UNIVERSIDAD POLITÉCNICA DE MADRID
ESCUELA TÉCNICA SUPERIOR DE INGENIERÍA AERONÁUTICA
Y DEL ESPACIO

Instabilities in vibrated fluids and manipulation of liquids in microgravity: theory and experiment

---

TESIS DOCTORAL

Pablo SALGADO SÁNCHEZ
Máster Universitario en Ingeniería Aeronáutica

Septiembre 2020

DEPARTAMENTO DE AERONAVES Y VEHÍCULOS ESPACIALES
ESCUELA TÉCNICA SUPERIOR DE INGENIERÍA AERONÁUTICA
Y DEL ESPACIO

Instabilities in vibrated fluids and manipulation of liquids in microgravity: theory and experiment

---

Autor:

Pablo SALGADO SÁNCHEZ
Máster Universitario en Ingeniería Aeronáutica

Directores:

Dr. Jeff PORTER
Dr. Ignacio TINAO PÉREZ-MIRAVETE

Septiembre 2020

POLITÉCNICA

Tribunal nombrado por el Sr. Rector Magfco. de la Universidad Politécnica de Madrid, el día.....de.....de 20....

Presidente:_____

Vocal: _____

Vocal:_____

Vocal:_____

Secretario:_____

Suplente:_____

Suplente:_____

Realizado el acto de defensa y lectura de la Tesis el día.....de.....de 20 ...
en la E.T.S.I. /Facultad.....

Calificación .....

EL PRESIDENTE

LOS VOCALES

EL SECRETARIO

A mis padres,

Contents

Resumen	
Abstract	
I INTRODUCTION AND METHODOLOGY	1
1 Introduction	3
2 Methodology	13
II PUBLICATIONS	29
3 Dynamics of weakly coupled parametrically forced oscillators	31
4 The CFVib experiment: Control of Fluids in microgravity with Vibrations	79
5 Vibroequilibria in microgravity: comparison of experiments and theory	103
6 Simulating vibrated fluids and vibroequilibria in the CFVib microgravity experiment	127
7 Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity	149
8 Finite-size effects on pattern selection in immiscible fluids subjected to horizontal vibrations in weightlessness	185
9 Effect of initial interface orientation on patterns produced by vibrational forcing in microgravity	211
10 Influence of gravity on the frozen wave instability in immiscible liquids	239
11 Symmetry breaking in large columnar frozen wave patterns in weightlessness	269
III DISCUSSION AND CONCLUSIONS	293
12 Discussion	295
13 Conclusions and future work	319

BIBLIOGRAPHY 325

LIST OF PUBLICATIONS..... 335

List of Figures

2.1	Bifurcation sets in the case of $a = -0.3$: pitchfork (P: thick solid curve), saddle-node (SN: thin solid line), Hopf (H: lower dashed curve), saddle-node of periodic orbit (SN Per: upper dashed curve), homoclinic gluing (Hom Gluing: thin solid curve above pitchfork), homoclinic (Hom: lower thin solid curve below pitchfork), heteroclinic (Het: upper thin solid curve below pitchfork), saddle-node heteroclinic (SN Het), and Bogdanov-Takens point (BT: solid dot). Reproduced from Salgado Sánchez et al. (2016), Chapter 3.	16
2.2	Contours of first Lyapunov coefficient l_1 in the (ν, a) plane. This coefficient is always negative, showing that the Hopf bifurcation, which occurs for $\nu < (a^2 - 1)/(2a)$, is supercritical. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.	17
2.3	Sketch illustrating the two-layer system of immiscible liquids. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7. ...	19
2.4	Pictures of the CFVib experimental cells and the bender beam: (a) cuboid and (b) cylinder. The aluminium bender beam has two piezoelectric ceramics attached to opposite horizontal faces. Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.	23
2.5	Sketch of experimental setup with: (a) cuboid and (b) cylinder. Shown are the (partially filled) experimental cell, bender beam, piezoelectric ceramics, mirror and CMOS camera. Selected optical paths indicate the direct and reflected images. Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.	24
2.6	(a) Schematic view of the optical setup (left) and the experimental cell (right). (b,d) Images of the experimental cells holding immiscible and miscible liquids, respectively. (c,e) Unprocessed cropped images showing the interface in reduced gravity: (c) immiscible combinations I_a (upper container) and I_b (lower container), (e) miscible liquids under horizontal forcing: mixture M_a (upper snapshot) and M_b (lower snapshot). Reproduced from Salgado Sánchez et al. (2020), Chapter 9.	25
3.1	Pitchfork (P: thick curve) and saddle-node (SN: thin curve) bifurcation sets for a single parametrically forced oscillator in the case of conservative nonlinearity $a = 0$	36
3.2	Contours of the normal form (first Lyapunov) coefficient l_1 (see Kuznetsov (2004)) in the (ν, a) plane. This coefficient is always negative, showing that the Hopf bifurcation, which occurs for $\nu < (a^2 - 1)/(2a)$, is supercritical.	37

3.3	Bifurcation sets in the case of $a = -0.3$: pitchfork (P: thick solid curve), saddle-node (SN: thin solid line), Hopf (H: lower dashed curve), saddle-node of periodic orbit (SN Per: upper dashed curve), homoclinic gluing (Hom Gluing: thin solid curve above pitchfork), homoclinic (Hom: lower thin solid curve below pitchfork), heteroclinic (Het: upper thin solid curve below pitchfork), saddle-node heteroclinic (SN Het), and Bogdanov-Takens point (BT: solid dot).....	37
3.4	Bifurcation diagrams for $a = -0.3$ and (a) $\nu = -0.5$, (b) $\nu = 1$, showing the (time-averaged) norm of the nontrivial steady states (A_s), periodic modulated solutions (P_m) and periodic rotating solution (P_r). Solid (dashed) curves denote stable (unstable) solutions. With increasing ν , the primary bifurcation transitions from supercritical to subcritical and the gluing bifurcation splits into distinct homoclinic and heteroclinic bifurcations. A selection of periodic orbits at particular points labelled $a-h$ in these diagrams are shown in (c) and (d), respectively. a, e : rotating orbit at $f = 0$, b : limit point at $f = 2.747$, c : symmetric homoclinic orbit at $f = 2.678$, d : small modulated orbits at $f = 2.3$, f : heteroclinic orbit at $f = 1.275$, g : homoclinic orbits at $f = 1.071$, h : small modulated orbits at $f = 0.9$	38
3.5	Bifurcation sets for $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$: primary pitchfork in $\Sigma_{\pm}$ subspaces (P $_{\pm}$: thick solid curves), saddle-node in $\Sigma_{\pm}$ (SN $_{\pm}$: thin solid lines), symmetry-breaking (pitchfork) from $\Sigma_{\pm}$ to mixed mode state (SB $_{\pm}$: dash dotted curves), saddle-node on mixed mode state (SN $_{MM}$: thin solid curves). In the larger regions are indicated in parentheses the types of solutions that exist there.....	46
3.6	Bifurcation diagram for $\nu = -1.5$, $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$ showing $ A + B $ (projection onto Σ_+) and $ A - B $ (projection onto Σ_-). Solid (dashed) curves denote stable (unstable) solutions. Solutions in the $\Sigma_{\pm}$ subspaces are labelled with $S_{\pm}$ and asymmetric mixed modes with MM	46
3.7	Phase space portraits corresponding to the case of Fig. 3.6 with $f = 2.5$. There are two stable steady states S_+ (solid dots) in Σ_+ , two stable steady states S_- (asterisks) in Σ_- and four unstable asymmetric steady states MM (open circles); one representative solution on each group orbit is labelled. (a) and (b) show projections onto $A = A_r + iA_i$ and $B = B_r + iB_i$, respectively, while the projections in (c) and (d) illustrate the interchange symmetry (3.21). The trivial state is not shown.....	47
3.8	Bifurcation diagram for $\nu = 1.5$, $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$ showing $ A + B $ and $ A - B $. Solid (dashed) curves denote stable (unstable) solutions. Solutions in the $\Sigma_{\pm}$ subspaces are labelled with $S_{\pm}$ and asymmetric mixed modes with MM	48

3.9	Bifurcation sets for $\phi = \pi$, $a = 0$, $\mu = 0.2i$: Hopf (H: dashed curves) and saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), homoclinic (Hom), saddle-node homoclinic (SN Hom), Bogdanov-Takens points (BT). In the larger regions, the types of solutions that exist there are indicated in parentheses. In (a) the regions of one and two C_4 -symmetric periodic orbits (P_{C_4}) are further marked with “1” and “2”, respectively. In (b) a close-up of the complex region between two BT points near the cusp of SN bifurcations is shown and, in the inset, a further zoom showing the cusp region.	50
3.10	Bifurcation diagrams for $\phi = \pi$, $a = 0$, $\mu = 0.2i$ showing the norm $ \mathbf{A} = (A, B) $ of steady states (S), C_4 -symmetric periodic orbits (P_{C_4}), and asymmetric periodic orbits (P_a): (a) $\nu = -0.5$, (b) $\nu = 0.5$, (c) $\nu = 1.4$, (d) $\nu = 2$. For $\nu = 1.4$, there is a secondary supercritical Hopf bifurcation on the isola of steady states that generates stable asymmetric periodic orbits, shown in the inset. Stable (unstable) solutions are shown with solid (dashed) lines.	52
3.11	Phase space portraits corresponding to the case of Fig. 3.10(c) with $f = 1.5$. There are four stable steady states S_s (solid dots) and four unstable steady states S_u (open circles) related by the symmetry (3.22); one representative solution on each group orbit is labelled. The C_4 -symmetric periodic orbit, P_{C_4} , is also shown. (a) and (b) show projections onto $A = A_r + iA_i$ and $B = B_r + iB_i$, respectively, while the projections in (c) and (d) illustrate the C_4 symmetry. The trivial state is not shown.	53
3.12	Bifurcation sets for $\phi = \pi$, $a = 0$, $\mu = 0.2 + 0.04i$: Hopf (H: dashed curves) and saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), and double zero point. In the larger regions, the types of solutions that exist there are indicated in parentheses.	53
3.13	(a) Close-up of the double zero bifurcation point and (b) cusp region of Fig. 3.12 showing bifurcation sets: Hopf (H: dashed curves), saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), saddle-node heteroclinic (SN Het), and heteroclinic (Het). The heteroclinic bifurcation set near the cusp does not collide there, but with the upper branch, although this cannot be distinguished on the scale of the plot. In the larger regions, the types of solutions that exist there are indicated in parentheses.	54
3.14	Contours showing the transition from supercritical to subcritical Hopf bifurcation for coupling values with $\mu_r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. The transition is independent of μ_i .	55
3.15	Contour map showing the location of the Z_2 -symmetric Bogdanov-Takens point with $\nu > 0$ in the conservative and purely dissipative cases, as given by Eq. (3.41) and Eq. (3.47). Curves for $\nu < 0$ are obtained by reflection. Curves are shown for constant μ , in steps of $0.2i$ ($\mu \in \text{Im}$) and 0.2 ($\mu \in \text{Re}$), and for constant ϕ (radial lines), in steps of $\pi/6$. The central curve intersecting $(\nu, f) = (0, 1)$ corresponds to the uncoupled case $\mu = 0$.	57

3.16	Primary Hopf (H) and pitchfork (P) bifurcation sets for (a) $\mu = 0.2i$, (b) $\mu = 0.04 + 0.2i$, (c) $\mu = 0.2$, and (d) $\mu = 0.2 + 0.04i$. $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points (not marked) lie on the intersection.	59
3.17	(a) Bifurcation sets for $\phi = \pi/2$, $a = 0$, $\mu = 0.2i$: Hopf (H: dashed curves), saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), homoclinic (Hom), Bogdanov-Takens points (BT). Some bifurcation sets (SN Per, homoclinic, and gluing bifurcations, for example) are hard to see on the scale of the plot and are not labelled. Panel (b) shows a close-up of the complex transitions for positive detuning. In the larger regions, the types of solutions that exist there are indicated in parentheses.	60
3.18	Close-up of the (a) left and (b) right transition between pitchfork and Hopf bifurcation of Fig. 3.17 showing bifurcation sets: pitchfork (P), Hopf (H: dashed curves), saddle-node (SN), saddle-node heteroclinic (SN Het), heteroclinic (Het), homoclinic gluing (Hom Gluing), and Bogdanov-Takens points (BT). Some bifurcation sets (homoclinic and heteroclinic, near the lower, asymmetric BT point in the lower panel) are not visible on the scale of the plot and are not labelled. In the larger regions, the types of solutions that exist there are indicated in parentheses...	61
3.19	Bifurcation diagrams for $\phi = \pi/2$, $a = 0$, $\mu = 0.2i$ showing the norm $ \mathbf{A} = (A, B) $ of steady states (S), $\mathbf{Z}_2$ -symmetric periodic orbits ($P_{\mathbf{Z}_2}$), and asymmetric periodic orbits (P_a): (a) $\nu = -1.5$, (b) $\nu = 0$, (c) $\nu = 1$, (d) $\nu = 2$. For $\nu = 1$ there is a secondary supercritical Hopf bifurcation on the isola of steady states that generates stable asymmetric periodic orbits, shown in the inset. Stable (unstable) solutions are shown with solid (dashed) lines.	62
3.20	Phase space portraits corresponding to the case of Fig. 3.19(c) with $f = 1.374$, at the saddle-node heteroclinic bifurcation. There are two stable steady states (upper branch denoted by solid dots, and the lower by asterisks) and two (larger amplitude) unstable steady states (upper branch denoted by open circles, and the lower by open squares). The $\mathbf{Z}_2$ -symmetric periodic orbit, which encloses the origin (trivial state not shown) is colliding with two smaller amplitude unstable steady states (open triangles). Projections onto (a) (A_r, A_i) , (b) (B_r, B_i) , (c) (A_r, B_i) , (d) (A_i, B_r)	64
3.21	Modulated (quasiperiodic) solution found experimentally in a 20 cm square container with 5 cSt oil driven at 13.95 Hz. The crosswise mode number is $N = 11$ and the modulation period is about 45.6 s. The snapshots (a)-(d) are separated by approximately 1/8 of this period, or 5.7 s. Reproduced from Tinao et al. (2014)....	65

3.22	Modulation (Hopf) frequency (in Hz) of the solution shown in Fig. 3.21 over an interval of forcing frequencies ω/π (in Hz) around the resonance frequency of 13.52 Hz. The experimentally measured Hopf frequency Ω is compared with that predicted by the model obtained in Tinao et al. (2014) with the derived boundary conditions (solid curve) and with Dirichlet boundary conditions (dashed curve); remaining parameters for these calculations are: viscosity 5 cSt, surface tension 19.7 dyne/cm, density 0.913 g/cm ³ , container depth 5 cm, and width 9 cm. Reproduced from Tinao et al. (2014).	66
3.23	Primary Hopf bifurcation for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (3.48) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$. (a) Critical forcing $f(\nu)$ (solid curve) compared to the pitchfork bifurcation of a single oscillator (dashed curve). (b) Hopf frequency $\Omega(\nu)$; this should be compared with Fig. 3.22.	67
3.24	Selected bifurcation sets for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (3.48) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$: Hopf (H: dashed curve), saddle-node heteroclinic (SN Het), saddle-node of periodic orbit (SN Per, solid curves for $\nu \gtrsim 2.15$). Dotted vertical lines mark ν values where the Hopf frequency vanishes. Additional bifurcation sets near these double zero points, which may include saddle-node, pitchfork, asymmetric Hopf, homoclinic, and heteroclinic bifurcations (Marqués et al., 2013), are not shown in detail.	68
3.25	Harmonic (black curve) and subharmonic (grey curve) components as forcing (in g 's) is increased for frequencies of (a) 30 Hz, (b) 40 Hz, (c) 50 Hz, and (d) 60 Hz. A transient has been removed from each simulation. Note the modulation of the subharmonic waves in all but (d) the 60 Hz case.	70
3.26	Snapshot of the change in fluid surface height (in cm) for the 50 Hz simulation at 0.9088 g 's. Above onset the subharmonic waves are dominant throughout the domain.	71
3.27	(a) Close-up of the Hopf bifurcation to modulated subharmonic waves in the case of 50 Hz in Fig. 3.25. Peak modulation amplitude (maximum over one modulation period) of the filtered subharmonic signal is shown for increasing forcing (open circles, up arrow) and decreasing forcing (open squares, down arrow). (b) Linear fit of the growth rate (in s^{-1}) of perturbations at fixed values of the forcing predicts an onset of $f = 0.804 g$'s.	71
3.28	Time series showing (signed) modulation amplitude of time series for points equidistant from the left (solid curves) and right (dashed curves) boundaries for the case of 50 Hz forcing. The transition from sinusoidal modulation near the Hopf bifurcation to slower irregular modulation near the global bifurcation can be seen with increasing forcing values (in g 's): (a) 1, (b) 1.2, (c) 1.25, (d) 1.27. A 30 s transient has been removed from each simulation.	73
3.29	The increase in modulation period (T_m) as the global bifurcation is approached. The data from the simulation (open circles) fit well with the scaling expected for a saddle-node heteroclinic bifurcation: the solid line shows the fitting function: $0.059 + 0.835(1.296 - f)^{-1/2}$.	74

3.30	Harmonic (black curve) and subharmonic (gray curve) components as forcing (in g 's) is increased with 60.4 Hz vibration.....	75
3.31	(a) Estimated location of onset (triangles) and saddle-node heteroclinic bifurcation (circles); the parameters shown are applied forcing f (in g 's) and applied frequency ω/π (in Hz). (b) Estimated modulation frequency Ω (in Hz) near onset.	75
4.1	Relaxation of a 2D fluid in a 4 cm wide container (fluid area/volume 6 cm ²) subjected to horizontal vibrations at 100 Hz in the absence of gravity; forcing starts at $t = 2$ s, viscosity is 50 cSt, surface tension 20 dyne/cm, and contact angle 60°. (a) Surface profile $h(x)$ at $t = 0$ (initial flat surface; dashed line), at $t = 2$ s (hydrostatic surface; dash-dotted curve), and for large time (vibroequilibria surface; solid curve). (b) Heights $h(t)$ of the two contact points (upper curves) and midpoint (lower curve) showing relaxation to the hydrostatic solution ($t < 2$ s) and to the vibroequilibria solution ($t > 2$ s). The inset shows the out-of-phase harmonic waves at the two walls (one grey, one black).	86
4.2	Acceleration (in g 's) during the microgravity portion of the parabolic flight in a 3-axis reference frame attached to the airplane showing the z -axis ("vertical" direction, g_z , dark grey), y -axis ("lateral" direction, g_y , medium grey), x -axis ("longitudinal" direction, g_x , light grey) and total magnitude ($ g $, black).	87
4.3	Vibroequilibria solution with 88 Hz forcing (near the lowest resonance) and 60° contact angle for a liquid with properties like water. The vibroequilibria surface is shown after 20 s while it takes about 4 s, starting from the hydrostatic equilibrium, for transients to decay.	88
4.4	Sketch of CFVib experiment showing just one cuboidal cell. Rack #1 holds the experiment samples and main elements of the acquisition system. Rack #2 holds the control and electronic components.	90
4.5	Prototypes for (a) rectangular cuboidal container and (b) cylindrical container.	91
4.6	Accelerometer measurements taken from the cuboidal cell with water during the third flight in parabolas 5 and 15, using excitation frequencies of 79.8 Hz (first resonance) and 846.5 Hz (third resonance). (a) Time series of the accelerometer signals a (in g 's) for forcing at 79.8 Hz (upper panel) and 846.5 Hz (lower panel). Unfiltered (grey curves) and filtered signals (black curves) are shown. (b) Absolute value $ A $ of the fast Fourier transform of both signals (lower frequency signal rendered with a thicker line) showing peaks at the corresponding excitation frequencies.	94
4.7	Resonance behaviour in the cuboidal container with 50 cSt silicone oil. Here V_{pp} (applied to the piezoelectrics) is held at 40 V while the frequency f is swept between 70 Hz and 12 kHz.	96
4.8	Vertical acceleration a (absolute value of harmonic component) measured during ground tests at the midpoint of the external rear wall for increasing (peak-to-peak) piezoelectric excitation voltages V_{pp} , at the first three resonance frequencies: 81 Hz (circles), 217 Hz (squares) and 841 Hz (triangles) of the cuboidal container with water. A linear fit is shown in each case.	97

4.9	Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container: (a) At the beginning of microgravity, (b) Just before forcing is applied, (c) About 2 s after forcing of amplitude 18.5 cm/s is initiated, (d) About 4 s after forcing is initiated.	99
5.1	Pictures of the experimental cells and the bender beam: (a) cuboid and (b) cylinder. The cells are held closed by four bolts, which compress the PMMA body and cover against an O-ring. The aluminium bender beam has two piezoelectric ceramics attached to opposite horizontal faces. Cables associated with electrical excitation of the ceramics and the clamping bolt can also be seen.	107
5.2	Sketch of experimental setup with: (a) cuboid and (b) cylinder. Shown are the (partially filled) experimental cell, bender beam, piezoelectric ceramics, mirror and CMOS camera. Selected optical paths indicate the direct and reflected images referred to in later sections. The inset in (b) shows a lateral view of the setup for a cylindrical cell. Shaded curves illustrate the excited motion of the beam for the first (b, inset) and second (a) resonant modes.	109
5.3	(a) Midplane cut of the experimental setup for a cuboidal container showing the bender beam, piezoelectrics (PZTs), fluid volume Ω , free surface S and solid boundaries W . The container and lab reference frames are illustrated; since the displacement is very small compared to the container size, we assume hereafter that $(x, y, z) \simeq (x_{\text{lab}}, y_{\text{lab}}, z_{\text{lab}})$. This geometry is used for the numerical results shown in Sec. 5.5.3. (b,c) Illustrations of the motion produced in the first two resonant modes.	110
5.4	Snapshots (front and lateral views) showing the evolution of the surface of silicone oil in a cylindrical cell. The upper row of panels (a-c) shows the fluid behaviour in the reference cell (experiencing residual acceleration of the airplane) at the given times, paired with corresponding snapshots of a vibrated experiment. The vibrated experiments are excited at $f = 59.3$ Hz with vibrational velocities: (a-c) $v_z = 73.9$, (d-f) 84.6, and (g-i) 94.8 mm/s, which correspond to applied (peak) voltages of 80, 90, and 100 V, respectively. To help visualise the free surface deformation, contact lines at “front” and “rear” halves of the lateral wall are highlighted with solid and dashed lines.	113
5.5	Evolution in time of the highest visible point of the contact line (solid dots) and the minimum visible point of the fluid surface (diamonds) in vibrated (black) and reference (grey) cells. Vibrations are delivered at $f = 59.3$ Hz and different applied vibrational velocities: (a) $v_z = 73.9$, (b) 84.6, and (c) 94.8 mm/s, which correspond to applied (peak) voltages of 80, 90, and 100 V, respectively. Measurements of the residual gravity level are shown below each panel, with the labelled components measured in the airplane reference frame. Vertical lines mark the times when the snapshots in Fig. 5.4 are taken [labelled as 4(a-i)].	114

5.6	Snapshots (top and front views) showing the evolution in time of experiments with silicone oil in a cuboidal container. The upper two rows show the reference cell, which is subjected to the airplane's residual gravity, while the lower two rows show the cell excited at $f = 83.07$ Hz and $v_\eta = 161.6$ mm/s, corresponding to a peak voltage $V = 100$ V. When feasible, the contact lines along the front and rear walls (front view) are highlighted using solid and dashed lines, respectively, as is the contact line in the upper views.	116
5.7	(a–c) Snapshots of vibroequilibria in a cuboidal container vibrated at 79.8 Hz in the first resonant mode. Vibrational velocities are: (a) 70.7, (b) 122.6, and (c) 161.6 mm/s, corresponding to peak voltages of 50, 80 and 100 V, respectively. The location of the “front” and “rear” contact lines is outlined with solid and dashed curves, respectively. Images of the reference cell at identical times are included for comparison. (d) Illustration of the measurement points used for quantitative comparison with the theory of Sec. 5.5.3.	119
5.8	Snapshots (top and front views) showing the time evolution of experiments with water a cuboidal container when vibrated in the second resonant mode. The upper rows show the reference cell, while the lower rows show the results of vibrating at $f = 215.4$ Hz and $v_\eta = 49.19$ mm/s, corresponding to an applied voltage of 100 V..	120
5.9	Comparison of experimental results (solid black dots) with direct numerical simulation of Eqs. (5.2)–(5.5) (open grey circles with solid fitting curve), with analogous simulations that ignore the rotational motion (open grey squares with dotted fitting curve), and with the predictions of vibroequilibria theory based on the minimisation of Eq. (5.10) (solid black curve). The contact angle used in these calculations is 50° , while the residual gravity is set to a representative value of $g_z = 0.05 g_0$	121
5.10	Influence of viscosity and gravity level for a fixed voltage $V = 50$ V, $\eta = 0.57$ on the predicted contact point separation used to quantify the vibroequilibria effect. (a) Simulations of the Navier-Stokes equations with $g_z = 0.01 g_0$ and $\nu = 10$ cSt (light grey curves), 50 cSt (medium grey curves), and 200 cSt (black curves). (b) Simulations using $\nu = 10$ cSt with $g_z = 0$ (light grey curves) and $g_z = 0.05 g_0$ (black curve). The remaining parameters are as in Fig. 5.9.....	122
5.11	Measurements (performed on ground) of the maximum linear and angular velocities at the centre of the side wall for the first resonant mode. (a) Horizontal velocity $ v_x $ (solid dots) and vertical velocity $ v_z $ (solid squares). (b) Angular velocity ϖ_0 . .	123
6.1	Images of the experimental cells: (a) cuboid and (b) cylinder. The piezoelectric ceramic attached on the upper side of the beam (the other is attached underneath) is visible in each assembly along with the associated cabling.	131
6.2	Sketches showing a midplane cut of the cuboid assembly. The domains of the mathematical formulation for (a) fluid dynamics and (b) fluid-structure-piezoelectric interaction are included. Throughout this paper, simulations assume the midplane reflection symmetry illustrated here.....	132

