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SHM VIA TOPOLOGICAL DERIVATIVE

Doctoral Thesis

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M.Sc. in Aerospace Engineering

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Dedicated to the readers.

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Abstract

This thesis focuses on the investigation of the usage of a relatively novel mathematical tool, the so-called *topological derivative*, for Structural Health Monitoring. In essence, the topological derivative approach minimizes a scalar objective function that measures the least squares difference between response to excitation of healthy and damaged structures. In contrast to conventional SHM methods, which usually draw on the processing of the time-lags between the transmitted and received pulse signals, the topological derivative requires to solve the full elasto-dynamic equations; this permits locating flaws in fairly demanding scenarios due to the necessity of considering the whole physics of the problem. In the chapters of the present work, the new method will be applied to aluminum plates with different kinds of shape complexities such as material inclusions, air slits, and thickness variations, in order to demonstrate its effectiveness in somewhat intricate situations. Furthermore, these plates will be excited by means of transmitters and receivers configurations positioned very close to the boundaries, which introduces even more complicatedness to an already challenging problem.

Resumen

Esta tesis se centra en la investigación del uso de una herramienta matemática relativamente nueva, conocida como *derivada topológica*, en la monitorización de la salud estructural. En esencia, la derivada topológica minimiza una función objetivo escalar que mide la menor diferencia cuadrática entre la respuesta a la excitación de las estructuras sanas y dañadas. En contraposición a los métodos convencionales para la monitorización de la salud estructural, los cuales normalmente se basan en el procesamiento de desfases de tiempo entre las señales transmitidas y recibidas, la derivada topológica requiere resolver completamente las ecuaciones elastodinámicas; esto permite localizar defectos en escenarios considerablemente exigentes debido a la necesidad de considerar la física del problema en su totalidad. En los capítulos del presente trabajo, el nuevo método será aplicado en placas de aluminio con diferentes tipos de complejidades en su forma tales como inclusiones de material, ranuras de aire, y variaciones de espesor, con el objetivo de demostrar su efectividad en situaciones en cierto modo intrincadas. Además, dichas placas serán excitadas por medio de configuraciones de transmisores y receptores posicionados muy cerca de los bordes, lo cual introduce si cabe más dificultad a un problema ya de por sí complicado.

General introduction

1.1 An overview of SHM

Structure Health Monitoring has been a growing technology within many engineering fields over the past years. SHM can be defined as the integration of sensing and possibly also actuation devices to allow the loading and damaging conditions of a structure to be recorded, analyzed, localized, and predicted in a way that Non Destructive Testing (NDT) [67] becomes an integral part of the structure and a material [11]. Assessing the condition and integrity of structures in real time is a crucial objective in the aerospace, mechanical, and civil engineering industry [31]. Besides the traditional non destructive testing methods, in the recent years, a large number of inspection methods which aim at giving continuous information about domain and material changes have been developed [60]. The main motivation of these new techniques, which in the end is the same motivation of SHM, is to increase safety and to reduce the cost of maintenance of the structures [6]. For instance, in the aerospace industry, costly periodic inspections are carried out which may be cheapen and implemented in a more optimal manner. Furthermore, reducing the human action in this assessment is a key feature of SHM.

As already stated, SHM aims at assessing the state of structural health by trying to detect the presence and evaluate of structural damage. This approach is shared with the traditional NDT techniques, however, SHM attempts to go further by taking profit of integrated sensor networks which can provide real time structural health reports [18]. These new approaches utilize an integrated sensor network to measure response changes at same locations at different times. Their main objective is to operate as standalone systems and to make a thorough diagnose of the structure with the minimal human action. For instance, one of the booming damage detection method in the recent years has been the

NDT via guided waves [30, 50]. These guided waves (i.e., Lamb waves if we are doing the inspection of a plate) can travel long distances with small amplitude losses, generated via very affordable piezoelectric transducers. The major limitation of the traditional NDT techniques was the size and the cost of the regular NDT transducers which made to integrate them in the structures not practical. Nonetheless, the recent significant development of sensing technologies in combination with powerful microprocessors have enabled to accomplish the transition from NDT techniques into SHM strategies.

As anticipated, SHM tries to be an autonomous system which implements a suitable strategy to determine damaged structures. These strategies normally rely on the comparison between two states of the assessed structure: a reference state considered the healthy state (e.g., a pristine plate), and another state to be tested so as to confirm that it is damaged (e.g., a plate with a small hole). However, in order to fulfil the autonomy condition, SHM should not only be limited to verify whether a structure is healthy or not, but also to perform further relevant tasks. Regarding these tasks, we can summarize the global steps that a every SHM system should have [25]:

1. Confirm that there is a discrepancy between the reference healthy state and the state to be verified, in order words, to ensure that the structure is damaged.
2. Locate where is the damage within the structure.
3. Characterize the damage to determine its severity.
4. Perform prognosis of the damage to assess when it may be critical for the structure.
5. Healing of the damage.

The previous enumerated procedures should be performed by any SHM system operating during the whole life cycle of a structure, retrieving information about it at every moment and at every needed location, in order to take appropriate decisions. Furthermore, it is also convenient to enumerate the constitutive elements that a SHM has [34]:

1. Structure to be monitored.
2. Type of damage to be detected.
3. Network of sensors permanently attached to the structure; this is the main difference (and improvement) between traditional NDT procedures and novel SHM.
4. On-board data handling and computing facilities.

5. Algorithms to process data retrieved at the network sensors. Cleaning data from environmental effects, comparing it to previous data from the pristine structure, and informing about occurrence, localization, and damage type, is the main scope of these algorithms.

We can even synthesize the points above in terms of a (somewhat simplistic) mathematical equation in order to better illustrate and highlight again the fundamental relation between NDT techniques and SHM,

$$\text{SHM} = \text{NDT} + \text{data handling} + \text{detection algorithm}.$$

As pointed out above, SHM is applied in many fields of engineering, each of them with their own kind of objectives and difficulties. We can classify these application fields based on the type of structure to be monitored,

- Civil engineering. The typical infrastructures to be monitored are dams, bridges, buildings and towers, tunnels, excavations, etc [13]. Dams are historically the first kind of structures on which SHM was applied, and everything learned has been applied to other structures. The most important part to monitor in dams is the dynamic response of the structure in order to calibrate seismic models (which is one of the main problems of civil SHM), or to estimate dynamic characteristics from ambient monitoring. Bridges are another relevant civil infrastructures whose monitoring programmes have been implemented for the purpose of understanding and eventually calibrating models of the load-structure-response chain [76]. One of the earliest monitoring experience was conducted over the famed Golden Gate and Bay bridges in San Francisco in order to gain experience against possible earthquakes. Another significant event which led to SHM development was the collapse of the Takoma bridge due to an unstable wind-induced response [20]. Besides, corrosion caused by salinity is a significant possible damage to be monitored in sea bridges. With respect to towers and buildings, SHM is motivated by the necessity of understanding the structural loading and response mechanisms under earthquake and wind-storm conditions for example in high-rise apartments [7]. Concerning tunnels [59], monitoring aims at determining whether the structure deformation is within appropriate limits either in construction or operation. For this reason, although stresses and strains are measured, in this case, a big effort must be done monitoring displacements. SHM in civil engineering is not limited to fixed structures,

for instance, high speed trains need to be monitor in order to ensure its correct performance [42].

- Energy engineering. The typical infrastructures to be monitored are nuclear plants, thermal plants, wind turbines, etc. For obvious reasons, the most critical installations to be monitored are nuclear facilities [2]; if their performance is severely affected, energy production may stop, or even worse, the so-called nuclear catastrophe could occur. In particular, the critical structural component to monitor is the prestressed concrete pressure vessel of the reactor (PCPV). The healthy state of the PCPV can be compromised by means of an anomalous performance or due to natural disasters such as earthquakes or tsunamis, as occurred in Chernobyl and Fukushima disasters respectively [72].
- Marine engineering. The typical infrastructures to be monitored can be any offshore structure such as ports, containment dikes, oil platforms, etc [22, 56]. All of the previous structures operate in considerable depths and subjected to the (hostile) marine environment which is strongly non-stationary and corrosive. Hence, the motivation for SHM is clear in this case, however, accurate monitoring systems are complicated to implement due to the poor accessibility to the majority of the offshore structures.
- Aerospace engineering. In this case, the monitoring is performed over every kind of aircraft and space vehicle [70, 82]. SHM applied to aerospace engineering has been one of the most developed fields in the recent years, and a field in which SHM has a very promising future [34]. Regarding aircrafts, monitoring their structures is essential to obtain optimal flight performances by reducing costs and increasing safety. Aircraft structures are subjected to numerous loading cycles which usually lead to fatigue damages [77], but also to accidental, environmental, or operational damages. Naturally, these kind of damages could happen all together (e.g., corrosion and fatigue may occur at the same place), however, classifying them it is convenient in order to somehow address specific SHM techniques to a type of damage. Moreover, the whole structure should not be monitored applying the same SHM procedures, but each part must have their own approach, provided that the vast majority of the damage is produced in the fuselage and in the thin components of the aircrafts [1]. The first developed monitoring systems were the Health and Usage Monitoring Systems (HUMS), implemented over helicopters [21]. The HUMS sensors record the status of critical systems and components (i.e., engines, gearboxes, shafts, fans, ro-

tor systems, etc) so that the early detection of progressive defects, or indications of them, is possible and thus rectification can be achieved before they have an immediate effect on operational safety. Concerning airplanes, SHM is less developed being the fatigue monitoring one of the most important assessments performed, specially in the military field [55]. In the recent years, the Operational Loads Monitoring programs applied to military airplanes are being brought to the civil field [65]. Also, a big amount of innovative damage-diagnose SHM patents for civil aviation have emerged particularly due to the higher stress levels of the structures or problems like the Widespread Fatigue Damage [63]. With regard to space vehicles, SHM is as well a key component of their systems [52]; by far, they are the most stressed structures, and subjected to many diverse operational situations. Meteorite and general impacts [66] are one of the significant damages that must be monitor so as to avoid situations such as the Columbia Space Shuttle disaster in which the shuttle disintegrated upon atmospheric entry as a result of a flaw in an insulation panel. Another shared feature among space vehicles that clearly motivates the SHM usage, is that very often these structures remain long periods of time without any possibility of direct human inspection.

As is evident, in the remain of this Thesis we will focus in SHM methods applied to aerospace structures.

1.2 NDT techniques

As mentioned, the great success of the monitoring procedures applied to the maintenance of helicopters plainly boosted the development of new SHM approaches in the rest of the aerospace engineering branches. Particularly, helicopter drive train is a complex system operating in highly variable and adverse conditions; bearings and gears are key components of it, and it is a must to maintain their condition against the stresses induced by the high rotational frequency. By means of appropriate accelerometers, it was sufficient to localize distortions in the frequency spectrum and set warning levels for possible threats. This methodology is known as Condition Monitoring and it is widely implemented in the area of rotatory machines [43].

Historically, the SHM techniques utilized in aerospace engineering (and also in the other fields) have been classified considering them as an evolution of the typical NDT techniques. Provided that each of them has its own specific characteristics, they are diversely applied in many scenarios. However, in practice, there is one technique in

aerospace engineering that stand out among the rest, ultrasonic testing. We can classify the most relevant in the following manner:

- Visual inspection.
- Liquid penetrant.
- Eddy currents.
- Imaging methods (radiography and thermography).
- Laser vibrometry.
- Fiber optic sensors.
- Ultrasonic testing.

Let us briefly comment the previous techniques:

Visual inspection

Visual inspection is the simplest approach to NDT [23]. It involves the operator to do checks utilizing the human senses such as vision, hearing and touch. Even though it does not require specific equipment, in some situations, the use of lenses and mirrors could be used if the access to a structure is complicated. It is a common technique in quality control and as a preliminary assessment since it principally allows only to do superficial inspections. One of the main disadvantages is that small defects are difficult to be detected. In aircrafts, it still used a very subjective technique named as *coin tap test* [79], in which an operator taps with a coin-like light tool on the structure to be inspected, feeling the subtle difference of impact force and hearing the resulting sound to discriminate defective regions from normal ones. Indeed, visual inspection procedures should be eliminated from completely autonomous SHM systems.

Liquid penetrant inspection

The liquid penetrant inspection is another low-cost technique which allows to make fast superficial inspections. This technique is used to detect casting, forging and welding surface defects such as cracks, surface porosity, leaks in new products, and fatigue cracks on components [64]. After a cleaning stage, the liquid penetrant is then applied to the surface of the item being tested. The liquid penetrant is drawn into the surface-breaking

crack by capillary action and excess surface penetrant is then removed. After excess penetrant has been removed, a white developer is applied to the sample, to draw out the penetrant in the crack and produce a surface indication. The minimal typical size of the cracks that this technique can detect is 150 nanometres. Recently, this process has been fully automated by means of programmed robotics procedures and television cameras in order to do flaw pattern recognition.

Eddy current testing

The Eddy current testing [10] is a non destructive technique utilized to assess conductive materials. It is based in the electromagnetic induction. To perform the testing, a coil excited with an alternating electrical current is situated close to the specimen to inspect. This coil generates a magnetic field as well. The coil produces currents in the specimen, the so-called Eddy currents, which will affect the current in the coil due to mutual induction. Thus, when the specimen suffers changes, the Eddy currents will vary and the presence of defects will be detected by measuring the impedance variations in the coil. This technique can detect either sub-surface cracks, size variations or material variations.

Industrial radiography

Industrial radiography is a NDT technique which takes advantage of two different forms of electromagnetic radiation: gamma and X-rays [39]. These two forms have the shortest wavelength which enables the wave to penetrate within the structure. Differences in the density of the materials entail a different radiation absorption. This leads shadows to appear in the film generated with the rays camera. It can detect either defects on the surface or inside structures. Its drawback is the high risk for the operator's health.

Thermography

The infrared thermography is a contactless technique which can detect either surface or sub-surface flaws [74]. After a convenient heating of a body, the temperature on its surface is measured, and if any part of this surface exhibits changes in its thermal conductivity (i.e., if there is a flaw), it will locally show a different response when cooling. Active thermography is a more convenient approach in contrast to passive thermography. In active thermographies, pulsed excitation sources such as heat lamps are utilized to induce thermal contrasts. At the same time, an infrared radiation camera is used to obtain the

heat distribution over the surface of the specimen. It is a relevant technique applied in the assessment of CFRP laminates and honeycomb bonded panels. The main limitation of this technique is that the surface condition of the specimen not always provides an optimal surface emissivity in order to perform accurate heat distribution images.

Laser vibrometry

Laser vibrometry is one of the contactless procedures utilized in non destructive evaluations for crack detection and delaminations in thin-walled structures [48, 68]. It utilizes a laser Doppler vibrometer, which is essentially a two beam interferometer that measures the frequency/phase shift between a reference beam and a test beam directly emitted to the surface of interest. In particular, the amplitude and the frequency of the points of the surface can be obtained by means of the Doppler shift of the scattered beam frequency. The output of the laser Doppler vibrometer is commonly a continuous analog voltage directly proportional to the velocity component of the surface point along the beam direction. This relation between voltage and velocity facilitates significantly its conversion and the general implementation of the procedure. One of its advantages in contrast to its competitors such as conventional accelerometers, is that, as mentioned, it is a contactless technique that can access fairly intricate targets. Furthermore, it is a convenient method in situations in which the structure cannot be overloaded with sensors.

Fiber optic sensors

Fiber optic sensors are widely used for monitoring purposes in the aerospace industry, typically to measure temperature or mechanical strain [36]. As its name indicates, these sensors benefit from optical fibers which are very thin filaments of near pure silica (0.125 mm of diameter), that keep the light confined inside its core even when they are bent. The general principle of such devices is that white light as a mixture of frequencies, either from monochromatic (laser) or LED (light emitting diode), is sent through an optical fiber. In its travel, the light experiences subtle changes of its parameters either in the fiber or in one of the several sensors (fiber Bragg gratings are the most utilized [46]), and a portion of light then reaches a detector arrangement which measures these changes. They have the useful capability of collecting information along the fiber allowing the design of distributed sensing systems within the structures. Concerning the aerospace field, many research works have arisen investigating its application in composite structures for strain measurements [35]. The main limitation of these procedures is that the damage can

only be detected when it crosses the optical fiber path (local damage detection). Global methods are still under development.

Ultrasonic testing

Ultrasonic testing is one of the most utilized methods to inspect aerospace structures, either metallic or composite materials [32, 47]. It permits not only to localize the position of flaws but also their size. An ultrasonic transducer is usually attached to the specimen by a couplant. These transducers generate mechanical waves of several types such as longitudinal, shear and surface waves. These waves have a very short wavelength due to the high frequencies they oscillate, namely from 0.1 MHz to 15 MHz. Two methods are used to perform ultrasonic testing: pulse-echo and through-transmission. In the first one, the transducer make both transmitting and receiving the pulsed waves. In the second one, a transducer transmits the pulses and they are received by separated sensors over the specimen. Taking profit of the time of arrival of the traveling waves, the position of the defects can be triangulated as the velocity is known. Concerning the through-transmission methods, the generation of ultrasonic guided waves via piezoelectric transducers is the approach par excellence in the present times [17, 80]. One of the main disadvantages of this technique is that inhomogeneous structures, irregular in shape or with variable thickness are difficult to inspect [19].

Technique	Application	Advantage	Limitation
Visual inspection.	Detection of large cracks and impact damage.	No sensors required.	Time-consuming, costly, highly subjective, limited to surface observation.
Liquid penetrant.	Detection of cracks, leaks and porosity.	Speed and low cost.	Superficial detection, very clean surface needed.
Eddy current.	Detection of small cracks.	Low cost, well established.	Cannot detect all types of damage, not suitable for on-board use.

