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ON THE USE OF QUASIORDERS IN FORMAL LANGUAGE THEORY

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ABSTRACT OF THE DISSERTATION

In this thesis we use *quasiorders* on words to offer a new perspective on two well-studied problems from *Formal Language Theory*: deciding language inclusion and manipulating the finite automata representations of regular languages.

First, we present a generic quasiorder-based framework that, when instantiated with different quasiorders, yields different algorithms (some of them new) for deciding *language inclusion*. We then instantiate this framework to devise an efficient algorithm for *searching with regular expressions on grammar-compressed text*. Finally, define a framework of quasiorder-based automata constructions to offer a new perspective on *residual automata*.

The Language Inclusion Problem

First, we study the *language inclusion problem* $L_1 \subseteq L_2$ where L_1 is regular or context-free and L_2 is regular. Our approach relies on checking whether an over-approximation of L_1 , obtained by successively over-approximating the Kleene iterates of its least fixpoint characterization, is included in L_2 . We show that a language inclusion problem is decidable whenever the over-approximating function satisfies a completeness condition (i.e. its loss of precision causes no false alarm) and prevents infinite ascending chains (i.e. it guarantees termination of least fixpoint computations).

Such over-approximation of L_1 can be defined using *quasiorder* relations on words where the over-approximation gives the language of all words “greater than or equal to” a given input word for that quasiorder. We put forward a range of quasiorders that allow us to systematically design decision procedures for dif-

ferent language inclusion problems such as regular languages into regular languages or into trace sets of one-counter nets and context-free languages into regular languages.

Some of the obtained inclusion checking procedures correspond to well-known algorithms like the so-called *antichains* algorithms. On the other hand, our quasiorder-based framework allows us to derive an equivalent greatest fix-point language inclusion check which relies on quotients of languages and which, to the best of our knowledge, was *not previously known*.

Searching on Compressed Text

Secondly, we instantiate our quasiorder-based framework for the scenario in which L_1 consists on a single word generated by a context-free grammar and L_2 is the regular language generated by an automaton. The resulting algorithm can be used for deciding whether a grammar-compressed text contains a match for a regular expression.

We then extend this algorithm in order to count the number of lines in the uncompressed text that contain a match for the regular expression. We show that this extension runs in time *linear* in the size of the *compressed* data, which might be exponentially smaller than the uncompressed text.

Furthermore, we propose efficient data structures that yield *optimal* complexity bounds and an implementation –*zsearch*– that outperforms the state of the art, offering up to 40% speedup with respect to *highly optimized* implementations of the decompress and search approach.

Residual Finite-State Automata

Finally, we present a framework of finite-state automata constructions based on quasiorders over words to provide new insights on residual finite-state automata (RFA for short).

We present a new residualization operation and show that the residual equivalent of the double-reversal method holds, i.e. our residualization operation applied to a co-residual automaton generating the language L yields the canonical RFA for L . We then present a generalization of the double-reversal method for RFAs along the lines of the one for deterministic automata.

Moreover, we use our quasiorder-based framework to offer a new perspective on NL^* , an on-line learning algorithm for RFAs.

We conclude that *quasiorders* are fundamental to *residual automata* in the same way *congruences* are fundamental for *deterministic automata*.

RESUMEN DE LA TESIS DOCTORAL

En esta tesis, usamos *preórdenes* para dar un nuevo enfoque a dos problemas fundamentales en *Teoría de Lenguajes Formales*: decidir la inclusión entre lenguajes y manipular la representación de lenguajes regulares como autómatas finitos.

En primer lugar, presentamos un esquema que, dado un preorden que satisface ciertos requisitos, nos permite derivar de manera sistemática algoritmos de decisión para la inclusión entre diferentes tipos de lenguajes. Partiendo de este esquema desarrollamos un algoritmo de búsqueda con expresiones regulares en textos comprimidos mediante gramáticas. Por último, presentamos una serie de autómatas, cuya definición depende de un preorden, que nos permite ofrecer un nuevo enfoque sobre la clase de autómatas residuales.

El Problema de la Inclusión de Lenguajes

En primer lugar, estudiamos el problema de decidir $L_1 \subseteq L_2$, donde L_1 es un lenguaje independiente de contexto y L_2 es un lenguaje regular. Para resolver este problema, sobre-aproximamos los sucesivos pasos de la iteración de punto fijo que define el lenguaje L_1 . Con ello, obtenemos una sobre-aproximación de L_1 y comprobamos si está incluida en el lenguaje L_2 . Esta técnica funciona siempre y cuando la sobre-aproximación sea completa (es decir, la imprecisión de la aproximación no produzca falsas alarmas) y evite cadenas infinitas ascendentes (es decir, garantice que la iteración de punto fijo termina).

Para definir una sobre-aproximación que cumple estas condiciones, usamos un preorden. De este modo, la aproximación del lenguaje L_1 contiene todas las palabras “mayores o iguales que” alguna palabra de L_1 . En concreto, definimos

una serie de preórdenes que nos permiten derivar, de manera sistemática, algoritmos de decisión para diferentes problemas de inclusión de lenguajes como la inclusión entre lenguajes regulares o la inclusión de lenguajes independientes de contexto en lenguajes regulares.

Algunos de los algoritmos obtenidos mediante esta técnica coinciden con algoritmos bien conocidos como los llamados *antichains algorithms*. Por otro lado, nuestra técnica también nos permite derivar algoritmos de punto fijo que, hasta donde sabemos, *no han sido descritos anteriormente*.

Búsqueda en textos comprimidos

En segundo lugar, aplicamos nuestro algoritmo de decisión de inclusión entre lenguajes al problema $L_1 \subseteq L_2$, donde L_1 es un lenguaje descrito por una gramática que genera una única palabra y L_2 es un lenguaje regular definido por un autómata o expresión regular. De esta manera, obtenemos un algoritmo que nos permite decidir si un texto comprimido mediante una gramática contiene, o no, una coincidencia de una expresión regular dada.

Posteriormente, modificamos este algoritmo para contar las líneas del texto comprimido que contienen coincidencias de la expresión regular. De este modo, obtenemos un algoritmo que opera en tiempo *linear* respecto del tamaño del texto *comprimido* el cual, por definición, puede ser exponencialmente más pequeño que el texto original.

Además, describimos las estructuras de datos necesarias para que nuestro algoritmo opere en tiempo *óptimo* y presentamos una implementación *-zsearch-* que resulta hasta un 40% más rápida que las mejores implementaciones del método estándar de descompresión y búsqueda.

Autómatas Residuales

Finalmente presentamos una serie de autómatas parametrizados por preórdenes que nos permiten mejorar nuestra compresión de la clase de autómatas residuales (abreviados como RFA).

Estos autómatas parametrizados nos permiten definir una nueva operación de residualization y demostrar que el método de *double-reversal* funciona para RFAs, es decir, residualizar un autómata cuyo reverso es residual da lugar al canonical RFA (un RFA de tamaño mínimo). Tras esto, generalizamos este método de forma similar a su generalización para el caso de autómatas deterministas. Por último, damos un nuevo enfoque a NL^* , un algoritmo de aprendizaje de RFAs.

Como conclusión, encontramos que los *preórdenes* juegan el mismo papel para los *autómatas residuales* que las *congruencias* para los *deterministas*.

To my parents and my wife, for their endless love and support

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LIST OF PUBLICATIONS

This thesis comprises the following four papers for which I am the main author. The first two have been published in top peer-reviewed academic conferences while the last two have recently been submitted and have not been published yet:

1. Pedro Valero and Dr. Pierre Ganty
Regular Expression Search on Compressed Text
Published in *Data Compression Conference*, March 2019.
2. Pedro Valero, Dr. Pierre Ganty and Prof. Francesco Ranzato
Language Inclusion Algorithms as Complete Abstract Interpretations
Published in *Static Analysis Symposium*, October 2019.
3. Pedro Valero, Dr. Pierre Ganty and Elena Gutiérrez
A Quasiorder-based Perspective on Residual Automata
Submitted to the *International Symposium on Mathematical Foundations of Computer Science*, May 2020.
4. Pedro Valero, Dr. Pierre Ganty and Prof. Francesco Ranzato
Complete Abstractions for Checking Language Inclusion
Submitted to *Transactions on Computational Logic*, May 2020.

Using the techniques presented in first of the above mentioned papers, I developed a tool for searching with regular expressions in compressed text. The implementation is available on-line at <https://github.com/pevalme/zearch>.

I have also contributed to the following papers which are not part of this thesis.

1. Elena Gutiérrez, Pedro Valero and Dr. Pierre Ganty
A Congruence-based Perspective on Automata Minimization Algorithms
Published in *International Symposium on Mathematical Foundations of Computer Science*, August 2019
2. Pedro Valero, Dr. Pierre Ganty and Boris Köpf
A Language-theoretic View on Network Protocols
Published in *Automated Technology for Verification and Analysis*, October 2017

INTRODUCTION

Formal languages, i.e. languages for which we have a *finite formal description*, are used to model possibly infinite sets so that their finite descriptions can be used to reason about these sets. As a consequence, *Formal Language Theory*, i.e. the study of formal languages and the techniques for manipulating their finite representations, finds applications in several domains in computer science.

For example, the possibly infinite set of assignments that satisfy a given formula in some logic can be seen as a formal language whose finite description is the formula itself. In some logics, the set of values that satisfy any formula is regular and, therefore, it can be described by means of a finite-state automaton (automaton for short). When this is the case, it is possible to reason in that logic by manipulating automata as shown in Example 1.1.

Example 1.1. Consider the formulas $f_2 : "x \bmod 2 = 0"$ and $f_4 : "x \bmod 4 = 0"$. Next we show how to reason about the formula $f_{42} : "f_4 \wedge f_2"$ by means of automata.

A binary sequence “ x ” encodes a number divisible by 4 iff the last two digits are 0’s. Similarly, “ x ” encodes a number divisible by 2 iff the last digit is 0. Therefore, the automata A_4 and A_2 from Figure 1.1 accept the binary encodings of numbers “ x ” that satisfy the formulas $f_4 : "x \bmod 4 = 0"$ and $f_2 : "x \bmod 2 = 0"$, respectively.

Since the numbers satisfying the formula f_{42} are, by definition, the ones satisfying both f_4 and f_2 , the automaton for f_{42} is $A_{42} = A_2 \times A_4$, shown in Figure 1.1, which recognizes exactly the encodings accepted by both A_2 and A_4 . Therefore, there exists a number satisfying f_{42} iff the language accepted by A_{42} is not empty.

On the other hand, since the automaton A_4 accepts a language that is included in the one of A_2 , we conclude that the encodings satisfying f_4 also satisfy f_2 . Therefore, the automaton for A_4 is equivalent to, i.e. it accepts the same language as, the automaton A_{42} and both are automata for f_{42} . $\diamond$

Figure 1.1: Automata accepting the set of binary encodings of numbers divisible by 4 (top left), divisible by two (top right) and the product of these two automata (bottom).

This idea led to the development of *automata-based decision procedures* for logical theories such as Presburger arithmetic [Wolper and Boigelot 1995] and the Weak Second-order theory of One or Two Successors (WS1S/WS2S) [Henriksen et al. 1995; Klarlund 1999] among others [Allouche et al. 2003; Schaeffer 2013].

A similar idea is used in *regular model checking* [To and Libkin 2008; Abdulla 2012; Clarke et al. 2018], where formal languages are used to describe the possibly infinite sets of states that a system might reach during its execution.

A different use of formal languages in computer science is the *lossless compression of textual data* [Charikar et al. 2005; Hucke et al. 2016]. In this scenario the data is seen as a language consisting of a single word and its finite formal description as a grammar is seen as a *succinct representation* of the language it generates. As the following example evidences, the grammar might be exponentially smaller than the data.

Example 1.2. Let k be an integer greater than 1 and let $\mathcal{G}$ be the grammar with the set of variables $\{X_i \mid 0 \leq i \leq k\}$, alphabet $\{a\}$, axiom X_k and the set of rules $\{X_i \rightarrow X_{i-1}X_{i-1} \mid 1 \leq i \leq k\} \cup \{X_0 \rightarrow a\}$.

Clearly, $\mathcal{G}$ has size linear in k and produces the word a^{2^k} . Therefore, the grammar is exponentially smaller than the word it generates. $\diamond$

The idea of using grammars to compress textual data has led to the development of several grammar-based compression algorithms [Ziv and Lempel 1978; Nevill-Manning and Witten 1997; Larsson and Moffat 1999]. These algorithms offer some advantages with respect to other classes of compression techniques, such as the ones based on the well-known LZ77 algorithm [Ziv and Lempel 1977],

in terms of the structure of the compressed representation of the data (which is a grammar). In particular, they allow us to analyze the uncompressed text, i.e. the language, by looking at the compressed data, i.e. the grammar [Lohrey 2012].

1.1 The Contributions of This Dissertation

In this dissertation we focus on three problems from *Formal Language Theory*: deciding language inclusion, searching on grammar-compressed text and building residual automata. As we describe next, these are well-studied and important problems in computer science for which there are still challenges to overcome.

The Language Inclusion Problem

In the first two scenarios described before, i.e. *automata-based decision procedures* and *regular model checking*, the *language inclusion problem*, i.e. deciding whether the language inclusion $L_1 \subseteq L_2$ holds, is a fundamental operation.

For instance, in Example 1.1, deciding the language inclusion between the languages generated by automata A_4 and A_2 allows us to infer that all values satisfying f_4 also satisfy f_2 . Similarly, in the context of regular model checking, we can define a possibly infinite set of “good” states that the system should never leave and solve a language inclusion problem to decide whether the system is confined to the set of good states.

As a consequence, the *language inclusion problem* is a fundamental and classical problem in computer science [Hopcroft and Ullman 1979, Chapter 11]. In particular, language inclusion problems of the form $L_1 \subseteq L_2$, where both L_1 and L_2 are regular languages, appear naturally in different scenarios as the ones previously described.

The standard approach for solving such problems consists on reducing them to *emptiness* problems using the fact that $L_1 \subseteq L_2 \Leftrightarrow L_1 \cap L_2^c = \emptyset$. However, algorithms implementing this approach suffer from a worst case exponential blowup when computing L_2^c since it requires determinizing the automaton for L_2 . The state of the art approach to overcome this limitation is to keep the computation of the automaton for L_2^c implicit, thus preventing the exponential blowup for many instances of the problem.

For instance, Wulf et al. [2006] developed an algorithm for deciding language inclusion between regular languages that uses *antichains*, i.e. sets of incomparable elements, to reduce the blowup resulting from building the complement of a given automaton. Their work was later improved by Abdulla et al. [2010] and Bonchi and Pous [2013] who used *simulations* between the states of the automata to further reduce the blowup associated to the complementation step. Then,

Holík and Meyer [2015] adapted the use of antichains to decide the inclusion of context-free languages into regular ones.

However, even though these algorithms have a common foundation, i.e. they all reduce the language inclusion problem to an emptiness one through complementation and use antichains to keep the complementation implicit, the relation between them is not well understood. This is evidenced by the fact that the generalization by Holík and Meyer [2015] of the antichains algorithm of Wulf et al. [2006] was obtained by rephrasing the inclusion problem as a data flow analysis problem over a relational domain.

Our Contribution. We use *quasiorders*, i.e. reflexive and transitive relations, to define a framework from which we systematically derive algorithms for deciding language inclusion such as the ones of Wulf et al. [2006] and Holík and Meyer [2015]. Indeed, we show that these two algorithms are conceptually equivalent and correspond to two instantiations of our framework using different quasiorders. Moreover, by using a quasiorder based on simulations between the states of an automata, we derive an improved antichains algorithm that partially matches the one of Abdulla et al. [2010].

Furthermore, our framework goes beyond inclusion into regular languages and allows us to derive an algorithm for deciding the language inclusion $L_1 \subseteq L_2$ when L_1 is regular and L_2 is the set of traces of a *one counter net*, i.e. an automaton equipped with a counter that cannot test for 0. Finally, we also derive a *novel* algorithm for deciding inclusion between regular languages.

Searching on Compressed Text

The growing amount of information handled by modern systems demands for efficient techniques both for compression, to reduce the storage cost, and for regular expression searching, to speed up querying.

Therefore, the problem of searching on compressed text is of practical interest as evidenced by the fact that state of the art tools for searching with regular expressions, such as `grep`¹ and `ripgrep`², provide a method for searching on compressed files by decompressing them on-the-fly.

Due to the high performance of state of the art compressors such as `zstd`³ and `lz4`⁴, the performance of searching on the decompressed data as it is recovered by the decompressor is comparable with that of searching on the un-

¹<https://www.gnu.org/software/grep/manual/grep.html>.

²<https://github.com/BurntSushi/ripgrep>.

³<https://github.com/facebook/zstd>

⁴<https://github.com/lz4/lz4>

compressed data. Therefore, the parallel decompress-and-search approach is the state of the art for searching on compressed text.

However, when using a grammar-based compression technique it is possible to manipulate the compressed data, i.e. the grammar, to analyze the uncompressed data, i.e. the language generated by the grammar. Intuitively, this means that the information about repetitions in the text present in its compressed version can be used to enhance the search. Therefore, *searching on grammar-compressed text* could be even faster than searching on the uncompressed text.

This idea is exploited by multiple algorithms that perform certain operations directly on grammar-compressed text, i.e. without having to recover the uncompressed data, such as finding given words [Navarro and Tarhio 2005], finding words that match a given regular expression [Navarro 2003; Bille et al. 2009] or finding approximate matches [Navarro 2001].

Nevertheless, the implementations of Navarro [2003] and Navarro and Tarhio [2005] (to the best of our knowledge, the only existing tools for searching on compressed text) are not faster than the state of the art decompress and search approach. Partly, this due to the fact that these algorithms only apply to data compressed with one specific grammar-based compressor, namely LZ78 [Ziv and Lempel 1978], which, as shown by Hucke et al. [2016], cannot achieve exponential compression ratios⁵.

Our Contribution. We improve this situation by rephrasing the problem of searching on compressed text as a language inclusion problem between a context-free language (the text) and a regular one (the expression). Then, we instantiate our quasiorder-based framework for solving language inclusion and adapt it to the specifics of this scenario, where the context-free grammar generates a single word: the uncompressed text. The resulting algorithm is not restricted to any class of grammar-based compressors and it reports the number of lines in the text containing a match for a given expression in time *linear* with respect to the size of the compressed data.

We implement this algorithm in a tool `-zearch`⁶ – for searching with regular expressions in grammar-compressed text. The experiments evidence that compression can be used to enhance the search and, therefore, the performance of `zearch` improves with the compression ratio of the data. Indeed, our tool is as fast as searching on the uncompressed data when the data is well-compressible, i.e. it results in compression ratio above 13, which occurs, for instance, when considering automatically generated *log files*.

⁵The compression ratio for a file of size T compressed into size t is T/t .

⁶<https://github.com/pevalme/zearch>

Building Residual Automata

Clearly, the problem of finding a concise representation of a regular language is also a fundamental problem in computer science.

There exists two main classes of automata representations for regular languages, both having the same expressive power: non-deterministic (NFA for short) and deterministic (DFA for short) automata. While DFAs are simpler to manipulate than NFAs⁷ they are, in the worst case, exponentially larger.

Example 1.3. *The minimal DFA for the set of words of length $2n+2$ with two 1's separated by n symbols has size exponential in n since any DFA for that language must have one state for each of the 2^n possible prefixes of length n . Figure 1.2 shows the minimal DFA and an exponentially smaller NFA for the case $n = 2$.*

Figure 1.2: Minimal DFA (left) and NFA (right) accepting the words in the alphabet $\{0, 1\}$ of length 6 that contains two 1's separated by two symbols. For clarity, we use colors red, blue and black for transitions with labels “0”, “1” and “0,1”, respectively.

Therefore, algorithms relying on determinized automata, such as the standard algorithm for building the complement of an NFA, do not scale despite the existence of different techniques for reducing the size of DFAs [Hopcroft 1971; Moore 1956] and for building DFAs of minimal size [Sakarovitch 2009; Adámek et al. 2012; Brzozowski and Tamm 2014].

This has led to the introduction of *residual automata* [Denis et al. 2001; 2002] (RFA for short) as a generalization of DFAs that breaks determinism in favor of conciseness of the representation. Therefore, RFAs are easier to manipulate

⁷For instance, in order to build the complement of a DFA it suffices to switch final and non-final states while complementing an NFA requires determinizing it.

than NFAs (there exists a canonical minimal RFA for every regular language, which makes learning easier) and more concise than DFAs (both automata from Figure 1.2 are RFAs). These properties make RFAs specially appealing in certain domains such as Grammatical Inference [Denis et al. 2004; Bollig et al. 2009].

There exists a clear relationship between RFAs and DFAs as evidenced by the similarities between the *residualization* and *determinization* operations and the fact that a straightforward modification of the double-reversal method for building minimal DFAs yields a method for building minimal RFAs. However, the connection between these two formalisms is not fully understood as evidenced by the fact that the relation between the generalization of the double-reversal methods for DFAs [Brzozowski and Tamm 2014] and RFAs [Tamm 2015] is not immediate.

Our Contribution. We present a framework of quasiorder-based automata constructions that yield residual and co-residual automata. We find that one of these constructions defines a residualization operation that produces smaller automata than the one of Denis et al. [2002] and for which the double-reversal method holds: residualizing a co-residual automaton yields the canonical RFA. Moreover, we derive a generalization of this double-reversal method for RFAs, along the lines of the one of Brzozowski and Tamm [2014] for DFAs that is more general than the one of Tamm [2015].

Incidentally, we also evidence the connection between the generalized double-reversal method for RFAs of Tamm [2015] and the one of Brzozowski and Tamm [2014] for DFAs. Finally, we offer a new perspective of the NL^* algorithm of Bollig et al. [2009] for learning RFAs as an algorithm that iteratively refines a quasiorder and uses our automata constructions to build RFAs.

1.2 Methodology

The contributions of this thesis, described in the previous section, are the result of using *monotone well-quasiorders*, i.e. quasiorders that satisfy certain properties with respect to concatenation of words and for which there is no infinite decreasing sequence of elements, as building blocks for tackling problems from *Formal Language Theory*.

Monotone well-quasiorders have proven useful for reasoning about formal languages from a theoretical perspective (see the survey of D'Alessandro and Varricchio [2008]). For instance, Ehrenfeucht et al. [1983] showed that a language is regular iff it is closed for a monotone well-quasiorder and de Luca and Varricchio [1994] extended this result by showing that a language is regular iff it is closed for a left monotone and for a right monotone well-quasiorders. On the

other hand, Kunc [2005] used well-quasiorders to show that all maximal solutions of certain systems of inequalities on languages are regular.

Our work evidences that monotone well-quasiorders also have practical applications by placing them at the core of some well-known algorithms.

Monotone Well-Quasiorders

Quasiorders are binary relations that are *reflexive*, i.e. every word is related to itself, and *transitive*, i.e. if a word “u” is related to “v” which is related to “w” then “u” is related to “w”.

Intuitively, we use quasiorders to group words that behave “similarly” (in a certain way) with respect to a given regular language. This naturally leads to the use of *monotone quasiorders* so that “similarity” between words is preserved by concatenation, i.e. when concatenating two “similar” words with the same letter the resulting words remain “similar”.

Example 1.4. Consider the length quasiorder, which says that “u” is related to “v” iff $|u| \leq |v|$ where $|u|$ denotes the length of a word “u”.

It is straightforward to check that this is a monotone quasiorder since

- (i) $|u| \leq |u|$ for every word u , hence it is reflexive;
- (ii) if $|u| \leq |v|$ and $|v| \leq |w|$ then $|u| \leq |w|$, hence it is transitive;
- (iii) if $|u| \leq |v|$ then $|ua| \leq |va|$ for every letter a , hence it is monotone. $\diamond$

The most basic sets of words that can be formed by using a quasiorder are the so called *principals*, i.e. sets of words that are related to a single one which we refer to as the *generating word* of the principal. For example, given the length quasiorder, the principal with generating word “u” is the set of all words “w” with $|u| \leq |w|$.

Finally, when considering *well-quasiorders* we find that the union of the principals of any (possibly infinite) set of words coincides with the union of the principals of a *finite* subset of words. For instance, the quasiorder from Example 1.4 is a monotone well-quasiorder since the union of the principals of any infinite set of words coincides with the principal of the shortest word in the set.

Next, we offer a high-level description on how we use *monotone well-quasiorders* and their induced principals in each of the contributions of this thesis.

1.2.1 Quasiorders for Deciding Language Inclusion

Consider the language inclusion problem $L_1 \subseteq L_2$ where L_1 is context-free and L_2 is regular. The principals of a given monotone well-quasiorder can be used to compute an *over-approximation* of L_1 that consists of a *finite* number of elements.

If the quasiorder is such that a principal is included in L_2 iff its generating word is in L_2 , then we can reduce the language inclusion problem $L_1 \subseteq L_2$ to the simpler problem of deciding a finite number of membership queries for L_2 . To do that it suffices to compute the over-approximation of L_1 and check membership in L_2 for the generating words of the finitely many principals that form the over-approximation. This approach is illustrated in Figure 1.3.

Given a monotone well-quasiorder whose principals are the dashed squares shown on the image on the left, we compute over-approximations (colored areas) of the languages L_1 and L_3 . Since L_2 is a union of principals, the over-approximation of a language is included in L_2 iff the language is included in L_2 . Therefore, we find that $L_1 \subseteq L_2$ but $L_3 \not\subseteq L_2$.

Figure 1.3: Illustration of our quasiorder-based approach for deciding the language inclusion problems $L_1 \subseteq L_2$ and $L_3 \subseteq L_2$.

In order to compute the over-approximation of L_1 we successively over-approximate the Kleene iterates of its least fixpoint characterization. The following example shows the language equations for a context-free language and the first steps of the Kleene iteration, which converges to the least fixpoint of the equations.

Example 1.5. Consider the language equations $\{X = aX \cup Ya \cup bY, Y = a\}$, whose Kleene iterates converge to their least fixpoint:

$$\begin{cases} X = \emptyset \\ Y = \emptyset \end{cases} \Rightarrow \begin{cases} X = \emptyset \\ Y = \{a\} \end{cases} \Rightarrow \begin{cases} X = \{aa, ba\} \\ Y = \{a\} \end{cases} \Rightarrow \dots \Rightarrow \begin{cases} X = a^*(aa \mid ba) \\ Y = \{a\} \end{cases} \quad \diamond$$

This approach for solving language inclusion problems is studied in Chapter 4. In that chapter we present a quasiorder-based framework which, by instantiating it with different monotone well-quasiorders, allows us to systematically derive well-known decision procedures for different language inclusion problems such as the antichains algorithms of Wulf et al. [2006] and Holík and Meyer [2015].

Moreover, by switching from least fixpoint equations for computing the over-approximation of L_1 to greatest fixpoint equations, we are able to obtain a *novel* algorithm for deciding language inclusion between regular languages.

1.2.2 Quasiorders for Searching on Compressed Text

Searching with a regular expression in a grammar-compressed text⁸ amounts to deciding whether the language generated by a grammar, which consists of a single word, is included in a regular language. Therefore, we can apply the quasiorder-based framework described in the previous section, i.e. we can compute an over-approximation of the language generated by the grammar and check inclusion of the over-approximation into the regular language.

However this approach would only indicate whether there is a subsequence in the text that matches the expression and it would not produce enough information to count the matches let alone recover them.

In order to report the exact lines⁹ that contain a match (either count them or recover the actual lines), we need to compute some extra information for each variable of the grammar, beyond the over-approximation of the generated language. Indeed, we need to compute the following information regarding the language generated by each variable, which consists of a single word¹⁰, w :

- (i) The number of lines that contain a match.
- (ii) Whether there is a “new line” symbol in w .
- (iii) Whether the prefix of w contains a match.
- (iv) Whether the suffix of w contains a match.

This quasiorder-based approach is presented in Chapter 5 where we show that the above mentioned extra information for each variable of the grammar is trivially computed for the terminals and then propagated through all the variables until the axiom. Furthermore, Chapter 5 includes a detailed description of the implementation and evaluation of the resulting algorithm which, as the experiments show, outperforms the state of the art.

1.2.3 Quasiorders for Building Residual Automata

It is well-known that the construction of the minimal DFA for a language is related to the use of *congruences*, i.e. symmetric monotone quasiorders [Büchi 1989; Khoussainov and Nerode 2001].

⁸By “searching” we mean finding subsequences of the uncompressed text that match a regular expression, i.e. that are included in a given regular language.

⁹We use the standard definition of *line* as a sequence of characters delimited by “new line” symbols.

¹⁰Recall that, in the context of grammar-based compression, the grammar is a compressed representation of a text, hence it generates a single word: the text. As a consequence, each variable of the grammar generates a single word.

Recently, Ganty et al. [2019] generalized this idea and offered a congruence-based perspective on minimization algorithms for DFAs. Intuitively, they build automata by using the principals induced by congruences as states and define the transitions according to inclusions between the principals and the sets obtained by concatenating them with letters. When the congruence has finite index then it induces a finite number of principals and, therefore, the resulting automata have finitely many states. Figure 1.4 illustrates this automata construction.

Figure 1.4: The image on the left shows the principals induced by a quasiorder. Each arrow of the form $\rho(x) \xrightarrow{a} \rho(y)$ indicates that $\rho(x)a \subseteq \rho(y)$. For clarity, we show on the right the automaton resulting from the relation between the principals.

Let $\rho(u)$ denote the principal for a word u . The monotonicity of congruences ensures that every set $\rho(u)a$ is included in a principal $\rho(v)$ and, since congruences are symmetric, the principals induced by a congruence are disjoint and, therefore, the resulting automata is deterministic. By switching from congruences to quasiorders we obtain possibly overlapping principals which enables non-determinism and allows us to obtain *residual automata* which, recall, are a generalization of DFAs. Clearly, the principals shown in Figure 1.4 correspond to a quasiorder rather than a congruence since they are not disjoint.

This quasiorder-based perspective on RFAs is presented in Chapter 6 where we define quasiorder-based automata constructions that yield RFAs or co-RFAs, depending on the properties of the input quasiorder. Moreover, given two comparable quasiorders, our automata construction instantiated with the coarser quasiorder yields a smaller automaton. This is to be expected since a coarser quasiorder induces fewer principals which, recall, are the states of the automata.

As a consequence, building the canonical minimal RFA for a given language amounts to instantiating our automata construction with the coarsest quasi-

order that satisfies certain requirements. Interestingly, building the minimal DFA amounts to instantiating the framework of [Ganty et al. \[2019\]](#) with the coarsest congruence that satisfies the same requirements. As we shall see in Chapter 6, the congruence and the quasiorder used for building the minimal DFA and RFA, respectively, are closely related.

We conclude that *monotone quasiorders* are fundamental for RFAs as *congruences* are fundamental for DFAs, which evidences the relationship between these two classes of automata.

STATE OF THE ART

In this dissertation, we present two quasiorder-based frameworks that allow us to systematically derive algorithms for solving different language inclusion problems and manipulating residual automata, respectively. Moreover, we show that our algorithms for deciding language inclusion can be adapted for searching on compressed text.

Our theoretical framework allows us to devise some novel algorithms and offer new insights on existing ones. Therefore, most of the works related to ours are briefly discussed within the following chapters, when explaining them within our quasiorder-based perspective. This is the case, specially, in Chapters 4 and 6.

However, we present in this chapter a detailed description of some previous works in order to provide an overview of the state of the art for these problems before writing this Ph.D. Thesis.

2.1 The Language Inclusion Problem

Consider the language inclusion problem $L_1 \subseteq L_2$. When the underlying representations of L_1 and L_2 are regular expressions, one can check language inclusion using some rewriting techniques [Antimirov 1995; Keil and Thiemann 2014], thus avoiding the translation of the regular expression into an equivalent automaton.

On the other hand, when the languages are given through finite automata, a well known and standard method to solve the language inclusion problem is to reduce it to a disjointness problem via the construction of the language complement: $L_1 \subseteq L_2$ iff $L_1 \cap L_2^c = \emptyset$. The bottleneck of this approach is the language complementation since it involves a determinization step which entails a worst case exponential blowup.

In order to alleviate this bottleneck, Wulf et al. [2006] put forward a breakthrough result where complementation was sidestepped by a lazy construction of the determinized NFA, which provided a huge performance gain in practice. Their algorithm, deemed the *antichains* algorithm, was subsequently enhanced with simulation relations by Abdulla et al. [2010]. The current state of the art for solving the language inclusion problem between regular languages is the bisimulation up-to approach proposed by Bonchi and Pous [2013], of which the antichains algorithm and their enhancement with simulations can be viewed as particular cases.

2.1.1 Antichains Algorithms

The *antichains* algorithm of Wulf et al. [2006] was originally designed as an algorithm for solving the universality problem for regular languages, i.e. deciding whether $\Sigma^* \subseteq L$ holds when L is regular.

Before the introduction of this algorithm, the standard approach for deciding universality of a regular language given its automaton was to determinize the automaton and check whether all states are final. The *antichains* algorithm improved this situation by keeping the determinization step implicit.

In their work, Wulf et al. [2006] also adapted their antichains algorithm for solving the language inclusion problem $L_1 \subseteq L_2$ when both L_1 and L_2 are regular. Next, we describe this antichains algorithm for solving language inclusion.

Consider the inclusion problem $L_1 \subseteq L_2$ and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be finite-state automata generating the languages L_1 and L_2 respectively. The intuition behind the *antichains* algorithm is to compute, for each state q of $\mathcal{N}_1$, the set S_q of sets of states of $\mathcal{N}_2$ from which no final state of $\mathcal{N}_2$ is reachable by reading words generated from q in $\mathcal{N}_1$.¹ Clearly, the inclusion $L_1 \subseteq L_2$ holds iff none of the sets of states computed for the initial states of $\mathcal{N}_1$ contain some initial state of $\mathcal{N}_2$.

In order to prevent the computation of all possible subsets of $\mathcal{N}_2$ from which the final states are non-reachable, which would be equivalent to determinizing $\mathcal{N}_2$, the *antichains* algorithm ensures that the set S_q for each state q in $\mathcal{N}_1$ is an antichain, i.e. $\forall s, s' \in S_q, s \not\subseteq s' \wedge s' \not\subseteq s$. The idea behind the use of *antichains* is that, given two sets of states of $\mathcal{N}_2$, namely s and s' , if $s \subseteq s'$ then if no final state of $\mathcal{N}_2$ is reachable from s' by reading words in a certain set then the same holds for s . Therefore, discarding the set s and keeping the set s' preserves the correctness of the algorithm. The resulting algorithm is referred to as the *backward antichains algorithm*.

¹Note that this is equivalent to finding states of the complement of the determinized version of $\mathcal{N}_2$ from which a final state is reachable by reading a word generated from q in $\mathcal{N}_1$.

Furthermore, [Wulf et al. \[2006\]](#) also defined a dual of the antichains algorithm described above. In this case, the algorithm computes the set $\widetilde{S}_q$ of sets of states of $\mathcal{N}_2$ reachable from an initial state by reading a word generated from q in $\mathcal{N}_1$. In this case, the inclusion $L_1 \subseteq L_2$ holds iff for every initial state q of $\mathcal{N}_1$, all the sets in $\widetilde{S}_q$ contain a final state. Again, by ensuring that $\widetilde{S}_q$ is an *antichain*, we can reduce the number of sets of states of $\mathcal{N}_2$ that need to be computed since, whenever $s \subseteq s'$, if a final state is reachable from s by a word in a given language, the same holds for s' and, therefore, it is possible to discard s' . The resulting algorithm is referred to as the *forward antichains algorithm*.

The proof of the correctness of the *antichains* algorithm, as presented by [Wulf et al. \[2006\]](#), heavily depends on the automata representation of the languages. We believe that our quasiorder-based framework, presented in Chapter 4, offers a better understanding on the *antichains* algorithm and its correctness proof by offering a new explanation of the algorithm from a language perspective.

Improvements on the Antichains Algorithm

The *antichains* algorithm of [Wulf et al. \[2006\]](#) was later improved by [Abdulla et al. \[2010\]](#), who used simulations (between states and between sets of states) for reducing the amount of sets of states considered by the algorithm.

In particular, they found that, for the *forward antichains algorithm*, there is no need to add the set s of states of $\mathcal{N}_2$ to the set $\widetilde{S}_q$ for a certain state q of $\mathcal{N}_1$ if there exists a state q' of $\mathcal{N}_1$ such that q simulates q' and whose associated set $\widetilde{S}_{q'}$ contains a set s' that simulates s . The idea behind this approach is that simulation is a sufficient condition for language inclusion to hold, i.e. if the set of states s' simulates the set s then the language generated from s' is a subset of the language generated from s .

As we show in Chapter 4, this improvement on the *antichains* algorithm can be partially accommodated by our quasiorder-based framework by using simulations in the definition of the quasiorder. By doing so, the resulting algorithm matches the behavior of the one of [Abdulla et al. \[2010\]](#) when $q = q'$.

On the other hand, [Bonchi and Pous \[2013\]](#) defined a new type of relation between sets of states, denoted *bisimulation up to congruence*, and used it to define a new algorithm for deciding language equivalence between sets of states of a given automaton.

Intuitively, bisimulations up to congruence are enhanced bisimulations (and, therefore, if they relate two sets of states then both sets generate the same language) that might relate sets of states that are not explicitly related by the underlying bisimulation but are related by its implicit congruence closure. Since $L_1 \subseteq L_2 \Leftrightarrow L_1 \cup L_2 = L_2$, the algorithm of [Bonchi and Pous \[2013\]](#) can be used to

decide the inclusion $L_1 \subseteq L_2$ by considering the union automaton $\mathcal{N}_1 \cup \mathcal{N}_2$ and checking whether the bisimulations up to congruence holds between the union of the initial states of $\mathcal{N}_1$ and $\mathcal{N}_2$, which generate $L_1 \cup L_2$, and the initial states of $\mathcal{N}_2$, which generate L_2 .

Finally, [Holík and Meyer \[2015\]](#) used *antichains* to solve the language inclusion problem $L_1 \subseteq L_2$ when L_1 is a context-free language and L_2 is regular. To do that, they reduced the language inclusion problem to a data flow analysis one. This allowed them to rephrase the language inclusion problem as an inclusion problem between sets of relations on the states of the automaton. Then, they applied the antichains principle to reduce the number of relations that need to be manipulated.

As we show in Chapter 4, our quasiorder-based framework for deciding the language inclusion $L_1 \subseteq L_2$ also applies to the case in which L_1 is a context-free grammar. Indeed, when L_1 is regular we instantiate our framework with left or right monotone quasiorders and obtain the antichains algorithm of [Wulf et al. \[2006\]](#) and its variants, among other algorithms. Similarly, when L_1 is context-free, we use a left and right monotone quasiorders and obtain the antichains algorithm of [Holík and Meyer \[2015\]](#), among others.

Therefore, our framework allows us to offer a more direct presentation of the *antichains* algorithm for grammars of [Holík and Meyer \[2015\]](#) as an straightforward extension of the *antichains* algorithm for regular languages.

2.1.2 Solving Language Inclusion through Abstractions

Our approach draws inspiration from the work of [Hofmann and Chen \[2014\]](#), who considered the language inclusion problem on *infinite words* $L_1 \subseteq L_2$ where L_1 is represented by a Büchi automata and L_2 is regular.

They defined a language inclusion algorithm based on fixpoint computations and a language abstraction based on an equivalence relation between states of the underlying automata representation. Although the equivalence relation is folklore (you find it in several textbooks on language theory [[Khoussainov and Nerode 2001](#); [Sakarovitch 2009](#)]), [Hofmann and Chen \[2014\]](#) were the first, to the best of our knowledge, to use it as an abstraction and, in particular, as a complete domain in abstract interpretation.

As we show in Chapter 4, our framework for solving the language inclusion problem also relies on computing the language abstraction of a fixpoint computation. However, we focus on languages on finite words and generalize the language abstractions by relaxing their equivalence relations to quasiorders. Moreover, by considering quasiorders instead of equivalences, we are able to generalize the fixed point-based approach to check $L_1 \subseteq L_2$ when L_2 is non-regular.

2.2 Searching on Compressed Text

The problem of searching with regular expressions on grammar-compressed text has been extensively studied for the last decades. Results in this topic can be divided in two main groups:

- a) Characterization of the problem's complexity from a theoretical point of view [Plandowski and Rytter 1999; Markey and Schnoebelen 2004; Abboud et al. 2017].
- b) Development of algorithms and data structures to efficiently solve different versions of the problem such as pattern matching [Navarro and Tarhio 2005; de Moura et al. 1998; Mäkinen and Navarro 2006], approximate pattern matching [Bille et al. 2009; Kärkkäinen et al. 2003], multi-pattern matching [Kida et al. 1998; Gawrychowski 2014], regular expression matching [Navarro 2003; Bille et al. 2009] and subsequence matching [Bille et al. 2014].

To characterize the complexity of search problems on grammar-compressed text it is common to use *straight line programs* (grammars generating a single string) to represent the output of the compression. Straight line programs are a natural model for algorithms such as LZ78 [Ziv and Lempel 1978], LZW [Welch 1984], Recursive Pairing [Larsson and Moffat 1999] or Sequitur [Nevill-Manning and Witten 1997] and, as proven by Rytter [2004], polynomially equivalent to LZ77 [Ziv and Lempel 1977].

However, algorithms for searching with regular expressions on grammar-compressed text are typically designed for a particular compression scheme [Navarro and Tarhio 2005; Navarro 2003; Bille et al. 2009].

The first algorithm to solve this problem is due to Navarro [2003] and it is defined for LZ78/LZW compressed text. His algorithm reports all positions in the uncompressed text at which a substring that matches the expression ends and exhibits $O(2^s + s \cdot T + \text{occ} \cdot s \cdot \log s)$ worst case time complexity using $O(2^s + t \cdot s)$ space, where “occ” is the number of occurrences, s is the size of the expression and T is the length of the text compressed to size t . To the best of our knowledge this is the only algorithm for regular expression searching on compressed text that has been implemented and evaluated in practice.

Bille et al. [2009] improved the result of Navarro by defining a relationship between the time and space required to perform regular expression searching on compressed text. They defined a data structure of size $o(t)$ to represent LZ78 compressed texts and an algorithm that, given a parameter τ , finds all occurrences of a regular expression in a LZ78 compressed text in $O(t \cdot s \cdot (s + \tau) + \text{occ} \cdot s \cdot \log s)$ time using $O(t \cdot s^2 / \tau + t \cdot s)$ space. To the best of our knowledge, no implementation

of this algorithm was carried out.

We tackle the problem of searching in grammar-compressed text by using our algorithms for deciding language inclusion *adapted* to efficiently handle straight line programs and *enhanced* with additional information that is computed for each variable of the grammar in order to find the exact matches.

Our approach, presented in Chapter 5, differs from the previous ones in the generality of its definition since, by working on straight line programs, our algorithm and its complexity analysis apply to any grammar-based compression scheme. This is a major improvement since, as shown by [Hucke et al. \[2016\]](#), the LZ78 representation of a text of length T has size $t = \Theta((T/\log(T))^{2/3})$ while its representation as a straight line program has size $t = \Omega(\log(T)/(\log \log(T)))$ and $t = O((T/\log(T))^{2/3})$. Therefore, our approach allows us to handle much more concise representations of the data.

Moreover, the definition of “occurrence” used in previous works, i.e. positions in the uncompressed text from which we can read a substring matching the expression, is of limited practical interest. As an evidence, state of the art tools for regular expression searching, such as `grep` or `ripgrep`, define an occurrence as a line of text containing a match of the expression and so do us.

As a consequence, our algorithm reports the number of occurrences of a *fixed* regular expression in a compressed text in $O(t)$ time while previous algorithms require $O(T)$ since $\text{occ} = O(T)$. Even when there are no matches ($\text{occ} = 0$), so previous approaches operate in $O(t)$ time, the result of [Hucke et al. \[2016\]](#) shows that our algorithm behaves potentially better than the others.

Deciding the Existence of a Match

It is worth to remark that the problem of deciding language inclusion between the languages generated by a straight line program and an automaton has been studied before. In particular [Plandowski and Rytter \[1999\]](#) reduced this problem to a series of matrix multiplications, showing that it can be solved in $O(t \cdot s^3)$ time ($O(t \cdot s)$ for deterministic automata) where t is the size of the grammar and s is the size of the automaton. Note that this problem corresponds to deciding whether a grammar-compressed text contains a match for a given regular expression.

On the other hand, [Esparza et al. \[2000\]](#) defined an algorithm to solve a number of decision problems involving automata and context-free grammars which, when restricted to grammars generating a single word, results in a particular implementation of Plandowsky’s approach. Indeed, this implementation coincides with our Algorithm `SLPIncS`, presented in Chapter 5 as a straightforward adaptation of the algorithm presented in Chapter 4 for deciding the inclusion of a context-free language into a regular one.

2.3 Building Residual Automata

Residual automata (RFA for short) were first introduced by Denis et al. [2000; 2001; 2002]. We deliberately use the notation RFA for residual automata, instead of the standard RFSA, in order to be consistent with the notation used in this thesis for deterministic (DFA) and non-deterministic (NFA) automata.

When introducing RFAs, Denis et al. [2000] also defined an algorithm for *residualizing* an automaton, which is an adaptation of the well-known subset construction used for determinization. Moreover, they showed that there exists a *unique canonical* RFA, which is minimal in number of states, for every regular language. Finally, they showed that the residual-equivalent of the double-reversal method holds, i.e. residualizing an automaton N whose reverse is residual yields the canonical RFA for the language generated by N .

Later, Tamm [2015] generalized the double-reversal method for RFAs by giving a sufficient and necessary condition that guarantees that the residualization operation defined by Denis et al. [2002] yields the canonical RFA. This generalization comes in the same lines as that of Brzowski and Tamm [2014] for the double-reversal method for DFAs.

In Chapter 6, we present a quasiorder-based framework of automata constructions inspired by the work of Ganty et al. [2019], who defined a framework of automata constructions based on *equivalences* over words to provide new insights on the relation between well-known methods for computing the minimal *deterministic* automaton of a language. Intuitively, the shift from equivalences to quasiorders allows us to move from deterministic automata to residual ones.

In their work, Ganty et al. [2019] used *congruences*, i.e. monotone equivalences, over words that induce finite partitions over Σ^* . Then, they used well-known automata constructions that yield automata generating a given language L [Büchi 1989; Khoussainov and Nerode 2001] to derive new automata constructions parametrized by a congruence. As a result, when using the Nerode's congruence for L , their automata construction yields the minimal DFA for L [Büchi 1989; Khoussainov and Nerode 2001] while, when using the so-called *automata-based equivalence* relative to an NFA their construction yields the determinized version of the input NFA. They also obtained counterpart automata constructions that yield, respectively, the minimal co-deterministic and a co-deterministic automaton for the language.

The relation between the automata constructions resulting from the Nerode's and the automata-based congruences allowed them to relate determinization and minimization operations. Finally, they re-formulated the generalization of the double-reversal method presented by Brzowski and Tamm [2014], which gives a sufficient and necessary condition that guarantees that determinizing an NFA

yields the minimal DFA for the language generated by the NFA.

Our quasiorder-based framework allows us to extend the work of [Ganty et al. \[2019\]](#) and devise automata constructions that result in residual automata. Moreover, we derive a residual-equivalent of the generalized double-reversal method from [Brzozowski and Tamm \[2014\]](#) that is more general than the one presented by [Tamm \[2015\]](#).

BACKGROUND

In this section we introduce all the concepts and notation that will be used throughout the rest of the thesis.

3.1 Words and Languages

Let Σ be a finite nonempty *alphabet* of symbols. A *string* or *word* w is a finite sequence of symbols of Σ where the empty sequence is denoted ε . We denote w^R the *reverse* of w and use $|w|$ to denote the *length* of w that we abbreviate to $\dagger$ when w is clear from the context. We define $(w)_i$ as the i -th symbol of w if $1 \leq i \leq \dagger$ and ε otherwise. Similarly, $(w)_{i,j}$ denotes the substring, also called *factor*, of w between the i -th and the j -th symbols, both included. Clearly, $w = (w)_{1,\dagger}$.

We write Σ^* to denote the set of all finite words on Σ and write $\wp(S)$ to denote the set of all subsets of S , i.e. $\wp(S) \stackrel{\text{def}}{=} \{S' \mid S' \subseteq S\}$. Given a language $L \in \wp(\Sigma^*)$, $L^R \stackrel{\text{def}}{=} \{w^R \mid w \in L\}$ denotes the *reverse* of L while $L^c \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid w \notin L\}$ denotes its *complement*. Concatenation in Σ^* is simply denoted by juxtaposition, both for concatenating words uv , languages L_1L_2 and words with languages such as uLv . We sometimes use the symbol $\cdot$ to refer explicitly to a concatenation operation.

Definition (Quotient). Let $L \subseteq \Sigma^*$ and $u \in \Sigma^*$. The left quotient of L by the word u is the set of suffixes of the word u in L , i.e.

$$u^{-1}L \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid uw \in L\} .$$

Similarly, the right quotient of L by the word u is the set of all prefixes of u in L , i.e.

$$Lu^{-1} \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid wu \in L\} .$$

Finally, we lift the notions of left and right quotients by a word to sets $S \subseteq \Sigma^*$ as:

$$S^{-1}L \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \forall s \in S, sw \in L\} \text{ and } LS^{-1} \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \forall s \in S, ws \in L\} \blacksquare$$

Note that the definition of quotient by a set is unconventional as it uses the universal quantifier instead of existential. We use this definition since it guarantees that the quotient by a set is the adjoint of concatenation, i.e.

$$XY \subseteq L \Leftrightarrow Y \subseteq X^{-1}L \Leftrightarrow X \subseteq LY^{-1}.$$

Definition (Composite and Prime Quotients). A left (resp. right) quotient $u^{-1}L$ is composite iff it is the union of all the left (resp. right) quotients that it strictly contains, i.e.

$$u^{-1}L = \bigcup_{x \in \Sigma^*, x^{-1}L \subsetneq u^{-1}L} x^{-1}L \quad (\text{resp. } Lu^{-1} = \bigcup_{x \in \Sigma^*, Lx^{-1} \subsetneq Lu^{-1}} Lx^{-1}).$$

When a quotient is not composite, we say it is prime. ■

3.2 Finite-state Automata

Throughout this dissertation we consider three different classes of automata: non-deterministic, deterministic and residual. Next, we define these classes of automata and introduce some basic notions related them.

Non-Deterministic Finite-State Automata

Definition (NFA). A non-deterministic finite-state automaton (NFA for short) is a tuple $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ where Σ is the alphabet, Q is the finite set of states, $I \subseteq Q$ is the subset of initial states, $F \subseteq Q$ is the subset of final states, and $\delta: Q \times \Sigma \rightarrow \wp(Q)$ is the transition relation. ■

We sometimes use the notation $q \xrightarrow{a} q'$ to denote that $q' \in \delta(q, a)$. If $u \in \Sigma^*$ and $q, q' \in Q$ then $q \xrightarrow{u} q'$ means that the state q' is reachable from q by following the string u . Formally, by induction on the length of $u \in \Sigma^*$:

- (i) if $u = \epsilon$ then $q \xrightarrow{\epsilon} q'$ iff $q = q'$;
- (ii) if $u = av$ with $a \in \Sigma, v \in \Sigma^*$ then $q \xrightarrow{av} q'$ iff $\exists q'' \in \delta(q, a), q'' \xrightarrow{v} q'$.

The language generated by an NFA $\mathcal{N}$, often referred to as the language accepted by $\mathcal{N}$ is $\mathcal{L}(\mathcal{N}) \stackrel{\text{def}}{=} \{u \in \Sigma^* \mid \exists q_i \in I, \exists q_f \in F, q_i \xrightarrow{u} q_f\}$. We define the successors and the predecessors of a set $S \subseteq Q$ by a word $w \in \Sigma^*$ as:

$$\text{post}_w^{\mathcal{N}}(S) \stackrel{\text{def}}{=} \{q \in Q \mid \exists q' \in S, q' \xrightarrow{w} q\}$$

$$\text{pre}_w^{\mathcal{N}}(S) \stackrel{\text{def}}{=} \{q \in Q \mid \exists q' \in S, q \xrightarrow{w} q'\} .$$

In general, we omit the automaton $\mathcal{N}$ from the superscript when it is clear from the context. Figure 3.1 shows an example of an NFA.

Figure 3.1: An NFA $\mathcal{N}$ with $\Sigma = \{a, b\}$ and $\mathcal{L}(\mathcal{N}) = \Sigma^* a \Sigma a \Sigma^*$.

Given $S, T \subseteq Q$, define

$$W_{S,T}^{\mathcal{N}} \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \exists q \in S, q' \in T, q \xrightarrow{w} q'\} .$$

When S or T are singletons, we abuse of notation and write $W_{q,T}^{\mathcal{N}}$, $W_{S,q'}^{\mathcal{N}}$ or even $W_{q,q'}^{\mathcal{N}}$. In particular, when $S = \{q\}$ and $T = F$, we say that $W_{q,F}^{\mathcal{N}}$ is the *right language* of q . Likewise, when $S = I$ and $T = \{q\}$, we say that $W_{I,q}^{\mathcal{N}}$ is the *left language* of q . We say that a state q is *unreachable* iff $W_{I,q}^{\mathcal{N}} = \emptyset$ and we say that q is *empty* iff $W_{q,F}^{\mathcal{N}} = \emptyset$. Finally, note that

$$\mathcal{L}(\mathcal{N}) = \bigcup_{q \in I} W_{q,F}^{\mathcal{N}} = \bigcup_{q \in F} W_{I,q}^{\mathcal{N}} = W_{I,F}^{\mathcal{N}} .$$

Definition (Sub-automaton). Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. A sub-automaton of $\mathcal{N}$ is an NFA $\mathcal{N}' = \langle Q', \Sigma, \delta', I', F' \rangle$ for which $Q' \subseteq Q$, $F' \subseteq F$, $I' \subseteq I$ and for every $q, q' \in Q$ and $a \in \Sigma$ we have that $q' \in \delta'(q, a) \Rightarrow q' \in \delta(q, a)$. ■

Clearly, if $\mathcal{N}'$ is a sub-automaton of $\mathcal{N}$ then $\mathcal{L}(\mathcal{N}') \subseteq \mathcal{L}(\mathcal{N})$.

Definition (Reverse Automaton). Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. The reverse of $\mathcal{N}$ is the NFA $\mathcal{N}^R \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta^R, F, I \rangle$ where for every $q, q' \in Q$ and $a \in \Sigma$ we have that $q \in \delta^R(q', a)$ iff $q' \in \delta(q, a)$. ■

It is straightforward to check that $\mathcal{L}(\mathcal{N})^R = \mathcal{L}(\mathcal{N}^R)$.

Deterministic Finite-State Automata

Definition (DFA and co-DFA). A deterministic finite-state automaton (DFA for short) is an NFA such that $I = \{q_0\}$ and, for every state $q \in Q$ and every symbol $a \in \Sigma$, there exists at most one state $q' \in Q$ such that $\delta(q, a) = q'$.

A co-deterministic finite-state automaton (*co-DFA for short*) is an NFA $\mathcal{N}$ such that $\mathcal{N}^R$ is deterministic. ■

Definition (Subset Construction). Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. The subset construction builds a DFA $\mathcal{N}^D \stackrel{\text{def}}{=} \langle Q^D, \Sigma, \delta^D, I^D, F^D \rangle$ where

$$Q^D \stackrel{\text{def}}{=} \{\text{post}_u^{\mathcal{N}}(I) \mid u \in \Sigma^*\}$$

$$I^D \stackrel{\text{def}}{=} \{I\}$$

$$F^D \stackrel{\text{def}}{=} \{S \in \wp(Q) \mid S \cap F \neq \emptyset\}$$

$$\delta^D(S, a) \stackrel{\text{def}}{=} \{q' \mid \exists q \in S, q' \in \delta(q, a)\} \text{ for every } S \in Q \text{ and } a \in \Sigma$$

Given an NFA $\mathcal{N}$, we denote by $\mathcal{N}^D$ the DFA that results from applying the subset construction to $\mathcal{N}$ where only subsets that are reachable from the initial states of $\mathcal{N}^D$ are used. As shown by Hopcroft et al. [2001], $\mathcal{L}(\mathcal{N}^D) = \mathcal{L}(\mathcal{N})$ for every automaton $\mathcal{N}$. Figure 3.2 shows the DFA obtained when applying the subset construction to the NFA from Figure 3.1.

Figure 3.2: DFA $\mathcal{N}^D$ obtained by determinizing the NFA from Figure 3.1.

A DFA for the language $\mathcal{L}(\mathcal{N})$ is *minimal*, denoted by $\mathcal{N}^{DM}$, if it has no unreachable states and no two states have the same right language. For instance, the DFA from Figure 3.2 is not minimal since the states $\{0, 1, 3\}$, $\{0, 3\}$, $\{0, 1, 2, 3\}$ and $\{3\}$ all have the same right language. The minimal DFA for a regular language is *unique* modulo isomorphism and is determined by the right quotients of the generated language.

Definition (Minimal DFA). Let L be a regular language. The minimal DFA for L is the DFA $\mathcal{D} \stackrel{\text{def}}{=} \langle Q^D, \Sigma, \delta^D, I^D, F^D \rangle$ where

$$Q^D \stackrel{\text{def}}{=} \{u^{-1}L \mid u \in \Sigma^*\}$$

$$\begin{aligned}
 I^D &\stackrel{\text{def}}{=} \{u^{-1}L \in Q \mid u^{-1}L \subseteq L\} \\
 F^D &\stackrel{\text{def}}{=} \{u^{-1}L \in Q \mid \varepsilon \in u^{-1}L\} \\
 \delta^D(u^{-1}L, a) &\stackrel{\text{def}}{=} \{v^{-1}L \in Q \mid v^{-1}L = a^{-1}(u^{-1}L)\} \text{ for every } u^{-1}L \in Q \text{ and } a \in \Sigma \quad \blacksquare
 \end{aligned}$$

Residual Finite-State Automata

Definition (RFA and co-RFA). A residual finite-state automaton (RFA for short) is an NFA such that the right language of each state is a left quotient of the language generated by the automaton.

A co-residual automaton (co-RFA for short) is an NFA $\mathcal{N}$ such that $\mathcal{N}^R$ is residual, i.e. the left language of each state is a right quotient of the language generated by the automaton. $\blacksquare$

Formally, an RFA is an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ satisfying

$$\forall q \in Q, \exists u \in \Sigma^*, W_{q,F} = u^{-1}\mathcal{L}(\mathcal{N}) .$$

Similarly, $\mathcal{N}$ is a co-RFA iff it satisfies

$$\forall q \in Q, \exists u \in \Sigma^*, W_{I,q} = Lu^{-1} .$$

The right quotients of the form $u^{-1}L$, where $L \subseteq \Sigma^*$ is a language and $u \in \Sigma^*$, are also known as *residuals*, which gives name to RFAs. We say $u \in \Sigma^*$ is a *characterizing word* for $q \in Q$ iff $W_{q,F}^{\mathcal{N}} = u^{-1}\mathcal{L}(\mathcal{N})$ and we say $\mathcal{N}$ is *consistent* iff each state q is reachable by a characterizing word for q . Moreover, $\mathcal{N}$ is *strongly consistent* iff every state q is reachable by every characterizing word of q .