6.3	(a) Algorithm using the timescales separation approach, showing the complete iterative scheme (solid lines) and the simplified scheme (dashed lines) followed here. Here T is the forcing period and T_f the stopping time. (b) The complete (symmetric) geometry used in the model. The relevant physical properties of water (at room temperature) and of structural materials, reproduced from the COMSOL Library, are included: density ρ , surface tension Γ , dynamic viscosity μ , Young's modulus E and Poisson ratio ν , when applicable.	137
6.4	Magnitude of nondimensional harmonic velocity $ \widehat{w}_h $ at the midpoint of the free surface (grey) and the midpoint of the right lateral wall (black) for different damping values $\eta_s = 0.01$ (solid) and 0.02 (dashed) with forcing $V_{pp} = 100$ V. A close-up view of the first resonance in the inset shows the effect of increased damping. The three first resonances near 81 Hz, 220 Hz and 842 Hz are illustrated at the instant of maximum deformation (displacement scale not preserved between illustrations)..	140
6.5	(a,b) Vibroequilibria calculated numerically for the (a) first and (b) second resonances after 20 s of applied vibrations at 200 V. The parameters used are those of water, with the contact angle set to $\beta = 80^\circ$. (c) Nondimensional height of the front (grey squares) and rear (black squares) contact points at the midplane for increasing vertical vibrational velocity $\widehat{w}_h$ in the first resonance mode. Note the approximately linear dependence on the squared velocity for $ \widehat{w}_h ^2 \lesssim 3$	141
6.6	(a) Measured magnitude of the vertical harmonic velocity $ \widehat{w}_h $ at the midpoint of the right endwall for different excitation voltages V_{pp} , at the three first natural frequencies: 81 Hz (solid circles), 220 Hz (solid squares) and 842 Hz (solid triangles); with associated linear fittings (grey lines). Values calculated numerically using $\eta_s(f)$ as in Eq. (6.20) are shown with grey open markers. (b) Magnitude of the vertical harmonic velocity $ w_h $ for frequencies up to 1 kHz with a driving voltage of 200 V: experimental measurements (black curve) and numerical calculations (grey curves). Two damping models are used: $\eta_s(f)$ [grey solid line, Eq. (6.20)] and with Rayleigh coefficients [grey dashed line, Eq. (6.22)]. Close-up views of the three resonances are provided in the insets.	143
6.7	Simultaneous images of <i>Vibrated</i> and unforced <i>Reference</i> microgravity experiments in cuboidal containers partially filled with water: (a) vibroequilibria effect when vibrated at $f \simeq 81$ Hz (the first resonant mode) and $V_{pp} = 200$ V, (b) reference experiment, experiencing only the effect of airplane disturbances.	145
7.1	(a) General view of the experiment cell assembly, consisting of two identical cells filled with immiscible fluids of different viscosity and density ratios. The lower liquid layers in both cells are FC-40, while the upper layers are 2 cSt silicone oil (top cell) and 20 cSt silicone oil (bottom cell). (b) Behaviour of the interface in microgravity in the absence of vibrational forcing. Raw (cropped) image of fluid cells: heavier FC-40 (lighter region); lighter silicone oils (darker region).....	154

7.2	Gravity profiles from one of the parabolic flights showing $g_x(t)/g_0$, $g_y(t)/g_0$ and $g_z(t)/g_0$ (red, black and grey curves, respectively); see Fig. 7.3 for axes. (a) Gravity profiles together with the experiment intervals, both when forcing begins in hypergravity (yellow) and in microgravity (blue); the overlap region is shown in light green. (b) Close-up of the microgravity interval.	156
7.3	Sketch illustrating the (two-dimensional) numerical model.	157
7.4	Mesh convergence test for numerical simulations. Results for meshes #1–4 are shown in blue, red, green and black, respectively. (a) Volume average of the level-set function $\langle\phi\rangle_V$ during the vibrated part of the simulation. (b) Horizontal velocity $v_x(y)$ at $x = 4$ mm. (c) Vertical velocity $v_y(x)$ at $y = 3.75$ mm.....	161
7.5	Idealised two-dimensional gravity profiles $ g_x(t) /g_0$ and $ g_y(t) /g_0$ (dotted and dashed curves, respectively) during the initial period of a parabolic manoeuvre. Vibrational forcing (shown as solid curves, normalised to unity) may be started during microgravity (a–c) or hypergravity (d–f): (a,d) profile I - ideal parabola with $g_x = 0$, (b,e) profile II - quality parabola with final $g_x = 0.001 g_0$ and (c,f) profile III - generic parabola with final $g_x = 0.01 g_0$. In all profiles g_y goes from $1.5 g_0$ to $0.05 g_0$. Shaded regions in (a) and (d) show the transition (blue, duration 1.5 s) from hypergravity to microgravity and the forcing initialisation (green, duration 0.25 s), both smoothed in the manner of (7.3).....	162
7.6	Interface between FC-40 and 20 cSt silicone oil (the lower experiment cell) when subjected to residual gravity. (a–c) Snapshots showing the temporal evolution of the interface when subjected solely to residual gravity during an experiment. (d–f) Simulations when subjected to the gravity profiles of Fig. 7.5 prior to initialising forcing in microgravity: (d) initial ($t = 0.5$ s, dashed-dotted line), intermediate ($t = 1.5$ s, dashed) and final ($t = 2.25$ s, solid) interface locations for profile III [see Fig. 7.5(c)], and (e,f) final quasi-steady equilibria selected with gravity profiles I and III, respectively.	163
7.7	Superimposed out-of-phase images for different surface wave modes with FC-40 and 2 cSt silicone oil: (a) harmonic mode at $f = 40$ Hz and $A = 1.06$ mm, (b) subharmonic mode at $f = 5$ Hz and $A = 4.1$ mm (approximately odd with respect to midplane reflection), and (c) subharmonic mode at $f = 18$ Hz and $A = 1.43$ mm (approximately even). The time between frames is labelled under each picture while solid and dashed lines outline the interfaces in each image.....	165
7.8	Successive snapshots showing the evolution of subharmonic Faraday waves on opposite sides of the frozen wave columnar structure for $f = 22$ Hz and $A = 1.76$ mm. Times are selected to best show the $\pi/2$ phase difference between these two subharmonic wave fields (1/4 period of these 11 Hz subharmonic waves is approximately 0.0227 s).....	166

7.9	Instability diagram of the subharmonic mode for FC-40 and 20 cSt silicone oil under different gravity profiles: (a) profiles I and (b) profile III. Experimental (filled circles) and numerical (open circles) results exhibit various dynamical modes, marked with alternating blue tones. The solid curves are drawn for visual reference to distinguish different average vertical wavenumbers. The dashed curve indicates that all these modes belong to the main subharmonic instability tongue. Illustrative snapshots in the upper-left corner show the initial condition, while the smaller images show the interface after $t \simeq 2$ s of applied vibration in microgravity.	167
7.10	Instability diagram for the subharmonic mode with FC-40 and 2 cSt silicone oil under gravity profile I. Experimental data (filled circles) and numerical simulations (open circles) reveal various dynamical modes, marked with alternating blue tones. The solid curves are drawn for visual reference to distinguish different average vertical wavenumbers. The dashed curve indicates that all these modes belong to the main subharmonic instability tongue. Illustrative snapshots in the upper-left corner show the initial condition, while the smaller images show the interface after $t \simeq 2$ s of applied vibration in microgravity.	168
7.11	Representative snapshots in time (values provided) showing the different drop ejection mechanisms observed in the FC-40 and 2 cSt silicone oil cell during microgravity: (a–c) mechanism A (near onset), (d–f) mechanism A (far from onset), (g–i) mechanism B.	169
7.12	Drop ejection instability thresholds for FC-40 and 2 cSt silicone oil as a function of frequency f and (a) amplitude A , (b) vibrational velocity $A\omega$. Circles mark experimental data points, coloured in red if drop ejection is found. Fits using different scaling for the drop ejection threshold [mechanism A, dashed black line] are given in colour: $A \propto f^{-1}$ (red), $A \propto f^{-2/3}$ (green) and $A \propto f^{-1/2}$ (blue). The threshold curve for mechanism B (solid black line) is estimated from experiments (no scaling assumptions).	170
7.13	Interfacial energies, measured by $\mathcal{P}_\sigma$ (in pixels), obtained for FC-40 and 2 cSt silicone oil, as a function of the $\delta_c = A\omega - A\omega _c$, where $A\omega _c$ is the drop ejection threshold found to be 0.1743 m/s from the fit. The interfacial energy of the drops (red circles) and the main interface (green squares) are included. The curves are fitted to quadratic and constant functions, respectively. The number of observed drops is marked in parentheses, while outlier data points marked with crosses are excluded from the fit.	172
7.14	Snapshots showing the temporal evolution of the FC-40 and 20 cSt silicone oil interface below the frozen wave threshold for both experiments and simulations. (a–c) Experiment response at $f = 22$ Hz and $A = 0.87$ mm. (d–f) Simulation performed at $f = 20$ Hz and $A = 1.5$ mm under gravity profile III. A comparison of experiment and simulations in time cannot be done quantitatively due to the definition of $\mathbf{g}(t)$ (see Sec. 7.4) so times values are not labelled.	175

7.15	Snapshots showing the temporal evolution of the FC-40 and 20 cSt silicone oil interface above the frozen wave threshold for both experiments and simulations. (a–e) Experiment response at $f = 22$ Hz and $A = 1.76$ mm (above frozen wave onset) showing the growth of the instability (a,c) and the secondary Faraday waves excited on the lateral boundaries of the columns. The snapshots (d) and (e) separated by $\Delta t = 0.05$ s $\simeq 1/(22$ Hz), showing the (nearly) out-of-phase subharmonic behaviour; waves on the right column are shifted in phase by $\pi/2$. (f–h) Simulation performed at $f = 20$ Hz and $A = 2$ mm under gravity profile II. A comparison of experiment and simulations in time cannot be done quantitatively due to the definition of $\mathbf{g}(t)$ (see Sec. 7.4) so the times values are not labelled.	176
7.16	(a) Frozen wave instability diagram showing wavenumber for both combinations (less viscous, green open squares) under gravity profile I, when vibrations are applied in microgravity. (b) Frozen wave instability diagram for FC-40 and 20 cSt silicone oil under different gravity profiles and forcing initiation times: profile I initiation in microgravity (red open squares), profile I initiation in hypergravity (pink open squares), profile III initiation in hypergravity (cyan open squares). An experimental measurement (black solid square) is also shown. Illustrations are included to help visualise the features of the instability.	178
7.17	Three snapshots, separated by 0.0125 s in time for FC-40 and 2 cSt silicone oil (a–c) and FC-40 and 20 cSt silicone oil (d–f) subjected to vibrations at $f = 50$ Hz and $A = 3.9$ mm. (a,d) Initial high-amplitude surface waves in hypergravity, (b,e) transient frozen wave pattern (in hypergravity), (c,f) transition from vibrated to residual gravity equilibria.	179
7.18	Response of the fluid interface to vibrations in microgravity as a function of frequency f and amplitude A : (a) FC-40 and 2 cSt silicone oil, (b) FC-40 and 20 cSt silicone oil. Solid circles denote experimental points in three regimes: harmonic waves (yellow), subharmonic Faraday waves (blue) and drop ejection (red). Open markers denote numerical results for the Faraday wave instability (blue open circles) and the frozen wave instability (green open squares). These thresholds are fitted to obtain the dark blue and green solid lines, respectively. The Faraday instability scales as $f^{-1/2}$ for high frequencies ($\gtrsim 20$ Hz) and as f^{-1} at lower frequencies where finite-size effects are important. Theoretical onset curves are shown for Faraday waves [Eq. (7.14), dashed blue lines] and frozen waves [Eq. (7.17) using $g_y = 0.05 g_0$, dashed green line]. The drop ejection threshold discussed in Sec. 7.5.2 is shown in panel (a), and is extended (dashed curve) for visual reference to higher frequencies. Panel (b) includes a presumed curve (dashed red line) for drop ejection, although supported by just one experimental data point.	181

8.1	Sketch illustrating the numerical model: two horizontal superposed layers of immiscible liquids confined in a $L \times 2H$ rectangular container. The cell is subjected to vibrations of frequency ω and amplitude A parallel to the initial flat interface (dashed line). Fluid behaviour is analysed for a pair of fluids similar to recent parabolic flight experiments (Salgado Sánchez et al., 2019d).	188
8.2	Time evolution, averaged over one excitation period, of the frozen wave instability under weightlessness in a container of size $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$ with $A = 1.5 \text{ mm}$ and two different frequencies: (a) $f = 45 \text{ Hz}$ and (b) $f = 60 \text{ Hz}$, corresponding to $\mathcal{F} \approx 28.48$ and $\mathcal{F} \approx 37.97$, respectively. The time is shown between the columns. The mean velocity field is shown using scaled arrows (scale not preserved between snapshots). The representative wavelength λ in the defect region of each pattern is defined using the final snapshot at $t = 0.9 \text{ s}$. The appearance of subharmonic Faraday waves (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) on the lateral boundaries of the columnar structure is also seen at that time in column (b).	193
8.3	Instability diagram in terms of the vibrational parameter $\mathcal{F}$ for a $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$ container showing the nondimensional averaged wavenumber $\langle \mathcal{K} \rangle$ (solid vertical lines), equal to the number of column pairs along the container length, and the wavenumber $\mathcal{K}$ (solid dots) near the defect in the central part of the container. The frozen wave threshold (see Sec. 8.4.2) is distinguished by a circular gray marker. Insets illustrate the final pattern for selected points on each branch, while dashed lines provide a visual guide for the forcing interval associated with each mode.	195
8.4	(a,b) Instability diagrams in terms of the vibrational parameter $\mathcal{F}$ for fixed vertical dimension $2H = 7.5 \text{ mm}$ and different container lengths (labelled in each curve): $L = 7.5 \text{ mm}$ (red, upright triangles), 15 mm (green, inverted triangles), 30 mm (black, squares) and 60 mm (blue, circles). (a) Average wavenumber $\langle \mathcal{K} \rangle$. (b) Wavenumber $\mathcal{K}$ near the defect region. Mode transitions are shown with dashed lines for visual reference. (c) Experiment showing a frozen wave pattern in microgravity in a $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$ container of FC-40 and 20 cSt silicone oil subjected to horizontal vibrations at $f = 22 \text{ Hz}$, $A = 1.755 \text{ mm}$, $\mathcal{F} = 11.519$	196
8.5	(a) Time evolution of the interface depth d (in mm) at the centre of a $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$ container for decreasing vibrational forcing (1–7) near threshold showing (at least) three distinct regions: initial quasi-exponential growth, a transition to nearly constant growth rate, and a transition to exponential growth when the column approaches the wall. (b) Inverse square of the frozen wave growth time, τ^{-2} (grey circles), for the smallest wavenumber $\langle \mathcal{K} \rangle = 1$ as a function of the vibrational parameter $\mathcal{F}$. The linear dependence allows straightforward estimation of the instability threshold (black circle). Numbers refer to the different forcings used, as in (a). (c) Frozen wave instability threshold $\mathcal{F}_c$ as a function of the container length L for fixed height $2H = 7.5 \text{ mm}$. Fitting curves (dashed) assuming L^{-1} , $L^{-0.5}$ and $L^{-0.25}$ scalings are shown for comparison.	198

8.6	Frozen wave instability diagram in terms of the vibrational parameter $\mathcal{F}$ for a $L \times 2H = 60 \text{ mm} \times 7.5 \text{ mm}$ container. The average wavenumber $\langle \mathcal{K} \rangle$ (vertical lines) gives the number of column pairs in the container, while the local wavenumber $\mathcal{K}$ (solid dots) near the defect region reflects the growth of this thin column (defect) into a regular column. Dashed lines are included for reference to delineate the mode transitions. A change from symmetric to asymmetric column growth (both regions are labelled) is observed near $\mathcal{F} \simeq 40$	200
8.7	Density distribution (silicone oil shaded) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a rectangular container with $L \times 2H = 60 \text{ mm} \times 7.5 \text{ mm}$. Darker lines show the resolution used, which is higher for $\mathcal{F} \lesssim 50$. The column formation cycle becomes asymmetric at $\mathcal{F} \simeq 40$	201
8.8	Nondimensional wavenumber $\langle \mathcal{K} \rangle$ of the selected columnar pattern versus the vibrational parameter $\mathcal{F}$ for $L = 30 \text{ mm}$ and different container heights (labelled): $2H = 3.75 \text{ mm}$ (red, medium thickness), 7.5 mm (grey, thicker) and 15 mm (black, thinner). Container heights are labelled for reference.	203
8.9	Evolution of the frozen wave instability (averaged over one excitation period) in weightlessness in a $L \times 2H = 30 \text{ mm} \times 1.875 \text{ mm}$ rectangular container vibrated with $f = 12 \text{ Hz}$ and $A = 1.5 \text{ mm}$, corresponding to $\mathcal{F} \approx 7.59$. Snapshots are labelled by time after initiation. The mean velocity field is shown in all but the final snapshot (scaling of arrow length is not constant between snapshots). Note the markedly asymmetric evolution of the interface, especially after the first (right) corner of the cell is covered by FC-40.....	204
8.10	Density distribution (silicone oil shaded; darker lines show resolution used) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for $L = 30 \text{ mm}$ and container heights (a) $2H = 3.75 \text{ mm}$ and (b) $2H = 1.875 \text{ mm}$. The thinner layer not only displays more asymmetry, but frustrates the development of patterns with large wavelengths: modes with $\langle \mathcal{K} \rangle = 1, 2$ are not observed when $2H = 1.875 \text{ mm}$	204
8.11	Time evolution of the frozen wave instability in weightlessness in a $L \times 2H = 30 \text{ mm} \times 30 \text{ mm}$ rectangular container at $f = 60 \text{ Hz}$, $A = 1.5 \text{ mm}$, resulting in $\mathcal{F} \approx 37.97$. Snapshot times are labeled. The mean velocity field — averaged over one excitation period — is shown by scaled arrows (scaling not preserved between snapshots) for all but the final time. Asymmetric evolution driven by the harmonic motion of the lateral column crests is observed.	205
8.12	Density distribution (silicone oil shaded; darker lines show resolution used) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a deep container with $L \times 2H = 30 \text{ mm} \times 30 \text{ mm}$. Asymmetric behavior is observed quite generally for $\mathcal{F} \gtrsim 20$	206

8.13	Frozen wave threshold as a function of the inverse aspect ratio Γ^{-1} . Numerically obtained onset values (solid dots with dashed line for visual reference) were calculated for a capillary number $\text{Ca}_H = 0.0317$. The simulation values are slightly higher than the theoretical prediction (solid line) of Eq. (8.19) for $\Gamma^{-1} \lesssim 0.5$ but deviate substantially beyond that as finite size effects play a dominant role.	207
9.1	(a) Schematic view of the optical setup (left) and the experimental cell (right). (b,d) Images of the experimental cells holding immiscible and miscible liquids, respectively. (c,e) Unprocessed cropped images showing the interface in reduced gravity: (c) unforced immiscible combinations (see Table 9.1) I_a (upper container) and I_b (lower container); the lighter regions of the image correspond to FC-40 and the darker regions to silicone oil, (e) miscible liquids under horizontal (x axis) forcing: mixture M_a (upper snapshot) and M_b (lower snapshot).	215
9.2	Snapshots (times labelled) showing the temporal evolution of the frozen wave instability in miscible layers subjected to horizontal vibrations starting from a slightly tilted interface. (a–d) Mixture M_a of Table 9.1 with $f = 14$ Hz, $A = 2.5$ mm ($A\omega = 219.9$ mm/s) and initial angle $\alpha \simeq 4.5^\circ$. (e–h) Mixture M_b of Table 9.1 with $f = 8$ Hz, $A = 11$ mm ($A\omega = 552.9$ mm/s) and initial angle $\alpha \simeq 5.0^\circ$. Microgravity begins at $t \simeq 0.75$ s and $t \simeq 1.2$ s, respectively, for these experiments.	219
9.3	Snapshots (times labelled) showing the temporal evolution of the frozen wave instability in miscible layers subjected to horizontal vibrations starting from a more tilted interface. (a–d) Mixture M_a of Table 9.1 with $f = 12$ Hz, $A = 2.91$ mm ($A\omega = 219.4$ mm/s) and initial angle $\alpha \simeq 27^\circ$. (e–h) Mixture M_b of Table 9.1 with $f = 8$ Hz, $A = 11$ mm ($A\omega = 552.9$ mm/s) and initial angle $\alpha \simeq 15^\circ$. These initial angles are measured in panels (b) and (e), respectively. The apparent breaks in the interface in (c,f) are optical effects.	221
9.4	Miscible simulations are performed in a domain of length L and height $2H$. Vibrations of frequency ω and amplitude A are applied in the x direction. A solution is shown for illustration (the interface appears dark).	223
9.5	Sketch illustrating the immiscible numerical model: two superposed layers of immiscible liquids confined in a $L \times 2H$ rectangular container, displaying a finite contact angle β with the solid walls. The cell is subjected to vibrations of frequency ω and amplitude A in the x direction.	226
9.6	Numerical simulations showing the temporal evolution of the frozen wave instability in mixture M_b with $f = 8$ Hz and $A = 11$ mm ($A\omega = 552.9$ mm/s) for increasing initial interface angles: (a) 0° , (b) 5° , (c) 10° , (d) 20° and (e) 30° . The columns show snapshots at $t = 0.2$ s, 0.5 s, 1 s and 3 s. The mean velocity field (averaged over two periods) is shown for the first three snapshots (arrow scale not maintained between frames); the solution at $t = 3$ s is quasisteady. The initial interface condition is shown in the leftmost panels at $t = 0.2$ s by a solid black line, while the interface is plotted in grey.	228

9.7	Diagram showing three measured nondimensional wavelengths λ/L of the final quasisteady state versus initial interface angle α (in degrees): minimum (blue), average (red) and maximum (black). The vibrational velocity is held constant at $A\omega = 552.9$ mm/s ($f = 8$ Hz, $A = 11$ mm). Snapshots of the selected patterns are included, with the colourmap showing the concentration field (red for $C = 0$, blue for $C = 1$), as is a sketch illustrating how wavelengths are measured from the contour $C = 0.5$. Note that the snapshots are taken after a longer time of $t = 10$ s so are slightly different than those at $t = 3$ s in Fig. 9.6.	230
9.8	Numerical simulations illustrating the development of the frozen wave instability in weightless in a $L \times 2H = 30$ mm $\times$ 7.5 mm container with $f = 45$ Hz and $A = 1.5$ mm. The time of each snapshot is given under the corresponding column. The mean velocity (averaged over two excitation periods) is illustrated by scaled arrows (scale not preserved between panels) during the transient phase. The angle α of the initial interface is varied: (a) 0° , (b) 7° , (c) 7.2° and (d) 14.16° . The manner of calculating maximum ($\lambda_{\max}$) and minimum ($\lambda_{\min}$) wavelengths along the midline $y = 0$, used in Fig. 9.9, is illustrated for the final state of (c). The initial interface is shown (solid black curve) for comparison in the rightmost panels.	231
9.9	Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ (measured at $y = 0$) for two applied vibrational velocities: (a) $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm), (b) $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). The domain measures 30 mm $\times$ 7.5 mm, with aspect ratio $\Gamma = 4$. Three different wavenumbers are shown: minimum (black), average (red) and maximum (blue). Snapshots of the final selected patterns are included for illustration. The simulations used for (a) include those shown in Fig. 9.8.	233
9.10	Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ (measured at $y = 0$) for different container aspect ratios: (a) $\Gamma = 3$, (b) $\Gamma = 4$ and (c) $\Gamma = 6$. The minimum (black), average (red) and maximum (blue) wavenumbers are shown for each container at two different vibrational velocities: $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm) and $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). Snapshots of the final selected patterns are included for illustration.	234
9.11	Angle $\alpha_{\max}$ of the final mode transition in the formation of columnar structures triggered by the frozen wave instability for different container aspect ratios $\Gamma = L/(2H)$. This angle is only weakly dependent on forcing, and the values plotted are the average of those found with $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm) and 565 mm/s ($f = 60$ Hz, $A = 1.5$ mm). Examples of patterns observed before and after the transition are shown in the insets. The predicted curve based on idealised geometry, $\tan^{-1}(\Gamma^{-1})$, is included (dashed line) for comparison.	235