Imaging.	Detection of cracks and impact damage.	Low to moderate cost, well established, identifies through thickness defects.	Not suitable for on-board use.
Laser vibrometry.	Wide-area scanning for cracks and delamination.	Contactless, it can scan large surface areas.	Bulky laser excitation system, expensive.
Fiber optics.	Measurement of strain and temperature.	High sensor density, and low cost.	Fibers are fragile, global damage detection.
Ultrasonic inspection.	Detection of small cracks, impact damage, and delamination.	Can be used on-board, low cost, identifies through thickness defects.	Limited surface range, actuation source needed.

Table 1.1: NDT techniques in aerospace engineering.

Essentially, NDT serves to detect discontinuities in solids, either at the surface or internal, which are identified as damages. SHM detect local changes in the structure, either of the material properties or its connectivities, which are identified by comparing the response of the structure to a stimuli with the response of the pristine structure. SHM cannot fulfil *first article* inspection.

NDT	SHM
Inspection is done by means of external probes and equipments.	Sensors are permanently attached at fixed locations in the structure.
Off-line monitoring, parts need to be disassembled for inspection.	On-line monitoring, aircraft inspection may be done in flight or during overnight stops.
Time based maintenance, checks must be regularly spaced.	Condition based maintenance. Disassembly only when required for repair.

Labour intensive.	Evaluation done without human intervention.
Mature technologies are available.	Still under development for real structures.

Table 1.2: Comparison between NDT and SHM.

In this sense, as already mentioned, SHM aims at going well beyond traditional NDT techniques, and for this reason, the previous classification is somewhat incomplete since it is not considering relevant aspects related with the analysis and processing of the retrieved data. Another classification may be established making difference between three groups, namely,

- **Model-based SHM.** Commonly, these approaches rely on a initial Finite Element model representing the analyzed structure. Then, by means of the experimental data retrieved in the sensors, the initial dynamic properties (mass, stiffness, and damping) of the FE model are updated in order to enhance its computed solution [73]. After having obtained an accurate approximation of the structure, the damage can be located solving an appropriate inverse problem. Needless to say, that the main limitation of this approach lie in the complexity of the structures to be modeled (cracks, joints, non-linear materials, etc).
- **Data-driven SHM.** In this case, the model created is statistically rather than based on the governing physics of the problem (no numerical model has to be validated). The model is usually established via pattern recognition and machine learning utilizing the experimental data obtained in the sensors [78]. The main limitation within data-driven approaches is the exhaustive data analysis that has to be done so as to locate and quantify the damage.
- **Hybrid models.** These models combine both previous approaches; predictive modeling to supplement experimental data, along with statistical modeling to discriminate damage [8].

Concerning the manner the structure is monitored, another useful classification for SHM may be established [71]. If no energy is intentionally transmitted into the testing

structure, namely, sensors just work in a pure detection mode for retrieving data, we are talking about *passive* SHM. Instead, if our testing structure is equipped with either transmitters and receivers, we have the possibility to perturbing the structure, and then, collect data in the receivers. Hence, now we can refer to it as *active* SHM. The passive scenario is encountered in situations monitoring the progression of damage in loaded structures (e.g., the growth of a crack) or when the monitoring of impacts on the structure is needed [53]. On the other hand, active techniques are found in the typical ultrasonic tests with piezoelectric devices, which may act in dual mode, this is, transmitting and receiving. Note that in pure NDT techniques, the active case is usually excited by an external source (in contrast to the SHM embedded transmitters), but the philosophy is clearly the same.

Apart from the above discussed classifications, SHM methods may be classified in many other forms, for instance, based on the size of the damage, on the governing physics behind the method, on its characteristic scales (time and length scales), or on the parts of the structure taken into account (global or local) [29].

1.3 Inverse problems in SHM

In the present work, we firmly draw on a definition of SHM claiming to be seen as an inverse problem [40]. Indeed, it aims at locating any type of damage within the structures by analyzing the response to excitation, i.e., determining if their present shape coincide with the shape given in the designing stage. Mathematically, processing data from sensors to identify any type of damage such as cracks, inclusions, corrosion, etc (which affect material parameters such as the Young modulus, the Poisson ratio, and the density) is clearly an inverse problem. These problems have been formulated and treated in mathematics for years and give rise to formulations that can be set in stimulating/provocative terms [45]. The name inverse problem comes as opposite to the direct problem, which consists in calculating the system response when its physical properties are known. At the same time, it is convenient to note that inverse problems are usually ill-posed from the mathematical point of view, and a regularization of the problem should be carried out in this regard [44, 24].

There is a variety of mathematical methods to solve inverse problems [5]. An appealing approach is to formulate the problem in terms of an optimization problem, in which the objective function accounts for the difference between the experimentally measured and computed (for varying values of the material properties) responses of the system. The

guessed values of the material properties are those that minimize the objective function. Hence, in the context of SHM, this formulation could give location and shape of all types of damage.

The above mentioned optimization problem can be treated by iterative methods [41]. However, these methods either require a good initial condition (which, in turn, necessitate a prior knowledge of the number of defects) or involve a high number of iterations. A promising alternative to plain optimization methods is based on the computation of the so-called topological derivative [9, 16, 26], which is a scalar function already used in the contexts of acoustic waves and elastic waves [4, 12, 27, 38], electromagnetism [49], thermographies [62], et cetera, that somehow gives the sensitivity of the objective function to local perturbations (from healthy conditions) in the material properties. Hybrid approaches has been utilized as well in literature, namely, a primary topological derivative stage to provide a reasonable first approximation of the damage to further iterative methods [14, 15].

Although it has been proved that the topological derivative is a powerful tool to solve optimization problems in many diverse contexts, it is important to note that its first implementation was related with shape optimization of structures in which unnecessary material was progressively eliminated [69]. Regarding its usage for damage detection, the idea of the topological derivative is that localized defects are expected to be located in those regions where the sensitivity is larger, namely near the topological derivative peaks. The topological derivative can be seen as the gradient of the objective function, and may be computed by solving an appropriate adjoint problem [58] which is an extremely affordable alternative in terms of computational time.

One of the crucial advantages of the topological derivative is that it could be applied to any type of sensing system configuration (piezoelectric transducers, fiber optics, thermographies, laser vibrometry, etc) and to any emitted signal. This is because this method takes into account whole physics of the problem instead of just using the propagation velocity as it happens, for example, in pulse-echo methods. Furthermore, despite the larger the number of (uncorrelated) available data the better the obtained results, the topological derivative has the capacity to detect damage accurately with a very reduced number of transmitters and receivers [28]. A fairly affordable practice to compensate the reduced sources of excitation and data acquisition is to force the structures at different frequencies, and afterwards, combine the calculated topological derivatives in an appropriate manner. Many works has determined that the behaviour of the topological derivative is affected by the frequency of the exited waves. For instance, in scattering problems for the Helmholtz

equation in unbounded media, the topological derivative somehow locates the inside of the damage when low/medium frequencies are considered, while for high frequencies, are the boundaries of this damage which are located [16, 26]. Needless to say, that emitting at low and high frequencies in a range, both position and shape (if it is really needed) may be obtained.

In contrast to previous investigations on topological-based methods in which several time domain incident waves at a single frequency are considered, very few works explore the topological derivative approach for multi-frequency data. As an example, a preliminary study for elastic wave-imaging in unbounded media is presented in [37], where a product of cut-off topological derivatives computed for different frequencies is utilized. Multi-frequency topological derivatives are also presented in [54] for the inspection of soils in order to look for metallic objects. More recently, multi-frequency topological derivatives conveniently combined (and weighted) for pure electric permittivity and for pure magnetic permeability contrast have been used for the reconstruction of defects in electromagnetic problems [3, 61]. In particular, in [3] is presented a study dealing with limited aperture angles for planar incident waves in which the number of incident directions is rather large and measurements are taken on the whole boundary of the media that contains the defects.

As anticipated, designing an optimal form of combining and weighing the single topological derivatives in multi-frequency approaches is fundamental so as to correctly locate damage. To this end, in [28] is illustrated a method which rescales the topological derivative of each transmitter and frequency with its minimum giving very good results. In a similar way, in [75] it is also presented a possible combination of the topological derivatives, in this case with time dependent pulses though.

1.4 Motivation of this Thesis ---

The scope of this Thesis is to present an innovative method for SHM based on the computation of the topological derivative which can overcome some of the difficulties and limitations that the traditional ultrasonic methods have. Although the topological derivative concept was developed in the early nineties, at that time was not feasible its implementation in SHM due to the limited computing power. However, at the present times, modern computers (with fast RAM and CPU) permit the computation of the rather big numerical models involved, motivating the application of the topological derivative for damage detection purposes.

Another fundamental fact that motivates the investigation of the topological derivative for SHM is that it does not need any information about the possible damage within the structure. Historically, this was a severe limitation of iterative methods since they needed good initial approximations of the damage. For this reason, the application of iterative methods to identify damaged was not practical at all (indeed, we already know that the structure is somehow damaged). Furthermore, the topological derivative algorithm is non-iterative, meaning that (in theory) in just one step we could guess the damage.

Regarding SHM via elastic waves, the general idea is to transmit and receive pulses, and utilize the time of flight in order to locate damage as the velocity is known. Due to the fact that guided waves methods solely take into account the wave propagation velocity, they are very flexible and fairly good results can be achieved by using them. However, in the plate assessment context, these methods present weaknesses in the following scenarios:

- The mixing of the primary emitted waves and their reflection at the lateral walls, may lead to very complex wave patterns. In order to triangulate the damage, neat scattered pulses should be received, otherwise these times of flight can not be transformed in the correct distance. Particularly, this situation is often originated by:
 1. The transmitters and/or receivers are close to the boundaries, which strongly mixes the primary and reflected waves near the transmitters and/or receivers, thus widening the width of the received signals.
 2. The cross section of the plate is curved and complex, which produces very complex reflected waves that may even produce caustics.
 3. The defects are beyond elongated through-slits, elongated inclusions of a different material, or attached elongated structural strips (such as stringers in an aircraft wing), which necessarily diffract/scatter the primary waves before these reach the defects and receivers.
- The variation of the propagation velocity in the plate when its thickness is not constant. In theory, since the thickness is known, the wave propagation is known too, but accounting for the effect of the variable wave propagation, the above mentioned relation between time-lags and distances is very problematic, namely, triangularization of the damage is not feasible.
- Intricate structures require a high number of transmitters and receivers to locate damage, and in some instances, even placing them in a certain manner.

Hence, in this research work, we propose our compelling alternative whose strengths are precisely those the traditional guided waves methods lack of. In particular, we will cover the enumerated weakness above, presenting the results of the topological derivative approach. To this end, our testing specimens will be aluminum plates with different kind of defects, air slits, material inclusions, thickness variation or complex geometry. For the sake of simplicity, we have utilized aluminum plates to illustrate the strengths of the method, however, it can be applied to any kind of material (e.g., composite materials) and structure (i.e., not only plates). It is worthwhile remembering that the potency of this method lies on the profound understanding of the physics of the problem, i.e., in our particular case, the elasticity equation which models the generated waves. As long as accurate numerical models are used for its computation, the topological derivative may be an optimal option for SHM.

Naturally, the method presented is supposed to locate damage utilizing the general case of Lamb waves where both symmetric and antisymmetric modes appear. However, due to the fact that antisymmetric Lamb waves exhibit a lower velocity (at least two times for the usual thicknesses and frequencies), this mode requires a fairly fine mesh to be modeled correctly. Therefore, for the sake of having reasonable computational times, we will calculate the results exciting just the symmetric mode. Nonetheless, in the final chapter, a test case in which symmetric and antisymmetric Lamb waves appear will be detailed in order to demonstrate that the method is valid as well, showing a suitable robustness for SHM.

In addition, several models of transmitters and receivers will be tested and combined in our case studies. This is motivated by the new approaches that mix different kind of transmitting and receiving methods [81]. Concretely, we calculate successful test cases in which the wave generation is performed by piezoelectric transducers, and then, a laser Doppler vibrometer receives the signals.

Finally, it is relevant to comment that because of the limited time to doing this Thesis, the experimental data has been computed numerically. Experimental noise will be mimicked by using fairly different numerical grids in the numerical computations and the calculation of the topological derivative. This Thesis intends to present its results as strong a proof of the concept in the SHM context for industrial case scenarios, motivated by already experimental validations arisen in literature [75].

Mathematical models and topological derivative

2.1 Introduction

In this chapter the mathematical models of the physics involved in this research are detailed. Furthermore, an explanation of the topological derivative will be done within the framework of the approaches to solve inverse problems.

2.2 Elastic waves

Elasticity is the most important property of a solid [33]. It is the capacity of a solid to retake its initial size and volume after the action of any external force. Henceforth, elastic waves or acoustics waves are perturbations propagated in this so-called elastic medium. A particular case of acoustic (from the Greek *akustikos*, meaning to hear) waves is sound, which is perceptible to human hearing. A profound understanding of elastic wave propagation is necessary to comprehend and develop new SHM methods based on this phenomena.

2.2.1 General concepts

In contrast to liquids and gases, solids physical nature is somewhat different [57]. Solids not only can exhibit changes in their volume, but also can exhibit changes in their shape. Therefore, the concept of *stress* is introduced in solids to treat this *shear* elasticity, instead of the concept of *pressure* in liquids and gases (i.e., the force per unit area). In

solids, the stress state is represented by a third order symmetric tensor σ , and it contains normal components σ_{xx} , σ_{yy} , σ_{zz} , and shear components σ_{xy} , σ_{xz} , σ_{yz} , where x , y , and z are the coordinate values. Unlike solids, liquids and gases are not capable *to resist* these shear stresses (recall that shear stresses in fluids are related with the gradient of the velocity by means of the viscosity), and the normal components are equal to each other and equal to a signed pressure. The sign of the pressure merely determines whether the stress induces traction or compression. Another typical feature of solids is the concept of deformation, which compares the distance between points of a solid before and after the application of a force. Similarly to the stress case, it is represented by a third order symmetric tensor ε containing normal and shear components. Again, for liquids and gases shear deformations do not exist while the traction and compression deformations are the same in all directions. In order to relate the stress and strain in solids, the Hook law is utilized. For an isotropic medium,

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ii} + 2\mu \varepsilon_{ij}, \quad (2.1)$$

where λ and μ are the Lamé parameters, and δ_{ij} is the Kronecker delta.

2.2.2 Body waves

The wave motion in a solid is mathematically modeled using the following partial differential equation,

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla (\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot [\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)]. \quad (2.2)$$

It is possible to demonstrate that in an infinite solid medium two different types of waves are generated at distinct propagation velocities. From vector analysis, we can state that the displacement field can be represented as the summation of two terms; the first one with a scalar potential and the second one with a vector potential,

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t = \nabla \varphi + \nabla \times \boldsymbol{\Psi}. \quad (2.3)$$

Considering that $\nabla \times \mathbf{u}_l = \nabla \cdot \mathbf{u}_t = 0$ and substituting (2.3) in (2.2) we obtain,

$$\frac{\partial^2 \mathbf{u}_l}{\partial t^2} - c_l^2 \nabla^2 \mathbf{u}_l = 0 \quad c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (2.4)$$

and

$$\frac{\partial^2 \mathbf{u}_t}{\partial t^2} - c_t^2 \nabla^2 \mathbf{u}_t = 0 \quad c_t = \sqrt{\frac{\mu}{\rho}}. \quad (2.5)$$

The wave u_l is called a pressure or longitudinal wave, since the particles of the solid vibrate in the same direction as the propagation of the wave. In the wave u_t , the particles of the solid vibrate in a direction perpendicular to the direction of the wave propagation motion, and it is known as shear or transversal wave. When a transversal wave propagates in an infinite isotropic medium, all directions of transversal vibrations are equal. However, if there is a surface to which the wave propagates parallel or at an angle, it is possible to make difference between two different kinds of transversal waves: horizontal polarized and vertical polarize waves. In the first one, the particles vibrate parallel to the above mentioned surface. In the vertical polarized, the vibrations of the particles occur in the plane perpendicular to the surface. Typically in literature, when a transversal wave is mentioned, generally it refers to a vertical polarized one.

Figure 2.1: Longitudinal wave and transversal wave.

In ultrasonic testing, piezoelectric transducers usually generate an expansion-compression in a solid surface, which leads to excite the longitudinal waves. Then, consequently, the excitation of a transversal wave due to shear deformation also occurs in the solid.

2.2.3 Rayleigh surface waves

The Rayleigh waves, are elastic waves which propagate along the surface of solids. Rayleigh waves consist in a linear combination of both longitudinal and transverse motions that decrease exponentially in amplitude as distance from the surface increases. Also, there is a phase difference between these wave motions. The vibrating particles of the

solid in a Rayleigh wave have a retrograde motion. These solid particles, revolve around ellipsis against the propagation direction. Regarding the velocity of the Rayleigh waves, it can be approximated by the following relation,

$$c_r = c_t \frac{0.87 + 1.12\nu}{1 + \nu}, \quad (2.6)$$

where c_t is the velocity of the transversal waves and ν the Poisson ratio. It can be seen that the velocity of the Rayleigh waves in the metals case ($\nu \approx 0.3$) is slightly less than the velocity of transversal waves. Regarding the penetration underneath the surface, it is relatively small, being almost unnoticeable when the depth is more than a wavelength.

2.2.4 Love surface waves

The Love waves, are a kind of elastic waves which propagate along the solid surface. They are formed due to the interaction of many transversal waves. These waves appear when there is a layer of a solid material above the solid half-space, and the velocity of transversal waves in this layer is less than in the half-space. The solid particles acquire an entirely horizontal motion perpendicular to the direction of propagation of the wave. These waves are slower than the body waves, but faster than the Rayleigh ones. The depth penetration in the half-space increases when the layer thickness becomes smaller.

Figure 2.2: Rayleigh wave and Love wave.

