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Symbolic Computation of Drazin inverses by specializations

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Abstract

In this paper, we show how to reduce the computation of Drazin inverses over certain computable fields to the computation of Drazin inverses of matrices with rational functions as entries. As a consequence we derive a symbolic algorithm. The algorithm is applied to matrices over the field of meromorphic functions, in several complex, on a connected domain and to matrices over the field of Laurent formal power series.

keywords Drazin inverse, analytic perturbation, Gröbner bases, symbolic computation, meromorphic functions, Laurent formal power series.

1 Introduction

This paper shows how symbolic computation can be applied to compute Drazin inverses of matrices whose coefficients are rational functions of finitely transcendental elements over the field of rational functions $\mathbb{C}(w_1, \ldots, w_r)$.

1.1 The interest of Drazin Inverses

Motivated on the notion of generalized inverse introduced by Moore and Penrose, Drazin, in [12], introduced the notion of Drazin inverse in the more general context of rings and semigroups. For matrices, Drazin inverse is defined as follows. Let A be an square matrix over a field, then the Drazin index of A is the smallest non-negative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$; let us denote it by $\mathrm{index}(A)$. In this situation, the Drazin inverse of A is the unique matrix satisfying the following matrix equations:

$$\begin{cases} A^{\mathrm{index}(A)+1} \cdot X = A^{\mathrm{index}(A)}, \\ A \cdot X = X \cdot A, \\ X \cdot A \cdot X = X. \end{cases} \tag{1}$$

Many authors have analyzed the properties of Drazin inverses (see e.g. Chapter 4 in [3], Chapters 7 in [7], [11], [17]) as well as their applications (see e.g. Chapters 8 and 9 in [7], [8], [16], [23]); particularly interesting are the application to singular differential equations and difference equations, and to finite Markov chains.

1.2 Computing Drazin Inverses: the state of the art

An important alternative issue in the topic is the computation of the Drazin inverse (see e.g. [3], [7], [16], [17], [18], [19]). The problem has been approached mainly for matrices with complex numbers. Nevertheless, in a second stage, different authors have addressed the problem of computing Drazin inverses of matrices over other coefficients domains as rational function fields (see [5], [14], [20], [21]). Furthermore, symbolic, techniques have proven to be a suitable tools for this goal.

The next challenging step is the computation of the Darzin inverses over more general domains, as for instance the field of formal Laurent series (see connection to the study of analytically perturbed matrices in [16]) or the field of meromorphic functions; in [6] one can read about the difficulties of reasoning with transcendental functions. In this paper, we show how to reduce the symbolic computation of the Drazin inverses over such domains to the computation over the field of rational functions, and hence the already available algorithms, in particular the one based on Gröbner bases developed in [21], can be applied.

1.3 Main contributions of the paper

In general terms, the main contribution of the paper is the establishment of an algorithmic criterium to symbolically reduce the computation of Darzin inverses of matrices over a field $\mathbb{C}(t_1 \dots, t_r)$, where $t_1, \dots, t_r$ are transcendental elements over $\mathbb{C}$, to the computation of Darzin inverses of matrices over the field of rational functions

$\mathbb{C}(w_1,\dots,w_r)$, where $w_1,\dots,w_r$ are independent complex variables. More precisely, the main contributions of the paper can be summarized as follows:

- we prove the existence, and actual computation, of a multivariate polynomial in $\mathbb{C}[w_1,\dots,w_r]$ (that we call the evaluation polynomial, see Def. 7) such that if it does not vanish at $(t_1,\dots,t_r)$ the computation of the Drazin inverse over $\mathbb{C}(t_1\,\dots,t_r)$ is reduced to the computation of the Drazin inverse over $\mathbb{C}(w_1\,\dots,w_r)$.
- As an application, we show how to compute the Drazin inverse of matrices whose entries are meromorphic functions or Laurent formal power series.
- Furthermore, we show how to relate the specialization of the Drazin inverse of a matrix, with meromorphic function entries, and the Drazin inverse of the specialization.
- Also, as a consequence of these ideas, one gets a method to compute the Laurent expansion of the Drazin inverses of analytically perturbed matrices.

1.4 Intuitive idea of the novel proposed solution method

Given a matrix A, the idea consists in the following three steps:

1. [Specialization step] first we associate to A a matrix A^*, whose entries are rational functions in several variables;
2. [Inverse computation step] secondly we compute the Drazin inverse of A^*, for instance using the algorithm in [21];
3. [Evaluation step] finally, from the Drazin inverse of A^* we get the Drazin inverse of A.

More precisely, we consider the chain of fields

$$\mathbb{K} \subset \mathbb{K}(t_1,\dots,t_r) \subset \mathbb{F}$$

where $\mathbb{K}$ is computable, and $\mathbb{K}(t_1,\dots,t_r)$ is the field extension of $\mathbb{K}$ via the adjunction of finitely many elements $t_1,\dots,t_r \in \mathbb{F}\setminus\mathbb{K}$. For instance, we may take

$$\mathbb{C} \subset \mathbb{C}(\sin(z),\cosh(z)) \subset \mathrm{Mer}(\mathbf{z},\Omega)$$

where $\mathrm{Mer}(\mathbf{z},\Omega)$ is the field of meromorphic functions on a connected domain Ω of $\mathbb{C}$, or we may take

$$\mathbb{C} \subset \mathbb{C}\left(\sum_{n=-2}^{\infty}\frac{1}{n^4+1}z^n, \sum_{n=0}^{\infty}\frac{n}{n+1}z^n\right) \subset \mathbb{C}((z))$$

where $\mathbb{C}((z))$ is the field of Laurent formal power series. Now we are given an $n \times n$ matrix $A \in \mathcal{M}_{n\times n}(\mathbb{K}(t_1,\ldots,t_r))$ with entries in the intermediate field $\mathbb{K}(t_1,\ldots,t_r)$, and we want to compute the Drazin inverse of A. For this purpose, we replace A by a matrix $A^* \in \mathcal{M}_{n\times n}(\mathbb{K}(w_1,\ldots,w_r))$ with entries in the field of rational functions $\mathbb{K}(w_1,\ldots,w_r)$. Then, the Drazin inverse of A is achieved by specializing the Drazin inverse of A^* at $(w_1,\ldots,w_r) := (t_1,\ldots,t_r)$. We prove the existence, and we actually show how to compute, of a multivariate polynomial in $\mathbb{K}[w_1,\ldots,w_r]$ such that if it does not vanish at $(t_1,\ldots,t_r)$ the correctness of the method is guaranteed.

1.5 Structure of the paper

The paper is structured as follows. In Section 2, we fix the notation and we introduce the theoretical framework. In Section 3 we present the method itself and we prove the main theorems. In Sections 4 and 5 we apply the method to the case of meromorphic functions and to the case Laurent formal power series, respectively.

2 Theoretical framework

In this section, we fix the notation to be used throughout the paper and we introduce the theoretical framework of the paper.

Let $\mathbb{K}$ be a subfield of a field $\mathbb{F}$, let $\mathbf{w} = (w_1\ldots,w_r)$ a tuple of variables, and let $\mathbf{t} = (t_1,\ldots,t_r) \in (\mathbb{F}\setminus\mathbb{K})^r$. We will assume w.l.o.g. that each t_i is transcendental over $\mathbb{K}$. Note that if this is not the case, we can always replace $\mathbb{K}$ by $\mathbb{K}(t_i)$. In addition, we also assume w.l.o.g. that $t_i \neq t_j$ if $i \neq j$. We assume that $\mathbf{t}$ is fixed. For instance, $\mathbb{K}$ could be $\mathbb{C}$, $\mathbb{F}$ the meromorphic functions on a connected domain of $\mathbb{C}$, and $\mathbf{t} = (z, \sin(z), \mathrm{e}(z))$.