9.12	Snapshots showing initial (left) and final (right) interfacial patterns in experiments (a–d) with combination I_b in a parabolic flight. Vibrations at 9 Hz with $A = 4.4$ mm begin during hypergravity in panels (a,b) and during microgravity in (c,d). Images showing the final states are averaged over (approximately) two excitation periods.	236
10.1	Sketch illustrating the two-layer system of immiscible liquids. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis and parallel to the initial flat interface (dashed line).	242
10.2	Time evolution (averaged over one excitation period) of the interface and mean velocity field in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$) under normal gravity ($g = g_0$, $Bo = 81.57$) with different forcing amplitudes: (a) $v = 0.380$ m/s, (b) 0.432 m/s, (c) 0.456 m/s and (d) 0.577 m/s, which correspond to $\mathcal{F}$ values of 12.65, 14.38, 15.19 and 19.22, respectively. The forcing frequency is fixed at $f = 55$ Hz while the amplitude A is increased (1.1, 1.25, 1.32 and 1.67 mm). The critical forcing is $v_c = 0.429$ m/s ($\mathcal{F}_c = 14.25$) as shown in Sec. 10.3.2. The velocity field during the transient portion of the simulation is shown with scaled arrows (scale differs between snapshots). The selected time of the intermediate snapshot in (d) is less than that of (a–c) due to the more rapid growth of that pattern. The final pattern in (d) also illustrates how the amplitude Δh is measured from the contour $\phi = 0.5$	247
10.3	Bifurcation diagram showing the trough-to-crest amplitude Δh of the frozen wave patterns versus the vibrational velocity v for $g = g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The numerically-obtained points are fit to (a) the perturbed pitchfork bifurcation of Eq. (10.17) or (b) the square-root dependence of Eq. (10.15), both shown as solid curves, and to the linear function (10.19), shown with a dotted line. These fits correspond to the two regimes identified by Jalikop and Juel (2009), with the transition between them marked by a vertical line. The points corresponding to the snapshots of Fig. 10.2(b–d) are distinguished by solid markers and labelled. .	249
10.4	Radius of curvature r_k at the wave crests as a function of the applied velocity v for (a) $g = g_0$ and (b) $g = 0.5 g_0$. Data points are fit with a solid curve for visual reference. Asymptotic behavior is seen at large v when r_k is comparable to the capillary length (shown with a dashed line) given by Eq. (10.21). Points corresponding to the snapshots of Fig. 10.2 are labelled and distinguished by solid markers in (a), while the same is done for (b) with regard to Fig. 10.6 (shown below).	251

10.5	(a) Bifurcation diagrams showing $\Delta h(v)$ with $g = 2g_0$ ($\text{Bo} = 163.14$, squares) and $g = g_0$ ($\text{Bo} = 87.57$, circles) in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$). The initial capillary-gravity regime of each diagram, with the interval given in Table 10.2 by $(v^* - v_c)$, is shaded. (b) Bifurcation diagrams under normal gravity, $g = g_0$, in containers of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$, circles) and $L \times 2H = 60 \times 15$ mm ($\Gamma = 4$, triangles). The insets show representative patterns, to be compared with Fig. 10.2, for (a) the hypergravity case and (b) the longer container with $\Gamma = 4$	253
10.6	Time evolution (averaged over one excitation period) of the interface and mean velocity field in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$) under reduced gravity $g = 0.5g_0$ ($\text{Bo} = 40.79$). Increasing vibrational forcing is applied: (a) $v = 0.282$, (b) 0.344 , (c) 0.364 and (d) 0.410 m/s, corresponding to $\mathcal{F} = 9.41$, 11.49 , 12.14 and 13.65 , respectively. The frequency is fixed at 45 Hz with the amplitude increased from 1 to 1.45 mm. The critical forcing is $v_c = 0.347$ m/s ($\mathcal{F}_c = 11.55$); see Sec. 10.4.2. The velocity field during the transient portion of the simulation is shown with scaled arrows (scale differs between snapshots). The selected times of series (a–c) are labelled at the top while, due to more rapid growth, the intermediate snapshot in (d) is taken at 0.7 s instead of 0.9 s.	255
10.7	Bifurcation diagram showing the trough-to-crest distance Δh of the frozen wave patterns versus vibrational velocity v for $g = 0.5g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The measurements in the gravity-capillary regime are fit to the branching equation (10.17) of a perturbed pitchfork (dashed curve) and of the unperturbed ($\epsilon = 0$) case (solid curve). The gravity regime is fit to the linear function (10.19) (dotted curve). The transitions between different regimes (labelled above) are highlighted by vertical lines, including the new finite-depth regime where the crest approaches the upper boundary. Points corresponding to the snapshots of Figs. 10.6(b–d) and Fig. 10.8(a,b) are labelled. The diagram of Fig. 10.3 for normal gravity is included for comparison (gray open circles).	257
10.8	Time evolution (averaged over one excitation period) of the interface and mean velocity field illustrating the frozen wave instability for $g = 0.5g_0$ in the finite-depth regime in a container of size $L \times 2H = 30 \times 15$ mm vibrated at $f = 45$ Hz and amplitudes: (a) $A = 1.52$ mm ($v = 0.430$ m/s, $\mathcal{F} = 14.31$), (b) $A = 1.55$ mm ($v = 0.438$ m/s, $\mathcal{F} = 14.59$). The time of the snapshot is labelled between the columns. The velocity field is shown using scaled arrows (scale differs between snapshots). The Δh values obtained from these two simulations are labelled in Fig. 10.7.	259

10.9	Time evolution (averaged over one excitation period) of the frozen wave instability with $g = 0.5 g_0$ in a container of size $L \times 2H = 30 \times 15$ mm vibrated at $f = 45$ Hz with $A = 1.6$ mm ($v = 0.452$ m/s, $\mathcal{F} = 15.06$). A mode transition between the $\mathcal{K} = 3$ and $\mathcal{K} = 2$ patterns is observed. The snapshot times are labelled and the mean velocity field during the transient period is shown using scaled arrows (scale differs between snapshots). For $t \lesssim 0.5$ s the evolution is similar to that of Fig. 10.8(b).	261
10.10	(a) Evolution of the symmetric vibroequilibria state in microgravity with increasing velocity v in a container of size $L \times 2H = 30 \times 15$ mm. The difference Δh between the height of the contact points and the midpoint of the interface is shown and fitted (dashed line) to a quadratic function as per the theoretical prediction. The inset illustrates a typical vibroequilibria state and the Δh measurement. (b) Inverse square of the frozen wave growth time, τ^{-2} (open squares), for the smallest wavenumber $\mathcal{K} = 1$ versus v . The observed linear dependence allows for an estimation of the instability threshold (black marker). The inset shows the final columnar pattern resulting from the $\mathcal{K} = 1$ mode.	262
10.11	Bifurcation diagrams obtained for the $L \times 2H = 30 \times 15$ mm container with varying gravity levels (in units of g_0): 0.25, 0.5, 0.75, 1, 1.5, 2. The data sets are distinguished with different markers and labelled with the gravity level used. Note that for $g = 0.25 g_0$ and $0.5 g_0$, the finite-depth regime discussed in Sec. 10.4.2 is visible at large amplitude.	263
10.12	Dimensionless fitting parameters characterising the frozen wave bifurcation diagram as a function of the Bond number Bo: (a) critical forcing $\mathcal{F}_c$ (open circles for $\hat{\epsilon} = 0$, crosses for $\hat{\epsilon} \neq 0$) and the point of transition to the gravity regime $\mathcal{F}^*$ (solid circles), (b) branching coefficient $\hat{K}$ of the gravity-capillary regime (open circles for $\hat{\epsilon} = 0$, crosses for $\hat{\epsilon} \neq 0$), (c) slope $\hat{\mathcal{M}}$ of the gravity regime, (d) symmetry-breaking parameter $\hat{\epsilon}$. The predicted value of Eq. (10.12) for the critical forcing is shown with a solid curve in (a) while the theoretical formula for $\hat{K}$ from Eq. (10.16) is shown in (b). The dashed curve in (c) shows a fit to Bo^{-1} while the dotted curve in (d) is intended only as a visual aid. The values of $\mathcal{F}^*$ and $\hat{\mathcal{M}}$ change imperceptibly between the $\hat{\epsilon} = 0$ and $\hat{\epsilon} \neq 0$ cases.	264
11.1	Sketch illustrating the two-layer system of immiscible liquids, with the selection of FC-40 and silicone oil indicated. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis and parallel to the initial flat interface (dashed line).	272
11.2	Evolution of the frozen wave instability in zero gravity in a container of length $L = 60$ mm vibrated with $\mathcal{F} = 8.95$ (40 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots). This column growth is characteristic of the medium frequency symmetric regime discussed in Sec. 11.4.2.	277

11.3	Frozen wave instability diagram in terms of $\mathcal{F}$ for a container with $L = 60$ mm. The nondimensional average wavenumber $\langle \mathcal{K} \rangle$ (solid vertical lines) gives the number of column pairs along the midline $y = 0$. The insets illustrate the final pattern at selected points on each branch, while the dashed line marks the value $\mathcal{F} \simeq 10.7$ where the symmetry-breaking transition occurs.....	281
11.4	Evolution of the frozen wave instability in zero gravity in a container of length $L = 60$ mm vibrated with $\mathcal{F} = 10.74$ (48 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots). This column growth is characteristic of the high frequency asymmetric regime discussed in Sec. 11.4.2 with $\mathcal{F}$ close to the symmetry-breaking transition.	283
11.5	(a) Density distribution of the final steady state (lighter liquid shaded) along the horizontal midline $y = 0$ as a function of $\mathcal{F}$ for a container with $L = 60$ mm. (b) The asymmetry $\mathcal{A}$ of these same solutions. The column growth and defect locations become asymmetric at $\mathcal{F} \simeq 10.7$ (marked by the horizontal line), which is reflected in the increase of $\mathcal{A}$. For visual guidance, a moving average of $\mathcal{A}$ is shown in (b) with the dashed curve.	284
11.6	Evolution of the frozen wave instability in zero gravity in the low frequency asymmetric regime. The container of length $L = 60$ mm is vibrated with $\mathcal{F} = 1.79$ (8 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots).	286
11.7	Evolution of the frozen wave instability in zero gravity in the high frequency asymmetric regime (well above the symmetry-breaking transition). The container of length $L = 60$ mm is vibrated with $\mathcal{F} = 22.38$ (100 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots). .	287
11.8	Distribution of asymmetry $\mathcal{A}$ along the x axis, calculated using 3 mm windows, for the final columnar states in a $L = 60$ mm container beyond the symmetry-breaking transition at $\mathcal{F} \simeq 10.7$. Simulations were performed with $\mathcal{F} = 11.19$ (50 Hz, circles with solid lines), $\mathcal{F} = 13.43$ (60 Hz, squares with dashed lines), $\mathcal{F} = 20.14$ (90 Hz, diamonds with dash-dotted lines) and $\mathcal{F} = 26.85$ (120 Hz, triangles with dotted lines). The maximum asymmetry is first found in the middle, then is displaced outward at higher forcing.	288
11.9	(a,b) Density distribution of the final steady state (lighter liquid shaded) along the horizontal midline $y = 0$ as a function of $\mathcal{F}$ for rectangular containers of lengths (a) $L = 90$ mm and (b) $L = 120$ mm. The column growth becomes asymmetric at $\mathcal{F} \simeq 7.4$ (solid line) and $\mathcal{F} \simeq 6.4$ (dashed line), respectively. (c) The asymmetry $\mathcal{A}$ of these solutions (black circles for $L = 90$ mm, grey squares for $L = 120$ mm). Note that the vertical scale is logarithmic.	288

11.10	Threshold of the symmetry-breaking transition as a function of container aspect ratio Γ . The relevant asymmetric mode is indicated under the corresponding portion of the curve. The onset value for frozen waves, obtained as in Salgado Sánchez et al. (2019c), is shown for reference.	289
12.1	Selected bifurcation sets for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (12.2) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$: Hopf (H: dashed curve), saddle-node heteroclinic (SN Het), saddle-node of periodic orbit (SN Per, solid curves for $\nu \gtrsim 2.15$). Dotted vertical lines mark ν values where the Hopf frequency vanishes. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.	297
12.2	(a) Estimated location of onset (triangles) and saddle-node heteroclinic bifurcation (circles); the parameters shown are applied forcing f (in g 's) and applied frequency ω/π (in Hz). (b) Estimated modulation frequency Ω (in Hz) near onset. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.	299
12.3	(a–c) Front and lateral view snapshots showing the position of silicone oil in a cylinder vibrated at $f = 59.3$ Hz and $V = 100$ V (measured vertical velocity v_z at the accelerometer of 94.8 mm/s). Observation times are labelled with respect to the initiation of the forcing in microgravity: (a) $t = 0$ s, (b) $t = 2.021$ s and (c) $t = 2.492$ s. The contact lines along the front and rear portions of the cylinder wall are outlined in solid and dashed lines, respectively, and the location of points $\mathcal{A}$ (front contact line maximum), $\mathcal{B}$ (interior minimum), $\mathcal{C}$ (rear contact line maximum) are indicated. (d) Time evolution of contact points $\mathcal{B}$ (solid diamonds; open diamonds for the reference cell) and $\mathcal{C}$ (solid dots; open dots for the reference cell) when vibrated at 80 V ($v_z = 73.9$ mm/s; grey) and 100 V ($v_z = 94.8$ mm/s; black). The times used in panels (b,c) are marked with vertical dashed lines. (e) The vertical gravity level of the corresponding experiments.	301
12.4	(a–d) Water surface in a cuboidal container vibrated at (a) $V = 0$ V, $\eta = 0$; (b) $V = 50$ V, $\eta = 0.5201$; (c) $V = 80$ V, $\eta = 1.565$ and (d) $V = 100$ V, $\eta = 2.72$. The contact lines are highlighted (front, solid; rear, dashed). (e) Comparison of experimental results (solid dots) with vibroequilibria theory (solid black curve) and direct numerical simulations using $Bo = 6.1$. Simulations are performed including the complete motion of the cuboid (open grey circles with solid fitting curve) and ignoring the rotational motion (open grey squares with dashed fitting curve). Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.	303
12.5	(a) Mode diagram in terms of the vibrational forcing $\mathcal{F}$ for a container of length $L = 30$ mm and depth $2H = 7.5$ mm showing the number of column pairs (solid vertical lines) and the wavenumber (solid dots) near the defect in the central part of the container. Small insets illustrate the final pattern for selected points on each branch. (b) Density distribution along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a rectangular container with $L \times 2H = 60$ mm $\times$ 7.5 mm. The column formation cycle becomes asymmetric at $\mathcal{F} \simeq 40$. Reproduced from Salgado Sánchez et al. (2019c), Chapter 8.	306

12.6	(a) Frozen wave threshold as a function of the inverse aspect ratio Γ^{-1} . Numerically obtained onset values (solid dots with dashed line for visual reference) are slightly higher than the theoretical prediction (solid line) of Eq. (12.7) for $\Gamma^{-1} \lesssim 0.5$ but deviate substantially beyond that as finite size effects play a dominant role. Reproduced from Salgado Sánchez et al. (2019c), Chapter 8. (b) Threshold of the symmetry-breaking transition as a function of container aspect ratio Γ . The relevant asymmetric mode is indicated under the corresponding portion of the curve. The onset value for frozen waves, obtained as in Salgado Sánchez et al. (2019c), is shown for reference. Reproduced from Troitiño et al. (2020), Chapter 11.	307
12.7	Bifurcation diagram showing the trough-to-crest distance Δh of the frozen wave patterns versus vibrational velocity v for $g = 0.5 g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The measurements in the gravity-capillary regime are fit to the branching equation (12.8) of a perturbed pitchfork (dashed curve) and of the unperturbed ($\epsilon = 0$) case (solid curve). The gravity regime is fit to the linear function (dotted curve). The transitions between different regimes (labelled above) are highlighted by vertical lines, including the new finite-depth regime where the crest approaches the upper boundary. The diagram for normal gravity is included for comparison (gray open circles). Reproduced from Gligor et al. (2020), Chapter 10.	309
12.8	Dimensionless fitting parameters characterising the frozen wave bifurcation diagram as a function of the Bond number Bo: (a) critical forcing $\mathcal{F}_c$ (open circles for $\epsilon = 0$, crosses for $\epsilon \neq 0$) and the point of transition to the gravity regime $\mathcal{F}^*$ (solid circles), (b) branching coefficient $\hat{K}$ of the gravity-capillary regime (open circles for $\epsilon = 0$, crosses for $\epsilon \neq 0$). The predicted value of Eq. (12.5) for the critical forcing is shown with a solid curve in (a) while the theoretical formula for $\hat{K}$ from Eq. (12.9) is shown in (b). Reproduced from Gligor et al. (2020), Chapter 10.	310
12.9	Patterns generated by (horizontal) vibrational forcing in miscible water-isopropanol layers of different densities (i.e., different initial concentrations) in microgravity. (a) Typical frozen wave columnar pattern excited when the initial interface is (mostly) parallel to the forcing axis. (b) Frozen wave pattern with irregular shape, observed for a small misalignment of the initial interface. (c) Simple two-column state observed for a large inclined initial interface.	311
12.10	Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ for two applied vibrational velocities: (a) $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm), (b) $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). The domain measures 30 mm $\times$ 7.5 mm, with aspect ratio $\Gamma = 4$. Three different wavenumbers are shown: minimum (black), average (red) and maximum (blue). Snapshots of the final selected patterns are included for illustration. Reproduced from Salgado Sánchez et al. (2020), Chapter 9.	312

12.11	Successive snapshots showing the evolution of subharmonic Faraday waves on opposite sides of the frozen wave columnar structure for $f = 22$ Hz and $A = 1.76$ mm. Times are selected to best show the $\pi/2$ phase difference between these two subharmonic wave fields (1/4 period of these 11 Hz subharmonic waves is approximately 0.0227 s). Reproduced from Salgado Sánchez et al. (2019d), Chapter 7.	313
12.12	(a) Snapshot of an immiscible liquids combination (FC-40 and 2 cSt silicone oil) subjected to vibrations at $f = 50$ Hz and $A = 3.9$ mm: a complex interfacial pattern consisting of a frozen wave column, Faraday waves and drops is shown. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7. (b) Snapshots showing out-of-phase subharmonic waves oscillating over a vibroequilibria state in a half-filled cylinder holding 50 cSt silicone oil subjected to vibrations in microgravity. ...	315
12.13	Response of the fluid interface to vibrations in microgravity as a function of frequency f and amplitude A : (a) FC-40 and 2 cSt silicone oil, (b) FC-40 and 20 cSt silicone oil. Solid circles denote experimental points in three regimes: harmonic waves (yellow), Faraday waves (blue) and drop ejection (red). Open markers denote numerical results for the Faraday wave instability (blue open circles) and the frozen wave instability (green open squares). The Faraday instability scales as $f^{-1/2}$ for high frequencies ($\gtrsim 20$ Hz) and as f^{-1} at lower frequencies where finite-size effects are important. Theoretical onset curves are shown for Faraday waves [Eq. (12.10), dashed blue lines] and frozen waves [Eq. (1.2), dashed green line]. The drop ejection threshold is shown in panel (a), and is extended (dashed curve) for visual reference to higher frequencies. Panel (b) includes a presumed curve (dashed red line) for drop ejection, although supported by just one experimental data point. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7.....	317
13.1	Preliminary results of the COMFLU experiment: (a) ground test, (b) simulation. ...	322
13.2	Simulations of the MAVFLU experiment showing particle adsorption events on (a) ground and (b) microgravity.	323

List of Tables

2.1	Physical properties of the experiment fluids at room temperature: density ρ , surface tension σ , and kinematic viscosity ν . Reproduced from Fernández et al. (2017a), Chapter 4.	23
2.2	Physical properties of the experimental fluids at 25°C. (a) Miscible water–IPA combinations: density ρ , kinematic viscosity ν , and diffusion coefficient D (Mialdun et al., 2012). (b) Immiscible combinations of FC-40 and silicone oil: density ρ , kinematic viscosity ν , and surface tension σ (Someya and Munakata, 2005). Reproduced from Salgado Sánchez et al. (2020), Chapter 9.	26
3.1	Diagram showing the three ways that the $\mathbf{D}_4$ symmetry of the uncoupled problem is broken for $\mu \neq 0$	43
4.1	Characteristic physical properties of the experiment fluids at room temperature: density (ρ), surface tension (Γ), and kinematic viscosity (γ).....	90
4.2	Comparison of relevant characteristics for alternative piezoelectric actuator set-ups.	92
4.3	Camera, lens and sensor characteristics. (★) Approximate numbers that vary somewhat in final configuration.	93
5.1	Characteristic physical properties of the experimental liquids at room temperature: density ρ , surface tension Γ , and kinematic viscosity ν . These values are used in the mathematical simulations using the formulation of Sec. 5.3.	107
6.1	Results of the mesh convergence test against experimental measurements at the first three resonance frequencies. The selected mesh #4 is highlighted in bold.	139
7.1	Physical properties of FC-40 (Flourinert Liquids ^{TM}), 2 cSt silicone oil, 20 cSt silicone oil [from Someya and Munakata (2005)], and the pairings used here. Provided are: density ρ , kinematic viscosity ν and surface tension σ at 25 °C; thickness of the vibrational viscous boundary layers $\delta_\nu = \sqrt{\nu/\omega}$ for the range of parameters used; contact angle with the aluminium alloy endwalls β_A (used for numerical simulations) and with the quartz glass β_G . Both contact angles are estimated from image processing.....	155
7.2	Nondimensional parameters corresponding to the experiments of Sec. 7.2.....	159

7.3	Results of a mesh convergence test for numerical simulations of the FC-40 and 20 cSt silicone oil cell with $A = 2$ mm and $f = 20$ Hz (see Sec. 7.6.1 for the relevance of this experiment). The selected mesh #3 is in bold, with deviations measured with respect to the maximum velocity for mesh #4 over the selected profiles: $v_x(y)$ at $x = 4$ mm and $v_y(x)$ at $y = 3.75$ mm. The interface thickness is $\varepsilon = 1.5 (H/12.5)$.	160
8.1	Physical properties of 20 cSt silicone oil, FC-40, and the interface between them: density ρ , kinematic viscosity ν , surface tension σ (reproduced from Someya and Munakata, 2005), and contact angle β (selected to achieve an initial flat interface). ¹ The real kinematic viscosity is $2.2\text{e-}6$ m ² /s. It is increased to delay the appearance of subharmonic Faraday waves at high frequency.	190
9.1	Physical properties of the experimental fluids at 25°C. (a) Miscible water–IPA combinations: density ρ , kinematic viscosity ν , and diffusion coefficient D [from Mialdun et al. (2012)]. (b) Immiscible combinations of FC-40 (Flourinert Liquids™) and silicone oil: density ρ , kinematic viscosity ν , and surface tension σ [from Someya and Munakata (2005)].	216
10.1	Physical and interfacial properties of FC-40 and 100 cSt silicone oil: density ρ , kinematic viscosity ν , interfacial tension σ and contact angle β , which has been selected to achieve an initially flat interface. According to Someya and Munakata (2005), interfacial tension measurements between silicone oils and FC-40 vary between 4 – 7 mN/m. Due to the lack of experimental measurements for 100 cSt silicone oils and FC-40, we use the same interfacial tension value as in Salgado Sánchez et al. (2019c,d, 2020). We also note that the real kinematic viscosity of FC-40 is 2.2 cSt but is increased here to 5 cSt in order to delay the secondary Faraday wave instability.	244
10.2	Parameters characterising the bifurcation diagrams obtained at different gravity levels: symmetry-breaking parameter ϵ , critical vibrational velocity v_c , transition point v^* , width of square root regime $v_c - v^*$, branching coefficient K , linear fitting slope $\mathcal{M}$. Values are obtained from a least squares fit of the data to Eq. (10.17), in one case with $\epsilon = 0$, and to Eq. (10.19).	250
11.1	Selected physical properties of FC-40, 20 cSt silicone oil, and their interface: density ρ , kinematic viscosity ν , surface tension σ (reproduced from Someya and Munakata, 2005), and contact angle β (chosen for an initial flat interface). We note that the real kinematic viscosity of FC-40 is 2.2 cSt but this is increased here to 5 cSt to delay the secondary Faraday wave instability.	273

Agradecimientos, Acknowledgements

Dedico esta Tesis a mis padres. Agradezco su apoyo, confianza, y el haberme educado en la importancia de aprender, que ha sido la semilla de este trabajo.