2.2.5 Guided waves

Guided waves, also known as Lamb waves, are elastic waves originated in bodies with two parallel free surfaces. These waves are *guided* between the parallel surfaces which is the usual situation that occurs in plates. The multiple reflections on both surfaces provoked by longitudinal and transversal waves, and their constructive and destructive interference, lead to the arising of the guided waves. Guided waves consist essentially of a pattern of propagating waves (transporting energy) along the direction parallel to the surfaces, and a pattern of standing waves in the perpendicular direction to the surfaces. The wave equation solution, with suitable boundary conditions such as stresses equal to zero in both surfaces (free surface condition) yields a system with two characteristic equations for the wave number k_p . Depending on the product of a plate thickness and frequency f , it has two or more positive real roots, which correspond to the different wave mode.

It is important to understand the physical essence behind the guided waves. For this purpose, we first introduce an explanatory example where normal waves are generated in a liquid layer. We suppose that a plane longitudinal wave gets onto a layer of thickness h at an angle β as depicted in figure 2.3. Then, the wave starts propagating at an angle α

Figure 2.3: Formation of waves in a liquid.

and reflects many times in the layer. At a certain angle of incidence, the wave reflected from the lower surface coincides in a phase with a direct wave that propagates from the upper surface. It is possible to define the angles β and α as

$$\frac{\sin(\beta)}{c_1} = \frac{\sin(\alpha)}{c_2}, \quad (2.7)$$

where c_1 and c_2 are the velocities of sound in the considered medium. Furthermore, we

can define the difference of the phases of the incident wave and the reflected wave as

$$2\pi \left(\frac{AEB}{\lambda_2} - \frac{DB}{\lambda_1} \right) = 2\pi \left(\frac{2h}{\lambda_2 \cos(\alpha)} - \frac{2htg(\alpha) \sin(\alpha)}{\lambda_1} \right) = \varphi, \quad (2.8)$$

where λ_1 and λ_2 are the wavelengths in an upper medium and in a layer. The condition of phase coincidence is obtained if $\varphi = 2n\pi$, where n is an integer. Thus,

$$h \cos(\alpha) = n\lambda_2/2. \quad (2.9)$$

Consequently, as stated, a wave is generated in the layer due to the interference of multiple propagating waves in different directions. A monochromatic wave that spreads zigzag-like along a layer under condition (2.9) can be considered to be a wave that envelopes all the sections of a layer and moves along it. Besides, one of the most important features of these waves is dispersion. The velocity of the guided waves changes with the frequency.

At this moment, it is also convenient to treat the concepts of phase velocity and group velocity. The velocity of the propagation of the energy in the direction of the wave motion is characterized by the group velocity, and a wave pulse is the typical energy carrier. This pulse propagates zigzag-like in the layer. Thus, the group velocity of this wave in the layer can be define in accordance with the representation in figure 2.4, namely,

$$c_g = c_2 \sin(\alpha). \quad (2.10)$$

On the other hand, the phase velocity c_{ph} indicates the velocity of phase propagation in

Figure 2.4: Relationship between c_g , c_2 , and c_{ph} .

the direction of wave propagation. It is equal to the velocity of the phase variation of an incident wave along the layer, and it can be determined using the sine law,

$$\frac{\sin(\beta)}{c_1} = \frac{\sin(\alpha)}{c_2} = \frac{1}{c_{ph}}, \quad c_{ph} = \frac{c_2}{\sin(\alpha)}. \quad (2.11)$$

The refraction angle of this new generated wave is $\pi/2$ radians. Recalling equation (2.9), we obtain

$$c_g = c_2 \sqrt{1 - \left(\frac{n\lambda_2}{2h}\right)^2} \quad \text{and} \quad c_{ph} = \frac{c_2}{\sqrt{1 - \left(\frac{n\lambda_2}{2h}\right)^2}}. \quad (2.12)$$

Hence, we can see that the phase and group velocity of this wave depend on the thickness of the layer, and on the frequency of the elastic vibrations as stated before. The curves which represent this dependency are called the dispersion curves. Phase velocity tend to infinity when $h/\lambda_2 = 1/2, 1, 3/2, \dots$. This means that the entire surface vibrate simultaneously and c_g is equal to zero. Likewise, when h/λ_2 tends to infinity, the both phase and group velocities tend to the spatial wave velocity c_2 . When n takes odd numbers, the waves are called symmetric, as the motion of the particles is symmetric with respect to the horizontal axe of the layer. If n takes even numbers, these waves will be called antisymmetric. Recalling (2.11) and (2.12), we can also define a relation between the phase and group velocity, namely,

$$\frac{1}{c_g} - \frac{1}{c_{ph}} = -\frac{f}{c_{ph}^2} \frac{dc_{ph}}{df}. \quad (2.13)$$

Regarding the case of solid layers, i.e., plates, the essence of the phenomenon is the same, although the conditions of formation of the new waves are more complicated because of the presence of longitudinal and transversal waves. Due to reflection, these waves are partly transformed into one another, and the wave phase varies proportionally to the π number.

In figure 2.6, it is shown the typical dispersion curves for a metal. These curves are obtained via the two characteristic equations mentioned in the beginning of this section. These equations are the so-called Rayleigh-Lamb equations, and they reveal the relationship between wave velocities and frequency. They are derived from the wave equation applying the suitable boundary conditions. There is one characteristic for the symmetric modes,

$$\frac{tg(ph/2)}{tg(qh/2)} = -\frac{(k_p^2 - q^2)}{4k_p^2 pq}, \quad (2.14)$$

and another one for the antisymmetric,

$$\frac{tg(ph/2)}{tg(qh/2)} = -\frac{4k_p^2 pq}{(k_p^2 - q^2)}, \quad (2.15)$$

where

$$p^2 = \frac{\omega^2}{c_l^2} - k_p^2, \quad q^2 = \frac{\omega^2}{c_t^2} - k_p^2, \quad \text{and} \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \quad (2.16)$$

Figure 2.5: Dispersion curves in a liquid layer.

These are transcendental equations since p and q also depend on k_p , and numerical methods are needed in order to solve them. Concerning the zero indices, they indicate the modes of the wave that are transformed into a Rayleigh surface wave when the thickness of the plate increases. These modes exist at any vibration frequency and for any thickness of the plate. The zero symmetric mode s_0 corresponds to an expansion and compression wave as long as $h \ll f$. Under the same condition, on the other hand, the zero anti-symmetric mode corresponds to a bending (also called flexural) wave. Furthermore, it is important to remark that when $fh \rightarrow 0$, the phase velocity and the group velocity in a s_0 wave are equal since there is no dispersion.

2.3 Piezoelectricity

Piezoelectricity accounts for the generation of an electric field when a material is subjected to a mechanical stress, and, conversely, for the generation of a mechanical strain within the material when an electric field is applied on it [31]. The piezoelectric effect observed in some crystal-like materials is a key property seized in SHM. Piezoelectric devices are widely utilized in ultrasonic testing since they permit the generation of high frequency induced deformations in the structures. Because of this local induced deforma-

Figure 2.6: Dispersion curves of the Lamb waves in a steel plate.

tions, different types of waves start to propagate in the inspected structure. Then, taking profit of the inverse effect, another piezoelectric devices can receive this propagated strain and transform it into voltage signal. Let us formulate the mathematical expression which model this physical phenomena. The Hooks law for linear elastic materials is

$$S_{ij} = s_{ijkl}T_{kl}, \quad (2.17)$$

where S is the strain, s is compliance and T is the stress. Furthermore, the material has a linear electrical behavior modeled as

$$D_i = \varepsilon_{ij}E_j = \varepsilon_0 E_i + P_i, \quad (2.18)$$

where D is the electric displacement, ε is the permittivity of the material, ε_0 is the permittivity of the vacuum, E is the electric field and P is the polarization. Combining equations (2.17) and (2.18), we can establish a constitutive relation between mechanical and electrical variables, namely,

$$S_{ij} = s_{ijkl}^E T_{kl} + d_{kij} E_k \quad (2.19)$$

and

$$D_i = d_{ijk} T_{jk} + \varepsilon_{ij}^T E_j, \quad (2.20)$$

where the second order tensors, S and T , represent the strain and stress respectively, and s_{ijkl}^E is the compliance. Regarding d_{ijk} and d_{kij} , they are the coupling coefficients between the elasticity and electric equations, namely, the charge per unit of stress and the strain per unit of electric field. The superscripts T and E signify that the magnitudes are measured at zero stress (i.e., no forces applied on the material), and at zero electric field (i.e., no voltage across the electrodes).

Let us write the previous equations in a matrix form. Taking advantage of Voigt notation we have,

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ \frac{1}{2}S_6 \end{bmatrix}, \quad \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}, \quad (2.21)$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & 0 & 0 & 0 \\ s_{21}^E & s_{22}^E & s_{23}^E & 0 & 0 & 0 \\ s_{31}^E & s_{32}^E & s_{33}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44}^E & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{55}^E \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} + \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \\ d_{14} & d_{24} & d_{34} \\ d_{15} & d_{25} & d_{35} \\ d_{16} & d_{26} & d_{36} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (2.22)$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}^T & \varepsilon_{12}^T & \varepsilon_{13}^T \\ \varepsilon_{21}^T & \varepsilon_{22}^T & \varepsilon_{23}^T \\ \varepsilon_{31}^T & \varepsilon_{32}^T & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \quad (2.23)$$

Hence,

$$[S] = s^E [T] + [d]^t [E], \quad (2.24)$$

and

$$[D] = d [T] + [\varepsilon^T] [E], \quad (2.25)$$

where superscript t merely defines transpose.

In general, due to the fact that the piezoelectric materials respond preferentially along certain directions depending on their intrinsic spontaneous polarization, it can be stated that certain values of the piezoelectric coefficients d are negligible. In figure 2.7, is represented the effect of different directions of the spontaneous polarization and electric field on the motion of the piezoelectric material. If the direction of the electric field is parallel to the direction of the spontaneous polarization, the piezoelectric exhibits a expansion along the thickness direction, i.e., $\partial u_3/\partial x_3 = d_{33}E_3$. Then, as a result, a lateral contraction occurs, i.e., $\partial u_1/\partial x_1 = d_{31}E_3$, and $\partial u_2/\partial x_2 = d_{32}E_3$. This is the usual situation

Figure 2.7: Spontaneous polarization and electric field on the motion of the piezoelectric material.

that takes place in common piezoelectric transmitters in which thickness and in-plane wise vibrations are generated. Furthermore, these piezoelectric materials usually possess transverse isotropy, i.e., $d_{32} = d_{31}$, $d_{24} = d_{15}$, and $e_{22} = e_{11}$. Thus, these two specificities of common piezoelectric materials lead to rewrite equations (2.24), and (2.25),

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & 0 & 0 & 0 \\ s_{21}^E & s_{22}^E & s_{23}^E & 0 & 0 & 0 \\ s_{31}^E & s_{32}^E & s_{33}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44}^E & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{55}^E \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{32} \\ 0 & 0 & d_{33} \\ 0 & d_{15} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (2.26)$$

and

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} + \begin{bmatrix} \varepsilon_{11}^T & 0 & 0 \\ 0 & \varepsilon_{22}^T & 0 \\ 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}. \quad (2.27)$$

2.4 Inverse Problems

Inverse problems deal with determining the input of a system that produces an specific output, or at least, determining the input which gives the closest solution to the desired output. We can find inverse problems in all branches of science such as seismology, medicine, astronomy, quantum physics, et cetera. The recent advent of powerful computers has contributed to increase the interest in inverse problems, since solving the mathematical models involved in these scientific fields is often computationally expensive.

Direct problems attempt to obtain the solutions of the governing equations of a physical system knowing every input parameters of the system in advance. In these problems, the media properties (i.e., the equation coefficients), the initial state of the system or its boundary conditions are assumed to be known. However, often occurs the following: we precisely necessitate to obtain these media properties which indeed become our unknowns. As stated, the unknown equation coefficients can represent media properties such as density, Young modulus, electrical conductivity, magnetic permeability, heat conductivity, et cetera. Hence, this leads to the necessity of specific methods to solve the above-mentioned inverse problems in order to determine the equation coefficients from the solution of direct problem.

Inverse problems are ill-posed. Ill-posedness may refer either to the absence of solution of the desired class, to the absence of an unique solution (i.e., two or more solutions) or to the fact that the solution is unstable (i.e., arbitrarily small errors in the measurement data may lead to indefinitely large errors in the solutions). Indeed, most difficulties in solving ill-posed problems are usually caused by the solution instability. Let consider the following example to illustrate the direct and inverse problems, and the concept of ill-posedness.

Suppose that a particle of unit mass is moving along a straight line. The motion is caused by a time dependent force $q = q(t)$. The particle is at the origin $x = 0$ and has zero velocity at the initial instant $t = 0$. According to Newton law, the motion of the particle is described by the function $u = u(t)$ satisfying the Cauchy problem

$$\frac{d^2 u}{dt^2} = q(t) \quad \text{with} \quad t \in [0, T], \quad (2.28)$$

$$u(0) = 0 \quad \frac{du}{dt}(0) = 0. \quad (2.29)$$

If we aim at obtaining the motion of the particle (i.e., the position $u = u(t)$) by knowing the force which acts on it, we are trying to solve the direct problem. On the other hand, if we try to determine the force $q = q(t)$ which caused an obtained motion $u = u(t)$, then, we are dealing with an inverse problem. Regarding the ill-posedness of this inverse problem, we can prove it is unstable. We now consider perturbations of the solution for the direct problem of the following type

$$u_n(t) = u(t) + \frac{1}{n} \cos(nt), \quad (2.30)$$

and the corresponding perturbations for the acting force

$$q_n(t) = q(t) - n \cos(nt). \quad (2.31)$$

As $\|q - q_n\| \rightarrow \infty$ while $\|u - u_n\| \rightarrow 0$ for $n \rightarrow \infty$, by means of stability theory, we can conclude that the solution is unstable.

To overcome ill-posedness, inverse problems require specific robust methods. They are usually posed in a less demanding way in order to regularize the problem, e.g., not only trying to achieve the exact value of a parameter which causes a specific solution, but trying to determine a close value instead. In the context of SHM, an inverse problem can be formulated as an optimization problem, in which the objective function accounts for the difference between the experimentally measured and computed responses of the system. The unknown values of the material properties are those that minimize the objective

function. Taking into account the physics of the problem is crucial in this situation, as it is needed a good model to obtain accurate synthetic solutions. This optimization problem could be treated by iterative methods. However, these methods either require a good initial condition (which, ultimately, necessitate a prior knowledge of the number of defects) or involve a high number of iterations. In this respect, a promising alternative to plain optimization methods is based on the computation of the so-called *topological derivative*, which has been already used in the contexts of acoustic waves and elastic waves, and will be detailed in the next section.

2.5 Topological derivative

This section is devoted to the core algorithm of the new SHM method developed in this research work, based on the computation of the topological derivative. Firstly, a description of our test problem is elaborated, and progressively, main concepts involved in the computation of the topological derivative such as the transmitting and receiving configurations, the objective function and the adjoint formulation are detailed. In this research, for the sake of clarity, the cases to study will be plates, and the defects to locate through-holes. Notwithstanding, due to the nature of this innovative SHM, it could be implemented in any kind of structure.

2.5.1 Formulation of the problem

Let us consider a rectangular metallic plate D of size $(2L_1) \times (2L_2) \times d$. Using a Cartesian coordinate system with the origin at the center of the plate and the x and y axes parallel to the plate sides, and z perpendicular to x and y , the plate is the prism $D : -L_1 < x < L_1, -L_2 < y < L_2, -d/2 < z < d/2$. With the usual notation, the *displacement vector* $\mathbf{u} = (u, v, w)$, assumed to be conveniently small, obeys the following linear elastodynamics equation

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla (\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot [\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^\top)] = \mathbf{f}(x, y, z, t), \quad (2.32)$$

in the domain D , with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{at } (x, y, z) = (\pm L_1, \pm L_2, 0), \quad (2.33)$$

$$\lambda (\nabla \cdot \mathbf{u}) \mathbf{n} + \mu (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) \cdot \mathbf{n} = \mathbf{0} \quad \text{at the rest of } \partial D \text{ except for the corners.} \quad (2.34)$$

These impose pure reflection at the lateral walls and eliminate rigid-body displacements. In this formulation, ρ is the *density*, t denotes time, λ and μ are the *Lamé coefficients*,[†]

denotes the transpose, the forcing term $\mathbf{f}$ depends on the transmitters configuration, and $\mathbf{n}$ is the outward unit normal to the boundary of D . Note that eqs. (2.32), and (2.34) can also be written as $\rho \partial^2 \mathbf{u} / \partial t^2 - \nabla \cdot \bar{\bar{\sigma}} = \mathbf{f}$ and $\mathbf{u} = \bar{\bar{\sigma}} \cdot \mathbf{n} = \mathbf{0}$, respectively, where the *stress tensor* $\bar{\bar{\sigma}}$ is defined as

$$\bar{\bar{\sigma}} = \lambda (\nabla \cdot \mathbf{u}) \bar{\bar{I}}_2 + 2\mu \bar{\bar{\varepsilon}}, \quad (2.35)$$

where, $\bar{\bar{I}}_2$ is the 3×3 unit tensor and $\bar{\bar{\varepsilon}} \equiv (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) / 2$ is the *strain tensor*.