Similarly to the case of DFAs, there exists a *residualization* operation [Denis et al. 2002] that, given an NFA $\mathcal{N}$, builds an RFA $\mathcal{N}^{\text{res}}$ such that $\mathcal{L}(\mathcal{N}^{\text{res}}) = \mathcal{L}(\mathcal{N})$. This construction can be seen as a determinization followed by the removal of coverable states and the addition of new transitions. We say that the set $\text{post}_u^{\mathcal{N}}(I)$ is *coverable* iff

$$\text{post}_u^{\mathcal{N}}(I) = \bigcup_{x \in \Sigma^*, \text{post}_x^{\mathcal{N}}(I) \subseteq \text{post}_u^{\mathcal{N}}(I)} \text{post}_x^{\mathcal{N}}(I) .$$

Definition (Residualization). Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. Then the residualization operation builds the RFA $\mathcal{N}^{\text{res}} \stackrel{\text{def}}{=} \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$ with

$$\begin{aligned}
 \tilde{Q} &\stackrel{\text{def}}{=} \{\text{post}_u^{\mathcal{N}}(I) \mid u \in \Sigma^* \wedge \text{post}_u^{\mathcal{N}}(I) \text{ is not coverable}\} \\
 \tilde{I} &\stackrel{\text{def}}{=} \{S \in \tilde{Q} \mid S \subseteq I\} \\
 \tilde{F} &\stackrel{\text{def}}{=} \{S \in \tilde{Q} \mid S \cap F \neq \emptyset\} \\
 \tilde{\delta}(S, a) &= \{S' \in \tilde{Q} \mid S' \subseteq \delta(S, a)\} \text{ for every } S \in \tilde{Q} \text{ and } a \in \Sigma \quad \blacksquare
 \end{aligned}$$

Figure 3.3 shows the RFA obtained by applying the residualization operation to the NFA from Figure 3.1.

Figure 3.3: RFA N^{res} obtained when residualizing to the NFA from Figure 3.1.

Similarly, to the case of DFAs, there exists an RFA for every regular language that is minimal in the number of states and is *unique* modulo isomorphism: the *canonical RFA*.

Definition (Canonical RFA). Let L be a regular language. The canonical RFA for L is the RFA $C \stackrel{\text{def}}{=} \langle Q^C, \Sigma, \delta^C, I^C, F^C \rangle$ with

$$Q^C \stackrel{\text{def}}{=} \{u^{-1}L \mid u \in \Sigma^*, u^{-1}L \text{ is prime}\}$$

$$I^C \stackrel{\text{def}}{=} \{u^{-1}L \in Q \mid u^{-1}L \subseteq L\}$$

$$F^C \stackrel{\text{def}}{=} \{u^{-1}L \in Q \mid \varepsilon \in u^{-1}L\}$$

$$\delta^C(u^{-1}L, a) \stackrel{\text{def}}{=} \{v^{-1}L \in Q \mid v^{-1}L \subseteq a^{-1}(u^{-1}L)\} \text{ for every } u^{-1}L \in Q \text{ and } a \in \Sigma \quad \blacksquare$$

The canonical RFA is a strongly consistent RFA and it is the *minimal* (in number of states) RFA such that $\mathcal{L}(C) = L$ [Denis et al. 2002]. Moreover, by definition, the canonical RFA has the *maximal* number of transitions.

Finally, it is straightforward to check that any DFA $\mathcal{D}$ is also an RFA since $W_{q,F}^{\mathcal{D}} = u^{-1}L$ for all $u \in W_{I,q}^{\mathcal{D}}$. Therefore, we have the following relations between these classes of automata:

$$\text{DFA} \subsetneq \text{RFA} \subsetneq \text{NFA} .$$

3.3 Context-free Grammars

Definition (CFG). A context-free grammar (grammar or CFG for short) is a tuple $\mathcal{G} \stackrel{\text{def}}{=} \langle \mathcal{V}, \Sigma, P \rangle$ where $\mathcal{V} = \{X_0, \dots, X_n\}$ is a finite set of variables including the start symbol X_0 (also denoted axiom), Σ is a finite alphabet of terminals and P is the set of rules $X_i \rightarrow \beta$ where $\beta \in (\mathcal{V} \cup \Sigma)^*$ ■

In the following we assume, for simplicity and without loss of generality, that grammars are always given in Chomsky Normal Form (CNF) [Chomsky 1959], that is, every rule $X_i \rightarrow \beta \in P$ is such that $\beta \in (\mathcal{V} \times \mathcal{V}) \cup \Sigma \cup \{\epsilon\}$ and if $\beta = \epsilon$ then $i = 0$. We also assume that for all $X_i \in \mathcal{V}$ there exists a rule $X_i \rightarrow \beta \in P$, otherwise X_i can be safely removed from $\mathcal{V}$.

Given two strings $w, w' \in (\mathcal{V} \cup \Sigma)^*$ we write $w \Rightarrow w'$ iff there exists two strings $u, v \in (\mathcal{V} \cup \Sigma)^*$ and a grammar rule $X \rightarrow \beta \in P$ such that $w = uXv$ and $w' = u\beta v$. We denote by $\Rightarrow^*$ the reflexive-transitive closure of $\Rightarrow$.

The language generated by a $\mathcal{G}$ is $\mathcal{L}(\mathcal{G}) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid X_0 \Rightarrow^* w\}$.

Straight-line Programs

In the context of grammar-based compression we are interested in straight line programs, i.e. grammars generating exactly one word.

Definition (SLP). A straight line program (SLP for short), is a CFG $\mathcal{P} = \langle \mathcal{V}, \Sigma, P \rangle$ where the set of rules is of the form

$$P \stackrel{\text{def}}{=} \{X_i \rightarrow \alpha_i \beta_i \mid 1 \leq i \leq |V|, \alpha_i, \beta_i \in (\Sigma \cup \{X_1, \dots, X_{i-1}\})\} .$$

We refer to $X_{|V|} \rightarrow \alpha_{|V|} \beta_{|V|}$ as the axiom rule. ■

It is straightforward to check that the language generated by an SLP consists of a single string $w \in \Sigma^*$ and, by definition, $|w| > 1$. Since $\mathcal{L}(P) = \{w\}$ we identify w with $\mathcal{L}(P)$.

3.4 Quasiorders

Let $f : X \rightarrow Y$ be a function between sets and let $S \in \wp(X)$. We denote the image of f on S by $f(S) \stackrel{\text{def}}{=} \{f(x) \in Y \mid x \in S\}$. The composition of two functions f and g is denoted both by fg and $f \circ g$.

A *quasiordered set* (qoset for short) is a tuple $\langle D, \leq \rangle$ such that $\leq$ is a *quasiorder* (qo for short) relation on D , i.e. a reflexive and transitive binary relation. Given a qoset $\langle D, \leq \rangle$ we denote by $\sim_D$ the equivalence relation induced by $\leq$:

$$d \sim_D d' \stackrel{\text{def}}{\Leftrightarrow} d \leq d' \wedge d' \leq d, \quad \text{for all } d, d' \in D .$$

Moreover, given a qo $\leq$ we denote its strict version by $<$:

$$u < v \stackrel{\text{def}}{\Leftrightarrow} u \leq v \wedge v \not\leq u .$$

We say that a qoset satisfies the *ascending* (resp. *descending*) *chain condition* (ACC, resp. DCC) if there is no countably infinite sequence of distinct elements $\{x_i\}_{i \in \mathbb{N}}$ such that, for all $i \in \mathbb{N}$, $x_i \leq x_{i+1}$ (resp. $x_{i+1} \leq x_i$). If a qoset satisfies the ACC (resp. DCC) we say it is ACC (resp. DCC).

Definition (Closure and Principals). *Let $\leq$ be a quasiorder on Σ^* and let $S \subseteq \Sigma^*$. The closure of S is the set*

$$\rho_{\leq}(S) \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \exists x \in S, x \leq w\}.$$

We say $\rho_{\leq}(S)$ is a principal if S is a singleton. In that case, we abuse of notation and write $\rho_{\leq}(u)$ instead of $\rho_{\leq}(\{u\})$. ■

Given two quasiorders $\leq$ and $\leq'$ we say that $\leq$ is finer than $\leq'$ (or $\leq'$ is coarser than $\leq$) and write $\leq \subseteq \leq'$ iff $\rho_{\leq}(S) \subseteq \rho_{\leq'}(S)$ for every set $S \subseteq \Sigma^*$.

Definition (Left and Right Quasiorders). *Let $\leq$ be a quasiorder. We say $\leq$ is right monotone (or equivalently, $\leq$ is a right quasiorder), and denote it by $\leq^r$, iff*

$$u \leq^r v \Rightarrow ua \leq^r va, \quad \text{for all } u, v \in \Sigma^* \text{ and } a \in \Sigma.$$

Similarly, we say $\leq$ is a left quasiorder, and denote it by $\leq^l$, iff

$$u \leq^l v \Rightarrow au \leq^l av, \quad \text{for all } u, v \in \Sigma^* \text{ and } a \in \Sigma$$

A qoset $\langle D, \leq \rangle$ is a *partially ordered set* (poset for short) when $\leq$ is antisymmetric. A subset $X \subseteq D$ of a poset is *directed* iff X is nonempty and every pair of elements in X has an upper bound in X .

Definition (Least Upper Bound). *Let $\langle D, \leq \rangle$ be a partially ordered set and let $x, y \in D$. The least upper bound of x and y is the element $z \in D$ such that*

$$x \leq z \wedge y \leq z \wedge (\forall d \in D, (x \leq d \wedge y \leq d) \Rightarrow z \leq d).$$

Definition (Greatest Lower Bound). *Let $\langle D, \leq \rangle$ be a partially ordered set and let $x, y \in D$. The greatest lower bound of x and y is the element $z \in D$ such that*

$$z \leq x \wedge z \leq y \wedge (\forall d \in D, (d \leq x \wedge d \leq y) \Rightarrow d \leq z).$$

A poset $\langle D, \leq \rangle$ is a *directed-complete partial order* (CPO for short) iff it has the least upper bound (lub for short) of all its directed subsets. A poset is a *join-semilattice* iff it has the lub of all its nonempty finite subsets (therefore binary lubs are enough). A poset is a *complete lattice* iff it has the lub of all its arbitrary (possibly empty) subsets; in this case, let us recall that it also has the greatest lower bound (glb for short) of all its arbitrary subsets.

Well-quasiorders

Definition (Antichain). Let $\langle D, \leq \rangle$ be a qoset. A subset $X \subseteq D$ is an antichain iff any two distinct elements in X are incomparable. ■

We denote the set of antichains of a qoset $\langle D, \leq \rangle$ by

$$\text{AC}_{\langle D, \leq \rangle} \stackrel{\text{def}}{=} \{X \subseteq D \mid X \text{ is an antichain}\} .$$

Definition (Well-quasiorder). Let $\langle D, \leq \rangle$ be a quasiordered set. We say it is a well-quasiordered set (wqoset for short), and $\leq$ is a well-quasiorder (wqo for short), iff for every countably infinite sequence of elements $\{x_i\}_{i \in \mathbb{N}}$ there exist $i, j \in \mathbb{N}$ such that $i < j$ and $x_i \leq x_j$.

Equivalently, we say $\langle D, \leq \rangle$ is a well-quasiordered set iff D is DCC and D has no infinite antichain. ■

For every qoset $\langle D, \leq \rangle$, we shift the quasiorder $\leq$ to a binary relation $\sqsubseteq$ on the powerset as follows. Given $X, Y \in \wp(D)$,

$$X \sqsubseteq Y \stackrel{\text{def}}{\iff} \forall x \in X, \exists y \in Y, y \leq x .$$

When the quasiorder is clear from the context, we drop the subindex and write simply $\sqsubseteq$. Given a qoset $\langle D, \leq \rangle$, we define the set of *minimal elements* of a subset $X \subseteq D$:

$$\min_{\leq}(X) \stackrel{\text{def}}{=} \{x \in X \mid \forall y \in X, y \leq x \Rightarrow y = x\} .$$

Definition (Minor). Let $\langle D, \leq \rangle$ be a qoset. A minor of a subset $X \subseteq D$, denoted by $\lfloor X \rfloor$, is a subset of the minimal elements of X w.r.t. $\leq$, i.e. $\lfloor X \rfloor \subseteq \min_{\leq}(X)$, such that $X \sqsubseteq \lfloor X \rfloor$ holds. ■

Clearly, a minor $\lfloor X \rfloor$ of some set X is always an antichain.

Let us recall that every subset X of a wqoset $\langle D, \leq \rangle$ has at least one minor set, all minor sets of X are finite, $\lfloor \{x\} \rfloor = \{x\}$, $\lfloor \emptyset \rfloor = \emptyset$, and if $\langle D, \leq \rangle$ is additionally a poset then there exists exactly one minor set of X . It turns out that $\langle \text{AC}_{\langle D, \leq \rangle}, \sqsubseteq \rangle$ is a qoset which is ACC if $\langle D, \leq \rangle$ is a wqoset and is a poset if $\langle D, \leq \rangle$ is a poset.

Nerode Quasiorders

Definition (Nerode's Quasiorders). Let $L \subseteq \Sigma^*$ be a language. The left and right Nerode's quasiorders on Σ^* are, respectively

$$u \leq_L^\ell v \stackrel{\text{def}}{\iff} Lu^{-1} \subseteq Lv^{-1}, \quad u \leq_L^r v \stackrel{\text{def}}{\iff} u^{-1}L \subseteq v^{-1}L \quad \blacksquare$$

As shown by [de Luca and Varricchio \[1994\]](#), $\leq_L^\ell$ and $\leq_L^r$ are, respectively, left and right monotone and, if L is regular then both $\leq_L^\ell$ and $\leq_L^r$ are wqos [[de Luca and Varricchio 1994](#), Theorem 2.4].

Furthermore, [de Luca and Varricchio \[1994\]](#) showed that $\leq_L^\ell$ is maximum in the set of all left monotone quasiorders $\leq^\ell$ that satisfy $\rho_{\leq^\ell}(L) = L$. Therefore, for every left quasiorder $\leq^\ell$, if $\rho_{\leq^\ell}(L) = L$ then $x \leq^\ell y \Rightarrow x \leq_L^\ell y$. Similarly holds for right quasiorders and the right Nerode quasiorder.

3.5 Kleene Iterates

Let $\langle X, \leq \rangle$ be a qoset and $f : X \rightarrow X$ be a function. The function f is *monotone* iff $x \leq y$ implies $f(x) \leq f(y)$. Given $b \in X$, the trace of values of the variable $x \in X$ computed by the following iterative procedure:

$$\text{KLEENE}(f, b) \stackrel{\text{def}}{=} \begin{cases} x := b; \\ \textbf{while } f(x) \neq x \textbf{ do } x := f(x); \\ \textbf{return } x; \end{cases}$$

provides the possibly infinite sequence of so-called *Kleene iterates* of the function f starting from the basis b .

Whenever $\langle X, \leq \rangle$ is an ACC (resp. DCC) CPO, $b \leq f(b)$ (resp. $f(b) \leq b$) and f is monotone then, by Knaster-Tarski-Kleene fixpoint theorem, $\text{KLEENE}(f, b)$ terminates and returns the least (resp. greatest) fixpoint of the function f which is greater (resp. lower) than or equal to b . In particular, if $\perp_X$ (resp. $\top_X$) is the least (resp. greatest) element of X then $\text{KLEENE}(f, \perp_X)$ (resp. $\text{KLEENE}(f, \top_X)$) computes the sequence of Kleene iterates that finitely converges to the least (resp. greatest) fixpoint of f , denoted by $\text{lfp}(f)$ (resp. $\text{gfp}(f)$).

THEOREM 3.5.1. *Let $\langle X, \leq \rangle$ be an ACC CPO and let $f : X \rightarrow X$ be a monotone function. Then $\text{KLEENE}(f, \perp_X)$ terminates and returns the least fixpoint of f .*

Proof. To simplify the notation, we use $\perp$ to denote the least element of X , $\perp_X$. Next, we show by induction that $f^n(\perp) \leq f^{n+1}(\perp)$ for all $n \geq 0$.

- *Base case:* The relation $\perp \leq f(\perp)$ holds since $\perp$ is the least element in X .
- *Inductive step:* Assume $f^n(\perp) \leq f^{n+1}(\perp)$ for some value n . Then, since f is a monotone function, we have that that $f^{n+1}(\perp) \leq f^{n+2}(\perp)$.

We conclude that $f^n(\perp) \leq f^{n+1}(\perp)$ holds for all $n \geq 0$. Since the qoset $\langle X, \leq \rangle$ is an ACC, there is no infinite sequence of ascending elements and, as a consequence, $\text{KLEENE}(f, \perp)$ terminates and returns a fixpoint of function f .

Next, we show that if $f^n(\perp) = f^{n+1}(\perp)$ for some n then $f^n(\perp) = \text{lfp}(f)$.

To do that, we show that $f^i(\perp) \leq p$ for every $i \geq 0$ and for every fixpoint p of f . Therefore, the fixpoint $f^n(\perp)$ is below (for the quasiorder $\leq$) than any other fixpoint, hence $f^n(\perp)$ is the least fixpoint of f , i.e. $f^n(\perp) = \text{lfp}(f)$.

Again, we proceed by induction in n . Let p be a fixpoint of f , i.e. $f(p) = p$.

- *Base case:* The relation $\perp \leq p$ trivially holds by definition of $\perp$.
- *Inductive step:* Assume $f^n(\perp) \leq p$ for some value n . Then, since f is a monotone function, we have that $f^{n+1}(\perp) \leq f(p) = p$, where the last equality follows from the fact that p is a fixpoint.

Clearly, $f^n(\perp) \leq p$ for all $n \geq 0$ and for all fixpoint p of f . We conclude that $\text{KLEENE}(f, \perp) = \text{lfp}(f)$.

For the sake of clarity, we overload the notation and use the same symbol for a function/relation and its componentwise (i.e. pointwise) extension on product domains. For instance, if $f : X \rightarrow Y$ then f also denotes the standard product function $f : X^n \rightarrow Y^n$ defined by $\lambda \langle x_1, \dots, x_n \rangle \in X^n. \langle f(x_1), \dots, f(x_n) \rangle$. A vector $\vec{Y}$ in some product domain $D^{|S|}$ is also denoted by $\langle Y_i \rangle_{i \in S}$ and, for some $i \in S$, $\vec{Y}_i$ denotes its component Y_i .

3.6 Closures and Galois Connections

We conclude this chapter by recalling some basic notions on closure operators and Galois Connections commonly used in abstract interpretation (see, e.g., [Cousot and Cousot 1979; Miné 2017]).

Closure operators and Galois Connections are equivalent notions [Cousot 1978] and, therefore, they are both used for defining the notion of *approximation* in abstract interpretation, where closure operators allow us to define and reason on abstract domains independently of a specific representation which is required by Galois Connections.

Definition (Upper Closure Operator). *Let $\langle C, \leq_C, \vee, \wedge \rangle$ be a complete lattice, where $\vee$ and $\wedge$ denote, respectively, the lub and glb. An upper closure operator, or simply closure, on $\langle C, \leq_C \rangle$ is a function $\rho : C \rightarrow C$ which is:*

- (i) *monotone, i.e. $x \leq_C y \Rightarrow \rho(x) \leq_C \rho(y)$ for all $x, y \in C$;*
- (ii) *idempotent, i.e. $\rho(\rho(x)) = \rho(x)$ for all $x \in C$, and*
- (iii) *extensive, i.e. $x \leq_C \rho(x)$ for all $x \in C$.* ■

The set of all upper closed operators on C is denoted by $\text{uco}(C)$. We often write $c \in \rho(C)$, or simply $c \in \rho$, to denote that there exists $c' \in C$ such that

$c = \rho(c')$, and recall that this happens iff $\rho(c) = c$. If $\rho \in \text{uco}(C)$ then for all $c_1 \in C$, $c_2 \in \rho$ and $X \subseteq C$, it turns out that:

$$c_1 \leq_C c_2 \Leftrightarrow \rho(c_1) \leq_C \rho(c_2) \Leftrightarrow \rho(c_1) \leq_C c_2 \quad (3.1)$$

$$\rho(\vee X) = \rho(\vee \rho(X)) \quad \text{and} \quad \wedge \rho(X) = \rho(\wedge \rho(X)) . \quad (3.2)$$

In abstract interpretation, a closure operator $\rho \in \text{uco}(C)$ on a concrete domain C plays the role of abstraction function for objects of C . Given two closures $\rho, \rho' \in \text{uco}(C)$, ρ is a *coarser abstraction* than ρ' (or, equivalently, ρ' is a more precise abstraction than ρ) iff the image of ρ is a subset of the image of ρ' , i.e. $\rho(C) \subseteq \rho'(C)$, and this happens iff for any $x \in C$, $\rho'(x) \leq_C \rho(x)$.

Definition (Galois Connection). A Galois Connection (GC for short) or adjunction between two posets $\langle C, \leq_C \rangle$ (a concrete domain) and $\langle A, \leq_A \rangle$ (an abstract domain) consists of two monotone functions $\alpha: C \rightarrow A$ and $\gamma: A \rightarrow C$ such that

$$\alpha(c) \leq_A a \Leftrightarrow c \leq_C \gamma(a), \quad \text{for all } a \in A, c \in C .$$

A Galois Connection is denoted by $\langle C, \leq_C \rangle \xleftrightarrow[\alpha]{\gamma} \langle A, \leq_A \rangle$. ■

LEMMA 3.6.1. Let $\langle C, \leq_C \rangle \xleftrightarrow[\alpha]{\gamma} \langle A, \leq_A \rangle$ be a GC. The following properties hold:

- (a) $x \leq_C \gamma \circ \alpha(x)$ and $\alpha \circ \gamma(y) \leq_A y$.
- (b) α and γ are monotonic functions.
- (c) $\alpha = \alpha \circ \gamma \circ \alpha$ and $\gamma = \gamma \circ \alpha \circ \gamma$.

Proof.

- (a) Since $\leq_A$ is reflexive, we have that for all $x \in A$ $\alpha(x) \leq_A \alpha(x)$ holds and, by definition of GC, $\alpha(x) \leq_A \alpha(x) \Leftrightarrow x \leq_C \gamma(\alpha(x))$. Therefore, $x \leq_C \gamma(\alpha(x))$.

Similarly, since $\alpha(\gamma(y)) \leq_A y \Leftrightarrow \gamma(y) \leq_C \gamma(y)$ and $\gamma(y) \leq_C \gamma(y)$, we conclude that $\alpha(\gamma(y)) \leq_A y$.

- (b) Let $c, c' \in C$ be such that $c \leq_C c'$. Then, by Lemma 3.6.1 (a), we have that $c' \leq_C \gamma(\alpha(c'))$ and, by definition of GC, $c \leq_C \gamma(\alpha(c')) \Rightarrow \alpha(c) \leq_A \alpha(c')$.

Similarly, let $a, a' \in A$ be such that $a \leq_A a'$. Then, by Lemma 3.6.1 (a), we have that $\alpha(\gamma(a)) \leq_A a'$, hence $\alpha(\gamma(a)) \leq_A a' \Rightarrow \gamma(a) \leq_A \gamma(a')$.

- (c) Let $c \in C$. By Lemma 3.6.1 (a), $c \leq_C \gamma(\alpha(c))$ which, by Lemma 3.6.1 (b), implies that $\alpha(c) \leq_A \alpha(\gamma(\alpha(c)))$. Moreover, since $\gamma(\alpha(c)) \leq_C \gamma(\alpha(c))$, it follows from the definition of GC that $\alpha(\gamma(\alpha(c))) \leq_A \alpha(c)$. Therefore $\alpha(\gamma(\alpha(c))) = \alpha(c)$.

Similarly, let $a \in A$. By Lemma 3.6.1 (a) and (b), $\gamma(\alpha(\gamma(a))) \leq_A \gamma(a)$

and, since $\alpha(\gamma(a)) \leq_C \alpha(\gamma(a))$, it follows from the definition of GC that $\gamma(a) \leq_A \gamma(\alpha(\gamma(a)))$. Therefore $\gamma(a) = \gamma(\alpha(\gamma(a)))$.

The function α is called the *left-adjoint* of γ , and, dually, γ is called the *right-adjoint* of α . This terminology is justified by the fact that if a function $\alpha : C \rightarrow A$ admits a right-adjoint $\gamma : A \rightarrow C$ then this is unique (and this dually holds for left-adjoints).

It turns out that, in a GC, γ is always *co-additive*, i.e. it preserves arbitrary glb's, while α is always *additive*, i.e. it preserves arbitrary lub's. Moreover, an additive function $\alpha : C \rightarrow A$ uniquely determines its right-adjoint by

$$\gamma \stackrel{\text{def}}{=} \lambda a. \bigvee_C \{c \in C \mid \alpha(c) \leq_A a\}.$$

Dually, a co-additive function $\gamma : A \rightarrow C$ uniquely determines its left-adjoint by

$$\alpha \stackrel{\text{def}}{=} \lambda c. \bigwedge_A \{a \in A \mid c \leq_C \gamma(a)\}.$$

We conclude this chapter with the following lemma, which is folklore in abstract interpretation yet we provide a proof for the sake of completeness.

LEMMA 3.6.2. *Let $\langle C, \leq_C \rangle \xleftrightarrow[\alpha]{\gamma} \langle A, \leq_A \rangle$ be a GC between complete lattices and $f : C \rightarrow C$ be a monotone function. Then, $\gamma(\text{lfp}(\alpha f \gamma)) = \text{lfp}(\gamma \alpha f)$.*

Proof. Let us first show that $\gamma(\text{lfp}(\alpha f \gamma)) \geq_C \text{lfp}(\gamma \alpha f)$:

$$\begin{aligned} \gamma(\text{lfp}(\alpha f \gamma)) &\leq_C \gamma(\text{lfp}(\alpha f \gamma)) \Leftrightarrow [\text{Since } g(\text{lfp}(g)) = \text{lfp}(g)] \\ \gamma \alpha f(\gamma(\text{lfp}(\alpha f \gamma))) &\leq_C \gamma(\text{lfp}(\alpha f \gamma)) \Rightarrow [\text{Since } g(x) \leq x \Rightarrow \text{lfp}(g) \leq x] \\ \text{lfp}(\gamma \alpha f) &\leq_C \gamma(\text{lfp}(\alpha f \gamma)) \end{aligned}$$

Then, let us prove that $\gamma(\text{lfp}(\alpha f \gamma)) \leq_C \text{lfp}(\gamma \alpha f)$:

$$\begin{aligned} \text{lfp}(\gamma \alpha f) &\leq_C \text{lfp}(\gamma \alpha f) \Leftrightarrow [\text{Since } g(\text{lfp}(g)) = \text{lfp}(g)] \\ \gamma \alpha f(\text{lfp}(\gamma \alpha f)) &\leq_C \text{lfp}(\gamma \alpha f) \Rightarrow [\text{Since } \alpha \text{ is monotone}] \\ \alpha \gamma \alpha f(\text{lfp}(\gamma \alpha f)) &\leq_A \alpha(\text{lfp}(\gamma \alpha f)) \Leftrightarrow [\text{Since } \alpha \gamma \alpha = \alpha \text{ in GCs}] \\ \alpha f(\text{lfp}(\gamma \alpha f)) &\leq_A \alpha(\text{lfp}(\gamma \alpha f)) \Leftrightarrow [\text{Since } g(\text{lfp}(g)) = \text{lfp}(g)] \\ \alpha f \gamma(\alpha(\text{lfp}(\gamma \alpha f))) &\leq_A \alpha(\text{lfp}(\gamma \alpha f)) \Rightarrow [\text{Since } g(x) \leq x \Rightarrow \text{lfp}(g) \leq x] \\ \text{lfp}(\alpha f \gamma) &\leq_A \alpha(\text{lfp}(\gamma \alpha f)) \Rightarrow [\text{Since } \gamma \text{ is monotone}] \end{aligned}$$

$$\gamma(\text{lfp}(\alpha f \gamma)) \leq_C \gamma \alpha(\text{lfp}(\gamma \alpha f)) \Leftrightarrow [\text{Since } g(\text{lfp}(g)) = \text{lfp}(g)] \\ \gamma(\text{lfp}(\alpha f \gamma)) \leq_C \text{lfp}(\gamma \alpha f)$$

3.7 Complexity Notation

In this thesis we analyze the time and space complexity of some algorithms and constructions. To do that, we use the standard small-O, big-O and big-Omega notation to compare functions. Next, we define these notations for the sake of completeness, where, given a real number k , we write $|k|$ to denote its absolute value.

Definition (Small-O, Big-O, Big-Omega). *Let f and g be two functions on the real numbers. Then*

$$\begin{aligned} f(n) = o(g(n)) &\stackrel{\text{def}}{\Leftrightarrow} \forall k > 0, \exists n_0, \forall n > n_0, f(n) \leq k \cdot g(n) \stackrel{\text{def}}{\Leftrightarrow} \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \\ f(n) = O(g(n)) &\stackrel{\text{def}}{\Leftrightarrow} \exists k > 0, \exists n_0, \forall n > n_0, f(n) \leq k \cdot g(n) \stackrel{\text{def}}{\Leftrightarrow} \limsup_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \\ f(n) = \Omega(g(n)) &\stackrel{\text{def}}{\Leftrightarrow} \exists k > 0, \exists n_0, \forall n > n_0, f(n) \geq k \cdot g(n) \stackrel{\text{def}}{\Leftrightarrow} \liminf_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0 \end{aligned}$$

Intuitively, $f(n) = o(g(n))$ indicates that f is asymptotically dominated by g ; $f(n) = O(g(n))$ indicates that f is asymptotically bounded above by g and $f(n) = \Omega(g(n))$ indicates that f is asymptotically bounded below by g . ■

These notations allow us to simplify the complexity analysis by removing all components of low impact in a complexity function. For instance, let the number of operations performed by an algorithm on an input of size n be given by a function $f(n)$ that satisfies

$$n^2 + n \cdot \log n + k \leq f(n) \leq n^3 + n^2 + \log n + k' ,$$

where k and k' are constants. Since, by definition, $n^2 = o(n^3)$, $\log(n) = o(n^3)$ and $k' = o(n^3)$ we find that the components n^2 , $\log n$ and k' have low impact in the behavior of the upper bound of $f(n)$ for large values of n . Similarly, the components $n \cdot \log(n)$ and k have low impact in the lower bound of $f(n)$ for large values of n . Therefore, we find that $O(f(n)) = O(n^3 + n^2 + \log n + k') = O(n^3)$ and $\Omega(f(n)) = \Omega(n^2 + n \cdot \log n + k) = \Omega(n^2)$. Intuitively, this means that for large values of the parameter n the function $f(n)$ is *below* n^3 and *above* n^2 .

DECIDING LANGUAGE INCLUSION

In this chapter, we present a quasiorder-based framework for deciding language inclusion which is a fundamental and classical problem [Hopcroft and Ullman 1979, Chapter 11] with applications to different areas of computer science.

The basic idea of our approach for solving a language inclusion problem $L_1 \subseteq L_2$ is to leverage Cousot and Cousot’s abstract interpretation [Cousot and Cousot 1977; 1979] for checking the inclusion of an over-approximation (i.e. a superset) of L_1 into L_2 . This idea draws inspiration from the work of Hofmann and Chen [2014], who used abstract interpretation to decide language inclusion between languages of infinite words.

Assuming that L_1 is specified as least fixpoint of an equation system on $\wp(\Sigma^*)$, an over-approximation of L_1 is obtained by applying an over-approximating abstraction function for sets of words $\rho : \wp(\Sigma^*) \rightarrow \wp(\Sigma^*)$ at each step of the Kleene iterates converging to the least fixpoint L_1 . This abstraction map ρ is an upper closure operator which is used in standard abstract interpretation for approximating an input language by adding words (possibly none) to it.

This abstract interpretation-based approach provides an abstract inclusion check $\rho(L_1) \subseteq L_2$ which is always *sound* by construction because $L_1 \subseteq \rho(L_1)$. We then give conditions on ρ which ensure a *complete* abstract inclusion check, namely, the answer to $\rho(L_1) \subseteq L_2$ is always exact (no “false alarms” in abstract interpretation terminology). These conditions are: (i) $\rho(L_2) = L_2$ and (ii) ρ is a complete abstraction for symbol concatenation $\lambda X \in \wp(\Sigma^*)$. aX , for all $a \in \Sigma$, according to the standard notion of completeness in abstract interpretation [Cousot and Cousot 1977; Giacobazzi et al. 2000; Ranzato 2013]. This approach leads us to design in Section 4.2 two general algorithmic frameworks for language inclusion problems which are parameterized by an underlying language abstraction

(see Theorems 4.2.10 and 4.2.11). Intuitively, the first of these frameworks allows us to decide the inclusion $L_1 \subseteq L_2$ by manipulating finite sets of words, even if the languages L_1 and L_2 are infinite. On the other hand, the second framework allows us to decide the inclusion by working on an abstract domain.

We then focus on over-approximating abstractions ρ which are induced by a quasiorder relation $\leq$ on words in Σ^* . Here, a language L is over-approximated by adding all the words which are “greater than or equal to” some word of L for $\leq$. This allows us to instantiate the above conditions (i) and (ii) for having a complete abstract inclusion check in terms of the quasiorder $\leq$. Termination, which corresponds to having finitely many Kleene iterates in the fixpoint computations, is guaranteed by requiring that the relation $\leq$ is a well-quasiorder.

We define quasiorders satisfying the above conditions which are directly derived from the standard Nerode equivalence relations on words. These quasiorders have been first investigated by Ehrenfeucht et al. [1983] and have been later generalized and extended by de Luca and Varricchio [1994; 2011]. In particular, drawing from a result by de Luca and Varricchio [1994], we show that the language abstractions induced by the Nerode’s quasiorders are the most general ones (thus, intuitively optimal) which fit in our algorithmic framework for checking language inclusion.

While these quasiorder abstractions do not depend on some language representation (e.g., some class of automata), we provide quasiorders which instead exploit an underlying language representation given by a finite automaton. In particular, by selecting suitable well-quasiorders for the class of language inclusion problems at hand, we are able to systematically derive decision procedures for the inclusion problem $L_1 \subseteq L_2$ when: (i) both L_1 and L_2 are regular, (ii) L_1 is regular and L_2 is the trace language of a one-counter net and (iii) L_1 is context-free and L_2 is regular.

These decision procedures that we systematically derive here by instantiating our framework are then related to existing language inclusion checking algorithms. We study in detail the case where both languages L_1 and L_2 are regular and represented by finite-state automata. When our decision procedure for $L_1 \subseteq L_2$ is derived from a well-quasiorder on Σ^* by exploiting the automaton-based representation of L_2 , it turns out that we obtain the well-known “antichains algorithm” by Wulf et al. [2006]. Also, by including a simulation relation in the definition of the well-quasiorder we derive a decision procedure that partially matches the language inclusion algorithm by Abdulla et al. [2010], and in turn also that by Bonchi and Pous [2013]. For the case in which L_1 is regular and L_2 is the set of traces of a one-counter net we derive an alternative proof for the decidability of the language inclusion problem [Jancar et al. 1999]. Moreover, for the

case in which L_1 is context-free and L_2 is regular, we derive a decision procedure that matches the “antichains algorithm” for context-free languages presented by Holík and Meyer [2015].

Finally, we leverage a standard duality result [Cousot 2000] and put forward a *greatest* fixpoint approach (instead of the above *least* fixpoint-based procedures) for the case where both L_1 and L_2 are regular languages. In this case, we exploit the properties of the over-approximating abstraction induced by the quasiorder in order to show that the Kleene iterates of this greatest fixpoint computation are finitely many. Interestingly, the Kleene iterates of the greatest fixpoint are finitely many whether you apply the over-approximating abstraction or not, which we show by relying on so-called forward complete abstract interpretations [Giacobazzi and Quintarelli 2001].

4.1 Inclusion Check by Complete Abstractions

The language inclusion problem consists in checking whether $L_1 \subseteq L_2$ holds where L_1 and L_2 are two languages over a common alphabet Σ . In this section, we show how complete abstractions ρ of $\wp(\Sigma^*)$ can be used to compute an over-approximation $\rho(L_1)$ of L_1 such that $\rho(L_1) \subseteq L_2 \Leftrightarrow L_1 \subseteq L_2$.

Closure-based abstract interpretation can be applied to solve a generic inclusion problem by leveraging backward complete abstractions [Cousot and Cousot 1977; 1979; Giacobazzi et al. 2000; Ranzato 2013]. An upper closure $\rho \in \text{uco}(C)$ is called *backward complete* for a concrete monotone function $f : C \rightarrow C$ when $\rho f = \rho f \rho$ holds. Since $\rho f(c) \leq_C \rho f \rho(c)$ always holds for all $c \in C$, the intuition is that backward completeness models an ideal situation where no loss of precision is accumulated in the computations of ρf when its concrete input objects c are over-approximated by $\rho(c)$. It is well known [Cousot and Cousot 1979] that backward completeness implies completeness of least fixpoints, namely

$$\rho f = \rho f \rho \Rightarrow \rho(\text{lfp}(f)) = \text{lfp}(\rho f) = \text{lfp}(\rho f \rho) \quad (4.1)$$

provided that these least fixpoints exist (this is the case, for instance, when C is a CPO). Theorem 4.1.1 states how a concrete inclusion check $\text{lfp}(f) \leq_C c_2$ can be equivalently performed in a backward complete abstraction ρ when $c_2 \in \rho$.

THEOREM 4.1.1. *If C is a CPO, $f : C \rightarrow C$ is monotone, $\rho \in \text{uco}(C)$ is backward complete for f and $c_2 \in \rho$, then*

$$\text{lfp}(f) \leq_C c_2 \Leftrightarrow \text{lfp}(\rho f) \leq_C c_2 .$$

In particular, if $\langle C, \leq_C \rangle$ is ACC then the Kleene iterates of $\text{lfp}(\rho f)$ are finitely many.

Proof. First, we show that $\text{lfp}(f) \leq_C c_2 \Leftrightarrow \text{lfp}(\rho f) \leq_C c_2$.

$$\begin{aligned}
 \text{lfp}(f) \leq_C c_2 &\Leftrightarrow [\text{Since } c_2 \in \rho] \\
 \text{lfp}(f) \leq_C \rho(c_2) &\Leftrightarrow [\text{Since } x \leq \rho(y) \Leftrightarrow \rho(x) \leq \rho(y)] \\
 \rho(\text{lfp}(f)) \leq_C \rho(c_2) &\Leftrightarrow [\text{By Equation (4.1)}] \\
 \text{lfp}(\rho f) \leq_C \rho(c_2) &\Leftrightarrow [\text{Since } c_2 \in \rho] \\
 \text{lfp}(\rho f) &\leq_C c_2
 \end{aligned}$$

It remains to prove that the Kleene iterates of $\text{lfp}(\rho f)$ are finitely many. Observe that, since ρ and f are monotone and $\perp \leq_C \rho f(\perp)$, we have that

$$(\rho f)^n(\perp) \leq_C (\rho f)^{n+1}(\perp) \text{ for all } n \geq 1 .$$

If $\langle C, \leq_C \rangle$ is ACC then, by definition, there are no infinite ascending chains, hence the sequence of Kleene iterates

$$\perp \leq_C \rho f(\perp) \leq_C (\rho f)^2(\perp) \leq_C \dots \leq_C (\rho f)^n(\perp)$$

converges in finitely many steps.

In the following, we will apply this general abstraction scheme to a number of different language inclusion problems, by designing inclusion algorithms which rely on several different backward complete abstractions of $\wp(\Sigma^*)$.

4.2 An Algorithmic Framework for Language Inclusion

4.2.1 Languages as Fixed Points

Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. Recall that the language accepted by $\mathcal{N}$ is given by $\mathcal{L}(\mathcal{N}) \stackrel{\text{def}}{=} W_{I,F}^{\mathcal{N}}$ and, therefore,

$$\mathcal{L}(\mathcal{N}) = \bigcup_{q \in I} W_{q,F}^{\mathcal{N}} = \bigcup_{q \in F} W_{I,q}^{\mathcal{N}} \quad (4.2)$$

where, as usual, $\bigcup \emptyset = \emptyset$.

Let us recall how to define the language accepted by an automaton as a solution of a set of equations [Schützenberger 1963]. To do that, given a generic boolean predicate $p(x)$ (typically a membership predicate) on some set and two generic sets T and F , we define the following parametric choice function:

$$\psi_F^T(p(x)) \stackrel{\text{def}}{=} \begin{cases} T & \text{if } p(x) \text{ holds} \\ F & \text{otherwise} \end{cases} .$$

The NFA $\mathcal{N}$ induces the following set of equations, where the X_q 's are variables of type $X_q \in \wp(\Sigma^*)$ and are indexed by states $q \in Q$:

$$\text{Eqn}(\mathcal{N}) \stackrel{\text{def}}{=} \{X_q = \psi_{\emptyset}^{\{\epsilon\}}(q \in F) \cup \bigcup_{a \in \Sigma, q' \in \delta(q, a)} aX_{q'} \mid q \in Q\} . \quad (4.3)$$

Thus, the functions in the right-hand side of the equations in $\text{Eqn}(\mathcal{N})$ have type $\wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)$. Since $\langle \wp(\Sigma^*)^{|Q|}, \subseteq \rangle$ is a (product) complete lattice (because $\langle \wp(\Sigma^*), \subseteq \rangle$ is a complete lattice) and all the right-hand side functions in $\text{Eqn}(\mathcal{N})$ are clearly monotone, the least solution $\langle Y_q \rangle_{q \in Q} \in \wp(\Sigma^*)^{|Q|}$ of $\text{Eqn}(\mathcal{N})$ does exist and it is easy to check that for every $q \in Q$, $Y_q = W_{q, F}^{\mathcal{N}}$ holds, hence, by Equation (4.2), $\mathcal{L}(\mathcal{N}) = \bigcup_{q_i \in I} Y_{q_i}$.

It is worth noticing that, by relying on right concatenations rather than left ones $aX_{q'}$ used in $\text{Eqn}(\mathcal{N})$, one could also define a set of symmetric equations whose least solution coincides with $\langle W_{I, q}^{\mathcal{N}} \rangle_{q \in Q}$ instead of $\langle W_{q, F}^{\mathcal{N}} \rangle_{q \in Q}$.

Figure 4.1: An NFA $\mathcal{N}$ with $\mathcal{L}(\mathcal{N}) = (a + (b^+a))^*$.

Example 4.2.1. Let us consider the automaton $\mathcal{N}$ in Figure 4.1. The set of equations induced by $\mathcal{N}$ are as follows:

$$\text{Eqn}(\mathcal{N}) = \begin{cases} X_1 = \{\epsilon\} \cup aX_1 \cup bX_2 \\ X_2 = \emptyset \cup aX_1 \cup bX_2 \end{cases} . \quad \diamond$$

It is convenient to state the equations in $\text{Eqn}(\mathcal{N})$ by exploiting an “initial” vector $\vec{\epsilon}^F \in \wp(\Sigma^*)^{|Q|}$ and a predecessor function $\text{Pre}_{\mathcal{N}}: \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$ defined as follows:

$$\vec{\epsilon}^F \stackrel{\text{def}}{=} \langle \psi_{\emptyset}^{\{\epsilon\}}(q \in F) \rangle_{q \in Q}, \quad \text{Pre}_{\mathcal{N}}(\langle X_q \rangle_{q \in Q}) \stackrel{\text{def}}{=} \langle \bigcup_{a \in \Sigma, q' \in \delta(q, a)} aX_{q'} \rangle_{q \in Q} .$$

The intuition for the function $\text{Pre}_{\mathcal{N}}$ is that given the language $W_{q', F}^{\mathcal{N}}$ and a transition $q' \in \delta(q, a)$, we have that $aW_{q', F}^{\mathcal{N}} \subseteq W_{q, F}^{\mathcal{N}}$ holds, i.e. given a subset $X_{q'}$ of the language generated by $\mathcal{N}$ from some state q' , the function $\text{Pre}_{\mathcal{N}}$ computes a subset X_q of the language generated by $\mathcal{N}$ for its predecessor state q .

Since $\epsilon \in W_{q, F}^{\mathcal{N}}$ for all $q \in F$, the least fixpoint computation can start from the vector $\vec{\epsilon}^F$ and iteratively apply $\text{Pre}_{\mathcal{N}}$. Therefore

$$\langle W_{q, F}^{\mathcal{N}} \rangle_{q \in Q} = \text{lfp}(\lambda \vec{X}. \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) . \quad (4.4)$$

Together with Equation (4.2), it follows that $\mathcal{L}(\mathcal{N})$ equals the union of the component languages of the vector $\text{lfp}(\lambda \vec{X}. \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}))$ indexed by the initial states in I .

Example 4.2.2 (Continuation of Example 4.2.1). *The fixpoint characterization of the vector $\langle W_{q,F}^{\mathcal{N}} \rangle_{q \in Q}$ is:*

$$\begin{pmatrix} W_{q_1,q_1}^{\mathcal{N}} \\ W_{q_2,q_1}^{\mathcal{N}} \end{pmatrix} = \text{lfp} \left(\lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} . \begin{pmatrix} \{\epsilon\} \cup aX_1 \cup bX_2 \\ \emptyset \cup aX_1 \cup bX_2 \end{pmatrix} \right) = \begin{pmatrix} (a + (b^+a))^* \\ (a + b)^*a \end{pmatrix} . \quad \diamond$$

Fixpoint-based Inclusion Check

Consider the language inclusion problem $L_1 \subseteq L_2$, where $L_1 = \mathcal{L}(\mathcal{N})$ for some NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$. The language L_2 can be formalized as a vector in $\wp(\Sigma^*)^{|Q|}$ as follows:

$$\vec{L}_2^I \stackrel{\text{def}}{=} \langle \psi_{\Sigma^*}^{L_2}(q \in I) \rangle_{q \in Q} \quad (4.5)$$

whose components indexed by initial states are L_2 and those indexed by non-initial states are Σ^* . Then, as a consequence of Equations (4.2), (4.4) and (4.5), we have that

$$\mathcal{L}(\mathcal{N}) \subseteq L_2 \Leftrightarrow \text{lfp}(\lambda \vec{X}. \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) \subseteq \vec{L}_2^I . \quad (4.6)$$

4.2.2 Abstract Inclusion Check using Closures

In what follows we will apply Theorem 4.1.1 for solving the language inclusion problem where: $C = \langle \wp(\Sigma^*)^{|Q|}, \subseteq \rangle$, $f = \lambda \vec{X}. \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})$ and $\rho \in \text{uco}(\wp(\Sigma^*))$, so that $\rho \in \text{uco}(\wp(\Sigma^*)^{|Q|})$.

THEOREM 4.2.3. *Let $\rho \in \text{uco}(\wp(\Sigma^*))$ be backward complete for $\lambda X \in \wp(\Sigma^*). aX$ for all $a \in \Sigma$ and let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. Then the extension of ρ to vectors, $\rho \in \text{uco}(\wp(\Sigma^*)^{|Q|})$, is backward complete for $\text{Pre}_{\mathcal{N}}$ and $\lambda \vec{X}. \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})$.*

Proof. First, it turns out that:

$$\begin{aligned} \rho(\text{Pre}_{\mathcal{N}}(\langle X_q \rangle_{q \in Q})) &= [\text{By definition of } \text{Pre}_{\mathcal{N}}] \\ \rho(\bigcup_{a \in \Sigma, q' \in \delta(q,a)} aX_{q'}) &= [\text{By Equation (3.2)}] \\ \rho(\bigcup_{a \in \Sigma, q' \in \delta(q,a)} \rho(aX_{q'})) &= [\text{By backward completeness of } \rho \text{ for } \lambda X. aX] \\ \rho(\bigcup_{a \in \Sigma, q' \in \delta(q,a)} \rho(a\rho(X_{q'}))) &= [\text{By Equation (3.2)}] \\ \rho(\bigcup_{a \in \Sigma, q' \in \delta(q,a)} a\rho(X_{q'})) &= [\text{By definition of } \text{Pre}_{\mathcal{N}}] \\ \rho(\text{Pre}_{\mathcal{N}}(\rho(\langle X_q \rangle_{q \in Q}))) & . \end{aligned}$$

Next, we show backward completeness of ρ for $\lambda\vec{X}. \vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})$:

$$\begin{aligned} \rho(\vec{\epsilon}^F \cup \text{Pre}_N(\rho(\vec{X}))) &= \quad [\text{By Equation (3.2)}] \\ \rho(\rho(\vec{\epsilon}^F) \cup \rho(\text{Pre}_N(\rho(\vec{X})))) &= \quad [\text{By backward completeness of } \rho \text{ for } \text{Pre}_N] \\ \rho(\rho(\vec{\epsilon}^F) \cup \rho(\text{Pre}_N(\vec{X}))) &= \quad [\text{By Equation (3.2)}] \\ \rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})) &. \end{aligned}$$

Then, by Equation (4.1), we obtain the following result.

COROLLARY 4.2.4. *If $\rho \in \text{uco}(\wp(\Sigma^*))$ is backward complete for $\lambda X \in \wp(\Sigma^*). aX$ for all $a \in \Sigma$ then*

$$\rho(\text{lfp}(\lambda\vec{X}. \vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) = \text{lfp}(\lambda\vec{X}. \rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) .$$

Note that if ρ is backward complete for $\lambda X.aX$ for all $a \in \Sigma$ and $L_2 \in \rho$ then, as a consequence of Theorem 4.1.1 and Corollary 4.2.4, we find that Equivalence (4.6) becomes

$$\mathcal{L}(N) \subseteq L_2 \Leftrightarrow \text{lfp}(\lambda\vec{X}. \rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) \subseteq \vec{L}_2^I . \quad (4.7)$$

4.2.2.1 Right Concatenation

Let us consider the symmetric case of right concatenation. Recall that, given an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$, we have that

$$W_{I,q}^{\mathcal{N}} = \psi_{\emptyset}^{\{\epsilon\}}(q \in I) \cup \bigcup_{a \in \Sigma, a \in W_{q',q}^{\mathcal{N}}} W_{I,q'}^{\mathcal{N}} .$$

Correspondingly, we can define a set of fixpoint equations on $\wp(\Sigma^*)$ which is based on right concatenation and is symmetric to Equation (4.3):

$$\text{Eqn}^r(\mathcal{N}) \stackrel{\text{def}}{=} \{X_q = \psi_{\emptyset}^{\{\epsilon\}}(q \in I) \cup \bigcup_{a \in \Sigma, q \in \delta(q',a)} X_{q'} a \mid q \in Q\} .$$

In this case, if $\vec{Y} = \langle Y_q \rangle_{q \in Q}$ is the least fixpoint solution of $\text{Eqn}^r(\mathcal{N})$ then we have that $Y_q = W_{I,q}^{\mathcal{N}}$ for every $q \in Q$. Also, by defining $\vec{\epsilon}^I \in \wp(\Sigma^*)^{|Q|}$ and $\text{Post}_N: \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$ as follows:

$$\vec{\epsilon}^I \stackrel{\text{def}}{=} \langle \psi_{\emptyset}^{\{\epsilon\}}(q \in I) \rangle_{q \in Q}, \quad \text{Post}_N(\langle X_q \rangle_{q \in Q}) \stackrel{\text{def}}{=} \langle \bigcup_{a \in \Sigma, q \in \delta(q',a)} X_{q'} a \rangle_{q \in Q} ,$$

we have that

$$\langle W_{I,q} \rangle_{q \in Q} = \text{lfp}(\lambda\vec{X}. \vec{\epsilon}^I \cup \text{Post}_N(\vec{X})) . \quad (4.8)$$

Since, by Equation (4.2), we have that $\mathcal{L}(\mathcal{N}) = \bigcup_{q \in F} W_{I,q}$, it follows that $\mathcal{L}(\mathcal{N})$ is the union of the component languages of the vector $\text{lfp}(\lambda \vec{X}. \vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\vec{X}))$ indexed by the final states in F .

Example 4.2.5. Consider again the NFA $\mathcal{N}$ in Figure 4.1. The set of right equations for $\mathcal{N}$ is:

$$\text{Eqn}^r(\mathcal{N}) = \begin{cases} X_1 = \{\epsilon\} \cup X_1 a \cup X_2 a \\ X_2 = \emptyset \cup X_1 b \cup X_2 b \end{cases}$$

so that

$$\begin{pmatrix} W_{q_1, q_1} \\ W_{q_1, q_2} \end{pmatrix} = \text{lfp} \left(\lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} . \begin{pmatrix} \{\epsilon\} \cup X_1 a \cup X_2 a \\ \emptyset \cup X_1 b \cup X_2 b \end{pmatrix} \right) = \begin{pmatrix} (a + (b^+ a))^* \\ a^* b (b + a^+ b)^* \end{pmatrix} . \quad \diamond$$

Finally, given a language inclusion problem $\mathcal{L}(\mathcal{N}) \subseteq L_2$, the language L_2 can be formalized as the vector

$$\vec{L}_2^F \stackrel{\text{def}}{=} \langle \psi_{\Sigma^*}^{L_2}(q \in F) \rangle_{q \in Q} \in \wp(\Sigma^*)^{|Q|} ,$$

so that, by Equation (4.8), it turns out that

$$\mathcal{L}(\mathcal{N}) \subseteq L_2 \Leftrightarrow \text{lfp}(\lambda \vec{X}. \vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\vec{X})) \subseteq \vec{L}_2^F$$

We therefore have the following symmetric version of Theorem 4.2.3 for right concatenation.

THEOREM 4.2.6. Let $\rho \in \text{uco}(\wp(\Sigma^*))$ be backward complete for $\lambda X \in \wp(\Sigma^*). Xa$ for all $a \in \Sigma$ and let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA. Then the extension of ρ to vectors, $\rho \in \text{uco}(\wp(\Sigma^*)^{|Q|})$, is backward complete for $\text{Post}_{\mathcal{N}}(\vec{X})$ and $\lambda \vec{X}. \vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\vec{X})$.

Proof. First, it turns out that:

$$\begin{aligned} \rho(\text{Post}_{\mathcal{N}}(\langle X_q \rangle_{q \in Q})) &= \text{[By definition of } \text{Post}_{\mathcal{N}}\text{]} \\ \rho(\bigcup_{a \in \Sigma, q \in \delta(q', a)} X_{q'} a) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{a \in \Sigma, q \in \delta(q', a)} \rho(X_{q'} a)) &= \text{[By backward completeness of } \rho \text{ for } \lambda X. Xa\text{]} \\ \rho(\bigcup_{a \in \Sigma, q \in \delta(q', a)} \rho(\rho(X_{q'} a))) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{a \in \Sigma, q \in \delta(q', a)} \rho(X_{q'} a)) &= \text{[By definition of } \text{Post}_{\mathcal{N}}\text{]} \\ \rho(\text{Post}_{\mathcal{N}}(\rho(\langle X_q \rangle_{q \in Q}))) & . \end{aligned}$$

Next, we show backward completeness of ρ for $\lambda \vec{X}. \vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\vec{X})$:

$$\rho(\vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\rho(\vec{X}))) = \text{[By Equation (3.2)]}$$

$$\begin{aligned}
 \rho(\rho(\vec{\epsilon}^I) \cup \rho(\text{Post}_N(\rho(\vec{X})))) &= \quad [\text{By backward completeness of } \rho \text{ for } \text{Post}_N] \\
 \rho(\rho(\vec{\epsilon}^I) \cup \rho(\text{Post}_N(\vec{X}))) &= \quad [\text{By Equation (3.2)}] \\
 \rho(\vec{\epsilon}^I \cup \text{Post}_N(\vec{X})) &.
 \end{aligned}$$

4.2.3 Solving the Abstract Inclusion Check

In this section we present two techniques for solving the language inclusion problem $\mathcal{L}(N) \subseteq L_2$ by relying on Equivalence (4.7).

The first of these techniques leads to algorithms for solving the inclusion problem by using *finite languages*. Intuitively, given a closure ρ , we show that it is possible to work on the domain $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ while considering only languages S that are finite.

On the other hand, we present a second technique that relies on the use of Galois Connections in order to solve the language inclusion problem in a different domain. This technique allows us to decide the inclusion $\mathcal{L}(N) \subseteq L_2$ by manipulating the underlying automata representation of the language L_2 .

4.2.3.1 Using Finite Languages

The following result shows that the successive steps of the fixpoint iteration for $\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})))$ can be replicated by iterating on a function f , instead of $\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))$, and then abstracting the result, provided that f meets a set of requirements.

LEMMA 4.2.7. *Let $N = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA, let $\rho \in \text{uco}(\Sigma^*)$ be backward complete for $\lambda X \in \wp(\Sigma^*). aX$ for all $a \in \Sigma$ and let $f : \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$ be a function such that $\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})) = \rho(f(\vec{X}))$. Then, for all $0 \leq n$,*

$$(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))^n = \rho(f^n(\vec{X})) .$$

Proof. We proceed by induction in n .

- *Base case:* Let $n = 0$. Then $f^0(\vec{X}) = (\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})))^0 = \vec{\emptyset}$.
- *Inductive step:* Assume that $\rho(f^n(\vec{X})) = (\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})))^n$ holds for some value $n \geq 0$. To simplify the notation, let $\mathcal{P}(\vec{X}) = \vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})$ so that $\rho f^n = (\rho \mathcal{P})^n$. Then

$$\begin{aligned}
 \rho f^{n+1}(\vec{X}) &= \quad [\text{Since } f^{n+1} = f^n f] \\
 \rho f^n f(\vec{X}) &= \quad [\text{By Inductive Hypothesis}]
 \end{aligned}$$

$$\begin{aligned}
 (\rho\mathcal{P})^n f(\vec{X}) &= \text{[By Theorem 4.2.3, } \rho \text{ is bw. complete for } \mathcal{P}] \\
 (\rho\mathcal{P})^n \rho f(\vec{X}) &= \text{[Since } \rho f = \rho\mathcal{P}] \\
 (\rho\mathcal{P})^n \rho\mathcal{P}(\vec{X}) &= \text{[Since } (\rho\mathcal{P})^{n+1} = (\rho\mathcal{P})^n \rho\mathcal{P}] \\
 (\rho\mathcal{P})^{n+1}(\vec{X})
 \end{aligned}$$

We conclude that $(\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})))^n = \rho(f^n(\vec{X}))$ for all $0 \leq n$.

Lemma 4.2.7 shows that the iterates of $\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})))$ can be computed by abstracting the iterates of a function f , which might manipulate only finite languages. Moreover, it is straightforward to check that Lemma 4.2.7 remains valid when considering a different function f at each step of the iteration as long as all the considered functions satisfy the requirements.

To simplify the notation, given a set of functions $\mathcal{F}$ and a function f , we write $\mathcal{F}f$ to denote the composition of one arbitrary function from $\mathcal{F}$ with f . Similarly, $f\mathcal{F}$ denotes the composition of f with an arbitrary function from $\mathcal{F}$. Finally, we write $\mathcal{F}^2 = f$, for instance, to indicate that any composition of two functions in $\mathcal{F}$ equals f .

COROLLARY 4.2.8. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA, let $\rho \in \text{uco}(\Sigma^*)$ be backward complete for $\lambda X \in \wp(\Sigma^*). aX$ for all $a \in \Sigma$ and let $\mathcal{F}$ be a set of functions such that every function $f \in \mathcal{F}$ is of the form $f : \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$ and satisfies $\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) = \rho(f(\vec{X}))$. Then, for all $0 \leq n$,*

$$(\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})))^n = \rho(\mathcal{F}^n(\vec{X})) .$$

Observe that, in particular, Corollary 4.2.8 holds when considering the set $\mathcal{F} = \{f\}$ with $f = \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})$. Intuitively, this means that we can compute the least fixpoint for $\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}))$ by iterating on $\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})$ until we reach an *abstract fixpoint*, i.e. the abstraction of two consecutive steps coincide.