Además, quiero agradecer a mis compañeros del E-USOC el día a día. En especial, a Álvaro, Jose Miguel y Choche, el trabajo y el éxito personal y profesional de los vuelos parabólicos. A Juan Carlos (y Dani), el hacer (y haber hecho) amena la rutina. A Vicky, la ayuda y estímulo durante todos estos años. Y a Ana, la oportunidad y confianza desde el primer momento.

I wish to thank also the people that make my stay in Brussels personally and professionally enriching. I thank my residence mates, for the great daily conviviality. I also thank Yuri, Viktor and Sasha for sharing their valuable knowledge and experience, and especially, I thank Valentina for her continuous advice, and for transmitting me her passion for science and tireless spirit.

A mis directores. A Nacho, el haber compartido su cuaderno en cualquier momento, lugar o servilleta. Y en especial, a Jeff, a quien agradezco el haberme introducido en la investigación hace ahora 6 años. Desde entonces, su ayuda y ejercicio de mentor han sido claves para mi desarrollo como investigador y la consecución de esta Tesis.

Finalmente, quiero agradecer a mis compañeros de la Escuela el haber convertido este aprendizaje en algo más que ecuaciones. A Dan y Mario, su contribución y empeño. A mi familia, equipo de rugby y amigos del campamento, su apoyo. Y a Carmen, su positividad contagiosa y sonrisa perenne.

Gracias, thank you.

This work was supported by the Ministerio de Economía y Competitividad under the Projects No. ESP2013-45432-P and ESP2015-70458-P, by the European Space Agency (ESA) through the 59th and 62nd ESA Parabolic Flight Campaigns, and by the ESA Education Office Fly Your Thesis! programme through the 65th ESA Parabolic Flight Campaign. The author also thanks the Departamento de Aeronaves y Vehículos Espaciales at the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio, Universidad Politécnica de Madrid for supporting the stay at the Microgravity Research Centre, Université libre de Bruxelles.

Resumen

Esta Tesis presenta una investigación teórica y experimental detallada de fluidos vibrados en microgravedad. Los resultados describen distintos fenómenos interfaciales, así como su interacción y coexistencia. En microgravedad, es esta interacción y coexistencia el reflejo de la respuesta aumentada de sistemas fluidos a la aplicación de vibraciones, y sugiere su uso para control y manejo activo de estos sistemas, un problema fundamental en misiones espaciales.

Se estudia la dinámica de ondas de superficie subarmónicas, por su relevancia en experimentos de ondas cross-waves, confirmando su aparición vía bifurcación tipo Hopf y modulación temporal, y en experimentos en microgravedad, donde se manifiestan como inestabilidad secundaria. Desde un punto de vista de selección de patrón interfacial, se analiza la inestabilidad de frozen waves y las estructuras columnares asociadas en microgravedad. Resultados específicos muestran la influencia sobre distintas características de la inestabilidad (umbral, selección de modos y transiciones, simetría, etc.) de un tamaño finito de contenedor y la orientación del eje de vibración respecto a la interfaz inicial, efectos inherentes a cualquier experimento, y el nivel de gravedad. Además, se demuestra la excitación secundaria de ondas tipo Faraday wave, y su desestabilización y consiguiente aparición del fenómeno drop ejection.

En cuanto a control activo de fluidos, el alineamiento perpendicular al eje de vibración característico de las superficies de vibroequilibria es especialmente relevante y sugiere, en cierto modo, gravedad artificial. Experimentos en gravedad reducida indican evidencia clara del efecto de vibroequilibria en geometría cilíndrica y rectangular, y en fluidos con distintas propiedades interfaciales y de contacto. Los resultados demuestran la tendencia general de las soluciones de vibroequilibria hacia el alineamiento perpendicular mencionado anteriormente, y confirman las predicciones teóricas y numéricas en el espacio de parámetros experimentales explorado.

Esta Tesis contribuye a la mejora del conocimiento en fluidos vibrados en microgravedad. La evaluación conjunta de los resultados sugiere que el uso de vibraciones para control activo de fluidos es posible. En línea con esta conclusión y motivado por la crisis ambiental de los microplásticos, el grupo de investigación está inmerso actualmente en el análisis del transporte de partículas a través de interfaces fluidas vibradas. Este objetivo se enmarca dentro de un proyecto más amplio que pretende investigar la absorción de microplásticos por criaturas marinas o, el problema análogo de la absorción de partículas de polvo a través de la piel; de relevancia para el futuro de la exploración espacial tripulada.

Abstract

In this Thesis, an extensive theoretical and experimental investigation of vibrated fluids in microgravity is presented. The results describe various interfacial phenomena of different nature, as well as their coexistence and interaction. The observed dynamics reflects the ease with which fluid systems respond to applied vibrations in microgravity, and suggests the possibility of using vibrations for fluid control and management in weightlessness, which is one of the fundamental challenges in space exploration.

The dynamics of parametrically forced subharmonic waves is analysed due to its relevance to experiments on cross-waves, confirming the initial Hopf bifurcation to modulated solutions, and to microgravity experiments, where these waves are observed following a secondary instability. A detailed analysis of the frozen wave instability in reduced gravity and the associated columnar structures is conducted from a pattern formation perspective. The specific results show the influence on different features of the instability (threshold, mode selection and transitions, symmetry, etc.) of finite-size effects and variable forcing orientation, both of which are intrinsic features of real experiments, and the gravity level. In addition, the appearance of secondary Faraday waves is demonstrated and, depending on forcing, their subsequent instability to drop ejection.

From the point of view of fluid management and control, the degree of interface alignment achieved by the vibroequilibria effect in weightlessness is especially suggestive of artificial gravity. Experiments show clear evidence of vibroequilibria in cylindrical and cuboidal geometries, for fluids of very different interfacial and contact properties. The results confirm the general tendency for vibroequilibria states to reorient large portions of the interface so as to be more perpendicular to the vibrational axis, and are further supported by theory and simulations over the explored range of parameters.

The present work contributes to the understanding of vibrated fluids in microgravity. Taken as a whole, the results strongly suggest that it should be possible to use vibrations for fluid control and management in microgravity environments. In line with this general conclusion, and motivated by the growing crisis of microplastics, the research group is currently analysing the use of vibrations for controlling the motion and transport of small particles through fluid interfaces. This objective is part of a more ambitious project aiming to investigate the adsorption of microplastics by marine creatures, or the analogous problem of dust adsorption through epithelia, which is relevant to manned space exploration missions.

Foreword

This Thesis is presented for the degree of Doctor of Philosophy, with International distinction, in Aerospace Engineering at the Universidad Politécnica de Madrid following the structure “Thesis by compendium of publications” (Tesis por compendio de publicaciones).

All of the publications related to the Thesis are collected in Part II in the form in which they were accepted by the journals. Although some effort was made to preserve the style and nomenclature (symbols, abbreviations, etc.) between these different articles during the publication process, this was not possible in all cases. Thus, some variations in nomenclature exist in the final accepted versions of these articles.

In the remaining Parts of the Thesis (Parts I and III), a uniform style is used with symbol definitions consistent with the majority of the collected articles. When this differs from that used in the corresponding published article, the reader is referred to the dedicated Chapter to avoid any misunderstanding.

Finally, it should be acknowledged that part of this Thesis (Chapters 7 to 10) is the result of a joint collaboration between the Aerospace Science and Operations Research Group at the Universidad Politécnica de Madrid, and the Nonlinear Phenomena in Fluids Research Group, led by Dr. Valentina Shevtsova, at the Microgravity Research Centre, Université libre de Bruxelles. The contributions of this Thesis to the corresponding Chapters were in the processing, analysis and interpretation of experimental results, and the development and application of the numerical model for vibrated immiscible fluid layers.

Part I

INTRODUCTION AND METHODOLOGY

Chapter 1

Introduction

By their nature, fluids move easily, adjusting their shape and position in response to external forces. This motion, for finite volumes, is defined by the dynamics of the interfaces that determine its free boundaries. These interfaces may separate one liquid from another, or from a gas such as air. The dynamics associated with an interface separating two fluids is important in a wide variety of disciplines ranging from biological systems, like the convective flow described by swimmers (bacteria) searching for oxygen at the free surface of an open container (Sokolov et al., 2009), to engineering applications, like liquid storage, solvent extraction, oil recovery, mixing or microfluidics, among others.

Vibrated fluids on Earth: the distinction between vertical and horizontal vibrations

On Earth, the most prominent (large-scale) external force is gravity, and fluid systems generally evolve so as to minimise their gravitational potential energy. Whether part of the natural landscape or held in a large container, a quiescent body of liquid minimises its height by pooling in the lowest configuration possible, maintaining the familiar sight of a (nearly) flat horizontal interface with the surrounding air (or another lighter fluid). Although this picture is complicated by surface tension on small length scales and by other physical processes like buoyancy, diffusion or evaporation, the basic flat interface that is selected in the most common situations is remarkably simple.

There are many different types of external forces that can perturb or disrupt the initial (static) equilibrium configuration of a fluid. Periodic forcing, in particular, is of interest both for its mathematical simplicity, and for its approximation of natural cycles and of vibrations as filtered through the resonance modes of adjacent structures. Even in the relatively simple case of ideal sinusoidal vibrations, a variety of distinct phenomena can occur on ground in two-fluid systems depending on the vibrational frequency and amplitude, and critically, on their orientation with respect to the flat (unperturbed) interface.

Vertical vibrations, which are (assumed to be perfectly) perpendicular to the interface, do not directly induce fluid motion. The liquid remains still until a certain forcing value is surpassed

that, as viewed from a reference frame moving with the container, can be treated as a parametric modulation of the effective gravity. They are able to both suppress instability — stabilising the corresponding Kapitza pendula (Krieger, 2017) or the Rayleigh-Taylor instability (Wolf, 1970) — and induce instability. At sufficient amplitude, this type of excitation leads to the phenomenon of Faraday waves, which are standing waves oscillating at half the excitation frequency (Faraday, 1831).

Under the simplifying assumptions of potential inviscid flow and the homogeneity of the forcing mechanism that follows from ignoring boundary effects for a large (flat) horizontal interface, Benjamin and Ursell (1954) described the Faraday instability by the well-known Mathieu equation. The complicating effect of viscosity was later accounted for by Kumar and Tuckerman (1994), who used a Floquet analysis of the linearised Navier-Stokes equations to determine the instability threshold and the natural frequency of oscillations,

$$\omega_0^2 = \frac{(\rho_1 - \rho_2)gk + \sigma k^3}{\rho_1 \coth(kH_1) + \rho_2 \coth(kH_2)}, \quad (1.1)$$

in a system of immiscible liquid layers, where $\rho_1, \rho_2 (< \rho_1)$ are the fluid densities, g is the gravitational acceleration, k is the wavenumber, σ is surface tension, and H_1, H_2 are the liquid layer depths. Equation (1.1) is generally known as the capillary-gravity dispersion relation and, in the limit of a liquid-gas system ($\rho_1 \gg \rho_2 \simeq 0$), it simplifies to the expression obtained in the seminal work of Benjamin and Ursell (1954) four decades before.

The subharmonic resonance condition $\omega_0 = \omega/2$, where ω is the applied forcing frequency, determines the approximate wavelength of the excited Faraday waves in sufficiently deep systems. Their onset occurs at a finite forcing value in the presence of viscosity and other sources of damping and, in an extended system, leads to a pattern selection problem as the (infinite number of) modes lying on the critical circle interact and compete with each other. For single-frequency forcing, this competition typically results in patterns of squares, rolls, or hexagons (Miles and Henderson, 1990; Prinnet, Juric, and Tuckerman, 2009), while for multi-frequency forcing, a much wider variety of patterns may be observed, including superlattices and quasipatterns (Arbell and Fineberg, 2002; Edwards and Fauve, 1994; Kudrolli, Pier, and Gollub, 1998). Weakly-nonlinear theory can be used to understand many aspects of this pattern forming transition (see, e.g, Skeldon and Rucklidge, 2015), which occurs via a symmetry-breaking pitchfork bifurcation.

Horizontal vibrations, defined as those parallel to the interface, are more complicated to treat since they immediately induce motion near the walls or other solid boundaries, which act as wavemakers to produce harmonic surface waves. Subharmonic waves, similarly to Faraday waves, may still arise via a parametric forcing mechanism (Garrett, 1970; Jones, 1984; Miles and Henderson, 1990; Tinao et al., 2014), but these are typically localised near the moving boundaries where this inhomogeneous forcing mechanism is strongest. Due to their cross-wise orientation with respect to the initial harmonic waves, they are referred to as cross-waves. In horizontally vibrated rectangular containers, the subharmonic wave patterns generated near opposite endwalls penetrate into the interior of the

domain and interact. The extent of this interaction for a given container length depends on amplitude, frequency and damping, and the interaction (coupling) can lead to interesting modulations and dynamics in the weakly-nonlinear regime (Perez-Gracia et al., 2014; Varas and Vega, 2007). In the limit of weak coupling, the dynamics of the system may be described by reduced-order amplitude equations (Perez-Gracia et al., 2014; Tinao et al., 2014) and, if the spatial dependence of these patterns is ignored at leading order, then the temporal behaviour (modulations) can be described by simple models of coupled parametrically forced oscillators.

In most vibrated fluid systems, the subharmonic interfacial (surface) waves, once excited, are dominant, and their nonlinear interaction becomes increasingly important as the vibrational amplitude increases. At a certain point, drop formation may follow from the destabilisation of these large-amplitude (subharmonic) waves, a process wherein the liquid layer absorbs energy from the inertial force and channels it into the ejection of drops from the interface. It is notable that the pulsation of the generated drops does not inherit the subharmonic character of the underlying wave, but oscillates harmonically at the frequency of the excitation.

There have been a number of studies of vibration-induced drop ejection from the surface of a liquid layer (see, e.g., Goodridge et al., 1997) or an initially large liquid drop on a vertically vibrating substrate (James, Smith, and Glezer, 2003). The work of Li and Umemura (2014) showed that in a deep liquid layer, the threshold condition for the formation of a spray of droplets is determined only by the forcing strength, and is independent of the initial conditions of the liquid surface.

In addition to interfacial waves and possible drop ejection, the harmonic motion of the interface near the system boundaries (one fluid penetrating into the other) that exists in stratified layers of liquids of different densities, is further associated, as per volume conservation, with a counterflow in the more quiescent interior. In a rectangular container, this shear counterflow causes a Kelvin-Helmholtz type instability that generates a spatially periodic wave pattern at the interface once a critical forcing amplitude is surpassed. Since this pattern is stationary in the noninertial reference frame of the container, it is referred to as a frozen wave.

In the limit of infinitely extended layers, Lyubimov and Cherepanov (1986) used linear theory to derive the critical onset of frozen waves for two inviscid immiscible liquids (of equal depth, H)

$$(A\omega)^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)^2} \left(\sigma k + \frac{(\rho_1 + \rho_2)g}{k} \right) \tanh(kH), \quad (1.2)$$

where A and ω are the forcing amplitude and frequency. Since this seminal work, the effect of horizontal vibrations on two superposed immiscible liquids has been investigated both experimentally (see, e.g., Jalikop and Juel, 2009; Talib, Jalikop, and Juel, 2007; Yoshikawa and Wesfreid, 2011) and theoretically (see, e.g., Khenner et al., 1999; Lyubimov et al., 1998). The recent experiments of Gaponenko et al. (2015a) and Shevtsova et al. (2016) demonstrated that frozen waves can also develop at the diffusive interface between miscible liquids of similar (but non-identical) viscosities and densities, where there exists an effective surface tension (Lacaze et al., 2010). The possibility

of stabilising the Rayleigh-Taylor instability in miscible liquids has been recently discussed by Wolf (2018).

Even in the absence of instabilities, vibrations affect the quasi-steady (time-average) interface between fluids, as has been known since the work of Wolf (1969, 1970). It is the inhomogeneous oscillatory velocity field, and the associated difference in dynamic pressure, that causes the interface to move towards a new quasiequilibrium. This phenomenon, known as the vibroequilibria effect, was first observed by Faraday (1831) in the flattening of drops beneath a vibrating plate.

The usual theoretical approach to vibroequilibria (Beyer et al., 2001; Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997) assumes a separation of timescales between the high-frequency forcing and the slower viscous diffusion of momentum. If the flow is further assumed to be potential, then the vibroequilibria states can be obtained as minima of a Lagrangian with terms describing vibrational, gravitational, interfacial and contact energies:

$$\mathcal{L} = \eta \int_{\mathcal{V}} |\nabla \phi|^2 d\mathcal{V} + \text{Bo} \int_{\mathcal{V}} z d\mathcal{V} + \int_S dS - \sin \beta \int_W dW + \dots \quad (1.3)$$

Here, ϕ is the velocity potential, $\mathcal{V}$ and S denote the fluid volume and interface, W denotes the solid boundaries, and β is the contact angle. The relative importance of each contribution to the system energy is captured by the dimensionless parameter η

$$\eta = \frac{(\rho_1 - \rho_2)l_c}{4\sigma} (A\omega)^2, \quad (1.4)$$

characterising the ratio of vibrational energy to surface energy, and the Bond number

$$\text{Bo} = \frac{(\rho_1 - \rho_2)l_c^2}{\sigma} g, \quad (1.5)$$

characterising the importance of gravitational to surface energy, where l_c refers to a characteristic length. The vibroequilibria solution that follows from minimising Eq. (1.3) reveals a global reorientation of the average interface that, in the limit of high forcing, favours an orientation that is more perpendicular to the axis of vibration. As measured by η , the vibroequilibria effect scales with the square of vibrational velocity (Fernández et al., 2017b). With high Bond numbers $\text{Bo} \gg 1$ (i.e., strong gravity) vibroequilibria effects are only significant at large forcing amplitudes.

Vibrated fluids in microgravity

Under reduced gravity conditions, fluids behave differently than they do on Earth. Surface tension effects are no longer masked by gravity and the preference for flat density contours (interfaces/surfaces) that results from a minimal gravitational energy is lost. Instead, a free-floating mass of liquid will minimise its surface energy by taking a spherical shape, while contact forces may cause a confined liquid to fully cover (wet) the solid boundaries, i.e., minimise its contact energy.

In the absence of gravity, therefore, the aforementioned distinction between horizontal and vertical vibrations is of minimal use since most surfaces will be curved. The forcing mechanisms driving interfacial phenomena will generally be inhomogeneous and, among other things, this means that the clear distinction between vertical and horizontal vibrations, typical of ground experiments, is blurred. Interfacial phenomena like the Faraday waves associated with uniform vertical forcing, the cross-waves appearing in horizontally forced systems, or frozen waves, are altered. Nonetheless, depending on frequency, amplitude, and the (local) orientation of certain portions of fluid interfaces with respect to the forcing axis, the variety of behaviours that vibrated fluids exhibit in reduced gravity can often be identified with the phenomena mentioned above and their interaction.

The experiments of Beysens et al. (2009) and Beysens (2014) in sounding rockets — sponsored by the European Space Agency — were the first to observe periodic stripe patterns perpendicular to the axis of vibration in supercritical CO₂. Many of these investigations were continued on ground in near-critical fluid using magnetic compensation of gravity, with the most recent among them aiming to study either the Faraday wave (Gandikota et al., 2014a) or frozen wave (Gandikota et al., 2014b) instability. These experimental efforts have been complemented by a significant amount of numerical and theoretical work (Amiroudine and Beysens, 2008; Jounet et al., 2000; Lyubimova et al., 2017; Sharma et al., 2019). It should be noted that the supercritical state in a liquid-vapour system is achieved by small modifications in temperature on the order of several millikelvins. In near-critical fluids, several thermophysical properties like the liquid-vapour density difference and the interfacial tension tend to zero, a situation analogous to that of miscible liquids.

Recent experiments using ordinary miscible liquids (combinations of water and isopropanol) carried out by Gaponenko et al. (2015b) showed that, under the reduced gravity environment of parabolic flights, a saw-tooth wave structure related to the frozen wave instability is initially formed on the interface, as occurs on ground. The frozen wave amplitude grows over time until the interface reaches the upper and lower walls. The final long-lived pattern consists of a series of vertical columns of alternating liquids. A further increase of vibrational forcing leads to Faraday waves being excited as a secondary instability on the primary columnar pattern (Shevtsova et al., 2016). These subharmonic waves appear at lower frequency when the driving amplitude is comparable with the height of the liquid layer. In this binary system, the increase of the initial diffusive interface width prior to the frozen wave development promotes the selection of a smaller wavenumber for column formation and, in the limit of large width, a transition to a vortical structure is observed (Gaponenko et al., 2018).

Therefore, the frozen wave instability leads to dramatically different patterns in microgravity compared to ordinary gravity. Some of these features are anticipated in the results of classical frozen wave theory. It follows from the work of Lyubimov and Cherepanov (1986) that a finite frozen wave threshold requires finite gravity; Eq. (1.2) predicts a vanishing threshold and wavenumber $A\omega$, $k \rightarrow 0$ for $g = 0$. Since the threshold of the longwave mode ($k = 0$) is much lower in reduced gravity, mode selection is also more complicated. A given applied forcing may be sufficient to excite an interval of unstable modes. This scenario was treated in the work of Lyubimova et al. (2017) by

maximising the growth rate over that interval and it was found that, although viscosity can have a significant effect, there is a monotonic increase of the observed wavenumber with forcing.

Even more recently, ground and parabolic flight experiments were carried out for a pair of immiscible liquids subjected to vertical vibrations, with the main objective of studying the influence of capillarity and gravity on the Faraday wave instability (Diwakar et al., 2018). The combination of liquids used, FC-72 and 1 cSt silicone oil, become miscible at an upper critical consolute point (Ancherbak et al., 2018). The low-gravity experiments, conducted with frequencies between 3 Hz and 8 Hz at several temperatures approaching the consolute point, showed that the critical amplitude increases with temperature, in contrast to what one would expect with a gradual reduction of the surface tension value. However, this effect can be partly explained by the simultaneous reduction in density that occurs when approaching the consolute point. These experiments also confirmed the existence of a crossover frequency, predicted by linear stability analysis (Batson, Zoueshtiagh, and Narayanan, 2013), where the interface in low gravity switches from being less to more stable than it is on ground.

The phenomenon of drop ejection, which was anticipated above to occur with the destabilisation of high-amplitude surface waves, is of much practical interest in weightlessness. Previous studies analysed threshold conditions and generally ignored gravitational effects despite being validated by ground experiments; drops were comparable in size to the capillary length. In microgravity, experiments with capillary tubes showed that drops are much larger than with gravity (Wollman et al., 2016), consistent with the increase of the capillary length. The numerical results of James, Smith, and Glezer (2003) also examined the dependence of drop volume on Bond number.

The absence of a strong gravitational force has direct implications for the vibroequilibria phenomenon as well, which is no longer suppressed by the high gravitational cost of changing surface shape (height), as reflected in the energy budget described by Eq. (1.3). In microgravity, vibrational energy primarily competes with surface and contact energy, and these may be comparable in magnitude for small to moderate forcing amplitudes in moderate to large systems.

Few experiments have directly studied vibroequilibria. Ganiev, Lakiza, and Tsapenko (1977) investigated the dynamic behaviour of water, oil, and air mixtures in partially filled cylindrical cells. The liquid-air interface was subjected to both low- and high-frequency vibrations using an electrodynamic shaker. At low frequencies (8 Hz, 12 Hz), relatively disordered behaviour, involving multiple modes and oscillating bubbles, was observed. High-frequency vibrations (600–800 Hz), on the other hand, led to the formation of deep funnels or craters. This phenomenon, which would now be attributed to the effect of vibroequilibria, was strongly resonant in character. In addition, although not directly investigating the effect of vibrations, the Capillary Flow Experiment (CFE) (Jenson et al., 2010; Weislogel et al., 2009) performed on the International Space Station (ISS) demonstrated the importance of boundary conditions at the contact line (i.e., contact energy) in microgravity, which are critical for determining the preferred static equilibria in confined fluids.

On the theoretical side, the predictions of vibroequilibria theory for horizontally vibrated containers were tested against direct numerical simulations by Fernández et al. (2017b) and very good agreement was found with large enough damping. In low viscosity fluids, vibrations excite more prominent surface waves that can destabilise the underlying vibroequilibria state through the appearance of sustained sloshing motion. Even without the complications of surface wave dynamics, an interesting dependence of the symmetric connected vibroequilibria state on fluid depth was found. This state develops as a large central trough (crater) with the fluid pushed towards the vibrating endwalls as forcing is increased. For shallow fluid layers, this central dip gradually approaches the container bottom, but for sufficiently deep layers, the symmetric vibroequilibria state disappears through a violent and unpredictable transition in which the central trough rapidly plunges towards the bottom.