Since the solution to the problem (2.32)-(2.34) is linear on the forcing term $\mathbf{f}$ (however, not on the material properties ρ , λ , and μ), for a multi-frequency excitation produced by M forcing devices, of the form

$$\mathbf{f}(x, y, z, t) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{F}_{km}(x, y, z) \cos(\omega_k t + \phi_k), \quad (2.36)$$

the dynamics of the displacement vector can be set into the form

$$\mathbf{u}(x, y, z, t) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{U}_{km}(x, y, z) \cos(\omega_k t + \phi_k). \quad (2.37)$$

Substituting (2.36)-(2.37) into (2.32)-(2.34) and setting to zero the various coefficients of $\cos(\omega_k t + \phi_k)$ yield the following purely spatial problems (for $k = 1, \dots, K$)

$$\nabla (\lambda \nabla \cdot \mathbf{U}_{km}) + \nabla \cdot [\mu (\nabla \mathbf{U}_{km} + \nabla \mathbf{U}_{km}^\top)] + \rho \omega_k^2 \mathbf{U}_{km} = -\mathbf{F}_{km}(x, y, z), \quad (2.38)$$

in D , with boundary conditions

$$\mathbf{U}_{km} = \mathbf{0} \quad \text{at } (x, y, z) = (\pm L_1, \pm L_2, 0). \quad (2.39)$$

$$\lambda (\nabla \cdot \mathbf{U}_{km}) \mathbf{n} + \mu (\nabla \mathbf{U}_{km} + \nabla \mathbf{U}_{km}^\top) \cdot \mathbf{n} = \mathbf{0} \quad \text{at the rest of } \partial D \text{ except for the corners.} \quad (2.40)$$

Now, if the forcing terms $\mathbf{F}_{km}$ appearing in (2.36) are all real, eqs.(2.38)-(2.40) show that the displacement fields $\mathbf{U}_{km}$ are also real, which means that each monochromatic component in the expansion (2.37) represents a *standing wave*. However, depending on the frequencies ω_k and the phases ϕ_k , the whole spatio-temporal pattern (2.37) may also be a standing wave or a *propagative wave*. For instance, by applying Fourier transform, the propagative pulses considered in conventional methods can be expanded in the form (2.36)-(2.37). In this Thesis, all computations will be based on the purely spatial problems (2.38)-(2.40), which means that the phases ϕ_k appearing in (2.36)-(2.37) will play no role. This makes a significant difference with conventional ultrasonic methods, in which the

phases do play an important role. A strategy based on a more general approach that takes the phases into account could improve the obtained results.

As anticipated in §2.4, inverse problems consist in guessing the properties of a physical system from its response to excitation. In the defect detection context, structural defects will manifest themselves in the values of the physical properties of the plate, namely the density and the Lamé coefficients, which are assumed to be of the form

$$\rho = \rho_0 - \tilde{\rho}, \quad \lambda = \lambda_0 - \tilde{\lambda}, \quad \mu = \mu_0 - \tilde{\mu}, \quad (2.41)$$

where the subscript $_0$ denotes the values of the material properties for the healthy plate and the perturbations, denoted with tildes, are due to the presence of localized defects. These are assumed to be appropriately small in the L_2 norm, namely to be such that

$$\begin{aligned} \int_D |\tilde{\rho}|^2 dx dy dz &\ll \int_D \rho_0^2 dx dy dz, \quad \int_D |\tilde{\lambda}|^2 dx dy dz \ll \int_D \lambda_0^2 dx dy dz, \\ \int_D |\tilde{\mu}|^2 dx dy dz &\ll \int_D \mu_0^2 dx dy dz. \end{aligned} \quad (2.42)$$

In the present context, these conditions will hold because of the localization of the defects, in spite of the fact that the perturbations in the coefficient values need not be small in the defects themselves. In this case, eq. (2.42) just means that the area occupied by the defects must be small compared to the area of the plate. For instance, circular defects with a diameter equal to one tenth of the plate sides, exhibit an area that is one hundred of that of the plate.

2.5.2 Transmitter - Receiver configurations

As stated, the forcing terms $\mathbf{F}_{km}$ in (2.38) depends on the transmitters configurations. In this research work, we have selected two different types which will induce an in-plane excitation on both sides of the plate at the same (x_m, y_m) , in order to obtain a symmetric elastodynamic state:

- *Point localized* (PL) transmitting, in which a concentrated force at a given point $(x_m, y_m, \pm d/2)$ is imposed. Thus, the forcing term appearing in (2.38) is given by

$$\mathbf{F}_{km}^{PL}(x, y, z) = \mathbf{a}_{km} \delta(x - x_m, y - y_m, \pm d/2), \quad (2.43)$$

where the vector $\mathbf{a}_{km}$ is the applied force and δ is the Dirac delta function. This excitation mimics the non-contact optical-laser actuators.

- *Rotationally symmetric radial* (RSR) transmitting, in which a rotationally symmetric radial force is applied in a circle with center at a given point $(x_m, y_m, \pm d/2)$ and radius r . Thus, the forcing term appearing in (2.38) is given by

$$\mathbf{F}_{km}^{\text{RSR}}(x, y, z) = a_{km} \mathbf{g}(x - x_m, y - y_m, \pm d/2, r), \quad (2.44)$$

where the scalar a_{km} is the overall intensity of the applied force and $\mathbf{g}(x - x_m, y - y_m, \pm d/2, r)$ denotes a forcing term consisting in applying a unit force along the outward normals to the circle of radius r , centered at $(x_m, y_m, \pm d/2)$. This mimics piezoelectric actuators.

Likewise, the response to excitation will be collected in N receivers at both sides of the plate at the same (x_m, y_m) . We will just retain the in-plane components of the displacement vector, thus, hereinafter we will consider $\mathbf{U}_{km}$ as two-dimensional vector. Each of the N receivers is of one of the following two types (which indeed, are the counterparts of the two types of excitation considered above):

- *Point localized* (PL) receiving, in which the displacement vector at a given point $(x_n, y_n, \pm d/2)$ is measured. This sensing device mimics laser Doppler sensors and produces two scalar measurements (namely, the two components of the displacement vector) at the sensor.
- *Rotationally symmetric radial* (RSR) receiving in a circle with center at a given point $(x_n, y_n, \pm d/2)$ and radius r , measuring the spatial average of the radial displacement vector along the circle. This mimics piezoelectric receivers of radius r centered at $(x_n, y_n, \pm d/2)$ and produces just one scalar measurement, the averaged radial displacement vector.

The transmitting and receiving types (either PL or RSR, as described above) will affect the computation of the topological derivative in the remaining chapters of this Thesis. According to our previous comments, these transmitting and receiving types will be combined in three ways, namely:

- PL transmitting and PL receiving.
- PL transmitting and RSR receiving.
- RSR transmitting and PL receiving.

We will consider M transmitter configurations, which will be common to the K considered frequencies. Then, for each of these configurations, appropriate data will be collected at N receivers, all them of the same type (either PL or RSR). Thus, the available scalar data is $K \times M \times (2N)$ for PL sensing and $K \times M \times N$ for RSR sensing. However, the unknowns in the inverse problem are the spatial distributions of the perturbations $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$, which involve an *infinite number of scalar unknowns* from the purely mathematical point of view. Upon discretization via a spatial mesh, with P nodes, the number of unknowns is $3P$, which is usually much larger than the number of available uncorrelated data. As stated in §2.4, this is one of the difficulties involved in inverse problems, which are generally ill-posed. An additional, even more important, difficulty is that the unknown material properties do not depend continuously on the given data.

2.5.3 Objective function and topological derivative

As already explained in §1.4, our approach will consist in somehow minimizing a scalar objective function, which accounts for the least squares difference between the measured data at the sensors and their computed counterpart for a given varying location and shape of the defects. As stated in the previous section, the defects manifest themselves in the present context through the distributions of the perturbations $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$, as defined in (2.41). In principle, minimizing the objective function defined that way should provide an approximation of $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$, and thus the location and shape of the defects when these are localized.

The above mentioned *objective function* is defined as the sum of the contributions from the K forcing frequencies and the M transmitters/receivers configurations, namely,

$$\mathcal{H}(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}) = \sum_{k=1}^K \sum_{m=1}^M \alpha_{km} \mathcal{H}_{km}^{\text{PL}} \quad \text{and} \quad \mathcal{H}(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}) = \sum_{k=1}^K \sum_{m=1}^M \alpha_{km} \mathcal{H}_{km}^{\text{RSR}}, \quad (2.45)$$

for PL and RSR receiving, respectively, which must be treated separately. Here, as above, $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$ denote the perturbations (from the healthy state, according to (2.41)) in these properties appearing in the damaged plate; the subscripts k and m label the frequencies and the transmitters/receivers configurations, respectively; the weights $\alpha_{km} > 0$ (explained in a paragraph below), measure the individual contribution of each frequency and each transmitting configuration. $\mathcal{H}_{km}^{\text{PL}}$ and $\mathcal{H}_{km}^{\text{RSR}}$ depend on the receiving mode.

For PL receiving,

$$\mathcal{H}_{km}^{\text{PL}}(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}) = \frac{1}{2} \sum_{n=1}^N \|\mathbf{U}_{km}^{\text{comp.}}(x_{mn}, y_{mn}, \pm d/2) - \mathbf{U}_{km}^{\text{meas.}}(x_{mn}, y_{mn}, \pm d/2)\|^2, \quad (2.46)$$

where $(x_{mn}, y_{mn}, \pm d/2)$ denote the receptors positions (which may depend on the transmitters set configuration), $\mathbf{U}_{km}$ is the k -th Fourier component of the displacement vector field produced by the m -th excitation, as defined in (2.37), and the superscripts ^{comp.} and ^{meas.} denote computed (by solving (2.38)-(2.40)) and measured data. Note that we are treating $\mathbf{U}_{km}$ as a 2-dimensional vector as we did not retain the out of plane component. In the present case, we are using synthetic data, mimicking actual measured data, which means that $U^{\text{meas.}}$ will also be computed by solving (2.38)-(2.40) for the actual damaged plate. Since (either actual or synthetic) measurements are known beforehand, dependence of $\mathcal{H}_{km}^{\text{PL}}$ on $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$ comes from $\mathbf{U}_{km}^{\text{comp.}}$, whose calculation requires solving (2.38)-(2.40), which in turn depend on $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$. In other words, $U^{\text{meas.}}$ will be computed by solving (2.38)-(2.40) for the known distributions of $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$, while $U^{\text{comp.}}$ will be computed by solving (2.38)-(2.40) for varying distributions of while $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$, which should be selected to minimize the objective function.

Similarly, for RSR sensing,

$$\mathcal{H}_{km}^{\text{RSR}}(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}) = \frac{1}{2} \sum_{n=1}^N \langle \mathbf{U}_{km}^{\text{comp.}} - \mathbf{U}_{km}^{\text{meas.}} \rangle_{x_{mn}, y_{mn}, \pm d/2, r}^2, \quad (2.47)$$

where $\mathbf{U}_{km}^{\text{comp.}}$ and $\mathbf{U}_{km}^{\text{meas.}}$ are as above and for a vector field $\mathbf{U}$, $\langle \mathbf{U} \rangle_{x_{mn}, y_{mn}, \pm d/2, r}$ denotes the average of the outward normal component of $\mathbf{U}$ on the circle of radius r centered at the point $(x_{mn}, y_{mn}, \pm d/2)$. Note that $\langle \mathbf{U} \rangle_{x_{mn}, y_{mn}, \pm d/2, r}$ is a scalar quantity in spite of the fact that $\mathbf{U}$ is a vector field. Again, dependence of $\mathcal{H}_{km}^{\text{RSR}}$ on $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$ comes through $\mathbf{U}_{km}^{\text{comp.}}$.

Minimizing the objective function (2.45), we expect good guesses of the unknowns $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$. However, the objective function depends nonlinearly on the unknowns and this approach is problematic. Thus, as previously stated, in this research work we present an approach based on a more direct, non-iterative process that uses the computation of the *topological derivative* of the objective function $\mathcal{H}$, which is a scalar vector field $\mathcal{T}(x, y)$ defined at each $(x, y) \in D$ as follows.

The topological derivative of a functional $\mathcal{H}$ defined in a region $\mathcal{R}$, can be seen as the sensitivity of $\mathcal{H}$ with respect the creation of a small cavity at each point $\mathbf{x} \in \mathcal{R}$. More precisely, we can define it as this limit

$$\mathcal{T}(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{H}(\mathcal{R} \setminus \mathcal{B}_\varepsilon(\mathbf{x})) - \mathcal{H}(\mathcal{R})}{h(\varepsilon)} = \lim_{r \rightarrow 0} \frac{\mathcal{H}^r - \mathcal{H}^0}{\frac{4}{3}\pi r^3}, \quad \mathbf{x} \in \mathcal{R}, \quad (2.48)$$

where $B_\varepsilon(\mathbf{x})$ is a ball with radius ε centered at $\mathbf{x}$ and $h(\varepsilon) > 0$ is a monotone decreasing function that tends to zero and selected such that the limit (2.48) exists and is non-zero.

Obviously, $\mathcal{H}^r$ and $\mathcal{H}^0$ denote the values of the objective function for the damaged and healthy plates, respectively.

Let rewrite the limit (2.48) as

$$\mathcal{H}(\mathcal{R} \setminus \mathcal{B}_\varepsilon(\mathbf{x})) = \mathcal{H}(\mathcal{R}) + h(\varepsilon)T(\mathbf{x}) + o(h(\varepsilon)), \quad \text{as } \varepsilon \rightarrow 0, \quad (2.49)$$

which gives the fundamental idea to the detection method: agglutinating points where the topological derivative is large and negative, since in such points $\mathcal{F}(\mathbb{R}e \setminus \mathcal{B}_\varepsilon(\mathbf{x})) < \mathcal{F}(\mathbb{R}e)$, i.e., damage is most likely to occur at those points of the plate where the topological derivative exhibits negative peaks. Note that the defects considered in (2.48) are infinitesimal, which means that considering them as circular is only a slight restriction and the actual defects that are to be localized may have an arbitrary shape.

The topological derivative is a linear operator. For a more precise reconstruction of the defects, combining excitations for different frequencies tends to be a powerful tool. Hence, the topological derivative of the corresponding overall objective functional can be defined as a linear combination of each independent derivative. Thus,

$$\mathcal{T}(x, y) = \sum_{k=1}^K \sum_{m=1}^M \alpha_{km} \mathcal{T}_{km}(x, y), \quad (2.50)$$

where $\mathcal{T}_{km}$ is the topological derivative of $\mathcal{H}_{km}^{\text{PL}}$ or $\mathcal{H}_{km}^{\text{RSR}}$ for PL receiving or RSR receiving, respectively, defined in (2.46) and (2.47), respectively. Note that we present the topological derivative as a two-dimensional scalar function, since the in-plane dimensions of the plate are much larger than the out of plane one.

Trying to calculate the topological derivative using the above limit definition it is not practical, or better said unaffordable. A large number of computations would have to be done in order to evaluate the limit (2.48) at each point. Thus, we are going to take profit of the so-called *adjoint formulation*. The adjoint formulation is a very computationally efficient method to calculate the sensitivity of a scalar observable under perturbations of a N-dimensional control variable when the evaluation of the observable requires solving a direct problem. Instead of solving the direct problem N times (e.g., in our case, the number of points in which we have to determine the functional), the adjoint formulation only requires solving once a linear problem whose dimension coincides with that of the direct problem. Note that the computational cost is reduced by a factor of N which represents a very good improvement as N is usually large in complex technical disciplines.

Let us consider a well-known expression in literature to calculate the topological derivative in harmonic elastic problems. It is formulated in terms of the solution of a direct and

an adjoint problem as follows,

$$\mathcal{T}_{km} = \rho \mathcal{S}_{km}^\rho + \frac{1}{\mu(1+\nu)} \left[\frac{\nu-1-2\nu^2}{2(1-\nu^2)} \mathcal{S}_{km}^\lambda + \mathcal{S}_{km}^\mu \right], \quad (2.51)$$

where $\nu = \lambda/[2(\lambda + \mu)]$ is the Poisson ratio and

$$\mathcal{S}_{km}^\rho = -\omega_k^2 \mathbf{U}_{km}^d \cdot \mathbf{U}_{km}^a, \quad \mathcal{S}_{km}^\lambda = (\nabla \cdot \mathbf{U}_{km}^d) (\nabla \cdot \mathbf{U}_{km}^a), \quad \text{and} \quad (2.52)$$

$$\mathcal{S}_{km}^\mu = \frac{1}{2} (\nabla \mathbf{U}_{km}^d + \nabla \mathbf{U}_{km}^{d\top}) : (\nabla \mathbf{U}_{km}^a + \nabla \mathbf{U}_{km}^{a\top}) \quad (2.53)$$

are the partial sensitivities. Here, $\mathbf{U}_{km}^d$ and $\mathbf{U}_{km}^a$ are the solutions of the *unperturbed direct and adjoint problems*, which are both given by (2.38)-(2.40) with $\rho = \rho_0$, $\lambda = \lambda_0$, $\mu = \mu_0$ (the unperturbed values of these properties for the healthy plate). Concerning the forcing term in the right hand side of eq. (2.38), namely, $\mathbf{F}_{km}$:

- For the *unperturbed direct problem*, $\mathbf{F}_{km}$ depends on the forcing type and is given by (2.43) and (2.44) for PL and RSR transmitting, respectively.
- For the *unperturbed adjoint problem*, $\mathbf{F}_{km}$ depends on the receiving type, either PL or RSR. For PL receiving at the points $(x_{mn}, y_{mn}, \pm d/2)$, for $n = 1, \dots, N$, we have

$$\mathbf{F}_{km} = \sum_{n=1}^N [\mathbf{U}_{km}^d(x_{mn}, y_{mn}, \pm d/2) - \mathbf{U}_{km}^{\text{meas.}}(x_{mn}, y_{mn}, \pm d/2)] \delta(x - x_{mn}, y - y_{mn}, \pm d/2), \quad (2.54)$$

where δ is the Dirac delta function, as above. For RSR sensing in devices of radius r , centered at the points $(x_{mn}, y_{mn}, \pm d/2)$, for $n = 1, \dots, N$, $\mathbf{F}_{km}$ is given by

$$\mathbf{F}_{km} = \sum_{n=1}^N \langle \mathbf{U}_{km}^d - \mathbf{U}_{km}^{\text{meas.}} \rangle_{x_{mn}, y_{mn}, \pm d/2, r} \mathbf{g}(x - x_{mn}, y - y_{mn}, \pm d/2, r), \quad (2.55)$$

where the rotationally symmetric vector field $\mathbf{g}$ and the mean value $\langle \cdot \rangle_{x_{mn}, y_{mn}, \pm d/2, r}$ are as defined after (2.44) and (2.47), respectively. Recall that the azimuthal average $\langle \cdot \rangle_{x_{mn}, y_{mn}, \pm d/2, r}$ is a scalar quantity.