The idea of recursively applying a function f until its abstraction reaches a fixpoint is captured by the following definition of the *abstract Kleene procedure*:

$$\widehat{\text{KLEENE}}(\text{AbsEq}, f, b) \stackrel{\text{def}}{=} \begin{cases} x := b; \\ \textbf{while } \neg \text{AbsEq}(f(x), x) \textbf{ do } x := f(x); \\ \textbf{return } x; \end{cases} ,$$

where $\text{AbsEq}(x, y)$ is a function that returns *true* iff the abstraction of x and y coincide, i.e. $\rho(x) = \rho(y)$. Clearly, $\widehat{\text{KLEENE}}(\text{id}, f, b) = \text{KLEENE}(f, b)$ where $\text{id}(x, y)$ returns *true* iff $x = y$. For simplicity, we abuse of notation and write

$\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, b)$ to denote the abstract $\widehat{\text{KLEENE}}$ iteration where, at each step, an arbitrary function from the set $\mathcal{F}$ is applied.

As the following lemma shows, whenever the domain $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is ACC and the abstraction ρ is backward complete for all the functions in the set $\mathcal{F}$, i.e. $\rho\mathcal{F} = \rho\mathcal{F}\rho$, the procedure $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, b)$ can be used to compute $\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})))$.

LEMMA 4.2.9. *Let $N = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA, let $\rho \in \text{uco}(\Sigma^*)$ be backward complete for $\lambda X \in \wp(\Sigma^*)$. aX for all $a \in \Sigma$ such that $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO. Let $\mathcal{F}$ be a set of monotone functions such that every $f \in \mathcal{F}$ is of the form $f : \wp(\Sigma^*)^{|\mathcal{Q}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{Q}|}$ and satisfies $\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})) = \rho(f(\vec{X}))$. Then,*

$$\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right) .$$

Moreover, the iterates of $\widehat{\text{KLEENE}}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})), \vec{\emptyset})$ coincide in lockstep with the abstraction of the iterates of $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset})$.

Proof. Since $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO, by Theorem 3.5.1, we have that

$$\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) = \widehat{\text{KLEENE}}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})), \vec{\emptyset})$$

On the other hand, by Corollary 4.2.8, the iterates of the above Kleene iteration coincide in lockstep with the abstraction of the iterates of $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset})$ and, therefore,

$$\widehat{\text{KLEENE}}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X})), \vec{\emptyset}) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right)$$

As a consequence,

$$\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}))) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right) .$$

The following result relies on the $\widehat{\text{KLEENE}}$ procedure to design an algorithm that solves the language inclusion problem $\mathcal{L}(N) \subseteq L_2$ whenever the abstraction ρ and the set of functions $\mathcal{F}$ satisfy a list of requirements in terms of backward completeness and computability.

THEOREM 4.2.10. *Let $N = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA, let L_2 be a regular language, let $\rho \in \text{uco}(\Sigma^*)$ and let $\mathcal{F}$ be a set of functions. Assume that the following properties hold:*

- (i) *The abstraction ρ is backward complete for $\lambda X \in \wp(\Sigma^*)$. aX for all $a \in \Sigma$ and satisfies $\rho(L_2) = L_2$.*

- (ii) The set $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO.
- (iii) Every function f in the set $\mathcal{F}$ is of the form $f : \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$, it is computable and satisfies $\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) = \rho(f(\vec{X}))$.
- (iv) There is an algorithm, say $\text{AbsEq}^\#(\vec{X}, \vec{Y})$, which decides the abstraction equivalence $\rho(\vec{X}) = \rho(\vec{Y})$, for all $\vec{X}, \vec{Y} \in \wp(\Sigma^*)^{|Q|}$.
- (v) There is an algorithm, say $\text{Incl}^\#(\vec{X})$, which decides the inclusion $\rho(\vec{X}) \subseteq \vec{L}_2^I$, for all $\vec{X} \in \wp(\Sigma^*)^{|Q|}$.

Then, the following is an algorithm which decides whether $\mathcal{L}(\mathcal{N}) \subseteq L_2$:

```

 $\langle Y_q \rangle_{q \in Q} := \widehat{\text{KLEENE}}(\text{AbsEq}^\#, \mathcal{F}, \vec{\emptyset});$ 
return  $\text{Incl}^\#(\langle Y_q \rangle_{q \in Q});$ 
    
```

Proof. It follows from hypotheses (i), (ii) and (iii), by Lemma 4.2.9, that

$$\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}))) = \rho\left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset})\right) \quad (4.9)$$

The function AbsEq can be replaced by function $\text{AbsEq}^\#$ due to hypothesis (iv). Moreover, by Equivalence (4.7), which holds by hypothesis (i), and Equation (4.9) we have that

$$\mathcal{L}(\mathcal{N}) \subseteq L_2 \Leftrightarrow \rho\left(\widehat{\text{KLEENE}}(\text{AbsEq}^\#, \mathcal{F}, \vec{\emptyset})\right) \subseteq \vec{L}_2^I.$$

Finally, hypotheses (iv) and (v) guarantee, respectively, the decidability of the inclusion $\rho\mathcal{F}(X) \subseteq \rho(X)$ performed at each step of the $\widehat{\text{KLEENE}}$ iteration and the decidability of the inclusion of the abstraction of the lfp in $\vec{L}_2^I$.

Note that Theorem 4.2.10 can also be stated in a symmetric version for right concatenation similarly to Theorem 4.2.6.

4.2.3.2 Using Galois Connections

The next result reformulates Equivalence (4.7) by using Galois Connections rather than closures, and shows how to design an algorithm that solves a language inclusion problem $\mathcal{L}(\mathcal{N}) \subseteq L_2$ on an abstraction D of the concrete domain $\langle \wp(\Sigma^*), \subseteq \rangle$ whenever D satisfies a list of requirements related to backward completeness and computability.

THEOREM 4.2.11. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA and L_2 be a language over Σ . Let $\langle \wp(\Sigma^*), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle D, \leq_D \rangle$ be a GC where $\langle D, \leq_D \rangle$ is a poset. Assume that the following properties hold:*

- (i) $L_2 \in \gamma(D)$ and for every $a \in \Sigma$ and $X \in \wp(\Sigma^*)$, $\gamma\alpha(aX) = \gamma\alpha(a\gamma\alpha(X))$.
- (ii) $(D, \leq_D, \sqcup, \perp_D)$ is an effective domain, meaning that: $(D, \leq_D, \sqcup, \perp_D)$ is an ACC join-semilattice with bottom $\perp_D$, every element of D has a finite representation, the binary relation $\leq_D$ is decidable and the binary lub $\sqcup$ is computable.
- (iii) There is an algorithm, say $\text{Pre}^\#(\vec{X}^\#)$, which computes $\alpha(\text{Pre}_N(\gamma(\vec{X}^\#)))$, for all $\vec{X}^\# \in \alpha(\wp(\Sigma^*))^{|\mathcal{Q}|}$.
- (iv) There is an algorithm, say $\epsilon^\#$, which computes $\alpha(\vec{\epsilon}^F)$.
- (v) There is an algorithm, say $\text{Incl}^\#(\vec{X}^\#)$, which decides the abstract inclusion $\vec{X}^\# \leq_D \alpha(\vec{L}_2^I)$, for all $\vec{X}^\# \in \alpha(\wp(\Sigma^*))^{|\mathcal{Q}|}$.

Then, the following is an algorithm which decides whether $\mathcal{L}(N) \subseteq L_2$:

```

 $\langle Y_q^\# \rangle_{q \in Q} := \text{KLEENE}(\lambda \vec{X}^\#. \epsilon^\# \sqcup \text{Pre}^\#(\vec{X}^\#), \vec{\perp}_D);$ 
return  $\text{Incl}^\#(\langle Y_q^\# \rangle_{q \in Q});$ 
    
```

Proof. Let $\rho \stackrel{\text{def}}{=} \gamma\alpha \in \text{uco}(\wp(\Sigma^*))$, so that hypothesis (i) can be stated as $L_2 \in \rho$ and $\rho(aX) = \rho(a\rho(X))$. It turns out that:

$$\begin{aligned}
 \mathcal{L}(N) \subseteq L_2 &\Leftrightarrow \quad [\text{By Equivalence (4.7)}] \\
 \text{lfp}(\lambda \vec{X}^\#. \rho(\vec{\epsilon}^F \cup \text{Pre}_N(\vec{X}^\#))) &\subseteq \vec{L}_2^I \Leftrightarrow \quad [\text{By Lemma 3.6.2}] \\
 \gamma(\text{lfp}(\lambda \vec{X}^\#. \alpha(\vec{\epsilon}^F \cup \text{Pre}_N(\gamma(\vec{X}^\#)))) &\subseteq \vec{L}_2^I \Leftrightarrow \quad [\text{By GC}] \\
 \gamma(\text{lfp}(\lambda \vec{X}^\#. \alpha(\vec{\epsilon}^F) \sqcup \alpha(\text{Pre}_N(\gamma(\vec{X}^\#)))) &\subseteq \vec{L}_2^I \Leftrightarrow \quad [\text{By GC since } L_2 \in \gamma(D)] \\
 \text{lfp}(\lambda \vec{X}^\#. \alpha(\vec{\epsilon}^F) \sqcup \alpha(\text{Pre}_N(\gamma(\vec{X}^\#)))) &\leq_D \alpha(\vec{L}_2^I)
 \end{aligned}$$

By hypotheses (ii), (iii) and (iv), $\text{KLEENE}(\lambda \vec{X}^\#. \epsilon^\# \sqcup \text{Pre}^\#(\vec{X}^\#), \vec{\perp}_D)$ is an algorithm computing the least fixpoint $\text{lfp}(\lambda \vec{X}^\#. \alpha(\vec{\epsilon}^F) \sqcup \alpha(\text{Pre}_N(\gamma(\vec{X}^\#))))$. In particular, the hypotheses (ii), (iii) and (iv) ensure that the Kleene iterates of $\lambda \vec{X}^\#. \epsilon^\# \sqcup \text{Pre}^\#(\vec{X}^\#)$ starting from $\vec{\perp}_D$ are in $\alpha(\wp(\Sigma^*))^{|\mathcal{Q}|}$, computable and finitely many and that it is decidable whether the iterates have reached the fixpoint.

Finally, hypothesis (v) ensures the decidability of the $\leq_D$ -inclusion check of this least fixpoint in $\alpha(\vec{L}_2^I)$.

It is also worth noticing that, analogously to what has been done in Theorem 4.2.6, the above Theorem 4.2.11 can be also stated in a symmetric version for right (rather than left) concatenation.

4.3 Instantiating the Framework

We instantiate the general algorithmic framework of Section 4.2 to the class of closure operators induced by quasiorder relations on words.

4.3.1 Word-based Abstractions

Let $\leq \subseteq \Sigma^* \times \Sigma^*$ be a quasiorder relation on words in Σ^* . Recall that the corresponding closure operator $\rho_{\leq} \in \text{uco}(\wp(\Sigma^*))$ is defined as follows:

$$\rho_{\leq}(X) \stackrel{\text{def}}{=} \{v \in \Sigma^* \mid \exists u \in X, u \leq v\} . \quad (4.10)$$

Thus, $\rho_{\leq}(X)$ is the $\leq$ -upward closure of X and it is easy to check that $\rho_{\leq}$ is indeed a closure on the complete lattice $\langle \wp(\Sigma^*), \subseteq \rangle$.

As described in Chapter 3, the quasiorder $\leq$ is left-monotone (resp. right-monotone) iff

$$\forall x_1, x_2 \in \Sigma^*, \forall a \in \Sigma, x_1 \leq x_2 \Rightarrow ax_1 \leq ax_2 \quad (\text{resp. } x_1a \leq x_2a) \quad (4.11)$$

In fact, if $x_1 \leq x_2$ then Equation (4.11) implies that for all $y \in \Sigma^*$, $yx_1 \leq yx_2$ since, by induction on the length $|y| \in \mathbb{N}$, we have that:

- (i) if $y = \epsilon$ then $yx_1 \leq yx_2$;
- (ii) if $y = av$ with $a \in \Sigma, v \in \Sigma^*$ then, by inductive hypothesis, $vx_1 \leq vx_2$, so that by (4.11), $yx_1 = avx_1 \leq avx_2 = yx_2$

Definition 4.3.1 (*L-Consistent Quasiorder*). Let $L \in \wp(\Sigma^*)$. A quasiorder $\leq_L$ on Σ^* is called *left* (resp. *right*) *L-consistent* iff

- (a) $\leq_L \cap (L \times \neg L) = \emptyset$;
- (b) $\leq_L$ is left-monotone (resp. right-monotone).

Also, $\leq_L$ is called *L-consistent* when it is both left and right *L-consistent*. ■

As the following lemma shows, it turns out that a quasiorder is *L-consistent* iff it induces a closure which includes *L* in its image and it is backward complete for concatenation.

Lemma 4.3.2. *Let $L \in \wp(\Sigma^*)$ and $\leq_L$ be a quasiorder on Σ^* . Then, $\leq_L$ is a left (resp. right) *L-consistent* quasiorder on Σ^* if and only if*

- (a) $\rho_{\leq_L}(L) = L$, and
- (b) $\rho_{\leq_L}$ is backward complete for $\lambda X. aX$ (resp. $\lambda X. Xa$) for all $a \in \Sigma$.

Proof. We consider the left case, the right case is symmetric.

- (a) The inclusion $L \subseteq \rho_{\leq_L}(L)$ always holds because $\rho_{\leq_L}$ is an upper closure. For the reverse inclusion we have that

$$\begin{aligned} \rho_{\leq_L}(L) \subseteq L &\Leftrightarrow \quad [\text{By definition of } \rho_{\leq_L}(L)] \\ \forall v \in \Sigma^*, (\exists u \in L, u \leq_L v) &\Rightarrow v \in L \Leftrightarrow \\ &\leq_L \cap (L \times \neg L) = \emptyset . \end{aligned}$$

Thus, $\rho_{\leq_L}(L) = L$ iff condition (a) of Definition 4.3.1 holds.

- (b) We first prove that if $\leq_L$ is left-monotone then for all $X \in \wp(\Sigma^*)$ we have that $\rho_{\leq_L}(aX) = \rho_{\leq_L}(a\rho_{\leq_L}(X))$ for all $a \in \Sigma$.

Monotonicity of concatenation together with monotonicity and extensivity of $\rho_{\leq_L}$ imply that the inclusion $\rho_{\leq_L}(aX) \subseteq \rho_{\leq_L}(a\rho_{\leq_L}(X))$ holds. For the reverse inclusion, we have that:

$$\begin{aligned} \rho_{\leq_L}(a\rho_{\leq_L}(X)) &= \quad [\text{By definition of } \rho_{\leq_L}] \\ \rho_{\leq_L}(\{ay \mid \exists x \in X, x \leq_L y\}) &= \quad [\text{By definition of } \rho_{\leq_L}] \\ \{z \mid \exists x \in X, y \in \Sigma^*, x \leq_L y \wedge ay \leq_L z\} &\subseteq \quad [\text{By monotonicity of } \leq_L] \\ \{z \mid \exists x \in X, y \in \Sigma^*, ax \leq_L ay \wedge ay \leq_L z\} &= \quad [\text{By transitivity of } \leq_L] \\ \{z \mid \exists x \in X, ax \leq_L z\} &= \quad [\text{By definition of } \rho_{\leq_L}] \\ \rho_{\leq_L}(aX) . \end{aligned}$$

Next, we show that if $\rho_{\leq_L}(aX) = \rho_{\leq_L}(a\rho_{\leq_L}(X))$ for all $X \in \wp(\Sigma^*)$ and $a \in \Sigma$ then $\leq_L$ is left-monotone.

Let $x_1, x_2 \in \Sigma^*$ and $a \in \Sigma$. If $x_1 \leq_L x_2$ then $\{x_2\} \subseteq \rho_{\leq_L}(\{x_1\})$, hence $a\{x_2\} \subseteq a\rho_{\leq_L}(\{x_1\})$. Then, by applying the monotone function $\rho_{\leq_L}$ we have that $\rho_{\leq_L}(a\{x_2\}) \subseteq \rho_{\leq_L}(a\rho_{\leq_L}(\{x_1\}))$, so that, by backward completeness, $\rho_{\leq_L}(a\{x_2\}) \subseteq \rho_{\leq_L}(a\{x_1\})$. Therefore, $a\{x_2\} \subseteq \rho_{\leq_L}(a\{x_1\})$, namely, $ax_1 \leq_L ax_2$. By (4.11), this shows that $\leq_L$ is left-monotone.

Since $\rho_{\leq}(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) = \rho_{\leq}(\lfloor \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}) \rfloor)$ for every quasiorder, it follows from Lemma 4.3.2 that we can apply Theorem 4.2.10 with the abstraction $\rho_{\leq_{L_2}}$ induced by a left L_2 -consistent well-quasiorder and $\mathcal{F} = \lfloor \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}) \rfloor$ interpreted as the set of functions of the form $f_i = \lfloor \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}) \rfloor_i$ where each $\lfloor \cdot \rfloor_i$ is a function mapping each set $X \in \wp(\Sigma^*)$ into a minor $\lfloor X \rfloor_i$. Intuitively, this means that we can manipulate $\leq$ -upward closed sets in $\wp(\Sigma^*)$ using their finite minors, as already shown by Abdulla et al. [1996].

As a consequence, we obtain Algorithm FAIncW which, given a left L_2 -

consistent well-quasiorder, solves the language inclusion problem $\mathcal{L}(\mathcal{N}) \subseteq L_2$ for any automaton $\mathcal{N}$. The algorithm is called “word-based” because the vector $\langle Y_q \rangle_{q \in Q}$ consists of finite sets of words in Σ^* .

FAIncW: Word-based algorithm for $\mathcal{L}(\mathcal{N}) \subseteq L_2$

Data: NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$; decision procedure for $u \in L_2$; decidable left L_2 -consistent wqo $\leq_{L_2}^\ell$.

- 1 $\langle Y_q \rangle_{q \in Q} := \overline{\text{KLEENE}}(\sqsubseteq_{\leq_{L_2}^\ell} \cap (\sqsupseteq_{\leq_{L_2}^\ell})^{-1}, \lambda \vec{X}. \lfloor \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}) \rfloor, \vec{\emptyset})$;
- 2 **forall** $q \in I$ **do**
- 3 **forall** $u \in Y_q$ **do**
- 4 **if** $u \notin L_2$ **then return false**;
- 5 **return true**;

THEOREM 4.3.3. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA and let $L_2 \in \wp(\Sigma^*)$ be a language such that: (i) membership in L_2 is decidable; (ii) there exists a decidable left L_2 -consistent wqo on Σ^* . Then, Algorithm FAIncW decides the inclusion problem $\mathcal{L}(\mathcal{N}) \subseteq L_2$.*

Proof. Let $\leq_{L_2}^\ell$ be a decidable left L_2 -consistent well-quasiorder on Σ^* . Then, we check that hypothesis (i)-(v) of Theorem 4.2.10 are satisfied.

- (a) It follows from hypothesis (ii) and Lemma 4.3.2 that $\leq_{L_2}^\ell$ is backward complete for left concatenation and satisfies $\rho_{\leq_{L_2}^\ell}(L_2) = L_2$.
- (b) Since $\leq_{L_2}^\ell$ is a wqo, then $\langle \{\rho_{\leq_{L_2}^\ell}(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO.
- (c) Let $\lfloor \vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}) \rfloor$ be the set of functions f_i each of which maps each set $X \in \wp(\Sigma^*)$ into a minor of $\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})$. Since $\rho_{\leq_{L_2}^\ell}(X) = \rho_{\leq_{L_2}^\ell}(\lfloor X \rfloor)$ for all $X \in \wp(\Sigma^*)^{|Q|}$, we have that all functions f_i satisfy

$$\rho_{\leq_{L_2}^\ell}(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X})) = \rho_{\leq_{L_2}^\ell}(f_i(\vec{X})) .$$

- (d) The equality $\rho_{\leq_{L_2}^\ell}(S_1) = \rho_{\leq_{L_2}^\ell}(S_2)$ is decidable for every $S_1, S_2 \in \wp(\Sigma^*)^{|Q|}$ since

$$\rho_{\leq_{L_2}^\ell}(S_1) = \rho_{\leq_{L_2}^\ell}(S_2) \Leftrightarrow S_1 \sqsubseteq_{\leq_{L_2}^\ell} S_2 \wedge S_2 \sqsubseteq_{\leq_{L_2}^\ell} S_1$$

and, by hypothesis (ii), $\leq_{L_2}^\ell$ is decidable.

- (e) Since $\vec{L}_2^I = \langle \psi_{\Sigma^*}^{L_2}(q \in I) \rangle_{q \in Q}$, the inclusion trivially holds for all components Y_q with $q \notin I$. Therefore, it suffices to check whether $Y_q \subseteq L_2$

holds for $q \in I$ which, since $Y_q = \lfloor S \rfloor$ with $S \in \wp(\Sigma^*)$, can be decided by performing finitely many membership tests as done by lines 2-5 of Algorithm [FAIncW](#). By hypothesis (i), this check is decidable.

4.3.1.1 Right Concatenation

Following Section 4.2.2.1, a symmetric version, called [FAIncWr](#), of Algorithm [FAIncW](#) and of Theorem 4.3.3 for *right* L_2 -consistent wqos can be easily derived as follows.

FAIncWr: Word-based algorithm for $\mathcal{L}(\mathcal{N}) \subseteq L_2$

Data: NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$; decision procedure for $u \in L_2$; decidable right L_2 -consistent wqo $\leq_{L_2}^r$.

```

1  $\langle Y_q \rangle_{q \in Q} := \widehat{\text{KLEENE}}(\sqsubseteq_{\leq_{L_2}^r} \cap (\sqsubseteq_{\leq_{L_2}^r})^{-1}, \lambda \vec{X}. \lfloor \vec{\epsilon}^I \cup \text{Post}_{\mathcal{N}}(\vec{X}) \rfloor, \vec{\emptyset})$ ;
2 forall  $q \in F$  do
3   forall  $u \in Y_q$  do
4     if  $u \notin L_2$  then return false;
5 return true;
```

THEOREM 4.3.4. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA and let $L_2 \in \wp(\Sigma^*)$ be a language such that (i) membership in L_2 is decidable; (ii) there exists a decidable right L_2 -consistent wqo on Σ^* . Then, Algorithm [FAIncWr](#) decides the inclusion problem $\mathcal{L}(\mathcal{N}) \subseteq L_2$.*

In the following, we will consider different quasiorders on Σ^* and we will show that they fulfill the requirements of Theorem 4.3.3, so that they yield algorithms for solving a language inclusion problem.

4.3.2 Nerode Quasiorders

Recall from Chapter 3 that the *left* and *right* Nerode's quasiorders on Σ^* are defined in the standard way:

$$u \leq_L^\ell v \stackrel{\text{def}}{\Leftrightarrow} Lu^{-1} \subseteq Lv^{-1}, \quad u \leq_L^r v \stackrel{\text{def}}{\Leftrightarrow} u^{-1}L \subseteq v^{-1}L.$$

The following result shows that Nerode's quasiorders are the weakest (i.e. greatest w.r.t. set inclusion of binary relations) L_2 -consistent quasiorders for which the algorithm [FAIncW](#) can be instantiated to decide a language inclusion $\mathcal{L}(\mathcal{N}) \subseteq L_2$.

LEMMA 4.3.5. *Let $L \in \wp(\Sigma^*)$. Then*

- (a) $\leq_L^\ell$ and $\leq_L^r$ are, respectively, left and right L -consistent quasiorders. If L is regular then, additionally, $\leq_L^\ell$ and $\leq_L^r$ are, respectively, decidable wqos.
- (b) Let $\leq^\ell$ (resp. $\leq^r$) be a left (resp. right) L -consistent quasiorder on Σ^* . Then $\rho_{\leq_L^\ell} \subseteq \rho_{\leq^\ell}$ (resp. $\rho_{\leq_L^r} \subseteq \rho_{\leq^r}$).

Proof.

- (a) As explained in Chapter 3, de Luca and Varricchio [1994, Section 2] show that $\leq_L^\ell$ and $\leq_L^r$ are, respectively, left and right monotone quasiorders. On the other hand, note that given $u \in L$ and $v \notin L$ we have that $\epsilon \in Lu^{-1}$ and $\epsilon \in u^{-1}L$ while $\epsilon \notin Lv^{-1}$ and $\epsilon \notin v^{-1}L$. Hence, $\leq_L^\ell$ (resp. $\leq_L^r$) is a left (resp. right) L -consistent quasiorder.

Finally, if L is regular then both relations are clearly decidable.

- (b) We consider the left case (the right case is symmetric).

As shown by de Luca and Varricchio [1994, Section 2, point 4], $\leq_L^\ell$ is maximum in the set of all the left L -consistent quasiorders, i.e. every left L -consistent quasiorder $\leq^\ell$ is such that $x \leq^\ell y \Rightarrow x \leq_L^\ell y$. As a consequence, $\rho_{\leq^\ell}(X) \subseteq \rho_{\leq_L^\ell}(X)$ holds for all $X \in \wp(\Sigma^*)$, namely, $\leq^\ell \subseteq \leq_L^\ell$.

We then derive a first instantiation of Theorem 4.3.3. Because membership is decidable for regular languages L_2 , Lemma 4.3.5 (a) for $\leq_{L_2}^\ell$ implies that the hypotheses (i) and (ii) of Theorem 4.3.3 are satisfied, so that Algorithm FAIncW instantiated to $\leq_{L_2}^\ell$ decides the inclusion $\mathcal{L}(\mathcal{N}) \subseteq L_2$ when L_2 is regular.

Furthermore, under these hypotheses, Lemma 4.3.5 (b) shows that $\leq_{L_2}^\ell$ is the weakest (i.e. greatest for set inclusion) left L_2 -consistent quasiorder for which the algorithm FAIncW can be instantiated for deciding the inclusion $\mathcal{L}(\mathcal{N}) \subseteq L_2$.

Example 4.3.6. *We illustrate the use of the left Nerode's quasiorder in the algorithm FAIncW for solving the language inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the automata shown in Figure 4.2. The equations for $\mathcal{N}_1$ are as follows:*

$$\text{Eqn}(\mathcal{N}_1) = \begin{cases} X_1 = \emptyset \cup aX_1 \cup aX_2 \cup bX_2 \cup cX_2 \\ X_2 = \{\epsilon\} \end{cases}.$$

We have the following quotients (among others) for $L = \mathcal{L}(\mathcal{N}_2)$.

$$\begin{aligned} L\epsilon^{-1} &= a^*(a(a+b)^*a + a^+c + ab + bb) & Lb^{-1} &= a^*(a+b) \\ La^{-1} &= a^*a(a+b)^* & Lc^{-1} &= a^*a^+ \\ Lw^{-1} &= a^* \text{ iff } w \in (a(a+b)^*a + ac + ab + bb) \end{aligned}$$

Figure 4.2: Two automata N_1 (left) and N_2 (right) generating the regular languages $\mathcal{L}(N_1) = a^*(a + b + c)$ and $\mathcal{L}(N_2) = a^*(a(a + b)^*a + a^+c + ab + bb)$.

It is straightforward to check that, among others, the following relations hold between different alphabet symbols: $b \leq_L^\ell a$, $c \leq_L^\ell a$ and $c \leq_L^\ell b$. Then, let us show the computation of the Kleene iterates performed by Algorithm [FAIncW](#).

$$\vec{Y}^{(0)} = \vec{\emptyset}$$

$$\vec{Y}^{(1)} = \vec{\epsilon}^F = \langle \emptyset, \{\epsilon\} \rangle$$

$$\begin{aligned} \vec{Y}^{(2)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(1)}) \rfloor = \langle \emptyset, \{\epsilon\} \rangle \sqcup \langle \lfloor \emptyset \cup a\emptyset \cup a\{\epsilon\} \cup b\{\epsilon\} \cup c\{\epsilon\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle \\ &= \langle \lfloor \{a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{c\}, \{\epsilon\} \rangle \end{aligned}$$

$$\begin{aligned} \vec{Y}^{(3)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(2)}) \rfloor = \langle \emptyset, \{\epsilon\} \rangle \sqcup \langle \lfloor \emptyset \cup a\{c\} \cup a\{\epsilon\} \cup b\{\epsilon\} \cup c\{\epsilon\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle \\ &= \langle \lfloor \{ac, a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{c\}, \{\epsilon\} \rangle \end{aligned}$$

The least fixpoint is thus $\vec{Y} = \langle \{c\}, \{\epsilon\} \rangle$. Since $c \in \vec{Y}_1$ and $c \notin \mathcal{L}(N_2)$, Algorithm [FAIncW](#) concludes that the language inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ does not hold. $\diamond$

4.3.2.1 On the Complexity of Nerode's quasiorders

For the inclusion problem between languages generated by finite automata, deciding the (left or right) Nerode's quasiorder can be easily shown to be as hard as the language inclusion problem, which is PSPACE-complete. In fact, given the automata $N_1 = \langle Q_1, \delta_1, I_1, F_1, \Sigma \rangle$ and $N_2 = \langle Q_2, \delta_2, I_2, F_2, \Sigma \rangle$, one can define the union automaton $N_3 \stackrel{\text{def}}{=} \langle Q_1 \cup Q_2 \cup \{q'\}, \delta_3, \{q'\}, F_1 \cup F_2 \rangle$ where δ_3 maps (q', a) to I_1 , (q', b) to I_2 and behaves like δ_1 or δ_2 elsewhere. Then, it turns out that

$$a \leq_{\mathcal{L}(N_3)}^r b \Leftrightarrow a^{-1} \mathcal{L}(N_3) \subseteq b^{-1} \mathcal{L}(N_3) \Leftrightarrow \mathcal{L}(N_1) \subseteq \mathcal{L}(N_2) .$$

It follows that deciding the right Nerode's quasiorder $\leq_{\mathcal{L}(N_3)}^r$ is as hard as deciding $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$.

Also, for the inclusion problem of a language generated by an automaton within the trace set of a one-counter net (see Section 4.3.4), the right Nerode's quasiorder is a right language-consistent well-quasiorder but it turns out to be undecidable (see Lemma 4.3.13).

4.3.3 State-based Quasiorders

Consider the inclusion problem $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$ where $\mathcal{N}_1$ and $\mathcal{N}_2$ are NFAs. In the following, we study a class of well-quasiorders based on $\mathcal{N}_2$, called state-based quasiorders. These quasiorders are strictly stronger (i.e. lower w.r.t. set inclusion of binary relations) than the Nerode's quasiorders and sidestep the untractability or undecidability of Nerode's quasiorders yet allowing to define an algorithm solving the language inclusion problem.

4.3.3.1 Inclusion in Regular Languages.

We define the quasiorders $\leq_N^\ell$ and $\leq_N^r$ induced by an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ as follows:

$$u \leq_N^\ell v \stackrel{\text{def}}{\Leftrightarrow} \text{pre}_u^{\mathcal{N}}(F) \subseteq \text{pre}_v^{\mathcal{N}}(F), \quad u \leq_N^r v \stackrel{\text{def}}{\Leftrightarrow} \text{post}_u^{\mathcal{N}}(I) \subseteq \text{post}_v^{\mathcal{N}}(I). \quad (4.12)$$

The superscripts in $\leq_N^\ell$ and $\leq_N^r$ stand, respectively, for left/right because they are, respectively, left and right well-quasiorders as the following result shows.

LEMMA 4.3.7. *The relations $\leq_N^\ell$ and $\leq_N^r$ are, respectively, decidable left and right $\mathcal{L}(\mathcal{N})$ -consistent wqos.*

Proof. Since, for every $u \in \Sigma^*$, $\text{pre}_u^{\mathcal{N}}(F)$ is a computable subset of a the finite set of states of $\mathcal{N}$, it turns out that $\leq_N^\ell$ is a decidable wqo. Let us check that $\leq_N^\ell$ is left $\mathcal{L}(\mathcal{N})$ -consistent according to Definition 4.3.1 (a)-(b).

- (a) Let $u \in \mathcal{L}(\mathcal{N})$ and $v \notin \mathcal{L}(\mathcal{N})$. We have that $\text{pre}_u^{\mathcal{N}}(F)$ contains some initial state while $\text{pre}_v^{\mathcal{N}}(F)$ does not, hence $u \not\leq_N^\ell v$. Therefore, $\leq_N^\ell \cap (L \times L^c) = \emptyset$.
- (b) Let us check that $\leq_N^\ell$ is left monotone. Observe that $\text{pre}_x^{\mathcal{N}}$ is a monotone function and that

$$\text{pre}_{uv}^{\mathcal{N}} = \text{pre}_u^{\mathcal{N}} \circ \text{pre}_v^{\mathcal{N}}. \quad (4.13)$$

Therefore, for all $x_1, x_2 \in \Sigma^*$ and $a \in \Sigma$,

$$\begin{aligned} x_1 \leq_N^\ell x_2 &\Rightarrow \quad [\text{By definition of } \leq_N^\ell] \\ \text{pre}_{x_1}^{\mathcal{N}}(F) \subseteq \text{pre}_{x_2}^{\mathcal{N}}(F) &\Rightarrow \quad [\text{Since } \text{pre}_a^{\mathcal{N}} \text{ is monotone}] \end{aligned}$$

$$\begin{aligned}
 \text{pre}_a^N(\text{pre}_{x_1}^N(F)) &\subseteq \text{pre}_a^N(\text{pre}_{x_2}^N(F)) \Leftrightarrow [\text{By Equation (4.13)}] \\
 \text{pre}_{ax_1}^N(F) &\subseteq \text{pre}_{ax_2}^N(F) \Leftrightarrow [\text{By definition of } \leq_N^\ell] \\
 ax_1 &\leq_N^\ell ax_2 \quad .
 \end{aligned}$$

The proof that $\leq_N^r$ is a decidable right $\mathcal{L}(N)$ -consistent quasiorder is symmetric.

As a consequence, Theorem 4.3.3 applies to the wqo $\leq_{N_2}^\ell$ (and $\leq_{N_2}^r$), so that one can instantiate Algorithm FAIncW with $\leq_{N_2}^\ell$ for deciding $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$.

Turning back to the left Nerode wqo $\leq_{\mathcal{L}(N_2)}^\ell$, it turns out that:

$$u \leq_{\mathcal{L}(N_2)}^\ell v \Leftrightarrow \mathcal{L}(N_2)u^{-1} \subseteq \mathcal{L}(N_2)v^{-1} \Leftrightarrow W_{I, \text{pre}_u^{N_2}(F)} \subseteq W_{I, \text{pre}_v^{N_2}(F)} \quad .$$

Since $\text{pre}_u^{N_2}(F) \subseteq \text{pre}_v^{N_2}(F) \Rightarrow W_{I, \text{pre}_u^{N_2}(F)} \subseteq W_{I, \text{pre}_v^{N_2}(F)}$, it follows that

$$u \leq_{N_2}^\ell v \Rightarrow u \leq_{\mathcal{L}(N_2)}^\ell v$$

Example 4.3.8. We illustrate the left state-based quasiorder by using it to solve the language inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ from Example 4.3.6. We have, among others, the following set of predecessors of F_{N_2} :

$$\begin{aligned}
 \text{pre}_\epsilon^{N_2}(F_{N_2}) &= \{5\} & \text{pre}_a^{N_2}(F_{N_2}) &= \{3\} & \text{pre}_b^{N_2}(F_{N_2}) &= \{4\} & \text{pre}_c^{N_2}(F_{N_2}) &= \{2\} \\
 \text{pre}_{aa}^{N_2}(F_{N_2}) &= \{1, 3\} & \text{pre}_{ab}^{N_2}(F_{N_2}) &= \{1\} & \text{pre}_{ac}^{N_2}(F_{N_2}) &= \{1, 2\} & \text{pre}_{aab}^{N_2}(F_{N_2}) &= \{1\}
 \end{aligned}$$

Recall from Example 4.3.6 that, for the Nerode's quasiorder, we have that $b \leq_{\mathcal{L}(N_2)}^\ell a$ and $c \leq_{\mathcal{L}(N_2)}^\ell a$ while none of these relations hold for $\leq_{N_2}^\ell$. Let us next show the Kleene iterates computed by Algorithm FAIncW when using $\leq_{N_2}^\ell$.

$$\begin{aligned}
 \vec{Y}^{(0)} &= \vec{0} \\
 \vec{Y}^{(1)} &= \vec{\epsilon}^F = \langle \emptyset, \{\epsilon\} \rangle \\
 \vec{Y}^{(2)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(1)}) \rfloor = \langle \lfloor \{a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{a, b, c\}, \{\epsilon\} \rangle \\
 \vec{Y}^{(3)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(2)}) \rfloor = \langle \lfloor \{aa, ab, ac, a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{ab, a, b, c\}, \{\epsilon\} \rangle \\
 \vec{Y}^{(4)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(3)}) \rfloor = \langle \lfloor \{aab, aa, ab, ac, a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle \\
 &= \langle \{ab, a, b, c\}, \{\epsilon\} \rangle
 \end{aligned}$$

The least fixpoint is therefore $\vec{Y} = \langle \{ab, a, b, c\}, \{\epsilon\} \rangle$. Since $c \in \vec{Y}_0$ and $c \notin \mathcal{L}(N_2)$, Algorithm FAIncW concludes that the inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ does not hold. $\diamond$

4.3.3.2 Simulation-based Quasiorders.

Recall that a *simulation* on an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ is a binary relation on the states of $\mathcal{N}$, i.e. $\leq \subseteq Q \times Q$, such that for all $p, q \in Q$ if $p \leq q$ then the following two conditions hold:

- (i) if $p \in F$ then $q \in F$;
- (ii) for every transition $p \xrightarrow{a} p'$, there exists a transition $q \xrightarrow{a} q'$ such that $p' \leq q'$.

It is well known that simulation implies language inclusion, i.e. if $\leq$ is a simulation on $\mathcal{N}$ then

$$q \leq q' \Rightarrow W_{q,F}^{\mathcal{N}} \subseteq W_{q',F}^{\mathcal{N}} .$$

A relation $\leq \subseteq Q \times Q$ can be lifted in the standard universal-existential way to a relation $\leq^{\forall\exists} \subseteq \wp(Q) \times \wp(Q)$ on sets of states as follows:

$$X \leq^{\forall\exists} Y \stackrel{\text{def}}{\Leftrightarrow} \forall x \in X, \exists y \in Y, x \leq y .$$

In particular, if $\leq$ is a qo then $\leq^{\forall\exists}$ is a qo as well. Also, if $\leq$ is a simulation relation then its lifting $\leq^{\forall\exists}$ is such that $X \leq^{\forall\exists} Y \Rightarrow W_{X,F}^{\mathcal{N}} \subseteq W_{Y,F}^{\mathcal{N}}$ holds. This suggests us to define a *right simulation-based quasiorder* $\leq_{\mathcal{N}}^r$ on Σ^* induced by a simulation $\leq$ on $\mathcal{N}$ as follows:

$$u \leq_{\mathcal{N}}^r v \stackrel{\text{def}}{\Leftrightarrow} \text{post}_u^{\mathcal{N}}(I) \leq^{\forall\exists} \text{post}_v^{\mathcal{N}}(I) . \quad (4.14)$$

LEMMA 4.3.9. *Let $\mathcal{N}$ be an NFA and let $\leq$ be a simulation on $\mathcal{N}$. Then the right simulation-based quasiorder $\leq_{\mathcal{N}}^r$ is a decidable right $\mathcal{L}(\mathcal{N})$ -consistent well-quasiorder.*

Proof. Since, for every $u \in \Sigma^*$, $\text{post}_u^{\mathcal{N}}(F)$ is a computable subset of a the finite set of states of $\mathcal{N}$, it turns out that $\leq_{\mathcal{N}}^r$ is a decidable wqo. Next, we show that $\leq_{\mathcal{N}}^r$ is right $\mathcal{L}(\mathcal{N})$ -consistent according to Definition 4.3.1 (a)-(b).

- (a) Let $u \in \mathcal{L}(\mathcal{N})$ and $v \notin \mathcal{L}(\mathcal{N})$. We have that $\text{post}_u^{\mathcal{N}}(I)$ contains some final state while $\text{post}_v^{\mathcal{N}}(I)$ does not. Let $q \in \text{post}_u^{\mathcal{N}}(I) \cap F$. We have that $q \leq_{\mathcal{N}}^r q'$ for no $q' \in \text{post}_v^{\mathcal{N}}(I)$ since, by simulation, this would imply $q' \in \text{post}_v^{\mathcal{N}}(I) \cap F$, which contradicts the fact that $F \cap \text{post}_v^{\mathcal{N}}(I) = \emptyset$. We conclude that $u \not\leq_{\mathcal{N}}^r v$, hence $\leq_{\mathcal{N}}^r \cap (L \times L^c) = \emptyset$.
- (b) Next we show that $\leq_{\mathcal{N}}^r$ is right monotone. By Equation (4.11), we check that for all $u, v \in \Sigma^*$ and $a \in \Sigma$, $u \leq_{\mathcal{N}}^r v \Rightarrow ua \leq_{\mathcal{N}}^r va$:

$$\begin{aligned} u \leq_{\mathcal{N}}^r v &\Leftrightarrow [\text{By def. of } \leq_{\mathcal{N}}^r] \\ \text{post}_u^{\mathcal{N}}(I) \leq^{\forall\exists} \text{post}_v^{\mathcal{N}}(I) &\Leftrightarrow [\text{By def. of } \leq^{\forall\exists}] \\ \forall x \in \text{post}_u^{\mathcal{N}}(I), \exists y \in \text{post}_v^{\mathcal{N}}(I), x \leq y &\Rightarrow [\text{By def. of } \leq] \end{aligned}$$

$$\begin{aligned}
& \forall x \xrightarrow{a} x', x \in \text{post}_u^N(I), \exists y \xrightarrow{a} y', y \in \text{post}_v^N(u), x' \leq y' \Leftrightarrow \\
& \quad [\text{Since } \text{post}_a^N \circ \text{post}_u^N = \text{post}_{ua}^N(I)] \\
& \forall x' \in \text{post}_{ua}^N(I), \exists y' \in \text{post}_{va}^N(I), x' \leq y' \Leftrightarrow [\text{By def. of } \leq^{\vee\exists}] \\
& \quad \text{post}_{ua}^N(I) \leq^{\vee\exists} \text{post}_{va}^N(I) \Leftrightarrow [\text{By def. of } \leq^r_N] \\
& \quad ua \leq^r_N va .
\end{aligned}$$

Thus, once again, Theorem 4.3.4 applies to $\leq^r_{N_2}$ and this allows us to instantiate the algorithm **FAIncWr** to the quasiorder $\leq^r_{N_2}$ for deciding the inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$.

On the other hand, note that it is possible to define a left simulation $\leq^{\vee\exists}_R$ on an automaton N by applying $\leq^{\vee\exists}$ on the reverse of N . This left simulation induces a *left simulation-based quasiorder* on Σ^* as follows:

$$u \leq^l_N v \stackrel{\text{def}}{\Leftrightarrow} \text{pre}_u^N(F) \leq^{\vee\exists}_R \text{pre}_v^N(F) . \quad (4.15)$$

It is straightforward to check that Theorem 4.3.3 applies to $\leq^\ell_{N_2}$ and, therefore, we can instantiate Algorithm **FAIncW** for deciding $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$.

Example 4.3.10. Finally, let us illustrate the use of the left simulation-based quasiorder to solve the language inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ of Example 4.3.6. For the set F_{N_2} of final states N_2 we have the same set of predecessors computed in Example 4.3.8 and, among others, the following left simulations between these sets (For clarity, we omit the argument of the function pre , which is always F_{N_2}):

$$\begin{aligned}
\text{pre}_c^{N_2}() &= \{2\} \leq^{\vee\exists}_R \{3\} = \text{pre}_a^{N_2}() & \text{pre}_b^{N_2}() &= \{4\} \not\leq^{\vee\exists}_R \{3\} = \text{pre}_a^{N_2}() \\
\text{pre}_{ac}^{N_2}() &= \{1\} \leq^{\vee\exists}_R \{4\} = \text{pre}_b^{N_2}() & \text{pre}_{ac}^{N_2}() &= \{1\} \not\leq^{\vee\exists}_R \{2\} = \text{pre}_c^{N_2}()
\end{aligned}$$

As expected, the simulation-based quasiorder lies in between the Nerode and the state-based quasiorders. As shown in Examples 4.3.6 and 4.3.8, we have $b \leq^\ell_{\mathcal{L}(N_2)} a$, $c \leq^\ell_{\mathcal{L}(N_2)} a$, $b \not\leq^\ell_{N_2} a$ and $c \not\leq^\ell_{N_2} a$ while $c \leq^\ell_{N_2} a$, but $b \not\leq^\ell_{N_2} a$.

Let us show the computation of the Kleene iterates performed by Algorithm **FAIncW** when using the quasiorder $\leq^\ell_{N_2}$.

$$\begin{aligned}
\vec{Y}^{(0)} &= \emptyset \\
\vec{Y}^{(1)} &= \vec{\epsilon}^F = \langle \emptyset, \{\epsilon\} \rangle \\
\vec{Y}^{(2)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(1)}) \rfloor = \langle \lfloor \{a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{c\}, \{\epsilon\} \rangle \\
\vec{Y}^{(3)} &= \lfloor \vec{\epsilon}^F \rfloor \sqcup \lfloor \text{Pre}_{N_1}(\vec{Y}^{(2)}) \rfloor = \langle \lfloor \{ac, a, b, c\} \rfloor, \lfloor \{\epsilon\} \rfloor \rangle = \langle \{c\}, \{\epsilon\} \rangle
\end{aligned}$$

The least fixpoint is therefore $\vec{Y} = \langle \{c\}, \{\epsilon\} \rangle$. Since $c \in \vec{Y}_0$ and $c \notin \mathcal{L}(N_2)$, Algorithm **FAIncW** concludes that the inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$ does not hold. $\diamond$

Let us observe that $u \leq_{N_2}^r v$ implies $W_{\text{post}_u^{N_2(I),F}} \subseteq W_{\text{post}_v^{N_2(I),F}}$, which is equivalent to the right Nerode's quasiorder $u \leq_{\mathcal{L}(N_2)}^r v$ for $\mathcal{L}(N_2)$. Furthermore, for the state-based quasiorder defined in (4.12), we have that $u \leq_{N_2}^r v \Rightarrow u \leq_{N_2}^r v$ trivially holds.

Summing up, given an NFA $\mathcal{N}$ with $\mathcal{L}(\mathcal{N}) = L$, the following containments relate the state-based, simulation-based and Nerode's quasiorders:

$$\leq_{\mathcal{N}}^r \subseteq \leq_{\mathcal{N}}^r \subseteq \leq_L^r, \quad \leq_{\mathcal{N}}^\ell \subseteq \leq_{\mathcal{N}}^\ell \subseteq \leq_L^\ell.$$

Recall that these are decidable $\mathcal{L}(N_2)$ -consistent well-quasiorders so that Algorithm [FAIncW](#) can be instantiated for each of them for deciding an inclusion $\mathcal{L}(N_1) \subseteq \mathcal{L}(N_2)$. Examples 4.3.6, 4.3.8 and 4.3.10 show how the algorithm behaves for each of the three quasiorders considered in this section. Despite their simplicity, the examples evidence the differences in the behavior of the algorithm when considering the different quasiorders. In particular, we observe that the fixpoint computation for $\leq_{\mathcal{L}(N_2)}^r$ coincides with the one for $\leq_{N_2}^r$ which, as expected, converge faster than the one for $\leq_{N_2}^r$.

As shown by [de Luca and Varricchio \[1994\]](#), $\leq_{\mathcal{L}(N_2)}^r$ is the coarsest well-quasiorder for which Algorithm [FAIncW](#) works (i.e. Theorem 4.3.3 holds), hence its corresponding fixpoint computation exhibits optimal behavior in terms of the number of closed sets considered. However, Nerode's quasiorder is not practical since it requires checking language inclusion, which is the PSPACE-complete problem we are trying to solve, in order to decide whether two words are related. Therefore, the coincidence of the fixpoint computations for $\leq_{\mathcal{L}(N_2)}^r$ and $\leq_{N_2}^r$ is of special interest since it evidences that Algorithm [FAIncW](#) might exhibit optimal behavior while using a "simpler" well-quasiorder such as $\leq_{N_2}^r$, which is a polynomial under-approximation of $\leq_{\mathcal{L}(N_2)}^r$.

4.3.4 Inclusion in Traces of One-Counter Nets.

We show that our framework can be instantiated to systematically derive an algorithm for deciding the inclusion $\mathcal{L}(\mathcal{N}) \subseteq L_2$ where L_2 is the trace set of a one-counter net. This is accomplished by defining a decidable L_2 -consistent quasiorder so that Theorem 4.3.3 can be applied.

Intuitively, a *one-counter net* is an NFA endowed with a nonnegative integer counter which can be incremented, decremented or left unchanged by a transition.

Definition (One-Counter Net). A *One-Counter Net (OCN)* [[Hofman and Totzke 2018](#)] is a tuple $\mathcal{O} = \langle Q, \Sigma, \delta \rangle$ where Q is a finite set of states, Σ is an alphabet and $\delta \subseteq Q \times \Sigma \times \{-1, 0, 1\} \times Q$ is a set of transitions. ■

A *configuration of an OCN* $O = \langle Q, \Sigma, \delta \rangle$ is a pair qn consisting of a state $q \in Q$ and a value $n \in \mathbb{N}$ for the counter. Given two configurations of an OCN, $qn, q'n' \in Q \times \mathbb{N}$, we write $qn \xrightarrow{a} q'n'$ and call it an a -step (or simply step) if there exists a transition $(q, a, d, q') \in \delta$ such that $n' = n + d$. Given $qn \in Q \times \mathbb{N}$, the *trace set* $T(qn) \subseteq \Sigma^*$ of an OCN is defined as follows:

$$T(qn) \stackrel{\text{def}}{=} \{u \in \Sigma^* \mid Z_u^{qn} \neq \emptyset\} \quad \text{where} \\ Z_u^{qn} \stackrel{\text{def}}{=} \{q_k n_k \in Q \times \mathbb{N} \mid qn = q_0 n_0 \xrightarrow{a_1} q_1 n_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} q_k n_k, a_1 \dots a_k = u\}.$$

Observe that $Z_\epsilon^{qn} = \{qn\}$ and Z_u^{qn} is a finite set for every word $u \in \Sigma^*$.

Let us consider the poset $\langle \mathbb{N}_\perp \stackrel{\text{def}}{=} \mathbb{N} \cup \{\perp\}, \leq_{\mathbb{N}_\perp} \rangle$ where $\perp \leq_{\mathbb{N}_\perp} n$ holds for all $n \in \mathbb{N}_\perp$, while for all $n, n' \in \mathbb{N}$, $n \leq_{\mathbb{N}_\perp} n'$ is the standard ordering relation between numbers. For a finite set of states $S \subseteq Q \times \mathbb{N}$ define the so-called macro state $M_S: Q \rightarrow \mathbb{N}_\perp$ as follows:

$$M_S(q) \stackrel{\text{def}}{=} \max\{n \in \mathbb{N} \mid qn \in S\},$$

where $\max \emptyset \stackrel{\text{def}}{=} \perp$. Define the following quasiorder on Σ^* :

$$u \leq_{qn}^r v \stackrel{\text{def}}{\Leftrightarrow} \forall q \in Q, M_{Z_u^{qn}}(q) \leq_{\mathbb{N}_\perp} M_{Z_v^{qn}}(q). \quad (4.16)$$

LEMMA 4.3.11. *Let O be an OCN. For any configuration qn of O , $\leq_{qn}^r$ is a right $T(qn)$ -consistent decidable well-quasiorder.*

Proof. It follows from Dickson's Lemma [Sakarovitch 2009, Section II.7.1.2] that $\leq_{qn}^r$ is a wqo. Next, we show that $\leq_{qn}^r$ is $T(qn)$ -consistent according to Definition 4.3.1 (a)-(b).

(a) Since Z_u^{qn} and Z_v^{qn} are finite sets, we have that the macro state functions $M_{Z_u^{qn}}$ and $M_{Z_v^{qn}}$ are computable, hence the relation $\leq_{qn}^r$ is decidable. Let $u \in T(qn)$ and $v \notin T(qn)$. Then $M_{Z_u^{qn}}(q') \neq \perp$ for some $q' \in Q$ and $M_{Z_v^{qn}}(q') = \perp$ since $Z_v^{qn} = \emptyset$. It follows that $u \not\leq_{qn}^r v$ and, therefore, $\leq_{qn}^r \cap (T(qn) \times (T(qn))^c) = \emptyset$.

(b) Next we show that $u \leq_{qn}^r v$ implies $ua \leq_{qn}^r va$ for all $a \in \Sigma$, since, by Equation (4.11), this is equivalent to the fact that $\leq_{qn}^r$ is right monotone. We proceed by contradiction.

Assume that $u \leq_{qn}^r v$ and $\exists q' \in Q, M_{Z_{ua}^{qn}}(q') \not\leq_{\mathbb{N}_\perp} M_{Z_{va}^{qn}}(q')$. Then we have that $m_1 \stackrel{\text{def}}{=} \max\{n \mid pn \in Z_{ua}^{qn}\} \not\leq_{\mathbb{N}_\perp} m_2 \stackrel{\text{def}}{=} \max\{n \mid pn \in Z_{va}^{qn}\}$, which implies, since $m_1 \neq \perp$, that $m_1, m_2 \in \mathbb{N}$ and $m_1 > m_2$.

On the other hand, for all $(q, a, d, q') \in \delta$ we have $q'(m_1 - d) \in Z_u^{qn}$ and $q'(m_2 - d) \in Z_v^{qn}$.

Observe that $\max\{n \mid pn \in Z_u^{qn}\} = m_1 - d$ since otherwise we would that have $\max\{n \mid pn \in Z_u^{qn}\} + d > m_1$ which contradicts the definition of m_1 . Similarly, $\max\{n \mid pn \in Z_v^{qn}\} = m_2 - d$.

Since $m_1 > m_2$ we have that $m_1 - d > m_2 - d$ and, as a consequence, $\max\{n \mid pn \in Z_u^{qn}\} > \max\{n \mid pn \in Z_v^{qn}\}$, which contradicts $u \leq_{qn}^r v$.

Thus, as a consequence of Theorem 4.3.3, Lemma 4.3.11 and the decidability of membership $u \in T(qn)$, the following known decidability result for language inclusion of regular languages into traces of OCNs [Jancar et al. 1999, Theorem 3.2] is systematically derived within our framework.

COROLLARY 4.3.12. *Let N be an NFA and O be an OCN. For any configuration qn of O , the language inclusion $\mathcal{L}(N) \subseteq T(qn)$ is decidable.*

Moreover, the following result closes a conjecture made by de Luca and Varricchio [1994, Section 6].

LEMMA 4.3.13. *Let O be an OCN. Then the right Nerode's quasiorder $\leq_{T(qn)}^r$ is an undecidable well-quasiorder.*

Proof. Recall that $\leq_{T(qn)}^r$ is maximum in the set of all right $T(qn)$ -consistent quasiorders [de Luca and Varricchio 1994, Section 2, point 4]. As a consequence, $u \leq_{qn}^r v \Rightarrow u \leq_{T(qn)}^r v$, for all $u, v \in \Sigma^*$. By Lemma 4.3.11, $\leq_{qn}^r$ is a wqo, so that $\leq_{T(qn)}^r$ is a wqo as well. Undecidability of $\leq_{T(qn)}^r$ follows from the undecidability of the trace inclusion problem for nondeterministic OCNs [Hofman et al. 2013, Theorem 20] since given the OCNs $O_1 = (Q_1, \Sigma, \delta_1)$ and $O_2 = (Q_2, \Sigma, \delta_2)$, we can define the union OCN $O_3 \stackrel{\text{def}}{=} (Q_1 \cup Q_2 \cup \{q\}, \Sigma, \delta_3)$ where δ_3 maps $(q, a, 0)$ to $q_1 \in Q_1$, $(q, b, 0)$ to $q_2 \in Q_2$ and behaves like δ_1 or δ_2 elsewhere. Then, it turns out that

$$a \leq_{T_3(qn)}^r b \Leftrightarrow a^{-1}T_3(q_1n) \subseteq b^{-1}T_3(q_2n) \Leftrightarrow T_1(q_1n) \subseteq T_2(q_2n) .$$

Therefore, deciding the right Nerode's quasiorder $\leq_{T_3(qn)}^r$ is as hard as deciding $T_1(q_1n) \subseteq T_2(q_2n)$.

It is worth to remark that, by Lemma 4.3.5 (a), the left and right Nerode's quasiorders $\leq_{T(qn)}^\ell$ and $\leq_{T(qn)}^r$ are $T(qn)$ -consistent. However, the left Nerode's quasiorder does not need to be a wqo, otherwise $T(qn)$ would be regular.

We conclude this section by conjecturing that our framework could be instantiated for extending Corollary 4.3.12 to traces of Petri Nets, a result which is already known to be true [Jancar et al. 1999].

4.4 A Novel Perspective on the Antichain Algorithm

Let $\mathcal{N}_1 = \langle Q_1, \delta_1, I_1, F_1, \Sigma \rangle$ and $\mathcal{N}_2 = \langle Q_2, \delta_2, I_2, F_2, \Sigma \rangle$ be two NFAs and consider the state-based left $\mathcal{L}(\mathcal{N}_2)$ -consistent wqo $\leq_{\mathcal{N}_2}^\ell$ defined by Equivalence (4.12). Theorem 4.3.3 shows that Algorithm FAIncW decides the language inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$ by manipulating finite sets of words.

Since $u \leq_{\mathcal{N}_2}^\ell v \Leftrightarrow \text{pre}_u^{\mathcal{N}_2}(F_2) \subseteq \text{pre}_v^{\mathcal{N}_2}(F_2)$, we could equivalently consider the set of states $\text{pre}_u^{\mathcal{N}_2}(F_2) \in \wp(Q_2)$ rather than a word $u \in \Sigma^*$. This observation suggests the design of an algorithm analogous to FAIncW but computing on the poset $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$ of antichains of sets of states of the complete lattice $\langle \wp(Q_2), \subseteq \rangle$.

In order to do this, the poset $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$ is viewed as an abstraction of the poset $\langle \wp(\Sigma^*), \subseteq \rangle$ by using the abstraction and concretization functions $\alpha: \wp(\Sigma^*) \rightarrow \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}$ and $\gamma: \text{AC}_{\langle \wp(Q_2), \subseteq \rangle} \rightarrow \wp(\Sigma^*)$ and using the abstract function $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}: (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \rightarrow (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|}$ defined as follows:

$$\begin{aligned} \alpha(X) &\stackrel{\text{def}}{=} \lfloor \{ \text{pre}_u^{\mathcal{N}_2}(F_2) \in \wp(Q_2) \mid u \in X \} \rfloor, \\ \gamma(Y) &\stackrel{\text{def}}{=} \{ v \in \Sigma^* \mid \exists u \in \Sigma^*, \text{pre}_u^{\mathcal{N}_2}(F_2) \in Y \wedge \text{pre}_u^{\mathcal{N}_2}(F_2) \subseteq \text{pre}_v^{\mathcal{N}_2}(F_2) \}, \quad (4.17) \\ \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\langle X_q \rangle_{q \in Q_1}) &\stackrel{\text{def}}{=} \langle \lfloor \{ \text{pre}_a^{\mathcal{N}_2}(S) \in \wp(Q_2) \mid \exists a \in \Sigma, q' \in Q_1, \\ &\quad q' \in \delta_1(q, a) \wedge S \in X_{q'} \} \rfloor \rangle_{q \in Q_1}. \end{aligned}$$

Observe that the functions α and $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}$ are well-defined because minors are antichains.

LEMMA 4.4.1. *The following properties hold:*

- (a) $\langle \wp(\Sigma^*), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$ is a GC.
- (b) $\gamma \circ \alpha = \rho_{\leq_{\mathcal{N}_2}^\ell}$.
- (c) For all $\vec{X} \in \alpha(\wp(\Sigma^*))^{|Q_1|}$, $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) = \alpha \circ \text{Pre}_{\mathcal{N}_1} \circ \gamma(\vec{X})$.