In addition to interfacial phenomena, the presence of (small-amplitude) vibrations under weightless conditions can substantially alter other physical processes too. Experiments using the ALICE-2 instrument on the MIR space station (Garrabos et al., 2007; Zyuzgin et al., 2001) examined the influence on heat propagation of low-frequency vibrations (0.2 Hz, 1.6 Hz) and observed a strong effect. Parabolic flight experiments (Mialdun et al., 2008; Shevtsova et al., 2010a,b) demonstrated vibration-induced convection (so-called thermovibrational convection) in non-uniformly heated fluid using excitation frequencies between 1–12 Hz.

An extension of these efforts was undertaken on board the ISS with the execution of the Influence of Vibrations on Diffusion in Liquids (IVIDIL) experiments (Shevtsova, 2010; Shevtsova et al., 2011, 2015b), which aimed to investigate the effects of vibration on diffusion and convection. The IVIDIL experiments measured diffusion and thermodiffusion coefficients for two binary mixtures. It was found that vibrations of constant frequency and amplitude can generate mean flows that influence both processes, and distinguished two mechanisms of mass transfer depending on the applied vibrational forcing. With low forcing, mass transport was controlled by the Soret effect. When the forcing was high, the transport mechanism was essentially convective and the Soret effect was localised in the boundaries generating the heat flux. The importance of g-jitter for diffusion experiments was also evaluated and found to be largely negligible; only events like docking or orbital corrections have a notable impact on such experiments.

The various results described above serve to highlight some of the interesting fluid (interfacial) phenomena that can be observed in the presence of vibrations in microgravity environments.

The potential of vibrations for fluid control and management

In reduced gravity, the selection of curved hydrostatic surfaces due to surface tension, combined with the ability of fluid masses to significantly adjust their shape in response to external forcing, means that fluid management is substantially more complicated than it is on Earth. Although vibrations are frequently undesired in microgravity, they may also be put to good use. Vibrations

are under consideration to help reduce muscle and bone loss in astronauts and can mimic the effects of gravity in other ways too (Beysens, 2006). Vibrations drive mean flows that affect heat and mass transfer, among other important physical processes. Further the understanding of the various effects of vibrations on fluids should facilitate using these effects for advantage in microgravity — not only to avoid unwanted phenomena, but to deliberately manipulate fluids, a critical challenge for space exploration.

On Earth, the ability to manipulate fluids is essential for a wide range of engineering and technological applications in areas like materials processing, biochemistry or medicine. The field of microfluidics, in particular, has attracted increasing attention due to the many advantages of automated lab-on-a-chip devices (Samiei, Tabrizian, and Hoorfar, 2016).

Active fluid control, via an applied force of some type, has a long history highlighted by the famous experiment of Millikan (1913) that measured the elementary electric charge by levitating charged droplets of oil in a static electric field. An example of more recent work in flow control is the use of electric discharge plasmas in jets or films (Audier et al., 2016).

Periodic forcing offers more potential for control due to frequency dependence and resonance effects. Acoustic levitation, for example, can overcome gravity for small liquid drops (and particles) and has progressed to the point that, with spatio-temporal modulation of the field, lateral transport can be achieved as well (Foresti et al., 2013); applications include containerless processing and substrate-free chemical reactions. In the context of oscillatory forcing, the direct mechanical vibration of the container or support can also be used to manipulate fluids. As mentioned above, not only can such excitation be used to generate or suppress interfacial phenomena like Faraday waves (Miles and Henderson, 1990), cross-waves, frozen waves, and the Rayleigh-Taylor instability (Wolf, 1969), but it can even be used to drive drops up an incline against the force of gravity (Brunet, Eggers, and Deegan, 2007).

When it is the force of gravity that needs to be overcome, most control approaches are restricted, in practice, to relatively small fluid masses. This situation changes drastically in microgravity environments, where surface and contact forces are dominant, and introduces the possibility for testing fluid control strategies on relatively large fluid volumes and with less applied forcing (energy). With appropriate container design and choice of excitation, it should be possible to use vibrations for control and management of fluids in space. The vibroequilibria effect, in particular, suggests artificial gravity since, as forcing amplitude increases, the interface typically evolves towards a more perpendicular alignment with respect to the axis of vibration, thereby restoring a preference for flat interfaces.

1.1 Thesis objectives

The principal objective of this Thesis is to study vibrated fluids and the dynamics they exhibit, and analyse, through both theory and experiments, the extent to which vibrations can be used to manipulate liquids in microgravity. The work described in the following sections is organised around three significant interfacial phenomena: subharmonic waves, vibroequilibria and frozen waves — and their interaction.

The primary objectives include:

1. A study of the dynamics of interfacial waves near the subharmonic resonance. The problem is analysed theoretically using amplitude equations (a reduced order model) and numerical simulations. Results are compared with available experiments.

This objective is fulfilled in Chapter 3.

2. A study of the vibroequilibria effect in microgravity, with particular objectives as follows:

- (a) Design, development and execution of the Control of Fluid in Microgravity with Vibrations (CFVib) experiment.
- (b) Analysis of experimental results to determine the feasibility of managing typical fluid configurations with small-amplitude vibrations in microgravity.
- (c) Development of theoretical and numerical models to predict the effect of vibroequilibria, and compare them against experiments.

This objective is fulfilled in Chapters 4 to 6.

3. A study of interfacial phenomena in immiscible liquids in microgravity, with particular objectives as follows:

- (a) Analysis of the Faraday instability and drop ejection in microgravity. Comparison of experiments and theory.
- (b) Analysis of the frozen wave instability: comparison of experiments and theory, analysis of the effects of finite container size, forcing orientation and gravity level.

This objective is fulfilled in Chapters 7 to 11

1.2 Thesis structure

This Thesis is organised as follows. First, the different methods used throughout this work are described in Chapter 2. We draw attention to the most relevant methods for the primary objectives of the Thesis. From the theoretical side, analytical methods are described with particular derivations and examples, while numerical methods are summarised presenting the level-set-based formulation of the Navier-Stokes equations. From the experimental side, the design and setup of the experiments used to obtain the associated results shown here are described.

The publications associated with the Thesis are collected in Part II.

Chapter 3 includes the article *Dynamics of weakly coupled parametrically forced oscillators*.

Chapters 4 to 6 includes the publications related to the vibroequilibria effect:

(Chap. 4) *The CFVib experiment: Control of Fluid in microgravity with Vibrations*

(Chap. 5) *Vibroequilibria in microgravity: comparison of experiments and theory*

(Chap. 6) *Simulating vibrated fluids and vibroequilibria in the CFVib microgravity experiment*.

Chapters 7 to 11 collect the articles related to the Faraday instability, drop ejection, and the frozen wave instability:

(Chap. 7) *Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity*

(Chap. 8) *Finite-size effects on pattern selection in immiscible fluids subjected to horizontal vibrations in weightlessness*

(Chap. 9) *Effect of initial interface orientation on patterns produced by vibrational forcing in microgravity*.

(Chap. 10) *Influence of gravity on the frozen wave instability in immiscible liquids*.

(Chap. 11) *Symmetry breaking in large columnar frozen wave patterns in weightlessness*.

Finally, the principal results of the Thesis are discussed together in Part III, where an integrated view of the main findings of this Thesis is provided with final conclusions. This final Part shows the thematic integrity of this work and suggests possibilities for future research.

Chapter 2

Methodology

In this Chapter, the scientific methods that are used in the Thesis are described. These are organised according to whether they are theoretical or experimental, and selected due to their relevance for each of the main objectives of the Thesis.

2.1 Theoretical methods

Two theoretical methods are included. First, the use of bifurcation theory, as applied to coupled oscillator models, is exemplified with analytical derivations. Following this, the formulation used to reproduce the dynamics at the interface between two immiscible layers, based on a level-set scheme, is described.

2.1.1 Stability and bifurcations of a parametrically forced oscillator

As detailed in Chapter 3, oscillator models are a powerful tool for modelling many physical systems with restoring forces in a simple and elegant manner. They apply to basic mechanical devices like a mass on a spring, but are also relevant to more complex problems, like the wave patterns generated by periodic forcing in fluid systems.

In the case of a parametrically forced oscillator, which is applicable to Faraday and cross-waves, the nonlinear dynamics can be approximated by a single (damped) Mathieu equation; see Eq. (3.1). By assuming that the oscillator is weakly damped and that it oscillates near the primary subharmonic resonance, the amplitude equation

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2A + \dots, \quad (2.1)$$

which has been rescaled to set the damping coefficient to unity, is obtained. Here, $A = A_r + iA_i$ is the (rescaled) complex amplitude, ν is the detuning, f is the forcing, and a refers to the real (dissipative) part of the cubic nonlinear term. Equation (2.1) describes the slow temporal dynamics of the oscillation, occurring over a timescale greater than the subharmonic period. To facilitate analysis, it may be written in terms of the norm and phase of A , $r = |A|$, $\theta = \tan^{-1}(A_i/A_r)$ to

obtain

$$\dot{r} = r \left(-1 + f \cos 2\theta - ar^2 \right), \quad (2.2a)$$

$$\dot{\theta} = \left(\nu - f \sin 2\theta - r^2 \right). \quad (2.2b)$$

Steady states and bifurcations

It is straightforward to show that the steady states of Eqs. (2.2) are the trivial solution $r = 0$ and the roots of

$$r^4 (a^2 + 1) + 2r^2 (a - \nu) + 1 + \nu^2 + f^2 = 0, \quad (2.3)$$

given by

$$r_s^2 = \frac{\nu - a}{1 + a^2} \pm \sqrt{\frac{f^2}{1 + a^2} - \left(\frac{1 + \nu a}{1 + a^2} \right)^2}. \quad (2.4)$$

Note that Eq. (2.4) requires $f^2 \geq (1 + a\nu)^2 / (1 + a^2)$.

The primary bifurcation of Eqs. (2.2) can be obtained directly from the linear problem (ignoring the cubic terms). The unstable eigenvalue is $\lambda = -1 + \sqrt{f^2 - \nu^2}$ and thus, the primary instability takes the form of a pitchfork bifurcation along the curve

$$\text{pitchfork : } f^2 = 1 + \nu^2. \quad (2.5)$$

This bifurcation destabilises the trivial solution and generates two symmetric steady states $\pm r_s$ given by Eq. (2.4).

The character of the pitchfork bifurcation changes with ν . If $\nu < a$, then only the solution associated with the '+' sign of the square root term of Eq. (2.4) is realised, and the pitchfork is supercritical. If $\nu > a$, the two solutions associated with the square root both exist, and the bifurcation is subcritical. The square root term vanishes along the half-line

$$\text{saddle-node: } f^2 = \frac{(1 + a\nu)^2}{1 + a^2}, \quad \nu > a \quad (2.6)$$

marking the location of a saddle-node bifurcation that stabilises the initial subcritical branch of unstable steady states created at the pitchfork.

To check for the existence of other instabilities, the system can be linearised around a generic (r_0, θ_0) steady solution. The Jacobian of the associated linear system

$$\mathbf{J} = \begin{pmatrix} f \cos 2\theta_0 - 1 - 3r_0^2 a & -2r_0 f \sin 2\theta_0 \\ -2r_0 & -2f \cos 2\theta_0 \end{pmatrix}, \quad (2.7)$$

then determines the stability. It is possible to demonstrate that, if $a \geq 0$, there are no additional bifurcations.

The case of $a < 0$ is qualitatively different. Even in the absence of forcing, with $f = 0$, it is evident from Eqs. (2.2) that a fixed-norm rotating solution with $r^2 = -1/a$ and frequency $\dot{\theta} = -\nu + 1/a$ exists. Furthermore, the pitchfork and saddle-node bifurcations derived above are accompanied by an additional secondary Hopf bifurcation and a set of global bifurcations (see Chapter 3 for details). The existence and character of this Hopf bifurcation is analysed below.

Secondary Hopf bifurcation: existence and character

The linear stability problem associated with $\mathbf{J}$ reads

$$|\mathbf{J} - \lambda \mathbf{I}| = \begin{vmatrix} J_{11} - \lambda & J_{12} \\ J_{21} & J_{22} - \lambda \end{vmatrix} = 0, \quad (2.8)$$

where $J_{11} = f \cos 2\theta - 1 - 3r^2 a$, $J_{12} = -2rf \sin 2\theta$, $J_{21} = -2r$ and $J_{22} = -2f \cos 2\theta$, and the eigenvalues determining the stability of the system are given by

$$\lambda = \frac{J_{11} + J_{22} \pm \sqrt{(J_{11} + J_{22})^2 - 4|\mathbf{J}|}}{2}. \quad (2.9)$$

The Hopf bifurcation requires $J_{11} + J_{22} = 0$ and $|\mathbf{J}| > 0$. Note that the subscript '0' on r and θ has been dropped for simplicity.

From the inequality on the determinant, it follows that $r^2 > (\nu - a)/(1 + a^2)$, which implies that the Hopf bifurcation always appears above the saddle-node bifurcation. The condition $J_{11} + J_{22} = 0$ can be expressed as $f \cos 2\theta = -1 - 3ar^2$ which, together with Eq. (2.2a), yields $r_H^2 = -1/(2a)$; the Hopf bifurcation occurs at a fixed norm r_H that depends only on a and requires $a < 0$.

The Hopf bifurcation occurs when the steady states given by Eq. (2.4) reach the amplitude $r_H^2 = -1/(2a)$. Thus, by equating r_H and Eq. (2.4), the location of the Hopf bifurcation is derived

$$\text{Hopf: } f^2 = \frac{1}{4} + \left(\frac{1 + 2a\nu}{2a} \right)^2. \quad (2.10)$$

The Hopf frequency is given by $\Omega = \text{Im}(\lambda) = \sqrt{|\mathbf{J}|}$.

Finally, a (co-dimension two) Bogdanov-Takens point lies on the intersection between the saddle-node and Hopf bifurcations

$$\text{Bogdanov-Takens: } f = \frac{\sqrt{1 + a^2}}{2}; \quad \nu = \frac{a^2 - 1}{2a}. \quad (2.11)$$

The bifurcations derived above are illustrated in Fig. 2.1 for $a = -0.3$. The primary pitchfork (P), saddle-node (SN), Hopf (H) and Bogdanov-Takens point (BT) are labelled and complemented by additional (numerically obtained) bifurcation sets (see Chapter 3 for details).

FIGURE 2.1: Bifurcation sets in the case of $a = -0.3$: pitchfork (P: thick solid curve), saddle-node (SN: thin solid line), Hopf (H: lower dashed curve), saddle-node of periodic orbit (SN Per: upper dashed curve), homoclinic gluing (Hom Gluing: thin solid curve above pitchfork), homoclinic (Hom: lower thin solid curve below pitchfork), heteroclinic (Het: upper thin solid curve below pitchfork), saddle-node heteroclinic (SN Het), and Bogdanov-Takens point (BT: solid dot). Reproduced from Salgado Sánchez et al. (2016), Chapter 3.

To determine whether this Hopf bifurcation is subcritical or supercritical, we calculate the first Lyapunov coefficient following Kuznetsov (2004). The system is linearised around (A_0, f_0) by writing $(A, f) \rightarrow (A_0 + A, f_0 + f)$, where A_0 refers to the steady solution at the Hopf bifurcation, and f_0 is given by Eq. (2.10).

The associated linear problem can be expanded in Taylor series as

$$\dot{A} = \mathbf{Q}_1(A) + \left(\frac{1}{2!}\right) \mathbf{Q}_2(A, A) + \left(\frac{1}{3!}\right) \mathbf{Q}_3(A, A, A) + \dots, \quad (2.12)$$

where $\mathbf{Q}_i$ are functions given by

$$\mathbf{Q}_1(A) = \mathbf{J}(A_0) \begin{pmatrix} A_r \\ A_i \end{pmatrix}, \quad (2.13)$$

$$\mathbf{Q}_2(A, A) = -2 \begin{pmatrix} 3aA_{r0} - A_{i0} & -A_{r0} + aA_{i0} & -A_{r0} + aA_{i0} & aA_{r0} - 3A_{i0} \\ 3A_{r0} + aA_{i0} & aA_{r0} + A_{i0} & aA_{r0} + A_{i0} & A_{r0} + 3aA_{i0} \end{pmatrix} \begin{pmatrix} A_r A_r \\ A_r A_i \\ A_i A_r \\ A_i A_i \end{pmatrix}, \quad (2.14)$$

$$\mathbf{Q}_3(A, A, A) = -2 \begin{pmatrix} 3a & -1 & -1 & a & -1 & a & a & 3 \\ 3 & a & a & 1 & a & 1 & 1 & 3a \end{pmatrix} \begin{pmatrix} A_r A_r A_r \\ A_r A_r A_i \\ A_r A_i A_r \\ A_r A_i A_i \\ A_i A_r A_r \\ A_i A_r A_i \\ A_i A_i A_r \\ A_i A_i A_i \end{pmatrix}. \quad (2.15)$$

FIGURE 2.2: Contours of first Lyapunov coefficient l_1 in the (ν, a) plane. This coefficient is always negative, showing that the Hopf bifurcation, which occurs for $\nu < (a^2 - 1)/(2a)$, is supercritical. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.

The Lyapunov coefficient l_1 can be expressed in terms of the following scalar products

$$\mathcal{G}_{20} = \langle p, \mathbf{Q}_2(q, q) \rangle, \quad \mathcal{G}_{11} = \langle p, \mathbf{Q}_2(q, \bar{q}) \rangle, \quad \mathcal{G}_{21} = \langle p, \mathbf{Q}_3(q, q, \bar{q}) \rangle, \quad (2.16)$$

as

$$l_1 = \frac{1}{2\omega_0^2} \text{Re} (i \mathcal{G}_{20} \mathcal{G}_{11} + \omega_0 \mathcal{G}_{21}), \quad (2.17)$$

where q, p are the complex eigenvectors associated with $\pm i \omega_0 = \pm i \Omega$, respectively.

The contours of l_1 are shown in Fig. 2.2 confirming that, in the region of existence of the Hopf bifurcation, $\nu > (a^2 - 1)/(2a)$, it is always supercritical; the Lyapunov coefficient is always negative.

Character of the primary Hopf bifurcation in the weakly coupled oscillators model

Following an analogous procedure, one can analyse the character of the primary Hopf bifurcation that occurs for two weakly coupled parametrically forced oscillators with out-of-phase forcing. The dynamics of this system is described by

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2 A + \mu B, \quad (2.18a)$$

$$\dot{B} = (-1 + i\nu)B - f\bar{B} - (a + i)|B|^2 B + \mu A, \quad (2.18b)$$

where A, B are complex amplitudes, and $\mu = \mu_r + i\mu_i$ is the (weak) coupling term, assumed to be small $|\mu| \ll 1$.

It can be shown that the primary bifurcation of Eqs. (2.18) is a Hopf bifurcation occurring along the curve

$$f^2 = (1 + \nu^2) (1 - \mu_r^2); \quad \nu \neq -\frac{\mu_i}{\mu_r}, \quad (2.19)$$

with the Hopf frequency given by

$$\Omega = \mu_i + \nu\mu_r. \quad (2.20)$$

In the particular case of conservative coupling $\mu_r = 0$ (see Chapter 3 for details), the first Lyapunov coefficient describing the character of the bifurcation has a relatively simple expression.

This problem can be expanded as before — now around the trivial solution — as

$$\dot{\mathbf{A}} = \mathbf{Q}_1(\mathbf{A}) + \left(\frac{1}{2!}\right) \mathbf{Q}_2(\mathbf{A}, \mathbf{A}) + \left(\frac{1}{3!}\right) \mathbf{Q}_3(\mathbf{A}, \mathbf{A}, \mathbf{A}) + \dots, \quad (2.21)$$

where $\mathbf{A} = (A_r, A_i, B_r, B_i)^T$ and $\mathbf{Q}_2 = \mathbf{0}$. The Lyapunov coefficient simplifies to

$$l_1 = \frac{1}{2\omega_0^2} \text{Re}(\langle \mathbf{p}, \mathbf{Q}_3(\mathbf{q}, \mathbf{q}, \bar{\mathbf{q}}) \rangle), \quad (2.22)$$

with $\omega_0 = \mu_i$, while the cubic term $\mathbf{Q}_3$ can be reduced (considering only nonzero terms) to the expression given by Eq. (2.15).

The eigenvectors of the linear problem, $\mathbf{q}, \mathbf{p}$, are given by the following compact expressions

$$\mathbf{q} = -\frac{1}{2\sqrt{1+f_0+\nu^2}} \begin{pmatrix} 1+f_0 \\ \nu \\ i\nu \\ -i(1+f_0) \end{pmatrix}, \quad \mathbf{p} = -\frac{\sqrt{1+f_0+\nu^2}}{2(1+f_0)} \begin{pmatrix} 1+f_0 \\ -\nu \\ -i\nu \\ -i(1+f_0) \end{pmatrix}, \quad (2.23)$$

with $f_0^2 = 1+\nu^2$ and, after calculating the scalar product and some algebraic manipulation, l_1 reads

$$l_1 = \frac{3}{8\mu_i(1+f_0)(1+f_0+\nu^2)} (a\nu^4 + 2\nu^3(1+f_0) + 2\nu(1+f_0)^3 - a(1+f_0)^4). \quad (2.24)$$

The transition from supercritical to subcritical, if it exists, occurs when l_1 changes from negative to positive. By imposing $l_1 = 0$, one can derive the following condition

$$(\nu - a)(2 + \nu^2) \left(1 + \sqrt{1 + \nu^2}\right) = 0, \quad (2.25)$$

whose unique solution lies on the line $\nu = a$. The primary Hopf bifurcation is supercritical (subcritical) if $\nu < a$ ($\nu > a$). This result is illustrated in Fig. 3.14 of Chapter 3.

The analytical techniques exemplified above are complemented with numerical methods for continuation and bifurcation analysis in Chapter 3, to study in detail the dynamics and stability of two weakly coupled parametrically forced oscillators near the primary subharmonic resonance. The results obtained from theory are later compared, to the extent possible, with experiments and

FIGURE 2.3: Sketch illustrating the two-layer system of immiscible liquids. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7.

two-dimensional simulations of the Navier-Stokes equations. Despite this numerical model being different than the level-set formulation presented below, they share the essential equations of mass and momentum conservation, and similarly describe the interfacial (stress balance) and contact line (fixed contact angle) boundary conditions.

2.1.2 Let-set-based formulation of the Navier-Stokes equations

We consider a rectangular container with interior dimensions $L \times 2H$ holding two layers of immiscible liquids, as sketched in Fig. 2.3. The corners of the container are rounded to avoid singular numerical behaviour and more closely match experiments. The container is subjected to horizontal (along the x -axis) vibrations of amplitude A and frequency $\omega = 2\pi f$. Fluid 1 is denser than fluid 2 ($\rho_1 > \rho_2$) so that, in the presence of vertical gravity, this configuration is stable.

The dynamical response to applied vibrations is solved using a level-set method (Olsson and Kreiss, 2005) that tracks the position of the interface via the definition of the (level-set) function ϕ obeying to

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right). \quad (2.26)$$

The tuning parameter γ is set equal to $A\omega$, ε is a small diffusive term added for numerical stability and taken to be on the order of the maximum mesh element size, and the flux term $-\phi(1 - \phi)\nabla \phi / |\nabla \phi|$ is added to counteract the widening of the interface due to numerical diffusion. This term was first introduced by Olsson and Kreiss (2005) and is part of the standard level-set implementation included in COMSOL Multiphysics.

The level-set function is used to define density and viscosity across both layers in a continuous way by introducing the Heaviside function $\mathcal{H}$

$$\rho = \rho_1 + (\rho_2 - \rho_1) \mathcal{H}, \quad (2.27a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1) \mathcal{H}, \quad (2.27b)$$

where

$$\mathcal{H}(\tilde{\phi}) = \begin{cases} 0 & \tilde{\phi} < -\varepsilon, \\ \frac{1}{2} + \frac{\tilde{\phi}}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\tilde{\phi}}{\varepsilon}\right) & |\tilde{\phi}| \leq \varepsilon, \\ 1 & \tilde{\phi} > \varepsilon. \end{cases} \quad (2.28)$$

Here $\tilde{\phi} = \phi - 0.5$ with the values $\phi = 0, 1$ corresponding, respectively, to fluid 1 and fluid 2. The interface is associated with the contour $\phi = 0.5$. The smoothing of $\mathcal{H}$ over a small interface width 2ε improves numerical stability compared to the use of the usual Heaviside step function.