Regarding the above-mentioned weights α_{km} in (2.50), they attempt to take into account the contribution of each emitter and each frequency. Following the ideas of [28] which gave very good results for the two-dimensional wave equation, we select

$$\alpha_{km} = \frac{1}{\left| \min_{(x,y) \in D_I} \mathcal{T}_{km}(x, y) \right|}, \quad (2.56)$$

where $D_I \subset D$ is an *interrogation window*, which is the subset of the plate where the topological derivative will be calculated. Such interrogation window must exclude a vicinity of

the actuators and sensors because the topological derivative takes very large (positive or negative) values in near these devices. This is because the solutions to the unperturbed direct and adjoint problems are very steep near the actuators and sensors, respectively, and the topological derivative includes terms (namely, the second and third terms in the right hand side of eq. (2.51)) that are proportional to the product of the gradients of the unperturbed direct and adjoint problems. The large values of the topological derivative near the transmitters/receivers could give negative peaks that would be spurious in the present context and must be avoided.

It is interesting to note that the fields $\mathcal{S}_{km}^\rho$, $\mathcal{S}_{km}^\lambda$, and $\mathcal{S}_{km}^\mu$, defined in (2.52)-(2.53) admit another interpretation, as sensitivities of $\mathcal{H}_{km}$ to perturbations in ρ , λ , and μ , respectively. In other words, if ρ , λ , and μ are perturbed as in (2.41), with conveniently small perturbations, according to (2.42), the individual components of the objective function defined in (2.46) can be linearized around $\tilde{\rho} = \tilde{\lambda} = \tilde{\mu} = 0$, which (using a standard continuous adjoint formulation) gives

$$\mathcal{H}_{km}(\tilde{\rho}, \tilde{\lambda}, \tilde{\mu}) - \mathcal{H}_{km}^0 \simeq \int_D \left[\mathcal{S}_{km}^\rho(x, y) \tilde{\rho}(x, y) + \mathcal{S}_{km}^\lambda(x, y) \tilde{\lambda}(x, y) + \mathcal{S}_{km}^\mu(x, y) \tilde{\mu}(x, y) \right] dx dy, \quad (2.57)$$

where $\mathcal{H}_{km}^0$ is the unperturbed value of $\mathcal{H}_{km}$, quadratic and higher order terms in the small quantities $\tilde{\rho}$, $\tilde{\lambda}$, and $\tilde{\mu}$ have been neglected, and the sensitivities $\mathcal{S}_{km}^\rho$, $\mathcal{S}_{km}^\lambda$, and $\mathcal{S}_{km}^\mu$ are precisely those given by (2.52)-(2.53), with the unperturbed direct and adjoint problems as defined above. Thus, eq. (2.51) represents a particular linear combination of these sensitivities. Other linear combinations could give better results, as could do a selection of the weights α_{km} different from that in (2.56). However, these possible improvements are beyond the scope of the present Thesis, where for the sake of simplicity and brevity, the somewhat standard theory described above will be used.

Test problems via 2D approximation

3.1 Introduction

In this chapter, the first results of method will be presented. As previously mentioned in our problem formulation, we are performing a symmetric excitation on both sides of the plate in order to have optimal computation times. In this sense, we will take advantage of a fundamental feature of elastic waves: if the wavelength of the generated waves is much larger than the thickness of the plate, the elastodynamic state admits a plane stress assumption, which allows for a two-dimensional approximation. In other words, the in-plane solution can be approximated by a 2D model, and then (if necessary), compute the out of plane solution by means of simple algebraic operations.

3.2 Results

Let us now apply the theory developed in the last section to an aluminium plate, with density $\rho = 2,700 \text{ kg/m}^3$ and Lamé coefficients $\lambda = 50.35 \text{ GPa}$ and $\mu = 25.94 \text{ GPa}$ (Young modulus $E = 69 \text{ GPa}$ and Poisson ratio $\nu = 0.33$). The horizontal size is $1 \times 1 \text{ m}^2$ and the thickness is assumed to be small (for instance, 1-2 mm), such that a two-dimensional approximation can be applied. Using the Cartesian coordinate system indicated in §2.5.1, the domain occupied by the plate is

$$D : -50 \text{ cm} \leq x \leq 50 \text{ cm}, \quad -50 \text{ cm} \leq y \leq 50 \text{ cm}. \quad (3.1)$$

A large number of cases have been considered, however, for the sake of brevity only the most representative ones will be presented. Concerning the healthy plate, two general

classes of plates will be presented, namely:

- A homogeneous plate, with constant density and Lamé coefficients.
- An inhomogeneous plate containing an elongated through-slit or an elongated inclusion of a different material.

Concerning the transmitting and receiving configurations, these will be classified into three types discussed in §2.5.2, namely:

- PL transmitting and PL receiving.
- RSR transmitting and RSR receiving.
- RSR transmitting and PL receiving.

For RSR transmitters and receivers, the radius of the device will always be $r = 5$ mm, which is similar to the radius of typical piezo-electric devices. In order to obtain comparable results for the various cases that will be considered, the centers of the actuators and sensors will always be located at 2 cm from the boundary, which has been selected on purpose as a very small distance compared to the plate size and would lead to strong difficulties if standard SHM methods were used. Location of the actuators and sensors along the lines parallel to the sides of the plate will be chosen with no particular care, which will emphasize robustness.

The topological derivative will be calculated in an *interrogation window* that will be a 70×70 cm² centered square, namely,

$$D_I : -35 \text{ cm} \leq x \leq 35 \text{ cm}, \quad -35 \text{ cm} \leq y \leq 35 \text{ cm}, \quad (3.2)$$

which means that a 15 cm gap will be ignored near the boundary of the plate. The defects will always be inside the interrogation window and will be circular or elliptic through-holes, with a very small size ~ 5 mm. As anticipated in §2.5.3, it is the negative peaks of the topological derivative that matter to localize defects. Thus, we shall consider plots of the topological derivative, which will always be rescaled as follows. Positive values of the topological derivative will be set to zero, and the negative values will be rescaled with the maximum of its absolute value. Therefore, all topological derivatives will vary in the range between 0 and -1.

The number of available data increases as the number of transmitter/receiver configurations, M , the number of receivers, N , and the number of involved frequencies K increase. However, increasing M and/or N requires increasing the number of devices.

This is more problematic in practice than increasing K , which only affects the complexity of the transmitting signal, according to (2.36). Thus, some emphasis will be put below in decreasing M and N (at the expense of increasing K). On the other hand, selecting transmitters and receivers that are somewhat close to each other (which makes the available data less uncorrelated) and close to the boundary of the plate favors compact designs of the experimental device and will be intended below.

Application of the method described in §2.5 requires solving elastic problems of the type (2.38)-(2.40) for both:

- The perturbed plate, to obtain the synthetic data (mimicking experimental data).
- The healthy plate, to solve the unperturbed direct and adjoint problems, which are needed to compute the topological derivative.

The solution to these problems will be obtained using the ANSYS Mechanical FEM solver [51], using the PLANE183 element which has a second degree polynomial as shape function. The PLANE183 element is the 2D counterpart of the SOLID186 element (a 3D one) which will be used in the next chapters of this Thesis. Concerning the computational mesh, the typical size of the individual elements will be ~ 2.5 mm, though they will concentrate near the through-holes when calculating synthetic data. All these will give a number of degrees of freedom ~ 0.96 Million (corresponding to 0.48 Million nodes).

An idea of the computational meshes that will be used to generate the synthetic data and to solve the unperturbed direct and adjoint problems for the case considered in figure 3.1. Note that the two meshes are quite different, both quantitatively and qualitatively. The main qualitative difference between them is that nodes concentrate near the defect in the left plot (because the defect is taken into account when computing synthetic data), but not in the right plot, which corresponds to computations of the unperturbed direct and adjoint problems in the healthy plate.

As stated, we will consider several transmitter and receiver types and configurations, homogeneous and inhomogeneous plates, and various defect locations. In order to emphasize the robustness of the method, the forcing frequencies will be chosen as equispaced in two different ranges, namely,

$$10 \text{ kHz} \leq \omega/2\pi \leq 25 \text{ kHz}, \quad (3.3)$$

and

$$10 \text{ kHz} \leq \omega/2\pi \leq 50 \text{ kHz}. \quad (3.4)$$

Figure 3.1: The meshes used to compute the synthetic data (left) and to solve the direct and adjoint problems (right) in the computations considered in figure 3.2-left, considering not the whole amount of nodes (which would make too dense plots), but only one out each of 4 nodes. The defect location is indicated with a red + symbol.

In the subsections below, we will refer to these ranges as the *low frequency range* (LFR) and the *low-high frequency range* (LHFR).

In the remaining of this chapter, we will present several test cases for each transmitting/receiving configuration. We will initiate the study with the PL-PL configuration, and then, we will test its fairly good results with the rest of the configurations, namely, the RSR-RSR and the RSR-PL.

3.2.1 PL emission and PL sensing

To begin with, we consider a number M of PL dual transmitting/receiving devices, which will be used to obtain (for each of the K driving frequencies) M transmitting/receiving configurations. In each of these, signals are transmitted from one of the devices, which acts as transmitter, and received at the remaining $N = M - 1$ devices, which act as receiver. Since two scalar data are produced at each PL receiver, the total number of available scalar data is $2K \times M \times (M - 1)$, which will be only moderately large (not larger than a few tens of thousands) in the applications below. In addition, these data are not completely uncorrelated, meaning that the effective number of data is even smaller. The number of unknowns, instead, are the values of ρ , λ , and μ at the nodes of the computational mesh, which is $\sim 3 \times 0.96 = 2.88$ Million unknowns, namely much larger than the number of data. This clearly illustrates the ill-posed nature of the inverse

problem. Using the topological derivative, described in §2.5 somehow circumvents this difficulty.

Homogeneous plate

The first considered emitting/receiving configuration consists of 32 evenly located PL

Figure 3.2: Color map of the topological derivative for the first transmitting/receiving configuration (using 32 transmitting/receiving devices indicated with black $\times$ symbols), considering a circular (left), elliptical (middle), and circular-plus-elliptical (right) defects, whose position is indicated with white $+$ symbols. In this case, the topological derivative is calculated in the LFR.

transmitting/receiving devices (as indicated with black $\times$ symbols in figure 3.2), which are fairly close to the boundary, namely 2 cm apart. The devices are equispaced along the lines parallel to the sides of the boundary. Concerning the driving frequencies, in this first case we use $K = 15$ equispaced values in the interval (3.3). For this transmitting/receiving configuration, we consider three sets of through-hole defects, namely (a) a circular defect of radius 2.5 mm, with its center at $(x, y) = (-0.1 \text{ m}, 0.2 \text{ m})$, (b) an elliptical defect with center at $(x, y) = (-0.15 \text{ m}, -0.2 \text{ m})$ and major and minor axes 4.84 mm and 1.29 mm, parallel to the x and y axes, respectively, and (c) the circular and elliptical defects simultaneously located in the plate. Note that the two defects exhibit the same area. In figure 3.2 (as in all plots of the below), the position of the defects is indicated with white $+$ symbols; note that the shapes of the defects cannot be seen in this plot due to their very small sizes.

The topological derivative for the three mentioned sets of defects is as plotted in figure 3.2, where only negative values of the topological derivative are considered in the color map; points with positive values and points outside the interrogation window are indicated

in grey. As can be seen, the negative peaks of the topological derivative locate fairly well the defects, especially in cases (a) and (b), in which only one defect is present; case (c), with two defects, is more demanding and thus the contrast of the negative peaks of the topological derivatives at the defects is less prominent. Also, the circular and elliptical defects promote comparable negative peaks in the topological derivative, which is due to the fact that both defects have been selected to exhibit the same area. If, instead, one of the through holes exhibited a smaller area, then the (negative) peak of the topological derivative at this defect would be less prominent.

The number of transmitting/receiving devices, $M = 32$, was chosen in the previous test problem to ensure obtaining good results, but most likely they produce highly correlated data and can be decreased. Thus, we drastically decrease this number from 32 to 4, maintaining the same defect sizes and locations as in the previous test problem. The four considered transmitting/receiving devices are located close to each other and in the upper side of the plate, as indicated with black $\times$ symbols in figure 3.3; in fact, they are a subset

Figure 3.3: Color map of the topological derivative for the same defects configurations considered in figure 3.2, but using only the four transmitting/receiving devices located at the positions indicated with black $\times$ symbols and using the indicated number of equispaced frequencies. The frequency range utilized is the LFR.

of those considered in figure 3.2. Concerning the number K of forcing frequencies, this is $K = 15$, as in the previous test problem, when only one defect is present, but this number is increased to $K = 35$ in the more demanding case in which both defects are present. In addition, the same test problem is presented in figure 3.4 for the LHRF. As can be seen in figure 3.3 and figure 3.4, the topological derivative locates the defects fairly well using only four compactly located transmitting/receiving devices (though it gives better results with the LFR). Thus, in the remaining of this subsection, we shall use only this reduced

Figure 3.4: Counterpart of the topological derivatives calculated in figure 3.3, using the second frequency range, namely, the LHFR.

set of four devices.

Inhomogeneous plate

The test problems considered here are much more demanding than the previous ones. Thus, they will require increasing the number of frequencies if the number of transmitting/receiving devices is maintained, as anticipated above. As a first test problem, we consider the plate with an elongated through-slit, considering two cases (see figure 3.5 and 3.6):

- A *centered parallel* (to the x axis) *slit* with length 0.8 m and width 1.6 cm located along the line $y = 0.17$ m, and a circular defect of radius 2.5 mm and center at $(x, y) = (0.1 \text{ m}, -0.2 \text{ m})$.
- An *oblique slit* with length 0.7 m and width 1.4 cm, located along a line with slope 40.17° and centered at $(x, y) = (0.0609 \text{ m}, 0.129 \text{ m})$. The defect is again circular, with radius 2.5 mm and center at $(x, y) = (0.15 \text{ m}, 0.1 \text{ m})$.

For this test problem, we consider $K = 30$ frequencies, equispaced in the intervals (3.3) and (3.4). The topological derivatives for the parallel and oblique slits, plotted in figures 3.5 and 3.6, show that the method locates the defects reasonably well in spite of the presence of the slits. In both cases, especially for the oblique slit, the defect is not reachable by the primary transmitted waves, but only by waves reflected at the boundary, which would make the application of standard SHM methods extremely problematic. In fact, through-slits are quite demanding because they completely reflect the waves, as the

Figure 3.5: Color map of the topological derivative for inhomogeneous plates showing the parallel (left) and oblique (right) through-slits indicated in black. As in previous figures, the transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols. The frequency range utilized is the LFR.

lateral walls do. In the present situation, the slits leave a space at the borders, which

Figure 3.6: Counterpart of the topological derivatives calculated in figure 3.6, using the second frequency range, namely, the LHFR.

allows the elastic waves to pass to the other side of the plate. If, instead, the through-slits were longer, as to completely cover a section of the plate, from side to side, it would be absolutely impossible to detect any defect beyond the slits. However, this does not happen with elongated inclusions of a different material, which are considered in the next test problem, because these inclusions allow for a part of the waves to trespass the inclusion (though they are subject to diffraction).

As a second test problem, we consider a plate with an elongated inclusion of a different material. The inclusion is parallel to the x axis and transversally covers the whole plate at $y = 16$ m from side to side. As for the thickness of the inclusion and the nature of the material, we consider two cases:

- *Titanium inclusion:* Titanium is a fairly dense and fairly rigid material, with density $\rho = 4500$ kg/m³ and Lamé coefficients $\lambda = 74.07$ GPa and $\mu = 41.67$ GPa (Young modulus $E = 110$ GPa and Poisson ratio $\nu = 0.32$). The thickness of the inclusion is 15 mm. In this case, we consider $K = 25$ forcing frequencies, equispaced in the intervals (3.3) and (3.4).
- *High-density Polyethylene inclusion:* Now, both the density ($\rho = 960$ kg/m³) and especially the rigidity ($\lambda = 1.777$ GPa and $\mu = 0.3901$ GPa, which correspond to Young modulus $E = 1.1$ GPa and Poisson ratio $\nu = 0.41$) are much smaller than that of aluminium. The thickness of the inclusion is 2.5 mm, much smaller than in the previous case since, as explained below, this is a more demanding case. Because of this, the number of forcing frequencies (again, equispaced in the intervals (3.4) and (3.3)) is larger than in the previous case, namely $K = 60$.

In both cases, the defect that is to be detected is circular and centered at $(x, y) = (0.1$ m, -0.2 m), with radius 2.5 mm.

The topological derivatives for these two cases are plotted in figures 3.7 and 3.8. As can be seen, the most demanding case is that of HDPE, which requires both a larger number of frequencies and a smaller thickness of the inclusion. The reason is that both the density and the Lamé coefficients are much smaller than that of the aluminium plate. This makes this case closer to the elongated through-slit, which as anticipated is the most demanding inhomogeneity. In any event, it is remarkable how the method is able to capture defects that are beyond the elongated through-slits covering a wide portion of the transversal section of the plate and elongated inclusions of different material covering the whole section of the plate, from side to side.