We consider the following rings of matrices

$$\mathrm{R}_{\mathbf{w}} = \mathcal{M}_{n\times n}(\mathbb{K}(\mathbf{w})), \text{ and } \mathrm{R}_{\mathbf{t}} = \mathcal{M}_{n\times n}(\mathbb{K}(\mathbf{t})).$$

Given $A \in \mathrm{R}_{\mathbf{w}}$, we define the denominator of A, denoted as $\mathrm{den}(A)$, as the least common multiple of the denominators of all entries, taken in reduced form, of A. And we define the numerator of A, denoted by $\mathrm{num}(A)$, as

$$\mathrm{num}(A) = \mathrm{den}(A)\cdot A.$$

Note that $\mathrm{num}(A) \in \mathcal{M}_{n\times n}(\mathbb{K}[\mathbf{w}])$. Now, we introduce the set

$$\mathcal{O}_{\mathbf{t}} = \{A(\mathbf{w}) \in \mathrm{R}_{\mathbf{w}} \mid \mathrm{den}(A)(\mathbf{t}) \neq 0\}.$$

$\mathcal{O}_{\mathbf{t}}$ is a subring of $\mathrm{R}_{\mathbf{w}}$. In this situation, we introduce the ring homomorphism

$$\begin{array}{rcl} \Phi_{\mathbf{t}}: \; \mathcal{M}_{n\times n}(\mathbb{K}[\mathbf{w}]) & \longrightarrow & \mathcal{M}_{n\times n}(\mathbb{K}[\mathbf{t}]) \\ A(\mathbf{w}) = (a_{i,j}(\mathbf{w}))_{1\leq i,j\leq n} & \longmapsto & A(\mathbf{t}) = (a_{i,j}(\mathbf{t}))_{1\leq i,j\leq n} \end{array}$$

Furthermore, we extend $\Phi_{\mathbf{t}}$ to $\mathcal{O}_{\mathbf{t}}$ as follows

$$\begin{array}{rcl} \Phi_{\mathbf{t}}: \; \mathcal{O}_{\mathbf{t}} \subset \mathrm{R}_{\mathbf{w}} & \longrightarrow & \mathrm{R}_{\mathbf{t}} \\ A(\mathbf{w}) & \longmapsto & \dfrac{1}{\mathrm{den}(A)(\mathbf{t})}\mathrm{num}(A)(\mathbf{t}). \end{array}$$

We want to work with the inverse of $\Phi_{\mathbf{t}}$. However, in general, although $\Phi_{\mathbf{t}}$ is surjective it is not injective (see Example 1). To overcome this difficulty, we consider the quotient set $\mathcal{O}_{\mathbf{t}}/\ker\,(\Phi_{\mathbf{t}})$. For $A \in \mathcal{O}_{\mathbf{t}}$, we denote its equivalence class as $[A]$. We recall that

$$[A] := A + \ker\,(\Phi_{\mathbf{t}}) = \{A + B \mid B \in \mathcal{O}_{\mathbf{t}} \text{ and } \Phi_{\mathbf{t}}(B) = \mathbf{O}\}. \tag{2}$$

Then, instead of working with $\Phi_{\mathbf{t}}$ we work with

$$\begin{array}{rcl} \overline{\Phi}_{\mathbf{t}}: \; \mathcal{O}_{\mathbf{t}}/\ker\,(\Phi_{\mathbf{t}}) \subset \mathrm{R}_{\mathbf{w}} & \longrightarrow & \mathrm{R}_{\mathbf{t}} \\ {[A(\mathbf{w})]} & \longmapsto & A(\mathbf{t}). \end{array}$$

Now, $\overline{\Phi}_{\mathbf{t}}$ is bijective and its inverse is

$$\begin{array}{rcl} \Psi_{\mathbf{w}}: \; \mathrm{R}_{\mathbf{t}} & \longrightarrow & \mathcal{O}_{\mathbf{t}}/\ker\,(\Phi_{\mathbf{t}}) \\ A = (a_{i,j}(\mathbf{t}))_{1\leq i,j\leq n} & \longmapsto & [(a_{i,j}(\mathbf{w}))_{1\leq i,j\leq n}]. \end{array}$$

Let us see a simple example.

Example 1 *Let $\mathbb{K} = \mathbb{C}$ and $\mathbb{F}$ the meromorphic functions in an open connected subset of $\mathbb{C}$. We take $\mathbf{t} = (\sin(z), \cos(z))$. Now, we consider the matrix*

$$A = \begin{pmatrix} 0 & \sin(z) \\ \cos(z) & 1 \end{pmatrix} \in \mathrm{R}_{\mathbf{t}} = \mathcal{M}_{2\times 2}(\mathbb{C}(\sin(z), \cos(z))) \subset \mathcal{M}_{2\times 2}(\mathbb{F}).$$

Then,

$$\begin{array}{rccc} \Phi_{\mathbf{t}}: & \mathcal{O}_{\mathbf{t}} & \longrightarrow & \mathrm{R}_{\mathbf{t}} \\ & \begin{pmatrix} a_{11}(w_1,w_2) & a_{12}(w_1,w_2) \\ a_{21}(w_1,w_2) & a_{22}(w_1,w_2) \end{pmatrix} & \longmapsto & \begin{pmatrix} a_{11}(\sin(z),\cos(z)) & a_{12}(\sin(z),\cos(z)) \\ a_{21}(\sin(z),\cos(z)) & a_{22}(\sin(z),\cos(z)) \end{pmatrix}. \end{array}$$

So,

$$ker\,(\Phi_{\mathbf{t}}) = \{(a_{i,j}(w_1,w_2))_{1\leq i,j\leq 2} \in \mathcal{O}_{\mathbf{t}} \mid (a_{i,j}(\sin(z),\cos(z)))_{1\leq i,j\leq 2} = (0)_{1\leq i,j\leq 2}\}.$$

Observe that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 - w_1^2 - w_2^2 \end{pmatrix} \in \mathit{ker}\,(\Phi_{\mathbf{t}}).$$

Thus, for instance,

$$B_1 = \begin{pmatrix} 0 & w_1 \\ w_2 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & w_1 \\ w_2 & w_1^2 + w_2^2 \end{pmatrix} \in \Psi_{\mathbf{w}}(A).$$

But, in both cases, $\overline{\Phi}_{\mathbf{t}}(\Psi_{\mathbf{w}}(A)) = \Phi_{\mathbf{t}}(B_1) = A = \Phi_{\mathbf{t}}(B_2) = \overline{\Phi}_{\mathbf{t}}(\Psi_{\mathbf{w}}(A))$. □

3 Drazin inverses under specializations

Given a square matrix A over a field, we denote by $\mathfrak{D}(A)$ the Drazin inverse of A, and by $\text{index}(A)$ the Drazin index of A; usually the Drazin inverse is denote by A^D or $A^{\#}$, here we have chosen $\mathfrak{D}(A)$ to avoid confusions when the matrix is specialized. We recall that the Drazin index of A is the smallest non-negative integer k such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$, and that $\mathfrak{D}(A)$ is the unique matrix satisfying the matrix equations (1) introduced in Subsection 1.1.