In the reference frame of the container, Eqs. (2.26)–(2.28) are resolved together with the continuous incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991):

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \nabla(\mu \nabla \mathbf{u}) + \delta \mathbf{f}_\sigma + \rho \mathbf{G}(t), \quad (2.29a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.29b)$$

where the effect of interfacial tension σ is included via $\mathbf{f}_\sigma$. Here,

$$\mathbf{f}_\sigma = (\sigma \kappa) \mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \quad (2.30)$$

where κ is the curvature of the interface and $\mathbf{n}$ the normal vector. The effect of interfacial tension is localised near the interface with the parameter δ , defined in terms of the level-set function as

$$\delta = 6\phi(1 - \phi)|\nabla \phi|. \quad (2.31)$$

Gravity (either constant or time-dependent) and the applied periodic forcing are included in the $\mathbf{G}(t)$ term of Eq. (2.29a)

$$\mathbf{G}(t) = \mathbf{g}(t) + A\omega^2 \cos \omega t \mathbf{i}, \quad (2.32)$$

where $\mathbf{i}$ is the unit vector along the x -axis.

Navier-slip boundary conditions are applied on the solid walls to maintain the contact angle β , while allowing the motion of the contact line as observed in the experiments (Salgado Sánchez et al., 2019d, 2020). These conditions are written as

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u} \cdot [\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta) \delta] = 0, \quad (2.33)$$

where $\mathbf{n}_w$ is a unit vector normal to the wall. They introduce an artificial frictional force $\mathbf{f}_v = (\rho\nu)/\delta_v \mathbf{u}$, where $\delta_v = \sqrt{\nu/\omega}$ is the thickness of the vibrational viscous boundary layer.

As initial conditions, the liquids are assumed to be motionless with $\mathbf{u} = 0$ and to be separated by a perfectly flat interface (the dashed line in Fig. 2.3). To improve numerical stability, the vibrational forcing is initiated smoothly in the manner of Eq. (2.28) over a 0.25 s transition time. If $\beta \neq \pi/2$, or particular time-dependent profiles are used for gravity $\mathbf{g}(t)$, the system can be evolved from this flat initial interface to a solution prior to the initiation of the forcing, and that new state used as the initial condition for the vibrated problem. This allows for the modelling of particular experiment sequences like the initiation of the forcing during microgravity in a parabolic flight, i.e., after the transition from hypergravity to microgravity during the parabolic manoeuvre is completed.

The system described by Eqs. (2.26)–(2.33) may be nondimensionalised by using H for the unit of distance, $\rho_1 - \rho_2$ for density and $\sqrt{(\rho_1 - \rho_2)H^3/\sigma}$ for time. Behaviour can then be characterised by the dimensionless velocity $\mathcal{F}_H$ and frequency Ω :

$$\mathcal{F}_H = A\omega\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad \Omega = \omega H\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad (2.34)$$

the density ratio $\hat{\rho}$ and viscosity ratio $\hat{\nu}$:

$$\hat{\rho} = \frac{\rho_2}{\rho_1}, \quad \hat{\nu} = \frac{\nu_2}{\nu_1}, \quad (2.35)$$

and the capillary number Ca, Bond number Bo and aspect ratio Γ :

$$\text{Ca} = \nu_1\sqrt{\frac{(\rho_1 - \rho_2)}{\sigma H}}, \quad \text{Bo} = \frac{(\rho_1 - \rho_2)H^2}{\sigma}g, \quad \Gamma = \frac{L}{2H}. \quad (2.36)$$

The simulation software COMSOL Multiphysics, which includes a standard implementation of the level-set method, is used for solving the governing equations (2.26)–(2.33) with the finite element method. The evolution in time is implemented with a Generalised- α integration scheme (Erlicher, Bonaventura, and Bursi, 2002) combined with streamline (Harari and Hughes, 1992) and cross-wind (Codina, 1993) stabilisation techniques. The maximum time step is set at $1/(10f)$, with automatic reduction as required for stability. Numerical convergence is tested against experiments to select the mesh size and the interface thickness.

This formulation is extensively used throughout this Thesis. All numerical results shown in Chapters 7 to 11 for immiscible liquids are obtained in this manner, including minor modifications to reproduce the experiments (i.e., time-dependent gravity profile, forcing initiation timing, fluid properties) when convenient. Further details can be found in the dedicated Chapters below.

2.2 Experimental methods

This section describes the experimental setups with which the majority of the experimental results included in this Thesis were obtained. Two experiments are included: the Control of Fluids in microgravity with Vibrations (CFVib) experiment, and the Vibrational Phenomena in Liquids (VIPIL) experiment.

Both setups are designed to be executed during parabolic flights, which benefit from the real-time interaction of *in-situ* operators. The residual acceleration during the reduced gravity periods provided during the flight (Pletser et al., 2016), which last approximately 20 s, is measured by the onboard accelerometers and provided by the airplane operators. The microgravity level roughly satisfies

$$|\mathbf{g}| \lesssim 0.05 g_0, \quad (2.37)$$

where $g_0 = 9.81 \text{ m/s}^2$.

The CFVib experiment¹ was selected as part of the ESA Fly your Thesis! programme² and conducted during the 65th ESA Parabolic Flight Campaign. The contribution to the CFVib project documented in this Thesis is extensive. The author collaborated in the experiment design, development, testing and manufacturing from the earliest stages, and later participated in the execution of the experiment as an operator during the parabolic flight campaign. Following that, the experimental results were analysed and complemented using different numerical models. The publications associated with this part of the Thesis are collected in Chapters 4 to 6.

The VIPIL experiment³, on the other hand, was conducted earlier during the 59th and 62nd ESA Parabolic Flight campaigns. The contribution of this Thesis to that project consisted in the processing and interpretation of results, the development of a numerical model for immiscible layers, and a detailed comparison of experiments and simulations. The associated experimental results are collected in Chapters 7 to 9.

2.2.1 The Control of Fluids in Microgravity with Vibrations (CFVib) experiment

The CFVib experiment was designed to compare pairs of cuboidal and cylindrical cells containing a liquid-air or immiscible two-liquid combination; see Fig. 2.4. One of the cells is subjected to vibrations at various amplitudes and frequencies, while the reference cell is left unforced and subjected only to the residual acceleration of the flight. The fluids tested during the experimental campaign are listed in Table 2.1, including their relevant physical properties.

¹<http://eea.spaceflight.esa.int/portal/exp/?id=9556>

²http://www.esa.int/Education/Fly_Your_Thesis/Meet_the_teams_CFVib

³<http://eea.spaceflight.esa.int/portal/exp/?id=9420> (2013), <http://eea.spaceflight.esa.int/portal/exp/?id=9478> (2015)

FIGURE 2.4: Pictures of the CFVib experimental cells and the bender beam: (a) cuboid and (b) cylinder. The aluminium bender beam has two piezoelectric ceramics attached to opposite horizontal faces. Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.

	ρ (kg/m ³)	σ (mN/m)	$\nu \times 10^{-6}$ (m ² /s)
Water	1000	72	1
5 cSt silicone oil	910	19.7	5
50 cSt silicone oil	960	20.8	50
Krytox 1506	1900	19	250

TABLE 2.1: Physical properties of the experiment fluids at room temperature: density ρ , surface tension σ , and kinematic viscosity ν . Reproduced from Fernández et al. (2017a), Chapter 4.

The containers are made of transparent polymethyl methacrylate (PMMA) to allow for visualisation of the liquid interface. The cuboids are 30 mm $\times$ 40 mm $\times$ 30 mm (length $\times$ width $\times$ height), while the cylinders have an interior radius of 15 mm and a height of 60 mm. These two simple geometries led to different dynamics. The cuboidal cells exhibited (quasi) two-dimensional behaviour in some cases, which facilitated the comparison with numerical simulations, while the selection of cylindrical cells was motivated by the experiments of Ganiev, Lakiza, and Tsapenko (1977).

Each experimental cell is fixed to an aluminium beam (cantilever) that, in the case of the excited cells, has two piezoelectric ceramics (PZTs) attached on opposite horizontal faces. An alternating counterphase voltage is applied to each PZT after being boosted by an amplifier. The periodic response of the ceramics is transmitted as a periodic torque to the beam, driving the periodic flexural motion that shakes the cell.

Both the amplitude V (with a maximum value determined by the operating envelope of the amplifier) and frequency f of the applied voltage are varied to control the magnitude and characteristics of the vibration. The experiments used amplitudes up to 100 V, while the frequencies were selected to match resonant modes of the assembly. The first and second resonances (at approximately 80 Hz and 215 Hz, respectively, for the cuboids) were found to be associated with the largest displacements and, therefore, the most pronounced vibroequilibria effects. Accelerometers were placed on the containers to directly measure the (vertical) acceleration on each vibrated sample.

FIGURE 2.5: Sketch of experimental setup with: (a) cuboid and (b) cylinder. Shown are the (partially filled) experimental cell, bender beam, piezoelectric ceramics, mirror and CMOS camera. Selected optical paths indicate the direct and reflected images. Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.

The optical system that monitored the position of the fluid included a camera, a mirror and source of illumination. For the vibrated cells, the camera sensor sampled at (approximately) 20 frames per second, providing excellent resolution of the slow timescale dynamics of the surface; the reference cells were recorded at (approximately) 10 frames per seconds. The behaviour of the liquid surface was observed along the axis of the beam, and after reflection by a mirror. Depending on the cell geometry, this mirror provided either a top view (for the cuboidal cell) or a lateral view (for the cylindrical cell). Both experimental setups are sketched in Fig. 2.5, which shows the container, bender beam, piezoelectric ceramics (PZTs), camera and mirror. Selected optical paths are drawn to illustrate the direct and reflected views.

The results have been analysed using various tools. In particular, we relied on the open software ImageJ for much of the image processing (contrast enhancement, interface tracking, etc.) and on MATLAB.

2.2.2 The Vibrational Phenomena in Liquids (VIPIL) experiment

The heart of the VIPIL experiment is a cuboidal cell of interior dimension $L \times 2H \times D = 15 \text{ mm} \times 7.5 \text{ mm} \times 5 \text{ mm}$ that is subjected to vibrations; see Fig. 2.6(a). The two large faces of the cell with area $L \times 2H$ were made of transparent quartz glass to allow observation of the fluids in the interior. The remaining boundaries were made from aluminium, with a hole drilled through each; the holes

FIGURE 2.6: (a) Schematic view of the optical setup (left) and the experimental cell (right). (b,d) Images of the experimental cells holding immiscible and miscible liquids, respectively. (c,e) Unprocessed cropped images showing the interface in reduced gravity: (c) immiscible combinations I_a (upper container) and I_b (lower container), (e) miscible liquids under horizontal forcing: mixture M_a (upper snapshot) and M_b (lower snapshot). Reproduced from Salgado Sánchez et al. (2020), Chapter 9.

(a) Miscible combinations				
	$\tilde{c}_0$, water (mass fraction)	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)	D (10 ⁻¹⁰ m ² /s)
M _a (1):	0.9	980.9	1.44	1.76
M _a (2):	0.5	902.4	3.39	1.76
M _b (1):	0.67	941.0	2.88	2.85
M _b (2):	0.27	847.5	3.21	2.85
(b) Immiscible combinations				
		ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)	σ (10 ⁻³ N/m)
FC-40 (1)		1855	2.2	16.63
2 cSt silicone oil (2)		873	2	18.63
20 cSt silicone oil (2)		950	20	20.85
Interfacial properties:				
I _a : FC-40 / 2 cSt oil	–	–	–	3.927
I _b : FC-40 / 20 cSt oil	–	–	–	6.021

TABLE 2.2: Physical properties of the experimental fluids at 25°C. (a) Miscible water–IPA combinations: density ρ , kinematic viscosity ν , and diffusion coefficient D (Mialdun et al., 2012). (b) Immiscible combinations of FC-40 and silicone oil: density ρ , kinematic viscosity ν , and surface tension σ (Someya and Munakata, 2005). Reproduced from Salgado Sánchez et al. (2020), Chapter 9.

in the upper and lower walls were used to fill the container, while those in the lateral walls were for drainage.

The container holds selected combinations of miscible or immiscible liquids, whose relevant physical properties are given in Table 2.2. In the miscible experiments, two layers of water-isopropanol (IPA) with different concentrations were used. The initial concentrations (mass fractions) of water in the two miscible layers were either 0.9 and 0.6 (mixture M_a), or 0.67 and 0.27 (mixture M_b). In the immiscible experiments, two different silicone oils (2 cSt and 20 cSt) were used in combination with FC-40. These immiscible combinations (respectively, I_a and I_b) have similar density ratios and interfacial tension, but differ by a factor of 10 in the viscosity ratio.

The position of the interface was monitored through (one of) the transparent sides of the container. In the immiscible case, the silicone oils were coloured so that their transparency was different from that of the FC-40; see Fig. 2.6(b). The two liquids thus appeared with different shading when illuminated from behind, as seen in the greyscale images of Fig. 2.6(c). In the miscible experiments, the interface was revealed by a shadowgraphy technique. Refraction caused by the large gradient across the interface made it appear dark, as seen in the images of Fig. 2.6(e).

The image acquisition system used light from a collimated LED source, which was directed towards the experimental cell by a series of three mirrors. The image of the liquids and their interface

was redirected by a final mirror before passing through a set of lenses to adjust the field of view and resolution; see Fig. 2.6(a). The CCD camera was set to record images at 48 or 80 frames per second. The higher rate was used for all experiments with immiscible liquids and provided very good resolution of interfacial waves at low or moderate frequencies. Even at higher forcing frequencies, there was no practical difficulty with stroboscopic effects since the frequencies of the relevant interfacial modes were known.

The experimental cell, along with the lenses and camera, was mounted on a translation assembly attached to a linear motor that drove oscillations along the x -axis (the lengthwise dimension L) at the selected frequency $\omega = 2\pi f$ and amplitude A . The experiments using miscible liquids explored frequency and amplitude ranges of 2–24 Hz and 1.5–16 mm, respectively, while the experiments with immiscible liquids explored frequency and amplitude ranges of 5–50 Hz and 1–5.3 mm. These forcing characteristics, together with the (properties of the) liquids used and the start time of the forcing, defined particular experiments. The forcing start time refers to whether vibrations begin during the hypergravity or microgravity portion of the relevant parabolic manoeuvre.

With miscible liquids, the diffusive interface existing between layers naturally thickened during the experiments. Gaponenko et al. (2018) considered the effect of initial interface thickness on pattern selection after the frozen wave instability and found that a more diffuse interface promotes patterns of smaller wavenumber due to a reduced shear force. In the absence of convection, the mixing of the two liquids is slow and driven by diffusion. Vibrations strongly enhance mixing, to the extent that the interface thickens noticeably on a timescale of several seconds in the present experiments. In order to have nearly constant initial conditions between different experiments, a filling procedure was used to reinitialise the interface. Both liquids were injected simultaneously at an optimised flow rate by two identical pumps through the inlets on the upper and lower walls. Upon completion of the experiment, the liquids were drained through the lateral outlets.

Conditions were nearly isothermal during the experimental campaign, as confirmed by a negative temperature coefficient (NTC) thermistor placed in the experimental cells. Over a (daily) set of experiments (31 parabolas), the temperature only increased by approximately 2°C.

Again, the analysis of results benefited substantially from the software ImageJ and, also, from dedicated tools developed with MATLAB.

Part II

PUBLICATIONS

Chapter 3

Dynamics of weakly coupled parametrically forced oscillators

Salgado Sánchez, P., Porter, J., Tíñao, I. and Laverón-Simavilla, A.

Physical Review E **94**, 022216 (2016).

<https://doi.org/10.1103/PhysRevE.94.022216>

(selected as Editor's suggestion)

The dynamics of two weakly coupled parametric oscillators are studied in the neighbourhood of the primary subharmonic instability. The nature of both primary and secondary instabilities depends in a critical way on the permutation symmetries, if any, that remain after coupling is considered, and this depends on the relative phases of the parametric forcing terms. Detailed bifurcation sets, revealing a complex series of transitions organised in part by Bogdanov-Takens points, are calculated for representative sets of parameters. In the particular case of out-of-phase forcing the predictions of the coupled oscillator model are compared with direct numerical simulations and with recent experiments on modulated cross-waves. Both the initial Hopf bifurcation and subsequent saddle-node heteroclinic bifurcation are confirmed.

3.1 Introduction

Oscillations arise in an enormous variety of systems, from basic mechanical devices like a mass on a spring, to surface or pressure waves in continuous media, to population (Turchin, 2003) and climate (Scafetta, 2010) cycles in the natural environment. In dissipative systems, such periodic variation typically involves both a restoring force, which determines the frequency of natural oscillations, and a driving force, which sustains them against energy losses. If the driving mechanism depends on time, its interaction with the restoring force can lead to interesting resonance phenomena. Further possibilities for resonance arise if the oscillatory system is not isolated, but interacts with other dynamical systems. In particular, there has been considerable recent interest in how an oscillatory system couples to other identical (or similar) systems and the collective behaviours that ensue.

The study of coupled oscillators, which can be traced to Huygens' work with pendulum clocks in the seventeenth century (Huygens, 1669), is concerned with collective phenomena such as synchronisation, phase locking, and amplitude death. These features, and others, depend both on the individual dynamics of the component oscillators and the properties of the coupling between them.

In many cases, it is reasonable to assume the oscillators are equivalent in the absence of coupling — identical fireflies, for example, in a population undergoing synchronous flashing (Mirollo and Strogatz, 1990). This *permutation symmetry* between individual oscillators may or may not persist when coupling is included, and the extent to which it does is a key factor in determining the structure of invariant subspaces and the possible classes of solutions. The type of coupling that applies (global all-to-all, nonlocal, local, etc.) plays a central role in determining the residual exchange symmetry.

The type of forcing mechanism that drives individual oscillations has an important effect as well, particularly with regard to phase locking. If the forcing mechanism is independent of time (like a steady wind blowing waves across a water surface) then the response of the system will inherit this time translation invariance and there will be no preferred phase of the uncoupled oscillation. Collections of such oscillators can, in the limit of weak coupling, be reduced to phase models (Kuramoto, 1984) where each oscillator is influenced only by the phase (not amplitude) of the other oscillators that are coupled to it.

If individual oscillations occur in response to periodic forcing then, because there is no continuous time translation symmetry, phases in the uncoupled problem are not equivalent. Each oscillator has (one or more) preferred phases that will be selected when coupling is weak; in this limit, synchronisation depends not on coupling but on the relationship between the individual forcing functions. If the forcing is direct in character (i.e., it excites synchronous oscillations for any forcing amplitude) then a typical oscillator will have, in the absence of coupling, a single preferred phase. If, on the other hand, the time-dependent forcing is parametric in nature, then the typical response is subharmonic, at half the forcing frequency, and there will be (at least) *two* preferred phases related by time translation through one forcing period. Among other things, this means that collections of weakly coupled parametrically forced oscillators have more possible solutions than their directly

forced counterparts, and hence the potential for more complicated dynamics.

Coupled parametrically forced oscillators have received less attention than autonomous or directly forced systems despite potential applications in microelectromechanical systems (MEMS) (Buks and Roukes, 2002; Lifshitz and Cross, 2003) and optical parametric oscillators (Phua et al., 1998), for example. In this paper, we are particularly interested in a novel application: the dynamics of parametrically excited surface waves such as cross-waves (Miles and Henderson, 1990). These subharmonic waves occur regularly in wavemaker experiments with a vibrating surface or boundary (Barnard and Pritchard, 1972; Lin and Howard, 1960; McGoldrick, 1968; Moisy et al., 2012; Taneda, 1994). Horizontally vibrated containers of fluid, in particular, are often used to study surface waves (Faltinsen, Rognebakke, and Timokha, 2006; Feng, 1997; Funakoshi and Inoue, 1988; Nobili et al., 1988) and, if the container is relatively large compared to the decay length of the waves, then the excitation patterns generated at the left and right endwalls (or wavemakers) can be considered to be weakly interacting. This is the situation in several recent experiments (Porter et al., 2012; Tinao et al., 2014).

There are a number of studies of coupled parametrically forced oscillators in the literature to which the current paper is related. Bena and Van den Broeck (1999) considered a large collection of linear oscillators, in the mean-field limit, driven by square-wave excitation with randomly distributed phases for each oscillator. They found complex stability boundaries for collective behaviour with a wider region of instability than for individual parametric excitation.

Goldobin and Pikovsky (2002) studied the collective excitations in a population of globally coupled weakly nonlinear oscillators parametrically driven by sinusoidal forcing, with uniformly distributed phases. A travelling wave regime was found and, in general, a primary instability that could be either steady or oscillatory. Copelli and Lindenberg (2001) studied a pair of linear oscillators driven by piecewise constant (square-wave) parametric forcing and found complicated stability boundaries with sensitive dependence on the phase difference between the two forcing functions. They observed that collective behaviour (i.e., similarity to mean-field models describing many oscillators) is reflected to a greater extent in the dimer with out-of-phase forcing; an analytical solution for this case was presented in Kawai, Lindenberg, and Van den Broeck (2002).

In the context of bifurcation theory, the interaction of two parametrically forced oscillators, which each undergoes steady state (pitchfork) bifurcations in the absence of coupling, is associated with the unfolding of a double zero bifurcation. This was studied, with both Huygens (exchange) symmetry and odd-Huygens symmetry by Kitanov, William, and Willms (2013). In the case of odd-Huygens symmetry, which is the relevant one for subharmonic modes like those considered here, both primary and secondary pitchfork bifurcations were found, and with certain unfolding parameters, Hopf and heteroclinic bifurcations.

Danzl and Moehlis (2010) considered networks of nonlinear parametrically forced oscillators with weak coupling, used the Implicit Function Theorem to prove the persistence of the uncoupled

solution classes, and investigated this numerically. Far from the weak coupling limit, interesting dynamics such as antisynchronised chaotic behaviour were found.

The current paper shares with Danzl and Moehlis (2010) an emphasis on symmetries and the assumption of weak coupling, and with Copelli and Lindenberg (2001) and Kawai, Lindenberg, and Van den Broeck (2002) an interest in the effect of the forcing phases. Like Copelli and Lindenberg (2001), we utilise a bifurcation theory (unfolding) approach. The principal contributions of the current paper are as follows.

In Sec. 3.2, we briefly review the dynamics of a single parametrically forced oscillator in the neighbourhood of the dominant subharmonic instability, and note the secondary Hopf and global bifurcations that can occur with a nonconservative nonlinear term. We consider a pair of coupled oscillators in Sec. 3.4. Here, the residual symmetry is critical in organising the structure of solutions near the initial instability, where the hyperbolicity assumption of Danzl and Moehlis (2010) fails. Unlike Copelli and Lindenberg (2001) and Kawai, Lindenberg, and Van den Broeck (2002), we do not calculate multiple resonance tongue boundaries, but instead provide a complete bifurcation picture for the nonlinear problem in a neighbourhood of the dominant subharmonic instability; this includes secondary local and global bifurcations, and organising Bogdanov-Takens points. Bifurcation sets are calculated as a function of forcing and detuning for representative sets of parameters. The detailed bifurcation scenario obtained for two coupled oscillators is compared, to the extent possible, with the recent experiments of Tinao et al. (2014) in Sec. 3.5, and with direct numerical simulations of a corresponding two-dimensional vibrated fluid system in Sec. 3.6. Conclusions are presented in Sec. 3.7.

3.2 Dynamics of a parametrically forced oscillator

A wide range of parametrically forced oscillators can be approximated by a damped nonlinear Mathieu equation of the form

$$\ddot{x} + 2\gamma\dot{x} + [1 + 4f \cos(2\omega t - \phi)]x + G(x, \dot{x}) = 0, \quad (3.1)$$

where $\gamma > 0$ is the damping coefficient, $f > 0$ is the forcing amplitude, $G(x, \dot{x})$ denotes nonlinear terms, and time has been scaled with the natural frequency of the oscillator. For a single oscillator, the phase ϕ is of no consequence and can be removed by a simple shift in time, but it is left in Eq. (3.1) to anticipate the general case of coupled oscillators.

It is well-known that the linear stability problem leads to a series of resonance tongues (Benjamin and Ursell, 1954; McLachlan, 1947) centred at integer values of ω . We assume here that the system is weakly damped and near the primary subharmonic ($\omega = 1$) instability by writing $\omega = 1 + \nu$. Using a perturbation expansion of the form

$$x(t) = A(t)e^{-i(\omega t + \pi/4)} + \text{c.c.} + \dots, \quad (3.2)$$

where c.c. denotes the complex conjugate, leads to the amplitude equation

$$\dot{A} = (-\gamma + i\nu)A + fe^{i\phi}\bar{A} - (a_r + ia_i)|A|^2A + \dots, \quad (3.3)$$

which applies when $|A|, \gamma, f, |\nu| \ll 1$. If when $\gamma = f = 0$ the nonlinearity $G(x, \dot{x}) = G(x)$ is conservative, then the cubic coefficient is imaginary at leading order and $a_r = 0$.