Proof.

- (a) Let us first observe that α and γ are well-defined. First, $\alpha(X)$ is an antichain of $\langle \wp(Q_2), \subseteq \rangle$ since it is a minor for the well-quasiorder $\subseteq$ and, therefore, it is finite. On the other hand, $\gamma(Y)$ is clearly an element of $\langle \wp(\Sigma^*), \subseteq \rangle$ by definition.

Then, for all $X \in \wp(\Sigma^*)$ and $Y \in \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}$, it turns out that:

$$\alpha(X) \sqsubseteq Y \Leftrightarrow \quad [\text{By definition of } \sqsubseteq]$$

$$\begin{aligned}
 \forall z \in \alpha(X), \exists y \in Y, y \subseteq z &\Leftrightarrow [\text{By definition of } \alpha \text{ and } [\cdot]] \\
 \forall v \in X, \exists y \in Y, y \subseteq \text{pre}_v^{N_2}(F_2) &\Leftrightarrow [\text{By definition of } \gamma] \\
 \forall v \in X, v \in \gamma(Y) &\Leftrightarrow [\text{By definition of } \subseteq] \\
 X &\subseteq \gamma(Y) .
 \end{aligned}$$

(b) For all $X \in \wp(\Sigma^*)$ we have that

$$\begin{aligned}
 \gamma(\alpha(X)) &= [\text{By definition of } \alpha, \gamma] \\
 \{v \in \Sigma^* \mid \exists u \in \Sigma^*, \text{pre}_u^{N_2}(F_2) \in [\{\text{pre}_w^{N_2}(F_2) \mid w \in X\}] \wedge \text{pre}_u^{N_2}(F_2) \subseteq \text{pre}_v^{N_2}(F_2)\} \\
 &= [\text{By definition of minor}] \\
 \{v \in \Sigma^* \mid \exists u \in X, \text{pre}_u^{N_2}(F_2) \subseteq \text{pre}_v^{N_2}(F_2)\} &= [\text{By definition of } \leq_{N_2}^\ell] \\
 \{v \in \Sigma^* \mid \exists u \in X, u \leq_{N_2}^\ell v\} &= [\text{By definition of } \rho_{\leq_{N_2}^\ell}] \\
 \rho_{\leq_{N_2}^\ell}(X) &.
 \end{aligned}$$

(c) For all $\vec{X} \in \alpha(\wp(\Sigma^*))^{|\mathcal{Q}_1|}$ we have that

$$\begin{aligned}
 \alpha(\text{Pre}_{N_1}(\gamma(\vec{X}))) &= \\
 [\text{By definition of } \text{Pre}_{N_1}] \\
 \langle \alpha(\bigcup_{a \in \Sigma, q \xrightarrow{a}_{N_1} q'} a\gamma(\vec{X}_{q'})) \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{By definition of } \alpha] \\
 \langle [\{\text{pre}_u^{N_2}(F_2) \mid u \in \bigcup_{a \in \Sigma, q \xrightarrow{a}_{N_1} q'} a\gamma(\vec{X}_{q'})\}] \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{By } \text{pre}_{av}^{N_2} = \text{pre}_a^{N_2} \circ \text{pre}_v^{N_2}] \\
 \langle [\{\text{pre}_a^{N_2}(\{\text{pre}_u^{N_2}(F_2) \mid u \in \bigcup_{q \xrightarrow{a}_{N_1} q'} \gamma(\vec{X}_{q'})\}) \mid a \in \Sigma\}] \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{By rewriting}] \\
 \langle [\{\text{pre}_a^{N_2}(S) \mid a \in \Sigma, q \xrightarrow{a}_{N_1} q', S \in \{\text{pre}_u^{N_2}(F_2) \mid u \in \gamma(\vec{X}_{q'})\}\}] \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{By } [\text{pre}_a^{N_2}(X)] = [\text{pre}_a^{N_2}([X])]] \\
 \langle [\{\text{pre}_a^{N_2}(S) \mid a \in \Sigma, q \xrightarrow{a}_{N_1} q', S \in [\{\text{pre}_u^{N_2}(F_2) \mid u \in \gamma(\vec{X}_{q'})\}]\}] \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{By definition of } \alpha] \\
 \langle [\{\text{pre}_a^{N_2}(S) \mid a \in \Sigma, q \xrightarrow{a}_{N_1} q', S \in \alpha(\gamma(\vec{X}_{q'}))\}] \rangle_{q \in \mathcal{Q}_1} &= \\
 [\text{Since } \vec{X} \in \alpha, \alpha(\gamma(\vec{X}_{q'})) = \vec{X}_{q'}]
 \end{aligned}$$

$$\begin{aligned} \langle \lfloor \{\text{pre}_a^{\mathcal{N}_2}(S) \mid a \in \Sigma, q \xrightarrow{\mathcal{N}_1} q', S \in \vec{X}_{q'}\} \rfloor \rangle_{q \in Q_1} = \\ \text{[By definition of } \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}] \\ \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) . \end{aligned}$$

It follows from Lemma 4.4.1 that the GC $\langle \wp(\Sigma^*), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$ and the abstract function $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}$ satisfy the hypotheses (i)-(iv) of Theorem 4.2.11. Thus, in order to obtain an algorithm for deciding $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$ it remains to show that requirement (v) of Theorem 4.2.11 holds, i.e. there is an algorithm to decide whether $\vec{Y} \sqsubseteq \alpha(\vec{L}_2^{I_2})$ for every $\vec{Y} \in \alpha(\wp(\Sigma^*))^{|Q_1|}$. In order to do that, we first provide some intuitions on how the resulting algorithm works.

First, observe that the Kleene iterates of the function $\lambda \vec{X}. \alpha(\vec{\epsilon}^{F_1}) \sqcup \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X})$ of Theorem 4.2.11 are vectors of antichains in $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$, where each component is indexed by some $q \in Q_1$ and represents (through its minor set) a set of sets of states that are predecessors of F_2 in $\mathcal{N}_2$ by a word u generated by $\mathcal{N}_1$ from that state q , i.e. $\text{pre}_u^{\mathcal{N}_2}(F_2)$ with $u \in W_{q, F_1}^{\mathcal{N}_1}$. Since $\epsilon \in W_{q, F_1}^{\mathcal{N}_1}$ for all $q \in F_1$ and $\text{pre}_\epsilon^{\mathcal{N}_2}(F_2) = F_2$ the iterations of the procedure KLEENE begin with the initial vector $\alpha(\vec{\epsilon}^{F_1}) = \langle \psi_\emptyset^{F_2}(q \in F_1) \rangle_{q \in Q_1}$.

On the other hand, note that by taking the minor of each vector component, we are considering smaller sets which still preserve the relation $\sqsubseteq$ since

$$A \sqsubseteq B \Leftrightarrow [A] \sqsubseteq B \Leftrightarrow A \sqsubseteq [B] \Leftrightarrow [A] \sqsubseteq [B] .$$

Let $\langle Y_q \rangle_{q \in Q_1}$ be the fixpoint computed by the KLEENE procedure. It turns out that, for each component $q \in Q_1$, $Y_q = \lfloor \{\text{pre}_u^{\mathcal{N}_2}(F_2) \mid u \in W_{q, F_1}^{\mathcal{N}_1}\} \rfloor$ holds. Whenever the inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$ holds, all the sets of states in Y_q for some initial state $q \in I_1$ are predecessors of F_2 in $\mathcal{N}_2$ by words in $\mathcal{L}(\mathcal{N}_1)$, so that they all contain at least one initial state in I_2 . As a result, we obtain Algorithm FAInCS, that is, a “state-based” inclusion algorithm for deciding $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$.

FAInCS: State-based algorithm for $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$

Data: NFAs $\mathcal{N}_1 = \langle Q_1, \delta_1, I_1, F_1, \Sigma \rangle$ and $\mathcal{N}_2 = \langle Q_2, \delta_2, I_2, F_2, \Sigma \rangle$.

- 1 $\langle Y_q \rangle_{q \in Q_1} := \text{KLEENE}(\lambda \vec{X}. \alpha(\vec{\epsilon}^{F_1}) \sqcup \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}), \vec{\emptyset})$;
 - 2 **forall** $q \in I_1$ **do**
 - 3 **forall** $S \in Y_q$ **do**
 - 4 **if** $S \cap I_2 = \emptyset$ **then return false**;
 - 5 **return true**;
-

THEOREM 4.4.2. *Let $\mathcal{N}_1, \mathcal{N}_2$ be NFAs. The algorithm **FAIncS** decides the inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$.*

Proof. We show that all the conditions (i)-(v) of Theorem 4.2.11 are satisfied for the abstract domain $\langle D, \leq_D \rangle = \langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq \rangle$ as defined by the Galois Connection of Lemma 4.4.1 (a).

- (a) Since, by Lemma 4.4.1 (b), $\rho_{\leq_{\mathcal{N}_2}}^\ell(X) = \gamma(\alpha(X))$ it follows from Lemmas 4.3.2 and 4.3.7 that $\gamma(\alpha(L_2)) = L_2$. Moreover, for all $a \in \Sigma, X \in \wp(\Sigma^*)$ we have that:

$$\begin{aligned} \gamma\alpha(aX) &= [\text{In GCs } \gamma = \gamma\alpha\gamma] \\ \gamma\alpha\gamma\alpha(aX) &= [\text{By Lemma 4.3.2 (b) with } \rho_{\leq_{\mathcal{N}_2}}^\ell = \gamma\alpha] \\ \gamma\alpha\gamma\alpha(a\gamma\alpha(X)) &= [\text{In GCs } \gamma = \gamma\alpha\gamma] \\ \gamma\alpha(a\gamma\alpha(X)) &. \end{aligned}$$

- (b) $(\text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \sqsubseteq)$ is effective because Q_2 is finite.
 (c) By Lemma 4.4.1 (c) we have that $\alpha(\text{Pre}_{\mathcal{N}_1}(\gamma(\vec{X}))) = \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X})$ for all vectors $\vec{X} \in \alpha(\wp(\Sigma^*))^{|Q_1|}$.
 (d) $\alpha(\{\epsilon\}) = \{F_2\}$ and $\alpha(\emptyset) = \emptyset$, hence $\lfloor \alpha(\vec{\epsilon}^{F_1}) \rfloor$ is trivial to compute.
 (e) Since $\alpha(\vec{L}_2^{I_1}) = \langle \alpha(\psi_{\Sigma^*}^{L_2}(q \in I_1)) \rangle_{q \in Q_1}$, for all $\vec{Y} \in \alpha(\wp(\Sigma^*))^{|Q_1|}$ the relation $\vec{Y} \sqsubseteq \alpha(\vec{L}_2^{I_1})$ trivially holds for all components $q \notin I_1$. For the components $q \in I_1$, it suffices to show that $Y_q \sqsubseteq \alpha(L_2) \Leftrightarrow \forall S \in Y_q, S \cap I_2 \neq \emptyset$, which is the check performed by lines 2-5 of algorithm **FAIncS**.

$$\begin{aligned} Y_q \sqsubseteq \alpha(L_2) &\Leftrightarrow [\text{Because } Y_q = \alpha(U) \text{ for some } U \in \wp(\Sigma^*)] \\ \alpha(U) \sqsubseteq \alpha(L_2) &\Leftrightarrow [\text{By GC}] \\ U \subseteq \gamma(\alpha(L_2)) &\Leftrightarrow [\text{By L. 4.3.2, 4.3.7 and 4.4.1, } \gamma(\alpha(L_2)) = L_2] \\ U \subseteq L_2 &\Leftrightarrow [\text{By definition of } \text{pre}_u^{\mathcal{N}_2}] \\ \forall u \in U, \text{pre}_u^{\mathcal{N}_2}(F_2) \cap I_2 \neq \emptyset &\Leftrightarrow [\text{Since } Y_q = \alpha(U) = \lfloor \{\text{pre}_u^{\mathcal{N}_2}(F_2) \mid u \in U\} \rfloor] \\ \forall S \in Y_q, S \cap I_2 \neq \emptyset &. \end{aligned}$$

Thus, by Theorem 4.2.11, Algorithm **FAIncS** decides $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$.

4.4.1 Relationship to the Antichains Algorithm

Wulf et al. [2006] introduced two so-called antichains algorithms, denoted *forward* and *backward*, for deciding the universality of the language accepted by an

NFA, i.e. whether the language is Σ^* or not. Then, they extended the backward algorithm to decide the inclusion between the languages accepted by two NFAs.

In what follows we show that Algorithm **FAInCS** is equivalent to the corresponding extension of the forward algorithm and, therefore, dual to the backward antichains algorithm for language inclusion by [Wulf et al. \[2006\]](#) [Theorem 6].

To do that, we first define the poset of antichains in which the forward antichains algorithm computes its fixpoint. Then, we give a formal definition of the forward antichains algorithm for deciding language inclusion and show that this algorithm coincides with **FAInCS** when applied to the reverse automata. Since language inclusion between the languages generated by two NFAs holds iff inclusion holds between the languages generated by their reverse NFAs, we conclude that the algorithm **FAInCS** is equivalent to the forward antichains algorithm.

Finally, we show how the different variants of the antichains algorithm, including the original backward antichains algorithm [[Wulf et al. 2006](#)] [Theorem 6], can be derived within our framework by considering the adequate quasi-orders.

Forward Antichains Algorithm

Let $\mathcal{N}_1 = \langle Q_1, \Sigma, \delta_1, I_1, F_1 \rangle$ and $\mathcal{N}_2 = \langle Q_2, \Sigma, \delta_2, I_2, F_2 \rangle$ be two NFAs and consider the language inclusion problem $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$. Let us consider the following poset of antichains $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \widetilde{\sqsubseteq} \rangle$ where

$$X \widetilde{\sqsubseteq} Y \stackrel{\text{def}}{\iff} \forall y \in Y, \exists x \in X, x \subseteq y$$

and notice that $\widetilde{\sqsubseteq}$ coincides with the reverse relation $\sqsubseteq^{-1}$. As observed by [Wulf et al. \[2006, Lemma 1\]](#), it turns out that $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \widetilde{\sqsubseteq}, \widetilde{\sqcup}, \widetilde{\sqcap}, \{\emptyset\}, \emptyset \rangle$ is a finite lattice, where $\widetilde{\sqcup}$ and $\widetilde{\sqcap}$ denote, resp., lub and glb, and $\{\emptyset\}$ and $\emptyset$ are, resp., the least and greatest elements. This lattice $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \widetilde{\sqsubseteq} \rangle$ is the domain in which the forward antichains algorithm computes on for deciding universality [[Wulf et al. 2006, Theorem 3](#)]. The following result extends this forward algorithm in order to decide language inclusion.

THEOREM 4.4.3 ([[Wulf et al. 2006, Theorems 3 and 6](#)]). *Let*

$$\overrightarrow{\mathcal{FP}} \stackrel{\text{def}}{=} \widetilde{\sqcup} \{ \vec{X} \in (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \mid \vec{X} = \text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) \widetilde{\sqcap} \langle \psi_{\emptyset}^{\{I_2\}}(q \in I_1) \rangle_{q \in Q_1} \}$$

where

$$\text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2}(\langle X_q \rangle_{q \in Q_1}) \stackrel{\text{def}}{=} \langle \sqcup \{ \text{post}_a^{\mathcal{N}_2}(x) \in \wp(Q_2) \mid \exists a \in \Sigma, q' \in Q_1, \\ q \in \delta_1(q', a) \wedge x \in X_{q'} \} \rangle_{q \in Q_1} .$$

Then, $\mathcal{L}(\mathcal{N}_1) \not\subseteq \mathcal{L}(\mathcal{N}_2)$ if and only if there exists $q \in F_1$ such that $\overrightarrow{\mathcal{FP}}_q \not\subseteq \{F_2^c\}$.

Proof. Let us first introduce some notation necessary to describe the forward antichains algorithm by Wulf et al. [2006] for deciding $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$. Let us consider the poset $\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle$ where

$$(q_1, x_1) \subseteq_x (q_2, x_2) \stackrel{\text{def}}{\Leftrightarrow} q_1 = q_2 \wedge x_1 \subseteq x_2 .$$

Then, let $\langle \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle}, \widetilde{\sqsubseteq}_x, \widetilde{\sqcup}_x, \widetilde{\sqcap}_x \rangle$ be the lattice of antichains over the poset $\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle$ where:

$$\begin{aligned} X \widetilde{\sqsubseteq}_x Y &\stackrel{\text{def}}{\Leftrightarrow} \forall (q, y) \in Y, \exists (q, x) \in X, x \subseteq y \\ \min_x(X) &\stackrel{\text{def}}{=} \{(q, x) \in X \mid \forall (q', x') \in X, q = q' \Rightarrow x' \not\subseteq x\} \\ X \widetilde{\sqcup}_x Y &\stackrel{\text{def}}{=} \min_x(\{(q, x \cup y) \mid (q, x) \in X, (q, y) \in Y\}) \\ X \widetilde{\sqcap}_x Y &\stackrel{\text{def}}{=} \min_x(\{(q, z) \mid (q, z) \in X \cup Y\}) . \end{aligned}$$

Also, let $\text{Post} : \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle} \rightarrow \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle}$ be defined as follows:

$$\begin{aligned} \text{Post}(X) &\stackrel{\text{def}}{=} \min_x(\{(q, \text{post}_a^{\mathcal{N}_2}(x)) \in Q_1 \times \wp(Q_2) \mid \exists a \in \Sigma, q \in Q_1, \\ &\quad (q', x) \in X, q' \xrightarrow{a}_{\mathcal{N}_1} q\}) . \end{aligned}$$

Then, it turns out that the dual of the backward antichains algorithm of Wulf et al. [2006, Theorem 6] states that $\mathcal{L}(\mathcal{N}_1) \not\subseteq \mathcal{L}(\mathcal{N}_2)$ iff there exists $q \in F_1$ such that $\mathcal{FP} \widetilde{\sqsubseteq}_x \{(q, F_2^c)\}$ where

$$\mathcal{FP} = \widetilde{\sqcap}_x \{X \in \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle} \mid X = \text{Post}(X) \widetilde{\sqcap}_x (I_1 \times \{I_2\})\} .$$

We observe that for every $X \in \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle}$, a pair $(q, x) \in Q_1 \times \wp(Q_2)$ such that $(q, x) \in X$ is used by Wulf et al. [2006, Theorem 6] simply as a way to associate states q of $\mathcal{N}_1$ with sets x of states of $\mathcal{N}_2$. In fact, every antichain $X \in \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle}$ can be equivalently formalized by a vector

$$\langle \{x \in \wp(Q_2) \mid (q, x) \in X\} \rangle_{q \in Q_1} \in (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|}$$

indexed by states $q \in Q_1$ and whose components are antichains in $\text{AC}_{\langle \wp(Q_2), \subseteq \rangle}$.

Correspondingly, we consider the lattice $\langle \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}, \widetilde{\sqsubseteq} \rangle$, where for every pair of elements $X, Y \in \text{AC}_{\langle \wp(Q_2), \subseteq \rangle}$ we have that

$$X \widetilde{\sqsubseteq} Y \stackrel{\text{def}}{\Leftrightarrow} \forall y \in Y, \exists x \in X, x \subseteq y$$

$$\begin{aligned}\min(X) &\stackrel{\text{def}}{=} \{x \in X \mid \forall x' \in X, x' \not\subseteq x\} \\ X \sqcap Y &\stackrel{\text{def}}{=} \min(\{x \cup y \in \wp(Q_2) \mid x \in X, y \in Y\}) \\ X \widetilde{\sqcap} Y &\stackrel{\text{def}}{=} \min(\{z \in \wp(Q_2) \mid z \in X \cup Y\}) .\end{aligned}$$

Then, Post can be replaced by $\text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2} : (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \rightarrow (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|}$, its equivalent formulation on vectors defined as follows:

$$\begin{aligned}\text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2}(\langle X_q \rangle_{q \in Q_1}) &\stackrel{\text{def}}{=} \langle \min(\{\text{post}_a^{\mathcal{N}_2}(x) \in \wp(Q_2) \mid \exists a \in \Sigma, q' \in Q_1, \\ &\quad x \in X_{q'}, q' \xrightarrow{a}_{\mathcal{N}_1} q\}) \rangle_{q \in Q_1} .\end{aligned}$$

In turn, $\mathcal{FP} \in \text{AC}_{\langle Q_1 \times \wp(Q_2), \subseteq_x \rangle}$ is replaced by the following vector:

$$\overrightarrow{\mathcal{FP}} \stackrel{\text{def}}{=} \sqcap \{ \vec{X} \in (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \mid \vec{X} = \text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) \widetilde{\sqcap} \langle \psi_{\emptyset}^{\{I_2\}}(q \in I_1) \rangle_{q \in Q_1} \} .$$

Finally, the check $\exists q \in F_1, \mathcal{FP} \widetilde{\sqsubseteq}_x \{(q, F_2^c)\}$ becomes $\exists q \in F_1, \overrightarrow{\mathcal{FP}}_q \widetilde{\sqsubseteq} \{F_2^c\}$.

Let $\mathcal{N}^R$ denote the reverse automaton of $\mathcal{N}$, where arrows are flipped and the initial/final states become final/initial. Note that language inclusion can be decided by considering the reverse automata since

$$\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2) \Leftrightarrow \mathcal{L}(\mathcal{N}_1^R) \subseteq \mathcal{L}(\mathcal{N}_2^R) .$$

Furthermore, it is straightforward to check that $\text{Post}_{\mathcal{N}_1}^{\mathcal{N}_2} = \text{Pre}_{\mathcal{N}_1^R}^{\mathcal{N}_2^R}$. We therefore obtain the following result as a consequence of Theorem 4.4.3.

COROLLARY 4.4.4. *Let*

$$\overrightarrow{\mathcal{FP}} \stackrel{\text{def}}{=} \sqcap \{ \vec{X} \in (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \mid \vec{X} = \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) \widetilde{\sqcap} \langle \psi_{\emptyset}^{\{F_2\}}(q \in F_1) \rangle_{q \in Q_1} \} .$$

Then, $\mathcal{L}(\mathcal{N}_1) \not\subseteq \mathcal{L}(\mathcal{N}_2)$ iff there exists $q \in I_1$ such that $\overrightarrow{\mathcal{FP}}_q \widetilde{\sqsubseteq} \{I_2^c\}$.

From the Forward Antichains Algorithm to FAIncS

Since $\widetilde{\sqsubseteq} = \sqsubseteq^{-1}$, we have that $\widetilde{\sqcap} = \sqcup$, $\widetilde{\sqcup} = \sqcap$ and the greatest element $\emptyset$ for $\widetilde{\sqsubseteq}$ is the least element for $\sqsubseteq$. Moreover, by (4.17), $\alpha(\vec{\epsilon}^{F_1}) = \langle \psi_{\emptyset}^{\{F_2\}}(q \in F_1) \rangle_{q \in Q_1}$. Therefore, we can rewrite the vector $\overrightarrow{\mathcal{FP}}$ of Corollary 4.4.4 as

$$\overrightarrow{\mathcal{FP}} = \sqcap \{ \vec{X} \in (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|} \mid \vec{X} = \text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) \sqcup \alpha(\vec{\epsilon}^{F_1}) \}$$

which is precisely the lfp in $\langle (\text{AC}_{\langle \wp(Q_2), \subseteq \rangle})^{|Q_1|}, \sqsubseteq \rangle$ of $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}$ above $\alpha(\vec{\epsilon}^{F_1})$.

Hence, it turns out that the Kleene iterates of the least fixpoint computation that converge to $\overrightarrow{\mathcal{FP}}$ exactly coincide with the iterates computed by the KLEENE procedure of the state-based algorithm FAInCS. In particular, if $\vec{Y}$ is the output vector of the call to KLEENE at line 1 of FAInCS then $\vec{Y} = \overrightarrow{\mathcal{FP}}$. Furthermore,

$$\exists q \in I_1, \overrightarrow{\mathcal{FP}}_q \sqsubseteq \{I_2^c\} \Leftrightarrow \exists q \in I_1, \exists S \in \overrightarrow{\mathcal{FP}}_q, S \cap I_2 = \emptyset .$$

Summing up, the $\sqsubseteq$ -lfp algorithm FAInCS coincides with the $\sqsubseteq$ -gfp antichains algorithm given by Corollary 4.4.4.

Backward Antichains Algorithm

We can also derive an antichains algorithm for deciding language inclusion fully equivalent to the backward one of Wulf et al. [2006][Theorem 6] by considering the lattice $\langle \text{AC}_{\langle \wp(Q_2), \supseteq \rangle}, \sqsubseteq \rangle$ for the dual lattice $\langle \wp(Q_2), \supseteq \rangle$ and by replacing the functions α, γ and $\text{Pre}_{\mathcal{N}_1}^{\mathcal{N}_2}$ of Lemma 4.4.1, respectively, with:

$$\begin{aligned} \alpha^c(X) &\stackrel{\text{def}}{=} \lfloor \{ \text{cpre}_u^{\mathcal{N}_2}(F_2^c) \in \wp(Q_2) \mid u \in X \} \rfloor , \\ \gamma^c(Y) &\stackrel{\text{def}}{=} \{ u \in \Sigma^* \mid \exists y \in Y, y \supseteq \text{cpre}_u^{\mathcal{N}_2}(F_2^c) \}, \\ \text{CPre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\langle X_q \rangle_{q \in Q_1}) &\stackrel{\text{def}}{=} \langle \lfloor \{ \text{cpre}_a^{\mathcal{N}_2}(S) \in \wp(Q_2) \mid \exists a \in \Sigma, q' \in Q_1, \\ &\quad q' \in \delta_1(q, a) \wedge S \in X_{q'} \} \rfloor \rangle_{q \in Q_1} . \end{aligned}$$

where $\text{cpre}_u^{\mathcal{N}_2}(S) \stackrel{\text{def}}{=} (\text{pre}_u^{\mathcal{N}_2}(S^c))^c$ for $u \in \Sigma^*$.

When instantiating Theorem 4.2.11 using these functions, we obtain an lfp algorithm computing on the lattice $\langle \text{AC}_{\langle \wp(Q_2), \supseteq \rangle}, \sqsubseteq \rangle$. Indeed, it turns out that

$$\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2) \Leftrightarrow \text{KLEENE}(\lambda \vec{X}. \text{CPre}_{\mathcal{N}_1}^{\mathcal{N}_2}(\vec{X}) \sqcup \alpha^c(\vec{\epsilon}^{F_1}), \vec{\emptyset}) \sqsubseteq \alpha^c(\vec{L}_2^{I_1}) .$$

It is easily seen that this algorithm coincides with the backward antichains algorithm defined by Wulf et al. [2006, Theorem 6] since both compute on the same lattice, $\lfloor X \rfloor$ corresponds to the maximal (w.r.t. set inclusion) elements of X , $\alpha^c(\{\epsilon\}) = \{F_2^c\}$ and for all $X \in \alpha^c(\wp(\Sigma^*))$, $X \sqsubseteq \alpha^c(L_2) \Leftrightarrow \forall S \in X, I_2 \not\subseteq S$.

Variants of the Antichains Algorithm

We have shown that the two forward/backward antichains algorithms introduced by Wulf et al. [2006] can be systematically derived by instantiating our framework and (possibly) considering the reverse automata. Similarly, we can derive within our framework an algorithm equivalent to the backward antichains algorithm applied to the reverse automata and an algorithm equivalent to the

forward antichains algorithm (without reverting the automata). Table 4.1 summarizes the relation between our framework and the antichains algorithms given (explicitly or implicitly) by Wulf et al. [2006].

	Backward	Forward
$\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$	$\text{cpre}_u^{N_2}(F_2^c) \subseteq \text{cpre}_v^{N_2}(F_2^c)$	$\text{pre}_u^{N_2}(I_2) \subseteq \text{pre}_v^{N_2}(I_2)$
$\mathcal{L}(\mathcal{N}_1^R) \subseteq \mathcal{L}(\mathcal{N}_2^R)$	$\text{cpost}_u^{N_2}(I_2^c) \subseteq \text{cpost}_v^{N_2}(I_2^c)$	$\text{post}_u^{N_2}(I_2) \subseteq \text{post}_v^{N_2}(I_2)$

Table 4.1: Summary of the quasiorders that should be used within our framework, i.e. using Theorem 4.2.11, to derive the different antichains algorithms that are (explicitly or implicitly) given by Wulf et al. [2006]. Each cell of the form $f(u) \subseteq f(v)$ is the definition of the quasiorder $u \leq v \stackrel{\text{def}}{=} f(u) \subseteq f(v)$ that should be used to derive the antichains algorithm given by the column for solving the language inclusion given by the row.

The original antichains algorithms were later improved by Abdulla et al. [2010] and, subsequently, by Bonchi and Pous [2013]. Among their improvements, they showed how to exploit a precomputed binary relation between pairs of states of the input automata such that language inclusion holds for all the pairs in the relation. When that binary relation is a simulation relation, our framework allows to partially match their results by using the quasiorder $\leq_{\mathcal{N}}^r$ defined in Section 4.3.3. However, this quasiorder relation $\leq_{\mathcal{N}}^r$ does not consider pairs of states $Q_1 \times Q_1$ whereas the aforementioned algorithms do.

4.5 Inclusion for Context Free Languages

In Section 4.2 we used the general abstraction scheme presented in Section 4.1 to derive two techniques (Theorems 4.2.10 and 4.2.11) for defining algorithms for solving language inclusion problems. Then, in Sections 4.3 and 4.4 we applied these techniques on different scenarios and derived algorithms for solving language inclusion problems $L_1 \subseteq L_2$ where L_1 and L_2 are regular languages.

In this section, we show that the abstraction scheme from Section 4.1 is general enough to cover language inclusion problems $L_1 \subseteq L_2$ where L_1 is context-free. In particular, we replicate the developments from Sections 4.2, 4.3 and 4.4 in order to extend our quasiorder-based framework for deciding the inclusion $L_1 \subseteq L_2$ where L_1 is a context-free language and L_2 is regular.

4.5.1 Extending the Framework to CFGs

Similarly to the case of automata, a CFG $\mathcal{G} = (\mathcal{V}, \Sigma, P)$ in CNF induces the following set of equations:

$$\text{Eqn}(\mathcal{G}) \stackrel{\text{def}}{=} \{X_i = \bigcup_{X_i \rightarrow \beta_j \in P} \beta_j \mid i \in [0, n]\} .$$

Given a subset of variables $S \subseteq \mathcal{V}$ of a grammar, the set of words generated from some variable in S is defined as

$$W_S^{\mathcal{G}} \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \exists X \in S, X \rightarrow^* w\} .$$

When $S = \{X\}$ we slightly abuse the notation and write $W_X^{\mathcal{G}}$. Also, we drop the superscript $\mathcal{G}$ when the grammar is clear from the context. The language generated by $\mathcal{G}$ is therefore $\mathcal{L}(\mathcal{G}) = W_{X_0}^{\mathcal{G}}$.

Next, we define the function $\text{Fn}_{\mathcal{G}} : \wp(\Sigma^*)^{|\mathcal{V}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{V}|}$ and the vector $\vec{b} \in \wp(\Sigma^*)^{|\mathcal{V}|}$, which are used to formalize the equations in $\text{Eqn}(\mathcal{G})$, as follows:

$$\begin{aligned} \vec{b} &\stackrel{\text{def}}{=} \langle b_i \rangle_{i \in [0, n]} \in \wp(\Sigma^*)^{|\mathcal{V}|} \quad \text{with } b_i \stackrel{\text{def}}{=} \{\beta \mid X_i \rightarrow \beta \in P, \beta \in \Sigma \cup \{\epsilon\}\}, \\ \text{Fn}_{\mathcal{G}}(\vec{X}) &\stackrel{\text{def}}{=} \langle \beta_1^{(i)} \cup \dots \cup \beta_{k_i}^{(i)} \rangle_{i \in [0, n]} \quad \text{with } \beta_j^{(i)} \in \mathcal{V}^2 \text{ and } X_i \rightarrow \beta_j^{(i)} \in P . \end{aligned}$$

Notice that function $\lambda \vec{X}. \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})$ is a well-defined monotone function in $\wp(\Sigma^*)^{|\mathcal{V}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{V}|}$, which therefore has the least fixpoint

$$\langle Y_i \rangle_{i \in [0, n]} = \text{lfp}(\lambda \vec{X}. \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) \quad (4.18)$$

It is known [Ginsburg and Rice 1962] that the language $\mathcal{L}(\mathcal{G})$ accepted by $\mathcal{G}$ is such that $\mathcal{L}(\mathcal{G}) = Y_0$.

Example 4.5.1. Consider the following grammar in CNF:

$$\mathcal{G} = \langle \{X_0, X_1\}, \{a, b\}, \{X_0 \rightarrow X_0X_1 \mid X_1X_0 \mid b, X_1 \rightarrow a\} \rangle .$$

The corresponding equation system is

$$\text{Eqn}(\mathcal{G}) = \begin{cases} X_0 = X_0X_1 \cup X_1X_0 \cup \{b\} \\ X_1 = \{a\} \end{cases}$$

so that

$$\begin{pmatrix} W_{X_0} \\ W_{X_1} \end{pmatrix} = \text{lfp} \left(\lambda \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} . \begin{pmatrix} X_0X_1 \cup X_1X_0 \cup \{b\} \\ \{a\} \end{pmatrix} \right) = \begin{pmatrix} a^*ba^* \\ a \end{pmatrix} .$$

Moreover, we have that $\vec{b} \in \wp(\Sigma^*)^2$ and $\text{Fn}_{\mathcal{G}} : \wp(\Sigma^*)^2 \rightarrow \wp(\Sigma^*)^2$ are given by

$$\vec{b} = \langle \{b\}, \{a\} \rangle \quad \text{Fn}_{\mathcal{G}}(\langle X_0, X_1 \rangle) = \langle X_0X_1 \cup X_1X_0, \emptyset \rangle \quad \diamond$$

Thus, it follows from Equation (4.18) that

$$\mathcal{L}(\mathcal{G}) \subseteq L_2 \Leftrightarrow \text{lfp}(\lambda \vec{X}. \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) \subseteq \vec{L}_2^{X_0} \quad (4.19)$$

where $\vec{L}_2^{X_0} \stackrel{\text{def}}{=} \langle \psi_{\Sigma^*}^{L_2}(i = 0) \rangle_{i \in [0, n]}$.

As we did for the automata case in Section 4.2, we next apply Theorem 4.1.1 in order to derive algorithms for solving the language inclusion problem $\mathcal{L}(\mathcal{G}) \subseteq L_2$ by using backward complete abstractions of $\wp(\Sigma^*)$.

THEOREM 4.5.2. *Let $\rho \in \text{uco}(\wp(\Sigma^*))$ be backward complete for both $\lambda X.Xa$ and $\lambda X.aX$, for all $a \in \Sigma$ and let $\mathcal{G} = (\mathcal{V}, \Sigma, P)$ be a CFG in CNF. Then ρ is backward complete for $\text{Fn}_{\mathcal{G}}$ and $\lambda \vec{X}. \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})$.*

Proof. Let us first show that backward completeness for left and right concatenation can be extended from letter to words. We give the proof for the concatenation to the left, the case of the concatenation to the right is symmetric. We prove that $\rho(wX) = \rho(w\rho(X))$ for every $w \in \Sigma^*$. We proceed by induction on the length of w .

The base case is trivial because ρ is idempotent. For the inductive case $|w| > 0$ let $w = au$ for some $u \in \Sigma^*$ and $a \in \Sigma$, so that:

$$\begin{aligned} \rho(auX) &= \text{[By backward completeness for } \lambda X.aX\text{]} \\ \rho(a\rho(uX)) &= \text{[By inductive hypothesis]} \\ \rho(a\rho(u\rho(X))) &= \text{[By backward completeness for } \lambda X.aX\text{]} \\ \rho(a\rho(X)) &. \end{aligned}$$

Next we turn to the binary concatenation case, i.e. we prove that $\rho(YZ) = \rho(\rho(Y)\rho(Z))$ for all $Y, Z \in \wp(\Sigma^*)$:

$$\begin{aligned} \rho(\rho(Y)\rho(Z)) &= \text{[By definition of concatenation]} \\ \rho(\bigcup_{u \in \rho(Y)} u\rho(Z)) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{u \in \rho(Y)} \rho(u\rho(Z))) &= \text{[By backward completeness of } \lambda X.wX\text{]} \\ \rho(\bigcup_{u \in \rho(Y)} \rho(uZ)) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{u \in \rho(Y)} uZ) &= \text{[By definition of concatenation]} \\ \rho(\rho(Y)Z) &= \text{[By definition of concatenation]} \\ \rho(\bigcup_{v \in Z} \rho(Y)v) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{v \in Z} \rho(Y)v) &= \text{[By backward completeness of } \lambda X.Xw\text{]} \\ \rho(\bigcup_{v \in Z} \rho(Yv)) &= \text{[By Equation (3.2)]} \end{aligned}$$

$$\begin{aligned}\rho(\bigcup_{v \in Z} Yv) &= \text{[By definition of concatenation]} \\ \rho(YZ) &.\end{aligned}$$

Then, the proof follows the same lines of the proof of Theorem 4.2.3. Indeed, it follows from the definition of $\text{Fn}_{\mathcal{G}}(\langle X_i \rangle_{i \in [0, n]})$ that:

$$\begin{aligned}\rho(\bigcup_{j=1}^{k_i} \beta_j^{(i)}) &= \text{[By definition of } \beta_j^{(i)}\text{]} \\ \rho(\bigcup_{j=1}^{k_i} X_j^{(i)} Y_j^{(i)}) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{j=1}^{k_i} \rho(X_j^{(i)} Y_j^{(i)})) &= \text{[By backward comp. of } \rho \text{ for concatenation]} \\ \rho(\bigcup_{j=1}^{k_i} \rho(\rho(X_j^{(i)}) \rho(Y_j^{(i)}))) &= \text{[By Equation (3.2)]} \\ \rho(\bigcup_{j=1}^{k_i} \rho(X_j^{(i)}) \rho(Y_j^{(i)})) &.\end{aligned}$$

Hence, by a straightforward componentwise application on vectors in $\wp(\Sigma^*)^{|\mathcal{V}|}$, we obtain that ρ is backward complete for $\text{Fn}_{\mathcal{G}}$. Finally, ρ is backward complete for $\lambda \vec{X}. (\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))$, because:

$$\begin{aligned}\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\rho(\vec{X}))) &= \text{[By Equation (3.2)]} \\ \rho(\rho(\vec{b}) \cup \rho(\text{Fn}_{\mathcal{G}}(\rho(\vec{X})))) &= \text{[By backward comp. for } \text{Fn}_{\mathcal{G}}\text{]} \\ \rho(\rho(\vec{b}) \cup \rho(\text{Fn}_{\mathcal{G}}(\vec{X}))) &= \text{[By Equation (3.2)]} \\ \rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) &.\end{aligned}$$

As a consequence, by backward completeness of ρ for $\lambda \vec{X}. (\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))$, by (4.1) it turns out that:

$$\rho(\text{lfp}(\lambda \vec{X}. \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) = \text{lfp}(\lambda \vec{X}. \rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) .$$

Note that if ρ is backward complete for both left and right concatenation and $\rho(L_2) = L_2$ then, as a straightforward consequence of Equivalence (4.19) and Theorems 4.1.1 and 4.5.2, we have that:

$$\mathcal{L}(\mathcal{G}) \subseteq L_2 \Leftrightarrow \text{lfp}(\lambda \vec{X}. \rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) \subseteq \vec{L}_2^{X_0} . \quad (4.20)$$

Next, we present two techniques for solving the language inclusion problem $\mathcal{L}(\mathcal{G}) \subseteq L_2$ by relying on Equivalence (4.20). As with the two techniques presented in Section 4.2.3, the first of these techniques allows us to define algorithms for deciding the inclusion by working on finite languages while the second one relies on the use of Galois Connections.

4.5.2 Solving the Abstract Inclusion Check using Finite Languages

The following result, which is an adaptation of Corollary 4.2.8 for grammars, shows that the fixpoint iteration for $\text{lfp}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))$ can be replicated by iterating on a set of functions $\mathcal{F}$, and then abstracting the result, provided that all functions in $\mathcal{F}$ meet a set of requirements.

LEMMA 4.5.3. *Let $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ be a CFG in CNF, let $\rho \in \text{uco}(\Sigma^*)$ be backward complete for $\lambda X \in \wp(\Sigma^*)$. aX and $\lambda X \in \wp(\Sigma^*)$. Xa for all $a \in \Sigma$ and let $\mathcal{F}$ be a set of functions such that every $f \in \mathcal{F}$ is of the form $f : \wp(\Sigma^*)^{|\mathcal{V}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{V}|}$ and satisfies $\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) = \rho(f(\vec{X}))$. Then, for all $0 \leq n$,*

$$(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))^n = \rho(\mathcal{F}^n(\vec{X})) .$$

Proof. We proceed by induction in n .

- Base case: Let $n = 0$. Then $\mathcal{F}^0(\vec{X}) = (\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))^0 = \vec{\emptyset}$.
- Inductive step: Assume that $\rho(\mathcal{F}^n(\vec{X})) = (\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))^n$ holds for some value $n \geq 0$. To simplify the notation, let $\mathcal{P}(\vec{X}) = \vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})$ so that $\rho\mathcal{F}^n = (\rho\mathcal{P})^n$. Then

$$\begin{aligned} \rho\mathcal{F}^{n+1}(\vec{X}) &= [\text{Since } \mathcal{F}^{n+1} = \mathcal{F}^n\mathcal{F}] \\ \rho\mathcal{F}^n\mathcal{F}(\vec{X}) &= [\text{By Inductive Hypothesis}] \\ (\rho\mathcal{P})^n\mathcal{F}(\vec{X}) &= [\text{By Theorem 4.5.2, } \rho \text{ is bw. complete for } \mathcal{P}] \\ (\rho\mathcal{P})^n\rho\mathcal{F}(\vec{X}) &= [\text{Since } \rho f = \rho\mathcal{P}] \\ (\rho\mathcal{P})^n\rho\mathcal{P}(\vec{X}) &= [\text{By definition of } (\rho(\mathcal{P}))^n] \\ (\rho\mathcal{P})^{n+1}(\vec{X}) & \end{aligned}$$

We conclude that $(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))^n = \rho(\mathcal{F}^n(\vec{X}))$ for all $0 \leq n$.

We are now in position to show that the procedure $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, b)$ can be used to compute $\text{lfp}(\lambda \vec{X}. \rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})))$.

LEMMA 4.5.4. *Let $\rho \in \text{uco}(\Sigma^*)$ be backward complete for $\lambda X \in \wp(\Sigma^*)$. aX and $\lambda X \in \wp(\Sigma^*)$. Xa for all $a \in \Sigma$ such that $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO and let $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ be a CFG in CNF. Let $\mathcal{F}$ be a set of functions such that every $f \in \mathcal{F}$ is of the form $f : \wp(\Sigma^*)^{|\mathcal{V}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{V}|}$ and satisfies $\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) = \rho(f(\vec{X}))$. Then,*

$$\text{lfp}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right) .$$

Moreover, the iterates of $\text{KLEENE}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})), \vec{\emptyset})$ coincide in lockstep with the abstraction of the iterates of $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset})$

Proof. Since $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO, by Theorem 3.5.1, we have that

$$\text{lfp}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) = \text{KLEENE}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})), \vec{\emptyset})$$

On the other hand, by Lemma 4.5.3, the iterates of the above Kleene iteration coincide in lockstep with the abstraction of the iterates of $\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset})$ and, therefore,

$$\text{KLEENE}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})), \vec{\emptyset}) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right)$$

As a consequence,

$$\text{lfp}(\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right) .$$

We are now in position to introduce the equivalent of Theorem 4.2.10 for grammars.

THEOREM 4.5.5. *Let $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ be a CFG in CNF, let L_2 be a regular language, let $\rho \in \text{uco}(\Sigma^*)$ and let $\mathcal{F}$ be a set of functions. Assume that the following properties hold:*

- (i) *The abstraction ρ satisfies $\rho(L_2) = L_2$ and it is backward complete for both $\lambda X \in \wp(\Sigma^*). aX$ and $\lambda X \in \wp(\Sigma^*). Xa$ for all $a \in \Sigma$.*
- (ii) *The set $\langle \{\rho(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO.*
- (iii) *Every function f_i in the set $\mathcal{F}$ is of the form $f_i : \wp(\Sigma^*)^{|\mathcal{V}|} \rightarrow \wp(\Sigma^*)^{|\mathcal{V}|}$, it is computable and satisfies $\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})) = \rho(f_i(\vec{X}))$.*
- (iv) *There is an algorithm, say $\text{AbsEq}^\sharp(\vec{X}, \vec{Y})$, which decides the abstraction equivalence $\rho(\vec{X}) = \rho(\vec{Y})$, for all $\vec{X}, \vec{Y} \in \wp(\Sigma^*)^{|\mathcal{V}|}$.*
- (v) *There is an algorithm, say $\text{Incl}^\sharp(\vec{X})$, which decides the inclusion $\rho(\vec{X}) \subseteq \vec{L}_2^{X_0}$, for all $\vec{X} \in \wp(\Sigma^*)^{|\mathcal{V}|}$.*

Then, the following is an algorithm which decides whether $\mathcal{L}(\mathcal{G}) \subseteq L_2$:

$\langle Y_i \rangle_{i \in [0, n]} := \widehat{\text{KLEENE}}(\text{AbsEq}^\sharp, \mathcal{F}, \vec{\emptyset});$
return $\text{Incl}^\sharp(\langle Y_i \rangle_{i \in [0, n]});$

Proof. It follows from hypotheses (i), (ii) and (iii), by Lemma 4.5.4, that

$$\text{lfp}(\rho(\vec{\epsilon}^F \cup \text{Pre}_{\mathcal{N}}(\vec{X}))) = \rho \left(\widehat{\text{KLEENE}}(\text{AbsEq}, \mathcal{F}, \vec{\emptyset}) \right) \quad (4.21)$$

Observe that function AbsEq can be replaced by function AbsEq[#] due to hypothesis (iv). Moreover, it follows from Equivalence (4.20), which holds by hypothesis (i), and Equation (4.21) that

$$\mathcal{L}(N) \subseteq L_2 \Leftrightarrow \rho \left(\overline{\text{KLEENE}}(\text{AbsEq}^\#, \mathcal{F}, \vec{\emptyset}) \right) \subseteq \vec{L}_2^{X_0}.$$

Finally, hypotheses (iv) and (v) guarantee, respectively, the decidability of the inclusion $\rho\mathcal{F}(X) \subseteq \rho(X)$ performed at each step of the $\overline{\text{KLEENE}}$ iteration and the decidability of the inclusion of the lfp in $\vec{L}_2^{X_0}$.

4.5.3 Solving the Abstract Inclusion Check using Galois Connections

The following result is the equivalent of Theorem 4.2.11 for context-free languages. It shows that the language inclusion problem $\mathcal{L}(\mathcal{G}) \subseteq L_2$ can be solved by working on an abstract domain.

THEOREM 4.5.6. *Let $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ be a CFG in CNF and let L_2 be a language over Σ . Let $\langle \wp(\Sigma^*), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle D, \sqsubseteq \rangle$ be a GC where $\langle D, \leq_D \rangle$ is a poset. Assume that the following properties hold:*

- (i) $L_2 \in \gamma(D)$ and for every $a \in \Sigma$, $X \in \wp(\Sigma^*)$, $\gamma\alpha(aX) = \gamma\alpha(a\gamma\alpha(X))$ and $\gamma\alpha(Xa) = \gamma\alpha\gamma(\alpha(X)a)$.
- (ii) $(D, \leq_D, \sqcup, \perp_D)$ is an effective domain, meaning that: $(D, \leq_D, \sqcup, \perp_D)$ is an ACC join-semilattice with bottom $\perp_D$, every element of D has a finite representation, the binary relation $\leq_D$ is decidable and the binary lub $\sqcup$ is computable.
- (iii) There is an algorithm, say $\text{Fn}^\#(\vec{X}^\#)$, which computes $\alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X})))$, for all $\vec{X}^\# \in \alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$.
- (iv) There is an algorithm, say $b^\#$, which computes $\alpha(\vec{b})$.
- (v) There is an algorithm, say $\text{Incl}^\#(\vec{X}^\#)$, which decides the abstract inclusion $\vec{X}^\# \leq_D \alpha(\vec{L}_2^{X_0})$, for all $\vec{X}^\# \in \alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$.

Then, the following is an algorithm which decides whether $\mathcal{L}(\mathcal{G}) \subseteq L_2$:

```

 $\langle Y_i^\# \rangle_{i \in [0, n]} := \text{KLEENE}(\lambda \vec{X}^\#. b^\# \sqcup \text{Fn}^\#(\vec{X}^\#), \perp_D);$ 
return  $\text{Incl}^\#(\langle Y_i^\# \rangle_{i \in [0, n]});$ 

```

Proof. Let $\rho = \gamma\alpha \in \text{uco}(\wp(\Sigma^*))$. Then, it follows from property (i) that $L_2 \in \rho$, $\rho(aX) = \rho(a\rho(X))$ and $\rho(Xa) = \rho(\rho(X)a)$. Therefore

$$\begin{aligned} \mathcal{L}(\mathcal{N}) \subseteq L_2 &\Leftrightarrow [\text{By (4.20)}] \\ \text{lfp}(\lambda \vec{X}. \rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X}))) &\subseteq \vec{L}_2^{X_0} \Leftrightarrow [\text{By Lemma 3.6.2}] \\ \gamma(\text{lfp}(\lambda \vec{X}^\sharp. \alpha(\vec{b}) \sqcup \alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}^\sharp)))))) &\subseteq \vec{L}_2^{X_0} \Leftrightarrow [\text{By GC and since } L_2 \in \rho] \\ \text{lfp}(\lambda \vec{X}^\sharp. \alpha(\vec{b}) \sqcup \alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}^\sharp)))) &\leq_D \alpha(\vec{L}_2^{X_0}) . \end{aligned}$$

By hypotheses (ii), (iii) and (iv) it turns out that $\text{KLEENE}(\lambda \vec{X}^\sharp. b^\sharp \sqcup \text{Fn}^\sharp(\vec{X}^\sharp), \vec{1}_D)$ is an algorithm computing $\text{lfp}(\lambda \vec{X}^\sharp. \alpha(\vec{b}) \sqcup \alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}^\sharp))))$. In particular, these hypotheses ensure that the Kleene iterates of $\text{lfp}(\lambda \vec{X}^\sharp. \alpha(\vec{b}) \sqcup \alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}^\sharp))))$ starting from $\vec{1}_D$ are computable, finitely many and that it is decidable whether the iterates have reached the fixpoint. The hypothesis (v) ensures decidability of the required $\leq_D$ -inclusion check of this least fixpoint in $\alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$.

4.5.4 Instantiating the Framework

Let us instantiate the general algorithmic framework provided by Theorem 4.5.5 to the class of closure operators induced by quasiorder relations on words. Recall that a quasiorder $\leq$ on Σ^* is monotone if

$$\forall x_1, x_2 \in \Sigma^*, \forall a, b \in \Sigma, x_1 \leq x_2 \Rightarrow ax_1b \leq ax_2b . \quad (4.22)$$

It follows that $x_1 \leq x_2 \Rightarrow \forall u, v \in \Sigma^*, ux_1v \leq ux_2v$. The following result is the equivalent to Lemma 4.3.2 for L -consistent quasiorders and it allows us to characterize L -consistent quasiorders in terms of the induced closure.

LEMMA 4.5.7. *Let $L \in \wp(\Sigma^*)$ and $\leq_L$ be a quasiorder on Σ^* . Then, $\leq_L$ is an L -consistent quasiorder on Σ^* if and only if*

- (a) $\rho_{\leq_L}(L) = L$, and
- (b) $\rho_{\leq_L}$ is backward complete for $\lambda X. aXb$ for all $a, b \in \Sigma$.

Proof.

- (a) It follows from Lemma 4.3.2 (a) since, by Definition 4.3.1, a quasiorder is L -consistent iff it is left and right L -consistent.
- (b) We first prove that if $\leq_L$ is monotone. Then for all $X \in \wp(\Sigma^*)$ we have that $\rho_{\leq_L}(aXb) = \rho_{\leq_L}(a\rho_{\leq_L}(X)b)$ for all $a, b \in \Sigma$.
Monotonicity of concatenation together with monotonicity and extensivity of the closure $\rho_{\leq_L}$ imply that $\rho_{\leq_L}(aXb) \subseteq \rho_{\leq_L}(a\rho_{\leq_L}(X)b)$ holds. For

the reverse inclusion, we have that:

$$\begin{aligned}
 \rho_{\leq_L}(a\rho_{\leq_L}(X)b) &= [\text{By definition of } \rho_{\leq_L}] \\
 \rho_{\leq_L}(\{ayb \mid \exists x \in X, x \leq_L y\}) &= [\text{By definition of } \rho_{\leq_L}] \\
 \{z \mid \exists x \in X, y \in \Sigma^*, x \leq_L y \wedge ayb \leq_L z\} &\subseteq [\text{By monotonicity of } \leq_L] \\
 \{z \mid \exists x \in X, y \in \Sigma^*, axb \leq_L ayb \wedge ayb \leq_L z\} &= [\text{By transitivity of } \leq_L] \\
 \{z \mid \exists x \in X, axb \leq_L z\} &= [\text{By definition of } \rho_{\leq_L}] \\
 \rho_{\leq_L}(aXb) &.
 \end{aligned}$$

Next, we show that if $\rho_{\leq_L}(aXb) = \rho_{\leq_L}(a\rho_{\leq_L}(X)b)$ for all $X \in \wp(\Sigma^*)$ and $a, b \in \Sigma$ then $\leq_L$ is monotone. Let $x_1, x_2 \in \Sigma^*$, $a, b \in \Sigma$. If $x_1 \leq_L x_2$ then $\{x_2\} \subseteq \rho_{\leq_L}(\{x_1\})$, and in turn $a\{x_2\}b \subseteq a\rho_{\leq_L}(\{x_1\})b$. Since $\rho_{\leq_L}$ is monotone, we have that $\rho_{\leq_L}(a\{x_2\}b) \subseteq \rho_{\leq_L}(a\rho_{\leq_L}(\{x_1\})b)$, so that, by backward completeness, $\rho_{\leq_L}(a\{x_2\}b) \subseteq \rho_{\leq_L}(a\{x_1\}b)$. It follows that, $a\{x_2\}b \subseteq \rho_{\leq_L}(a\{x_1\}b)$, namely, $ax_1b \leq_L ax_2b$. By Equation (4.22), this shows that $\leq_L$ is monotone.

Analogously to the case of regular languages presented in Section 4.3, Theorem 4.5.5 induces an algorithm for deciding the language inclusion $\mathcal{L}(\mathcal{G}) \subseteq L_2$ for any CFG $\mathcal{G}$ and regular language L_2 . Indeed, we can apply Theorem 4.5.5 with $[\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})]$ interpreted as the set of functions $f_i \stackrel{\text{def}}{=} [\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})]_i$ where, again, each $[\cdot]_i$ is a function mapping each set $X \in \wp(\Sigma^*)$ into a minor $[X]_i$.

As a consequence, we obtain Algorithm **CFGIncW** which, given a language L_2 whose membership problem is decidable and a decidable L_2 -consistent well-quasiorder, determines whether $\mathcal{L}(\mathcal{G}) \subseteq L_2$ holds.

CFGIncW: Word-based algorithm for $\mathcal{L}(\mathcal{G}) \subseteq L_2$

Data: CFG $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$; decision procedure for $u \in L_2$; decidable

L_2 -consistent wqo $\leq_{L_2}$.

- 1 $\langle Y_i \rangle_{i \in [0, n]} := \widehat{\text{KLEENE}}(\sqsubseteq_{\leq_{L_2}} \cap (\sqsubseteq_{\leq_{L_2}})^{-1}, \lambda \vec{X}. [\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})], \vec{\emptyset})$;
 - 2 **forall** $u \in Y_0$ **do**
 - 3 **if** $u \notin L_2$ **then return false**;
 - 4 **return true**;
-

THEOREM 4.5.8. *Let $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ be a CFG in CNF and let $L_2 \in \wp(\Sigma^*)$ be a language such that: (i) membership $u \in L_2$ is decidable; (ii) there exists a decidable L_2 -consistent well-quasiorder on Σ^* . Then, Algorithm **CFGIncW** decides the inclusion $\mathcal{L}(\mathcal{G}) \subseteq L_2$.*

Proof. Let $\leq_{L_2}$ be a decidable L_2 -consistent well-quasiorder on Σ^* . Then, we check that hypotheses (i)-(v) of Theorem 4.5.5 are satisfied.

- (a) It follows from hypothesis (ii) and Lemma 4.5.7 that $\leq_{L_2}$ is backward complete for left and right concatenation and satisfies $\rho_{\leq_{L_2}}(L_2) = L_2$.
- (b) Since $\leq_{L_2}$ is a well-quasiorder, it follows that $\langle \{\rho_{\leq_{L_2}}(S) \mid S \in \wp(\Sigma^*)\}, \subseteq \rangle$ is an ACC CPO.
- (c) Let $[\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})]$ be the set of functions f_i each of which maps each set $X \in \wp(\Sigma^*)$ into a minor of $\vec{b} \cup \text{Fn}_{\mathcal{G}}(\vec{X})$. Since $\rho_{\leq_{L_2}}(X) = \rho_{\leq_{L_2}}(\lfloor X \rfloor)$ for all $X \in \wp(\Sigma^*)^{|\mathcal{V}|}$, we have that all functions f_i satisfy

$$\rho(\vec{b} \cup \text{Fn}_{\mathcal{G}}(\rho(\vec{X}))) = \rho(f_i(\vec{X})) .$$

- (d) The equality $\rho_{\leq_{L_2}}(S_1) = \rho_{\leq_{L_2}}(S_2)$ is decidable for every $S_1, S_2 \in \wp(\Sigma^*)^{|\mathcal{V}|}$ since $\rho_{\leq_{L_2}}(S_1) = \rho_{\leq_{L_2}}(S_2) \Leftrightarrow S_1 \sqsubseteq_{\leq_{L_2}} S_2 \wedge S_2 \sqsubseteq_{\leq_{L_2}} S_1$ and $\leq_{L_2}$ is decidable.
- (e) Since $\vec{L}_2^{X_0} = \langle \psi_{\Sigma^*}^{L_2}(i = 0) \rangle_{i \in [0, n]}$, the inclusion trivially holds for all components Y_i with $i \neq 0$. Therefore, it suffices to check whether $Y_0 \subseteq L_2$ holds. Since $Y_0 = \lfloor S \rfloor$ for some set $S \in \wp(\Sigma^*)$, the inclusion $Y_0 \subseteq L_2$ can be decided by performing finitely many membership tests, which is exactly the check performed by lines 2-4 of Algorithm CFGIncW. By hypothesis (i), this check is decidable.

4.5.4.1 Myhill and State-based Quasiorders

In the following, we will consider two quasiorders on Σ^* and we will show that they fulfill the requirements of Theorem 4.5.8, so that they yield algorithms for deciding the inclusion $\mathcal{L}(\mathcal{G}) \subseteq L_2$ for every CFG $\mathcal{G}$ and regular language L_2 .