In the following we analyze the behavior of the Drazin inverse of matrices in $\mathrm{R}_{\mathbf{t}}$ under specializations. More precisely, if $A \in \mathrm{R}_{\mathbf{t}}$, we study when there exists $A^* \in [A]$ (see formula (2)) such that

$$\Phi_{\mathbf{t}}\,(\mathfrak{D}(A^*)) = \mathfrak{D}(\Phi_{\mathbf{t}}\,(A^*)),$$

that is, when

$$\mathfrak{D}(A^*)(\mathbf{t}) = \mathfrak{D}(A(\mathbf{t})).$$

In this case, we could compute the Drazin inverse of $\mathfrak{D}(A)$ specializing $\mathfrak{D}(A^*)$. In Diagram (3) we illustrate the strategy.

$$\begin{array}{ccc} A(\mathbf{t}) \in \mathrm{R}_{\mathbf{t}} & \xrightarrow{\quad \Psi_{\mathbf{w}} \quad} & A^*(\mathbf{w}) \in [A] \subset \mathrm{R}_{\mathbf{w}} \\ \Big\downarrow \text{Drazin} & & \text{Drazin} \Big\downarrow \\ \mathfrak{D}(A(\mathbf{t})) \overset{?}{=} \mathfrak{D}(A^*)(\mathbf{t}) \in \mathrm{R}_{\mathbf{t}} & \xleftarrow{\quad \Phi_{\mathbf{t}} \quad} & \mathfrak{D}(A^*)(\mathbf{w}) \in \mathrm{R}_{\mathbf{w}} \end{array} \qquad (3)$$

This motivates the next definition.

Definition 2 *We say that* $A \in \mathrm{R}_{\mathbf{t}}$ behaves properly under specialization *if there exists* $A^* \in [A]$ *such that*

1. $\mathfrak{D}(A^*) \in \mathcal{O}_{\mathbf{t}}$ *and*

2. $\Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \mathfrak{D}(A)$.

We will say that A specializes properly *at $(A^*, \mathbf{t})$.*

The next results shows that the key is the invariance of the index.

Theorem 3 *Let $A \in \mathrm{R}_{\mathbf{t}}$. If there exists $A^* \in [A]$ such that $\mathfrak{D}(A^*) \in \mathcal{O}_{\mathbf{t}}$ and* $\mathrm{index}(A^*) = \mathrm{index}(A)$*, then A specializes properly at $(A^*, \mathbf{t})$.*

Proof: First we observe that, since $A^* \in [A]$, then $\Phi_{\mathbf{t}}(A^*) = A$. Since $\mathfrak{D}(A^*)$ is the Drazin inverse of A^*, it holds (1), that is

$$\begin{cases} {A^*}^{\mathrm{index}(A^*)+1} \cdot \mathfrak{D}(A^*) = {A^*}^{\mathrm{index}(A^*)}, \\ A^* \cdot \mathfrak{D}(A^*) = \mathfrak{D}(A^*) \cdot A^*, \\ \mathfrak{D}(A^*) \cdot A^* \cdot \mathfrak{D}(A^*) = \mathfrak{D}(A^*). \end{cases}$$

By assumption $A^* \in \mathcal{O}_{\mathbf{t}}$. Since $\mathrm{den}(\mathfrak{D}(A^*))(\mathbf{t}) \neq 0$, one has that $\mathfrak{D}(A^*) \in \mathcal{O}_{\mathbf{t}}$. So, we can apply $\Phi_{\mathbf{t}}$ to the equations above. Taking into account that $\Phi_{\mathbf{t}}(A^*) = A$, we get

$$\begin{cases} A^{\mathrm{index}(A^*)+1} \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = A^{\mathrm{index}(A^*)}, \\ A \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) \cdot A, \\ \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) \cdot A \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)). \end{cases}$$

By hypothesis, we have that $\mathrm{index}(A^*) = \mathrm{index}(A)$. So

$$\begin{cases} A^{\mathrm{index}(A)+1} \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = A^{\mathrm{index}(A)}, \\ A \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) \cdot A, \\ \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) \cdot A \cdot \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \Phi_{\mathbf{t}}(\mathfrak{D}(A^*)). \end{cases}$$

Using the uniqueness of the Drazin Inverse, it holds that $\Phi_{\mathbf{t}}(\mathfrak{D}(A^*)) = \mathfrak{D}(A)$. □

Let us interpret the previous theorem. Let $A \in \mathrm{R}_{\mathbf{t}}$. We consider the set

$$\Sigma = \{A^* \mid A^* \in [A] \text{ such that } \mathfrak{D}(A^*) \in \mathcal{O}_{\mathbf{t}} \text{ and } \mathrm{index}(A) = \mathrm{index}(A^*).\}$$

Because of Theorem 3, it holds that $\Sigma \subset [\mathfrak{D}(A)]$. So, the general strategy would be as follows.

General Strategy. Given $A = (a_{i,j}(\mathbf{t}))_{1 \leq i,j \leq n}$

- Take $A^* = (a_{i,j}(\mathbf{w}))_{1 \leq i,j \leq n} \in [A]$.
- Check whether $A^* \in \Sigma$. If so, compute $\mathfrak{D}(A^*)$, and return $\mathfrak{D}(A(\mathbf{t})) = \mathfrak{D}(A^*)(\mathbf{t})$.

We leave the case $A^* \notin \Sigma$ as future research work.

Because of the claim in Theorem 3, we search for sufficient conditions to guarantee that the Drazin index stays invariant. For this purpose, one can use the fact that the index is the multiplicity of the root zero at the minimal polynomial of the matrix (see e.g. [3] or [13]) or the expression of the Drazin inverse based on the characteristic polynomial (see e.g. [4] page 269). Alternatively, one can find sufficient conditions analyzing the Gaussian elimination of the matrix. More precisely, reasoning as in Theorem 6 in [21], for $A \in \mathrm{R}_{\mathbf{w}}$, we proceed as follows. For $j \in \{1, \ldots, \mathrm{index}(A)\}$, let

$$P_j A^j = L_j U_j$$

be the $PA = LU$ factorization of A^j, where P_j are permutation matrices and the diagonal entries of L_j are 1.

Definition 4 *We define the* index-polynomial *of $A \in \mathrm{R}_{\mathbf{w}}$ as the squarefree part of the polynomial*

$$\prod_{j=1}^{\mathrm{index}(A)} \mathrm{den}(L_j)\,\mathrm{den}(U_j) \prod_{g \in D_{U_j}} \mathrm{num}(g)$$

where D_{U_j} is the set consisting in the most left non-zero element in each row of U_j. We denote the index polynomial as $H_A(\mathbf{w})$.

In this situation, we have the next lemma that analyzes the Drazin index.

Lemma 5 *Let $A \in \mathrm{R}_{\mathbf{w}}$ an let H_M be its index-polynomial. If $H_A(\mathbf{t}) \neq 0$, it holds that*

$$\mathrm{index}(A) = \mathrm{index}(\Phi_{\mathbf{t}}(A)).$$

Proof: Since $\mathrm{den}(L_j)(\mathbf{t})\mathrm{den}(U_j)(\mathbf{t}) \neq 0$, then $L_j, U_j \in \mathcal{O}_{\mathbf{t}}$. Therefore, $\Phi_{\mathbf{t}}(L_j), \Phi_{\mathbf{t}}(U_j)$ are well defined. Hence

$$P_j \Phi_{\mathbf{t}}(A)^j = \Phi_{\mathbf{t}}(L_j)\,\Phi_{\mathbf{t}}(U_j).$$

Furthermore, since $\mathrm{num}(g)(\mathbf{t}) \neq 0$ for all $g \in D_{U_j}$, it holds that $\mathrm{rank}(\Phi_{\mathbf{t}}(U_j)) = \mathrm{rank}(U_j)$. Moreover, since $\Phi_{\mathbf{t}}(L_j)$ is regular, we get that $\mathrm{rank}(A^j) = \mathrm{rank}(U_j) = \mathrm{rank}(\Phi_{\mathbf{t}}(U_j)) = \mathrm{rank}(\Phi_{\mathbf{t}}(A)^j)$. Thus, $\mathrm{index}(A) = \mathrm{index}(\Phi_{\mathbf{t}}(A))$. □

Corollary 6 *Let $A \in \mathrm{R}_{\mathbf{t}}$ and $A^* \in [A]$. Let $K_{A^*}(\mathbf{w})$ be the square-free part of the polynomial*

$$H_{A^*}(\mathbf{w})\,\mathrm{den}(\mathfrak{D}(A^*))(\mathbf{w})$$

where $H_{A^*}(\mathbf{w})$ is the index-polynomial of A^*. If $K_{A^*}(\mathbf{t}) \neq 0$, then A specializes properly at $(A^*, \mathbf{t})$.