3.2.2 RSR emission and RSR sensing

Let us now consider dual transmitting/receiving devices of the RSR type. Since RSR receivers only capture one scalar quantity (instead of two), a larger (than in the PL receiving case) number of devices and a larger number of frequencies are needed in the present case. Thus, we will consider eight RSR devices of radius $r = 5$ mm and centers

Figure 3.7: Color map of the topological derivative for inhomogeneous plates showing horizontal inclusions (indicated in brown) of Titanium (left) and HDPE (right). As in previous figures, the transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols. The frequency range utilized is the LFR.

Figure 3.8: Counterpart of the topological derivatives calculated in figure 3.7, using the second frequency range, namely, the LHFR.

located at 2 cm from the border $y = 0.5$ m of the plate, equispaced in the x coordinate along the horizontal line $y = 0.48$ m.

For the homogeneous plate, we consider the same defects configurations as in figures 3.3 and 3.4, and an appropriate number K of forcing frequencies to be indicated below. The obtained topological derivatives are given in figures 3.9 and 3.10.

As can be seen, the topological derivative identifies the defects reasonably well, especially when only one defect is present, though the number of required frequencies is larger than in the case considered in figure 3.3, as expected. Note that two additional

Figure 3.9: Counterpart of figure 3.3, using the eight RSR transmitting/receiving devices located at the positions indicated with black asterisks and the indicated number of equispaced frequencies in the LFR.

Figure 3.10: Counterpart of the topological derivatives calculated in figure 3.9, using the second frequency range, namely, the LHFR.

blue regions appear in the case of two defects. However, the topological derivative peaks at these additional regions are less intense than near the actual position of the defects, though this is not clearly seen in the plots in figure 3.3.

Concerning the case of inhomogeneous plates considered in §3.2.1, with the same elongated slits or inclusions and defects as in figures 3.5, 3.6, 3.7 and 3.8, the topological derivatives for the present RSR transmitter/receiver configuration are given in figures 3.11 (for the LFR) and 3.12 (for the LHFR).

As can be seen in this figure, the topological derivative locates the defects quite well using an appropriate number of forcing frequencies, namely 60 and 40 frequencies for the parallel and oblique slits, respectively, and 40 and 60 frequencies for the titanium and

Figure 3.11: Color map of the topological derivative for inhomogeneous plates. Top plots: parallel (left) and oblique (right) through-slits indicated in black. Bottom plots: Titanium (left) and HDPE (right) inclusions. As in previous figures, the transmitting/receiving devices locations are indicated with black asterisks, and the defect locations, with white + symbols. The frequency range utilized is the LFR.

HDPE inclusions, respectively. Note that the number of frequencies is larger for HDPE than for titanium because, as already mentioned, this material is much more demanding in connection with inclusions.

Summarizing, the results obtained for this new type of dual RSR transmitters and RSR receivers are consistent with those obtained in §3.2.1 for PL transmitting and PL receiving. However, in this particular case, the LFR gives better results than the LHFR.

Figure 3.12: Counterpart of the topological derivatives calculated in figure 3.11, using the second frequency range, namely, the LHFR.

3.2.3 RSR emission and PL sensing

Let us now consider the case of RSR transmitters and PL receivers. Since, as anticipated, PL receivers collect a larger amount of data (namely, twice as much) than RSR receivers, the present case is less demanding than that considered in the last subsection. This means that we may decrease either the number of transmitters or the number of receivers, which must be located in different positions (but again 2 cms apart from the boundary) because they are of a different type among each other.

We select the same four RSR transmitters as in §3.2.2 and four PL receivers aligned with the actuators in the line $y = 0.48$ m and equispaced along the x coordinate in the right part of this line. Again, in the figures below, transmitters and receivers are indicated with black asterisks and black $\times$ symbols, respectively. Figures 3.13 and 3.14 show the topological derivative for the homogeneous plate, with the same defects as in figures 3.3

and 3.4. As can be seen, the topological derivative locates the defects reasonably well in the three considered cases, either for the LFR and the LHFR.

Figure 3.13: Counterpart of figure 3.3 for RSR emission and PL sensing. The frequency range utilized is the LFR.

Figure 3.14: Counterpart of the topological derivatives calculated in figure 3.13, using the second frequency range, namely, the LHFR.

On the other hand, figures 3.15 and 3.16 gives the topological derivative for the inhomogeneous plates with the same slits or inclusions and defects considered in the topological derivatives calculated in the previous subsections. Again, the topological derivative locates the defect quite well in the four considered cases, either for the LFR or the LHFR.

Summarizing the results obtained in this subsection, RSR transmitting and PL receiving give similar results to PL transmitting and PL receiving because, as anticipated, these two methods provide comparable number of data (which in fact, mainly depends

Figure 3.15: Counterpart of figure 3.11 for RSR emission and PL sensing. The frequency range utilized is the LFR.

on the selected receiving configuration). Concerning the utilized frequency ranges, either the LFR or the LHFR provide accurate results.

Figure 3.16: Counterpart of the topological derivatives calculated in figure 3.15, using the second frequency range, namely, the LHFR.

2D approximation and full 3D model comparison

4.1 Introduction

In this chapter, a comparison between the 2D approximation implemented in the previous chapter and the full 3D model is performed. The 2D approximation gave very good results in affordable computational times in contrast to the computational times that the 3D model would necessitate. Now, the objective is to assess whether the 2D approximation matches the solution of its 3D counterpart. To this end, we will recall the most demanding test cases studied in chapter 3, and obtain their solution by means of the mentioned 3D model. Furthermore, applying the topological derivative to 3D models will lay the foundations to cope with further types of scenarios such as plates with variable thickness in which the 2D approximation can not be utilized.

4.2 Results

Let us present now the results of this comparison. In this case, we will utilize the same aluminum plate D as in chapter 2 with density $\rho = 2,700 \text{ kg/m}^3$ and Lamé coefficients $\lambda = 50.35 \text{ GPa}$ and $\mu = 25.94 \text{ GPa}$. However, regarding plate horizontal size, in the present comparison we will reduce its size to the half, namely $0.5 \times 0.5 \text{ m}^2$, in order to obtain reasonable computational times for the 3D model. On the other hand, the thickness of the plate is 1 mm. As previously mentioned, we will compare the most demanding cases of chapter 3, namely:

- Plate with centered parallel through-slit.

- Plate with oblique parallel through-slit.
- Plate with titanium inclusion.
- Plate with HDPE inclusion.

Concerning the transmitting and receiving configurations, we will again classify them into three different types of configuration, namely:

- PL-PL.
- RSR-RSR.
- RSR-PL.

As in chapter 3, for the RSR configuration the radius of the devices will always be $r = 5$ mm. In the present comparison, in order to obtain comparable results in each of the studied cases, the centers of the transmitting and receiving devices will always be located at 1.5 cm from the boundary (for the PL configuration as well). This distance has been reduced with respect to the test cases studied in chapter 3 (it was 2 cm) due to the fact that the size of the plate has been reduced to the half. Accordingly, the dimensions of the interrogation window D_I had to be redefined as well, namely,

$$D_I : -17.5 \text{ cm} \leq x \leq 17.5 \text{ cm}, \quad -17.5 \text{ cm} \leq y \leq 17.5 \text{ cm}, \quad (4.1)$$

which means the interrogation window will be a $35 \times 35 \text{ cm}^2$ centered square, and a gap of 7.5 cm will be ignored near the boundary of the plate. The defect will be a circular through-hole with 5 mm of diameter in all the studied cases, and it will be always placed inside the interrogation window D_I . With regard to the interpretation of the plotted results of the topological derivative, we will follow exactly the ideas detailed in chapter 3, this is, positive values of the topological derivative will be set to zero, and the negative values will be rescaled with the maximum of its absolute value. Also, recall that it is the negative peaks (i.e., the blue regions in the plots) of the topological derivative that locate the possible position of the defect.

The numerical solutions to these cases have been obtained using the ANSYS Mechanical FEM solver. However, clearly, we had to use different types of elements and meshes for the 2D and 3D simulations, namely,

- For the 2D simulations we will use the PLANE183 8-node (or 6-node, if it degenerates in a triangle) element which has a second degree polynomial as shape function.

Concerning the computational meshes, they will have an element size ~ 2.5 cm, and the number of the degrees of freedom will be ~ 0.240 Million. Recall that this is the same element used for the simulations in chapter 3.

- For the 3D simulations, we will use the SOLID186 20-node element which has a second degree polynomial as shape function. Concerning the computational meshes, they will have an element size ~ 2.5 cm, and the number of the degrees of freedom will be ~ 0.560 Million. The element SOLID186 is an element widely utilized in industrial FEM simulations with an already proved accurate performance.

The frequency range at which the plates will be excited is the following:

$$20 \text{ kHz} \leq \omega/2\pi \leq 40 \text{ kHz}. \quad (4.2)$$

The frequencies will be chosen equispaced in the previous range either for the 2D or 3D simulations. The number of frequencies K will be increased depending on the complexity of the test case.

4.2.1 PL emission and PL sensing

Following the methodology presented in chapter 2, we consider a number $M = 4$ of PL dual transmitting/receiving devices, which will be used to obtain (for each of the K forcing frequencies) M transmitting/receiving configurations. In each of these, signals are transmitted from one of the devices (acting as transmitter) and received at the remaining $N = M - 1$ devices, which act as receivers. In the 3D test cases, these M PL dual transmitting/receiving devices will be placed in both faces of the plate in order to match the elastodynamic state generated by the 2D approximation. Accordingly, despite the fact that in the 3D simulations we are calculating the out of plane component of the displacement, it will not be considered for the computation of the topological derivative as previously remarked.

Let us describe the two first cases, which are the same plates with elongated through-slits considered in chapter 3. However, in this case, their dimensions have been reduced to the half, namely,

- The centered parallel (to the x axis) slit has a length of 0.4 m and a width of 0.8 cm located along the line $y = 0.085$. The defect in this plate is circular with radius 2.5 cm and center at $(x, y) = (0.05 \text{ m}, -0.1 \text{ m})$.

- The oblique slit has a length of 0.35 m and a width of 0.7 m located along a line with slope 40.17° and centered at $(x, y) = (0.0304 \text{ m}, 0.0645 \text{ m})$. Once more, the defect in this plate is circular with radius 2.5 cm and center at $(x, y) = (0.075 \text{ m}, 0.05 \text{ m})$

As usual, the PL transmitters and receivers will be indicated in the plots with black $\times$ symbols, and they will be equispaced along a parallel line to the boundary of the plate. In the other hand, the position of the defects will be indicated with $+$ white symbols; the size of the defect cannot be seen in the plots because of its very small size. Concerning the representation of the topological derivative, as already mentioned, only negative values will be plotted; the positive values of the topological derivative, and the points outside the interrogation window are indicated in grey.

At this moment, it is also convenient to explain how results we will presented in the figures of this chapter: the left plots correspond to the 2D approximation, and the right plots, to its 3D model counterpart. The number of frequencies will always be the same either for the 2D approximation or the 3D model for each type of plate case.

In figure 4.1, it is plotted the topological derivatives for the parallel slit case. The number of forcing frequencies is $K = 30$ equispaced in the range (4.2). The blue regions of the topological derivative indicate the functions negative peaks, which are the points where the defect probably is. It can be seen, that the topological derivative locates the defect fairly well for both models (2D and 3D). Furthermore, we can even see that the 2D plot and its 3D counterpart are very much alike.

Figure 4.1: Color map of the topological derivative for the parallel through-slit case. As in the previous figure, the 2D approximation case is presented in the left plot, and the 3D case in the right one. The transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols.

Very similar results were obtained for the oblique slit case which is represented in figure 4.2. Although this case seems to be more demanding than the previous one (the defect is completely shielded by the oblique slit), we have kept the same number of forcing frequencies (i.e., 30), obtaining again a good guessed position of the defect.

Figure 4.2: Color map of the topological derivative for the oblique through-slit case. The 2D approximation case is presented in the left plot, and the 3D case in the right one. The transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols.

In the two remaining cases of this subsection, we will use the same plates with elongated material inclusions as in chapter 3. Once more, some changes in their size were introduced for this comparison, namely,

- The titanium inclusion is parallel to the x axis of the plate, and it transversally covers the whole plate at $y = 0.08$ m from side to side, and it has a thickness of 7 mm. This titanium inclusion has a density $\rho = 4500$ kg/m³ and Lamé coefficients $\lambda = 74.07$ GPa and $\mu = 41.67$ GPa.
- The HDPE inclusion is parallel to the x axis of the plate, and it transversally covers the whole plate at $y=0.08$ m from side to side, and it has a thickness of 2.5 mm. This HDPE inclusion has a density $\rho = 960$ kg/m³ and Lamé coefficients $\lambda = 1.777$ GPa and $\mu = 0.3901$ GPa.

In both cases, the defect to be located is circular with radius 2.5 mm, and centered at $(x, y) = (0.05, -0.1)$. The material inclusion will be depicted in the plots with a dark brown strip.

In figure 4.3, it is depicted the topological derivatives for the titanium inclusion case. The number of frequencies utilized is $K = 25$, and it is clearly enough to perfectly locate the defect. It can be seen in both plots that the minimum value of the topological derivative is very isolated, and barely no relative minima appear in the plots.

Figure 4.3: Color map of the topological derivative for the titanium inclusion case. As in the previous figure, the 2D approximation case is presented in the left plot, and the 3D case in the right one. The transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols. The material inclusion is depicted with a brown strip.

The representation of the topological derivative for the HDPE inclusion is shown in figure 4.4. As already seen, this is the most demanding case due to the small values of the density and the Lamé coefficients of the HDPE, which make this inclusion close to an air slit. Consequently, as in chapter 3, the thickness of this inclusion must be quite narrow, namely 2.5 mm, in contrast with the titanium inclusion. Also, the number of frequencies has to be increased to $K = 60$ in order to get accurate results. In this case, it can be seen that the plot of the 3D model gives better results than its 2D counterpart, this is, the topological derivative for the 3D model (right plot) exhibits less relative minima.

4.2.2 RSR emission and RSR sensing

Let us present the comparison with RSR transmitters and RSR receivers. In this case, the number of transmitters and receivers, and the number of excitation frequencies have to be increased since the RSR receivers can only capture one scalar quantity (instead of two as in the PL type). The number of transmitters/receivers will be $M = 8$, and they

Figure 4.4: Color map of the topological derivative for the HDPE inclusion case. As in the previous figure, the 2D approximation case is presented in the left plot, and the 3D case in the right one. The transmitting/receiving devices locations are indicated with black $\times$ symbols, and the defect locations, with white $+$ symbols. The HDPE inclusion is depicted with a brown strip.

will be equispaced in a parallel line at 1.5 cm from the boundary of the plate. Concerning the cases to be studied, the same plates as in the previous subsection will be utilized.

In figure 4.5 is presented the parallel slit case which has been computed using 60 forcing frequencies. On the other hand, in figure 4.6 is presented the oblique slit case computed using 40 forcing frequencies. In contrast to the PL-PL case, in this comparison the number of forcing frequencies for the parallel slit case had to be increased considerably. Despite this, it can be seen that the oblique slit case is still neater than the parallel slit case. In any event, as for the PL-PL case, the results for the 2D approximation (left plots) and for the 3D model (right plots) are quite good, and, once more, both topological derivatives are very similar.

In figure 4.7 is shown the topological derivatives for the titanium inclusion. The number of forcing frequencies utilized in this case is 40. As expected, very good results are obtained for this case.

Finally, the topological derivatives for the HDPE inclusion are shown in figure 4.8. As usual, the number of frequencies needed for the HDPE case is always high, namely, 60. In both topological derivatives (2D and 3D), the defect is located without problems, however, just remark that in the present case the patterns in the plots are quite dissimilar.

Figure 4.5: Counterpart of the topological derivatives calculated in figure 4.1, using the RSR-RSR configuration. The transmitters and receivers are indicated with black asterisks, and the defect locations, with white + symbols.

Figure 4.6: Counterpart of the topological derivatives calculated in figure 4.2, using the RSR-PL configuration. As in the previous figure, the transmitters and receivers are indicated with black asterisks, and the defect locations, with white + symbols.

4.2.3 RSR emission and PL sensing

This last subsection is devoted to the RSR-PL configuration for transmitters and receivers respectively. As stated, the PL receivers can collect more data than the RSR receivers. Hence, the RSR-PL configuration should give good results with either a smaller number of forcing frequencies or a smaller number of transmitters/receivers, compared to the RSR-RSR configuration. The number of RSR transmitters will be 4, and the number of PL receivers 4 as well. These transmitters/receivers will be placed as usual, this is,

Figure 4.7: Counterpart of the topological derivatives calculated in figure 4.3, using the RSR-RSR configuration. As in the previous figures, the transmitters and receivers are indicated with black asterisks, and the defect locations, with white + symbols.

Figure 4.8: Counterpart of the topological derivatives calculated in figure 4.4, using the RSR-RSR configuration. As in the previous figures, the transmitters and receivers are indicated with black asterisks, and the defect locations, with white + symbols.

equispaced in a parallel line at 1.5 cm from the boundary of the plate. Note that the transmitters will be placed in the left side of the line (black asterisks in the plots), and the receivers in the right ($\times$ black symbols in the plots).

In figures 4.9 and 4.10 are shown the horizontal and oblique slit cases respectively. The defect is well located in the oblique cases using 30 forcing frequencies. However, in the horizontal case, the 3D plot (right) shows a local minimum somewhat intense that may lead to an incorrect location of a defect. The forcing frequencies used in the horizontal case is 40.

Figure 4.9: Counterpart of the topological derivative calculated in figure 4.1, using the RSR-PL configuration. Transmitters are indicated with black asterisks, and receivers with black $\times$ symbols. As usual, the white $+$ symbols indicate the position of the defect.