The *context* of a word $w \in \Sigma^*$ w.r.t a given language $L \in \wp(\Sigma^*)$ is defined as:

$$\text{ctx}_L(w) \stackrel{\text{def}}{=} \{(u, v) \in \Sigma^* \times \Sigma^* \mid uwv \in L\} .$$

Correspondingly, let us define the following quasiorder relation on Σ^* :

$$u \leq_L v \stackrel{\text{def}}{\Leftrightarrow} \text{ctx}_L(u) \subseteq \text{ctx}_L(v) . \quad (4.23)$$

de Luca and Varricchio [1994, Section 2] call $\leq_L$ the *Myhill quasiorder relative to L*. The following result is the analogue of Lemma 4.3.5 for L -consistent and Myhill's quasiorders: it shows that the Myhill's quasiorder is the weakest (i.e. greatest w.r.t. set inclusion between binary relations) L -consistent quasiorder for which Algorithm CFGIncW can be instantiated to decide the inclusion $\mathcal{L}(\mathcal{G}) \subseteq L$.

Figure 4.3: A finite automaton N with $\mathcal{L}(N) = (b + ab^*a)(a + b)^*$.

LEMMA 4.5.9. Let $L \in \wp(\Sigma^*)$.

- (a) $\leq_L$ is an L -consistent quasiorder. If L is regular then, additionally, $\leq_L$ is a decidable well-quasiorder.
- (b) If $\leq$ is an L -consistent quasiorder on Σ^* then $\rho_{\leq_L} \subseteq \rho_{\leq}$.

Proof. The proof follows the same lines of the proof of Lemma 4.3.5.

- (a) [de Luca and Varricchio \[1994, Section 3\]](#) observe that $\leq_L$ is monotone. Moreover, if L is regular then $\leq_L$ is a wqo [[de Luca and Varricchio 1994, Proposition 2.3](#)]. Let us observe that given $u \in L$ and $v \notin L$ we have that $(\epsilon, \epsilon) \in \text{ctx}_L(u)$ while $(\epsilon, \epsilon) \notin \text{ctx}_L(v)$. Hence, $\leq_L \cap (L \times L^c) = \emptyset$ and, therefore, $\leq_L$ is an L -consistent quasiorder. Finally, if L is regular then $\leq_L$ is clearly decidable.
- (b) As shown by [de Luca and Varricchio \[1994\]](#), $\leq_L$ is maximum in the set of all L -consistent quasiorders, i.e. every L -consistent quasiorder $\leq$ is such that $x \leq y \Rightarrow x \leq_L y$. As a consequence, $\rho_{\leq}(X) \subseteq \rho_{\leq_L}(X)$ holds for all $X \in \wp(\Sigma^*)$, namely $\leq \subseteq \leq_L$.

Example 4.5.10. Let us illustrate the use of the Myhill quasiorder $\leq_{\mathcal{L}(N)}$ in Algorithm [CFGIncW](#) for solving the language inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(N)$, where $\mathcal{G}$ is the CFG in Example 4.5.1 and N is the NFA depicted in Figure 4.3. Recall that the equations for $\mathcal{G}$ are:

$$\text{Eqn}(\mathcal{G}) = \begin{cases} X_0 = X_0X_1 \cup X_1X_0 \cup \{b\} \\ X_1 = \{a\} \end{cases}.$$

We write $\{(S, T)\} \cup \{(X, Y)\}$ to denote the set $\{(u, v) \mid (u, v) \in S \times T \cup X \times Y\}$. Then, we have the following contexts (among others) for $L = \mathcal{L}(N) = (b + ab^*a)(a + b)^*$:

$$\begin{aligned} \text{ctx}_L(\epsilon) &= \{(\epsilon, L)\} \cup \{(ab^*, b^*a\Sigma^*)\} \cup \{(L, \Sigma^*)\} \\ \text{ctx}_L(a) &= \{(\epsilon, b^*a\Sigma^*)\} \cup \{ab^*, \Sigma^*\} \cup \{(L, \Sigma^*)\} \end{aligned}$$

$$\begin{aligned}\text{ctx}_L(b) &= \{(\epsilon, \Sigma^*)\} \cup \{(ab^*, b^* a \Sigma^*)\} \cup \{(L, \Sigma^*)\} \\ \text{ctx}_L(ba) &= \{(\epsilon, \Sigma^*)\} \cup \{(ab^*, \Sigma^*)\} \cup \{(L, \Sigma^*)\}\end{aligned}$$

Moreover, $\text{ctx}_L(ab) = \text{ctx}_L(a)$ and $\text{ctx}_L(ba) = \text{ctx}_L(aa) = \text{ctx}_L(aaa) = \text{ctx}_L(aab) = \text{ctx}_L(aba)$ and, since $a \leq_L ba$ and $\epsilon \leq_L b$, it follows that $\lfloor \Sigma^* \rfloor = \{\epsilon, a\}$.

Recall that, as shown in Example 4.5.1, we have that $\vec{b} = \langle \{b\}, \{a\} \rangle$ and $\text{Fn}_{\mathcal{G}}(\langle X_0, X_1 \rangle) = \langle X_0 X_1 \cup X_1 X_0, \emptyset \rangle$. Next, we show the computation of the Kleene iterates according to Algorithm CFGIncW when using the quasiorder $\leq_L$.

$$\begin{aligned}\vec{Y}^{(0)} &= \vec{\emptyset} \\ \vec{Y}^{(1)} &= \lfloor \vec{b} \rfloor = \langle \{b\}, \{a\} \rangle \\ \vec{Y}^{(2)} &= \lfloor \vec{b} \rfloor \sqcup \lfloor \text{Fn}_{\mathcal{G}}(\vec{Y}^{(1)}) \rfloor = \langle \{b\}, \{a\} \rangle \sqcup \langle \lfloor \{ba, ab\} \rfloor, \lfloor \emptyset \rfloor \rangle \\ &= \langle \lfloor \{ba, ab, b\} \rfloor, \lfloor \{a\} \rfloor \rangle = \langle \{ab, b\}, \{a\} \rangle \\ \vec{Y}^{(3)} &= \lfloor \vec{b} \rfloor \sqcup \lfloor \text{Fn}_{\mathcal{G}}(\vec{Y}^{(2)}) \rfloor = \langle \{b\}, \{a\} \rangle \sqcup \langle \lfloor \{aba, ba, aab, ab\} \rfloor, \lfloor \emptyset \rfloor \rangle \\ &= \langle \lfloor \{aba, ba, aab, ab, b\} \rfloor, \lfloor \{a\} \rfloor \rangle = \langle \{ab, b\}, \{a\} \rangle\end{aligned}$$

The least fixpoint is therefore $\vec{Y} = \langle \{ab, b\}, \{a\} \rangle$. Since $ab \in \vec{Y}_0$ but $ab \notin \mathcal{L}(\mathcal{N})$, Algorithm CFGIncW concludes that the inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$ does not hold. $\diamond$

Similarly to Section 4.3, we also consider a state-based quasiorder that can be used with Algorithm CFGIncW. First, given an NFA $\mathcal{N} = \langle Q, \delta, I, F, \Sigma \rangle$ we define the state-based equivalent of the context of a word $w \in \Sigma^*$ as follows:

$$\text{ctx}_{\mathcal{N}}(w) \stackrel{\text{def}}{=} \{(q, q') \in Q \times Q \mid q \overset{w}{\rightsquigarrow} q'\}.$$

Then, the quasiorder $\leq_{\mathcal{N}}$ on Σ^* is defined as follows: for all $u, v \in \Sigma^*$,

$$u \leq_{\mathcal{N}} v \stackrel{\text{def}}{\iff} \text{ctx}_{\mathcal{N}}(u) \subseteq \text{ctx}_{\mathcal{N}}(v) \quad (4.24)$$

The following result is the analogue of Lemma 4.3.7 and shows that $\leq_{\mathcal{N}}$ is a $\mathcal{L}(\mathcal{N})$ -consistent well-quasiorder, hence it can be used with Algorithm CFGIncW to decide the language inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$.

LEMMA 4.5.11. *The relation $\leq_{\mathcal{N}}$ is a decidable $\mathcal{L}(\mathcal{N})$ -consistent wqo.*

Proof. For every $u \in \Sigma^*$, $\text{ctx}_{\mathcal{N}}(u)$ is a finite and computable set, so that $\leq_{\mathcal{N}}$ is a decidable wqo. Next, we show that $\leq_{\mathcal{N}}$ is $\mathcal{L}(\mathcal{N})$ -consistent according to Definition 4.3.1 (a)-(b).

- (a) By picking $u \in \mathcal{L}(\mathcal{N})$ and $v \notin \mathcal{L}(\mathcal{N})$ we have that $\text{ctx}_{\mathcal{N}}(u)$ contains a pair (q_i, q_f) with $q_i \in I$ and $q_f \in F$ while $\text{ctx}_{\mathcal{N}}(v)$ does not, hence $u \not\leq_{\mathcal{N}} v$. Therefore, $\leq_{\mathcal{N}} \cap (\mathcal{L}(\mathcal{N}) \times \mathcal{L}(\mathcal{N})^c) = \emptyset$.

(b) Let us check that $\leq_N$ is monotone. To do that, first observe that function $\text{ctx}_N : \langle \Sigma^*, \leq_N \rangle \rightarrow \langle \wp(Q^2), \subseteq \rangle$ is monotone. Therefore, for all $x_1, x_2 \in \Sigma^*$ and $a, b \in \Sigma$ we have that

$$\begin{aligned} x_1 \leq_N x_2 &\Rightarrow \quad [\text{By def. of } \leq_N] \\ \text{ctx}_N(x_1) \subseteq \text{ctx}_N(x_2) &\Rightarrow \quad [\text{Since } \text{ctx}_N \text{ is monotone}] \\ \text{ctx}_N(ax_1b) \subseteq \text{ctx}_N(ax_2b) &\Rightarrow \quad [\text{By def. of } \leq_N] \\ ax_1b \leq_N ax_2b &. \end{aligned}$$

For the Myhill wqo $\leq_{\mathcal{L}(N)}$, it turns out that for all $u, v \in \Sigma^*$,

$$\begin{aligned} u \leq_{\mathcal{L}(N)} v &\Leftrightarrow \begin{array}{ccc} \text{ctx}_{\mathcal{L}(N)}(u) & \subseteq & \{(x, y) \mid x \in W_{I,q} \wedge y \in W_{q',F} \wedge q \xrightarrow{u} q'\} \\ & \Leftrightarrow & \subseteq \\ \text{ctx}_{\mathcal{L}(N)}(v) & \subseteq & \{(x, y) \mid x \in W_{I,q} \wedge y \in W_{q',F} \wedge q \xrightarrow{v} q'\} \end{array} \end{aligned}$$

Therefore, $u \leq_N v \Rightarrow u \leq_{\mathcal{L}(N)} v$, hence $\leq_N \subseteq \leq_{\mathcal{L}(N)}$ holds.

Example 4.5.12. Let us illustrate the use of the state-based quasiorder $\leq_N$ to solve the language inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(N)$ of Example 4.5.10. Here, we have the following contexts (among others):

$$\begin{aligned} \text{ctx}_N(\epsilon) &= \{(q_1, q_1), (q_2, q_2), (q_3, q_3)\} & \text{ctx}_N(a) &= \{(q_1, q_2), (q_2, q_3), (q_3, q_3)\} \\ \text{ctx}_N(b) &= \{(q_1, q_3), (q_2, q_2), (q_3, q_3)\} & \text{ctx}_N(aa) &= \{(q_1, q_3), (q_2, q_3), (q_3, q_3)\} \end{aligned}$$

Moreover, $\text{ctx}_N(ab) = \text{ctx}_N(a)$ and $\text{ctx}_N(ba) = \text{ctx}_N(aa) = \text{ctx}_N(baa) = \text{ctx}_N(aab) = \text{ctx}_N(aba)$. Recall from Example 4.5.10 that for the Myhill wqo we have that $a \leq_{\mathcal{L}(N)} ba$, while for the state-based qo $a \not\leq_N ba$. Next, we show the Kleene iterates computed by Algorithm [CFGIncW](#) when using the wqo $\leq_N$.

$$\begin{aligned} \vec{Y}^{(0)} &= \vec{\emptyset} \\ \vec{Y}^{(1)} &= \lfloor \vec{b} \rfloor = \langle \{b\}, \{a\} \rangle \\ \vec{Y}^{(2)} &= \lfloor \vec{b} \rfloor \sqcup \lfloor \text{Fn}_{\mathcal{G}}(\vec{Y}^{(1)}) \rfloor = \langle \lfloor \{ba, ab, b\} \rfloor, \lfloor \{a\} \rfloor \rangle = \langle \{ba, ab, b\}, \{a\} \rangle \\ \vec{Y}^{(3)} &= \lfloor \vec{b} \rfloor \sqcup \lfloor \text{Fn}_{\mathcal{G}}(\vec{Y}^{(2)}) \rfloor = \langle \lfloor \{aba, aab, ab, baa, aba, ba, b\} \rfloor, \lfloor \{a\} \rfloor \rangle \\ &= \langle \{ba, ab, b\}, \{a\} \rangle \end{aligned}$$

The least fixpoint is therefore $\vec{Y} = \langle \{ba, ab, b\}, \{a\} \rangle$. Since $ab \in \vec{Y}_0$ but $ab \notin \mathcal{L}(N)$, Algorithm [CFGIncW](#) concludes that the inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(N)$ does not hold. $\diamond$

4.5.5 A Systematic Approach to the Antichain Algorithm

Consider a CFG $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ and an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ and let $\leq_{\mathcal{N}}$ be the $\mathcal{L}(\mathcal{N})$ -consistent wqo defined in (4.24). Theorem 4.5.3 shows that the algorithm **CFGIncW** solves the inclusion problem $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$ by working with finite languages.

Similarly to the case of the quasiorder $\leq_{\mathcal{N}}^{\ell}$ (Section 4.4) it suffices to keep the sets $\text{ctx}_{\mathcal{N}}(u)$ of pairs of states of Q for each word u instead of the words themselves. Therefore, we can systematically derive a “state-based” algorithm analogous to **CFGIncW** but working on the antichain poset $\langle \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}, \sqsubseteq \rangle$ viewed as an abstraction of $\langle \wp(\Sigma^*), \subseteq \rangle$. Let us define the abstraction and concretization maps $\alpha: \wp(\Sigma^*) \rightarrow \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}$ and $\gamma: \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle} \rightarrow \wp(\Sigma^*)$ and the abstract function $\text{Fn}_{\mathcal{G}}^{\mathcal{N}}(\langle X_i \rangle_{i \in [0, n]}): \wp(Q \times Q)^{|\mathcal{V}|} \rightarrow \wp(Q \times Q)^{|\mathcal{V}|}$ as follows:

$$\begin{aligned} \alpha(X) &\stackrel{\text{def}}{=} \lfloor \{ \text{ctx}_{\mathcal{N}}(u) \mid u \in X \} \rfloor \\ \gamma(Y) &\stackrel{\text{def}}{=} \{ u \in \Sigma^* \mid \exists y \in Y, y \subseteq \text{ctx}_{\mathcal{N}}(u) \} \\ \text{Fn}_{\mathcal{G}}^{\mathcal{N}}(\langle X_i \rangle_{i \in [0, n]}) &\stackrel{\text{def}}{=} \langle \lfloor \{ X_j \circ X_k \mid X_i \rightarrow X_j X_k \in P \} \rfloor \rangle_{i \in [0, n]} \end{aligned}$$

where $X \circ Y \stackrel{\text{def}}{=} \{ (q, q') \mid (q, q'') \in X \wedge (q'', q') \in Y \}$ is standard composition of two relations $X, Y \subseteq Q \times Q$.

LEMMA 4.5.13. *The following hold:*

- (a) $\langle \wp(\Sigma^*), \subseteq \rangle \xleftrightarrow[\alpha]{\gamma} \langle \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}, \sqsubseteq \rangle$ is a GC.
- (b) $\gamma \circ \alpha = \rho_{\leq_{\mathcal{N}}}$
- (c) $\text{Fn}_{\mathcal{G}}^{\mathcal{N}}(\vec{X}) = \alpha \circ \text{Fn}_{\mathcal{G}} \circ \gamma(\vec{X})$ for all $\vec{X} \in \alpha(\wp(\Sigma^*)^{|\mathcal{V}|})$

Proof.

- (a) Let us first observe that α and γ are well-defined. First, $\alpha(X)$ is an antichain of $\langle \wp(Q \times Q), \subseteq \rangle$ since it is a minor for the well-quasiorder $\subseteq$ and, therefore, it is finite. On the other hand, $\gamma(Y)$ is clearly an element of $\langle \wp(\Sigma^*), \subseteq \rangle$ by definition.

Then, for all $X \in \wp(\Sigma^*)$ and $Y \in \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}$, it turns out that:

$$\begin{aligned} \alpha(X) \sqsubseteq Y &\Leftrightarrow \text{[By definition of } \sqsubseteq \text{]} \\ \forall z \in \alpha(X), \exists y \in Y, y \subseteq z &\Leftrightarrow \text{[By definition of } \alpha \text{ and } \lfloor \cdot \rfloor \text{]} \\ \forall v \in X, \exists y \in Y, y \subseteq \text{ctx}_{\mathcal{N}}(v) &\Leftrightarrow \text{[By definition of } \gamma \text{]} \\ \forall v \in X, x \in \gamma(Y) &\Leftrightarrow \text{[By definition of } \sqsubseteq \text{]} \\ X &\subseteq \gamma(Y) . \end{aligned}$$

(b) For all $X \in \wp(\Sigma^*)$ we have that:

$$\begin{aligned}
 \gamma(\alpha(X)) &= \text{[By definition of } \alpha, \gamma\text{]} \\
 \{v \in \Sigma^* \mid \exists u \in \Sigma^*, \text{ctx}_N(u) \in \lfloor \{\text{ctx}_N(w) \mid w \in X\} \rfloor \wedge \text{ctx}_N(u) \subseteq \text{ctx}_N(v)\} \\
 &= \text{[By definition of minor]} \\
 \{v \in \Sigma^* \mid \exists u \in X, \text{ctx}_N(u) \subseteq \text{ctx}_N(v)\} &= \text{[By definition of } \leq_N\text{]} \\
 \{v \in \Sigma^* \mid \exists u \in X, u \leq_N v\} &= \text{[By definition of } \rho_{\leq_N}\text{]} \\
 \rho_{\leq_N}(X) &.
 \end{aligned}$$

(c) First, we show that $\text{ctx}_N(uv) = \text{ctx}_N(u) \circ \text{ctx}_N(v)$ for every pair of words $u, v \in \Sigma^*$.

$$\begin{aligned}
 \text{ctx}_N(uv) &= \text{[By def. of } \text{ctx}_N\text{]} \\
 \{(q, q') \in Q^2 \mid q \xrightarrow{uv} q'\} &= \\
 \text{[Since } q \xrightarrow{uv} q' \Leftrightarrow \exists q'' \in Q, q \xrightarrow{u} q'' \wedge q'' \xrightarrow{v} q'\text{]} \\
 \{(q, q') \in Q^2 \mid \exists q'' \in Q, q \xrightarrow{u} q'' \wedge q'' \xrightarrow{v} q'\} &= \\
 \text{[By definition of } \circ \text{ for binary relations]} \\
 \{(q, q'') \in Q^2 \mid q \xrightarrow{u} q''\} \circ \{(q'', q') \in Q^2 \mid q'' \xrightarrow{v} q'\} &= \\
 \text{[By definition of } W_{q, q'} \text{ and } \text{ctx}_N\text{]} \\
 \text{ctx}_N(u) \circ \text{ctx}_N(v) &
 \end{aligned}$$

Secondly, we show that $\lfloor X \circ Y \rfloor = \lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor$ for every $X, Y \in \wp(Q \times Q)$. It is straightforward to check that $\lfloor X \rfloor \circ \lfloor Y \rfloor \subseteq X \circ Y$ and, therefore, $\lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor \subseteq \lfloor X \circ Y \rfloor$. Next, we prove the reverse inclusion by contradiction.

Let $x \circ y \in \lfloor X \circ Y \rfloor$ with $x \in X$ and $y \in Y$. Assume $x \circ y \notin \lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor$. Then, there exists $\tilde{x} \in \lfloor X \rfloor$ and $\tilde{y} \in \lfloor Y \rfloor$ such that $\tilde{x} \circ \tilde{y} \in \lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor$ and $\tilde{x} \circ \tilde{y} \subseteq x \circ y$ which contradicts the fact that $x \circ y \in \lfloor X \circ Y \rfloor$ unless $\tilde{x} \circ \tilde{y} = x \circ y$, in which case $x \circ y \in \lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor$. Therefore, $\lfloor X \circ Y \rfloor \subseteq \lfloor \lfloor X \rfloor \circ \lfloor Y \rfloor \rfloor$.

Finally, we show that $\alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}))) = \text{Fn}_{\mathcal{G}}^N(\vec{X})$ for all $\vec{X} \in \alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$.

$$\begin{aligned}
 \alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}))) &= \\
 \text{[By definition of } \text{Fn}_{\mathcal{G}}\text{]} \\
 \langle \alpha(\bigcup_{X_i \rightarrow X_j, X_k \in P} \gamma(\vec{X}_j) \gamma(\vec{X}_k)) \rangle_{i \in [0, n]} &=
 \end{aligned}$$

$$\begin{aligned}
 & \text{[By definition of } \alpha \text{]} \\
 & \langle \llbracket \{\text{ctx}_{\mathcal{N}}(w) \mid w \in \bigcup_{X_i \rightarrow X_j X_k \in P} \gamma(\vec{X}_j) \gamma(\vec{X}_k)\} \rrbracket \rangle_{i \in [0, n]} = \\
 & \langle \llbracket \{\text{ctx}_{\mathcal{N}}(w) \mid \exists X_i \rightarrow X_j X_k \in P, w \in \gamma(\vec{X}_j) \gamma(\vec{X}_k)\} \rrbracket \rangle_{i \in [0, n]} = \\
 & \text{[By definition of concatenation]} \\
 & \langle \llbracket \{\text{ctx}_{\mathcal{N}}(uv) \mid \exists X_i \rightarrow X_j X_k \in P, u \in \gamma(\vec{X}_j) \wedge v \in \gamma(\vec{X}_k)\} \rrbracket \rangle_{i \in [0, n]} = \\
 & \text{[Since } \text{ctx}_{\mathcal{N}}(uv) = \text{ctx}_{\mathcal{N}}(u) \circ \text{ctx}_{\mathcal{N}}(v) \text{]} \\
 & \langle \llbracket \{\text{ctx}_{\mathcal{N}}(u) \circ \text{ctx}_{\mathcal{N}}(v) \mid \exists X_i \rightarrow X_j X_k \in P, u \in \gamma(\vec{X}_j) \wedge v \in \gamma(\vec{X}_k)\} \rrbracket \rangle_{i \in [0, n]} \\
 & \text{[By definition of } X \circ Y \text{]} \\
 & \langle \llbracket \{\text{ctx}_{\mathcal{N}}(u) \mid u \in \gamma(\vec{X}_j), X_i \rightarrow X_j X_k\} \circ \{\text{ctx}_{\mathcal{N}}(v) \mid v \in \gamma(\vec{X}_k), X_i \rightarrow X_j X_k\} \rrbracket \rangle_{i \in [0, n]} \\
 & \text{[Since } \llbracket X \circ Y \rrbracket = \llbracket \llbracket X \rrbracket \circ \llbracket Y \rrbracket \rrbracket \text{]} \\
 & \langle \llbracket \llbracket \{\text{ctx}_{\mathcal{N}}(u) \mid u \in \gamma(\vec{X}_j), X_i \rightarrow X_j X_k\} \rrbracket \circ \llbracket \{\text{ctx}_{\mathcal{N}}(v) \mid v \in \gamma(\vec{X}_k), X_i \rightarrow X_j X_k\} \rrbracket \rrbracket \rangle_{i \in [0, n]} = \\
 & \text{[Since } \alpha(\gamma(X)) = \llbracket X \rrbracket \text{]} \\
 & \langle \llbracket \llbracket \{\vec{X}_j \mid X_i \rightarrow X_j X_k\} \rrbracket \circ \llbracket \{\vec{X}_k \mid X_i \rightarrow X_j X_k\} \rrbracket \rrbracket \rangle_{i \in [0, n]} \\
 & \text{[Since } \llbracket X \circ Y \rrbracket = \llbracket \llbracket X \rrbracket \circ \llbracket Y \rrbracket \rrbracket \text{]} \\
 & \langle \llbracket \{\vec{X}_j \mid X_i \rightarrow X_j X_k\} \circ \{\vec{X}_k \mid X_i \rightarrow X_j X_k\} \rrbracket \rangle_{i \in [0, n]} \\
 & \text{[By definition of } \circ \text{]} \\
 & \langle \llbracket \{\vec{X}_j \circ \vec{X}_k \mid X_i \rightarrow X_j X_k\} \rrbracket \rangle_{i \in [0, n]} = \\
 & \text{[By definition of } \text{Fn}_{\mathcal{G}}^{\mathcal{N}} \text{]} \\
 & \text{Fn}_{\mathcal{G}}^{\mathcal{N}}(\vec{X}) .
 \end{aligned}$$

CFGIncS: State-based algorithm for $L(\mathcal{G}) \subseteq L(\mathcal{N})$

Data: CFG $\mathcal{G} = \langle \mathcal{V}, \Sigma, P \rangle$ and NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$

- 1 $\langle Y_i \rangle_{i \in [0, n]} := \text{KLEENE}(\lambda \vec{X}. \llbracket \vec{b} \rrbracket \sqcup \text{Fn}_{\mathcal{G}}^{\mathcal{N}}(\vec{X}), \vec{\emptyset})$;
 - 2 **forall** $y \in Y_0$ **do**
 - 3 **if** $y \cap (I \times F) = \emptyset$ **then return false**;
 - 4 **return true**;
-

THEOREM 4.5.14. *Let $\mathcal{G}$ be a CFG and $\mathcal{N}$ be an NFA. The algorithm **CFGIncS** decides $L(\mathcal{G}) \subseteq L(\mathcal{N})$.*

Proof. We show that all the hypotheses (i)-(v) of Theorem 4.5.6 are satisfied for the abstract domain $\langle D, \leq_D \rangle = \langle \text{AC}_{\langle \emptyset(Q \times Q), \subseteq \rangle}, \sqsubseteq \rangle$ as defined by the GC of Lemma 4.5.13 (a).

- (i) Since, by Lemma 4.5.13 (b), we have that $\rho_{\leq_N}(X) = \gamma(\alpha(X))$, it follows from Lemmas 4.5.7 (a) and 4.5.11 that $\gamma(\alpha(L_2)) = L_2$. Moreover, for every $a \in \Sigma$ and $X \in \wp(\Sigma^*)$ we have $\gamma\alpha(aX) = \gamma\alpha(a\gamma\alpha(X))$:

$$\begin{aligned}\gamma\alpha(aX) &= [\text{In GCs } \gamma = \gamma\alpha\gamma] \\ \gamma\alpha\gamma\alpha(aX) &= [\text{By Lemma 4.5.7 (b) with } \rho_{\leq_N} = \gamma\alpha] \\ \gamma\alpha\gamma\alpha(a\gamma\alpha(X)) &= [\text{In GCs } \gamma = \gamma\alpha\gamma] \\ \gamma\alpha(a\gamma\alpha(X)) &.\end{aligned}$$

- (ii) $(\text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}, \sqsubseteq)$ is effective because Q is finite.

- (iii) By Lemma 4.5.13 (c) we have that $\alpha(\text{Fn}_{\mathcal{G}}(\gamma(\vec{X}))) = \text{Fn}_{\mathcal{G}}^N(\vec{X})$ for all vectors $\vec{X} \in \alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$.

- (iv) $\alpha(\{b\}) = \{(q, q') \mid q \xrightarrow{b} q'\}$ and $\alpha(\emptyset) = \emptyset$, hence $\lfloor \alpha(\vec{b}) \rfloor$ is trivial to compute.

- (v) Since $\alpha(\vec{L}_2^{X_0}) = \langle \alpha(\psi_{\Sigma^*}^{L_2}(i = 0)) \rangle_{i \in [0, n]}$, for all $\vec{Y} \in \alpha(\wp(\Sigma^*))^{|\mathcal{V}|}$ the relation $\vec{Y} \sqsubseteq \alpha(\vec{L}_2^{X_0})$ trivially holds for all components Y_i with $i \neq 0$. For Y_0 , it suffices to show that $Y_0 \sqsubseteq \alpha(L_2) \Leftrightarrow \forall S \in Y_q, S \cap (I \times F) \neq \emptyset$, which is the check performed by lines 2-5 of algorithm CFGIncS.

$$\begin{aligned}Y_0 \sqsubseteq \alpha(L_2) &\Leftrightarrow [\text{Since } Y_0 = \alpha(U) \text{ for some } U \in \wp(\Sigma^*)] \\ \alpha(U) \sqsubseteq \alpha(L_2) &\Leftrightarrow [\text{By GC}] \\ U \subseteq \gamma(\alpha(L_2)) &\Leftrightarrow \\ &[\text{By Lemmas 4.5.7, 4.5.11 and 4.5.13, } \gamma(\alpha(L_2)) = L_2] \\ U \subseteq L_2 &\Leftrightarrow [\text{Since } Y_0 = \alpha(U) = \lfloor \{\text{ctx}_{\mathcal{N}}(u) \mid u \in U\} \rfloor] \\ \forall u \in U, \text{ctx}_{\mathcal{N}}(u) \cap (I \times F) \neq \emptyset &\Leftrightarrow [\text{By definition of } \text{ctx}_{\mathcal{N}}(u)] \\ \forall S \in Y_0, S \cap I \neq \emptyset &.\end{aligned}$$

Thus, by Theorem 4.5.6, Algorithm CFGIncS decides $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$.

The resulting algorithm CFGIncS shares some features with two previous works. On the one hand, it is related to the work of Hofmann and Chen [2014] which defines an abstract interpretation-based language inclusion decision procedure similar to ours.

Even though Hofmann and Chen's algorithm and ours both manipulate sets of pairs of states of an automaton, their abstraction is based on equivalence relations and not quasiorders. Since quasiorders are strictly more general than equivalences our framework can be instantiated to a larger class of abstractions, most importantly coarser ones. Finally, it is worth pointing out that the approach of

[Hofmann and Chen \[2014\]](#) aims at including languages of finite and also infinite words.

A second related work is that of [Holík and Meyer \[2015\]](#) who define an anti-chain like algorithm manipulating sets of pairs of states. [Holík and Meyer \[2015\]](#) tackle the language inclusion problem $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$, where $\mathcal{G}$ is a grammar and $\mathcal{N}$ and automaton, by rephrasing the problem as a data flow analysis problem over a relational domain. In this scenario, the solution of the problem requires the computation of a least fixpoint on the relational domain, followed by an inclusion check between sets of relations. Then, they use the “antichains principle” to improve the performance of the fixpoint computation and, finally, they move from manipulating relations to manipulating pairs of states. As a consequence, [Holík and Meyer \[2015\]](#) obtain an antichains algorithm for checking the inclusion $\mathcal{L}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{N})$.

By contrast, our approach is direct and systematic, since we derive [CFGIncS](#) starting from the well-known Myhill quasiorder. We believe our approach evidences the relationship between the original antichains algorithm of [Wulf et al. \[2006\]](#) for regular languages and the one of [Holík and Meyer \[2015\]](#) for context-free languages, which is the relation between Algorithms [FAIncS](#) and [CFGIncS](#). Specifically, we show that these two algorithms are conceptually identical and differ in the quasiorder used to define the abstraction in which the computation takes place.

4.6 An Equivalent Greatest Fixpoint Algorithm

Let us recall from [Cousot \[2000, Theorem 4\]](#) that if $g: C \rightarrow C$ is a monotone function on a complete lattice $\langle C, \leq, \vee, \wedge \rangle$ which admits its unique *right-adjoint* $\tilde{g}: C \rightarrow C$, i.e. $\forall c, c' \in C, g(c) \leq c' \Leftrightarrow c \leq \tilde{g}(c')$ holds, then the following equivalence holds for all $c, c' \in C$

$$\text{lfp}(\lambda x. c \vee g(x)) \leq c' \Leftrightarrow c \leq \text{gfp}(\lambda y. c' \wedge \tilde{g}(y)) . \quad (4.25)$$

This property has been used by [Cousot \[2000\]](#) to derive equivalent least/greatest fixpoint-based invariance proof methods for programs.

In the following, we use Equivalence (4.25) to derive an algorithm for deciding the language inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$, which relies on the computation of a greatest fixpoint rather than a least fixpoint. This can be achieved by exploiting the following simple observation, which provides an adjunction between concatenation and quotients of sets of words.

LEMMA 4.6.1. *For all $X, Y \in \wp(\Sigma^*)$ and $w \in \Sigma^*$, $wY \subseteq Z \Leftrightarrow Y \subseteq w^{-1}Z$ and $Yw \subseteq Z \Leftrightarrow Y \subseteq Zw^{-1}$.*

Proof. By definition, for all $u \in \Sigma^*$, $u \in w^{-1}Z$ iff $wu \in Z$. Hence,

$$Y \subseteq w^{-1}Z \Leftrightarrow \forall u \in Y, wu \in Z \Leftrightarrow wY \subseteq Z .$$

Symmetrically, $Yw \subseteq Z \Leftrightarrow Y \subseteq Zw^{-1}$ holds.

Given an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$, we define $\widetilde{\text{Pre}}_{\mathcal{N}} : \wp(\Sigma^*)^{|Q|} \rightarrow \wp(\Sigma^*)^{|Q|}$ as a function on Q -indexed vectors of sets of words as follows:

$$\widetilde{\text{Pre}}_{\mathcal{N}}(\langle X_q \rangle_{q \in Q}) \stackrel{\text{def}}{=} \langle \bigcap_{a \in \Sigma, q' \in \delta(q, a)} a^{-1}X_{q'} \rangle_{q \in Q} ,$$

where, as usual, $\bigcap \emptyset = \Sigma^*$. It turns out that $\widetilde{\text{Pre}}_{\mathcal{N}}$ is the usual weakest liberal precondition which is right-adjoint of $\text{Pre}_{\mathcal{N}}$.

LEMMA 4.6.2. For all $\vec{X}, \vec{Y} \in \wp(\Sigma^*)^{|Q|}$, $\text{Pre}_{\mathcal{N}}(\vec{X}) \subseteq \vec{Y} \Leftrightarrow \vec{X} \subseteq \widetilde{\text{Pre}}_{\mathcal{N}}(\vec{Y})$.

Proof. For all $\vec{X}, \vec{Y} \in \wp(\Sigma^*)^{|Q|}$,

$$\begin{aligned} \text{Pre}_{\mathcal{N}}(\langle X_q \rangle_{q \in Q}) &\subseteq \langle Y_q \rangle_{q \in Q} \Leftrightarrow && [\text{By definition of } \text{Pre}_{\mathcal{N}}] \\ \forall q \in Q, \bigcup_{q \xrightarrow{a} q'} aX_{q'} &\subseteq Y_q \Leftrightarrow \\ \forall q, q' \in Q, q \xrightarrow{a} q' &\Rightarrow aX_{q'} \subseteq Y_q \Leftrightarrow && [\text{By Lemma 4.6.1}] \\ \forall q, q' \in Q, q \xrightarrow{a} q' &\Rightarrow X_{q'} \subseteq a^{-1}Y_q \Leftrightarrow && [(\forall i \in I, X \subseteq Y_i) \Leftrightarrow X \subseteq \bigcap_{i \in I} Y_i] \\ \forall q' \in Q, X_{q'} &\subseteq \bigcap_{q \xrightarrow{a} q'} a^{-1}Y_q \Leftrightarrow && [\text{By definition of } \widetilde{\text{Pre}}_{\mathcal{N}}] \\ \langle X_q \rangle_{q \in Q} &\subseteq \widetilde{\text{Pre}}_{\mathcal{N}}(\langle Y_q \rangle_{q \in Q}) \end{aligned}$$

Hence, from Equivalences (4.6) and (4.25) we obtain:

$$\mathcal{L}(\mathcal{N}_1) \subseteq L_2 \Leftrightarrow \vec{\epsilon}^{F_1} \subseteq \text{gfp}(\lambda \vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{\mathcal{N}_1}(\vec{X})) . \quad (4.26)$$

The following algorithm **FAINC_{GFP}** decides the inclusion $\mathcal{L}(\mathcal{N}_1) \subseteq L_2$ by implementing the greatest fixpoint computation from Equivalence (4.26). The intuition behind algorithm **FAINC_{GFP}** is that

$$\mathcal{L}(\mathcal{N}_1) \subseteq L_2 \Leftrightarrow \epsilon \in \bigcap_{w \in \mathcal{L}(\mathcal{N}_1)} w^{-1}L_2 .$$

Therefore, **FAINC_{GFP}** computes the set $\bigcap_{w \in \mathcal{L}(\mathcal{N}_1)} w^{-1}L_2$ by using the automaton $\mathcal{N}_1$ and by considering prefixes of $\mathcal{L}(\mathcal{N}_1)$ of increasing lengths. This means that

FAIncGFP: Greatest fixpoint algorithm for $\mathcal{L}(\mathcal{N}_1) \subseteq L_2$

Data: NFA $\mathcal{N}_1 = \langle Q_1, \delta_1, I_1, F_1, \Sigma \rangle$; regular language L_2 .

```

1  $\langle Y_q \rangle_{q \in Q} := \text{KLEENE}(\lambda \vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{\mathcal{N}_1}(\vec{X}), \vec{\Sigma}^*)$ ;
2 forall  $q \in F_1$  do
3   if  $\epsilon \notin Y_q$  then return false;
4 return true;
```

after n iterations of the KLEENE procedure, Algorithm **FAIncGFP** has computed, for every state $q \in Q_1$, the set

$$\bigcap_{wu \in \mathcal{L}(\mathcal{N}_1), |w| \leq n, q_0 \in I_1, q_0 \xrightarrow{w} q} w^{-1} L_2,$$

The regularity of L_2 together with the property of regular languages of being closed under intersections and quotients show that each iterate computed by $\text{KLEENE}(\lambda \vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{\mathcal{N}_1}(\vec{X}), \vec{\Sigma}^*)$ is a (computable) regular language. To the best of our knowledge, this language inclusion algorithm **FAIncGFP** has never been described in the literature before.

Next, we discharge the fundamental assumption on which the correctness of this algorithm **FAIncGFP** depends on: the Kleene iterates computed by **FAIncGFP** are finitely many. In order to do that, we consider an abstract version of the greatest fixpoint computation exploiting a closure operator which guarantees that the abstract Kleene iterates are finitely many. This closure operator $\rho_{\leq \mathcal{N}_2}$ will be defined by using an ordering relation $\leq_{\mathcal{N}_2}$ induced by an NFA $\mathcal{N}_2$ such that $L_2 = \mathcal{L}(\mathcal{N}_2)$ and will be shown to be *forward complete* for the function $\lambda \vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{\mathcal{N}_1}(\vec{X})$ used by **FAIncGFP**.

Forward completeness of abstract interpretations [Giacobazzi and Quintarelli 2001], also called exactness [Miné 2017, Definition 2.15], is different from and orthogonal to backward completeness introduced in Section 4.1 and crucially used in Sections 4.2-4.5. In particular, a remarkable consequence of exploiting a forward complete abstraction is that the Kleene iterates of the concrete and abstract greatest fixpoint computations coincide.

The intuition here is that this forward complete closure $\rho_{\leq \mathcal{N}_2}$ allows us to establish that all Kleene iterates of $\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{\mathcal{N}_1}(\vec{X}))$ belong to the image of the closure $\rho_{\leq \mathcal{N}_2}$. More precisely, every Kleene iterate is a language which is upward closed for $\leq_{\mathcal{N}_2}$. Interestingly, a similar phenomenon occurs in well-structured transition systems [Abdulla et al. 1996; Finkel and Schnoebelen 2001].

Let us now describe in detail this abstraction. A closure $\rho \in \text{uco}(C)$ on a

concrete domain C is forward complete for a monotone function $f : C \rightarrow C$ iff $\rho f \rho = f \rho$ holds. The intuition here is that forward completeness means that no loss of precision is accumulated when the output of a computation of $f \rho$ is approximated by ρ , or, equivalently, f maps abstract elements of ρ into abstract elements of ρ . Dually to the case of backward completeness, forward completeness implies that $\text{gfp}(f) = \text{gfp}(f \rho) = \text{gfp}(\rho f \rho)$, when these greatest fixpoints exist (this is the case, e.g., when C is a complete lattice).

It turns out that forward and backward completeness are related by the following duality on function f .

LEMMA 4.6.3 ([Giacobazzi and Quintarelli 2001, Corollary 1]). *Let $\langle C, \leq \rangle$ be a complete lattice and assume that $f : C \rightarrow C$ admits the right-adjoint $\tilde{f} : C \rightarrow C$, i.e. $f(c) \leq c' \Leftrightarrow c \leq \tilde{f}(c')$ holds. Then, ρ is backward complete for f iff ρ is forward complete for $\tilde{f}$.*

Thus, by Lemma 4.6.3, in the following result instead of assuming the hypotheses implying that a closure ρ is forward complete for the right-adjoint Pre_{N_1} we state some hypotheses which guarantee that ρ is backward complete for its left-adjoint, which, by Lemma 4.6.2, is Pre_{N_1} .

THEOREM 4.6.4. *Let $N_1 = \langle Q_1, \delta_1, I_1, F_1, \Sigma \rangle$ be an NFA, let L_2 be a regular language and let $\rho \in \text{uco}(\langle \wp(\Sigma^*), \subseteq \rangle)$. Let us assume that:*

- (a) $\rho(L_2) = L_2$;
- (b) ρ is backward complete for $\lambda X. aX$ for all $a \in \Sigma$.

Then

$$\mathcal{L}(N_1) \subseteq L_2 \Leftrightarrow \vec{\epsilon}^{F_1} \subseteq \text{gfp}(\vec{X}. \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\rho(\vec{X})))) .$$

Moreover, the Kleene iterates computed by $\text{gfp}(\vec{X}. \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\rho(\vec{X}))))$ coincide in lockstep with those of $\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\vec{X}))$.

Proof. Theorem 4.2.3 shows that if ρ is backward complete for $\lambda X. aX$ for every $a \in \Sigma$ then it is backward complete for Pre_{N_1} . Thus, by Lemma 4.6.3, ρ is forward complete for $\widetilde{\text{Pre}_{N_1}}$, hence it is forward complete for $\lambda \vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\vec{X})$ since:

$$\begin{aligned} \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\rho(\vec{X}))) &= \text{[By forward comp. for } \widetilde{\text{Pre}_{N_1}} \text{ and } \rho(L_2) = L_2\text{]} \\ \rho(\rho(\vec{L}_2^{I_1}) \cap \rho(\widetilde{\text{Pre}_{N_1}}(\rho(\vec{X})))) &= \text{[Since } \rho(\cap \rho(X)) = \cap \rho(X)\text{]} \\ \rho(\vec{L}_2^{I_1}) \cap \rho(\widetilde{\text{Pre}_{N_1}}(\rho(\vec{X}))) &= \text{[By forward comp. for } \widetilde{\text{Pre}_{N_1}} \text{ and } \rho(L_2) = L_2\text{]} \\ \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}_{N_1}}(\rho(\vec{X})) & . \end{aligned}$$

Since, by forward completeness, we have that

$$\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\vec{X})) = \text{gfp}(\vec{X}. \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho(\vec{X})))) ,$$

by Equivalence (4.26), we conclude that

$$\mathcal{L}(N_1) \subseteq L_2 \Leftrightarrow \vec{\epsilon}^{F_1} \subseteq \text{gfp}(\vec{X}. \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho(\vec{X})))) .$$

Finally, we observe that the Kleene iterates computing $\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\vec{X}))$ and those computing $\text{gfp}(\vec{X}. \rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho(\vec{X}))))$ coincide in lockstep since $\rho(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho(\vec{X}))) = \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho(\vec{X}))$ and $\rho(\vec{L}_2^{I_1}) = \vec{L}_2^{I_1}$.

We can now establish that the sequence of Kleene iterates computed by $\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\vec{X}))$ is finite. Let $L_2 = \mathcal{L}(N_2)$, for some NFA N_2 , and consider the corresponding left state-based quasiorder $\leq_{N_2}^l$ on Σ^* as defined by (4.12).

Lemma 4.3.7 tells us that $\leq_{N_2}^l$ is a left L_2 -consistent wqo. Furthermore, since Q_2 is finite we have that both $\leq_{N_2}^l$ and $(\leq_{N_2}^l)^{-1}$ are wqos, so that, in turn, $\langle \rho_{\leq_{N_2}^l}, \subseteq \rangle$ is a poset which is both ACC and DCC. In particular, the definition of $\leq_{N_2}^l$ implies that every chain in $\langle \rho_{\leq_{N_2}^l}, \subseteq \rangle$ has at most $2^{|Q_2|}$ elements, so that if we compute $2^{|Q_2|}$ Kleene iterates then we have surely computed the greatest fixpoint.

Moreover, as a consequence of the DCC we have that the Kleene iterates of $\text{gfp}(\vec{X}. \rho_{\leq_{N_2}}(\vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\rho_{\leq_{N_2}}(\vec{X}))))$ are finitely many, hence so are the iterates of $\text{gfp}(\vec{X}. \vec{L}_2^{I_1} \cap \widetilde{\text{Pre}}_{N_1}(\vec{X}))$ because they go in lockstep as stated by Theorem 4.6.4.

COROLLARY 4.6.5. *Let N_1 be an NFA and let L_2 be a regular language. Then, the algorithm **FAIncGFP** decides the inclusion $\mathcal{L}(N_1) \subseteq L_2$*

Finally, it is worth citing that Fiedor et al. [2015] put forward an algorithm for deciding WS1S formula which relies on the same lfp computation used in **FAIncS**. Then, they derive a dual gfp computation by relying on Park's duality [Park 1969]: $\text{lfp}(\lambda X. f(X)) = (\text{gfp}(\lambda X. (f(X^c))^c))^c$. Their approach differs from ours since we use the Equivalence (4.25) to compute a gfp, different from the lfp, which still allows us to decide the inclusion problem. Furthermore, their algorithm decides whether a given automaton accepts ϵ and it is not clear how their algorithm could be extended for deciding language inclusion.

SEARCHING ON COMPRESSED TEXT

In this chapter, we show how to instantiate the quasiorder-based framework presented in Chapter 4 to search on compressed text. Specifically, we adapt Algorithm CFGIncS to report the number of lines in a grammar-compressed text containing a match for a given regular expression.

The problem of searching in compressed text is of practical interest due the growing amount of information handled by modern systems, which demands efficient techniques both for compression, to reduce the storage cost, and for regular expression searching, to speed up querying. As an evidence of the importance of this problem, note that state of the art tools for searching with regular expressions, such as `grep` and `ripgrep`, provide a method to search on compressed files by decompressing them on-the-fly.

In the following, we focus on the problem of *counting*, i.e. finding the number of lines of the input text that contain a match for the expression. This type of query is supported out of the box by many tools¹, which evidences its practical interest. However, when the text is given in compressed form, the fastest approach in practice is the *decompress and search approach*, i.e. querying the uncompressed text as it is recovered by the decompressor. In this chapter, we challenge this approach.

Lossless compression of textual data is achieved by finding repetitions in the input text and replacing them by references. We focus on grammar-based compression schemes in which each tuple “reference $\rightarrow$ repeated text” is considered as a rule of a context-free grammar. The resulting grammar, produced as the

¹Tools such as `grep`, `ripgrep`, `awk` and `ag`, among others, can be used to report the number of matching lines in a text.

output of the compression, generates a language consisting of a single word: the uncompressed text. Figure 5.1 depicts the output of a grammar-based compression algorithm.

Figure 5.1: List of grammar rules (left) generating the string “ab\$a\$bab\$a\$b” (and no other) as evidenced by the parse tree (right).

Intuitively, the decompress and search approach prevents the searching algorithm from taking advantage of the repetitions in the data found by the compressor. For instance, in the grammar shown in Figure 5.1, the decompress and search approach results in processing the subsequence “ab\$a\$b” twice. By working on the compressed data, our algorithm would process that subsequence once and reuse the information each time it finds the variable X_5 .

Given a grammar-compressed text and a regular expression, deciding whether the compressed text matches the expression amounts to deciding the emptiness of the intersection of the languages generated by the grammar and an automaton built for the regular expression.

In order to solve this emptiness problem, we reduce it to an inclusion problem. Note that this reduction is possible since the grammar generates a single word and, therefore, $\{w\} \cap L \neq \emptyset \Leftrightarrow w \in L$, where L is the language generated by the regular expression. Then, we could instantiate the quasiorder-based framework described in Chapter 4 with different quasiorders to decide the inclusion.

However, in order to go beyond a yes/no answer and report or count the exact matches, we need to compute some extra information for each variable of the grammar. This extra information is computed for the terminals of the grammar and then propagated through the variables according to the grammar rules in a bottom-up fashion. To do that, we iterate thorough the grammar rules and compose, for each of them, the information previously computed for the variables on the right hand side. For example, when processing rule $X_3 \rightarrow X_1\$$ of Figure 5.1 our algorithm composes the information for X_1 with the one for $\$$. The information computed for the string “ab\$”, will be reused every time the variable X_3 appears in the right hand side of a rule.

Following this idea, we present an algorithm for counting the lines in a grammar-compressed text containing a match for a regular expression whose runtime does not depend on the size T of the uncompressed text. Instead, it runs in time *linear* in the size of its *compressed version*. Furthermore, the information computed for counting can be used to perform an *on-the-fly, lazy* decompression to recover the matching lines from the compressed text. Note that, for reporting the matching lines, the dependency on T is unavoidable.

The salient features of our approach are:

Generality

Our algorithm is not tied to any particular grammar-based compressor. Instead, we consider the compressed text is given by a straight line program (SLP for short), i.e. a grammar generating the uncompressed text and nothing else.

Finding the smallest SLP g generating a text of length T is an NP-hard problem, as shown by Charikar et al. [2005], for which grammar-based compressors such as LZ78 [Ziv and Lempel 1978], LZW [Welch 1984], RePair [Larsson and Moffat 1999] and Sequitur [Nevill-Manning and Witten 1997] produce different approximations. For instance, Hucke et al. [2016] showed that the LZ78 algorithm produces a representation of size $\Omega(|g| \cdot (T/\log T)^{2/3})$ and the representation produced by the RePair algorithm has size $\Omega(|g| \cdot \log T / \log(\log T))$.

Since it is defined over SLPs, our algorithm applies to all such approximations, including g itself.

Nearly optimal data structures

We define data structures enabling the algorithm to run in time linear in the size of the compressed text. With these data structures our algorithm runs in $O(t \cdot s^3)$ time using $O(t \cdot s^2)$ space where t is the size of the compressed text, i.e. the grammar, and s is the size of the automaton built from the regular expression. When the automaton is deterministic, the complexity drops to $O(t \cdot s)$ time and $O(t \cdot s)$ space.

As shown by Abboud et al. [2017], there is no combinatorial² algorithm improving these time complexity bounds beyond *polylog* factors, hence our algorithm is *nearly optimal*.

²Interpreted as any *practically efficient* algorithm that does not suffer from the issues of Fast Matrix Multiplication such as large constants and inefficient memory usage.

Efficient implementation

We present **zsearch**, a purely *sequential* implementation of our algorithm which uses the above mentioned data structures.³

The experiments show that **zsearch** requires up to 25% less time than the state of the art: running **hyperscan** on the uncompressed text as it is recovered by **1z4** (in *parallel*). Furthermore, when the grammar-based compressor achieves high compression ratio (above 13:1), running **zsearch** on the compressed text is as fast as running **hyperscan** directly on the uncompressed text. Such compression ratios are achieved, for instance, when working with automatically generated log files.

5.1 Finding the Matches

Recall that the problem of deciding whether a grammar-compressed text contains a match for a regular expression can be reduced to an emptiness problem for the intersection of the languages generated by a grammar and an automaton. Indeed, given an SLP $\mathcal{P}$ generating a text T over an alphabet Σ , i.e. $\mathcal{L}(\mathcal{P}) = \{T\}$ where $T \in \Sigma^*$, and an automaton $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ representing a regular expression, we find that:

There exists a substring of T in $\mathcal{L}(\mathcal{N}) \Leftrightarrow \mathcal{L}(\mathcal{P}) \cap (\Sigma^* \cdot \mathcal{L}(\mathcal{N}) \cdot \Sigma^*) \neq \emptyset$.

On the other hand, since $\mathcal{L}(\mathcal{P})$ contains exactly one word we have that

$$\mathcal{L}(\mathcal{P}) \cap (\Sigma^* \cdot \mathcal{L}(\mathcal{N}) \cdot \Sigma^*) \neq \emptyset \Leftrightarrow \mathcal{L}(\mathcal{P}) \subseteq (\Sigma^* \cdot \mathcal{L}(\mathcal{N}) \cdot \Sigma^*) \text{ .}$$

As a consequence, the problem of deciding whether a grammar-compressed text contains a match for a regular expression can be solved by using Algorithm **CFGIncS** with the quasiorder $\leq_{\mathcal{N}}$ as described in Chapter 4.

Observe that, as the following example evidences, when restricting to SLPs the iteration of the **KLEENE** procedure updates the abstraction for each variable of the grammar *exactly once* since there are no loops in SLPs. As a consequence, it is enough to process the rules in an orderly manner and compute the abstraction for each variable, i.e. $\alpha(X)$, exactly once.

Example 5.1.1. Let $\mathcal{P}$ be the SLP from Figure 5.1 and let $\mathcal{N}$ and $\mathcal{N}'$ be the automata from Figure 5.2. Next, we show the Kleene iterates computed by Algorithm **CFGIncS** which, as shown in Chapter 4, works on the abstract domain $\langle \text{AC}_{\langle \emptyset(Q \times Q), \subseteq \rangle}, \sqsubseteq \rangle$ with the abstraction function defined as $\alpha(X) = \lfloor \{\text{ctx}_{\mathcal{N}}(u) \mid u \in X\} \rfloor$.

³**zsearch** can optionally report the matching lines.

Figure 5.2: NFAs $\mathcal{N}'$ (left) and $\mathcal{N}$ (right) on $\Sigma = \{a, b, \$\}$ with $\mathcal{L}(\mathcal{N}') = \{ab, bb\}$ and $\mathcal{L}(\mathcal{N}) = \Sigma^* \cdot \mathcal{L}(\mathcal{N}') \cdot \Sigma^*$.

To simplify the notation, we denote the pair (q_i, q_j) by ij .

$$\begin{pmatrix} \alpha(W_{X_6}^{\mathcal{P}}) \\ \alpha(W_{X_5}^{\mathcal{P}}) \\ \alpha(W_{X_4}^{\mathcal{P}}) \\ \alpha(W_{X_3}^{\mathcal{P}}) \\ \alpha(W_{X_2}^{\mathcal{P}}) \\ \alpha(W_{X_1}^{\mathcal{P}}) \end{pmatrix} = \begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \end{pmatrix} \Rightarrow \begin{pmatrix} \emptyset \\ \emptyset \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \end{pmatrix} \Rightarrow \begin{pmatrix} \emptyset \\ \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \end{pmatrix} \Rightarrow \begin{pmatrix} \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \\ \lfloor \{11, 33\} \rfloor \\ \lfloor \{11, 33, 13\} \rfloor \end{pmatrix}$$

Since for every variable X_n the value of $\alpha(X_n)$ is computed by combining the values of $\alpha(X_i)$ and $\alpha(X_j)$ for some $i, j < n$, the KLEENE procedure is equivalent to computing, sequentially, the values of $\alpha(X_1), \alpha(X_2), \dots, \alpha(X_5)$.

In this case, since $(q_1, q_3) \in \alpha(X_5)$ Algorithm [CFGIncS](#) concludes that the inclusion $\mathcal{L}(\mathcal{P}) \subseteq \mathcal{L}(\mathcal{N})$ holds, i.e. there exists a substring w of the uncompressed such that $w \in \mathcal{L}(\mathcal{N}')$. $\diamond$

Furthermore, in an SLP each variable generates exactly one word and, therefore, the abstraction of a variable consists of a single set, i.e. $\alpha(X) \in \text{AC}_{\langle \wp(Q \times Q), \subseteq \rangle}$ is a singleton as shown in Example 5.1.1. As a consequence, we can drop the $\lfloor \cdot \rfloor$ from function $\text{Fn}_{\mathcal{P}}^{\mathcal{N}}$ defined in Section 4.5.5, since $\lfloor \{\text{ctx}_{\mathcal{N}}(w)\} \rfloor = \{\text{ctx}_{\mathcal{N}}(w)\}$ for any word, and write:

$$\text{Fn}_{\mathcal{P}}^{\mathcal{N}}(\langle X_i \rangle_{i \in [0, n]}) \stackrel{\text{def}}{=} \langle \{X_j \circ X_k \mid X_i \rightarrow X_j X_k \in P\} \rangle_{i \in [0, n]}$$

Recall that, by definition, for all $X_j, X_k \in \wp(Q \times Q)^{|\mathcal{V}|}$,

$$X_j \circ X_k = \{(q_1, q_2) \mid \exists q' \in Q, (q_1, q') \in X_j \wedge (q', q_2) \in X_k\}.$$

Finally, given an NFA $\mathcal{N}'$ it is straightforward to build an automaton $\mathcal{N}$ generating the language $\Sigma^* \cdot \mathcal{L}(\mathcal{N}') \cdot \Sigma^*$ by adding self-loops reading each letter of the alphabet to every initial and every final state of $\mathcal{N}'$ as shown in Figure 5.2. Instead of adding these transitions to $\mathcal{N}$, which, as shown in Example 5.1.1, results in adding the pairs $\{(q, q) \mid q \in I \cup F\}$ to $\text{ctx}_{\mathcal{N}}(w)$ for every word $w \in \Sigma^*$, we consider them as implicit.

As a consequence, when the input grammar is an SLP and we are interested in deciding whether $\mathcal{L}(\mathcal{P}) \subseteq \Sigma^* \cdot \mathcal{L}(\mathcal{N}) \cdot \Sigma^*$, Algorithm **CFGInCS** can be written as Algorithm **SLPInCS**. Observe that Algorithm **SLPInCS** uses the transition function δ to store and manipulate the sets $\text{ctx}_{\mathcal{N}}(X_i)$ for each variable X_i of the grammar, i.e.

$$(q_1, X_i, q_2) \in \delta \Leftrightarrow (q_1, q_2) \in \text{ctx}_{\mathcal{N}}(X_i) .$$

SLPInCS: Algorithm for deciding $\mathcal{L}(\mathcal{P}) \subseteq \Sigma^* \cdot \mathcal{L}(\mathcal{N}) \cdot \Sigma^*$.

Data: An SLP $\mathcal{P} = \langle V, \Sigma, P \rangle$ and an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$.

```

1 Procedure MAIN
2   forall  $\ell = 1, 2, \dots, |V|-1$  do
3     let  $(X_\ell \rightarrow \alpha_\ell \beta_\ell) \in P$ ;
4     forall  $q_1, q' \in Q$  s.t.  $(q_1, \alpha_\ell, q') \in \delta$  or  $q_1 = q' \in I$  do
5       forall  $q_2 \in Q$  s.t.  $(q', \beta_\ell, q_2) \in \delta$  or  $q' = q_2 \in F$  do
6          $\delta := \delta \cup \{(q_1, X_\ell, q_2)\}$ ;
7   return  $((q_1, X_{|V|}, q_2) \in \delta \wedge q_1 \in I \wedge q_2 \in F \text{ ? } \text{true} : \text{false})$ ;
```

5.2 Counting Algorithm

State of the art tools for regular expression search are equipped with a number of features⁴ to perform different operations beyond deciding the existence of a match in the text. Among the most relevant of these features we find *counting*. Tools like `grep`⁵, `rg`⁶, `ack`⁷ or `ag`⁸ report the number of lines containing a match, ignoring matches across lines. Next we extend Algorithm **SLPInCS** to perform this sort of counting.