Proof: It follows from Theorem 3 and Lemma 5. □

Definition 7 *Let $A \in \mathrm{R}_{\mathbf{t}}$ and $A^* \in [A]$. We call the polynomial $K_{A^*}(\mathbf{w})$, in Corollary 6, the* evaluation polynomial *of A^*.*

Corollary 8 *Let $\mathbf{t}$ be algebraically independent over $\mathbb{K}$. Then, every $A \in \mathrm{R}_{\mathbf{t}}$ specializes properly at $(A^*, \mathbf{t})$, for every $A^* \in [A]$.*

Proof: Since $\mathbf{t}$ are algebraically independent over $\mathbb{K}$, $K(\mathbf{t}) \neq 0$ for every $A \in \mathrm{R}_{\mathbf{t}}$. So, the result follows from Corollary 6. □

4 Application to meromorphic functions

Let $\mathbb{K} = \mathbb{C}$ and $\mathbb{F} = \mathrm{Mer}(\mathbf{z}, \Omega)$ be the field of meromorphic functions, in the complex variables $\mathbf{z} = (z_1, \ldots, z_\ell)$, on a connected domain Ω of $\mathbb{C}^\ell$ (see e.g. [15]). We illustrate the computation of Drazin inverses using the ideas developed in this paper.

Example 9 *We consider the matrix*

$$A = \begin{pmatrix} 0 & 0 & 2\,\mathrm{e}^{z}\cos(z) \\ \dfrac{2\,\cos(z)}{\mathrm{e}^{z}} & \dfrac{2}{e^{z}} & \dfrac{2}{e^{z}} \\ \dfrac{3\,\cos(z)}{\mathrm{e}^{z}} & \dfrac{3}{e^{z}} & \dfrac{6}{e^{z}} \end{pmatrix},$$

and we want to compute $\mathfrak{D}(A)$. We take $\mathbf{t} = (\cos(z), e^z)$, and we consider the fields $\mathbb{C} \subset \mathbb{C}(\mathbf{t}) \subset \mathrm{Mer}(z, \mathbb{C})$, as well as the matrix rings $\mathrm{R}_{\mathbf{t}} := \mathcal{M}_{3\times 3}(\mathbb{C}(\mathbf{t}))$ and $\mathrm{R}_{\mathbf{w}} := \mathcal{M}_{3\times 3}(\mathbb{C}(\mathbf{w}))$, with $\mathbf{w} = (w_1, w_2)$. The input matrix A belongs to $\mathrm{R}_{\mathbf{t}}$. Let

$$A^* = \begin{pmatrix} 0 & 0 & 2\,w_2 w_1 \\ \dfrac{2\,w_1}{w_2} & \dfrac{2}{w_2} & \dfrac{2}{w_2} \\ \dfrac{3\,w_1}{w_2} & \dfrac{3}{w_2} & \dfrac{6}{w_2} \end{pmatrix} \in [A].$$

$\mathrm{index}(A^*) = 1$ *and the $PA = LU$ decomposition of A^* is*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{3}{2} & \dfrac{3}{2\,{w_2}^2 w_1} & 1 \end{pmatrix},\ U = \begin{pmatrix} \dfrac{2\,w_1}{w_2} & \dfrac{2}{w_2} & \dfrac{2}{w_2} \\ 0 & 0 & 2\,w_2 w_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the index polynomial of A^* *(see Def. 4) is*

$$H_{A^*}(\mathbf{w}) = w_2 w_1.$$

Furthermore,

$$\mathfrak{D}(A^*) = \begin{pmatrix} \frac{-4\,{w_2}^3{w_1}^2}{3\,({w_1}^4{w_2}^4 - 2\,{w_1}^2{w_2}^2 + 1)} & \frac{-4\,{w_2}^3 w_1}{3\,({w_1}^2{w_2}^2 - 1)^2} & \frac{(3\,{w_1}^2{w_2}^2 + 5)\,{w_2}^3 w_1}{9\,({w_1}^2{w_2}^2 - 1)^2} \\ \frac{({w_1}^2{w_2}^2 + 3)\,w_2 w_1}{3\,({w_1}^2{w_2}^2 - 1)^2} & \frac{({w_1}^2{w_2}^2 + 3)\,w_2}{3\,({w_1}^2{w_2}^2 - 1)^2} & \frac{-(5\,{w_1}^2{w_2}^2 + 3)\,w_2}{9\,({w_1}^2{w_2}^2 - 1)^2} \\ \frac{w_2 w_1}{2\,({w_1}^2{w_2}^2 - 1)} & \frac{w_2}{2\,({w_1}^2{w_2}^2 - 1)} & \frac{-w_2}{3\,({w_1}^2{w_2}^2 - 1)} \end{pmatrix}.$$

Thus, the evaluation polynomial $K_{A^*}(\mathbf{w})$ *(see Def. 7) is*

$$K_{A^*}(\mathbf{w}) = w_2 w_1 (w_1 w_2 - 1)(w_1 w_2 + 1).$$

Now, we observe that

$$K_{A^*}(\mathbf{t}) = \mathrm{e}^z \cos(z)(\cos(z)\mathrm{e}^z - 1)(\cos(z)\mathrm{e}^z + 1).$$

Substituting $z = \pi$ *in* $K_{A^*}(\mathbf{t})$ *we get* $\mathrm{e}^\pi(\mathrm{e}^\pi + 1)(\mathrm{e}^\pi - 1) \neq 0$*. So,* $K_{A^*}(\mathbf{t})$ *is not identically zero. Therefore, the Drazin inverse* $\mathfrak{D}(A)$ *of* A *is* $\mathfrak{D}(A^*)(\mathbf{t})$*, that is*

$$\mathfrak{D}(A) = \begin{pmatrix} \frac{-4\,\mathrm{e}^{3z}\cos(z)^2}{3\,\left(\cos(z)^4\,\mathrm{e}^{4z} - 2\,\cos(z)^2\,\mathrm{e}^{2z} + 1\right)} & \frac{-4\,\mathrm{e}^{3z}\cos(z)}{3\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)^2} & \frac{\left(3\,\cos(z)^2\,\mathrm{e}^{2z} + 5\right)\mathrm{e}^{3z}\cos(z)}{9\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)^2} \\ \frac{\left(\cos(z)^2\,\mathrm{e}^{2z} + 3\right)\mathrm{e}^{z}\cos(z)}{3\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)^2} & \frac{\left(\cos(z)^2\,\mathrm{e}^{2z} + 3\right)\mathrm{e}^{z}}{3\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)^2} & \frac{-\left(5\,\cos(z)^2\,\mathrm{e}^{2z} + 3\right)\mathrm{e}^{z}}{9\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)^2} \\ \frac{\mathrm{e}^{z}\cos(z)}{2\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)} & \frac{\mathrm{e}^{z}}{2\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)} & \frac{-\mathrm{e}^{z}}{3\,\left(\cos(z)^2\,\mathrm{e}^{2z} - 1\right)} \end{pmatrix}$$