Equation (3.3) can be simplified with the rescaling

$$t \rightarrow t/\gamma, \quad (f, \nu) \rightarrow (f, \nu)\gamma, \quad A \rightarrow \sqrt{\frac{\gamma}{|a_i|}} A, \quad (3.4)$$

and, if $a_i < 0$, with the additional substitutions $A \rightarrow \bar{A}$ and $(\nu, \phi) \rightarrow -(\nu, \phi)$:

$$\dot{A} = (-1 + i\nu)A + fe^{i\phi}\bar{A} - (a + i)|A|^2A + \dots, \quad (3.5)$$

where $a = a_r/|a_i|$.

The primary subharmonic instability of a parametrically forced oscillator is a pitchfork bifurcation that produces two symmetry-related steady states $\pm A_s$. This bifurcation occurs along the curve

$$\text{pitchfork} : f^2 = 1 + \nu^2. \quad (3.6)$$

Since the nontrivial steady states A_s satisfy

$$|A_s|^2 = \frac{\nu - a}{a^2 + 1} \pm \sqrt{\frac{f^2}{a^2 + 1} - \left(\frac{1 + a\nu}{a^2 + 1}\right)^2}, \quad (3.7)$$

the primary bifurcation is supercritical if $\nu < a$ and subcritical if $\nu > a$. In the subcritical case, the initially unstable steady branch is stabilised in a saddle-node bifurcation along the half-line

$$\text{saddle-node} : f^2 = \frac{(1 + a\nu)^2}{1 + a^2}; \quad \nu > a. \quad (3.8)$$

These two bifurcation sets are illustrated in Fig. 3.1 for the case $a = 0$ corresponding to conservative nonlinearity. In this case, there are no additional bifurcations nor are there any with dissipative nonlinearity $a > 0$, although the saddle-node bifurcation is delayed in that case.

The case $a < 0$, however, is qualitatively different because the nonlinearity can provide energy to the oscillator and sustain motion, even without parametric forcing. With $f = 0$, Eq. (3.5) contains an unstable rotating solution with constant amplitude $|A| = \sqrt{-1/a}$ and frequency $\nu + 1/a$. Furthermore, the two (symmetry-related) stable steady states, created in a pitchfork bifurcation for $\nu < a$ and in a saddle-node bifurcation for $\nu > a$, undergo Hopf bifurcation along the curve

$$\text{Hopf} : f^2 = \frac{1}{4} + \left(\frac{1 + 2a\nu}{2a}\right)^2; \quad \nu < \frac{a^2 - 1}{2a}. \quad (3.9)$$

FIGURE 3.1: Pitchfork (P: thick curve) and saddle-node (SN: thin curve) bifurcation sets for a single parametrically forced oscillator in the case of conservative nonlinearity $a = 0$.

This Hopf bifurcation set terminates at a Bogdanov-Takens point when it intersects the line of saddle-node bifurcations.

$$\text{Bogdanov-Takens : } f = \frac{\sqrt{1+a^2}}{2}; \quad \nu = \frac{a^2-1}{2a}. \quad (3.10)$$

The Hopf bifurcation occurs when the stable steady state branch (Eq. (3.7) with the “+” sign) reaches a fixed amplitude A_H , independent of ν :

$$|A_H|^2 = -\frac{1}{2a}, \quad (3.11)$$

while the Hopf frequency ω_H increases with distance from the Bogdanov-Takens point

$$\omega_H = \sqrt{\frac{2}{a} \left(\nu + \frac{1-a^2}{2a} \right)}. \quad (3.12)$$

The normal form (first Lyapunov) coefficient that determines whether the Hopf bifurcation is supercritical or subcritical can be calculated following Kuznetsov (2004), for example, and is found to be negative throughout the region of existence; contours of this coefficient are shown in Fig. 3.2. This secondary Hopf bifurcation is therefore always supercritical.

The stable modulated states generated here grow in amplitude and are destroyed by a homoclinic bifurcation. As shown in Fig. 3.3, the nature of this homoclinic bifurcation changes with the stability of the trivial state. If $f^2 > 1 + \nu^2$ the trivial state is unstable and the periodic modulated states collide with it in a homoclinic gluing bifurcation. This gluing bifurcation transforms the symmetry-related pair of stable modulated states into the unique rotating (unbounded phase) state — the one supported at $f = 0$ purely by nonlinearity. Note that the stable rotating branch created in the gluing bifurcation soon undergoes a saddle-node bifurcation that renders it unstable (as it is for

FIGURE 3.2: Contours of the normal form (first Lyapunov) coefficient l_1 (see Kuznetsov (2004)) in the (ν, a) plane. This coefficient is always negative, showing that the Hopf bifurcation, which occurs for $\nu < (a^2 - 1)/(2a)$, is supercritical.

FIGURE 3.3: Bifurcation sets in the case of $a = -0.3$: pitchfork (P: thick solid curve), saddle-node (SN: thin solid line), Hopf (H: lower dashed curve), saddle-node of periodic orbit (SN Per: upper dashed curve), homoclinic gluing (Hom Gluing: thin solid curve above pitchfork), homoclinic (Hom: lower thin solid curve below pitchfork), heteroclinic (Het: upper thin solid curve below pitchfork), saddle-node heteroclinic (SN Het), and Bogdanov-Takens point (BT: solid dot).

most parameters).

For forcing below the primary pitchfork bifurcation, $f^2 \leq 1 + \nu^2$, the trivial state is stable and cannot participate in a global bifurcation. The gluing bifurcation therefore splits when crossing below the (subcritical) pitchfork bifurcation set (see Fig. 3.3) into two separate global bifurcations, a (nonsymmetric) homoclinic bifurcation where the modulated states created in the Hopf bifurcation separately collide with the finite amplitude saddle fixed points created in the subcritical pitchfork bifurcation, and a heteroclinic bifurcation, at slightly larger f , where the rotating periodic state simultaneously collides with these same two nontrivial steady states. This heteroclinic bifurcation set eventually merges with the saddle-node bifurcation set to create (for $\nu > 2.67$) a saddle-node heteroclinic bifurcation, i.e., a heteroclinic cycle connecting nonhyperbolic stationary solutions with a simple zero eigenvalue (see, e.g., Kuznetsov, 2004). We note that this is a symmetric version of

FIGURE 3.4: Bifurcation diagrams for $a = -0.3$ and (a) $\nu = -0.5$, (b) $\nu = 1$, showing the (time-averaged) norm of the nontrivial steady states (A_s), periodic modulated solutions (P_m) and periodic rotating solution (P_r). Solid (dashed) curves denote stable (unstable) solutions. With increasing ν , the primary bifurcation transitions from supercritical to subcritical and the gluing bifurcation splits into distinct homoclinic and heteroclinic bifurcations. A selection of periodic orbits at particular points labelled $a - h$ in these diagrams are shown in (c) and (d), respectively. a, e : rotating orbit at $f = 0$, b : limit point at $f = 2.747$, c : symmetric homoclinic orbit at $f = 2.678$, d : small modulated orbits at $f = 2.3$, f : heteroclinic orbit at $f = 1.275$, g : homoclinic orbits at $f = 1.071$, h : small modulated orbits at $f = 0.9$.

what is also known as Shilnikov' saddle-node bifurcation (Shilnikov, 1969), SNIC (saddle-node on invariant circle) bifurcation, or SNIPER (saddle-node infinite period) bifurcation. We will refer to the usual asymmetric case as a saddle-node homoclinic bifurcation, following the terminology of Kuznetsov (2004).

Figure 3.4 shows one bifurcation diagram obtained with $\nu = -0.5$, when the primary pitchfork bifurcation is supercritical, and one obtained with $\nu = 1$, when this bifurcation is subcritical. For each of these detuning values, a supercritical Hopf bifurcation occurs on the stable nontrivial steady solution branch. With $\nu = -0.5$ the periodic modulated states P_m created here collide

with the origin in a gluing bifurcation at $f = 2.678$ that creates a periodic rotating state P_r . For $\nu = 1$, in contrast, there are distinct homoclinic and heteroclinic bifurcations involving the small-amplitude unstable steady states. Regardless of the detuning value, when $a < 0$ stable solutions can only be found relatively close to onset (and for initial conditions not too large); otherwise (most) trajectories diverge due to the destabilising nonlinearity considered. In this case, a more realistic version of Eq. (3.5) would include stabilising fifth order terms.

3.3 Weakly coupled parametrically forced oscillators

If two or more parametrically forced oscillators are coupled together, their dynamics are potentially much richer than that described above. These dynamics depends both on the coupling between the oscillators and on the individual forcing terms, which act to select particular phases at each oscillator. Motivated by its relevance to recent experiments on horizontally vibrated fluids (Tinao et al., 2014; see Sec. 3.5) we will consider below the case of two coupled oscillators in detail. Since the symmetry arguments are quite general, however, we first describe these in a broader context before focusing on the case of two coupled oscillators, which may be compared with experiments.

A collection of n linearly coupled parametrically forced oscillators can be described by the equation

$$\dot{\mathbf{A}} = \mathcal{L}\mathbf{A} + f\Phi\bar{\mathbf{A}} - \mathcal{G}(|\mathbf{A}|^2)\mathbf{A} + \Lambda\mathbf{A}, \quad (3.13)$$

where $\mathbf{A} = (A_1, A_2, \dots, A_n)^T$. The diagonal matrices $\mathcal{L}$, Φ , and $\mathcal{G}$ represent the uncoupled problem of Eq. (3.5): $\mathcal{L}_{jj} = -1 + i\nu$, $\Phi_{jj} = e^{i\phi_j}$, $\mathcal{G}_{jj} = (a + i)|A_j|^2$.

The coupling matrix Λ is assumed to be symmetric (the j th and k th oscillator affect each other in the same way) and we further assume that the forcing differs only in phase (i.e., each oscillator loses stability at the same critical value of the bifurcation parameter f).

3.3.1 General symmetry considerations

If there is no coupling, $\Lambda = 0$, each of the oscillator equations is individually equivariant under reflection $A_j \rightarrow -A_j$ due to the subharmonic nature of the critical modes. In addition, the identical oscillators may be exchanged, or permuted, in $n!$ distinct ways. These permutations form a representation of the symmetric group $\mathbf{S}_n$. For example, the j th and k th oscillators can be exchanged with the operation

$$(A_j, A_k) \rightarrow (e^{i\delta_{jk}} A_k, e^{-i\delta_{jk}} A_j), \quad (3.14)$$

where $\delta_{jk} = (\phi_j - \phi_k)/2 = -\delta_{kj}$. Given a set of possible limit sets for each oscillator (trivial state, stable steady states, unstable steady states, periodic orbits, etc.) there is a larger set of solution classes formed by different combinations of these individual oscillator states. The members of each symmetry (isotropy) class are related by a combination of reflections and permutations.

As discussed in Danzl and Moehlis (2010), the Implicit Function Theorem guarantees that these various uncoupled solution branches persist with the same stability properties in the presence of sufficiently weak coupling, provided they are hyperbolic. Bifurcation points, however, where solutions are nonhyperbolic, will be strongly influenced by the nature of the residual symmetry of the coupled problem and this includes the primary bifurcation, which can not only be shifted by the coupling terms, but may also be transformed from steady to oscillatory.

Coupling breaks the symmetry of the uncoupled problem for two reasons. First, unless the coupling is trivial ($\Lambda = 0$) or uniform all-to-all, it will not commute with all permutations of the oscillators even in the case of identical forcing phases ($\Phi \propto \mathbb{1}$). Second, a nontrivial set of phase differences ($\phi_j \neq \phi_k$ for some j, k) means that exchanging oscillators while preserving the parametric forcing terms requires a phase shift as in Eq. (3.14); Φ only commutes with ordinary permutations if $\Phi \propto \mathbb{1}$. The phase matrix Φ can be formally eliminated with the transformation $A_j \rightarrow e^{i\phi_j/2} A_j$ but this, in general, introduces asymmetric phase shifts into the coupling matrix Λ . Generally, whether or not a subset of exchange symmetries is preserved by the coupling depends both on the structure of the coupling matrix Λ and on the choice of forcing phases ϕ_j .

The case of maximum symmetry occurs with uniform all-to-all coupling and identical forcing phases. In this case, arbitrary permutations are permitted, although the subharmonic reflection symmetry can only be applied collectively with

$$R : \mathbf{A} \rightarrow -\mathbf{A}. \quad (3.15)$$

This case, with the maximum number of exchange (permutation) symmetries and corresponding fixed point subspaces, is most similar to the uncoupled problem.

The case of minimum symmetry occurs if the coupling and/or forcing phases are general enough that no permutation symmetries remain. The only symmetry is then the basic subharmonic reflection (3.15).

3.3.2 Motivation from vibrated fluids experiments

Vibrated fluid experiments can produce a wide range of surface wave patterns (Arbell and Fineberg, 2002; Miles and Henderson, 1990), some of the most studied being due to a parametric forcing mechanism (Faraday, 1831; Jones, 1984). Vertically forced Faraday waves, which are (usually assumed to be) excited uniformly throughout the container, cannot easily be described as a system of weakly coupled oscillators. Cross-waves, on the other hand, whether driven by wavemakers or by direct (horizontal) motion of a rigid container, are excited by a spatially nonuniform oscillatory field (Porter et al., 2012; Tinao et al., 2014; Varas and Vega, 2007) that can, in many cases, be considered localised. If more than one wavemaker is used, the situation may naturally arise where different parametrically forced surface wave patterns interact “weakly” in the far-field (subcritically forced) region of the surface or interface.

If, for example, two identical wavemakers are spatially separated and driven in phase, then the surface wave patterns they excite after a subharmonic instability may be approximated as weakly coupled parametrically forced oscillators with synchronised forcing. If, instead, these same two wavemakers are driven with distinct phases, then the coupled system behaves differently (it is equivalent to introducing asymmetry in the coupling between the two patterns/oscillators). We note that this second situation automatically occurs when the subharmonic surface waves are excited by rigid horizontal motion of an open fluid container (Perez-Gracia et al., 2014; Porter et al., 2012, 2013). In this case, while the left endwall is moving into the fluid (leading to upward displacement near the surface), the right endwall is pulling away from the fluid (causing downward displacement near the surface). If the aspect ratio is large enough (the patterns are sufficiently localised) such a system may be approximated by two weakly coupled parametrically forced oscillators with out-of-phase forcing.

Each of the two cases mentioned above will be considered in detail in the following section, and the case of out-of-phase forcing will be compared to recent experimental results in Sec. 3.5 and to two-dimensional simulations of the Navier-Stokes in Sec. 3.6. To the best of our knowledge, there are no experiments to compare with in the case of synchronous forcing, or more general forcing phase difference, although it is easy to imagine a fluid experiment with independent wavemakers exciting localised cross-waves that could test the predictions of the following section in these cases too. We may undertake such an experiment in the near future.

3.4 Two coupled parametrically forced oscillators

We now consider in detail the case of two weakly coupled parametrically forced oscillators. This situation constitutes the simplest interaction possible, and allows a relatively complete analysis. As mentioned above, and discussed in more detail in Sec. 3.5, it is relevant to experiments on horizontally vibrated fluids in rigid containers (Porter et al., 2012; Tino et al., 2014). In this system, the parametric forcing mechanism (Perez-Gracia et al., 2014; Porter et al., 2013; Varas and Vega, 2007) is concentrated near opposing boundaries (endwalls or wavemakers) and, in containers of sufficiently large aspect ratio, this leads to relatively isolated patterns of subharmonic surface waves that can be considered as weakly interacting.

The governing equations for the complex amplitudes, A and B , characterising the two oscillators are

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2A + \mu B, \quad (3.16a)$$

$$\dot{B} = (-1 + i\nu)B + fe^{i\phi}\bar{B} - (a + i)|B|^2B + \mu A, \quad (3.16b)$$

where the forcing phase of the first equation has, without loss of generality, been set to zero.

It was noted earlier that if the oscillator is affected only by conservative nonlinearities, then the nonlinear coefficient is purely imaginary ($a = 0$). The complex coupling coefficient $\mu = \mu_r + i\mu_i$

similarly splits into conservative and nonconservative parts, as seen with the energy function

$$E = \frac{1}{2}(|A|^2 + |B|^2), \quad (3.17)$$

which satisfies

$$\dot{E} = -(|A|^2 + |B|^2) - a(|A|^4 + |B|^4) + f \operatorname{Re}(A^2 + e^{-i\phi} B^2) + 2\mu_r \operatorname{Re}(A\bar{B}). \quad (3.18)$$

The first two terms reflect linear and nonlinear dissipation, respectively, while the forcing may provide or extract energy depending on the phases of A and B . The final term shows that only the real part μ_r of the coupling term can affect the energy E while the imaginary part μ_i is conservative. The sign of the contribution to $\dot{E}$ from the coupling depends on the phase of $A\bar{B}$.

3.4.1 Symmetries

Uncoupled problem

Due to the odd character of the critical subharmonic modes under translation through one period of the forcing, the uncoupled problem possesses a separate reflection symmetry for each oscillator,

$$R_A : A \rightarrow -A, \quad R_B : B \rightarrow -B.$$

Permutation symmetry with two oscillators is represented by the single operation

$$\kappa : (A, B) \rightarrow (e^{-i\phi/2} B, e^{i\phi/2} A). \quad (3.19)$$

These symmetries of the uncoupled problem generate a representation of the dihedral group (of order eight) D_4 ; one may take as generators, for example, R_A and $\rho = R_A \cdot \kappa$.

Coupled problem

Coupling implies that the modes can no longer be reflected independently and only the single (subharmonic) reflection symmetry holds:

$$R : (A, B) \rightarrow -(A, B). \quad (3.20)$$

Since a pair of (identical) oscillators is described by a single coupling coefficient, the residual symmetry of the coupled problem depends entirely on the phase difference ϕ . It is easy to show that only two choices of ϕ permit any type of permutation symmetry.

Symmetric forcing

If $\phi = 0$ there is an interchange symmetry:

$$\kappa_0 : (A, B) \rightarrow (B, A) \quad (3.21)$$

TABLE 3.1: Diagram showing the three ways that the D_4 symmetry of the uncoupled problem is broken for $\mu \neq 0$.

which, together with R generates the group $\mathbf{Z}_2 \times \mathbf{Z}_2$. There are two invariant subspaces:

$$\begin{aligned}\text{Fix}(\kappa_0) &= \{(A, B) \mid A = B\} \\ \text{Fix}(\kappa_0 \cdot R) &= \{(A, B) \mid A = -B\}\end{aligned}$$

in which the system is equivalent to a single oscillator.

Antisymmetric forcing

If $\phi = \pi$ there is a cyclic exchange symmetry generated by:

$$\rho : (A, B) \rightarrow i(B, A). \quad (3.22)$$

The corresponding symmetry group, $\mathbf{C}_4$, has no nontrivial fixed point subspaces. Note that $\rho^2 = R$.

General forcing

If $\phi \neq 0, \pi$ then the only symmetry is the reflection R , representing the group $\mathbf{Z}_2$.

Table 3.1 summarises the three ways, depending on phase, that coupling breaks the D_4 symmetry of the uncoupled problem.

3.4.2 Parameter symmetries

Parameter symmetries can be used to reduce the range of parameters that needs to be considered.

The system (3.16) is equivariant under the transformation

$$(A, B) \rightarrow e^{i\phi/2}(B, A), \quad \phi \rightarrow -\phi \quad (3.23)$$

which means that ϕ can be restricted, without loss of generality to $0 \leq \phi \leq \pi$. Physically, this symmetry reflects the fact that the sign of ϕ , which determines whether the parametric forcing of A is leading or lagging that of B , is not important. Equivalent solutions exist for $\phi \rightarrow -\phi$ and an appropriately phase shifted exchange of A and B .

The equivariance of Eqs. (3.16) under the parameter symmetry

$$A \rightarrow -A, \quad \mu \rightarrow -\mu \quad (3.24)$$

means that μ can be restricted, without loss of generality, to $\mu_r > 0$.

3.4.3 Bifurcation analysis

As shown above, the manner in which the D_4 symmetry of a pair of identical uncoupled parametrically forced oscillators is broken by (symmetric) coupling depends only on the relative phase of the parametric forcing at each oscillator. We consider now in more detail the three distinct cases.

The coupling coefficient μ is assumed to be small but, at the same time, we use illustrative values that are large enough to clearly show the different behaviour in each case. Specifically, we choose $\mu = 0.2i$ or $\mu = 0.05 + 0.2i$ for most of the calculations below, selecting $\mu_r \ll \mu_i$ for a nearly conservative coupling term. The bifurcations, when not available analytically, are obtained numerically¹.

Symmetric forcing, $\phi = 0$

In this case, the equations governing the two subharmonic modes A and B are

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2A + \mu B, \quad (3.25a)$$

$$\dot{B} = (-1 + i\nu)B + f\bar{B} - (a + i)|B|^2B + \mu A. \quad (3.25b)$$

Recall that the exchange symmetry $\kappa_0 : (A, B) \rightarrow (B, A)$ and subharmonic symmetry $R : (A, B) \rightarrow -(A, B)$ generate a representation of the group $\mathbf{Z}_2 \times \mathbf{Z}_2$ and that there are two nontrivial fixed point subspaces

$$\Sigma_{\pm} = \{(A, B) \mid A = \pm B\}$$

corresponding to $\text{Fix}(\kappa_0)$ and $\text{Fix}(\kappa_0 \cdot R)$, respectively, and representing in-phase (synchronous) and out-of-phase (antisynchronous) motion. Within each subspace, the system behaves as a single oscillator and the coupled linear problem separates into corresponding 2-by-2 blocks. Primary Hopf bifurcations cannot occur.

Within the subspaces $\Sigma_{\pm}$, the real and imaginary parts of the coupling coefficient μ contribute to damping and detuning, respectively. Primary bifurcations in $\Sigma_{\pm}$ therefore occur along curves that are shifted with respect to Eq. (3.5). Pitchfork bifurcation occurs on

$$f^2 = (1 \mp \mu_r)^2 + (\nu \pm \mu_i)^2, \quad (3.26)$$

¹Numerical computations were performed with XPPAUT (<http://www.math.pitt.edu/~bard/xpp/xpp.html>), AUTO-07P (<http://sourceforge.net/projects/auto-07p/>) and MATCONT (<http://sourceforge.net/projects/matcont/>)

while saddle-node bifurcations appear when

$$f^2 = \frac{[(1 \mp \mu_r) + a(\nu \pm \mu_i)]^2}{1 + a^2}; \quad \nu > (1 \mp \mu_r)a \mp \mu_i. \quad (3.27)$$

These bifurcations produce $\mathbf{Z}_2$ -symmetric steady state solutions of Eqs. (3.25) denoted by $S_{\pm}$.

In the case of $a < 0$, secondary Hopf bifurcations within the $\Sigma_{\pm}$ subspaces occur when a critical amplitude, depending only on a and μ_r , is reached

$$|A|^2 = |B|^2 = \frac{1 \pm \mu_r}{2a}. \quad (3.28)$$

These secondary Hopf instabilities are located on the curves

$$f^2 = \left(\frac{1 \mp \mu_r}{2}\right)^2 + \left(\frac{1 \mp \mu_r + 2a(\nu \pm \mu_i)}{2a}\right)^2; \quad \nu < (1 \mp \mu_r)\frac{a^2 - 1}{2a} \mp \mu_i, \quad (3.29)$$

which terminate at the Bogdanov-Takens points

$$f = \frac{1 \mp \mu_r}{2} \sqrt{1 + a^2}; \quad \nu = (1 \mp \mu_r)\frac{a^2 - 1}{2a} \mp \mu_i. \quad (3.30)$$

The Hopf frequency is given by

$$\omega_H = \sqrt{\frac{2(1 \mp \mu_r)}{a} \left[\nu \pm \mu_i + (1 \mp \mu_r)\frac{1 - a^2}{2a} \right]}. \quad (3.31)$$

Note that taking $\mu \rightarrow -\mu$ switches the roles of the in-phase and out-of-phase solutions, as indicated in Eq. (3.24).

Nonsymmetric solutions not contained in $\Sigma_{\pm}$ can arise in saddle-node bifurcations or in secondary pitchfork bifurcations. These solutions, with $A \neq \pm B$, are neither synchronous nor antisynchronous and are referred to here as mixed modes (MM). They do not arise in primary bifurcations.

Figure 3.5 shows the bifurcation sets for conservative nonlinearity $a = 0$, and a representative value of the coupling, $\mu = 0.05 + 0.2i$. There are two pitchfork bifurcations within the invariant subspaces $\Sigma_{\pm}$ (labelled $P_{\pm}$) defined by Eq. (3.26) and, in the subcritical regime, associated saddle-node bifurcations (labelled $SN_{\pm}$). In addition to these single oscillator instabilities, there is a smooth curve of symmetry-breaking (pitchfork) bifurcations from the Σ_{-} subspace to mixed mode (asymmetric) states and two more symmetry-breaking bifurcations, one from Σ_{+} and one from Σ_{-} emerging from the mode interaction point where $P_{\pm}$ cross; these three dash-dotted curves are labelled $SB_{\pm}$ according to the subspace of the (pure) mode undergoing the instability. Finally, near the mode interaction point is a cusp that forms the leftmost boundary of two saddle-node bifurcations on the mixed mode branch.