Let $\hookrightarrow$ denote the new-line delimiter and let $\widehat{\Sigma} = \Sigma \setminus \{\hookrightarrow\}$. Given a string $w \in \Sigma^+$ compressed as an SLP $\mathcal{P} = \langle V, \Sigma, P \rangle$ and an automaton $\mathcal{N} = \langle Q, \widehat{\Sigma}, \delta, I, F \rangle$ built from a regular expression, Algorithm **COUNTLINES** reports the number of lines in w containing a match for the expression. Note that, as the tools mentioned in the previous paragraph, we deliberately ignore matches across lines.

As an overview, our algorithm computes some *counting information* for each alphabet symbol of the grammar (procedure **INIT_AUTOMATON**) which is then propagated, in a bottom-up manner, to the axiom rule. Such propagation is

⁴<https://beyondgrep.com/feature-comparison/>

⁵<https://www.gnu.org/software/grep>

⁶<https://github.com/BurntSushi/ripgrep>

⁷<https://github.com/beyondgrep/ack2>

⁸<https://geoff.greer.fm/ag/>

achieved by iterating through the grammar rules (loop in line 14) and combining, for each rule, the information for the symbols on the right hand side to obtain the information for the variable on the left (procedure COUNT). Finally, the output of the algorithm is computed from the information propagated to the axiom symbol (line 22).

COUNTLINES: Algorithm for counting the lines in $\mathcal{L}(\mathcal{P})$ that contain a word in $\mathcal{L}(\mathcal{N})$.

Data: An SLP $\mathcal{P} = \langle V, \Sigma, P \rangle$ and an NFA $\mathcal{N} = \langle Q, \widehat{\Sigma}, \delta, I, F \rangle$.

```

1  Procedure COUNT( $X, \alpha, \beta, m$ )
2     $N_X := N_\alpha \vee N_\beta$ ;
3     $L_X := (\neg N_\alpha ? L_\alpha \vee L_\beta \vee m : L_\alpha)$ ;
4     $R_X := (\neg N_\beta ? R_\alpha \vee R_\beta \vee m : R_\beta)$ ;
5     $M_X := M_\alpha + M_\beta + (N_\alpha \wedge N_\beta \wedge (R_\alpha \vee L_\beta \vee m) ? 1 : 0)$ ;
6  Procedure INIT_AUTOMATON()
7    forall  $a \in \Sigma$  do
8       $N_a := (a = \prec \triangleright)$ ;
9       $M_a := 0$ ;
10      $L_a := ((q_0, a, q_f) \in \delta, q_0 \in I, q_f \in F)$ ;
11      $R_a := L_a$ ;
12  Procedure MAIN
13    INIT_AUTOMATON();
14    forall  $\ell = 1, 2, \dots, |V|$  do
15      let  $(X_\ell \rightarrow \alpha_\ell \beta_\ell) \in P$ ;
16      new_match := false;
17      forall  $q_1, q' \in Q$  s.t.  $(q_1, \alpha_\ell, q') \in \delta$  or  $q_1 = q' \in I$  do
18        forall  $q_2 \in Q$  s.t.  $(q', \beta_\ell, q_2) \in \delta$  or  $q' = q_2 \in F$  do
19           $\delta := \delta \cup \{(q_1, X_\ell, q_2)\}$ ;
20          new_match := new_match  $\vee (q_1 \in I \wedge q' \notin (I \cup F) \wedge q_2 \in F)$ ;
21      COUNT( $X_\ell, \alpha_\ell, \beta_\ell, \text{new\_match}$ );
22  return  $M_{X_{|V|}} + (N_{X_{|V|}} ? L_{X_{|V|}} + R_{X_{|V|}} : L_{X_{|V|}})$ ;

```

Define a *line* as a maximal factor of w each symbol of which belongs to $\widehat{\Sigma}$, a *closed line* as a line which is not a prefix nor a suffix of w and a *matching line* as a line in $\mathcal{L}(\mathcal{N})$, where $\mathcal{L}(\mathcal{N}) = \widehat{\Sigma}^* \cdot \mathcal{L}(\mathcal{N}) \cdot \widehat{\Sigma}^*$.

Example 5.2.1. Consider the word $w = "ab\prec\triangleright a\prec\triangleright bab\prec\triangleright"$ and an NFA $\mathcal{N}$ with $\mathcal{L}(\mathcal{N}) = \{ba\}$. Then the strings "ab", "a" and "bab" are lines of which only the strings "ab" and "a" are closed lines and "bab" is the only matching line. $\diamond$

Definition 5.2.2 (Counting Information). Let $\mathcal{N}$ be an NFA and let $\mathcal{P} = \langle \mathcal{V}, \Sigma, P \rangle$ be an SLP. The counting information of $\tau \in (V \cup \Sigma)$, with $\tau \Rightarrow^* u$ and $u \in \Sigma^+$, is the tuple $C_\tau \stackrel{\text{def}}{=} \langle N_\tau, L_\tau, R_\tau, M_\tau \rangle$ where

$$\begin{aligned} N_\tau &\stackrel{\text{def}}{=} \exists k; (u)_k = \leftarrow \quad & L_\tau &\stackrel{\text{def}}{=} \exists i, (u)_{1,i} \in \widehat{\Sigma}^* \cdot \mathcal{L}(\mathcal{N}) \\ R_\tau &\stackrel{\text{def}}{=} \exists j, (u)_{j,\dagger} \in \mathcal{L}(\mathcal{N}) \cdot \widehat{\Sigma}^* \quad & M_\tau &\stackrel{\text{def}}{=} |\{(i+1, j-1) \mid (u)_{i,j} \in \leftarrow \cdot \widehat{\mathcal{L}(\mathcal{N})} \cdot \leftarrow\}| \quad \blacksquare \end{aligned}$$

Note that N_τ , L_τ and R_τ are boolean values while M_τ is an integer. It follows from the definition that the number of *matching lines* in u , with $\tau \Rightarrow^* u$, is given by the number of *closed matching lines* (M_τ) plus the prefix of u iff it is a *matching line* (L_τ) and the suffix of u iff it is a *matching line* (R_τ) different from the prefix (N_τ). Since whenever $N_\tau = \text{false}$ we have $L_\tau = R_\tau$, it follows that

$$\# \text{matching lines in } u = M_\tau + \begin{cases} 1 & \text{if } L_\tau \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & \text{if } N_\tau \wedge R_\tau \\ 0 & \text{otherwise} \end{cases}$$

Computing the counting information of τ requires deciding membership of certain factors of u in $\widehat{\mathcal{L}(A)}$. As explained before, we reduce these membership queries to language inclusion checks which are solved by Algorithm [SLPINC5](#). This operation corresponds to lines [17](#) to [19](#) of Algorithm [COUNTLINES](#). As a result, after processing the rule for τ , we have $(q_1, \tau, q_2) \in \delta$ iff the automaton moves from q to q' reading (a) u , (b) a suffix of u and $q_1 \in I$, or (c) a prefix of u and $q_2 \in F$.

Procedures COUNT and INIT_AUTOMATON are quite straightforward, the main difficulty being the computation of M_X which we explain next. Let $x, y \in \Sigma^+$ be the strings generated by α and β , respectively. Given rule $X \rightarrow \alpha\beta$, X generates all the matching lines generated by α and β plus, possibly, a “new” matching line of the form $z = (x)_{i,\dagger}(y)_{1,j}$ with $1 < i \leq |x|$ and $1 \leq j < |y|$. Such an extra matching line appears iff both α and β generate a $\leftarrow$ symbol and one of the following holds:

- (a) The suffix of x matches the expression.
- (b) The prefix of y matches the expression.
- (c) There is a new match $m \in z$ with $m \notin x$, $m \notin y$ (line [20](#)).

Example 5.2.3. Let $\mathcal{N}$ be an automaton with $\mathcal{L}(\mathcal{N}) = \{ab, ba\}$ and let $X \rightarrow \alpha\beta$ be a grammar rule with $\alpha \Rightarrow^* ba\leftarrow a$ and $\beta \Rightarrow^* b\leftarrow aba$. Then $X \Rightarrow^* ba\leftarrow ab\leftarrow aba$.

The matching lines generated by α , β and X are, respectively, $\{ba\}$, $\{aba\}$ and $\{ba, ab, aba\}$. Furthermore

$$C_\alpha = \langle \text{true}, \text{true}, \text{false}, 0 \rangle \quad \text{and} \quad C_\beta = \langle \text{true}, \text{false}, \text{true}, 0 \rangle .$$

Therefore, applying function *COUNT* we find that $C_X = \langle \text{true}, \text{true}, \text{true}, 1 \rangle$. Therefore the number of matching lines is $1+1+1=3$, as expected. $\diamond$

Note that the counting information computed by Algorithm *COUNTLINES* can be used to uncompress *only* the matching lines by performing a top-down processing of the SLP. For instance, given $X \rightarrow \alpha\beta$ with $C_X = \langle \text{true}, \text{true}, \text{false}, 0 \rangle$ and $C_\alpha = \langle \text{true}, \text{true}, \text{false}, 0 \rangle$, there is no need to decompress the string generated by β since we are certain it is not part of any matching line (otherwise we should have $M_X > 0$ or $R_X = \text{true}$).

Next, we describe the data structures that we use to implement Algorithm *COUNTLINES* with *nearly optimal* complexity.

5.2.1 Data Structures

We assume the alphabet symbols, variables and states are indexed and use the following data structures, illustrated in Figure 5.3: an array $\mathcal{A}$ with $t+|\Sigma|$ elements, where t is the number of rules of the SLP, and two $s \times s$ matrices $\mathcal{M}$ and $\mathcal{K}$ where s is the number of states of the automaton.

Figure 5.3: Data structures enabling nearly optimal running time for Algorithm *COUNTLINES*. The image shows the contents of $\mathcal{M}$ after processing rule $X_i \rightarrow \alpha_i \beta_i$ and the contents of $\mathcal{K}$ after processing $X_\ell \rightarrow \alpha_\ell \beta_\ell$ with $\beta_\ell = X_i$.

Each element $\mathcal{A}[i]$ contains the information related to variable X_i , i.e. C_{X_i} and the list of transitions labeled with X_i , denoted E_{X_i} . We store C_X using one bit for each N_X , L_X and R_X and an integer for M_X .

For each rule $X_\ell \rightarrow \alpha_\ell \beta_\ell$ the matrix $\mathcal{K}$ is set so that row i contains the set of states reachable from the state q_i by reading the string generated by β_ℓ , i.e. $\mathcal{K}[i] = \{q_j \mid (q_i, \beta_\ell, q_j) \in \delta\}$. If there are less than s such states we use a sentinel value (-1 in Figure 5.3).

Finally, each element $\mathcal{M}[i][j]$ stores the index ℓ of the last variable for which (q_i, X_ℓ, q_j) was added to δ . Note that since rules are processed one at a time, matrices $\mathcal{K}$ and $\mathcal{M}$ can be reused for all rules.

Observe that it is straightforward to update the matrices $\mathcal{M}$ and $\mathcal{K}$ in $O(s^2)$ time for each rule $X_\ell \rightarrow \alpha_\ell \beta_\ell$ since there are up to s^2 transitions $(q_i, \beta_\ell, q_j) \in \delta$. These data structures provide $O(1)$ runtime for the following operations:

- Accessing the information corresponding to α_ℓ and β_ℓ at line 15 (using $\mathcal{A}$).
- Accessing the list of pairs (q, q') with $(q, \alpha_\ell, q') \in \delta$ at line 17 (using E_{X_i}).
- Accessing the list of states q_2 with $(q', \beta_\ell, q_2) \in \delta$ at line 18 (using $\mathcal{K}$).
- Inserting a pair (q, q') in E_{X_i} (avoiding duplicates) at line 19 (using $\mathcal{M}$).

As a result, Algorithm **COUNTLINES** runs in $O(t \cdot s^3)^9$ time using $O(t \cdot s^2)$ space when the automaton built from the regular expression is an NFA and it runs in $O(t \cdot s)$ time and $O(t \cdot s)$ space when the automaton is a DFA (each row of $\mathcal{K}$ stores up to one state, hence the loop in line 18 results in, at most, one iteration).

Abboud et al. [2017, Thm. 3.2] proved that, under the Strong Exponential Time Hypothesis, there is no combinatorial algorithm for deciding whether a grammar-compressed text contains a match for a DFA running in $O((t \cdot s)^{1-\varepsilon})$ time with $\varepsilon > 0$. For NFAs, they proved [Abboud et al. 2017, Thm. 4.2] that, under the k -Clique Conjecture, there is no combinatorial algorithm running in $O((t \cdot s^3)^{1-\varepsilon})$ time. Therefore, our algorithm is *nearly optimal* in both scenarios.

5.3 Implementation

We implemented Algorithm **COUNTLINES**, using the data structures described in the previous section, in a tool named `zearch`¹⁰. Our tool works on `repair`¹¹-compressed text and, beyond counting the matching lines, it can also report them by partially decompressing the input file. The implementation consists of less than 2000 lines of C code.

The choice of this particular compressor, which implements the RePair algorithm of Larsson and Moffat [1999], is due to the little effort required to adapt Algorithm **COUNTLINES** to the specifics of the grammar built by `repair` and the compression it achieves (see Table 5.1). However `zearch` can handle any grammar-based compression scheme by providing a way to recover the SLP from the input file.

Recall that we assume the alphabet symbols, variables and states are indexed. For `repair`-compressed text, the indexes of the alphabet symbols are $0 \dots 255$ (Σ is fixed¹²) and the indexes of the variables are $256 \dots t+256$. Typically, grammar-based compressors such as `repair` encode the grammar so that rule $X \rightarrow \alpha\beta$ appears always after the rules with α and β on the left hand side. Thus, each iteration of the loop in line 15 reads a subsequent rule from the compressed input.

⁹The algorithm performs t iterations of loop in line 15, up to s^2 iterations of loop in line 17 and up to s iterations for loop in line 18.

¹⁰<https://github.com/pevalme/zearch>

¹¹<https://storage.googleapis.com/google-code-archive-downloads/v2/code.google.com/repair/repair110811.tar.gz>

¹²Our algorithm also applies to larger alphabets, such as UTF8, without altering its complexity.

We translate the input regular expression into an ϵ -free NFA using the automata library `libfa`¹³ which applies Thompson’s algorithm [Thompson 1968] with on-the-fly ϵ -removal.

5.4 Empirical Evaluation

Next we present a *summary* of the experiments carried out to assess the performance of `zsearch`. The details of the experiments, including the runtime and number of matching lines reported for each expression on each file and considering more tools, file sizes and regular expressions are available on-line¹⁴, where we report graphs as the ones shown in Figure 5.4. The following explanations about how the experiments reported in this thesis were carried out also apply to the larger set of experiments available on-line.

All tools for regular expression searching considered in this benchmark are used to count the matching lines without reporting them. As expected, all tools report the exact same result for all benchmarks. To simplify the terminology, we refer to counting the matching lines as *searching*, unless otherwise stated.

5.4.1 Tools

Our benchmark compares the performance of `zsearch` against the fastest implementations we found for the following operations:

- (i) Searching the compressed text without decompression.
- (ii) Searching the uncompressed text.
- (iii) Decompressing the text without searching.
- (iv) Searching the uncompressed text as it is recovered by the decompressor.

For searching the compressed text we consider `GNgrep`, the tool developed by Navarro [2003] for searching on text compressed with the grammar-based compressor LZW defined by Welch [1984]. To the best of our knowledge, this is the only existing tool departing from the *decompress and search* approach.

For searching uncompressed text we consider `grep` and `hyperscan`. We improve the performance of `grep` by compiling it without *perl regular expression* compatibility, which is not supported by `zsearch`. We used the library `hyperscan` by means of the tool (provided with the library) `simplegrep`, which we modified¹⁵ to *efficiently* read data either from `stdin` or an input file. These tools are

¹³<http://augeas.net/libfa/index.html>

¹⁴<https://pevalme.github.io/zsearch/graphs/index.html>

¹⁵<https://gist.github.com/pevalme/f94bedc9ff08373a0301b8c795063093>

Regular Expressions

r1: ".", r2: "wosel", r3: "port", r4: "20[0-9]{2}", r5: "([0-9]{3}\.){3}[0-9]", r6: "[0-9]{4}",
r7: "([a-z]\.)+[a-z]+ - -", r8: "GET .*" ([13-9][21-9][2-1-9])",
r9: "([0-9])([0-2][0-9])([3][0-1])/(Jan|Feb|Mar|Apr|May|Jun|Jul|Aug|Sep|Oct|Nov|Dec)/[0-9]{4}",
r10: "([0-9])([0-2][0-9])([3][0-1])-(Jan|Feb|Mar|Apr|May|Jun|Jul|Aug|Sep|Oct|Nov|Dec)-[0-9]{4}",

Graphs

Figure 5.4: The first graph shows the time required to report the number of lines in a log file matching a regular expression. All tools are fed with the same regular expression. The decompress and search approach is implemented in parallel i.e. searching on the output uncompressed text as it is recovered by the decompressor. As a reference, we show the time required for decompressing the file with different tools (horizontal lines). The second graph is the cactus plot corresponding the the data from the first graph. In this case, we observe that search is faster than any other tool, except grep.

top of the class¹⁶ for regular expression searching.

For (de)compressing the files we use zstd and lz4 which are among the best lossless compressors¹⁷, being lz4 considerably faster while zstd achieves better compression. We use both tools with the highest compression level, which has little impact on the time required for decompression.

We use versions `grep v3.3`, `hyperscan v5.0.0`, `lz4 v1.8.3` and `zstd v1.3.6` running in an Intel Xeon E5640 CPU 2.67 GHz with 20 GB RAM which supports SIMD instructions up to SSE4-2. We restrict to ASCII inputs and set `LC_ALL=C` for all experiments, which significantly improves the performance of `grep`. Since both `hyperscan` and `GNgrep` count positions of the text where a match ends, we extend each regular expression (when used with these tools) to match the whole line. We made this decision to ensure all tools solve the same counting problem and therefore produce the *same output*.

5.4.2 Files and Regular Expressions

Our benchmark consists of an automatically generated *Log*¹⁸ of HTTP requests, English *Subtitles* [Lison and Tiedemann 2016], and a concatenation of English *Books*¹⁹. Table 5.1 shows how each compressor behaves on these files.

	File	Compressed size				Compression time				Decompression time			
		LZW	repair	zstd	lz4	LZW	repair	zstd	lz4	LZW	repair	zstd	lz4
Uncompressed	1 MB	Logs	0.19	0.08	0.07	0.12	0.04	0.19	0.51	0.03	0.02	0.01	0.01
		Subtitles	0.36	0.13	0.11	0.15	0.04	0.25	0.3	0.03	0.02	0.01	0.01
		Books	0.42	0.34	0.27	0.43	0.04	0.29	0.42	0.08	0.02	0.02	0.01
	500 MB	Logs	96	38	33	65	16.9	123.2	819.1	13.3	7.8	5.5	1.1
		Subtitles	191	66	55	114	19.9	169.3	415.2	22.8	8.6	8.2	1.2
		Books	206	153	129	216	20.2	198.6	646.3	40.6	8.6	9.7	2.0

Table 5.1: Sizes (in MB) of the compressed files and (de)compression times (in seconds). Maximum compression levels enabled. (Blue = best; bold black = second best; red = worst).

We first run each experiment 3 times as warm up so that the files are loaded in memory. Then we measure the running time 30 times and compute the *confidence interval* (with 95% confidence) for the running time required to count the number of matching lines for a regular expression in a certain file using a certain tool.

We consider the *point estimate* of the confidence interval and omit the *margin of error* which never exceeds the 9% of the point estimate for the reported experiments. The on-line version of these experiment *does report* the margin

¹⁶<https://rust-leipzig.github.io/regex/2017/03/28/comparison-of-regex-engines/>

¹⁷<https://quixdb.github.io/squash-benchmark/>

¹⁸<http://ita.ee.lbl.gov/html/contrib/NASA-HTTP.html>

¹⁹https://web.eecs.umich.edu/~lahiri/gutenberg_dataset.html

of error as a black mark on the top of each bar. The height of the bar is the point estimate computed for the given experiment while the black mark denotes the confidence interval (see Figure 5.4). Figure 5.5 summarizes the obtained results when considering, for all files, the regular expressions: “what”, “HTTP”, “.”, “I . * you ”, “ [a-z]{4} ”, “ [a-z]*[a-z]{3} ”, “[0-9]{4}”, “[0-9]{2}/(Jun|Jul|Aug)/[0-9]{4}”.

For clarity, we report only on the most relevant tools among the ones considered. For lz4 and zstd, we report the time required to decompress the file and send the output to /dev/null.

Figure 5.5: Average running time required to report the number of lines matching a regular expression in a file and time required for decompression. Colors indicate whether the tool performs the search on the uncompressed text (blue); the compressed text (black); the output of the decompressor (green); or decompresses the file without searching (red).

5.4.3 Analysis of the Results.

Figure 5.5 and Table 5.1 show that the performance of `zsearch` improves with the compression ratio. This is to be expected since `zsearch` processes each grammar rule exactly once and better compression results in less rules to be processed. In consequence, `zsearch` is the fastest tool for counting matching lines in compressed *Log* files while it is the second slowest one for the *Books*.

In particular, `zsearch` is more than 25% faster than any other tool working on compressed *Log* files. Actually `zsearch` is competitive with `grep` and `hyperscan`, even though these tools operate on the uncompressed text. These results are remarkable since `hyperscan`, unlike `zsearch`, uses algorithms specifically designed to take advantage of SIMD parallelization.²⁰

Finally, the fastest tool for counting matching lines in compressed *Subtitles* and *Books*, `lz4|hyperscan`, applies to files larger than the ones obtained when

²⁰According to the documentation, `hyperscan` requires, at least, support for SSE3.

compressing the data with `repair` (see Table 5.1). However, when considering a better compressor such as `zstd`, which achieves slightly more compression than `repair`, the decompression becomes slower. As a result, `zsearch` outperforms `zstd|hyperscan` by more than 7% for *Subtitles* files and 50% for *Logs*.

Contrived Example

Next, we discharge the full potential of our approach by considering a contrived experiment in which the data is highly repetitive. In particular, we consider a file where all lines are identical and consist of the sentence “This is a contrived experiment.”. Table 5.2 shows the compression achieved on this data for each of the compressors.

File size	Compressed size				Compression time				Decompression time			
	LZW	repair	zstd	lz4	LZW	repair	zstd	lz4	LZW	repair	zstd	lz4
1MB	13	0.072	0.135	4.1	0.01	0.08	0.01	0.004	0.01	0.01	0.003	0.003
500MB	950	0.09	44	2000	14.5	53.1	0.99	0.28	3.9	3.3	0.24	0.2

Table 5.2: Sizes (in KB) of the compressed files and (de)compression times (in seconds). Maximum compression levels enabled. (Blue = best; bold black = second best; red = worst).

As expected this contrived file results in really high compression ratios. As we show next, this scenario evidences the virtues of `zsearch` which is capable of searching in 500MB of data by processing a grammar consisting of 57 rules.

Table 5.3 summarizes the results obtained when searching the 500 MB contrived file for different regular expressions.²¹ For each expression, we report the time required to (i) search on the compressed data without decompression, (ii) search on the uncompressed data and (iii) search with the best implementation of the parallel decompress and search approach.²²

As shown by Tables 5.2 and 5.3, our tool is about *10 times faster* at searching than `lz4` at decompression. Therefore, `zsearch` clearly outperforms any decompress and search approach, even if decompression and search are done in parallel. This is to be expected since `zsearch` only needs to process 90 Bytes of data (the size of the grammar) while the rest of the tools need to process 500 MB.

Similarly, `GNgrep` processes 950 KB of data (the size of the LZW-compressed data). As a consequence, when there are no matches of the expression, `GNgrep` is faster than decompression as evidenced by the last two rows of Table 5.3. However, `GNgrep` reports the number of matching lines by explicitly finding the pos-

²¹We run each experiment 30 times and report the point estimate of the confidence interval with 95% confidence.

²²The best implementation might vary depending on the expression.

Expression	zearch	GNgrep	grep	hyperscan	decompress and search
"experiment"	2.267	14K	1352	1784	1652
"This"	2.533	14K	764	2166	959
"."	2.467	14K	703	1276	886
"[a-z]{4}"	2.667	14K	1138	1270	1360
"[a-z]{11}"	2.233	37	1690	1312	1397
"That"	2.433	37.2	607	239	444

Table 5.3: Time (ms) required to report the number of lines matching a regular expression in the 500 MB large contrived file. (Blue = fastest; bold black = second fastest; red = slowest).

itions in the data where the match begins, which results rather inefficient when all lines of the file contain a match, as evidenced by the first 4 rows of Table 5.3.

5.5 Fine-Grained Analysis of the Implementation

The grammars produced by `repair` break the definition of SLP in behalf of compression by allowing the axiom rule to have more than two symbols on the right hand side. This is due to the fact that the axiom rule is built with the remains of the input text after creating all grammar rules.

Typically, the length of the axiom is larger or equal than the number of rules in the SLP so the way in which the axiom is processed heavily influences the performance of `zearch`.

On the other hand, our experiments show that the performance of `zearch` is typically far from its worst case complexity. This is because the worst case scenario assumes each string generated by a grammar variable labels a path between each pair of states of the automaton. However, we only observed such behavior in contrived examples.

5.5.1 Processing the Axiom Rule.

Algorithm `COUNTLINES` could process the axiom rule $X_{|V|} \rightarrow \sigma$ by building an SLP with the set of rules

$$\{S_1 \rightarrow (\sigma)_1(\sigma)_2\} \cup \{S_i \rightarrow S_{i-1}(\sigma)_{i+1} \mid i = 2 \dots |\sigma|-2\} \cup \{X_{|V|} \rightarrow S_{|\sigma|-2}(\sigma)_\dagger\} .$$

However it is more efficient to compute the set of states reachable from the initial ones when reading the string generated with S_1 and update this set for each symbol $(\sigma)_i$. To perform the counting note that C_{S_i} is only used to compute $C_{S_{i+1}}$ and can be discarded afterwards. This yields an algorithm running in

$\mathcal{O}(|V| \cdot s^3 + |\sigma| \cdot s^2)$ time using $\mathcal{O}(|V| \cdot s^2)$ space where $|V|$ is the number of rules of the input grammar and $X_{|V|} \rightarrow \sigma$ its axiom.

5.5.2 Number of Operations Performed by the Algorithm

Define $s_{\tau,q} = |\{q' \mid (q, \tau, q') \in \delta\}|$ and $s_\tau = \sum_{q \in Q} s_{\tau,q}$ and let us recall the complexity of Algorithm [COUNTLINES](#) according to the data structures described in Section 5.2.1. The algorithm iterates over the $|V|$ rules of the grammar and, for each of them:

- (i) Initializes matrix $\mathcal{K}$ with s_{β_ℓ} elements²³
- (ii) Iterates through $\mathcal{K}[q'][[0 \dots s_{\beta_\ell, q'}]]$ for each pair $(q_1, q') \in E_{\alpha_\ell}$.

Then it processes the axiom rule iterating, for each symbol $(\sigma)_i$, through $s_{(\sigma)_i}$ transitions.

These are all the operations performed by the algorithm with running time dependent on the size of the input. Hence, Algorithm [COUNTLINES](#) runs in

$$\mathcal{O}\left(\sum_{\ell=1}^{|V|} \tilde{s}_\ell + \sum_{i=1}^{|\sigma|} s_{(\sigma)_i}\right) \text{ time, where } \tilde{s}_\ell = s_{\beta_\ell} + s + \sum_{(q_1, q') \in E_{\alpha_\ell}} (1 + s_{\beta_\ell, q'}) p.$$

Note that $\tilde{s}_\ell \leq s^3$, $s_{(\sigma)_i} \leq s^2$ so the worst case time complexity of the algorithm is $\mathcal{O}(|V| \cdot s^3 + |\sigma| \cdot s^2)$. However, in the experiments we observed that $\tilde{s}_\ell$ and $s_{(\sigma)_i}$ are usually much smaller than s^3 and s^2 , respectively, as reported in Table 5.4.

Expression	s	s ³	percentiles for $\tilde{s}_\ell$					s ²	percentiles for $s_{(\sigma)_i}$				
			50%	75%	95%	98%	100%		50%	75%	95%	98%	100%
"what"	5	125	0	0	1	1	9	25	0	0	1	1	2
"HTTP"	5	125	0	0	0	0	10	25	0	0	0	0	2
" "	2	8	0	0	0	0	4	4	0	0	0	0	1
"I .* you"	9	729	3	13	16	18	29	81	3	3	5	5	9
" [a-z]4 "	7	343	2	10	11	12	18	49	1	1	2	2	4
" [a-z]*[a-z]3 "	7	343	3	11	14	18	31	49	1	3	3	4	8
" [0-9]{4}"	6	216	8	8	8	8	18	36	1	1	1	1	5
" *[A-Za-z]{5}"	7	343	14	25	48	48	48	49	11	14	14	14	14
" *[A-Za-z]{10}"	12	1728	29	51	86	95	98	144	16	26	29	29	29
" *[A-Za-z]{20}"	22	10648	57	87	132	153	198	484	23	38	52	58	59
" (((.)*.)*.)*"	6	216	12	29	209	209	209	36	29	29	29	29	29
" (((((.)*.)*.)*.)*.)*"	7	343	14	34	249	249	249	49	34	34	34	34	34

Table 5.4: Analysis of the values $\tilde{s}_\ell$ and $s_{(\sigma)_i}$ obtained when considering different regular expressions to search Subtitles (100 MB uncompressed long). The fifth column of the fourth row indicates that when considering the expression "I .* you", for 75% of the grammar rules we have $\tilde{s}_\ell \leq 13$ while $s^3 = 729$.

²³We need to set up to s sentinel values for the rows in $\mathcal{K}$ not used for storing s_{β_ℓ}

As the experiments show, `zsearch` exhibits almost linear behavior with respect to the size of the automaton built from the expression. Nevertheless, there are regular expressions that trigger the worst case behavior (last two rows in Table 5.4), which cannot be avoided due to the result of Abboud et al. [2017] described before.

5.6 Fine-Grained Complexity

In Section 5.2 we obtained upper bounds for the worst-case time complexity of our algorithm depending on whether the automata built from the expression is an NFA or a DFA.

However, we observed in Section 5.5 that the actual behavior of our implementation is, in general, far from its worst-case scenario (see Table 5.4). This is due to the fact that the worst-case scenario assumes an NFA where each pair of states are connected by a transition for each symbol in the alphabet but this is rarely the type of automata obtained from non-contrived regular expressions.

This difference between the worst-case time complexity of the algorithm and its behavior in practice also appears when considering the problem of searching with regular expressions on plain text. Indeed, this problem led Backurs and Indyk [2016] to analyze the complexity of searching on plain text for different classes of regular expressions.

In their work, Backurs and Indyk [2016] restrict themselves to *homogeneous regular expressions*, i.e. regular expressions in which operators at the same level of the formula are equal²⁴, which are grouped in classes depending on the sequence of the operators involved. Then, they obtain a lower bound for the search complexity for each class of expressions by building reductions from the *Orthogonal Vector Problem* (OVP for short) which, given two sets of vectors $A, B \subseteq \{0, 1\}^d$ in d dimensions, with N and M elements respectively, asks whether there exists $a \in A$ and $b \in B$ such that $a \cdot b = 0$.

Conjecture (OV Conjecture [Bringmann and Künnemann 2015]). *There are no reals $\varepsilon, d > 0$ such that the OVP in $d < N^{o(1)}$ dimensions with $M = \Theta(N^\alpha)$ for $\alpha \in (0, 1]$ can be solved in $O((N \cdot M)^{1-\varepsilon})$ time.*

The idea behind the conjecture is that any algorithm defying it would yield an algorithm for SAT violating the Strong Exponential Time Hypothesis.

Backurs and Indyk [2016] relied on the OV conjecture to determine whether a search problem is *easy*, i.e. there is an algorithm running in $O(T+s)$ time where T

²⁴Write the regular expression as a tree. The expression is homogeneous if all non-leaf nodes at the same depth have the same label.

is the size of the input text and s is the number of states of the automaton, or *hard*, i.e. assuming the Strong Exponential Time Hypothesis (SETH) any algorithm has $\Omega((T \cdot s)^{1-\epsilon})$ time complexity with $\epsilon > 0$.

This analysis can be extended to consider searching on compressed text and decide whether our implementation is optimal on different classes of homogeneous regular expressions. To do that, we apply the following remark, inherited from Abboud et al. [2017], who used the OVP to analyze whether the decompress and solve approach can be outperformed by manipulating the compressed text for different problems.

Remark 5.6.1. *Let $A = \{a_1, \dots, a_N\} \subseteq \{0, 1\}^d$ and $B = \{b_1, \dots, b_M\} \subseteq \{0, 1\}^d$ be an instance of the OVP in $d \leq N^{o(1)}$ dimensions with $M = O(N)$. We define a string T with a representation as an SLP of size $t = O(N \cdot d)$ and a regular expression π of size $s = O(M \cdot d)$ such that the string contains a match for the expression iff we have a solution for the OVP.*

If there is an algorithm for regular expression searching on compressed text that operates, for a class of regular expressions that includes π , in $O((t \cdot s)^{1-\epsilon})$ with $\epsilon > 0$ then it would solve the OVP in $O((N \cdot M)^{1-\epsilon})$ (since the dimension is fixed) which contradicts the OV conjecture.

5.6.1 Complexity of Searching on Compressed Text

Given a regular expression, we say it is *homogeneous of type “|+”* iff the regular expression is a disjunction of + operators and terminals. We extend this notation to any combination of operators. For instance, the expressions “a+b+” and “a+b” are homogeneous of type “.+” while “a+b*” is not homogeneous. Recall that the size of a regular expression is the number of operators and terminals used to define the expression. For instance, “a+b+” and “a+b” have size 4 and 3, respectively.

The following three results use Remark 5.6.1 to show that the time complexity of regular expression searching on compressed text is $\Omega(t \cdot s)$, where t is the size of the SLP and s is the size of the expression, when the regular expression is homogeneous of type “.+”, “.*” or “.|”.

THEOREM 5.6.2. *There is no algorithm for searching with a regular expression on grammar-compressed text that operates in $O((t \cdot s)^{1-\epsilon})$ time with $\epsilon > 0$, where t is the size of the compressed text and s is the size of the regular expression, when the expression is homogeneous of type “.+”.*

Proof. Let $A = \{a_1, \dots, a_N\} \subseteq \{0, 1\}^d$ and $B = \{b_1, \dots, b_M\} \subseteq \{0, 1\}^d$ be an instance of the OVP in $d \leq N^{o(1)}$ dimensions with $M = O(N)$. Without loss of generality, assume the dimension is even. Consider the regular expression

$$\pi \stackrel{\text{def}}{=} "F(b_1)zF(b_2)z \dots zF(b_M)z"$$

on the alphabet $\Sigma = \{x, y, z\}$ with $F(b_i) \stackrel{\text{def}}{=} f(b_i, 1)f(b_i, 2), \dots, f(b_i, d)$ and

$$f(b, j) \stackrel{\text{def}}{=} \begin{cases} xx^+ & \text{if } (b)_j = 1 \text{ and } j \text{ is even} \\ x^+ & \text{if } (b)_j = 0 \text{ and } j \text{ is even} \\ yy^+ & \text{if } (b)_j = 1 \text{ and } j \text{ is odd} \\ y^+ & \text{if } (b)_j = 0 \text{ and } j \text{ is odd} \end{cases},$$

where $(b)_j$ is the j -th component of the vector b . Clearly, π is homogeneous of type “ $\cdot$ ” and has size $s = O(M \cdot d)$.

Now, we define an SLP $\mathcal{P}$ on $\Sigma = \{x, y, z\}$ such that $\mathcal{L}(\mathcal{P}) = \{w\}$ with

$$w \stackrel{\text{def}}{=} \left((xxyy)^{\frac{d}{2}} z \right)^{M-1} \tilde{F}(a_1)z \dots \left((xxyy)^{\frac{d}{2}} z \right)^{M-1} \tilde{F}(a_N)z \left((xxyy)^{\frac{d}{2}} z \right)^{M-1}$$

where $\tilde{F}(a_i) \stackrel{\text{def}}{=} \tilde{f}(a_i, 1)\tilde{f}(a_i, 2), \dots, \tilde{f}(a_i, d)$ and

$$\tilde{f}(a, j) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } (a)_j = 1 \text{ and } j \text{ is even} \\ xx & \text{if } (a)_j = 0 \text{ and } j \text{ is even} \\ y & \text{if } (a)_j = 1 \text{ and } j \text{ is odd} \\ yy & \text{if } (a)_j = 0 \text{ and } j \text{ is odd} \end{cases}.$$

The substring $\left((xxyy)^{\frac{d}{2}} z \right)^{M-1}$ can be generated with an SLP of size $O(d + \log M)$, hence w can be compressed as an SLP $\mathcal{P}$ of size $t = O(N \cdot d + d + \log M)$ and, since $d \leq N^{o(1)}$ is a constant and $M = O(N)$, we find that $t = O(N)$.

Clearly, π and $\mathcal{P}$ can be built in $O(M \cdot d)$ and $O(N \cdot d)$ time, respectively.

Finally, we show that there exists $a \in A$, $b \in B$ such that $a \cdot b = 0$ iff there is a factor of w that matches π . Let $a_{i_1} \in A$ and $b_{i_2} \in B$. Then $a_{i_1} \cdot b_{i_2} = 0$ iff

- (i) The factor $\left((xxyy)^{\frac{d}{2}} z \right)^{i_2-1}$ of w that *precedes* $\tilde{F}(a_{i_1})z$ matches the subexpression “ $F(b_1)z \dots F(b_{i_2-1})z$ ”.
- (ii) The factor $\tilde{F}(a_{i_1})z$ of w matches the subexpression “ $F(b_{i_2})z$ ”.
- (iii) The factor $\left((xxyy)^{\frac{d}{2}} z \right)^{M-i_2}$ of w that *succeeds* $\tilde{F}(a_{i_1})z$ matches the subexpression “ $F(b_{i_2+1})z \dots F(b_M)z$ ”.

It follows from Remark 5.6.1 that there is no algorithm for searching with an homogeneous regular expression of type “ $\cdot +$ ” working on $O((t \cdot s)^{1-\varepsilon})$ time.

Finally, note that if the dimension of the OVP is odd it suffices to replace the factors $\left((xxyy)^{\frac{d}{2}}z\right)^{M-1}$ from w by $\left((xxyy)^{\frac{d-1}{2}}xxz\right)^{M-1}$.

Note that for any homogeneous regular expression of type “ $\cdot +$ ” of size s , we can build in $O(s)$ time an equivalent homogeneous regular expression of type “ $\cdot \cdot$ ” and size $O(s)$. Therefore, we obtain the following corollary from Theorem 5.6.2.

COROLLARY 5.6.3. *There is no algorithm for searching with a regular expression on grammar-compressed text that operates in $O((t \cdot s)^{1-\varepsilon})$ with $\varepsilon > 0$, where t is the size of the compressed text and s is the size of the regular expression, when the expression is homogeneous of type “ $\cdot \cdot$ ”.*

THEOREM 5.6.4. *There is no algorithm for searching with regular expressions on grammar-compressed text that operates in $O((t \cdot s)^{1-\varepsilon})$ with $\varepsilon > 0$, where t is the size of the compressed text and s is the size of the regular expression, when the expression is homogeneous of type “ $\cdot |$ ”.*

Proof. The proof is identical to that of Theorem 5.6.2 but considering the expression

$$\pi \stackrel{\text{def}}{=} “F(b_1)zF(b_2)z \dots zF(b_M)z”$$

on the alphabet $\Sigma = \{0, 1, z\}$ with

$$F(b_i) = f(b_i, 1)f(b_i, 2), \dots, f(b_i, d) \text{ and } f(b, j) = \begin{cases} 0 & \text{if } (b)_j = 1 \\ 0|1 & \text{if } (b)_j = 0 \end{cases}$$

and the word

$$w \stackrel{\text{def}}{=} \left(0^d z\right)^{M-1} a_1 z \left(0^d z\right)^{M-1} a_2 z \dots \left(0^d z\right)^{M-1} a_N z \left(0^d z\right)^{M-1}.$$

Note that, unlike the proof of Theorem 5.6.2, this proof does not depend on the parity of the dimension of the OVP.

5.6.2 Complexity of Our Implementation

In the following, we analyze the complexity of the implementation of Algorithm COUNTLINES described in Section 5.3 when the input regular expression is homogeneous of type “ $\cdot +$ ”, “ $\cdot \cdot$ ” or “ $\cdot |$ ”

As explained in Section 5.3, `zsearch` uses `libfa`, which applies Thompson’s algorithm [Thompson 1968] with on-the-fly ε -removal, to build an NFA for the input regular expression.

However, given a regular expression of size s we can decide in $O(s)$ time whether a expression is homogeneous of type “ $\cdot+$ ”, “ $\cdot*$ ” or “ $\cdot|$ ” and, as we show next, use an specialized algorithm for building a DFA with $O(s)$ states in $O(s)$ time for the given expression. Therefore, `zsearch` admits a straightforward modification that allows it to search on grammar-compressed text with homogeneous regular expressions of type “ $\cdot+$ ”, “ $\cdot*$ ” or “ $\cdot|$ ” in $O(t \cdot s)$ time, where s is the size of the expression. Next, we show how to build such DFAs from the given regular expressions.

First, observe that every homogeneous regular expression of type “ $\cdot+$ ” of size s such that it contains no concatenation of the form “ $a+a+$ ” can be captured by a DFA with $s+1$ states as we show next. Let $a_1, \dots, a_n$ be the sequence of letters that appear in an homogeneous expression of type “ $\cdot+$ ”. Then,

$$\mathcal{D} = \langle \{q_i \mid 0 \leq i \leq n\}, \Sigma, \{(q_i, a_i, q_i), (q_{i-1}, a_i, q_i) \mid 1 \leq i \leq n\}, \{q_1\}, \{q_n\} \rangle$$

is a DFA for the given expression.

If the expression contains a concatenation of the form “ $a+a+$ ” then $\mathcal{D}$ is no longer deterministic. In that case, we can replace “ $a+a+$ ” by “ $aa+$ ” and, therefore, remove from $\mathcal{D}$ the self-loop corresponding to the first a . It is straightforward to check that this change results in a deterministic automaton and it does not alter the generated language, hence it does not alter the result of the search. Figure 5.6 shows the DFA for an homogeneous regular expression of type “ $\cdot+$ ”.

Figure 5.6: DFA for the regular expression “ $a+b+b+a+c+$ ” = “ $a+bb+a+c+$ ”.

On the other hand, let $a_1, \dots, a_n$ be the sequence of letters that appear in an homogeneous expression of type “ $\cdot*$ ”. For every a , let j_a^k be the smallest index such that $k \leq j_a^k \leq n$ and $a = a_{j_a^k}$. Then, the DFA obtained by making every state of $\mathcal{D}$ final and adding the transitions $\{(q_{i-1}, a_k, q_{j_a^k}^i) \mid 1 \leq i \leq k \leq n\}$, is an automaton for the given expression. Note that, if the expression contains a concatenation of the form “ $a*a*$ ”, which will break the determinism of our automata, then we can safely replace it by “ $a*$ ”. Figure 5.7 shows the DFA for an homogeneous regular expression of type “ $\cdot*$ ”.

Figure 5.7: DFA for the regular expression “ $a^*b^*b^*a^*c^*$ ” = “ $a^*b^*a^*c^*$ ”.

Finally, it is straightforward to build a DFA for an homogeneous regular expression of the type “ $\cdot |$ ” with $n+1$ states where n is the number of concatenations. Figure 5.8 shows the DFA for an homogeneous regular expression of type “ $\cdot |$ ”.

Figure 5.8: DFA for the regular expression “ $(a|b)(a|c)(b|c)(a|c)$ ”.

It is worth to remark that the DFA of Figure 5.8 is the result of applying Thompson’s construction on the input expression. As a consequence, `zsearch` already builds a DFA when the input expression is homogeneous of type “ $\cdot |$ ” and, therefore, it performs the search in $O(t \cdot s)$.

We conclude that `zsearch` admits a straightforward modification to exhibit $O(t \cdot s)$ time complexity when working on homogeneous regular expressions of types “ $\cdot +$ ”, “ $\cdot *$ ” and “ $\cdot |$ ” and, therefore, be nearly optimal for these classes of regular expressions.

BUILDING RESIDUAL AUTOMATA

As shown in Chapter 3, residual automata (RFAs for short) are a class of automata that lies between deterministic (DFAs) and nondeterministic automata (NFAs). They share with DFAs a significant property: the existence of a canonical minimal form for any regular language. On the other hand, they share with NFAs the existence of automata that are exponentially smaller (in the number of states) than the corresponding minimal DFA for the language. These properties make RFAs specially appealing in certain areas of computer science such as Grammatical Inference [Denis et al. 2004; Kasprzik 2011].

RFAs were first introduced by Denis et al. [2000; 2002] who defined an algorithm for *residualizing* an automaton (see Section 3.2), showed that there exists a *unique canonical* RFA for every regular language and proved that the residual-equivalent of double-reversal method for DFAs [Brzozowski 1962] holds for RFAs, i.e. residualizing an automaton $\mathcal{N}$ whose reverse is residual yields the canonical RFA for $\mathcal{L}(\mathcal{N})$. Later, Tamm [2015] generalized the double-reversal method for RFAs in the same lines as that of Brzozowski and Tamm [2014] for the double-reversal method for DFAs.

The similarities between the determinization and residualization (see Section 3.2) operations and between the double-reversal methods for DFAs and RFAs evidence the existence of a relationship between these two classes of automata. However, the connection between them is not clear and, as a consequence, the relation between the generalization by Brzozowski and Tamm [2014] of the double-reversal method for DFAs and the one by Tamm [2015] for RFAs is not immediate.

In this chapter, we show that *quasiorders* are fundamental to RFAs as *congruences* are for DFAs, which evidences the relation between these two classes of automata. To do that, we define a framework of finite-state automata construc-

tions based on *quasiorders* over words.

As explained in Chapter 2, Ganty et al. [2019] studied the problem of building DFAs using congruences, i.e., equivalence relations over words with good properties w.r.t. concatenation, and derived several well-known results about minimization of DFAs, including the double-reversal method and its generalization by Brzozowski and Tamm [2014]. While the use of congruences over words suited for the construction of a subclass of residual automata, namely, *deterministic* automata, these are no longer useful to describe the more general class of *nondeterministic* residual automata. By moving from *congruences* to *quasiorders*, we are able to introduce nondeterminism in our automata constructions.

We consider quasiorders with good properties w.r.t. *right* and *left* concatenation. In particular, we define the so-called *right language-based* quasiorder, whose definition relies on a given regular language; and the *right automata-based* quasiorder, whose definition relies on a finite representation of the language, i.e., an automaton. We also give counterpart definitions for quasiorders that behave well with respect to *left* concatenation.

When instantiating our automata constructions using the *right language-based* quasiorder, we obtain the canonical RFA for the given language; while using the *right automata-based* quasiorder yields an RFA for the language generated by the automaton that has, at most, as many states as the RFA obtained by the residualization operation defined by Denis et al. [2002]. Similarly, *left automata-based* and *language-based* quasiorders yield co-residual automata, i.e., automata whose reverse is residual.

Our quasiorder-based framework allows us to give a simple correctness proof of the double-reversal method for building the canonical RFA. Moreover, it allows us to generalize this method in the same fashion as Brzozowski and Tamm [2014] generalized the double-reversal method for building the minimal DFA. Specifically, we give a characterization of the class of automata for which our automata-based quasiorder construction yields the canonical RFA.

We compare our characterization with the class of automata, defined by Tamm [2015], for which the residualization operation of Denis et al. [2002] yields the canonical RFA and show that her class of automata is strictly contained in the class we define. Furthermore, we highlight the connection between the generalization of Brzozowski and Tamm [2014] and the one of Tamm [2015] for the double-reversal methods for DFAs and RFAs, respectively.

Finally, we revisit the problem of learning RFAs from a quasiorder-based perspective. Specifically, we observe that the NL^* algorithm defined by Bollig et al. [2009], inspired by the popular Angluin's L^* algorithm for learning DFAs [Angluin 1987], can be seen as an algorithm that starts from a quasiorder

and refines it at each iteration. At the end of each iteration, the automaton built by NL^* coincides with our quasiorder-based automata construction applied to the refined quasiorder.

6.1 Automata Constructions from Quasiorders

In this chapter, we consider monotone quasiorders on Σ^* (and their corresponding closures) and we use them to define RFAs constructions for regular languages. The following lemma gives a characterization of right and left quasiorders.

LEMMA 6.1.1. *The following properties hold:*

- (a) $\leq^r$ is a right quasiorder iff $\rho_{\leq^r}(u) v \subseteq \rho_{\leq^r}(uv)$, for all $u, v \in \Sigma^*$.
- (b) $\leq^\ell$ is a left quasiorder iff $v \rho_{\leq^\ell}(u) \subseteq \rho_{\leq^\ell}(vu)$, for all $u, v \in \Sigma^*$.

Proof.

- (a) To simplify the notation, we denote $\rho_{\leq^r}$, the closure induced by $\leq^r$, by ρ .
 - ($\Rightarrow$) Let $x \in \rho(v)u$, i.e. $x = \tilde{v}u$ with $v \leq^r \tilde{v}$. Since $\leq^r$ is a right quasiorder and $v \leq^r \tilde{v}$ then $vu \leq^r \tilde{v}u$. Therefore $x \in \rho(vu)$.
 - ($\Leftarrow$) Assume that for each $u, v \in \Sigma^*$ and $\tilde{v} \in \rho(v)$ we have that $\tilde{v}u \in \rho(vu)$. Then, $v \leq^r \tilde{v} \Rightarrow vu \leq^r \tilde{v}u$.
- (b) To simplify the notation we denote $\rho_{\leq^\ell}$, the closure induced by $\leq^\ell$, by ρ .
 - ($\Rightarrow$) Let $x \in u\rho(v)$, i.e. $x = u\tilde{v}$ with $v \leq^\ell \tilde{v}$. Since $\leq^\ell$ is a left quasiorder and $v \leq^\ell \tilde{v}$ then $uv \leq^\ell u\tilde{v}$. Therefore $x \in \rho(uv)$.
 - ($\Leftarrow$) Assume that for each $u, v \in \Sigma^*$ and $\tilde{v} \in \rho(v)$ we have that $u\tilde{v} \in \rho(uv)$. Then $v \leq^\ell \tilde{v} \Rightarrow uv \leq^\ell u\tilde{v}$.

Given a regular language L , we are interested in left and right L -consistent quasiorders. We will use the principals of these quasiorders as states of automata constructions that yield RFAs and co-RFAs generating the language L . Therefore, in the sequel, we will only consider quasiorders that induce a finite number of principals, i.e., quasiorders $\leq$ such that the induced equivalence $\sim \stackrel{\text{def}}{=} \leq \cap (\leq)^{-1}$ has finite index.

Next, we introduce the notion of *L-composite principals* which, intuitively, correspond to states of our automata constructions that can be removed without altering the generated language.

Definition 6.1.2 (*L-Composite Principal*). *Let L be a regular language and let $\leq^r$ (resp. $\leq^\ell$) be a right (resp. left) quasiorder on Σ^* . Given $u \in \Sigma^*$, the principal $\rho_{\leq^r}(u)$*

(resp. $\rho_{\leq^{\ell}}(u)$) is L -composite iff

$$u^{-1}L = \bigcup_{x \in \Sigma^*, x <^r u} x^{-1}L \quad (\text{resp. } Lu^{-1} = \bigcup_{x \in \Sigma^*, x <^{\ell} u} Lx^{-1})$$

If $\rho_{\leq^r}(u)$ (resp. $\rho_{\leq^{\ell}}(u)$) is not L -composite then it is L -prime. ■

We sometimes use the terms *composite* and *prime principal* when the language L is clear from the context. Observe that, if $\rho_{\leq^r}(u)$ is L -composite, for some $u \in \Sigma^*$, then so is $\rho_{\leq^r}(v)$, for every $v \in \Sigma^*$ such that $u \sim^r v$. The same holds for a left quasiorder $\leq^{\ell}$.

Given a regular language L and a right L -consistent quasiorder $\leq^r$, the following automata construction yields an RFA that generates exactly L .

Definition 6.1.3 (Automata construction $H^r(\leq^r, L)$). Let $\leq^r$ be a right quasiorder and let $L \subseteq \Sigma^*$ be a language. Define the automaton $H^r(\leq^r, L) \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, F \rangle$ where $Q = \{\rho_{\leq^r}(u) \mid u \in \Sigma^*, \rho_{\leq^r}(u) \text{ is } L\text{-prime}\}$, $I = \{\rho_{\leq^r}(u) \in Q \mid \varepsilon \in \rho_{\leq^r}(u)\}$, $F = \{\rho_{\leq^r}(u) \in Q \mid u \in L\}$ and $\delta(\rho_{\leq^r}(u), a) = \{\rho_{\leq^r}(v) \in Q \mid \rho_{\leq^r}(u) \cdot a \subseteq \rho_{\leq^r}(v)\}$ for all $\rho_{\leq^r}(u) \in Q, a \in \Sigma$. ■

LEMMA 6.1.4. Let $L \subseteq \Sigma^*$ be a regular language and let $\leq^r$ be a right L -consistent quasiorder. Then, $H^r(\leq^r, L)$ is an RFA such that $\mathcal{L}(H^r(\leq^r, L)) = L$.

Proof. To simplify the notation, we denote $\rho_{\leq^r}$, the closure induced by the quasiorder $\leq^r$, simply by ρ . Let $\mathcal{H} = H^r(\leq^r, L) = \langle Q, \Sigma, \delta, I, F \rangle$. We first show that $\mathcal{H}$ is an RFA, i.e.

$$W_{\rho(u), F}^{\mathcal{H}} = u^{-1}L, \quad \text{for each } \rho(u) \in Q. \quad (6.1)$$

Let us prove that $w \in u^{-1}L \Rightarrow w \in W_{\rho(u), F}^{\mathcal{H}}$. We proceed by induction on the length of w .

– *Base case:* Assume $w = \varepsilon$. Then,

$$\varepsilon \in u^{-1}L \Rightarrow u \in L \Rightarrow \rho(u) \in F \Rightarrow \varepsilon \in W_{\rho(u), F}^{\mathcal{H}}.$$

– *Inductive step:* Assume that the hypothesis holds for each word $x \in \Sigma^*$ with $|x| \leq n$, where $n \geq 1$, and let $w \in \Sigma^*$ be such that $|w| = n+1$. Then $w = ax$ with $|x| = n$ and $a \in \Sigma$.

$$\begin{aligned} ax \in u^{-1}L &\Rightarrow \text{[By definition of quotient]} \\ x \in (ua)^{-1}L &\Rightarrow \end{aligned}$$

[By Def. 6.1.2, $\rho(ua)$ is L -prime (so $z \stackrel{\text{def}}{=} ua$) or $(ua)^{-1}L = \bigcup_{x_i <^r ua} x_i^{-1}L$ (so $z \stackrel{\text{def}}{=} x_i$)]

$\exists \rho(z) \in Q, x \in z^{-1}L \wedge \rho(ua) \subseteq \rho(z) \Rightarrow$ [By I.H., Lemma 6.1.1 and Def. 6.1.3]

$x \in W_{\rho(z), F}^{\mathcal{H}} \wedge \rho(z) \in \delta(\rho(u), a) \Rightarrow$ [By definition of $W_{S, T}$]

$$ax \in W_{\rho(u), F}^{\mathcal{H}} .$$

We now prove the other side of the implication, $w \in W_{\rho(u), F}^{\mathcal{H}} \Rightarrow w \in u^{-1}L$.

– *Base case:* Let $w = \varepsilon$. By Definition 6.1.3,

$$\varepsilon \in W_{\rho(u), F}^{\mathcal{H}} \Rightarrow \exists \rho(x) \in Q, x \in L \wedge \rho(u)\varepsilon \subseteq \rho(x) .$$

Since $\rho(L) = L$, we have that $u\varepsilon \in L$, hence $\varepsilon \in u^{-1}L$.

– *Inductive step:* Assume the hypothesis holds for each $x \in \Sigma^*$ with $|x| \leq n$, where $n \geq 1$, and let $w \in \Sigma^*$ be such that $|w| = n+1$. Then $w = ax$ with $|x| = n$ and $a \in \Sigma$.

$$ax \in W_{\rho(u), F}^{\mathcal{H}} \Rightarrow \text{[By Definition 6.1.3]}$$

$$x \in W_{\rho(y), F}^{\mathcal{H}} \wedge \rho(u)a \subseteq \rho(y) \Rightarrow \text{[By I.H. and since } \rho \text{ is induced by } \leq^r \text{]}$$

$$x \in y^{-1}L \wedge y \leq^r ua \Rightarrow \text{[By de Luca and Varricchio [1994]]}$$

$$x \in y^{-1}L \wedge y^{-1}L \subseteq (ua)^{-1}L \Rightarrow \text{[Since } x \in (ua)^{-1}L \Rightarrow ax \in u^{-1}L \text{]}$$

$$ax \in u^{-1}L .$$

We have shown that $\mathcal{H}$ is an RFA. Finally, we show that $\mathcal{L}(\mathcal{H}) = L$. First note that

$$\mathcal{L}(\mathcal{H}) = \bigcup_{\rho(u) \in I} W_{\rho(u), F}^{\mathcal{H}} = \bigcup_{\rho(u) \in I} u^{-1}L ,$$

where the first equality holds by definition of $\mathcal{L}(\mathcal{H})$ and the second by Equation (6.1). On one hand, we have that $\bigcup_{\rho(u) \in I} u^{-1}L \subseteq L$ since, by Definition 6.1.3, $\varepsilon \in \rho(u)$ for each $\rho(u) \in I$, hence $u \leq^r \varepsilon$ which, as shown by de Luca and Varricchio [1994], implies that $u^{-1}L \subseteq \varepsilon^{-1}L = L$.

Let us now show that $L \subseteq \bigcup_{\rho(u) \in I} u^{-1}L$. First, let us assume that $\rho(\varepsilon) \in I$. Then,

$$L = \varepsilon^{-1}L \subseteq \bigcup_{\rho(u) \in I} u^{-1}L .$$

Now suppose that $\rho(\varepsilon) \notin I$, i.e. $\rho(\varepsilon)$ is L -composite. Then,

$$L = \varepsilon^{-1}L = \bigcup_{u <^r \varepsilon} u^{-1}L = \bigcup_{\rho(u) \in I} u^{-1}L .$$

where the last equality follows from $\rho(u) \in I \Leftrightarrow \varepsilon \in \rho(u)$.

Given a regular language L and a left L -consistent quasiorder $\leq^\ell$, we can give a similar automata construction of a co-RFA that recognizes exactly L

Definition 6.1.5 (Automata construction $H^\ell(\leq^\ell, L)$). Let $\leq^\ell$ be a left quasiorder and let $L \subseteq \Sigma^*$ be a language. Define the automaton $H^\ell(\leq^\ell, L) \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, F \rangle$ where $Q = \{\rho_{\leq^\ell}(u) \mid u \in \Sigma^*, \rho_{\leq^\ell}(u) \text{ is } L\text{-prime}\}$, $I = \{\rho_{\leq^\ell}(u) \in Q \mid u \in L\}$, $F = \{\rho_{\leq^\ell}(u) \in Q \mid \varepsilon \in \rho_{\leq^\ell}(u)\}$, and $\delta(\rho_{\leq^\ell}(u), a) = \{\rho_{\leq^\ell}(v) \in Q \mid a \cdot \rho_{\leq^\ell}(v) \subseteq \rho_{\leq^\ell}(u)\}$ for all $\rho_{\leq^\ell}(u) \in Q, a \in \Sigma$. ■

Lemma 6.1.6. Let $L \subseteq \Sigma^*$ be a language and let $\leq^\ell$ be a left L -consistent quasiorder. Then $H^\ell(\leq^\ell, L)$ is a co-RFA such that $\mathcal{L}(H^\ell(\leq^\ell, L)) = L$.