□

Example 10 *We consider the matrix*

$$A = \begin{pmatrix} \frac{1}{2}\zeta(z) + \frac{1}{2}\Gamma(z) + \frac{1}{2}\Gamma(z)^{-1} + \frac{1}{2} & -\frac{1}{2}\zeta(z) - \frac{1}{2}\Gamma(z) - \frac{1}{2}\Gamma(z)^{-1} & \frac{1}{2} \\ -\frac{1}{2}\zeta(z) - \frac{1}{2}\Gamma(z) - \frac{1}{2}\Gamma(z)^{-1} + \frac{1}{2} & \frac{1}{2}\zeta(z) + \frac{1}{2}\Gamma(z) + \frac{1}{2}\Gamma(z)^{-1} & \frac{1}{2} \\ -\frac{1}{2}\zeta(z) - \frac{1}{2}\Gamma(z) - \frac{1}{2}\Gamma(z)^{-1} - \frac{1}{2} & \frac{1}{2}\zeta(z) + \frac{1}{2}\Gamma(z) + \frac{1}{2}\Gamma(z)^{-1} & -\frac{1}{2} \end{pmatrix},$$

where $\zeta(z)$ is the Riemann zeta function and $\Gamma(z)$ is the gamma function. We want to compute $\mathfrak{D}(A)$. We take $\mathbf{t} = (\zeta(z), \Gamma(z))$, and we consider the fields $\mathbb{C} \subset \mathbb{C}(\mathbf{t}) \subset \mathrm{Mer}(z, \mathbb{C})$, as well as the matrix rings $\mathrm{R}_{\mathbf{t}} := \mathcal{M}_{3\times 3}(\mathbb{C}(\mathbf{t}))$ and $\mathrm{R}_{\mathbf{w}} := \mathcal{M}_{3\times 3}(\mathbb{C}(\mathbf{w}))$, with $\mathbf{w} = (w_1, w_2)$. The input matrix A belongs to $\mathrm{R}_{\mathbf{t}}$. Let

$$A^* = \begin{pmatrix} \frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}{w_2}^{-1} + \frac{1}{2} & -\frac{1}{2}w_1 - \frac{1}{2}w_2 - \frac{1}{2}{w_2}^{-1} & \frac{1}{2} \\ -\frac{1}{2}w_1 - \frac{1}{2}w_2 - \frac{1}{2}{w_2}^{-1} + \frac{1}{2} & \frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}{w_2}^{-1} & \frac{1}{2} \\ -\frac{1}{2}w_1 - \frac{1}{2}w_2 - \frac{1}{2}{w_2}^{-1} - \frac{1}{2} & \frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}{w_2}^{-1} & -\frac{1}{2} \end{pmatrix} \in [A].$$

$\mathrm{index}(A^*) = 2$. *We take the decomposition of $P_1 A^* = L_1 U_1$ and $P_2 {A^*}^2 = L_2 U_2$:*

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{w_1 w_2 + {w_2}^2 - w_2 + 1}{w_1 w_2 + {w_2}^2 + w_2 + 1} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \; U_1 = \begin{pmatrix} \frac{w_1 w_2 + {w_2}^2 + w_2 + 1}{2\, w_2} & \frac{-(w_1 w_2 + {w_2}^2 + 1)}{2\, w_2} & \frac{1}{2} \\ 0 & \frac{w_1 w_2 + {w_2}^2 + 1}{w_1 w_2 + {w_2}^2 + w_2 + 1} & \frac{w_1 w_2 + {w_2}^2 + 1}{w_1 w_2 + {w_2}^2 + w_2 + 1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \; U_2 = \begin{pmatrix} \frac{\left(w_1 w_2 + {w_2}^2 + 1\right)^2}{2\, {w_2}^2} & -\frac{\left(w_1 w_2 + {w_2}^2 + 1\right)^2}{2\, {w_2}^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, the index-polynomial of A^* is

$$H_{A^*}(\mathbf{w}) = \left(w_1 w_2 + {w_2}^2 + w_2 + 1\right) w_2 \left(w_1 w_2 + {w_2}^2 + 1\right).$$

Furthermore,

$$\mathfrak{D}(A^*) = \begin{pmatrix} \dfrac{w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & \dfrac{-w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & 0 \\ \dfrac{-w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & \dfrac{w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & 0 \\ \dfrac{-w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & \dfrac{w_2}{2\,(w_1 w_2 + {w_2}^2 + 1)} & 0 \end{pmatrix}.$$

Thus, the evaluation polynomial K_{A^*} is

$$K_{A^*}(\mathbf{w}) = \left(w_1 w_2 + {w_2}^2 + w_2 + 1\right) w_2 \left(w_1 w_2 + {w_2}^2 + 1\right) \left(w_1 w_2 + {w_2}^2 - w_2 + 1\right).$$

Substituting $z = 2$ in $K_{A^*}(\mathbf{t})$ we get $\zeta(2)^3 + 6\zeta(2)^2 + 11\zeta(2) + 6 \neq 0$. So, $K_{A^*}(\mathbf{t})$ is not identically zero. Therefore, the Drazin inverse $\mathfrak{D}(A)$ of A is $\mathfrak{D}(A^*)(\mathbf{t})$, that is

$$\mathfrak{D}(A) = \begin{pmatrix} \frac{\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+1)} & \frac{-\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+1)} & 0 \\ \frac{-\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+1)} & \frac{\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+1)} & 0 \\ \frac{-\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+19} & \frac{\Gamma(z)}{2\,(\zeta(z)\Gamma(z)+\Gamma(z)^2+1)} & 0 \end{pmatrix}.$$

□

Besides the direct application of the results in the previous sections, we can consider the following additional application. Let $\mathbf{t} \in \mathrm{Mer}(\mathbf{z}, \Omega)$ and $A(\mathbf{t}) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t}))$. Then, $\mathfrak{D}(A)(\mathbf{t}) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t}))$. The functions in $\mathbf{t}$ depend on the complex variables $\mathbf{z} = (z_1, \ldots, z_\ell)$, and hence we can see $A(\mathbf{t})$ and $\mathfrak{D}(A)(\mathbf{t})$ as functions in $\mathbf{z}$. To emphasize this fact we write $A(\mathbf{t}(\mathbf{z}))$ and $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}))$. Now, for a particular value $\mathbf{z}_0 \in \mathbb{C}^\ell$ we analyze whether $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0))$ is $\mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0)))$; i.e. whether the evaluation of the Drazin inverse is the Drazin inverse of the evaluation. In Diagram (4) we illustrate the idea.

$$\begin{array}{ccc} A(\mathbf{t}(\mathbf{z})) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t})) & \xrightarrow{\text{Drazin}} & \mathfrak{D}(A)(\mathbf{t}(\mathbf{z})) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t})) \\ \Big\downarrow \text{Evaluation} & & \text{Evaluation} \Big\downarrow \\ A(\mathbf{t}(\mathbf{z}_0)) \in \mathcal{M}_{n\times n}(\mathbb{C}) & \xrightarrow{\text{Drazin}} & \mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0))) \stackrel{?}{=} \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) \in \mathcal{M}_{n\times n}(\mathbb{C}) \end{array} \tag{4}$$

Let $A^* \in [A]$, and let $K_{A^*}(\mathbf{w})$ be the polynomial associated to A^* via Corollary 6. We assume that $K_{A^*}(\mathbf{t})$ is not identically zero. Again, $K_{A^*}(\mathbf{t})$ can be seen as a function in the complex variables $\mathbf{z}$, so we denote it as $K_{A^*}(\mathbf{t}(\mathbf{z}))$. We observe that, since $K_{A^*}(\mathbf{w})$ is a polynomial in the variables $\mathbf{w}$ with complex coefficients, $K_{A^*}(\mathbf{t}(\mathbf{z})) \in \mathrm{Mer}(\mathbf{z}, \Omega)$.

We introduce the set

$$\Omega^* = \Omega \setminus \{\mathbf{z}_0 \in \Omega \,|\, K_{A^*}(\mathbf{t}(\mathbf{z}_0)) = 0\} \subset \mathbb{C}^\ell.$$

Then, following proposition holds.