Figure 4.10: Counterpart of the topological derivative calculated in figure 4.2, using the RSR-PL configuration. As in the previous figure, the transmitters are indicated with black asterisks, and receivers with black $\times$ symbols. As usual, the white $+$ symbols indicate the position of the defect.

The titanium inclusion case is presented in figure 4.11. As expected, the guess of the topological derivative is correct using only 30 frequencies. On the other hand, in figure 4.12, it is shown the HDPE case. As in the previous subsections, the required number of frequencies for this case is 60. The location of the defect provided by the topological derivative is fairly accurate. Again, simply highlight that in this case the topological derivatives have different patterns (between the 2D and the 3D) in contrast with the previous test cases studied in this subsection.

Figure 4.11: Counterpart of the topological derivative calculated in figure 4.3, using the RSR-PL configuration. As in previous figures, the transmitters are indicated with black asterisks, and receivers with black $\times$ symbols. As usual, the white $+$ symbols indicate the position of the defect.

Figure 4.12: Counterpart of the topological derivative calculated in figure 4.4, using the RSR-PL configuration. As in previous figures, the transmitters are indicated with black asterisks, and receivers with black $\times$ symbols. As usual, the white $+$ symbols indicate the position of the defect.

The thickness variation problem and complex geometry

5.1 Introduction

This section is devoted to the implementation of the topological derivative approach to aluminum plates in which the thickness is not constant, and in plates with complex geometry. As previously stated in section §2.5.1, guided waves are dispersive, this is, their velocity changes with the excitation frequency. Furthermore, this velocity also depends on the thickness of the plate. This dependence with thickness causes problems within the conventional pulse-echo techniques since due to the velocity fluctuations, it is very problematic to establish accurate relations between time lags and distances in order to accurately locate the defects. Regarding scenarios in which plates have a curved and rather complex domains, the resulting wave patterns are also very complicated to be assessed by means of standard pulse-echo techniques.

5.2 Results

In this section, we present some testing cases in order to illustrate the performance of the topological derivative in the above-mentioned situations. We will continue with the application of the theory developed in chapter 2, utilizing the same methodology as in the previous chapters.

Let us consider the usual aluminium plate, with constant density $\rho = 2,700 \text{ kg/m}^3$, and Lamé coefficients $\lambda = 50.35 \text{ GPa}$ and $\mu = 25.94 \text{ GPa}$ (Young modulus $E = 69 \text{ GPa}$ and Poisson ratio $\nu = 0.33$). As stated, we will use a similar notation to the one utilized

in the previous chapters, and the same Cartesian coordinate system anticipated in §2.5.1.

Two types of pristine plates will be examined:

- Rectangular plates with variable thickness, which have been considered in connection with the simulation of guided waves in [19], where comparison with experimental results has also been made. In the present case studies, the planform of the plates with variable thickness will always be a 50×50 cm² square, while the planform of the interrogation window D_I will be a 40×40 cm² centered square. Concerning the thickness, this will vary from 1 mm to 2.5 mm in various ways, from the transmitting/receiving side of the plate to the opposite side. Namely, thin/thick/thin, thick/thin/thick, thin/thick, and thick/thin. In all cases, the thickness variation will be fairly sharp.
- Plates with a more complex planform, which will be as sketched in figure 5.1-left, where $L = 50$ cm, $L_1 = 40$ cm, $L_2 = 30$ cm, $L_3 = 12.7$ cm, and $R = 0.8$ cm;

Figure 5.1: Left: sketch of the complex planform of the plate. Right: planform of the associated interrogation window (solid line) included in the plate (dashed line).

the interrogation window will be as sketched in figure 5.1-right, where $L' = 40$ cm. Concerning the thickness of this plate, this is $h = 1$ mm. Moreover, in order to increase the complexity of the configuration of the healthy plate, an elongated, centered through slit will be added parallel to the plate side with length L , at a distance 16.5 cm to this side of the plate. The slit length and width will be 38.8 cm and 7.88 cm, respectively.

Concerning the transmitting/receiving configurations, these will be of one of the usual types defined in previous chapters, namely: PL transmitting/PL receiving, RSR transmitting/RSR receiving, and RSR transmitting/PL receiving. In order to obtain comparable results for the various cases that will be considered, the centers of the transmitters and receivers will always be located at 1.5 cm distance from a straight boundary of the plate (the same part of the boundary for both transmitting and receiving devices). The small 1.5 cm distance has been selected on purpose to be very small compared to the plate size and would lead to strong difficulties if standard SHM methods were used. Location of the transmitters and receivers along the line parallel to the involved side of the plate will be chosen with no particular care, which will emphasize robustness as in previous chapters. RSR transmitters and receivers will always be circular, with a radius $r = 5$ mm, similar to the radius of typical piezo-electric devices.

The defects will always be inside the interrogation window D_I and will be circular through-holes, with a very small size, namely 2.5 mm of radius. For the sake of clarity, the topological derivative will be plotted in the mid plane of the interrogation window, D_I . Since it is the negative peaks of the topological derivative that matter to localize defects, in these plots, the topological derivative will be always rescaled as follows. The topological derivative outside D_I will be set to zero and the positive values of the topological derivative inside D_I will also be replaced by zero. The negative values of the topological derivative inside D_I will be rescaled with the maximum of its absolute value inside D_I . Therefore, all topological derivatives will vary in the range between 0 and -1. Recall that this way of plotting the topological derivative is the same used in §3.2 and in §4.2, however, in this chapter, the interrogation window is proportionally bigger. As it is quite obvious, inspecting plates with bigger inspection windows is more complicated, since a higher number of spurious peaks might be taken into account when rescaling. In fact, the usual source of generation of spurious negative peaks is the emitter itself, as near it, big gradients in the calculated solution of the plate appear.

As previously mentioned, while transmitters/receivers configurations, M , the number of sensors, N , and the number of involved frequencies, K , increase, the available data will also be bigger. However, increasing M and/or N is problematic as it requires increasing the number of devices. For this reason, in practice is more convenient increasing K , which only affects the complexity of the emitted signal. Thus, some emphasis will be put below in decreasing M and N , and compensating with an increase of K . Based on the experience gathered in the previous chapters, the number of transmitting/receiving devices will always be both limited, namely equal to four for PL receivers and eight for

RSR receivers; these numbers could be further decreased in some of the cases considered below, but this will not be done to emphasize robustness. On the other hand, selecting transmitters/receivers that are somewhat close to each other (which makes the available data less uncorrelated) and close to the boundary of the plate favors compact designs of the experimental device and will be intended below.

As anticipated in §2.5.1, the application of the method developed in this Thesis requires solving various elastic problems. Namely, the full problem (i.e., the experiment to simulate a damaged plate with the actual distributions of ρ , λ , and μ , taking the faults into account) to obtain synthetic data, and the unperturbed direct and adjoint problems with the material properties equal to their values for the healthy plate. Recall that in the adjoint problem, its forcing term is feeded with the simulated experimental data and with the solution of the unperturbed direct problem.

Concerning the computational mesh, the typical size of the individual elements along the plate planform will be ~ 2.5 mm and just one element throughout thickness. This will give a number of degrees of freedom ~ 0.85 Million (corresponding to 0.283 Million nodes). Again, these problems are numerically solved using the ANSYS Mechanical FEM solver with the 3D element SOLID186, which has a second degree polynomial as shape function resulting in 20 nodes with 3 degrees of freedom (the three components of the displacement vector) in each of them. An idea of the computational meshes that will

Figure 5.2: Sketch of the 3D element SOLID186. It is an homogeneous solid structural element well suited to modeling irregular meshes. The black dots represent the position of the nodes within the element.

be used to generate the synthetic data and to solve the unperturbed direct and adjoint problems involved is given in figure 5.3. As can be seen, the two meshes are fairly different

Figure 5.3: The meshes used to compute the synthetic data (left) and the direct and adjoint problems (right), showing not the whole amount of nodes (which would make too dense plots), but only one out each of 4 nodes. The defect location is indicated with a red + symbol.

among each other, both quantitatively and qualitatively. The main qualitative difference between them is that nodes concentrate around the defect in the left plot (because the defect is taken into account when computing synthetic data), but not in the right plot, which corresponds to computations of the unperturbed direct and adjoint problems in the pristine plate. Using different meshes avoids the so called (in the context of inverse problems) *inverse crime*, which occurs when as the same numerical ingredients are used to both synthetizing data and solving the inverse problem [44]. Note that the meshes do not have concentration of nodes in the position of the transmitters/receivers because we are benefiting from the fact that we already have a fine mesh to model them.

Again, in order to further emphasize the robustness of the method, the forcing frequencies will be chosen as equispaced in a range that will be the same for all cases, namely,

$$20 \text{ kHz} \leq \omega/2\pi \leq 40 \text{ kHz}. \quad (5.1)$$

Note that some of these frequencies can be close to natural frequencies of the plate, which will give disparate amplitudes of the generated waves. However, since we are integrating a linear equation we can profit from scaling, and the effect of these high amplitude waves will not mask the contribution of the remaining waves.

As anticipated, transmitting and receiving devices will be fairly close to the boundary,

namely 1.5 cm apart in all cases considered below. Also, these devices will be equispaced along the line parallel to the boundary where they are located. When considering RSR devices, their radius will be 5 mm. When performing PL transmitting, the excitation will be perpendicular to that boundary of the plate that is closest to the emitters. Also, in all cases, the considered fault will be a through-hole circular defect of radius 2.5 mm, centered at the point whose coordinates are $(x, y) = (0.05 \text{ m}, -0.1 \text{ m})$ in a Cartesian coordinate system with its origin at the center of the mid-section of the plate. The defect will be indicated with a white cross.

In the remaining of this subsection, we consider several distributions of the thickness of the plate.

5.2.1 Thin-thick-thin thickness distribution

The sketch of the transversal section of the plate, from the transmitting/receiving side to the opposite side, is as sketched in figure 5.4. The various lengths appearing in

Figure 5.4: Sketch of the thin-thick-thin thickness distribution along the lateral section of the plate.

this sketch are $L = 50 \text{ cm}$, $L_1 = 17.5 \text{ cm}$, $L_2 = 15 \text{ cm}$, $h_{\min} = 1 \text{ mm}$, and $h_{\max} = 2.5 \text{ mm}$. As can be seen in this figure, the thickness variation is very sharp, which represents a difficulty for standard methods due to reflection/refraction at the sharp edges. An additional difficulty is the different wave propagation velocity at those parts of the plate exhibiting different thickness.

The plot of the topological derivative for the various transmitting/receiving configurations is given in figure 5.5, where the sharp thickness increase/decrease is indicated with horizontal lines. Also, the location of the transmitting/receiving devices is indicated as follows. Dual PL-PL transmitting/receiving devices (in the left plot) are marked with

Figure 5.5: Thin-thick-thin thickness distribution. Topological derivatives with different types of emission/sensing devices.

black $\times$ -symbols and dual RSR-RSR devices (in the right plot), with black asterisks. Consistently, in the middle plot, RSR emitters are indicated with black asterisks and PL receivers, with black $\times$ -symbols. This rule will be followed in all plots of the topological derivative along the remaining of the chapter.

As anticipated, our prediction of the position of the defect is the negative peak of the topological derivative, which corresponds to the small blue region in figure 5.5, where the actual position of the defect is indicated with a white cross. As can be seen, the topological derivative locates the defect quite well. In fact, the negative peak of the topological derivative is extremely close to the position of the defect, which cannot be seen in the figure because of the plot is not precise enough. This will occur with the remaining cases considered in this chapter.

The results in figure 5.5 are consistent with our comments above about the efficiency of the method, which provides better results in the PL-PL transmitting/receiving configuration, since only four devices and 30 frequencies are needed. The method functions equally well for the RSR-PL configuration, since four transmitters and four receivers are again needed, and 30 frequencies are involved. The only difference is that transmitters and receivers are of different type and thus located at different positions. Instead, the performance of the method worsens in the RSR-RSR transmitting/receiving configuration, since, to obtain results that are comparable to their counterparts in the former two cases, (a) the number of transmitters and receivers must be doubled, and (b) the number of frequencies increased. This is because RSR sensing essentially halves the number of receiving data compared to PL receiving, namely one scalar instead of a vector at each sensor for each frequency.

Finally, figure 5.5 also shows that the absolute value of the topological derivative is smaller in the thick part of the plate (namely, in that region in between of the two parallel lines) than in the thin part of the plate. This is because the solutions of the unperturbed direct and adjoint problems are smaller in the thick part of the plate, where the energy of the waves per unit area of the plate planform is also smaller.

5.2.2 Thick-thin-thick thickness distribution

This is the dual case of that considered in the last sub-subsection. Now, the transversal section of the plate, from the transmitting/sensing side to the opposite side, is as sketched in figure 5.6, where the various indicated lengths are $L = 50$ cm, $L_1 = 17.5$ cm, $L_2 = 15$

Figure 5.6: Counterpart of figure 5.4 for the thick-thin-thick thickness distribution.

cm, $h_{min} = 1$ mm, and $h_{max} = 2.5$ mm. Note that, as in figure 5.4, the plate exhibits sharp thickness variations, which again is a very demanding case for standard methods.

The topological derivative for the present case (considering the same transmitter/receiver configurations as in the former case) is given in figure 5.7. As can be seen, the results are fairly good and the performance of the method follows a similar trend as in the previous case. Namely, the application of the method requires the same number of devices and the same number of involved frequencies as in the previous case. And, again, using PL-PL and RSR-PL transmitter/receiver distributions, the number of transmitter/receiver devices, and the number of required frequencies are both smaller than their counterparts using the RSR-RSR configuration.

5.2.3 Thin-thick thickness distribution

The transversal section of the plate in this case, from the transmitting/receiving side to the opposite side, is as sketched in figure 5.8, where the various indicated lengths are

Figure 5.7: Counterpart of figure 5.5 for the thick-thin-thick thickness distribution.

Figure 5.8: Counterpart of figure 5.4 for the thin-thick thickness distribution.

$L = 50$ cm, $L_1 = 22.5$ cm, $L_{\text{ramp}} = 5$ cm, $h_{\text{min}} = 1$ mm, and $h_{\text{max}} = 2.5$ mm. Note that the thickness variation is not sharp in the present case, but is imposed along a ramp of length L_{ramp} . Also, emitters and receivers are located in the thin side of the plate.

The topological derivative for the present case is given in figure 5.9, where it can be seen that the method performs well and shows similar trends as in the previous cases in connection with the transmitter/receiver configurations. However, the number of required frequencies is now larger. This is because the defect is located in the thick part of the plate, where according to our previous comment on figure 5.5, the contrast in the topological derivative is smaller.

5.2.4 Thick-thin thickness distribution

The lateral section of the plate in this case is the same as in the latest case, sketched in figure 5.8, except that now the transmitters and receivers are located in the thick side

Figure 5.9: Counterpart of figure 5.5 for the thin-thick thickness distribution.

of the plate. The topological derivative is given in figure 5.10. Comparison with figure 5.9 shows that the method gives comparable results as in the former case using a smaller number of frequencies. This is due to the fact that now the defect is located in the thin part of the plate, where the contrast in the topological derivative is larger.

Figure 5.10: Counterpart of figure 5.5 for the thick-thin thickness distribution.

5.2.5 Complex planform plate

Let us now consider the plate with a more complex planform sketched in figure 5.1, which exhibits a through slit sketched in that figure, and considering the interrogation window also sketched in that figure. The transmitting/receiving device configurations are as in the last section and located 1.5 cm apart from that part of the boundary with length L . As in the previous cases, the defect to be detected is a small circular through hole of

radius 2.5 mm. Its center is located at 35 cm and 20 cm from the upper and right sides of the plate, respectively. Note that the present complex geometry case is very demanding and hardly accessible to standard SHM methods.

The topological derivative for the present case is given in figure 5.11, where it can be seen that the method performs quite well in localizing the defect. However, the number

Figure 5.11: Counterpart of figure 5.5 for the plate with complex planform.

of required frequencies is larger than in the cases considered in the last section. This is due to the presence of the through slit, which partially hides the defect.

Symmetric plus antisymmetric scenario

6.1 Introduction

This last chapter is focused on the application of the topological derivative method to plates in which a non symmetric excitation is applied. Consequently, this kind of excitation will lead to a scenario with antisymmetric and symmetric waves. As already stated, the topological derivative approach is a fairly robust method which only takes into account final responses of the pristine and damaged plates irrespective of the way of excitation, however, in the previous chapters, symmetric excitations have been selected in order to have affordable computational meshes. In this chapter, we will implement this non symmetric excitation by means of piezoelectric devices attached to the upper face of the plate (instead of a radial model as in the previous chapters). Abundant works on how to model piezoelectric devices have arisen in the recent years due to the fact that they are a key element in the non destructive techniques; the FEM modeling of the piezoelectric devices will be discussed based on the theory developed in §2.3. As a result, a simple test case will be presented as a proof of the concept to show that the topological derivative could be also perfectly applied in these situations, in which a non symmetric excitation and 3D model for the transmitters are considered.

6.2 Piezoelectric transducer validation

Prior to implement the topological derivative, in this section we are presenting an experimental validation of a usual pitch and catch experiment utilizing a pair of piezoelectric devices on an aluminum plate: one is going to act as a transmitter, and the other one as a receiver. The objective of this validation is to test how accurate were the common

numerical models for the piezoelectric devices presented in literature, and to anticipate their usage as transmitters of a topological derivative case study. To this end, a real experiment and a numerical simulation of this experiment were carried out and eventually compared.

