each iteration.

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