Proposition 11 *Let $A^* \in [A]$ be such that $K_{A^*}(\mathbf{t})$ is not zero. Then, for $\mathbf{z}_0 \in \Omega^*$, it holds that $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = \mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0)))$.*

Proof: Since $\mathbf{z}_0 \in \Omega^* \subset \Omega$, then $A(\mathbf{t}(\mathbf{z}_0))$ is well defined as hence $\mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0)))$ exists. On the other hand, since we have assumed that $K_{A^*}(\mathbf{t}(\mathbf{z}))$ is not identically zero, by Corollary 6 we have that $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z})) = \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z}))$. Moreover, since $\mathbf{z}_0 \in \Omega^*$, then $K_{A^*}(\mathbf{t}(\mathbf{z}_0)) \neq 0$, and hence $\mathrm{den}(\mathfrak{D}(A^*))$ does not vanish at $\mathbf{t}(\mathbf{z}_0)$. Thus, $\mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z}_0))$ and $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0))$ are well defined. On the other hand, since $\mathfrak{D}(A^*)$ is the Drazin inverse of A^*, it holds (1), that is

$$\begin{cases} A^*(\mathbf{w}))^{\text{index}(A^*(\mathbf{W}))+1} \cdot \mathfrak{D}(A^*)(\mathbf{w}) = A^*(\mathbf{w})^{\text{index}(A^*(\mathbf{W}))}, \\ A^*(\mathbf{w}) \cdot \mathfrak{D}(A^*)(\mathbf{w}) = \mathfrak{D}(A^*)(\mathbf{w}) \cdot A^*(\mathbf{w}), \\ \mathfrak{D}(A^*)(\mathbf{w}) \cdot A^*(\mathbf{w}) \cdot \mathfrak{D}(A^*)(\mathbf{w}) = \mathfrak{D}(A^*)(\mathbf{w}). \end{cases}$$

Specializing at $\mathbf{t}(\mathbf{z})$ we get

$$\begin{cases} A^*(\mathbf{t}(\mathbf{z}))^{\text{index}(A^*(\mathbf{W}))+1} \cdot \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})) = A^*(\mathbf{t}(\mathbf{z}))^{\text{index}(A^*(\mathbf{W}))}, \\ A^*(\mathbf{t}(\mathbf{z})) \cdot \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})) = \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})) \cdot A^*(\mathbf{t}(\mathbf{z})), \\ \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})) \cdot A^*(\mathbf{t}(\mathbf{z})) \cdot \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})) = \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z})). \end{cases}$$

So, since $A^*(\mathbf{t}(\mathbf{z})) = A(\mathbf{t}(\mathbf{z}))$, specializing at $\mathbf{z}_0$, we get that

$$\begin{cases} A(\mathbf{t}(\mathbf{z}_0))^{\text{index}(A^*(\mathbf{W}))+1} \cdot \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = A(\mathbf{t}(\mathbf{z}_0))^{\text{index}(A^*(\mathbf{W}))}, \\ A(\mathbf{t}(\mathbf{z}_0)) \cdot \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) \cdot A(\mathbf{t}(\mathbf{z}_0)), \\ \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) \cdot A(\mathbf{t}(\mathbf{z}_0)) \cdot \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = \mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)). \end{cases}$$

Now, let $P_j \cdot A^{*j}(\mathbf{w}) = L_j(\mathbf{w}) \cdot U_j(\mathbf{w})$ be the $PA = LU$ decomposition of A^{*j} with $j = 1, \ldots, \text{index}(A^*)$. Since $K_{A^*}(\mathbf{t}(\mathbf{z})) \neq 0$, the index polynomial of A^* does not vanish at $\mathbf{t}(\mathbf{z})$ and hence the expressions $P_j \cdot A^{*j}(\mathbf{t}(\mathbf{z})) = L_j(\mathbf{t}(\mathbf{z})) \cdot U_j(\mathbf{t}(\mathbf{z}))$ are well defined. Furthermore, for $z_0 \in \Omega^*$ it holds that $K_{A^*}(\mathbf{t}(\mathbf{z}_0)) \neq 0$, and hence the expressions $P_j \cdot A^{*j}(\mathbf{t}(\mathbf{z}_0)) = L_j(\mathbf{t}(\mathbf{z}_0)) \cdot U_j(\mathbf{t}(\mathbf{z}_0))$ are also well defined. Therefore, since $K_{A^*}(\mathbf{t}(\mathbf{z}))K_{A^*}(\mathbf{t}(\mathbf{z}_0)) \neq 0$, it holds that

$$\text{rank}(A^{*j}(\mathbf{t}(\mathbf{z}_0)) = \text{rank}(U_j(\mathbf{t}(\mathbf{z}_0))) = \text{rank}(U_j(\mathbf{t}(\mathbf{z}))) = \text{rank}(A^{*j}(\mathbf{t}(\mathbf{z})).$$

Thus, $\text{index}(A^*(\mathbf{t}(\mathbf{z}))) = \text{index}(A^*(\mathbf{t}(\mathbf{z}_0)))$. Now, since $A^*(\mathbf{t}(\mathbf{z})) = A(\mathbf{t}(\mathbf{z}))$ then $\text{index}(A^*(\mathbf{t}(\mathbf{z}))) = \text{index}(A(\mathbf{t}(\mathbf{z})))$ and $\text{index}(A^*(\mathbf{t}(\mathbf{z}_0))) = \text{index}(A(\mathbf{t}(\mathbf{z}_0)))$. By Lemma 5 we have that $\text{index}(A^*(\mathbf{w})) = \text{index}(A^*(\mathbf{t}(\mathbf{z}))$. So, $\text{index}(A^*(\mathbf{w})) = \text{index}(A(\mathbf{t}(\mathbf{z}_0)))$. So, by the uniqueness of the inverse we get that $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = \mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0)))$. □

We finish this subsection with an example.

Example 12 *Let $\mathbf{t}(\mathbf{z}) = (\cosh(z_1), \sin(z_2)) \in \text{Mer}(\mathbf{z}, \mathbb{C}^2)$ and let A be the matrix*

$$A(\mathbf{t}(\mathbf{z})) = \begin{pmatrix} 0 & 0 & \sin(z_2) \\ \cosh(z_1) & 1 & 1 \\ \cosh(z_1) & 1 & 2 \end{pmatrix}.$$

The take

$$A^*(\mathbf{w}) = \begin{pmatrix} 0 & 0 & w_2 \\ w_1 & 1 & 1 \\ w_1 & 1 & 2 \end{pmatrix} \in [A].$$

Then,

$$\mathfrak{D}(A^*)(\mathbf{w}) = \begin{pmatrix} \frac{-3\,w_2w_1}{(w_2w_1-1)^2} & \frac{-3\,w_2}{(w_2w_1-1)^2} & \frac{(w_2w_1+2)\,w_2}{(w_2w_1-1)^2} \\ \frac{(w_2w_1+2)\,w_1}{(w_2w_1-1)^2} & \frac{w_2w_1+2}{(w_2w_1-1)^2} & \frac{-2\,w_2w_1-1}{(w_2w_1-1)^2} \\ \frac{w_1}{w_2w_1-1} & \frac{1}{(w_2w_1-1)} & \frac{-1}{(w_2w_1-1)} \end{pmatrix}.$$