Proof. To simplify the notation we denote $\rho_{\leq^\ell}$, the closure induced by the quasiorder $\leq^\ell$, simply by ρ . Let $\mathcal{H} = H^\ell(\leq^\ell, L) = \langle Q, \Sigma, \delta, I, F \rangle$. We first show that $\mathcal{H}$ is a co-RFA.

$$W_{I, \rho(u)}^{\mathcal{H}} = Lu^{-1}, \quad \text{for each } \rho(u) \in Q . \quad (6.2)$$

Let us prove that $w \in Lu^{-1} \Rightarrow w \in W_{I, \rho(u)}^{\mathcal{H}}$. We proceed by induction.

– *Base case:* Let $w = \varepsilon$. Then

$$\varepsilon \in Lu^{-1} \Rightarrow u \in L \Rightarrow \rho(u) \in I \Rightarrow \varepsilon \in W_{I, \rho(u)}^{\mathcal{H}} .$$

– *Inductive step:* Assume the hypothesis holds for all $x \in \Sigma^*$ with $|x| \leq n$, where $n \geq 1$, and let $w \in \Sigma^*$ be such that $|w| = n+1$. Then $w = xa$ with $|x| = n$ and $a \in \Sigma$.

$$\begin{aligned} xa &\in Lu^{-1} \Rightarrow \text{[By definition of quotient]} \\ x &\in L(au)^{-1} \Rightarrow \end{aligned}$$

[By Def. 6.1.2, $\rho(ua)$ is L -prime (so $z \stackrel{\text{def}}{=} au$) or $L(au)^{-1} = \bigcup_{x_i <^\ell au} Lx_i^{-1}$ (so $z \stackrel{\text{def}}{=} x_i$)

$\exists \rho(z) \in Q, x \in Lz^{-1} \wedge \rho(au) \subseteq \rho(z) \Rightarrow$ [By I.H., Lemma 6.1.1 and Def. 6.1.5]

$$\begin{aligned} x \in W_{I, \rho(z)}^{\mathcal{H}} \wedge \rho(u) \in \delta(\rho(z), a) &\Rightarrow \text{[By definition of } W_{S, T}] \\ xa &\in W_{I, \rho(u)}^{\mathcal{H}} . \end{aligned}$$

We now prove the other side of the implication, $w \in W_{I, \rho(u)}^{\mathcal{H}} \Rightarrow w \in Lu^{-1}$.

– *Base case:* Let $w = \varepsilon$. Then

$$\varepsilon \in W_{I, \rho(u)}^{\mathcal{H}} \Rightarrow \exists \rho(x) \in Q, x \in L \wedge \varepsilon \rho(u) \subseteq \rho(x) .$$

Since $\rho(L) = L$, we have that $\varepsilon u \in L$, hence $\varepsilon \in Lu^{-1}$.

– *Inductive step:* Assume the hypothesis holds for each $x \in \Sigma^*$ with $|x| \leq n$, where $n \geq 1$, and let $w \in \Sigma^*$ be such that $|w| = n+1$. Then $w = x \cdot a$ with $|x| = n$ and $a \in \Sigma$.

$$\begin{aligned} xa &\in W_{I, \rho(u)}^{\mathcal{H}} \Rightarrow \text{[By Definition 6.1.5]} \\ a \cdot \rho(u) \subseteq \rho(y) \wedge x \in W_{I, \rho(y)}^{\mathcal{H}} &\Rightarrow \text{[By I.H. and since } \rho \text{ is induced by } \leq^\ell] \\ y &\leq^\ell au \wedge x \in Ly^{-1} \Rightarrow \text{[By de Luca and Varricchio [1994]]} \\ Ly^{-1} \subseteq L(au)^{-1} \wedge x \in Ly^{-1} &\Rightarrow \text{[Since } x \in L(au)^{-1} \Rightarrow xa \in Lu^{-1}] \\ xa &\in u^{-1}L . \end{aligned}$$

We have shown that $\mathcal{H}$ is a co-RFA. Finally, we show that $\mathcal{L}(\mathcal{H}) = L$. First note that

$$\mathcal{L}(\mathcal{H}) = \bigcup_{\rho(u) \in F} W_{I, \rho(u)}^{\mathcal{H}} = \bigcup_{\rho(u) \in F} Lu^{-1} ,$$

where the first equality holds by definition of $\mathcal{L}(\mathcal{H})$ and the second by Equation (6.2). On one hand, we have that $\bigcup_{\rho(u) \in F} Lu^{-1} \subseteq L$ since, by Definition 6.1.5, $\varepsilon \in \rho(u)$ for each $\rho(u) \in F$, hence $u \leq^\ell \varepsilon$ which, as shown by de Luca and Varricchio [1994], implies that $Lu^{-1} \subseteq L\varepsilon^{-1} = L$.

Let us now show that $L \subseteq \bigcup_{\rho(u) \in F} Lu^{-1}$. First, let us assume that $\rho(\varepsilon) \in F$. Then,

$$L = L\varepsilon^{-1} \subseteq \bigcup_{\rho(u) \in F} Lu^{-1} .$$

Now suppose that $\rho(\varepsilon) \notin F$, i.e. $\rho(\varepsilon)$ is L -composite. Then,

$$L = L\varepsilon^{-1} = \bigcup_{u <^\ell \varepsilon} Lu^{-1} = \bigcup_{\rho(u) \in F} u^{-1}L .$$

where the last equality follows from $\rho(u) \in F \Leftrightarrow \varepsilon \in \rho(u)$.

Observe that the automaton $\mathcal{H}^r = H^r(\leq^r, L)$ (resp. $\mathcal{H}^\ell = H^\ell(\leq^\ell, L)$) is *finite*, since we assume $\leq^r$ (resp. $\leq^\ell$) induces a finite number of principals. Note also that $\mathcal{H}^r$ (resp. $\mathcal{H}^\ell$) possibly contains empty (resp. unreachable) states but no state is unreachable (resp. empty).

Moreover, notice that by keeping all principals of $\leq^r$ (resp. $\leq^\ell$) as states, instead of only the L -prime ones as in Definition 6.1.3 (resp. Definition 6.1.5), we would obtain an RFA (resp. a co-RFA) with (possibly) more states that also recognizes L .

Finally, Lemma 6.1.7 shows that $\mathcal{H}^\ell$ and $\mathcal{H}^r$ inherit the left-right duality between $\leq^\ell$ and $\leq^r$ through the reverse operation.

LEMMA 6.1.7. *Let $\leq^r$ and $\leq^\ell$ be a right and a left quasiorder, respectively, and let $L \subseteq \Sigma^*$ be a language. If the following property holds*

$$u \leq^r v \Leftrightarrow u^R \leq^\ell v^R \quad (6.3)$$

then $H^r(\leq^r, L)$ is isomorphic to $(H^\ell(\leq^\ell, L^R))^R$.

Proof. Let $H^r(\leq^r, L) = \langle Q, \Sigma, \delta, I, F \rangle$ and $(H^\ell(\leq^\ell, L^R))^R = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$. We will show that $H^r(\leq^r, L)$ is isomorphic to $(H^\ell(\leq^\ell, L^R))^R$.

Let $\varphi : Q \rightarrow \tilde{Q}$ be a mapping assigning to each state $\rho_{\leq^r}(u) \in Q$ with $u \in \Sigma^*$, the state $\rho_{\leq^\ell}(u^R) \in \tilde{Q}$. Next, we show that φ is an NFA isomorphism between $H^r(\leq^r, L)$ and $(H^\ell(\leq^\ell, L^R))^R$.

Observe that:

$$\begin{aligned} u^{-1}L &= \bigcup_{x <^r u} x^{-1}L \Leftrightarrow [\text{Since } \left(\bigcup S_i\right)^R = \bigcup S_i^R] \\ (u^{-1}L)^R &= \bigcup_{x <^r u} (x^{-1}L)^R \Leftrightarrow [\text{Since } (u^{-1}L)^R = L^R(u^R)^{-1}] \\ L^R(u^R)^{-1} &= \bigcup_{x <^r u} L^R(x^R)^{-1} \Leftrightarrow [\text{By Equation (6.3)}] \\ L^R(u^R)^{-1} &= \bigcup_{x^R <^\ell u^R} L^R(x^R)^{-1} . \end{aligned}$$

Therefore $\rho_{\leq^r}(u)$ is L -composite iff $\rho_{\leq^\ell}(u^R)$ is L^R -composite, hence $\varphi(Q) = \tilde{Q}$.

Since

$$\varepsilon \in \rho_{\leq^r}(u) \Leftrightarrow u \leq^r \varepsilon \Leftrightarrow u^r \leq^\ell \varepsilon \Leftrightarrow \varepsilon \in \rho_{\leq^\ell}(u^R) ,$$

we have that $\rho_{\leq^r}(u)$ is an initial state of $H^r(\leq^r, L)$ iff $\rho_{\leq^\ell}(u^R)$ is a final state of $H^\ell(\leq^\ell, L^R)$, i.e. an initial state of $(H^\ell(\leq^\ell, L^R))^R$. Therefore, $\varphi(I) = \tilde{I}$.

Since

$$\rho_{\leq^r}(u) \subseteq L \Leftrightarrow u \in L \Leftrightarrow u^r \in L^R ,$$

we have that $\rho_{\leq^r}(u)$ is a final state of $H^r(\leq^r, L)$ iff $\rho_{\leq^\ell}(u^R)$ is an initial state of $H^\ell(\leq^\ell, L^R)$, i.e. a final state of $(H^\ell(\leq^\ell, L^R))^R$. Therefore, $\varphi(F) = \widetilde{F}$.

It remains to show that $q' \in \delta(q, a) \Leftrightarrow \varphi(q') \in \widetilde{\delta}(\varphi(q), a)$, for all $q, q' \in Q$ and $a \in \Sigma$. Assume that $q = \rho_{\leq^r}(u)$ for some $u \in \Sigma^*$, $q' = \rho_{\leq^r}(v)$ for some $v \in \Sigma^*$ and $q' \in \delta(q, a)$ with $a \in \Sigma$. Then,

$$\begin{aligned} \rho_{\leq^r}(v) \in \delta(\rho_{\leq^r}(u), a) &\Leftrightarrow \text{[By Definition 6.1.3]} \\ \rho_{\leq^r}(u)a \subseteq \rho_{\leq^r}(v) &\Leftrightarrow \text{[By definition of } \rho_{\leq^r} \text{ and Lemma 6.1.1]} \\ v \leq^r ua &\Leftrightarrow \text{[By Equation (6.3) and } (ua)^R = au^R] \\ v^r \leq^\ell au^R &\Leftrightarrow \text{[By definition of } \rho_{\leq^\ell} \text{ and Lemma 6.1.1]} \\ a\rho_{\leq^\ell}(u^R) \subseteq \rho_{\leq^\ell}(v^R) &\Leftrightarrow \text{[By Definition 6.1.5]} \\ \rho_{\leq^\ell}(v^R) \in \widetilde{\delta}(\rho_{\leq^\ell}(u^R), a) &\Leftrightarrow \text{[Definition of } q, q' \text{ and } \varphi] \\ \varphi(q') \in \widetilde{\delta}(\varphi(q), a) &. \end{aligned}$$

6.1.1 On the Size of $H^r(\leq^r, L)$ and $H^\ell(\leq^\ell, L)$

We conclude this section with a note on the sizes of the automata constructions $H^r(\leq^r, L)$ and $H^\ell(\leq^\ell, L)$ when applied to quasiorders satisfying $\leq_1^r \subseteq \leq_2^r$ and $\leq_1^\ell \subseteq \leq_2^\ell$, respectively.

The following result establishes a relationship between the L -composite principals for two comparable right quasiorders $\leq_1^r \subseteq \leq_2^r$. This result is then used in Theorem 6.1.9 to show that the number of L -prime principals induced by $\leq_1^r$ is greater or equal than the number number of L -prime principals induced by $\leq_2^r$.

As a consequence, if $\leq_1^r \subseteq \leq_2^r$ then the automaton $H^r(\leq_1^r, L)$ has, at least, as many states as $H^r(\leq_2^r, L)$. The same holds for left quasiorders and H^ℓ .

LEMMA 6.1.8. *Let $L \subseteq \Sigma^*$ be a regular language and let $u \in \Sigma^*$. Let $\leq_1^r$ and $\leq_2^r$ be two L -consistent right quasiorders such that $\leq_1^r \subseteq \leq_2^r$. Then*

$$\rho_{\leq_1^r}(u) \text{ is } L\text{-composite} \Rightarrow \left(\rho_{\leq_2^r}(u) \text{ is } L\text{-composite} \vee \exists x <_1^r u, \rho_{\leq_2^r}(u) = \rho_{\leq_2^r}(x) \right) .$$

Similarly holds for left quasiorders.

Proof. Let $u \in \Sigma^*$ be such that $\rho_{\leq_1^r}(u)$ is L -composite. Then we have that $u^{-1}L = \bigcup_{x \in \Sigma^*, x <_1^r u} x^{-1}L$. On the other hand, since $\leq_2^r$ is a right L -consistent quasiorder, we have that $\leq_2^r \subseteq \leq_1^r$, as shown by [de Luca and Varricchio \[1994\]](#). Therefore $u^{-1}L \supseteq \bigcup_{x \in \Sigma^*, x <_2^r u} x^{-1}L$. There are now two possibilities:

- For all $x \in \Sigma^*$ such that $x <_1^r u$ we have that $x <_2^r u$. In that case we have that $u^{-1}L = \bigcup_{x \in \Sigma^*, x <_2^r u} x^{-1}L$, hence $\rho_{\leq_2^r}(u)$ is L -composite.
- There exists $x \in \Sigma^*$ such that $x <_1^r u$, hence $x \leq_2^r u$, but $x \not<_2^r u$. In that case, it follows that $\rho_{\leq_2^r}(x) = \rho_{\leq_2^r}(u)$.

The proof for left quasiorders is symmetric.

THEOREM 6.1.9. *Let $L \subseteq \Sigma^*$ and let $\leq_1$ and $\leq_2$ be two right or two left L -consistent quasiorders such that $\leq_1 \subseteq \leq_2$. Then*

$$|\{\rho_{\leq_1}(u) \mid u \in \Sigma^* \wedge \rho_{\leq_1}(u) \text{ is } L\text{-prime}\}| \geq |\{\rho_{\leq_2}(u) \mid u \in \Sigma^* \wedge \rho_{\leq_2}(u) \text{ is } L\text{-prime}\}|$$

Proof. We proceed by showing that for every L -prime $\rho_{\leq_2}(u)$ there exists an L -prime $\rho_{\leq_1}(x)$ such that $\rho_{\leq_2}(x) = \rho_{\leq_2}(u)$. Clearly, this entails that there are, at least, as many L -prime principals for $\leq_1$ as there are for $\leq_2$.

Let $\rho_{\leq_2}(u)$ be L -prime.

If $\rho_{\leq_1}(u)$ is L -prime, we are done. Otherwise, by Lemma 6.1.8, we have that there exists $x <_1 u$ such that $\rho_{\leq_2}(u) = \rho_{\leq_2}(x)$.

We repeat the reasoning with x . If $\rho_{\leq_1}(x)$ is L -prime, we are done. Otherwise, there exists $x_1 <_1 x$ such that $\rho_{\leq_2}(u) = \rho_{\leq_2}(x) = \rho_{\leq_2}(x_1)$.

Since $\leq_1$ induces finitely many principals, there are no infinite strictly descending chains and, therefore, there exists x_n such that $\rho_{\leq_2}(u) = \rho_{\leq_2}(x) = \rho_{\leq_2}(x_1) = \dots = \rho_{\leq_2}(x_n)$ and $\rho_{\leq_1}(x_n)$ is L -prime.

6.2 Language-based Quasiorders and their Approximation using NFAs

In this section we instantiate our automata constructions using two classes of quasiorders, namely, the so-called *Nerode's* quasiorders [[de Luca and Varricchio 1994](#)], whose definition is based on a given regular language; and the *automata-based* quasiorders, whose definition is based on a finite representation of the language, i.e., an automaton.

Both quasiorders have been used previously in Chapter 4 in order to derive algorithms for solving the language inclusion problem between regular languages.

We recall their definitions next:

$$u \leqslant_L^r v \stackrel{\text{def}}{\Leftrightarrow} u^{-1}L \subseteq v^{-1}L \quad \text{Right-language-based Quasiorder} \quad (6.4)$$

$$u \leqslant_L^\ell v \stackrel{\text{def}}{\Leftrightarrow} Lu^{-1} \subseteq Lv^{-1} \quad \text{Left-language-based Quasiorder} \quad (6.5)$$

$$u \leqslant_N^r v \stackrel{\text{def}}{\Leftrightarrow} \text{post}_u^N(I) \subseteq \text{post}_v^N(I) \quad \text{Right-Automata-based Quasiorder} \quad (6.6)$$

$$u \leqslant_N^\ell v \stackrel{\text{def}}{\Leftrightarrow} \text{pre}_u^N(F) \subseteq \text{pre}_v^N(F) \quad \text{Left-Automata-based Quasiorder} \quad (6.7)$$

As explained in Chapter 4, [de Luca and Varricchio \[2011\]](#) showed that for every regular language L there exists a finite number of quotients $u^{-1}L$ and, therefore, $\leqslant_L^r$ and $\leqslant_L^\ell$ are well-quasiorders. On the other hand, the automata-based quasiorders are also well-quasiorders. Therefore, all the quasiorders defined above induce a finite number of principals.

Remark 6.2.1. The pairs of quasiorders $\leqslant_L^r - \leqslant_L^\ell$ and $\leqslant_N^r - \leqslant_N^\ell$ are dual, i.e.

$$u \leqslant_L^r v \Leftrightarrow u^R \leqslant_L^\ell v^R \quad \text{and} \quad u \leqslant_N^r v \Leftrightarrow u^R \leqslant_N^\ell v^R .$$

The following result shows that the principals of $\leqslant_N^r$ and $\leqslant_N^\ell$ can be described, respectively, as intersections of left and right languages of the states of N while the principals of $\leqslant_L^r$ and $\leqslant_L^\ell$ can be described as intersections of left and right quotients of L .

LEMMA 6.2.2. Let $N = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA with $\mathcal{L}(N) = L$. Then, for every $u \in \Sigma^*$,

$$\begin{aligned} \rho_{\leqslant_N^r}(u) &= \bigcap_{q \in \text{post}_u^N(I)} W_{I,q}^N & \rho_{\leqslant_L^r}(u) &= \bigcap_{w \in \Sigma^*, w \in u^{-1}L} Lw^{-1} \\ \rho_{\leqslant_N^\ell}(u) &= \bigcap_{q \in \text{pre}_u^N(F)} W_{q,F}^N & \rho_{\leqslant_L^\ell}(u) &= \bigcap_{w \in \Sigma^*, w \in Lu^{-1}L} w^{-1}L . \end{aligned}$$

Proof. We prove the lemma for the principals induced by $\leqslant_L^r$ and $\leqslant_N^r$. The proofs for the left quasiorders are symmetric.

For each $u \in \Sigma^*$ we have that

$$\begin{aligned} \rho_{\leqslant_N^r}(u) &= \quad [\text{By definition of } \rho_{\leqslant_N^r}] \\ \{v \in \Sigma^* \mid \text{post}_u^N(I) \subseteq \text{post}_v^N(I)\} &= \quad [\text{By definition of set inclusion}] \\ \{v \in \Sigma^* \mid \forall q \in \text{post}_u^N(I), q \in \text{post}_v^N(I)\} &= \quad [\text{Since } q \in \text{post}_v^N(I) \Leftrightarrow v \in W_{I,q}^N] \\ \{v \in \Sigma^* \mid \forall q \in \text{post}_u^N(I), v \in W_{I,q}^N\} &= \quad [\text{By definition of intersection}] \\ \bigcap_{q \in \text{post}_u^N(I)} W_{I,q}^N . \end{aligned}$$

On the other hand,

$$\begin{aligned}
 v &\in \bigcap_{w \in \Sigma^*, w \in u^{-1}L} Lw^{-1} \Leftrightarrow \text{[By definition of intersection]} \\
 \forall w \in \Sigma^*, w \in u^{-1}L &\Rightarrow v \in Lw^{-1} \Leftrightarrow \text{[Since } \forall x, y \in \Sigma^*, x \in Ly^{-1} \Leftrightarrow y \in x^{-1}L\text{]} \\
 \forall w \in \Sigma^*, w \in u^{-1}L &\Rightarrow w \in v^{-1}L \Leftrightarrow \text{[By definition of set inclusion]} \\
 u^{-1}L &\subseteq v^{-1}L \Leftrightarrow \text{[By definition of } \rho_{\leq_L^\ell}(u)\text{]} \\
 v &\in \rho_{\leq_L^r}(u)
 \end{aligned}$$

As shown by Lemma 4.3.7, given an NFA $\mathcal{N}$ with $L = \mathcal{L}(\mathcal{N})$, the quasiorders $\leq_L^r$ and $\leq_{\mathcal{N}}^r$ are right L -consistent, while the quasiorders $\leq_L^\ell$ and $\leq_{\mathcal{N}}^\ell$ are left L -consistent. Therefore, by Lemma 6.1.4 and 6.1.6, our automata constructions applied to these quasiorders yield automata for L .

Finally, recall that, as shown by de Luca and Varricchio [1994], $\leq_{\mathcal{N}}^r$ is finer than $\leq_L^r$, i.e., $\leq_{\mathcal{N}}^r \subseteq \leq_L^r$. In that sense we say $\leq_{\mathcal{N}}^r$ *approximates* $\leq_L^r$. As the following lemma shows, the approximation is precise, i.e., $\leq_{\mathcal{N}}^r = \leq_L^r$, whenever $\mathcal{N}$ is a co-RFA.

LEMMA 6.2.3. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be a co-RFA with $L = \mathcal{L}(\mathcal{N})$. Then $\leq_L^r = \leq_{\mathcal{N}}^r$. Similarly, if $\mathcal{N}$ is an RFA then $\leq_L^\ell = \leq_{\mathcal{N}}^\ell$.*

Proof. It is straightforward to check that the following implication holds for every NFA $\mathcal{N}$ and $u, v \in \Sigma^*$.

$$\text{post}_u^{\mathcal{N}}(I) \subseteq \text{post}_v^{\mathcal{N}}(I) \Rightarrow W_{\text{post}_u^{\mathcal{N}}(I), F}^{\mathcal{N}} \subseteq W_{\text{post}_v^{\mathcal{N}}(I), F}^{\mathcal{N}}$$

Next we show that the reverse implication also holds when $\mathcal{N}$ is a co-RFA. Let $u, v \in \Sigma^*$ be such that $W_{\text{post}_u^{\mathcal{N}}(I), F}^{\mathcal{N}} \subseteq W_{\text{post}_v^{\mathcal{N}}(I), F}^{\mathcal{N}}$. Then,

$$\begin{aligned}
 q &\in \text{post}_u^{\mathcal{N}}(I) \Rightarrow \text{[Since } \mathcal{N} \text{ is co-RFA]} \\
 \exists x \in \Sigma^*, u &\in W_{I, q} = Lx^{-1} \Rightarrow \text{[Since } u \in Lx^{-1} \Rightarrow x \in u^{-1}L\text{]} \\
 x &\in W_{\text{post}_u^{\mathcal{N}}(I), F}^{\mathcal{N}} \Rightarrow \text{[Since } W_{\text{post}_u^{\mathcal{N}}(I), F}^{\mathcal{N}} \subseteq W_{\text{post}_v^{\mathcal{N}}(I), F}^{\mathcal{N}}\text{]} \\
 x &\in W_{\text{post}_v^{\mathcal{N}}(I), F}^{\mathcal{N}} \Rightarrow \text{[By definition of } W_{S, T}^{\mathcal{N}}\text{]} \\
 \exists q' \in Q, x &\in W_{q', F} \wedge v \in W_{I, q'} \Rightarrow \text{[Since } x \in W_{q', F} \Rightarrow W_{I, q'} \subseteq Lx^{-1}\text{]} \\
 v &\in Lx^{-1} \Rightarrow \text{[Since } Lx^{-1} = W_{I, q}\text{]} \\
 v &\in W_{I, q} \Rightarrow \text{[By definition of } \text{post}_v^{\mathcal{N}}(I)\text{]} \\
 q &\in \text{post}_v^{\mathcal{N}}(I) .
 \end{aligned}$$

Therefore, $W_{\text{post}_u^N(I), F}^N \subseteq W_{\text{post}_v^N(I), F}^N \Rightarrow \text{post}_u^N(I) \subseteq \text{post}_v^N(I)$.
The proof for RFAs and left quasiorders is symmetric.

Finally, the following lemma shows that, for the Nerode's quasiorders, the L -composite principals can be described as intersections of L -prime principals.

LEMMA 6.2.4. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA with $\mathcal{L}(\mathcal{N}) = L$. Then,*

$$u^{-1}L = \bigcup_{x \in \Sigma^*, x <_L^r u} x^{-1}L \implies \rho_{\leq_L^r}(u) = \bigcap_{x \in \Sigma^*, x <_L^r u} \rho_{\leq_L^r}(x) . \quad (6.8)$$

Similarly holds for the left Nerode's quasiorder $\leq_L^\ell$.

Proof. Observe that the inclusion $\rho_{\leq_L^r}(u) \subseteq \bigcap_{x \in \Sigma^*, x <_L^r u} \rho_{\leq_L^r}(x)$ always holds since $x <_L^r u \implies \rho_{\leq_L^r}(u) \subseteq \rho_{\leq_L^r}(x)$. Next, we prove the reverse inclusion.

Let $w \in \bigcap_{x \in \Sigma^*, x <_L^r u} \rho_{\leq_L^r}(x)$ and assume that the left hand side of Equation (6.8) holds. Then, by definition of intersection and $\rho_{\leq_L^r}$, we have that $x \leq_L^r w$ for every $x \in \Sigma^*$ such that $x <_L^r u$, i.e., $x^{-1}L \subseteq w^{-1}L$ for every $x \in \Sigma^*$ such that $x^{-1}L \subsetneq u^{-1}L$. Since, by hypothesis, $u^{-1}L = \bigcup_{x \in \Sigma^*, x <_L^r u} x^{-1}L$, it follows that $u^{-1}L \subseteq w^{-1}L$ and, therefore, $w \in \rho_{\leq_L^r}(u)$.

We conclude that $\bigcap_{x \in \Sigma^*, x <_L^r u} \rho_{\leq_L^r}(x) \subseteq \rho_{\leq_L^r}(u)$.

6.2.1 Automata Constructions

In what follows, we will use Can and Res to denote the construction H when applied, respectively, to the language-based quasiorders induced by a regular language and the automata-based quasiorders induced by an NFA.

Definition 6.2.5 (Res and Can). *Let $\mathcal{N}$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. Define:*

$$\begin{aligned} \text{Can}^r(L) &\stackrel{\text{def}}{=} H^r(\leq_L^r, L) & \text{Res}^r(\mathcal{N}) &\stackrel{\text{def}}{=} H^r(\leq_{\mathcal{N}}^r, L) \\ \text{Can}^\ell(L) &\stackrel{\text{def}}{=} H^\ell(\leq_L^\ell, L) & \text{Res}^\ell(\mathcal{N}) &\stackrel{\text{def}}{=} H^\ell(\leq_{\mathcal{N}}^\ell, L) . \quad \blacksquare \end{aligned}$$

Given an NFA $\mathcal{N}$ generating the language $L = \mathcal{L}(\mathcal{N})$, all constructions in the above definition yield automata generating L . However, while the constructions using the right quasiorders result in RFAs, those using left quasiorders result in co-RFAs. Furthermore, it follows from Remark 6.2.1 and Lemma 6.1.7 that $\text{Can}^\ell(L)$ is isomorphic to $(\text{Can}^r(L^R))^R$ and $\text{Res}^\ell(\mathcal{N})$ is isomorphic to $(\text{Res}^r(\mathcal{N}^R))^R$.

It follows from Theorem 6.1.9 that the automata $\text{Res}^r(\mathcal{N})$ and $\text{Res}^\ell(\mathcal{N})$ have, at least, as many states as $\text{Can}^r(L)$ and $\text{Can}^\ell(L)$, respectively. Intuitively, $\text{Can}^r(L)$

is the minimal RFA for L , i.e. it is isomorphic to the canonical RFA for L , since, as shown by [de Luca and Varricchio \[1994\]](#), $\leq_L^r$ is the coarsest right L -consistent quasiorder. On the other hand, as we shall see in [Example 6.2.9](#), $\text{Res}^r(\mathcal{N})$ is a sub-automaton of $\mathcal{N}^{\text{res}}$ [[Denis et al. 2002](#)] for every NFA $\mathcal{N}$.

Finally, it follows from [Lemma 6.2.3](#) that residualizing (Res^r) a co-RFA (for instance, $\text{Res}^\ell(\mathcal{N})$) results in the canonical RFA for $\mathcal{L}(\mathcal{N})$ ($\text{Can}^r(\mathcal{L}(\mathcal{N}))$).

We formalize all these notions in [Theorem 6.2.6](#).

THEOREM 6.2.6. *Let $\mathcal{N}$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. Then the following hold:*

- (a) $\mathcal{L}(\text{Can}^r(L)) = \mathcal{L}(\text{Can}^\ell(L)) = L = \mathcal{L}(\text{Res}^r(\mathcal{N})) = \mathcal{L}(\text{Res}^\ell(\mathcal{N}))$.
- (b) $\text{Can}^\ell(L)$ is isomorphic to $(\text{Can}^r(L^R))^R$.
- (c) $\text{Res}^\ell(\mathcal{N})$ is isomorphic to $(\text{Res}^r(\mathcal{N}^R))^R$.
- (d) $\text{Can}^r(L)$ is isomorphic to the canonical RFA for L .
- (e) $\text{Res}^r(\mathcal{N})$ is isomorphic to a sub-automaton of $\mathcal{N}^{\text{res}}$.
- (f) $\text{Res}^r(\text{Res}^\ell(\mathcal{N}))$ is isomorphic to $\text{Can}^r(L)$.

Proof. In the following, let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$.

- (a) By [Definition 6.2.5](#), $\text{Can}^r(L) = H^r(\leq_L^r, L)$ and $\text{Res}^r(\mathcal{N}) = H^r(\leq_{\mathcal{N}}^r, L)$. On the other hand, by [Lemma 6.1.4](#), $\mathcal{L}(H^r(\leq_L^r, L)) = \mathcal{L}(H^r(\leq_{\mathcal{N}}^r, L)) = L$. Therefore, $\mathcal{L}(\text{Can}^r(L)) = \mathcal{L}(\text{Res}^r(L)) = L$.
Similarly, it follows from [Lemma 6.1.6](#) that $\mathcal{L}(\text{Can}^\ell(L)) = \mathcal{L}(\text{Res}^\ell(L)) = L$.
- (b) For each $u, v \in \Sigma^*$:

$$\begin{aligned}
 u &\leq_L^\ell v \Leftrightarrow && [\text{By Definition (6.5)}] \\
 u^{-1}L &\subseteq v^{-1}L \Leftrightarrow && [A \subseteq B \Leftrightarrow A^R \subseteq B^R] \\
 (u^{-1}L)^R &\subseteq (v^{-1}L)^R \Leftrightarrow && [\text{Since } (u^{-1}L)^R = L^R(u^R)^{-1}] \\
 L^R(u^R)^{-1} &\subseteq L^R(v^R)^{-1} \Leftrightarrow && [\text{By Definition (6.4)}] \\
 u^R &\leq_{L^R}^r v^R.
 \end{aligned}$$

Therefore, by [Lemma 6.1.7](#), $\text{Can}^\ell(L)$ is isomorphic to $(\text{Can}^r(L^R))^R$.

- (c) For each $u, v \in \Sigma^*$:

$$\begin{aligned}
 u &\leq_{\mathcal{N}}^\ell v \Leftrightarrow && [\text{By Definition (6.7)}] \\
 \text{pre}_u^{\mathcal{N}^R}(F) &\subseteq \text{pre}_v^{\mathcal{N}^R}(F) \Leftrightarrow && [\text{Since } q \in \text{pre}_x^{\mathcal{N}^R}(F) \text{ iff } q \in \text{post}_{x^R}^{\mathcal{N}}(I)] \\
 \text{post}_{u^R}^{\mathcal{N}}(I) &\subseteq \text{post}_{v^R}^{\mathcal{N}}(I) \Leftrightarrow && [\text{By Definition (6.6)}] \\
 u^R &\leq_{\mathcal{N}^R}^\ell v^R.
 \end{aligned}$$

It follows from Lemma 6.1.7 that $\text{Res}^\ell(\mathcal{N})$ is isomorphic to $\text{Res}^r(\mathcal{N}^R)^R$.

- (d) Let ρ be the closure induced by $\leq_L^r$. Let $C = \langle \tilde{Q}, \Sigma, \eta, \tilde{I}, \tilde{F} \rangle$ be the canonical RFA for L and let $\text{Can}^r(L) = \langle Q, \Sigma, \delta, I, F \rangle$. Let $\varphi : \tilde{Q} \rightarrow Q$ be the mapping assigning to each state $\tilde{q}_i \in \tilde{Q}$ of the form $u^{-1}L$, the state $\rho(u) \in Q$, with $u \in \Sigma^*$.

We show that φ is an NFA isomorphism between C and $\text{Can}^r(L)$.

Since

$$u^{-1}L \subseteq L \Leftrightarrow u \leq_L^r \varepsilon \Leftrightarrow \varepsilon \in \rho(u) ,$$

we have that $u^{-1}L$ is an initial state of C iff $\rho(u)$ is an initial state of $\text{Can}^r(L)$, hence $\varphi(\tilde{I}) = I$.

On the other hand, since

$$\varepsilon \in u^{-1}L \Leftrightarrow u \in L ,$$

we have that $u^{-1}L$ is a final state of C iff $\rho(u)$ is a final state of $\text{Can}^r(L)$, hence $\varphi(\tilde{F}) = F$.

Moreover, since

$$\rho(u) \cdot a \subseteq \rho(v) \Leftrightarrow v \leq_L^r ua \Leftrightarrow v^{-1}L \subseteq (ua)^{-1}L ,$$

we have that $v^{-1}L = \eta(u^{-1}L, a)$ if and only if $\rho(v) \in \delta(\rho(u), a)$, for all $u^{-1}L, v^{-1}L \in \tilde{q}$ and $a \in \Sigma$.

Finally, we need to show that $\forall u \in \Sigma^*, \rho(u) \in Q \Leftrightarrow \exists q_i \in \tilde{Q}, q_i = u^{-1}L$. Observe that:

$$\begin{aligned} u^{-1}L &= \bigcup_{x <_L^r u} u^{-1}L \Leftrightarrow \quad [\text{By Definition (6.4)}] \\ u^{-1}L &= \bigcup_{x^{-1}L \subset u^{-1}L} x^{-1}L . \end{aligned}$$

Therefore, $\forall u \in \Sigma^*, \rho(u)$ is L -prime $\Leftrightarrow u^{-1}L$ is prime, hence $\varphi(\tilde{Q}) = Q$.

- (e) Recall that $\mathcal{N}^{\text{res}} = \langle Q_r, \Sigma, \delta_r, I_r, F_r \rangle$ is the RFA built by the residualization operation defined by Denis et al. [2002] (see Chapter 3). Let $\text{Res}^r(\mathcal{N}) = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$.

Next, we show that there is a surjective mapping φ that associates states and transitions of $\text{Res}^r(\mathcal{N})$ with states and transitions of $\mathcal{N}^{\text{res}}$. Moreover, if $q \in \tilde{Q}$ is initial (resp. final) then $\varphi(q) \in Q_r$ is initial (resp. final) and

$q' \in \widetilde{\delta}(q, a) \Leftrightarrow \varphi(q') \in \delta_r(\varphi(q), a)$. In this way, we conclude that $\text{Res}^r(\mathcal{N})$ is isomorphic to a sub-automaton of $\mathcal{N}^{\text{res}}$.

Finally, since $\mathcal{L}(\mathcal{N}^{\text{res}}) = \mathcal{L}(\mathcal{N})$ then it follows from Lemma 6.1.4 that $\mathcal{L}(\mathcal{N}^{\text{res}}) = \mathcal{L}(\mathcal{N}) = \mathcal{L}(\text{Res}^r(\mathcal{N}))$.

Let ρ be the closure induced by $\leq_N^r$ and let $\varphi : \widetilde{Q} \rightarrow Q_r$ be the mapping assigning to each state $\rho(u) \in \widetilde{Q}$, the set $\text{post}_u^N(I) \in Q_r$ with $u \in \Sigma^*$.

Since

$$\varepsilon \in \rho(u) \Leftrightarrow u \leq_N^r \varepsilon \Leftrightarrow \text{post}_u^N(I) \subseteq \text{post}_\varepsilon^N(I)$$

The initial states of $\text{Res}^r(\mathcal{N})$ are mapped into the set the initial states of $\mathcal{N}^{\text{res}}$, hence $\varphi(\widetilde{I}) = I_r$.

On the other hand, since

$$\rho(u) \subseteq L \Leftrightarrow u \in L \Leftrightarrow (\text{post}_u^N(I) \cap F) \neq \emptyset ,$$

we have that the final states of $\text{Res}^r(\mathcal{N})$, are mapped to the final states of $\mathcal{N}^{\text{res}}$, hence $\varphi(\widetilde{F}) = F_r$.

Moreover, since

$$\rho(u) \cdot a \subseteq \rho(v) \Leftrightarrow v \leq_N^r ua \Leftrightarrow \text{post}_v^N(I) \subseteq \text{post}_{ua}^N(I) ,$$

it follows that $\forall u, v \in \Sigma^*$ such that $\text{post}_u^N(I), \text{post}_v^N(I) \in Q_r$, we have

$$\text{post}_v^N(I) \in \delta_r(\text{post}_u^N(I), a) \Leftrightarrow \rho(v) \in \widetilde{\delta}(\rho(u), a) .$$

Finally, we show that $\forall u \in \Sigma^*$, $\rho(u) \in \widetilde{Q} \Rightarrow \text{post}_u^N(I) \in Q_r$. By definition of $\widetilde{Q}$ and Q_r , this is equivalent to showing that for every word $u \in \Sigma^*$, if $\text{post}_u^N(I)$ is coverable then $\rho(u)$ is L -composite. Observe that:

$$\begin{aligned} \text{post}_u^N(I) &= \bigcup_{\text{post}_x^N(I) \subset \text{post}_u^N(I)} \text{post}_x^N(I) \Leftrightarrow [x <_N^r u \Leftrightarrow \text{post}_x^N(I) \subset \text{post}_u^N(I)] \\ \text{post}_u^N(I) &= \bigcup_{x <_N^r u} \text{post}_x^N(I) \Rightarrow [\text{Since } W_{\text{post}_u^N(I), T}^N = u^{-1}L] \\ u^{-1}L &= \bigcup_{x <_N^r u} x^{-1}L . \end{aligned}$$

It follows that if $\text{post}_u^N(I)$ is coverable then $\rho(u)$ is L -composite, hence $\varphi(\widetilde{Q}) \subseteq Q_r$.

- (f) As shown by Lemma 6.1.6, $\text{Res}^\ell(\mathcal{N})$ is a co-RFA with $\mathcal{L}(\text{Res}^\ell(\mathcal{N})) = \mathcal{L}(\mathcal{N})$. Therefore, it follows from Lemma 6.2.3 that $\text{Res}^r(\text{Res}^\ell(\mathcal{N}))$ is isomorphic to $\text{Can}^r(\mathcal{L}(\text{Res}^\ell(\mathcal{N}))) = \text{Can}^r(\mathcal{L}(\mathcal{N}))$.

Figure 6.1 summarizes all the connections between the automata constructions given in Definition 6.2.5.

The upper part of the diagram follows from Theorem 6.2.6 (f), the squares follow from Theorem 6.2.6 (c) and the bottom curved arc follows from Theorem 6.2.6 (b). Incidentally, the diagram shows a new relation which is a consequence of the left-right dualities between $\leq_L^\ell$ and $\leq_L^r$, and $\leq_N^\ell$ and $\leq_N^r$: $\text{Can}^\ell(\mathcal{L}(\mathcal{N}^R))$ is isomorphic to $\text{Res}^\ell(\text{Res}^r(\mathcal{N}^R))$.

Figure 6.1: Relations between the constructions Res^ℓ , Res^r , Can^ℓ and Can^r . Note that constructions Can^r and Can^ℓ are applied to the language generated by the automaton in the origin of the labeled arrow while constructions Res^r and Res^ℓ are applied directly to the automaton.

It is well-known that determinizing a deterministic automata yields the same automaton, i.e. $\mathcal{D}^D = \mathcal{D}$ for every DFA $\mathcal{D}$. As a consequence, determinizing twice and automaton is the same as doing it once, i.e. $(\mathcal{N}^D)^D = \mathcal{N}^D$. However, it is not clear that the same holds for our residualization operation, i.e. it is not clear whether $\text{Res}^r(\text{Res}^r(\mathcal{N})) = \text{Res}^r(\mathcal{N})$.

The following lemma gives a sufficient condition on an RFA $\mathcal{H}$ built with our right automata construction so that applying our residualization operation results in the same automaton, i.e. $\text{Res}^r(\mathcal{H}) = \mathcal{H}$. In particular, we find that $\text{Can}^r(L)$ is invariant to our residualization operation Res^r .

LEMMA 6.2.7. *Let L be a regular language and let $\leq^r$ be a right L -consistent quasiorder. Let $\mathcal{H} = \text{H}^r(\leq^r, L)$. If $\mathcal{H}$ is a strongly consistent RFA then $\leq_{\mathcal{H}}^r = \leq^r$.*

Proof. Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ and $\mathcal{H} = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$. As shown by Lemma 6.1.4, $\mathcal{H} = \text{H}^r(\leq^r, L)$ is an RFA generating L , hence each state of $\mathcal{H}$ is an L -prime principal $\rho_{\leq^r}(u)$ whose right language is the quotient $u^{-1}L$ for some $u \in \Sigma^*$.

Observe that, by definition,

$$\leq_{\mathcal{H}}^r = \leq^r \Leftrightarrow \left(\forall u, v \in \Sigma^*, \text{post}_u^{\mathcal{H}}(\tilde{I}) \subseteq \text{post}_v^{\mathcal{H}}(\tilde{I}) \Leftrightarrow u \leq^r v \right) .$$

Next we show that:

$$\text{post}_u^{\mathcal{H}}(\tilde{I}) = \{\rho \leq^r(x) \in \tilde{Q} \mid x \leq^r u\} . \quad (6.9)$$

First, we prove that $\text{post}_u^{\mathcal{H}}(\tilde{I}) \subseteq \{\rho \leq^r(x) \in \tilde{Q} \mid x \leq^r u\}$. To simplify the notation, let ρ denote $\rho \leq^r$.

$$\begin{aligned} \rho(x) \in \text{post}_u^{\mathcal{H}}(\tilde{I}) &\Leftrightarrow [\text{By definition of } \text{post}_u^{\mathcal{H}}(\tilde{I})] \\ \exists \rho(x_0) \in \tilde{I}, u \in W_{\rho(x_0), \rho(x)}^{\mathcal{H}} &\Rightarrow [\text{By Definition 6.1.3}] \\ \exists \rho(x_0) \in \tilde{Q}, \varepsilon \in \rho(x_0) \wedge \rho(x_0) \cdot u \subseteq \rho(x) &\Leftrightarrow [\text{By definition of } \rho] \\ \exists \rho(x_0) \in \tilde{Q}, x_0 \leq^r \varepsilon \wedge x \leq^r u \cdot x_0 &\Rightarrow [\text{By mon. and trans. of } \leq^r] \\ x \leq^r u . \end{aligned}$$

We now prove the reverse inclusion. Let $\rho(u), \rho(x) \in \tilde{Q}$ be such that $x \leq^r u$. Then,

$$\begin{aligned} \rho(u) \in \tilde{Q} &\Rightarrow [\text{By Lemma 6.1.4}] \\ W_{\rho(u), F}^{\mathcal{H}} = u^{-1}L &\Rightarrow [\text{Since } \mathcal{H} \text{ is str. cons.}] \\ u \in W_{I, \rho(u)}^{\mathcal{H}} &\Rightarrow [\text{By def. } W_{S, T}^{\mathcal{H}}, u = za] \\ \exists \rho(y) \in \tilde{Q}, \rho(u_0) \in \tilde{I}, z \in W_{\rho(u_0), \rho(y)} \wedge a \in W_{\rho(y), \rho(u)} &\Rightarrow [\text{By Definition 6.1.3}] \\ z \in W_{\rho(u_0), \rho(y)} \wedge \rho(y) \cdot a \subseteq \rho(u) &\Rightarrow [\text{By def. } \rho = \rho \leq^r] \\ z \in W_{\rho(u_0), \rho(y)} \wedge u \leq^r y \cdot a &\Rightarrow [\text{Since } x \leq^r u] \\ z \in W_{\rho(u_0), \rho(y)} \wedge x \leq^r ya &\Rightarrow [\text{By Def. 6.1.3}] \\ z \in W_{\rho(u_0), \rho(y)} \wedge \rho(x) \in \delta(\rho(y), a) &\Rightarrow [\text{By def. } \text{post}_u(I)] \\ \rho(x) \in \text{post}_u(I) . \end{aligned}$$

It follows from Equation (6.9) that $\text{post}_u^{\mathcal{H}}(I) \subseteq \text{post}_v^{\mathcal{H}}(I) \Leftrightarrow u \leq^r v$, i.e. $\leq_{\mathcal{H}}^r = \leq^r$.

Finally, note that if $\leq_L^r = \leq_N^r$ then clearly the automata $\text{Can}^r(L)$ and $\text{Res}^r(\mathcal{N})$ coincide for any NFA $\mathcal{N}$ with $L = \mathcal{L}(\mathcal{N})$. The following result shows that the reverse implication also holds.

LEMMA 6.2.8. *Let $\mathcal{N}$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. Then $\leq_L^r = \leq_N^r$ iff $\text{Res}^r(\mathcal{N})$ is isomorphic to $\text{Can}^r(L)$.*

Proof. As shown by Theorem 6.2.6 (d), $\text{Can}^r(L)$ is the canonical RFA for L , hence it is strongly consistent and, by Lemma 6.2.7, we have that $\leq_{\text{Can}^r(L)}^r = \leq_L^r$. On the other hand, if $\text{Res}^r(\mathcal{N})$ is isomorphic to $\text{Can}^r(L)$ we have that $\leq_{\text{Res}^r(\mathcal{N})}^r = \leq_{\text{Can}^r(L)}^r$, and by Lemma 6.2.7, $\leq_{\text{Res}^r(\mathcal{N})}^r = \leq_{\mathcal{N}}^r$. It follows that if $\text{Res}^r(\mathcal{N})$ is isomorphic to $\text{Can}^r(L)$ then $\leq_L^r = \leq_{\mathcal{N}}^r$.

Finally, if $\leq_L^r = \leq_{\mathcal{N}}^r$ then $H^r(\leq_L^r, L) = H^r(\leq_{\mathcal{N}}^r, \mathcal{L}(\mathcal{N}))$, in other words, $\text{Can}^r(L) = \text{Res}^r(\mathcal{N})$.

The following example illustrates the differences between our residualization operation, $\text{Res}^r(\mathcal{N})$, and the one defined by Denis et al. [2001], $\mathcal{N}^{\text{res}}$, on a given NFA $\mathcal{N}$: the automaton $\text{Res}^r(\mathcal{N})$ has, at most, as many states as $\mathcal{N}^{\text{res}}$. This follows from the fact that for every $u \in \Sigma^*$, if $\text{post}_u^{\mathcal{N}}(I)$ is coverable then $\rho_{\leq_{\mathcal{N}}^r}^r(u)$ is composite but *not* vice-versa.

Example 6.2.9. Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be the automata on the left of Figure 6.2 and let $L = \mathcal{L}(\mathcal{N})$. In order to build $\mathcal{N}^{\text{res}}$ we compute $\text{post}_u^{\mathcal{N}}(I)$, for all $u \in \Sigma^*$. Let $C \stackrel{\text{def}}{=} L^c \setminus \{\varepsilon, a, b, c\}$.

$$\begin{array}{lll} \text{post}_{\varepsilon}^{\mathcal{N}}(I) = \{0\} & \text{post}_a^{\mathcal{N}}(I) = \{1, 2\} & \forall w \in L, \text{post}_w^{\mathcal{N}}(I) = \{5\} \\ \text{post}_c^{\mathcal{N}}(I) = \{1, 2, 3, 4\} & \text{post}_b^{\mathcal{N}}(I) = \{1, 3\} & \forall w \in C, \text{post}_w^{\mathcal{N}}(I) = \emptyset \end{array}$$

Since none of these sets is coverable by the others, they are all states of $\mathcal{N}^{\text{res}}$. The resulting RFA $\mathcal{N}^{\text{res}}$ is shown in the center of Figure 6.2.

On the other hand, let us denote $\rho_{\leq_{\mathcal{N}}^r}$ simply by ρ . In order to build $\text{Res}^r(\mathcal{N})$ we need to compute the principals $\rho(u)$, for all $u \in \Sigma^*$. By definition of $\leq_{\mathcal{N}}^r$, we have that $w \in \rho(u) \Leftrightarrow \text{post}_u^{\mathcal{N}}(I) \subseteq \text{post}_w^{\mathcal{N}}(I)$. Therefore, we obtain:

$$\begin{array}{lll} \rho(\varepsilon) = \{\varepsilon\} & \rho(a) = \{a, c\} & \forall w \in L, \rho(w) = L \\ \rho(c) = \{c\} & \rho(b) = \{b, c\} & \forall w \in C, \rho(w) = \Sigma^* \end{array}$$

Since $a <_{\mathcal{N}}^r c$, $b <_{\mathcal{N}}^r c$ and $\forall w \in \Sigma^*$, $cw \subseteq L \Leftrightarrow (aw \subseteq L \vee bw \subseteq L)$, it follows that $\rho(c)$ is L -composite. The resulting RFA $\text{Res}^r(\mathcal{N})$ is shown on the right of Figure 6.2. $\diamond$

6.3 Double-Reversal Method for Building the Canonical RFA

Denis et al. [2002] show that their residualization operation satisfies the residual-equivalent of the double-reversal method for building the minimal DFA. More

Figure 6.2: Form left to right: an NFA $\mathcal{N}$ and the RFAs $\mathcal{N}^{\text{res}}$ and $\text{Res}^r(\mathcal{N})$. We omit the empty states for clarity.

specifically, they prove that if an NFA $\mathcal{N}$ is a co-RFA, then their residualization operation applied to $\mathcal{N}$ results in the canonical RFA for $\mathcal{L}(\mathcal{N})$. As a consequence,

$$(((\mathcal{N}^R)^{\text{res}})^R)^{\text{res}} \text{ is the canonical RFA for } \mathcal{L}(\mathcal{N}) .$$

In this section we show that the residual-equivalent of the double-reversal method works when using our automata constructions based on quasiorders, i.e.

$$\text{Res}^r((\text{Res}^r(\mathcal{N}^R))^R) \text{ is isomorphic to } \text{Can}^r(\mathcal{N}) .$$

Then, we generalize this method along the lines of the generalization of the double-reversal method for building the minimal DFA given by [Brzozowski and Tamm \[2014\]](#).

To this end, we extend the work of [Ganty et al. \[2019\]](#) where they use congruences to offer a new perspective on the generalization of [Brzozowski and Tamm \[2014\]](#). By switching from congruences to monotone quasiorders, we are able to give a *necessary* and *sufficient* condition on an NFA $\mathcal{N}$ that guarantees that our residualization operation yields the canonical RFA for $\mathcal{L}(\mathcal{N})$. Finally, we compare our generalization with the one given by [Tamm \[2015\]](#).

6.3.1 Double-reversal Method

We give a simple proof of the double-reversal method for building the canonical RFA for the language generated by a given NFA $\mathcal{N}$.

THEOREM 6.3.1 (Double-Reversal). *Let $\mathcal{N}$ be an NFA. Then $\text{Res}^r((\text{Res}^r(\mathcal{N}^R))^R)$ is isomorphic to the canonical RFA for $\mathcal{L}(\mathcal{N})$.*

Proof. It follows from Theorem 6.2.6 (c), (d) and (f).

Note that Theorem 6.3.1 can be inferred from Figure 6.1 by following the path starting at $\mathcal{N}$, labeled with $R - \text{Res}^r - R - \text{Res}^r$ and ending in $\text{Can}^r(\mathcal{L}(\mathcal{N}))$.

6.3.2 Generalization of the Double-reversal Method

Next we show that residualizing an automaton yields the canonical RFA iff the left language of every state is closed w.r.t. the right Nerode quasiorder.

THEOREM 6.3.2. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. Then $\text{Res}^r(\mathcal{N})$ is the canonical RFA for L iff $\forall q \in Q, \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$.*

Proof. We first show that $\forall q \in Q, \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$ is a *necessary* condition, i.e. if $\text{Res}^r(\mathcal{N})$ is the canonical RFA for L then $\forall q \in Q, \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$ holds.

By Lemma 6.2.8 we have that if $\text{Res}^r(\mathcal{N})$ is the canonical RFA for L then $\leq_L^r = \leq_{\mathcal{N}}^r$. Moreover:

$$\begin{aligned} \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) &= [\text{By definition of } \rho_{\leq_L^r}] \\ \{w \in \Sigma^* \mid \exists u \in W_{I,q}^{\mathcal{N}}, u^{-1}L \subseteq w^{-1}L\} &= [\text{Since } \leq_L^r = \leq_{\mathcal{N}}^r] \\ \{w \in \Sigma^* \mid \exists u \in W_{I,q}^{\mathcal{N}}, \text{post}_u^{\mathcal{N}}(I) \subseteq \text{post}_w^{\mathcal{N}}(I)\} &\subseteq [\text{Since } u \in W_{I,q}^{\mathcal{N}} \Leftrightarrow q \in \text{post}_u^{\mathcal{N}}(I)] \\ \{w \in \Sigma^* \mid q \in \text{post}_w^{\mathcal{N}}(I)\} &= [\text{By definition of } W_{I,q}^{\mathcal{N}}] \\ W_{I,q}^{\mathcal{N}}. \end{aligned}$$

By reflexivity of $\leq_L^r$, we conclude that $\rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$.

Next, we show that $\forall q \in Q, \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$ is also a *sufficient* condition. By Lemma 6.2.2 and condition $\forall q \in Q, \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$, we have that

$$\rho_{\leq_{\mathcal{N}}^r}(u) = \bigcap_{q \in \text{post}_u^{\mathcal{N}}(I)} W_{I,q}^{\mathcal{N}} = \bigcap_{q \in \text{post}_u^{\mathcal{N}}(I)} \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}). \quad (6.10)$$

Since $u \in \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}})$ for all $q \in \text{post}_u^{\mathcal{N}}(I)$, it follows that $\rho_{\leq_L^r}(u) \subseteq \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}})$ for all $q \in \text{post}_u^{\mathcal{N}}(I)$ and, since $\rho_{\leq_{\mathcal{N}}^r}(u) = \bigcap_{q \in \text{post}_u^{\mathcal{N}}(I)} \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}})$, we have that $\rho_{\leq_L^r}(u) \subseteq \rho_{\leq_{\mathcal{N}}^r}(u)$ for every $u \in \Sigma^*$, i.e., $\leq_L^r \subseteq \leq_{\mathcal{N}}^r$.

On the other hand, as shown by de Luca and Varricchio [1994], we have that $\leq_{\mathcal{N}}^r \subseteq \leq_L^r$. We conclude that $\leq_{\mathcal{N}}^r = \leq_L^r$, hence $\text{Res}^r(\mathcal{N}) = \text{Can}^r(L)$.

It is worth to remark that Theorem 6.3.2 does not hold when considering the residualization operation $\mathcal{N}^{\text{res}}$ [Denis et al. 2002]. As a counterexample we have the automaton $\mathcal{N}$ in Figure 6.2 where $\text{Res}^r(\mathcal{N})$ is the canonical RFA for $\mathcal{L}(\mathcal{N})$, hence $\mathcal{N}$ satisfies the condition of Theorem 6.3.2, while $\mathcal{N}^{\text{res}}$ is not canonical.

6.3.2.1 Co-atoms and co-rests

The condition of Theorem 6.3.2 is analogue to the one Ganty et al. [2019, Theorem 16] give for building the minimal DFA, except that the later is formulated in terms of congruences instead of quasiorders. In that case they prove that, given an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ with $L = \mathcal{L}(\mathcal{N})$,

$$\mathcal{N}^D \text{ is the minimal DFA for } L \text{ iff } \forall q \in Q, \rho_{\sim_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}},$$

where $\sim_L^r \stackrel{\text{def}}{=} \leq_L^r \cap (\leq_L^r)^{-1}$ is the right Nerode's congruence.

Moreover, Ganty et al. [2019] show that the principals of $\sim_L^r$ coincide with the so-called *co-atoms* [Ganty et al. 2019], which are non-empty intersections of complemented and uncomplemented right quotients of the language. This allowed them to connect their result with the generalization of the double-reversal method for DFAs of Brzozowski and Tamm [2014], who establish that determinizing an NFA $\mathcal{N}$ yields the minimal DFA for $\mathcal{L}(\mathcal{N})$ iff the left languages of the states of $\mathcal{N}$ are unions of co-atoms of $\mathcal{L}(\mathcal{N})$.

Next, we give a formulation of the condition from Theorem 6.3.2 along the lines of the one given by Brzozowski and Tamm [2014] for their generalization of the double-reversal method for building the minimal DFA.

To do that, let us call the intersections used in Lemma 6.2.2 to describe the principals of $\leq_L^\ell$ and $\leq_L^r$ as *rests* and *co-rests* of L , respectively.

Definition 6.3.3 (Rest and Co-rest). *Let L be a regular language. A rest (resp. co-rest) is any non-empty intersection of left (resp. right) quotients of L .* ■

As shown by Theorem 6.3.2, residualizing an NFA $\mathcal{N}$ yields the canonical RFA for $\mathcal{L}(\mathcal{N})$ iff the left language of every state of $\mathcal{N}$ satisfies $\rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$. By definition, $\rho_{\leq_L^r}(S) = S$ iff S is a union of principals of $\leq_L^r$.

Therefore we derive the following statement, equivalent to Theorem 6.3.2, that we consider as the residual-equivalent of the generalization of the double-reversal for building the minimal DFA [Brzozowski and Tamm 2014].

COROLLARY 6.3.4. *Let $\mathcal{N}$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. Then $\text{Res}^r(\mathcal{N})$ is the canonical RFA for L iff the left languages of $\mathcal{N}$ are union of co-rests.*

6.3.2.2 Tamm's Generalization of the Double-reversal Method for RFAs

Tamm [2015] generalized the double-reversal method of Denis et al. [2002] by showing that $\mathcal{N}^{\text{res}}$ is the canonical RFA for $\mathcal{L}(\mathcal{N})$ iff the left languages of $\mathcal{N}$ are union of the left languages of the canonical RFA for $\mathcal{L}(\mathcal{N})$.

In this section, we compare the generalization of [Tamm \[2015\]](#) with ours. The two approaches differ in the definition of the residualization operation they consider and, as we show next, the sufficient and necessary condition from [Theorem 6.3.2](#) is more general than that of [Tamm \[2015, Theorem 4\]](#).