As expected, piezoelectric devices have been modeled in ANSYS Multiphysics utilizing the SOLID226 element. SOLID226 is a 3D coupled-field element which has a second degree polynomial as shape function resulting in 20 nodes. In fact, it could be seen as an extension of the SOLID186 element utilized in the previous chapter, but in this case, each node has an extra degree of freedom reserved to the voltage. Let us recall two equations from chapter 3 that describe the piezoelectric behaviour of a material,

$$[S] = s^E [T] + [d]^t [E] \quad (6.1)$$

and

$$[D] = d [T] + [\varepsilon^T] [E], \quad (6.2)$$

which lead to the fully coupled finite element matrix equation,

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ V \end{bmatrix} + \begin{bmatrix} K & K^Z \\ K^{Zt} & -K^d \end{bmatrix} \begin{bmatrix} \dot{u} \\ V \end{bmatrix} = \begin{bmatrix} F \\ Q \end{bmatrix} \quad (6.3)$$

where M is the mass matrix, K is the stiffness matrix, K^Z is the piezoelectric coupling matrix, K^d is the dielectric permittivity matrix, u is the displacement vector, V is voltage, F is the force vector, and Q is the electric charge. The superscripts t and $^{\cdot}$ merely indicates transpose and derived with respect to time respectively.

For this experimental validation, we have selected circular piezoelectric devices with 8.7 mm of diameter and 0.5 mm of thickness. These two devices are attached to a square aluminum plate of 25×25 cm² and 1 mm of thickness, and this plate simply rests in a foam pad in order to free its movements. Regarding the material parameters, they have a density $\rho = 7600$ kg/m³ and a stiffness matrix s^{-1} (inverse of the compliance)

$$s^{-1} = \begin{bmatrix} 97 & 49 & 49 & 0 & 0 & 0 \\ 49 & 97 & 49 & 0 & 0 & 0 \\ 49 & 49 & 84 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22 \end{bmatrix} \text{ GPa.} \quad (6.4)$$

In the other hand, the piezoelectric electromechanical coefficients, and the dielectric permittivity are

$$d = \begin{bmatrix} 0 & 0 & -5.2 \\ 0 & 0 & -5.2 \\ 0 & 0 & 15.1 \\ 0 & 0 & 0 \\ 0 & 12.7 & 0 \\ 12.7 & 0 & 0 \end{bmatrix} \times 10^{-12} \text{ C/N} \quad \text{and} \quad \varepsilon = \begin{bmatrix} 729 & 0 & 0 \\ 0 & 729 & 0 \\ 0 & 0 & 635 \end{bmatrix}. \quad (6.5)$$

The plate to which these devices are bonded is modeled with the ANSYS SOLID186 element. No additional layer simulating glue between plate and piezoelectric devices is modeled. In this particular bonding, the piezoelectric elements (i.e., SOLID226 elements), and the plate elements will share their nodes, this is, no relative displacement between them is possible. Concerning the mesh of the plate, we anticipate the element size is 0.7 mm and it was chosen based on the wavelength of the generated waves. The explanation of the selected element size will be discussed some lines below right after having determined the excitation frequency in the transmitter. Besides, taking into account that the thickness of the plate is discretized with two elements, the total number of nodes is ~ 1.43 Million. The number of elements utilized in the devices' discretization (~ 1500 elements) is very small compared to the number of elements of the plate (in figure 6.1 it is shown the mesh of the piezoelectric wafer).

Figure 6.1: Mesh of the piezoelectric wafer

As previously mentioned, the plate will be considered as a free plate which means Neumann boundary conditions are set to zero on the faces of the plate (i.e., no stresses in the boundaries of the plate). Regarding the boundary conditions applied to the piezoelectric devices, it is convenient to differentiate between the transmitter and the receiver. For the transmitter, we will apply the same voltage to all the nodes of the upper face of the device, namely, a Dirichlet boundary condition. Indeed, this voltage is going to be a function of the time, and this function will be detailed some lines below. For the receiver, let us recall that piezoelectric devices are not conductors, are capacitors, thus, the solution of the PDE modeling their physical behaviour will provide a different voltage value at each node within the piezoelectric material. For the purpose of simulating the way voltage is acquired with an oscilloscope, we will force the nodes of the upper face of the receiver device to provide us a uniform value for the voltage at any time. For both devices, transmitter and receiver, Neumann boundary conditions are set to zero in their lateral face.

Figure 6.2: Piezoelectric transducer.

The distance d between the transmitter and the receiver is 15 cm which is sufficient to not have early reflections from the boundaries (figure 6.3). With regard to the excitation applied to the device acting as transmitter, a 3-tone burst signal with 20 volts peak to peak and at a frequency of 240 kHz was selected. After this signal is applied, the voltage is set to zero for the rest of the simulation. Let us remark some aspects concerning the frequency, velocities, and wavelengths of the modes,

- Given an aluminum plate with 1 mm of thickness excited at a frequency of 240 kHz, the fastest mode originated inside it is the A_0 mode with a velocity $V_{A_0} = 5500$ m/s.

- Given an excitation frequency of 240 kHz and a wave velocity V_{A_0} of 5500 m/s, the wavelength ξ can be obtained with the following relation,

$$\xi = \frac{f}{V_{A_0}} = \frac{5500 \text{ m/s}}{240000 \text{ Hz}} = 0.023 \text{ m.} \quad (6.6)$$

- In order to have accurate results, it is advisable to have at least 12 elements to discretize the wavelength, thus, this leads to an element size of

$$e = \frac{\lambda}{12} = \frac{0.023 \text{ m}}{12} = 0.0019 \text{ m,} \quad (6.7)$$

which is consistent with the selected element size within the plate, namely 0.0007 m (recall that a middle node due to 2nd order polynomial is inserted).

At this point, let us clarify that in the real experiment the dimensions of the plate are bigger, and therefore, no problems related to early reflection arise. However, in the numerical simulation, we had to reduce the plate dimensions in order to have an affordable computational domain. The plate size in the numerical simulation was designed in such a way that we can have at least a first *clean* hit of the waves in the receiver. Naturally, to do this dimensioning, we took into account the velocity of the generated waves which depends on the frequency of the excitation source. The time step utilized in the numerical simulation in order to accurately capture the S_0 and A_0 modes is set to 1e-7 seconds.

In figure 6.4, the comparison between the experiment and the numerical computation is presented. The functions depicted in the graph represent the normalized voltage retrieved in the receiver. It can be seen that both output voltages (i.e., the voltage acquired in the experiment by means of an oscilloscope, and the voltage acquired in the numerical simulation from the calculated voltage in the piezoelectric elements) match quite well.

The total time of the simulation is 5e-5 seconds. It can be seen that the first wave packet (i.e., the S_0 mode) reaches the receiver in 2e-5 seconds which is coherent with the analytical calculation of the Lamb waves velocity in an aluminum plate discussed before. After second 3e-5, we can see that a small phase shift starts between the numerical and the experimental solution most probably due to the absence of damping in the computed simulation.

6.3 Results

Let us now present an illustrative case study to test the performance of the topological derivative in an antisymmetric plus symmetric scenario. In order to generate antisymmetric and symmetric wave modes, we will simply excite the testing plate from its the upper

Figure 6.3: Sketch of the experiment showing the plate and the position of the piezoelectric transducers.

face. This excitation will be performed by means of the piezoelectric devices discussed in the previous section.

We consider a aluminium plate, with constant density $\rho = 2,700 \text{ kg/m}^3$, and Young modulus $E = 72 \text{ GPa}$ and Poisson ratio $\nu = 0.33$. The horizontal size of the plate is $0.55 \times 0.55 \text{ m}$ and the thickness is 1 mm . We will use the same notation to the one utilized in the previous chapters and the same Cartesian coordinate system anticipated in §2.5.1.

The number of piezoelectric transmitters will be 4, with radius $r = 5 \text{ mm}$, and 0.4 mm of thickness. Regarding the receivers, we will perform a punctual acquisition of data as in the PL configuration developed in the previous chapters. In fact, we are using a RSR-PL configuration, however, in this case, the transmitters are modeled piezoelectric devices (instead of applying a radial force over nodes). The transmitters will be equispaced along a parallel line at 2.575 cm from the upper boundary of the plate, and they will be concentrated in the left side of it. On the other hand, receivers will be equispaced along a parallel line at 2.575 cm from the lower boundary of the plate, and they will be distributed as follows: two sets of four receivers separated 10 cm between them (the sets); the piezoelectric within these sets will be separated 5 cm .

Figure 6.4: Comparison of the experimental and synthetic voltage acquired in the receiver.

The topological derivative will be computed inside the interrogation window which in this case will be a $40 \times 40 \text{ cm}^2$ centered square, namely,

$$D_I : -20 \text{ cm} \leq x \leq 20 \text{ cm}, \quad -20 \text{ cm} \leq y \leq 20 \text{ cm}. \quad (6.8)$$

The defect to be located will be inside this interrogation window and it will be a circular through-hole of 6 mm of radius. As in the previous chapters, the negative peaks of the topological derivative inside the interrogation window indicate the probable location of the defects. Also, the topological derivative will be rescaled as usual in such a way that its values will vary in a range between 0 and -1.

Concerning the forcing frequencies at which we will excite the plate, they will be chosen as equispaced in the following range:

$$750 \text{ Hz} \leq \omega/2\pi \leq 4000 \text{ Hz}. \quad (6.9)$$

Note that the frequencies of this range have lower values than the frequencies utilized in the previous chapters. This is due to the fact that in this case the antisymmetric mode is also excited, and it has a slower velocity than the symmetric one, originating smaller wavelengths which necessitate much finer meshes. This range ensures wavelengths for the antisymmetric mode similar to the wavelengths obtained in the previous chapters for the symmetric scenario, and consequently, giving a similar ratio with regard to the defect characteristic size.

The numerical solutions will be computed with the ANSYS Mechanical FEM Solver. The piezoelectric devices are modeled with the SOLID226 coupled-field elements, and the aluminum plate with the SHELL281 elements. The element size is 2.5 mm, and the total number of nodes is ~ 0.149 Millions.

Test cases

Let us present the performance of the topological derivative in this antisymmetric plus symmetric scenario detailed above. As a proof of the concept, we will present two cases varying the position of the defect; the first one with the defect closer to the transmitters, and the second one, with the defect closer to the receivers.

In figure 6.5 is shown the first test case. Transmitters are indicated with black circles, the receivers with black $\times$ symbols, and the position of the defect with a white $+$. Recall that the blue regions in the plots are the regions where is more likely to find the defect (i.e., the topological derivative negative peaks), and the grey zone represents either the non inspected are outside the interrogation window or the positive values of the topological derivative. The number of frequencies utilized in the present case is $K = 40$. It can be seen that the defect is well located by the topological derivative in its the correct position, namely,

$$(x, y) = (-0.1 \text{ m}, 0.15 \text{ m}). \quad (6.10)$$

The second case is illustrated in figure 6.6. Now the defect has been shifted to the lower side of the plate, this is, to the point

$$(x, y) = (-0.0988 \text{ m}, -0.1513 \text{ m}), \quad (6.11)$$

which is farther from the transmitters and closer to the receivers. As in the previous case, the number of frequencies utilized is $K = 40$.

Figure 6.5: Topological derivative for an antisymmetric plus symmetric scenario. Transmitters are indicated with black circles and receivers with black $\times$ symbols.

Once more, the topological derivative detects the defect quite well. However, note that we can see relevant differences between the patterns in the topological derivatives in both cases.

Figure 6.6: Counterpart of figure 6.5 for a different position of the defect.

Conclusions

In this Thesis, it has been investigated an innovative approach to Structure Health Monitoring based on the multi-frequency *topological derivative*. This new method has clearly shown the possibility to overcome the typical scenarios in which the SHM conventional techniques (specially the ultrasonic techniques) have serious difficulties to assess the condition of the structures. These complicated scenarios range from situations in which the transmitters and receivers are placed close to reflective boundaries (or even to themselves) to structures with complex geometries, thickness variations or material inclusions. Undoubtedly, the strength of the topological derivative method against traditional SHM techniques is explained by the fact that it uses the whole physics of the problem, instead of only a part of it (e.g., only the propagation velocity of the waves).

From the beginning, the objective of this Thesis was to bring the already developed ideas within the topological derivative context into situations approaching real industrial scenarios. To this end, we started studying the performance of this new method over aluminium plates (although it applies to any kind of material) which is detailed in chapter 3. The initial case studies considered homogenous plates, and then, we progressively increased the level of complexity. In particular, through-hole defects have been detected with no problems in considerably big plates (i.e., $1 \times 1 \text{ m}^2$) containing air slits, and titanium and HDPE inclusions. Moreover, aspects like the proximity of the transmitters and receivers to the boundaries of the plate (i.e., 2 cm), and the selection of frequencies with no particular care (inside an appropriate range though), prove the robustness of this method. The transmitting and receiving of the signals was done by means of point load (PL) and rotationally symmetric radial (RSR) models, simulating a laser vibrometer and piezoelectric transducers respectively. Concerning the numerical simulations carried out, on account of the 2D approximation utilized to model very thin plates, the computation

times have been fairly affordable.

In chapter 4, a comparison between the 2D approximation utilized in chapter 3, and the full 3D model was studied. This comparison aimed at proving that the 2D approximation is a very efficient way to calculate thin plates in which a plane stress elastodynamic state occurs. As can be seen, the comparison gave very similar (and good) results for the topological derivative in both models, and therefore, verifying the 2D approximation. Mention that the computational times between these numerical models, for the usual plate size used in the test cases, have a difference of approximately two orders of magnitude. Furthermore, calculating the topological derivative in 3D models was a necessary step before implementing the method to cases in which the 2D approximation cannot be applied, namely, plates with thickness variations and plates excited non symmetrically. In any event, it has been plainly demonstrated that the 2D approximation should be applied against the 3D model in cases in which is possible.

The application of the topological derivative to aluminum plates with thickness variations was studied in chapter 5. Four testing cases were selected, trying to cover different types of changes in the thickness that may occur in the usual aerospace structures. We have studied either hard or gradual variations (i.e., ramps) in the thickness, locating the defects with excellent results for the same transmitting/receiving configurations (i.e., PL and RSR) employed in the previous chapters. Once more, the topological derivative has shown its power in a scenario which has been a *headache* over the years for the traditional methods due to the velocity shifts. Also, we noted that the computational effort, for a very limited number of transmitters and receivers, has not been increased so much in comparison with the test cases with constant thickness in order to achieve accurate results. As stated, in this situation a 3D model was used to simulate the plates which obviously incremented notably the computational time. On the other hand, also in this chapter 5, it is presented a case to demonstrate the excellent performance of the topological derivative detecting a defect in a plate with a very intricate geometry. These types of plate domains usually lead to chaotic wave patterns before the receiver collects the signals, making impossible the detection of damage by means of common triangularizations; as expected, the topological derivative did easily its job.

Finally, a last chapter was devoted to the application of the topological derivative to cases in which a non-symmetric excitation of the plate is performed. Because of this type of excitation, antisymmetric and symmetric wave modes are generated within the plate. The antisymmetric mode has a smaller velocity which leads to a smaller wavelength (for the same frequency), originating the necessity of finer meshes. Certainly, the non-symmetric

excitation of structures represent a more realistic situation that usually happen when doing active SHM, however, as mentioned, for the sake of simplicity (save computational time), the symmetric excitation was applied in the previous chapters. A pair of test cases, as proof of the concept, were studied to prove the accurate performance of the topological derivative for the non-symmetric excitation, whose results, as expected, are again fairly good. In addition, the results of an experimental validation of the piezoelectric devices used as transmitters are discussed in the first section of this chapter.

As previously noted, in order to calculate the topological derivative we have used synthetic data instead of actual experimental data; either the experiment, the unperturbed direct, and adjoint problems involved in this work have been computed via FEM models. This may leads to the belief that this method does require a big computational effort to be implemented, but it does not. In a real case, the experimental data will not be obtained by any numerical simulation (evidently), and with the regard to the direct and adjoint problems, a database with the results of the pristine structure can be created (i.e., solve numerically the structure only one time). Then, we can compute the topological derivative rescaling the solutions of the direct and adjoint problems using the experimental results as input, taking advantage of the linear behaviour of the elastic equations. Computing the topological derivative algorithm, provided that the FEM models are already calculated, can be achieved in a few minutes (or even in seconds), making feasible the development of an on-board software/hardware to perform a real-time monitoring of the structures.

With regard to the future work, the results of this Thesis originate several lines of research that may be investigated:

- Validate experimentally the test cases proposed in this Thesis. Despite the promising results presented in this Thesis, this novel SHM procedure must have a reliable experimental counterpart.
- Test the performance of the topological derivative in different kind of structures such as composite materials, riveted panels or welding seams. Normally, the topological derivative will not have any problem to detect damage in these structures, even when anisotropy exists; difficulties arise due to the accurate FEM models needed to calculate the pristine reference structure.
- Characterization of the damage by means of the topological derivative. In this Thesis, for the sake of simplicity, the damage was introduced in the plates via small through-holes (≈ 2.5 mm of radius), but, corrosion, cracks, or delaminations may be correctly detected as well. The key element to evaluate the type of damage is

the frequency of the waves. As stated, low frequencies locate better inner points of the defects, while higher ones, indicate the shape.

- Investigate different forms of combining the partial topological derivatives resulting from the Lamé parameters and the density. Finding optimal ways of combining these partial sensitivities may enhance results. In all likelihood, in some occasions the overall topological derivative is more sensible to certain physical parameters and not to others.
- Development of reduced-order models to compute the numerical simulations involved in the calculation of the topological derivative. Recall that this computational effort would be done only one time for each structure, though these structures may be considerably big and rather complex, resulting in unaffordable computational domains.

Admittedly, the generation of actual experimental data and to validate the topological derivative is the number one challenge for a future work. It involves additional difficulties that are beyond the scope of this Thesis, which aims at presenting the proposed methodology as a proof of the concept. At any event, we hope that the flattering results presented in this Thesis act as an inspiring *seed* to continue investigating the application of the topological derivative to more complex case studies.

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