In addition, $K_{A^*}(\mathbf{w}) = w_2(w_1w_2-1)$*, and* $K_{A^*}(\mathbf{t}(\mathbf{z})) = \sin(z_2)(\sin(z_2)\cosh(z_1)-1)$ *that is not zero, since e.g.* $K(\mathbf{t}(0,3\pi/2)) = 2 \neq 0$*. Therefore,* $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z})) = \mathfrak{D}(A^*)(\mathbf{t}(\mathbf{z}))$*, namely*

$$\mathfrak{D}(A) = \begin{pmatrix} \frac{-3\,\sin(z_2)\cosh(z_1)}{(\sin(z_2)\cosh(z_1)-1)^2} & \frac{-3\,\sin(z_2)}{(\sin(z_2)\cosh(z_1)-1)^2} & \frac{(\sin(z_2)\cosh(z_1)+2)\sin(z_2)}{(\sin(z_2)\cosh(z_1)-1)^2} \\ \frac{(\sin(z_2)\cosh(z_1)+2)\cosh(z_1)}{(\sin(z_2)\cosh(z_1)-1)^2} & \frac{\sin(z_2)\cosh(z_1)+2}{(\sin(z_2)\cosh(z_1)-1)^2} & \frac{-2\,\sin(z_2)\cosh(z_1)-1}{(\sin(z_2)\cosh(z_1)-1)^2} \\ \frac{\cosh(z_1)}{\sin(z_2)\cosh(z_1)-1} & \frac{-1}{\sin(z_2)\cosh(z_1)-1} & \frac{-1}{\sin(z_2)\cosh(z_1)-1} \end{pmatrix}.$$

On the other hand, for every $\mathbf{z}_0 \in \mathbb{C}^2$ *such that* $K_{A^*}(\mathbf{t}(\mathbf{z}_0)) \neq 0$ *it holds that* $\mathfrak{D}(A)(\mathbf{t}(\mathbf{z}_0)) = \mathfrak{D}(A(\mathbf{t}(\mathbf{z}_0)))$*. For instance, since* $K_{A^*}(\mathbf{t}(0,3\pi/2)) \neq 0$*, we get*

$$\mathfrak{D}(A(0,3\pi/2)) = \mathfrak{D}\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3/4 & 3/4 & -1/4 \\ 1/4 & 1/4 & 1/4 \\ -1/2 & -1/2 & 1/2 \end{pmatrix} = \mathfrak{D}(A)(0,3\pi/2).$$

□

5 Application to Laurent formal series

In this section, we illustrate how to compute Drazin inverses of matrices whose entries are formal Laurent series. Observe that this contains the case of analytically perturbed matrices (see [16]). So let $\mathbb{K} = \mathbb{C}$ and let $\mathbb{F} = \mathbb{C}((z))$ be the field of formal Laurent series in z with coefficients in $\mathbb{C}$ (see e.g. [2] pg. 65). Let $\mathbf{t}$ be a tuple of elements in $\mathbb{F} \setminus \mathbb{K}$, and $A \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t}))$, then $\mathfrak{D}(A) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{t}))$. Applying our ideas, one computes $A^* \in [A]$, as well as the evaluation polynomial, via Corollary 6, $K_{A^*}(\mathbf{w})$. Then, if $K_{A^*}(\mathbf{t}) \neq 0$ it holds that $\mathfrak{D}(A(\mathbf{t})) = \mathfrak{D}(A^*)(\mathbf{t})$. Nevertheless, in general, one works with finite truncations. So, we will consider truncations of order $m > 0$. More precisely, we consider the map

$$\begin{array}{cccc} \mathcal{T}_m : & \mathbb{C}((z)) & \longrightarrow & \mathbb{C}(z) \\ & \displaystyle\sum_{i=-r}^{\infty} a_i z^i & \longmapsto & \displaystyle\sum_{i=-r}^{m-1} a_i z^i. \end{array}$$

We consider also the natural extension of $\mathcal{T}_m$ to $\mathbb{C}((z))^r$ and $\mathcal{M}_{n\times n}(\mathbb{C}((z)))$; i.e. if $\mathbf{t} = (t_1, \ldots, t_r) \in \mathbb{C}((z))^r$ then $\mathcal{T}_m(\mathbf{t}) = (\mathcal{T}_m(t_1), \ldots, \mathcal{T}_m(t_r))$, and, if $A = (a_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{C}((z)))$ then $\mathcal{T}_m(A) = (\mathcal{T}_m(a_{ij}))$. Now, the idea is to analyze when the Drazin inverse of the truncation of A is the truncation of the Drazin inverse evaluated at the truncation of $\mathbf{t}$ (see Diagram (5)).

$$\begin{array}{ccc} A(\mathbf{t}) \in \mathcal{M}_{n\times n}(\mathbb{C}((z))) & \xrightarrow{\ \text{Drazin} \circ \Psi_{\mathbf{w}}\ } & \mathfrak{D}(A^*)(\mathbf{w}) \in \mathcal{M}_{n\times n}(\mathbb{C}(\mathbf{w})) \\ \big\downarrow \text{Drazin} & & \big\downarrow \mathbf{w} \leftarrow \mathcal{T}_\ell(\mathbf{t}) \\ \mathfrak{D}(A(\mathbf{t})) \in \mathcal{M}_{n\times n}(\mathbb{C}(z)) & & \\ \mathcal{T}_m \big\downarrow & & \\ \mathcal{T}_m(\mathfrak{D}(A(\mathbf{t}))) \stackrel{?}{=} \mathcal{T}_m(\mathfrak{D}(A^*)(\mathcal{T}_\ell(\mathbf{t}))) & \xleftarrow{\ \mathcal{T}_m\ } & \mathfrak{D}(A^*)(\mathcal{T}_\ell(\mathbf{t})) \in \mathcal{M}_{n\times n}(\mathbb{C}(z)) \end{array} \tag{5}$$

For $\mathbf{t} = (t_1, \ldots, t_r)$ we denote by $\operatorname{ord}(t_i)$ the order of t_i; note that, by assumption $t_i \in \mathbb{C}((z)) \setminus \mathbb{C}$, and hence $t_i \neq 0$. Now, if $P \in \mathbb{C}[\mathbf{w}]$ is such that $P(\mathbf{t}) \neq 0$, it holds that for every $\ell \geq m + \deg(P) \cdot \max_{i\in\{1,\ldots,r\}}\{|\operatorname{ord}(t_i)|\}$.

$$\mathcal{T}_m(P(\mathbf{t})) = \mathcal{T}_m(P(\mathcal{T}_\ell(\mathbf{t})).$$

Similarly, let $P/Q \in \mathbb{C}(\mathbf{w})$, with $\gcd(P, Q) = 1$, such that $Q(\mathbf{t}) \neq 0$. Let $m \geq \operatorname{ord}(Q(\mathbf{t}))$. If $\ell \geq m + \max\{\deg(P), \deg(Q)\} \cdot \max_{i\in\{1,\ldots,r\}}\{|\operatorname{ord}(t_i)|\}$ then $Q(\mathcal{T}_\ell(\mathbf{t})) \neq 0$ and

$$\mathcal{T}_m\left(\frac{P(\mathbf{t})}{Q(\mathbf{t})}\right) = \mathcal{T}_m\left(\frac{P(\mathcal{T}_\ell(\mathbf{t}))}{Q(\mathcal{T}_\ell(\mathbf{t}))}\right).$$

In this situation, let $A^* \in [A]$ and $K_{A^*}(\mathbf{w})$ be the evaluation polynomial of A^*; see Def. 7. Let us assume that $K_{A^*}(\mathbf{t}) \neq 0$. Then, $\mathfrak{D}(A)(\mathbf{t}) = \mathfrak{D}(A^*)(\mathbf{t})$. Note that $\mathfrak{D}(A^*)(\mathbf{w})$ is a matrix with entries in $\mathbb{C}(\mathbf{w})$. Let $\mathfrak{D}(A^*)$ be expresses as $\mathfrak{D}(A^*) = (a_{ij}/b_{ij})_{1\leq i,j\leq n}$ with $\gcd(a_{ij}, b_{ij}) = 1$. Then, we introduce the quantity

$$\omega = \max\{\max\{\deg(a_{ij}), \deg(b_{ij})\} \,|\, 1 \leq i, j \leq n\},$$

as well as

$$\hat{m} = \max\{\operatorname{ord}(b_{ij}(\mathbf{t})) \,|\, 1 \leq i, j \leq n\}.$$

Then, taking into account the previous reasoning one has the following result. Note that it provides a method to compute the Laurent series expansion of perturbed Drazin inverses (see Chapter 3 in [16]).