LEMMA 6.3.5. *Let $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$ be an NFA with $L = \mathcal{L}(\mathcal{N})$ and let $C = \text{Can}^r(\leq_L^r, L) = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$ be the canonical RFA for L . Then*

$$W_{I,q}^{\mathcal{N}} = \bigcup_{q \in \tilde{Q}} W_{I,q}^C \implies \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}} .$$

Proof. Since the canonical RFA, C , is strongly consistent then it follows from [Lemma 6.2.7](#) that $\leq_C^r = \leq_L^r$ and, consequently, $\text{Res}^r(C)$ is isomorphic to $\text{Can}^r(L)$. It follows from [Theorem 6.3.2](#) that $\rho_{\leq_L^r}(W_{I,q}^C) = W_{I,q}^C$ for every $q \in \tilde{Q}$. Therefore,

$$\begin{aligned} \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) &= [\text{Since } W_{I,q}^{\mathcal{N}} = \bigcup_{q \in \tilde{Q}} W_{I,q}^C \text{ and } \rho_{\leq_L^r}(\cup S_i) = \cup \rho_{\leq_L^r}(S_i)] \\ \bigcup_{q \in \tilde{Q}} \rho_{\leq_L^r}(W_{I,q}^C) &= [\text{Since } \rho_{\leq_L^r}(W_{I,q}^C) = W_{I,q}^C \text{ for every } q \in \tilde{Q}] \\ &\quad \bigcup_{q \in \tilde{Q}} W_{I,q}^C . \end{aligned}$$

Observe that, since the canonical RFA $C = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$ for a language L is strongly consistent, the left language of each state is a principal of $\leq_L^r$. In particular, if the right language of a state is $u^{-1}L$ then its left language is the principal $\rho_{\leq_L^r}(u)$. Therefore, if $W_{I,q}^{\mathcal{N}} = \bigcup_{q \in \tilde{Q}} W_{I,q}^C$ then $W_{I,q}^{\mathcal{N}}$ is a closed set in $\rho_{\leq_L^r}$. However, the reverse implication does not hold since *only the L -prime principals are left languages of states of C* .

On the other hand, [Lemma 6.2.4](#) shows that L -composite principals can be described as intersections of L -prime principals when we consider the Nerode's quasiorder $\leq_L^r$. As a consequence, our residualization operation applied on an NFA $\mathcal{N}$ yields the canonical RFA for $\mathcal{L}(\mathcal{N})$ iff the left languages of states of $\mathcal{N}$ are *union of non-empty intersections of left languages of the canonical RFA* while [Tamm \[2015\]](#) proves that $\mathcal{N}^{\text{res}}$ yields to the canonical RFA iff the left languages of states of $\mathcal{N}$ are *union of left languages of the canonical RFA*.

6.4 Learning Residual Automata

[Bollig et al. \[2009\]](#) devised the NL^* algorithm for learning the canonical RFA for a given regular language. The algorithm describes the behavior of a *Learner* that infers a language L by performing membership queries on L (which are answered

by a *Teacher*) and equivalence queries between the language generated by a candidate automaton and L (which are answered by an *Oracle*). The algorithm terminates when the *Learner* builds an RFA generating the language L .

For the sake of completeness, we offer an overview of the NL^* algorithm as presented by Bollig et al. [2009].

6.4.1 The NL^* Algorithm [Bollig et al. 2009]

The *Learner* maintains a prefix-closed finite set $\mathcal{P} \subseteq \Sigma^*$ and a suffix-closed finite set $\mathcal{S} \subseteq \Sigma^*$. The *Learner* groups the words in $\mathcal{P}$ by building a table $T = (\mathcal{T}, \mathcal{P}, \mathcal{S})$ where $T : (\mathcal{P} \cup \mathcal{P} \cdot \Sigma) \times \mathcal{S} \rightarrow \{+, -\}$ is a function such that for every $u \in \mathcal{P} \cup \mathcal{P} \cdot \Sigma$ and $v \in \mathcal{S}$ we have that $T(u, v) = + \Leftrightarrow uv \in L$. Otherwise $T(u, v) = -$.

For every word $u \in \mathcal{P} \cup \mathcal{P} \cdot \Sigma$, define the function $r(u) : \mathcal{S} \rightarrow \{+, -\}$ as $r(u)(v) \stackrel{\text{def}}{=} T(u, v)$. The set of all rows of a table $\mathcal{T}$ is denoted by $\text{Rows}(\mathcal{T})$.

The algorithm uses the table $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ to build an automaton whose states are some of the rows of $\mathcal{T}$. In order to do that, it is necessary to define the notions of *union* of rows, *prime* row and *composite* row.

Definition 6.4.1 (Join Operator). Let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be a table. For every pair of rows $r_1, r_2 \in \text{Rows}(\mathcal{T})$, define the join $r_1 \sqcup r_2 : \mathcal{S} \rightarrow \{+, -\}$ as:

$$\forall x \in \mathcal{S}, (r_1 \sqcup r_2)(x) \stackrel{\text{def}}{=} \begin{cases} + & \text{if } r_1(x) = + \vee r_2(x) = + \\ - & \text{otherwise} \end{cases} \quad \blacksquare$$

Note that the join operator is associative, commutative and idempotent. However, the join of two rows is not necessarily a row of $\mathcal{T}$.

Definition 6.4.2 (Covering Relation). Let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be a table. Then, for every pair of rows $r_1, r_2 \in \text{Rows}(\mathcal{T})$ we have that

$$r_1 \sqsubseteq r_2 \stackrel{\text{def}}{\Leftrightarrow} \forall x \in \mathcal{S}, r_1(x) = + \Rightarrow r_2(x) = +.$$

We write $r_1 \sqsubset r_2$ to denote $r_1 \sqsubseteq r_2$ and $r_1 \neq r_2$. ■

Definition 6.4.3 (Composite and Prime Rows). Let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be a table. We say a row $r \in \text{Rows}(\mathcal{T})$ is $\mathcal{T}$ -composite if it is the join of all the rows that it strictly covers, i.e. $r = \sqcup_{r' \in \text{Rows}(\mathcal{T}), r' \sqsubset r} r'$. Otherwise, we say r is $\mathcal{T}$ -prime. ■

Definition 6.4.4 (Closed and Consistent Table). Let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be a table. Then

(a) $\mathcal{T}$ is closed iff

$$\forall u \in \mathcal{P}, a \in \Sigma, r(ua) = \sqcup \{r(v) \mid v \in \mathcal{P}, r(v) \sqsubseteq r(ua) \wedge r(v) \text{ is } \mathcal{T}\text{-prime}\}.$$

(b) $\mathcal{T}$ is consistent iff

$$r(u) \sqsubseteq r(v) \Rightarrow r(ua) \sqsubseteq r(va) \text{ for every } u, v \in \mathcal{P} \text{ and } a \in \Sigma . \quad \blacksquare$$

At each iteration of the algorithm, the *Learner* checks whether the current table $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ is closed and consistent.

If $\mathcal{T}$ is not closed, then it finds $r(ua)$ with $u \in \mathcal{P}, a \in \Sigma$ such that $r(ua)$ is $\mathcal{T}$ -prime and it is not equal to some $r(v)$ with $v \in \mathcal{P}$. Then the *Learner* adds ua to $\mathcal{P}$ and updates the table $\mathcal{T}$.

Similarly, if $\mathcal{T}$ is not consistent, the *Learner* finds $u, v \in \mathcal{P}, a \in \Sigma, x \in \mathcal{S}$ such that $r(u) \sqsubseteq r(v)$ but $r(ua)(x) = + \wedge r(va)(x) = -$. Then the *Learner* adds ax to $\mathcal{S}$ and updates $\mathcal{T}$.

When the table $\mathcal{T}$ is closed and consistent, the *Learner* builds the RFA $R(\mathcal{T})$.

Definition 6.4.5 (Automata Construction $R(\mathcal{T})$). Let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be a closed and consistent table. Define the automaton $R(\mathcal{T}) \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, F \rangle$ with $Q = \{r(u) \mid u \in \mathcal{P} \wedge r(u) \text{ is } \mathcal{T}\text{-prime}\}$, $I = \{r(u) \in Q \mid r(u) \sqsubseteq r(\varepsilon)\}$, $F = \{r(u) \in Q \mid r(u)(\varepsilon) = +\}$ and $r(v) \in \delta(r(u), a) = \{r(v) \in Q \mid r(v) \sqsubseteq r(ua)\}$ for all $r(u) \in Q, a \in \Sigma$. $\blacksquare$

The *Learner* asks the *Oracle* whether $\mathcal{L}(R(\mathcal{T})) = L$. If the *Oracle* answers yes then the algorithm terminates. Otherwise, the *Oracle* returns a counterexample w for the language equivalence. Then the *Learner* adds every suffix of w to $\mathcal{S}$, updates the table $\mathcal{T}$ and repeats the process.

6.4.2 The $NL^{\leq}$ Algorithm

In this section we present a quasiorder-based perspective on the NL^* algorithm in which the *Learner* iteratively refines a quasiorder on Σ^* by querying the *Teacher* and uses an adaption of the automata construction from Definition 6.1.3 to build an RFA that is used to query the *Oracle*. We capture this approach in the so-called $NL^{\leq}$ algorithm.

Next we explain the behavior of algorithm $NL^{\leq}$ and give the necessary definitions in order to understand it and its relation with the algorithm NL^* .

The *Learner* maintains a prefix-closed finite set $\mathcal{P} \subseteq \Sigma^*$ and a suffix-closed finite set $\mathcal{S} \subseteq \Sigma^*$. The set $\mathcal{S}$ is used to approximate the principals in $\leq_L^r$ for the words in $\mathcal{P}$. In order to manipulate these approximations, we define the following two operators.

Definition 6.4.6. Let L be a language, $\mathcal{S} \subseteq \Sigma^*$ and $u \in \Sigma^*$. Define:

$$\begin{aligned} u^{-1}L =_{\mathcal{S}} v^{-1}L &\stackrel{\text{def}}{\Leftrightarrow} (u^{-1}L \cap \mathcal{S}) = (v^{-1}L \cap \mathcal{S}) \\ u^{-1}L \subseteq_{\mathcal{S}} v^{-1}L &\stackrel{\text{def}}{\Leftrightarrow} (u^{-1}L \cap \mathcal{S}) \subseteq (v^{-1}L \cap \mathcal{S}) . \quad \blacksquare \end{aligned}$$

Algorithm: $NL^{\leq_r}$: A quasiorder-based version of NL^*

Data: A *Teacher* that answers membership queries in L
Data: An *Oracle* that answers equivalence queries between the language generated by an RFA and L
Result: The canonical RFA for the language L .

```

1  $\mathcal{P}, \mathcal{S} := \{\varepsilon\};$ 
2 while True do
3   while  $\leq_{L_S}^r$  not closed or consistent: do
4     if  $\leq_{L_S}^r$  is not closed then
5       Find  $u \in \mathcal{P}, a \in \Sigma$  with  $\rho_{\leq_{L_S}^r}(u)$   $L_S$ -prime for  $\mathcal{P}$  and
          $\forall v \in \mathcal{P}, \rho_{\leq_{L_S}^r}(u) \neq \rho_{\leq_{L_S}^r}(v);$ 
6       Let  $\mathcal{P} := \mathcal{P} \cup \{ua\};$ 
7     if  $\leq_{L_S}^r$  is not consistent then
8       Find  $u, v \in \mathcal{P}, a \in \Sigma$  with  $u \leq_{L_S}^r v$  s.t.  $ua \not\leq_{L_S}^r va;$ 
9       Find  $x \in (ua)^{-1}L \cap ((va)^{-1}L)^c \cap \mathcal{S};$ 
10      Let  $\mathcal{S} := \mathcal{S} \cup \{ax\};$ 
11    Build  $R(\leq_{L_S}^r, \mathcal{P});$ 
12    Ask the Oracle whether  $L = \mathcal{L}(R(\leq_{L_S}^r, \mathcal{P}));$ 
13    if the Oracle replies with a counterexample  $w$  then
14      Let  $\mathcal{S} := \mathcal{S} \cup \{x \in \Sigma^* \mid w = w'x \text{ with } w \in \mathcal{S}, w' \in \Sigma^*\};$ 
15    else
16      return  $R(\leq_{L_S}^r, \mathcal{P});$ 

```

These operators allow us to define an over-approximation of Nerode's quasiorder that can be decided with finitely many membership tests.

Definition 6.4.7 (Right-language-based quasiorder w.r.t. $\mathcal{S}$). *Let L be a language, $\mathcal{S} \subseteq \Sigma^*$ and $u, v \in \Sigma^*$. Define $u \leq_{L_S}^r v \stackrel{\text{def}}{\Leftrightarrow} u^{-1}L \subseteq_{\mathcal{S}} v^{-1}L$.* ■

Recall that the *Learner* only manipulates the principals for the words in $\mathcal{P}$. Therefore, we need to adapt the notion of composite principal for $\leq_{L_S}^r$.

Definition 6.4.8 (L_S -Composite Principal w.r.t. $\mathcal{P}$). *Let $\mathcal{P}, \mathcal{S} \subseteq \Sigma^*$ with $u \in \mathcal{P}$ and let $L \subseteq \Sigma^*$ be a language. We say $\rho_{\leq_{L_S}^r}(u)$ is L_S -composite w.r.t. $\mathcal{P}$ iff*

$$u^{-1}L =_{\mathcal{S}} \bigcup_{x \in \mathcal{P}, x <_{L_S}^r u} x^{-1}L .$$

Otherwise, we say it is L_S -prime w.r.t. $\mathcal{P}$. ■

The *Learner* uses the quasiorder $\leq_{L_S}^r$ to build an automaton by adapting the construction from Definition 6.1.3 in order to use only the information that is available by means of the sets $\mathcal{S}$ and $\mathcal{P}$. Building such an automaton requires the quasiorder to satisfy two conditions: it must be *closed* and *consistent* w.r.t. $\mathcal{P}$.

Definition 6.4.9 (Closedness and Consistency of $\leq_{L_S}^r$ w.r.t. $\mathcal{P}$).

(a) $\leq_{L_S}^r$ is closed w.r.t. $\mathcal{P}$ iff

$$\forall u \in \mathcal{P}, a \in \Sigma, \rho_{\leq_{L_S}^r}(ua) \text{ is } L_S\text{-prime w.r.t. } \mathcal{P} \Rightarrow \exists v \in \mathcal{P}, \rho_{\leq_{L_S}^r}(ua) = \rho_{\leq_{L_S}^r}(v).$$

(b) $\leq_{L_S}^r$ is consistent w.r.t. $\mathcal{P}$ iff

$$\forall u, v \in \mathcal{P}, a \in \Sigma : u \leq_{L_S}^r v \Rightarrow ua \leq_{L_S}^r va . \quad \blacksquare$$

At each iteration, the *Learner* checks whether the quasiorder $\leq_{L_S}^r$ is closed and consistent w.r.t. $\mathcal{P}$. If $\leq_{L_S}^r$ is not closed w.r.t. $\mathcal{P}$, then it finds $\rho_{\leq_{L_S}^r}(ua)$ with $u \in \mathcal{P}, a \in \Sigma$ such that $\rho_{\leq_{L_S}^r}(ua)$ is L_S -prime w.r.t. $\mathcal{P}$ and it is not equal to some $\rho_{\leq_{L_S}^r}(v)$ with $v \in \mathcal{P}$. Then the *Learner* adds ua to $\mathcal{P}$.

Similarly, if $\leq_{L_S}^r$ is not consistent w.r.t. $\mathcal{P}$ then the *Learner* finds $u, v \in \mathcal{P}, a \in \Sigma, x \in \mathcal{S}$ such that $u \leq_{L_S}^r v$ but $uax \in L \wedge vax \notin L$. Then the *Learner* adds ax to $\mathcal{S}$. When the quasiorder $\leq_{L_S}^r$ is closed and consistent w.r.t. $\mathcal{P}$, the *Learner* builds the RFA $R(\leq_{L_S}^r, \mathcal{P})$.

Definition 6.4.10 is an adaptation of the automata construction H^r from Definition 6.1.3. Instead of considering all principals, it considers only those that correspond to words in $\mathcal{P}$. Moreover, the notion of L -primality is replaced by L_S -primality w.r.t. $\mathcal{P}$ since the algorithm does not manipulate quotients of L by words in Σ^* but the approximation through $\mathcal{S}$ of the quotients of L by words in $\mathcal{P}$ (see Definition 6.4.6).

Definition 6.4.10 (Automata construction $L(\leq_{L_S}^r, \mathcal{P})$). Let $L \subseteq \Sigma^*$ be a regular language and let $\mathcal{P}, \mathcal{S} \subseteq \Sigma^*$. Define the automaton $L(\leq_{L_S}^r, \mathcal{P}) = \langle Q, \Sigma, \delta, I, F \rangle$ with $Q = \{\rho_{\leq_{L_S}^r}(u) \mid u \in \mathcal{P}, \rho_{\leq_{L_S}^r}(u) \text{ is } L_S\text{-prime w.r.t. } \mathcal{P}\}$, $I = \{\rho_{\leq_{L_S}^r}(u) \in Q \mid \varepsilon \in \rho_{\leq_{L_S}^r}(u)\}$, $F = \{\rho_{\leq_{L_S}^r}(u) \in Q \mid u \in L\}$ and $\delta(\rho_{\leq_{L_S}^r}(u), a) = \{\rho_{\leq_{L_S}^r}(v) \in Q \mid \rho_{\leq_{L_S}^r}(u)a \subseteq \rho_{\leq_{L_S}^r}(v)\}$ for all $\rho_{\leq_{L_S}^r}(u) \in Q$ and $a \in \Sigma$. $\blacksquare$

Finally, the *Learner* asks the *Oracle* whether $\mathcal{L}(R(\leq_{L_S}^r, \mathcal{P})) = L$. If the *Oracle* answers yes then the algorithm terminates. Otherwise, the *Oracle* returns a counterexample w for the language equivalence. Then, the *Learner* adds every suffix of w to $\mathcal{S}$ and repeats the process.

Theorem 6.4.11 shows that the $NL^{\leq}$ algorithm exactly coincides with NL^* .

THEOREM 6.4.11. $NL^{\leq}$ builds the same sets $\mathcal{P}$ and $\mathcal{S}$, performs the same queries to the Oracle and the Teacher and returns the same RFA as NL^* , provided that both algorithms perform the same non-deterministic choices.

Proof. Let $\mathcal{P}, \mathcal{S} \subseteq \Sigma^*$ be a prefix-closed and a suffix-closed finite set, respectively, and let $\mathcal{T} = (T, \mathcal{P}, \mathcal{S})$ be the table built by algorithm NL^* . Observe that for every $u, v \in \mathcal{P}$:

$$\begin{aligned}
 u &\leq_{L_S}^r v \Leftrightarrow && \text{[By Definition 6.4.7]} \\
 u^{-1}L &\subseteq_S v^{-1}L \Leftrightarrow && \text{[By definition of quotient w.r.t } S] \\
 \forall x \in S, ux \in L &\Rightarrow vx \in L \Leftrightarrow && \text{[By definition of } \mathcal{T}] \\
 \forall x \in S, (r(u)(x) = +) &\Rightarrow (r(v)(x) = +) \Leftrightarrow && \text{[By Definition 6.4.2]} \\
 r(u) &\sqsubseteq r(v) . && (6.11)
 \end{aligned}$$

Moreover, for every $u, v \in \mathcal{P}$ we have that $u^{-1}L =_S v^{-1}L$ iff $r(u) = r(v)$.

Next, we show that the join operator applied to rows corresponds to the set union applied to quotients w.r.t S . Let $u, v \in \mathcal{P}$ and let $x \in \mathcal{S}$. Then,

$$\begin{aligned}
 (r(u) \sqcup r(v))(x) &= + \Leftrightarrow && \text{[By Definition 6.4.1]} \\
 (r(u)(x) = +) \vee (r(v)(x) = +) &\Leftrightarrow && \text{[By definition of row]} \\
 (ux \in L) \vee (vx \in L) &\Leftrightarrow && \text{[By definition of quotient w.r.t } S] \\
 (x \in u^{-1}L) \vee (x \in v^{-1}L) &\Leftrightarrow && \text{[By definition of } \cup] \\
 x &\in u^{-1}L \cup v^{-1}L . && (6.12)
 \end{aligned}$$

Therefore, we can prove that $r(u)$ is $\mathcal{T}$ -prime iff $\rho_{\leq_{L_S}^r}(u)$ is L_S -prime w.r.t. $\mathcal{P}$.

$$\begin{aligned}
 r(u) &= \bigsqcup_{v \in \mathcal{P}, r(v) \sqsubset r(u)} r(v) \Leftrightarrow && \text{[By Equation (6.11)]} \\
 r(u) &= \bigsqcup_{v \in \mathcal{P}, v^{-1}L \subsetneq_S u^{-1}L} r(v) \Leftrightarrow && \text{[By Equation (6.12)]} \\
 u^{-1}L &= \bigcup_{v \in \mathcal{P}, v^{-1}L \subsetneq_S u^{-1}L} v^{-1}L \Leftrightarrow && [v^{-1}L \subsetneq_S u^{-1}L \Leftrightarrow u <_{L_S}^r v] \\
 u^{-1}L &= \bigcup_{v \in \mathcal{P}, u <_{L_S}^r v} v^{-1}L .
 \end{aligned}$$

It follows from Definitions 6.4.9 (a) and 6.4.4 (a) and Equation (6.12) that $\mathcal{T}$ is closed iff $\leq_{L_S}^r$ is closed.

Moreover, it follows from Definitions 6.4.9 (b) and 6.4.4 (b) that $\mathcal{T}$ is consistent iff $\leq_{L_S}^r$ is consistent.

On the other hand, for every $u, v \in \mathcal{P}$, $a \in \Sigma$ and $x \in \mathcal{S}$ we have that:

$$\begin{aligned} (r(u) \subseteq r(v)) \wedge (r(ua)(x) = +) \wedge (r(va)(x) = -) &\Leftrightarrow \quad [\text{By Equation (6.11)}] \\ (u \leq_{L_S}^r v) \wedge (uax \in L) \wedge (vax \notin L) \end{aligned}$$

It follows that if $\mathcal{T}$ and $\leq_{L_S}^r$ are not consistent then both NL^* and $NL^{\leq}$ can find the same word $ax \in \Sigma \mathcal{S}$ and add it to $\mathcal{S}$. Similarly, it is straightforward to check that if $r(ua)$ with $u \in \mathcal{P}$ and $a \in \Sigma$ break consistency, i.e. it is $\mathcal{T}$ -prime and it is not equal to any $r(v)$ with $v \in \mathcal{P}$, then $\rho_{\leq_{L_S}^r}(ua)$ is L_S -prime for $\mathcal{P}$ and not equal to any $\rho_{\leq_{L_S}^r}(v)$ with $v \in \mathcal{P}$. Thus, if $\mathcal{T}$ and $\leq_{L_S}^r$ are not closed then both NL^* and $NL^{\leq}$ can find the same word ua and add it to $\mathcal{P}$.

It remains to show that both algorithms build the same automaton modulo isomorphism, i.e., $R(\mathcal{T}) = \langle \tilde{Q}, \Sigma, \tilde{\delta}, \tilde{I}, \tilde{F} \rangle$ is isomorphic to $R(\leq_{L_S}^r, \mathcal{P}) = \langle Q, \Sigma, \delta, I, F \rangle$. Define the mapping $\varphi : Q \rightarrow \tilde{Q}$ as $\varphi(\rho_{\leq_{L_S}^r}(u)) = r(u)$. Then:

$$\begin{aligned} \varphi(Q) &= \{\varphi(\rho_{\leq_{L_S}^r}(u)) \mid u \in \mathcal{P} \wedge \rho_{\leq_{L_S}^r}(u) \text{ is } L_S\text{-prime w.r.t. } \mathcal{P}\} \\ &= \{r(u) \mid u \in \mathcal{P} \wedge r(u) \text{ is } \mathcal{T}\text{-prime}\} = \tilde{Q} . \\ \varphi(I) &= \{\varphi(\rho_{\leq_{L_S}^r}(u)) \mid \varepsilon \in \rho_{\leq_{L_S}^r}(u)\} = \{r(u) \mid u \leq_{L_S}^r \varepsilon\} \\ &= \{r(u) \mid r(u) \subseteq r(\varepsilon)\} = \tilde{I} . \\ \varphi(F) &= \{\varphi(\rho_{\leq_{L_S}^r}(u)) \mid u \in L \cap \mathcal{P}\} = \{r(u) \mid u \in L \cap \mathcal{P}\} \\ &= \{r(u) \mid r(u)(\varepsilon) = +\} = \tilde{F} . \\ \varphi(\delta(\rho_{\leq_{L_S}^r}(u), a)) &= \varphi(\rho_{\leq_{L_S}^r}(ua)) = \{r(v) \mid \rho_{\leq_{L_S}^r}(u) \in Q \wedge \rho_{\leq_{L_S}^r}(u)a \subseteq \rho_{\leq_{L_S}^r}(v)\} \\ &= \{r(v) \mid r(v) \in \tilde{Q} \wedge v \leq_{L_S}^r ua\} \\ &= \{r(v) \mid r(v) \in \tilde{Q} \wedge r(v) \subseteq r(ua)\} \\ &= \tilde{\delta}(r(u), a) = \tilde{\delta}(\varphi(\rho_{\leq_{L_S}^r}(u)), a) . \end{aligned}$$

Finally, we show that φ is an isomorphism. Clearly, the function φ is surjective since, for every $u \in \mathcal{P}$, we have that $r(u) = \varphi(\rho_{\leq_{L_S}^r}(u))$. Moreover φ is injective since for every $u, v \in \mathcal{P}$, $r(u) = r(v) \Leftrightarrow u^{-1}L =_{\mathcal{S}} v^{-1}L$, hence $r(u) = r(v) \Leftrightarrow \rho_{\leq_{L_S}^r}(u) = \rho_{\leq_{L_S}^r}(v)$.

We conclude that φ is an NFA isomorphism between $R(\leq_{L_S}^r, \mathcal{P})$ and $R(\mathcal{T})$. Therefore NL^* and $NL^{\leq}$ exhibit the same behavior, provided that both algorithms perform the same non-deterministic choices, as they both maintain the same sets $\mathcal{P}$ and $\mathcal{S}$ and build the same automata at each step.

Termination of NL^* and $NL^{\leq}$

At each iteration of the $NL^{\leq}$ algorithm, it either terminates or the counterexample w given by the *Oracle* refines the quasiorder $\leq_{L_S}^r$ which results in having, at least, one new principal $\rho_{\leq_{L_S}^r}(w)$.

Since

$$\rho_{\leq_{L_S}^r}(u) \neq \rho_{\leq_{L_S}^r}(v) \Rightarrow \exists s \in \mathcal{S}, us \in L \wedge vs \notin L \Rightarrow \rho_{\leq_L^r}(u) \neq \rho_{\leq_L^r}(v) ,$$

we conclude that the number of principals for $\leq_{L_S}^r$ is smaller or equal than the number of principals for $\leq_L^r$. Given that $\leq_L^r$ induces finitely many principals, algorithm $NL^{\leq}$ can only add finitely many principals to $\leq_{L_S}^r$ and, therefore, the algorithm terminates.

It is worth to remark that, in order to prove the termination of the NL^* algorithm, [Bollig et al. \[2009\]](#) first had to show that the number of rows built during the computation of the NL^* algorithm is a lower bound for the number of rows computed during an execution of the L^* algorithm of [Angluin \[1987\]](#). Then, the termination of the NL^* algorithms follows from the termination of L^* .

Finally, observe that, by replacing the right quasiorder $\leq_{L_S}^r$ by the right congruence $\sim_{L_S} \stackrel{\text{def}}{=} \leq_{L_S}^r \cap (\leq_{L_S}^r)^{-1}$ in the above algorithm (precisely, in Definitions 6.4.9 and 6.4.10), the resulting algorithm corresponds to the L^* algorithm of [Angluin \[1987\]](#). Note that, in that case, all principals $\rho_{\sim_{L_S}}(u)$, with $u \in \Sigma^*$, are L_S -prime w.r.t. $\mathcal{P}$.

FUTURE WORK

We believe that we have only scratched the surface on the use of *well-quasiorders* on words for solving problems from *Formal Language Theory*.

In this section, we present some directions for further developments that show how our work could be extended to (i) take full advantage of simulation relations, (ii) better understand, and possibly improve, the performance of `zsearch` and (iii) develop new algorithms for building smaller residual automata.

7.1 The Language Inclusion Problem

Consider the inclusion problem $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are NFAs. Even though we have shown in Chapter 4 that simulations can be used to derive an algorithm for solving this language inclusion problem, we are not on par with the thoughtful use of simulation relations made by [Abdulla et al. \[2010\]](#) and [Bonchi and Pous \[2013\]](#). The main reason for which we are not able to accommodate within our framework their use of simulations is that our abstraction only manipulates sets of states of $\mathcal{N}_2$. As a consequence, any use of simulations that involves states of $\mathcal{N}_1$ is out of reach.

However, it is possible to overcome this limitation by using *alternating automata* as we show next. Intuitively, since alternating automata can be complemented without altering their number of states, we can reduce any language inclusion problem $\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are alternating automata, into a universality problem $\Sigma^* \subseteq \mathcal{L}(\mathcal{A}_3)$, where $\mathcal{A}_3 = \mathcal{A}_1^c \cup \mathcal{A}_2$. Since $\mathcal{A}_3$ is built by combining the two input automata its states are the union of the states of $\mathcal{A}_1$ and $\mathcal{A}_2$. Therefore, simulations applicable within our framework to decide $\Sigma^* \subseteq \mathcal{L}(\mathcal{A}_3)$, which only involve states of $\mathcal{A}_3$, now involve states of $\mathcal{A}_1$ and $\mathcal{A}_2$.

7.1.1 Language Inclusion Through Alternating Automata

Let S be a set. We denote by $\mathcal{B}^+(S)$ the set of *positive Boolean formulas* over S which are of the form $\Phi \stackrel{\text{def}}{=} s \mid \Phi_1 \vee \Phi_2 \mid \Phi_1 \wedge \Phi_2 \mid \text{false}$, where $s \in S$ and $\Phi_1, \Phi_2 \in \mathcal{B}^+(S)$. We say $S' \subseteq S$ *satisfies* a formula $\Phi \in \mathcal{B}^+(S)$ iff Φ is *true* when assigning the value *true* to all elements in S' and *false* to the elements in $S \setminus S'$. Given $\Phi \in \mathcal{B}^+(S)$, we denote $\llbracket \Phi \rrbracket$ the set of all subsets of S that satisfy Φ . Clearly, if S' satisfies a formula Φ , any set $S'' \subseteq S$ such that $S' \subseteq S''$ also satisfies Φ . Therefore, the set $\llbracket \Phi \rrbracket$ is an $\subseteq$ -upward closed set, i.e. $\rho_{\subseteq}(\llbracket \Phi \rrbracket) = \llbracket \Phi \rrbracket$. Finally, if a formula Φ is not satisfiable, i.e. no set $S' \subseteq S$ satisfies Φ , then $\llbracket \Phi \rrbracket = \emptyset$.

Definition (AFA). An alternating finite-state automata (AFA for short) is a tuple $\mathcal{A} \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta, I, F \rangle$ where Q is the finite set of states, Σ is the finite alphabet, $\delta: Q \times \Sigma \rightarrow \mathcal{B}^+(Q)$ is the transition function, $I \subseteq Q$ are the initial states and $F \subseteq Q$ are the final states. ■

Intuitively, given an active state $q \in Q$ and an alphabet symbol $a \in \Sigma$ an AFA can activate any set of states in $\llbracket \delta(q, a) \rrbracket$. Figure 7.1 shows an example of an AFA.

$$\begin{aligned} \mathcal{A} &= \langle Q, \Sigma, \delta, I, F \rangle \text{ with } Q = \{q_0, q_1, q_2, q_3\}, \\ \Sigma &= \{a, b\}, I = \{q_0\}, F = \{q_3\} \text{ and} \\ \delta(q_0, a) &= (q_1 \wedge q_2) \vee q_3 \\ \delta(q_1, b) &= q_2 \wedge q_3 \\ \delta(q_2, a) &= \delta(q_3, a) = \delta(q_3, b) = q_3 \\ \delta(q_0, b) &= \delta(q_1, a) = \delta(q_2, b) = \text{false} \end{aligned}$$

Figure 7.1: Alternating automaton $\mathcal{A}$ generating the language $\mathcal{L}(\mathcal{A}) = a(a + b)^*$.

Given an AFA $\mathcal{A} = \langle Q, \Sigma, \delta, I, F \rangle$, we extend the transition function δ to sets of states obtaining $\Delta: \wp(Q) \times \Sigma \rightarrow \mathcal{B}^+$ defined as $\Delta(S, a) \stackrel{\text{def}}{=} \bigwedge_{s \in S} \delta(s, a)$. Intuitively, $\Delta(S, a)$ indicates the states that will be activated after reading a when all states in S are active. Let $X \uplus Y \stackrel{\text{def}}{=} \{x \cup y \mid x \in X, y \in Y\}$. Then

$$\llbracket \Delta(S, a) \rrbracket = \begin{cases} \emptyset & \text{if } \exists s \in S \text{ s.t. } \delta(s, a) = \text{false} \\ \uplus_{s \in S} \llbracket \delta(s, a) \rrbracket & \text{otherwise} \end{cases} \quad (7.1)$$

We say a word w is accepted by an AFA $\mathcal{A} = \langle Q, \Sigma, \delta, I, F \rangle$ iff there exists a sequence of sets of active states $S_0, \dots, S_{|w|}$ such that $S_0 = \{q_i\}$ with $q_i \in I$, $S_n \subseteq F$, $S_n \neq \emptyset$ and $S_i \in \llbracket \Delta(S_{i-1}, (w)_i) \rrbracket$ for $1 \leq i \leq |w|$.

Example 7.1.1. Let us consider the alternating automaton $\mathcal{A}$ in Figure 7.1. In this case, we have that

$$\begin{aligned}
 \Delta(\{q_0\}, a) &= \delta(q_0, a) = (q_1 \wedge q_2) \vee q_3 \quad . \\
 \llbracket \Delta(\{q_0\}, a) \rrbracket &= \llbracket \delta(q_0, a) \rrbracket = \rho_{\subseteq}(\{\{q_1, q_2\}, \{q_3\}\}) \quad . \\
 \Delta(\{q_1, q_2\}, b) &= \delta(q_1, b) \wedge \delta(q_2, b) = q_2 \wedge q_3 \wedge \text{false} = \text{false} \\
 \llbracket \Delta(\{q_1, q_2\}, b) \rrbracket &= \llbracket \delta(q_1, b) \rrbracket \bigcup \llbracket \delta(q_2, b) \rrbracket = \rho_{\subseteq}(\{\{q_2, q_3\}\}) \uplus \emptyset = \emptyset \quad . \\
 \Delta(\{q_1, q_2\}, a) &= \delta(q_1, a) \wedge \delta(q_2, a) = \text{false} \wedge q_3 = \text{false} \\
 \llbracket \Delta(\{q_1, q_2\}, a) \rrbracket &= \llbracket \delta(q_1, a) \rrbracket \bigcup \llbracket \delta(q_2, a) \rrbracket = \emptyset \uplus \rho_{\subseteq}(\{\{q_3\}\}) = \emptyset \quad . \\
 \Delta(\{q_3\}, a) &= \Delta(\{q_3\}, b) = q_3 \\
 \llbracket \Delta(\{q_3\}, a) \rrbracket &= \llbracket \Delta(\{q_3\}, b) \rrbracket = \rho_{\subseteq}(\{\{q_3\}\}) \quad .
 \end{aligned}$$

Since q_0 is the only initial state and $F = \{q_3\}$, it follows that the language generated by the automaton is $\mathcal{L}(\mathcal{A}) = a(a+b)^*$. $\diamond$

We denote the reflexo-transitive closure of $\llbracket \Delta \rrbracket$ as $\sim$. Thus, the language of an AFA, $\mathcal{A}$, is $\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^* \mid \exists q_i \in I, S \subseteq F, S \neq \emptyset \wedge \{q_i\} \sim^w S\}$.

One of the most interesting properties of AFAs is that their complement, i.e. an AFA generating the complement language, can be built in polynomial time.

Definition 7.1.2 (Complement of an AFA). Let $\mathcal{A} = \langle Q, \Sigma, \delta, I, F \rangle$ be an AFA with $L = \mathcal{L}(\mathcal{A})$. Its complement AFA, denoted $\mathcal{A}^c$ is the AFA $\mathcal{A}^c \stackrel{\text{def}}{=} \langle Q, \Sigma, \delta^c, I, Q \setminus F \rangle$ where $\delta^c(q, a)$ is the result of switching $\wedge$ and $\vee$ operators in $\delta(q, a)$. $\blacksquare$

The simplicity of the computation of the complement for AFAs, which does not alter the number of states of the automaton, allows us to use them in order to solve the language inclusion problem $\mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2)$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are NFAs, by reducing it to universality of alternating automata as follows:

$$\begin{aligned}
 \mathcal{L}(\mathcal{N}_1) \subseteq \mathcal{L}(\mathcal{N}_2) &\Leftrightarrow [\text{Since NFAs} \subseteq \text{AFAs}] \\
 \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) &\Leftrightarrow [A \subseteq B \Leftrightarrow A \cap B^c = \emptyset] \\
 \mathcal{L}(\mathcal{A}_1) \cap (\mathcal{L}(\mathcal{A}_2))^c = \emptyset &\Leftrightarrow [(A \cap B)^c = A^c \cup B^c \text{ and } \emptyset^c = \Sigma^*] \\
 (\mathcal{L}(\mathcal{A}_1))^c \cup \mathcal{L}(\mathcal{A}_2) = \Sigma^* &\Leftrightarrow [\text{AFAs are closed under complement}] \\
 \mathcal{L}(\mathcal{A}_1^c) \cup \mathcal{L}(\mathcal{A}_2) = \Sigma^* &\Leftrightarrow [A = \Sigma^* \Leftrightarrow \Sigma^* \subseteq A] \\
 \Sigma^* \subseteq \mathcal{L}(\mathcal{A}_1^c) \cup \mathcal{L}(\mathcal{A}_2) &\Leftrightarrow [\text{With } \mathcal{A}_3 = \mathcal{A}_1^c \cup \mathcal{A}_2] \\
 \Sigma^* &\subseteq \mathcal{L}(\mathcal{A}_3)
 \end{aligned} \tag{7.2}$$

On the other hand, Σ^* is the lfp of the equation $\lambda X. \{\varepsilon\} \cup \bigcup_{a \in \Sigma} aX$. Therefore

$$\Sigma^* \subseteq \mathcal{L}(\mathcal{A}_3) \Leftrightarrow \text{lfp}(\lambda X. \{\varepsilon\} \cup \bigcup_{a \in \Sigma} aX) \subseteq \mathcal{L}(\mathcal{A}_3) .$$

We are now in position to leverage our quasiorder-based framework from Chapter 4 to derive an algorithm for deciding the universality of a regular language given by an AFA $\mathcal{A}$. To do that, we adapt our right state-based quasiorder from Equation 4.12, which requires defining the successor operator for AFAs $\text{post}_w^{\mathcal{A}} : \wp(\wp(Q)) \rightarrow \wp(\wp(Q))$, where $w \in \Sigma^*$, as follows:

$$\text{post}_w^{\mathcal{A}}(X) \stackrel{\text{def}}{=} \{S' \in \wp(Q) \mid \exists S \in X, S \xrightarrow{w} S'\} . \quad (7.3)$$

It is straightforward to check that $\text{post}_{wa}^{\mathcal{A}}(X) = \text{post}_a^{\mathcal{A}}(\text{post}_w^{\mathcal{A}}(X))$. The following example illustrates the behavior of the function $\text{post}_w^{\mathcal{A}}$ on the AFA from Figure 7.1.

Example 7.1.3. Consider again the AFA $\mathcal{A}$ from Figure 7.1. We have that

$$\begin{aligned} \text{post}_a^{\mathcal{A}}(\{\{q_0\}\}) &= \rho_{\subseteq}(\{\{q_1, q_2\}, \{q_3\}\}) \\ \text{post}_{aa}^{\mathcal{A}}(\{\{q_0\}\}) &= \text{post}_a^{\mathcal{A}}(\{\{q_1, q_2\}, \{q_3\}\}) = \rho_{\subseteq}(\{\{q_3\}\}) \\ \text{post}_{ab}^{\mathcal{A}}(\{\{q_0\}\}) &= \text{post}_b^{\mathcal{A}}(\{\{q_1, q_2\}, \{q_3\}\}) = \rho_{\subseteq}(\{\{q_3\}\}) . \end{aligned} \quad \diamond$$

Similarly to what we did in Section 4.3.3 for NFAs, we next define a state-based quasiorder for AFAs, $\leq_{\mathcal{A}}$. To do that, let $I_{\{ \}}$ be the set of singleton subsets of I , i.e. $I_{\{ \}} \stackrel{\text{def}}{=} \{\{q\} \mid q \in I\}$. Then

$$u \leq_{\mathcal{A}} v \Leftrightarrow \text{post}_u^{\mathcal{A}}(I_{\{ \}}) \subseteq \text{post}_v^{\mathcal{A}}(I_{\{ \}}) \quad (7.4)$$

LEMMA 7.1.4. Let $\mathcal{A} = \langle Q, \Sigma, \delta, I, F \rangle$ be an AFA with $L = \mathcal{L}(\mathcal{A})$. Then $\leq_{\mathcal{A}}$ is a right L -consistent well-quasiorder.

Proof. First, we show that $\leq_{\mathcal{A}}$ is right monotone. Let $u, v \in \Sigma^*$ and $a \in \Sigma$. Recall that $\text{post}_a^{\mathcal{A}}$ is a monotonic function and that

$$\text{post}_{uv}^{\mathcal{A}} = \text{post}_v^{\mathcal{A}} \circ \text{post}_u^{\mathcal{A}} . \quad (7.5)$$

Then

$$\begin{aligned} u \leq_{\mathcal{A}} v &\Rightarrow \text{[By definition of } \leq_{\mathcal{A}}\text{]} \\ \text{post}_u^{\mathcal{A}}(I_{\{ \}}) &\subseteq \text{post}_v^{\mathcal{A}}(I_{\{ \}}) \Rightarrow \text{[Since } \text{post}_a^{\mathcal{A}} \text{ is monotonic]} \\ \text{post}_a^{\mathcal{A}}(\text{post}_u^{\mathcal{A}}(I_{\{ \}})) &\subseteq \text{post}_a^{\mathcal{A}}(\text{post}_v^{\mathcal{A}}(I_{\{ \}})) \Leftrightarrow \text{[By Equation (7.5)]} \\ \text{post}_{ua}^{\mathcal{A}}(I_{\{ \}}) &\subseteq \text{post}_{va}^{\mathcal{A}}(I_{\{ \}}) \Leftrightarrow \text{[By definition of } \leq_{\mathcal{A}}\text{]} \\ ua &\leq_{\mathcal{A}} va . \end{aligned}$$

On the other hand, $\leq_{\mathcal{A}}$ is L -consistent since, by definition

$$\forall w \in \Sigma^*, w \in L \Leftrightarrow \exists S \in \text{post}_w^{\mathcal{A}}(I_{\{\}}), S \neq \emptyset \wedge S \subseteq F.$$

Therefore, if $u \in L$ and $u \leq_{\mathcal{A}} v$ then it follows that $v \in L$.

Finally, it is straightforward to check that $\leq_{\mathcal{A}}$ is a well-quasiorder since $\wp(\wp(Q))$ is finite.

Since membership in AFAs is decidable, it follows from Lemma 7.1.4 and Theorem 4.3.4 that Algorithm FAIncWr instantiated with the wqo $\leq_{\mathcal{A}}$ decides the inclusion $\Sigma^* \subseteq \mathcal{L}(\mathcal{A})$, where $\mathcal{A}$ is an AFA.

Following the developments of Chapter 4, given an AFA $\mathcal{A} = \langle Q, \Sigma, \delta, I, F \rangle$, we could define a Galois Connection $\langle \wp(\Sigma^*), \subseteq \rangle \xrightleftharpoons[\alpha]{\gamma} \langle \text{AC}_{\langle \wp(\wp(Q)), \subseteq \rangle}, \sqsubseteq \rangle$ that yields an antichains algorithm for deciding the universality of AFAs by manipulating sets of sets of states. By doing so, we would obtain an algorithm that computes the set $Y = \lfloor \{\text{post}_w^{\mathcal{A}}(I_{\{\}}) \mid w \in \Sigma^*\} \rfloor$ and checks whether all elements $y \in Y$ satisfy $\exists s \in y, s \neq \emptyset \wedge s \subseteq F$.

Moreover, we could enhance the state-based quasiorder for AFAs by using simulations between the states of $\mathcal{A}$ which, recall, are the union of the states of the input automata $\mathcal{N}_1$ and $\mathcal{N}_2$. This would allow us to use the simulations that relate states of both automata, similarly to Abdulla et al. [2010] and Bonchi and Pous [2013].

Therefore, we believe that the full development of an antichains algorithm for AFAs is an interesting line for future work since it will allow us to understand how close our framework can get to the results of Abdulla et al. [2010] and Bonchi and Pous [2013].

7.2 The Complexity of Searching on Compressed Text

We believe the good results obtained during the evaluation of `zsearch` (see Figure 5.5) invite for a deeper study of our algorithm in order to better understand its behavior and improve its performance.

For instance, it is yet to be considered how the performance of `zsearch` is affected by the choice of the grammar-based compression algorithm. By using different heuristics to build the grammar, the resulting SLP will have different properties, such as depth, width or length of the rules, which would definitely affect `zsearch`'s performance. Figure 7.2 shows the grammars built by different compression algorithms for the same string.

In particular, there are grammar-based compression algorithms such as Sequitur [Nevill-Manning and Witten 1997] that produce SLPs which are not

Figure 7.2: From left to right, grammars built by the compression algorithms sequitur [Nevill-Manning and Witten 1997], repair [Larsson and Moffat 1999] and LZW [Welch 1984] for “xabcd b c a b c d b c y”.

in CNF, i.e. in which rules might have more than two symbols on the right hand side. Processing such a grammar, instead of the one built by repair reduces the number of rules to be processed at the expense of a greater cost for processing each rule. It is worth considering whether adapting `zsearch` to work on such SLPs will have a positive impact on its performance.

On the other hand, Algorithm `COUNTLINES` allows for a conceptually simple parallelization since any set of rules such that no variable appearing on the left hand side of a rule appears on the right hand side of another, can be processed simultaneously. Indeed, a theoretical result by Ullman and Gelder [1988] on the parallelization of Datalog queries can be used to show that counting the number of lines in a grammar-compressed text containing a match for a regular expression is in NC^2 , i.e. it is solvable in *polylogarithmic time* on *parallel* computer with a polynomial number of processors, when the automaton built from the expression is acyclic. Therefore, we are optimistic about the possibilities of a parallel version of `zsearch`.

Finally, *patterns* are a commonly used subclass of regular expressions for which specific searching algorithms have been developed [Kida et al. 1998; Navarro and Tarhio 2005; Gawrychowski 2011; 2014]. Since the standard automata construction from regular expressions yields a DFA when the expression is a pattern, our algorithm allows us to search for patterns in $O(t \cdot s)$ time, where t is the size of the grammar and s is the length of the pattern. However, as shown by Gawrychowski [2011], it is possible to decide the existence of a pattern in an LZW-compressed text in $O(t + s)$ time. It is yet to be considered whether the algorithm of Gawrychowski [2011] can be adapted to the more general scenario of searching on grammar-compressed text and whether it can be extended to re-

port the number of matching lines without altering its complexity as we did with Algorithm [COUNTLINES](#).

7.3 The Performance of Residualization

In Chapter 6 we presented the automata construction $\text{Res}^r(\mathcal{N})$ as an alternative to $\mathcal{N}^{\text{res}}$, the residualization operation defined by [Denis et al. \[2002\]](#). We have shown in Theorem 6.2.6 (e) that given an NFA $\mathcal{N}$, the automaton $\text{Res}^r(\mathcal{N})$ is a sub-automaton of $\mathcal{N}^{\text{res}}$, meaning that our construction yields smaller automata.

On the other than, it is clear that, given an NFA $\mathcal{N} = \langle Q, \Sigma, \delta, I, F \rangle$, finding the coverable sets in $\{\text{post}_u^{\mathcal{N}}(I) \mid u \in \Sigma^*\}$ is easier than finding the L -composite principals in $\{\rho_{\leq_{\mathcal{N}}}^r(u) \mid u \in \Sigma^*\}$. However, it is yet to be considered the performance of both algorithms and the actual difference in size between the RFAs $\text{Res}^r(\mathcal{N})$ and $\mathcal{N}^{\text{res}}$.

7.3.1 Reducing RFAs with Simulations

Let $\mathcal{N}$ be an NFA with $L = \mathcal{L}(\mathcal{N})$. As shown by Lemma 4.3.9, the simulation-based quasiorder $\leq_{\mathcal{N}}^r$ is an L -consistent right well-quasiorder. Therefore, it follows from Lemma 6.1.4 that $\text{H}^r(\leq_{\mathcal{N}}^r, L)$ is an RFA generating the language L . Moreover, as shown in Section 4.3.3.2, we have the following relation between the state-based, the simulation-based and the Nerode's right quasiorders:

$$\leq_{\mathcal{N}}^r \subseteq \leq_{\mathcal{N}}^r \subseteq \leq_{\mathcal{L}(\mathcal{N})}^r .$$

Therefore, by Theorem 6.1.9, we have that

$$\begin{aligned} & |\{\rho_{\leq_{\mathcal{N}}}^r(u) \mid u \in \Sigma^* \text{ and } \rho_{\leq_{\mathcal{N}}}^r(u) \text{ is } L\text{-prime}\}| \\ & \quad \text{IV} \\ & |\{\rho_{\leq_{\mathcal{N}}}^r(u) \mid u \in \Sigma^* \text{ and } \rho_{\leq_{\mathcal{N}}}^r \text{ is } L\text{-prime}\}| \\ & \quad \text{IV} \\ & |\{\rho_{\leq_L}^r(u) \mid u \in \Sigma^* \text{ and } \rho_{\leq_L}^r(u) \text{ is } L\text{-prime}\}| . \end{aligned}$$

One promising direction for future work is to fully develop this idea of using simulation-based quasiorders to build even smaller RFAs. Such technique should be implemented and evaluated in practice in comparison with the residualization operations $\text{Res}^r(\mathcal{N})$ and $\mathcal{N}^{\text{res}}$.

CONCLUSIONS

In this thesis, we have shown that well-quasiorders are the right tool for addressing different problems from *Formal Language Theory*. Indeed, we presented two quasiorder-based frameworks in Chapters 4 and 6 that allowed us to offer a new perspective on *The Language Inclusion Problem* and *Residual Automata*, respectively. In both cases, our frameworks allowed us to (i) offer a *new perspective on known algorithms* that facilitates their understanding and evidences the relationships between them and (ii) *systematically derive new algorithms*, some of which proved to be of practical interest due to their performance.

The Language Inclusion Problem

We have been able to systematically derive well-known algorithms such as the antichains algorithms for regular languages of Wulf et al. [2006], with its multiple variants (see Section 4.4), and the antichains algorithm for grammars of Holík and Meyer [2015]. These systematic derivations result in a simpler presentation of the antichains algorithm for grammars of Holík and Meyer [2015] as a straightforward extension of the antichains algorithm for regular languages. Indeed, we have shown that the antichains algorithm for regular languages and for grammars are conceptually identical and correspond to two instantiations of our framework with different quasiorders. Recall that, previously, the use of antichains for grammars was justified through a reduction to data flow analysis.

Our framework has also allowed us to derive algorithms for deciding the inclusion of a regular language in the trace set of a one-counter net. In doing so, we have shown that the right Nerode quasiorder for the trace set of a one-counter net is an undecidable well-quasiorder, thereby closing a conjecture made by de Luca and Varricchio [1994, Section 6].

Finally, our quasiorder-based framework also allowed us to derive novel algorithms, such as gfp-based Algorithm [FAIncGFP](#), for deciding the inclusion between regular languages. It is yet to be considered the performance of this algorithm in order to decide whether it is of practical interest.

Searching on Compressed Text

We then adapted the antichains algorithm for grammars to the problem of *searching with regular expressions in grammar compressed text*. As a result, we have presented the first algorithm for *counting* the number of lines in a grammar-compressed text containing a match for a regular expression. It is worth to remark that our algorithm applies to any grammar-based compression scheme while being nearly optimal.

Together with the presentation of our algorithm, we described in Chapter [5](#) the data structures required to achieve nearly optimal complexity for searching in compressed text and used them to implement a *sequential* tool –*zsearch*– that significantly outperforms the *parallel* state of the art to solve this problem. Indeed, when the grammar-based compressor achieves high compression ratio, which is the case, for instance, for automatically generated *Log* files, *zsearch* uses up to 25% less time than *lz4|hyperscan*, even outperforming *grep* and being competitive with *hyperscan*.

Our results evidence that compression of textual data and regular expression searching, two problems considered independent in practice, are connected. Intuitively, the search can take advantage of the information about repetitions in the text, highlighted by the compressor, to skip parts of the uncompressed text.

Residual Automata

[Denis et al. \[2002\]](#) introduced the notion of RFA and canonical RFA for a regular language and devised a procedure, similar to the subset construction for DFAs, to build the RFA $\mathcal{N}^{\text{res}}$ from a given automaton $\mathcal{N}$. Furthermore, they showed that the *double-reversal method* holds for RFAs with their *residualization* operation, i.e. $\mathcal{N}^{\text{res}}$ is isomorphic to the canonical RFA C for $\mathcal{L}(\mathcal{N})$ for every co-RFA $\mathcal{N}$.

Later, [Tamm \[2015\]](#) proved the following result:

LEMMA 8.1 ([Tamm \[2015\]](#)). *Let $\mathcal{N}$ be an NFA and let C be the canonical RFA for $\mathcal{L}(\mathcal{N})$. Then, $\mathcal{N}^{\text{res}}$ is isomorphic to C iff the left language of every state of $\mathcal{N}$ is a union of left languages of states of C .*

The result of [Tamm \[2015\]](#) generalizes the double-reversal method for RFAs along the lines of the generalization by [Brzozowski and Tamm \[2014\]](#) of the double-reversal method for DFAs which we state next.

LEMMA 8.2 (Brzozowski and Tamm [2014]). *Let $\mathcal{N}$ be an NFA and let $\mathcal{M}$ be the minimal DFA for $\mathcal{L}(\mathcal{N})$. Then $\mathcal{N}^D$ is isomorphic to $\mathcal{M}$ iff the left language of each state of $\mathcal{N}$ is a union of co-atoms of $\mathcal{L}(\mathcal{N})$.*

Although the two generalizations have a common foundation, the connection between the two resulting characterizations is not immediate. Our work, together with the work of Ganty et al. [2019] allows us to clarify the relation between these two results and our Theorem 6.3.2. Indeed, Ganty et al. [2019] offered a congruence-based perspective of the generalized double-reversal method for building the minimal DFA which lead to the following result.

LEMMA 8.3 (Ganty et al. [2019]). *Let $\mathcal{N}$ be an NFA and let $\mathcal{M}$ be the minimal DFA for $\mathcal{L}(\mathcal{N})$. Then $\mathcal{N}^D$ is isomorphic to $\mathcal{M}$ iff*

$$\rho_{\sim_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}} ,$$

where $\sim_L^r \stackrel{\text{def}}{=} \leq_L^r \cap (\leq_L^r)^{-1}$ is the right Nerode's congruence.

We believe that the similarity between the generalizations of the double-reversal methods for the minimal DFA (Lemma 8.3) and for the canonical RFA (Theorem 6.3.2), which says that

$$\text{Res}^r(\mathcal{N}) \text{ is isomorphic to } C \Leftrightarrow \rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}} ,$$

evidences that *quasiorders are for RFAs as congruences are for DFAs*. Figure 8.1 summarizes the existing results about these double-reversal methods.

<p>Brzozowski and Tamm [2014]</p> <p>$\mathcal{N}^D \equiv \mathcal{M}$ iff $\forall q, W_{I,q}^{\mathcal{N}}$ is a union of co-atoms</p>	<p>Theorem 6.3.2</p> <p>$\mathcal{N}^D \equiv \mathcal{M}$ iff $\rho_{\sim_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$</p>	<p>In the diagram: $\mathcal{N}$ is an NFA with $L = \mathcal{L}(\mathcal{N})$; $\mathcal{N}^D$ is the result of determinizing $\mathcal{N}$ with the standard subset construction; $\mathcal{M}$ is the minimal DFA for L; $C = \text{Can}^r(L)$ is the canonical RFA for L and $\mathcal{N}_1 \equiv \mathcal{N}_2$ denotes that automaton $\mathcal{N}_1$ is isomorphic to $\mathcal{N}_2$.</p>
<p>Ganty et al. [2019]</p> <p>$\mathcal{N}^{\text{res}} \equiv C$ iff $\forall q, W_{I,q}^{\mathcal{N}}$ is a union of $W_{I,q'}^C$</p>	<p>Tamm [2015]</p> <p>$\text{Res}^r(\mathcal{N}) \equiv C$ iff $\rho_{\leq_L^r}(W_{I,q}^{\mathcal{N}}) = W_{I,q}^{\mathcal{N}}$</p>	

Figure 8.1: Summary of the existing results about the generalized double-reversal method for building the minimal DFA (first row) and the canonical RFA (second row) for a given language. The results on the first column are based on the notion of atoms of a language while the results on the second column are based on quasiorders.

Moreover, as shown by Lemma 6.2.8, our residualization operation $\text{Res}^r(\mathcal{N})$ offers a desirable property that $\mathcal{N}^{\text{res}}$ lacks: residualizing $\mathcal{N}$ yields the canonical RFA for $\mathcal{L}(\mathcal{N})$ iff $\leq_L^r = \leq_N^r$. Again, this property is equivalent to the one presented by Ganty et al. [2019] for DFAs.

LEMMA 8.4 (Ganty et al. [2019]). *Let $\mathcal{N}$ be an NFA and let $\mathcal{M}$ be the minimal DFA for $\mathcal{L}(\mathcal{N})$. Then $\mathcal{N}^D$ is isomorphic to $\mathcal{M}$ iff $\sim_N^r = \sim_L^r$, where $\sim_N^r \stackrel{\text{def}}{=} \leq_N^r \cap (\leq_N^r)^{-1}$ is the right state-based congruence.*

On the other hand, since Ganty et al. [2019] showed that the left languages of the minimal DFA for a regular language are the blocks of the partition $\rho_{\sim_L^r}$, Lemma 8.3 can be equivalently stated as follows.

LEMMA 8.5 (Ganty et al. [2019]). *Let $\mathcal{N}$ be an NFA and let $\mathcal{M}$ be the minimal DFA for $\mathcal{L}(\mathcal{N})$. Then $\mathcal{N}^D$ is isomorphic to $\mathcal{M}$ iff the left language of each state of $\mathcal{N}$ is a union of left languages of states of the minimal DFA.*

Therefore, Lemma 8.5 can be seen as the DFA-equivalent of Tamm’s condition for RFAs (Lemma 8.1). Therefore, Lemma 8.5 together with Lemma 8.3, evidence the connection between the generalization of the double reversal for RFAs of Tamm [2015] and the one for DFAs of Brzozowski and Tamm [2014].

Finally, we further support the idea that quasiorders are natural to residual automata by observing that the NL^* algorithm proposed by Bollig et al. [2009] for learning RFAs can be interpreted within our framework as an algorithm that, at each step, refines an approximation of the Nerode’s quasiorder and builds an RFA using our automata construction.

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