Proposition 13 *Let $A^* \in [A]$ and let $K_{A^*}(\mathbf{w})$ be the evaluation polynomial of A^*. Let $K_{A^*}(\mathbf{t}) \neq 0$. If $m \geq \hat{m}$ and $\ell \geq m + \omega \cdot \max_{i\in\{1,\ldots,r\}}\{|\mathrm{ord}(t_i)|\}$, then*

$$\mathcal{T}_m(\mathfrak{D}(A(\mathbf{t}))) = \mathcal{T}_m(\mathfrak{D}(A^*)(\mathcal{T}_\ell(\mathbf{t}))).$$

We illustrate these ideas by an example.

Example 14 *We consider the matrix $A \in \mathcal{M}_{4\times 4}(\sin(z), \sinh(z), \mathrm{e}^z)$*

$$A = \begin{pmatrix} 0 & \frac{(\sin(z)+2\,\sinh(z))\sinh(z)}{\mathrm{e}^z} & \sin(z) + 2\,\sinh(z) & 2\,\sin(z) + 5\,\sinh(z) \\ 2\,\frac{\sin(z)\sinh(z)}{\mathrm{e}^z} & 2\,\sin(z) & 2\,\sin(z) & 3\,\sin(z) + 3\,\sinh(z) \\ 3\,\frac{\sin(z)\sinh(z)}{\mathrm{e}^z} & 3\,\sin(z) + 3\,\frac{\sin(z)\sinh(z)}{\mathrm{e}^z} & 6\,\sin(z) & 6\,\sin(z) \\ 4\,\frac{\sin(z)\sinh(z)}{\mathrm{e}^z} & 4\,\sin(z) + 4\,\frac{\sin(z)\sinh(z)}{\mathrm{e}^z} & 8\,\sin(z) & 12\,\sin(z) \end{pmatrix}.$$

So, we can apply the results in Section 4 to compute $\mathfrak{D}(A)$. Alternatively, we consider the Laurent expansion of A; say

$$A = \begin{pmatrix} 0 & 0 & 3 & 7 \\ 0 & 2 & 2 & 6 \\ 0 & 3 & 6 & 6 \\ 0 & 4 & 8 & 12 \end{pmatrix} z + \begin{pmatrix} 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 4 & 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 0 & -3 & \frac{1}{6} & \frac{1}{2} \\ -2 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -3 & -\frac{7}{2} & -1 & -1 \\ -4 & -\frac{14}{3} & -\frac{4}{3} & -2 \end{pmatrix} z^3 +$$

$$+ \begin{pmatrix} 0 & \frac{13}{6} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} z^4 + \mathcal{O}(z^5),$$

and we want to compute the Laurent expanssion of the Drazin inverse of A, namely $\mathfrak{D}(A)$. Let us say that we want to compute the 5 truncation of $\mathfrak{D}(A)$, i.e. $\mathcal{T}_5(\mathfrak{D}(A))$. So, in the terminology of Proposition 13, $m = 5$ and $\mathbf{t} = (t_1, t_2, t_3)$ are the Laurent expansion of $(\sin(z), \sinh(x), \mathrm{e}^z)$, $\hat{m} = 5$ and $\ell \geq 14$. Therefore,

$$\mathcal{T}_5(\mathfrak{D}(A(\mathbf{t}))) = \mathcal{T}_5(\mathfrak{D}(A^*)(\mathcal{T}_{14}(\mathbf{t}))).$$

Now, we determine $A^* \in [A]$ such that $K_{A^*}(\sin(z), \sinh(z), \mathrm{e}^z) \neq 0$. We get

$$A^* = \begin{pmatrix} 0 & \frac{(w_2+2\,w_1)\,w_1}{w_3} & w_2+2\,w_1 & 2\,w_2+5\,w_1 \\ 2\,\frac{w_2 w_1}{w_3} & 2\,w_2 & 2\,w_2 & 3\,w_2+3\,w_1 \\ 3\,\frac{w_2 w_1}{w_3} & 3\,w_2+3\,\frac{w_2 w_1}{w_3} & 6\,w_2 & 6\,w_2 \\ 4\,\frac{w_2 w_1}{w_3} & 4\,w_2+4\,\frac{w_2 w_1}{w_3} & 8\,w_2 & 12\,w_2 \end{pmatrix}$$

as well as $\mathfrak{D}(A^*)$; $\mathfrak{D}(A^*)$ is big and we do not show it here. Finally, we get

$$\mathcal{T}_5(\mathfrak{D}(A(\mathbf{t}))) = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{7}{3} & \frac{37}{24} \\ 0 & 1 & 1 & -1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{4} \end{pmatrix} \frac{1}{z} + \begin{pmatrix} -\frac{1}{3} & -\frac{1}{6} & -\frac{31}{9} & \frac{37}{16} \\ 1 & \frac{17}{6} & \frac{17}{3} & -\frac{121}{24} \\ -\frac{1}{2} & -\frac{7}{4} & -\frac{13}{6} & \frac{9}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{13}{6} & \frac{14}{3} & \frac{385}{72} & -\frac{295}{48} \\ \frac{11}{6} & \frac{31}{12} & \frac{67}{9} & -\frac{311}{48} \\ -\frac{7}{4} & -\frac{73}{24} & -\frac{79}{12} & \frac{73}{12} \\ \frac{1}{3} & \frac{1}{2} & \frac{13}{18} & -\frac{19}{24} \end{pmatrix} z + \begin{pmatrix} \frac{53}{9} & \frac{715}{72} & \frac{761}{48} & -\frac{4723}{288} \\ -\frac{29}{12} & -\frac{359}{72} & -\frac{395}{72} & \frac{899}{144} \\ -\frac{17}{24} & -\frac{19}{48} & -\frac{193}{72} & \frac{109}{48} \\ \frac{1}{6} & \frac{1}{4} & \frac{7}{18} & -\frac{5}{12} \end{pmatrix} z^2$$

$$+ \begin{pmatrix} \frac{71}{36} & \frac{2617}{720} & \frac{35179}{4320} & -\frac{469}{60} \\ -\frac{193}{24} & -\frac{3007}{240} & -\frac{43313}{2160} & \frac{29857}{1440} \\ \frac{221}{48} & \frac{3497}{480} & \frac{529}{48} & -\frac{1837}{160} \\ -\frac{13}{36} & -\frac{31}{72} & -\frac{179}{360} & \frac{847}{1440} \end{pmatrix} z^3 + \begin{pmatrix} -\frac{791}{108} & -\frac{8363}{864} & -\frac{60581}{5184} & \frac{5779}{432} \\ -\frac{319}{144} & -\frac{983}{288} & -\frac{7069}{864} & \frac{63}{8} \\ \frac{485}{96} & \frac{1367}{192} & \frac{10441}{864} & -\frac{1801}{144} \\ -\frac{19}{72} & -\frac{49}{144} & -\frac{95}{216} & \frac{1}{2} \end{pmatrix} z^4 + \mathcal{O}(z^5).$$

□

Remark 1 *We observe that a similar reasoning can be done for Puiseux series (see e.g. [2] pg. 65) or for Gevrey asymptotic power series (see e.g. chapter 4 in [1]).*

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