FIGURE 3.5: Bifurcation sets for $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$: primary pitchfork in $\Sigma_{\pm}$ subspaces ($P_{\pm}$: thick solid curves), saddle-node in $\Sigma_{\pm}$ ($SN_{\pm}$: thin solid lines), symmetry-breaking (pitchfork) from $\Sigma_{\pm}$ to mixed mode state ($SB_{\pm}$: dash dotted curves), saddle-node on mixed mode state (SN_{MM} : thin solid curves). In the larger regions are indicated in parentheses the types of solutions that exist there.

FIGURE 3.6: Bifurcation diagram for $\nu = -1.5$, $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$ showing $|A+B|$ (projection onto Σ_+) and $|A-B|$ (projection onto Σ_-). Solid (dashed) curves denote stable (unstable) solutions. Solutions in the $\Sigma_{\pm}$ subspaces are labelled with $S_{\pm}$ and asymmetric mixed modes with MM .

The bifurcation diagram in Fig. 3.6 shows the three bifurcations that occur for $\nu = -1.5$. There is a primary pitchfork bifurcation in Σ_+ at $f = 1.61$, and in Σ_- at $f = 1.998$; these produce the two $\mathbf{Z}_2$ -symmetric branches, $S_{\pm}$. A symmetry-breaking bifurcation on the S_- branch at $f = 2.196$ stabilises that branch and gives rise to a branch of (four symmetry-related) asymmetric mixed modes (MM).

FIGURE 3.7: Phase space portraits corresponding to the case of Fig. 3.6 with $f = 2.5$. There are two stable steady states S_+ (solid dots) in Σ_+ , two stable steady states S_- (asterisks) in Σ_- and four unstable asymmetric steady states MM (open circles); one representative solution on each group orbit is labelled. (a) and (b) show projections onto $A = A_r + iA_i$ and $B = B_r + iB_i$, respectively, while the projections in (c) and (d) illustrate the interchange symmetry (3.21). The trivial state is not shown.

Phase portraits illustrating the different types of solutions are shown in Fig. 3.7 for the parameters of Fig. 3.6 and $f = 2.5$; here $A = A_r + iA_i$ and $B = B_r + iB_i$. The projections onto (A_r, B_r) and (A_i, B_i) show the invariant subspaces $\Sigma_{\pm}$ where the solutions $S_{\pm}$ are found, and the effect of the symmetry-breaking bifurcation from S_- that produces the (four) asymmetric MM states.

With positive detuning the scenario is more complex, as illustrated for $\nu = 1.5$ in Fig. 3.8. The primary pitchfork bifurcations in $\Sigma_{\pm}$, at $f = 1.947$ and $f = 1.671$, respectively, are subcritical. The nontrivial $\mathbf{Z}_2$ -symmetric branches created at these bifurcations, $S_{\pm}$, are stabilised (within the corresponding subspace) in saddle-node bifurcations at $f = 0.95$ and $f = 1.05$, respectively.

In the intermediate region between these saddle-node bifurcations and the original pitchfork bifurcations is a branch of asymmetric MM states connecting to S_+ at $f = 0.977$ and S_- at $f = 1.548$. This MM branch undergoes two saddle-node bifurcations, at $f = 1.011$ and $f = 1.52$; between these values are three distinct MM solutions, one of which is stable. A separate unstable asymmetric MM branch bifurcates at $f = 1.077$ from S_- and stabilizes it. Both $S_{\pm}$ branches are thus stable for large forcing values.

We remark that the case $a = \mu_r = 0$, where neither nonlinearity nor coupling contribute to dissipation, is degenerate in certain respects. The three leftmost saddle-node bifurcations, for example, on the $S_{\pm}$ and MM branches, occur exactly at $f = 1$, while the two symmetry-breaking bifurcations

FIGURE 3.8: Bifurcation diagram for $\nu = 1.5$, $\phi = 0$, $a = 0$, $\mu = 0.05 + 0.2i$ showing $|A + B|$ and $|A - B|$. Solid (dashed) curves denote stable (unstable) solutions. Solutions in the $\Sigma_{\pm}$ subspaces are labelled with $S_{\pm}$ and asymmetric mixed modes with MM .

nearest $f = 1$, one on S_+ and one on S_- , coincide as well.

It is also worth pointing out that, in the limit of large forcing, there are always four stable steady states and four unstable steady states. This fact, which can be seen in Fig. 3.6 and Fig. 3.8, and in the following two subsections, is independent of the forcing phase ϕ because, at large amplitude, the coupling term becomes negligible compared to the nonlinearity. In this limit the coupled system will increasingly resemble two uncoupled oscillators. The uncoupled system has four stable states, corresponding to the four possible phases with both oscillators excited, and four unstable states, corresponding to the four possible equilibria with only one oscillator excited. Thus, one does not expect complicated dynamics for large forcing regardless of the forcing phase — the residual symmetry that depends on ϕ is important near onset where the weak coupling is relevant, but not at large amplitude.

Antisymmetric forcing, $\phi = \pi$

With antisymmetric forcing the governing equations for the two coupled oscillators are

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2A + \mu B, \quad (3.32a)$$

$$\dot{B} = (-1 + i\nu)B - f\bar{B} - (a + i)|B|^2B + \mu A. \quad (3.32b)$$

Recall that these equations are equivariant under the cyclic permutation $\rho : (A, B) \rightarrow i(B, A)$, which generates a representation of the group C_4 . There are no nontrivial fixed point subspaces and, consequently, the primary bifurcation cannot be reduced to the case of a single oscillator. The

four eigenvalues describing stability of the trivial state are given by

$$\lambda_{\pm} = -1 \pm i\mu_i + \sqrt{f^2 - \nu^2 + \mu_r^2 \pm 2i\nu\mu_r}, \quad (3.33a)$$

$$\beta_{\pm} = -1 \pm i\mu_i - \sqrt{f^2 - \nu^2 + \mu_r^2 \pm 2i\nu\mu_r}, \quad (3.33b)$$

which are, in general, complex.

A Hopf bifurcation occurs along the curve

$$f^2 = (1 + \nu^2)(1 - \mu_r^2); \quad \nu \neq -\frac{\mu_i}{\mu_r}, \quad (3.34)$$

where $\lambda_{\pm} = \pm i\Omega$, creating a C_4 -symmetric periodic orbit (P_{C_4}). The Hopf frequency is given by

$$\Omega = \mu_i + \nu\mu_r. \quad (3.35)$$

We emphasize that this is a *primary* bifurcation, in contrast to the secondary Hopf bifurcations that already arise in a single oscillator system with $a < 0$; see Eq. (3.9) and Eq. (3.29). It results from the linear coupling and the symmetry properties of antisymmetric forcing, and not nonlinear effects. Note also that the Hopf frequency Ω is small if the coupling μ is small, as assumed in this analysis.

If the interaction is conservative ($\mu_r = 0$), then the location of the primary instability, which is now oscillatory rather than steady, is unchanged from the uncoupled problem and given by $f^2 = 1 + \nu^2$. If the interaction is not conservative then the Hopf bifurcation is shifted to lower forcing values, i.e., the interaction is always destabilising.

The periodic modulated solution P_{C_4} generated in the primary Hopf bifurcation is not expected to persist for large values of f where the influence of the coupling term diminishes compared to the nonlinear and forcing terms, but only an $\mathcal{O}(|\mu|)$ distance above onset. Beyond that point, this C_4 -symmetric limit cycle must be replaced by steady states (S) approximating the solutions of the uncoupled problem given by Eq. (3.7). We find that this transition generally takes the form of a saddle-node heteroclinic bifurcation. This heteroclinic bifurcation is codimension one because the four steady states involved lie on the same group orbit.

The bifurcation sets corresponding to the conservative case $a = \mu_r = 0$ are shown in Fig. 3.9. The primary Hopf (H) bifurcation is supercritical for $\nu < 0$ and subcritical for $\nu > 0$, just as the primary symmetry-breaking bifurcation is for a single oscillator (cf. Fig. 3.1). In the supercritical region, the C_4 -symmetric periodic orbit created in the primary bifurcation is stable and exists until a saddle-node heteroclinic (SN Het) bifurcation. In the subcritical region for $\nu \leq 1.322$, this stable periodic orbit exists between the saddle-node heteroclinic bifurcation and a saddle-node of periodic orbits (SN Per); the unstable C_4 -symmetric periodic orbit involved in this saddle-node is the one generated in the subcritical primary Hopf bifurcation. For $\nu \geq 1.322$ there is a curve of saddle-node (SN) bifurcations generating (four) asymmetric steady states and, for $\nu \geq 1.466$, a second such

FIGURE 3.9: Bifurcation sets for $\phi = \pi$, $a = 0$, $\mu = 0.2i$: Hopf (H: dashed curves) and saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), homoclinic (Hom), saddle-node homoclinic (SN Hom), Bogdanov-Takens points (BT). In the larger regions, the types of solutions that exist there are indicated in parentheses. In (a) the regions of one and two C_4 -symmetric periodic orbits (P_{C_4}) are further marked with "1" and "2", respectively. In (b) a close-up of the complex region between two BT points near the cusp of SN bifurcations is shown and, in the inset, a further zoom showing the cusp region.

curve originating in a cusp. Bogdanov-Takens (BT) points on each of these SN curves form the boundaries of associated Hopf and homoclinic (Hom) bifurcation sets (only visible in the lower plot) between which (four) asymmetric periodic solutions can be found. The termination of these two bifurcation sets at the BT point on the right is preceded by a couple of transitions. First, the homoclinic bifurcation set collides with the upper SN curve, marking the upper boundary of a set of saddle-node homoclinic (SN Hom) bifurcations. The homoclinic bifurcation then detaches again from the SN curve before reaching the cusp and moves towards the Bogdanov-Takens point on the lower SN curve.

As with the case of symmetric forcing, the bifurcation structure is quite complex for positive detuning and is characterised for $\phi = \pi$ by multiple saddle-node bifurcations, secondary Hopf bifurcations, and Bogdanov-Takens points. The periodic solutions characteristic of antisymmetric forcing exist in a finite region near the primary Hopf bifurcation and are destroyed in global bifurcations (heteroclinic or saddle-node heteroclinic).

Several bifurcation diagrams are shown in Fig. 3.10. For $\nu = -0.5$ and $\nu = 0.5$, the C_4 -symmetric periodic orbit (P_{C_4}) created in the primary bifurcation at $f = 1.118$ terminates in a saddle-node heteroclinic bifurcation at $f = 1.491$ and $f = 1.273$, respectively; for $\nu = 0.5$, the periodic orbit first undergoes a saddle-node bifurcation at $f = 1.011$. The stable and unstable branches of steady states (S) created in the saddle-node heteroclinic bifurcation persist for large f .

Phase portraits illustrating both steady and periodic solutions are shown in Fig. 3.11 for the parameters of Fig. 3.10(c) and $f = 1.5$. At this forcing value there are four stable steady states, related by the symmetry ρ of Eq. (3.22), and four unstable steady states, likewise related by the rotation ρ . The projections onto (A_r, B_i) and (A_i, B_r) best reveal this C_4 symmetry, and further show that the limit cycle itself is symmetric.

For $\nu = 1.4$, this scenario is interrupted by an isola of steady states existing for $1.057 \geq f \leq 1.271$. The initial (unstable) C_4 -symmetric periodic orbit created in via Hopf bifurcation at $f = 1.721$ is destroyed in a saddle-node heteroclinic bifurcation at the right boundary of this isola. A heteroclinic bifurcation on the lower branch of the isola generates a stable C_4 -symmetric periodic orbit, which terminates on the unbounded steady state branch in a saddle-node heteroclinic bifurcation at $f = 1.23$. A secondary Hopf bifurcation at $f = 1.0808$ on the upper branch of the isola produces stable asymmetric periodic orbits (P_a) that quickly collide with the lower branch in a homoclinic bifurcation at $f = 1.0818$.

With $\nu = 2$, the primary subcritical Hopf bifurcation occurs at $f = 2.236$ and the C_4 -symmetric periodic orbit created there again terminates in a saddle-node heteroclinic bifurcation, at $f = 1.944$. The lower branch of steady states created here is stabilised in a second saddle-node bifurcation at $f = 1.021$ and destabilised again in another saddle-node at $f = 1.368$, then once more at $f = 1.131$. The unbounded steady states appear in a saddle-node bifurcation at $f = 1.218$.

In the case of real coupling μ there are differences with the scenario described above, most notably the absence of Bogdanov-Takens points in the cusp region (cf. the lower panel of Fig. 3.9). Because of degeneracies that make the numerical calculations difficult in the purely real case, we show the bifurcation sets for $\mu = 0.2 + 0.04i$ in Fig. 3.12.

With complex coupling there is a double zero (degenerate Hopf) bifurcation at $\nu = -\mu_i/\mu_r$ where the Hopf frequency Ω vanishes; see Eq. (3.35). Note that we use the term “double-zero bifurcation” to refer to the case of a pair of zero eigenvalues described by a semisimple (diagonalizable) Jordan

FIGURE 3.10: Bifurcation diagrams for $\phi = \pi$, $a = 0$, $\mu = 0.2i$ showing the norm $|A| = |(A, B)|$ of steady states (S), C_4 -symmetric periodic orbits (P_{C_4}), and asymmetric periodic orbits (P_a): (a) $\nu = -0.5$, (b) $\nu = 0.5$, (c) $\nu = 1.4$, (d) $\nu = 2$. For $\nu = 1.4$, there is a secondary supercritical Hopf bifurcation on the isola of steady states that generates stable asymmetric periodic orbits, shown in the inset. Stable (unstable) solutions are shown with solid (dashed) lines.

FIGURE 3.11: Phase space portraits corresponding to the case of Fig. 3.10(c) with $f = 1.5$. There are four stable steady states S_s (solid dots) and four unstable steady states S_u (open circles) related by the symmetry (3.22); one representative solution on each group orbit is labelled. The C_4 -symmetric periodic orbit, P_{C_4} , is also shown. (a) and (b) show projections onto $A = A_r + iA_i$ and $B = B_r + iB_i$, respectively, while the projections in (c) and (d) illustrate the C_4 symmetry. The trivial state is not shown.

FIGURE 3.12: Bifurcation sets for $\phi = \pi$, $a = 0$, $\mu = 0.2 + 0.04i$: Hopf (H: dashed curves) and saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), and double zero point. In the larger regions, the types of solutions that exist there are indicated in parentheses.

FIGURE 3.13: (a) Close-up of the double zero bifurcation point and (b) cusp region of Fig. 3.12 showing bifurcation sets: Hopf (H: dashed curves), saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), saddle-node heteroclinic (SN Het), and heteroclinic (Het). The heteroclinic bifurcation set near the cusp does not collide there, but with the upper branch, although this cannot be distinguished on the scale of the plot. In the larger regions, the types of solutions that exist there are indicated in parentheses.

block, while the term “Bogdanov-Takens bifurcation” refers throughout this paper to the usual non-semisimple case; some authors use these terms synonymously (Kuznetsov, 2004). In this case, since the codimension-two point corresponds to the vanishing of the Hopf frequency along a curve of Hopf bifurcations, it may also be called a degenerate Hopf bifurcation.

The curve of saddle-node heteroclinic bifurcations that follows the primary Hopf bifurcation for negative ν appears to pinch off here at the double zero point ($\nu = -0.2$ for this choice of coupling) then continues to the right before separating into distinct saddle-node and heteroclinic bifurcations at about $\nu = -0.132$; this region is shown in more detail in Fig. 3.13(a). We note that the details of this double-zero point can be very complicated (Marqués et al., 2013) and include, in a small

FIGURE 3.14: Contours showing the transition from supercritical to subcritical Hopf bifurcation for coupling values with $\mu_r = 0, 0.1, 0.2, 0.3, 0.4, 0.5$. The transition is independent of μ_i .

neighbourhood, many other bifurcations: pitchfork, homoclinic, heteroclinic, gluing, Bogdanov-Takens, etc. We did not attempt to resolve these in Fig. 3.12 or Fig. 3.13.

The curve of heteroclinic bifurcations collides with the (smooth) curve of saddle-node bifurcations at $\nu = 0.98$, creating a short region of saddle-node heteroclinic bifurcations, before detaching again at $\mu = 0.982$ and heading towards the cusp on the other curve of saddle-nodes and merging (just beyond the cusp) with the upper branch. Figure 3.13(b) shows this scenario in more detail. Note the absence, compared to Fig. 3.9, of Bogdanov-Takens points and the associated asymmetric periodic states.

The bifurcation sets in the examples above were obtained for $a = 0$. In this case, the transition from supercritical to subcritical Hopf occurs at $\nu = 0$ and the complicated (cusp) region with multiple saddle-node bifurcations appears for $\nu \gtrsim 1$. If $a \neq 0$ these features can be shifted left ($a < 0$) or right ($a > 0$). Figure 3.14 shows the transition from supercritical to subcritical Hopf bifurcation in the (ν, a) plane, which does not depend on μ_i . If $\mu_r = 0$ the transition occurs along the line $a = \nu$, while for $\mu_r > 0$ there is a critical value of $|a| = a_c$ such that for $a > a_c$ the Hopf bifurcation is supercritical for all ν and for $a < -a_c$ the Hopf bifurcation is subcritical for all ν . For $\mu_r = 0.5$, for example, this critical value is $a_c \sim 1$.

General forcing phase

With $\phi \neq 0, \pi$ the governing equations (3.16) do not simplify. They possess only the subharmonic symmetry R of Eq. (3.20). This case is intermediate between symmetric forcing ($\phi = 0$), where the primary instabilities can be identified with the pitchfork bifurcations of single oscillators, and antisymmetric forcing ($\phi = \pi$), where the primary instability is a Hopf bifurcation. Transitions between steady and oscillatory behaviour are observed as parameters are varied.

In the limiting case of conservative ($\mu_r = 0$) and purely dissipative coupling ($\mu_i = 0$) the location of the primary bifurcations are given by relatively simple expressions.

In the case of conservative coupling, when $\mu_r = 0$, a primary Hopf bifurcation, which produces $\mathbf{Z}_2$ -symmetric periodic orbits ($P_{\mathbf{Z}_2}$), occurs for

$$|\nu| < \sqrt{1 + \mu_i^2} \tan \frac{\phi}{2}, \quad (3.36)$$

along the curve

$$f^2 = \frac{(1 + \mu_i^2)(1 + \nu^2)}{1 + \mu_i^2 \sin^2 \frac{\phi}{2}}, \quad (3.37)$$

and the Hopf frequency satisfies

$$\Omega = \mu_i \sqrt{\frac{(1 + \mu_i^2) \sin^2 \frac{\phi}{2} - \nu^2 \cos^2 \frac{\phi}{2}}{1 + \mu_i^2 \sin^2 \frac{\phi}{2}}}. \quad (3.38)$$

Primary pitchfork bifurcations, producing steady states (S), occur for

$$|\nu| \geq \tan \frac{\phi}{2} \sqrt{1 + \mu_i^2 \cos^2 \frac{\phi}{2}}, \quad (3.39)$$

on the curves

$$f^2 = 1 + \nu^2 + \mu_i^2 \cos \phi \pm 2 \sqrt{\nu^2 \mu_i^2 \cos^2 \frac{\phi}{2} - \mu_i^2 \sin^2 \frac{\phi}{2} - \frac{\mu_i^4}{4} \sin^2 \phi}. \quad (3.40)$$

The codimension-two points where the pitchfork and Hopf bifurcation curves meet are given by

$$(\nu, f) = \sqrt{1 + \mu_i^2} \left(\pm \tan \frac{\phi}{2}, \sec \frac{\phi}{2} \right). \quad (3.41)$$

Such transitions will be referred to here as Bogdanov-Takens bifurcations with $\mathbf{Z}_2$ symmetry (see, e.g., Marqués et al., 2013; Tuckerman, 2012) since the pair of zero eigenvalues corresponds to a non-semisimple block of the Jordan normal form. Note that a Bogdanov-Takens bifurcation with $\mathbf{Z}_2$ symmetry will occur when the equilibrium in question is the symmetric (trivial) state $A = B = 0$, while Bogdanov-Takens bifurcations occurring on finite amplitude branches (where the $\mathbf{Z}_2$ symmetry has already been broken) will be the usual (asymmetric) kind.

In the case of purely dissipative coupling, with $\mu_i = 0$, primary Hopf bifurcation occurs for

$$|\nu| > \sqrt{1 - \mu_r^2} \cot \frac{\phi}{2}, \quad (3.42)$$

along the curve

$$f^2 = \frac{(1 - \mu_r^2)(1 + \nu^2)}{1 - \mu_r^2 \cos^2 \frac{\phi}{2}}, \quad (3.43)$$

FIGURE 3.15: Contour map showing the location of the $\mathbf{Z}_2$ -symmetric Bogdanov-Takens point with $\nu > 0$ in the conservative and purely dissipative cases, as given by Eq. (3.41) and Eq. (3.47). Curves for $\nu < 0$ are obtained by reflection. Curves are shown for constant μ , in steps of $0.2i$ ($\mu \in \text{Im}$) and 0.2 ($\mu \in \text{Re}$), and for constant ϕ (radial lines), in steps of $\pi/6$. The central curve intersecting $(\nu, f) = (0, 1)$ corresponds to the uncoupled case $\mu = 0$.

and the Hopf frequency is

$$\Omega = \mu_r \sqrt{\frac{(\mu_r^2 - 1) \cos^2 \frac{\phi}{2} + \nu^2 \sin^2 \frac{\phi}{2}}{1 - \mu_r^2 \cos^2 \frac{\phi}{2}}}. \quad (3.44)$$

Primary pitchfork bifurcations occur for

$$|\nu| \leq \cot \frac{\phi}{2} \sqrt{1 - \mu_r^2 \sin^2 \frac{\phi}{2}}, \quad (3.45)$$

on the curves

$$f^2 = 1 + \nu^2 + \mu_r^2 \cos \phi \pm 2 \sqrt{\mu_r^2 \cos^2 \frac{\phi}{2} - \nu^2 \mu_r^2 \sin^2 \frac{\phi}{2} - \frac{\mu_i^4}{4} \sin^2 \phi}. \quad (3.46)$$

Bogdanov-Takens bifurcations with $\mathbf{Z}_2$ symmetry terminate the set of Hopf bifurcations at

$$(\nu, f) = \sqrt{1 - \mu_r^2} \left(\pm \cot \frac{\phi}{2}, \csc \frac{\phi}{2} \right). \quad (3.47)$$

Figure 3.15 shows the location of the $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points in the conservative and purely dissipative cases. Note that for μ imaginary (upper set of curves in Fig. 3.15), increasing ϕ moves the $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points outwards, while for μ real (lower set of curves)

the $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points move inwards with increasing ϕ . In both cases the region of Hopf bifurcation increases with ϕ . Note that, although Eq. (3.41) and Eq. (3.47) have well-defined limits as $\mu_i \rightarrow 0$ and $\mu_r \rightarrow 0$, respectively, there are no primary Hopf bifurcations in the uncoupled problem since the Hopf frequency Ω is zero; see Eq. (3.38) and Eq. (3.44). For $\mu = 0$ there is a double zero (dual pitchfork) bifurcation for any value of detuning.

Figure 3.16 shows the primary Hopf and pitchfork bifurcation sets corresponding to two sets of values from Fig. 3.15: $\phi = \pi/2$, $\mu = 0.2i$, shown in Fig. 3.16(a), with $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points at $(\nu, f) = (\pm 1.0198, 1.4422)$, and $\phi = \pi/2$, $\mu = 0.2$, shown in Fig. 3.16(c), with $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points at $(\nu, f) = (\pm 0.9798, 1.3856)$.

Observe that conservative coupling produces a region of Hopf bifurcations centred at $\nu = 0$ with pairs of pitchfork bifurcations outside of this, in accordance with Eq. (3.36) and Eq. (3.39). Purely dissipative coupling produces the opposite, a region of pitchfork bifurcations centred at $\nu = 0$ with Hopf bifurcations outside of this, in accordance with Eq. (3.45) and Eq. (3.42). Perturbations of these conservative and purely dissipative cases are similar, but with asymmetrically located $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points, as seen for $\mu = 0.04 + 0.2i$ (b) and for $\mu = 0.2 + 0.04i$ (d).

A more complete bifurcation picture is shown in Fig. 3.17 for the case of $a = 0$ and $\mu = 0.2i$, when the primary Hopf bifurcation set is centred around $\nu = 0$. We first note the broad similarities with Fig. 3.9. Primary periodic solutions are destroyed via saddle-node heteroclinic or (less often) heteroclinic bifurcation. The bifurcation structure is most complex for positive detuning where a series of saddle-node bifurcations leads to hysteresis and isolas of steady states. Again, there is a very complicated region near the cusp of saddle-node bifurcations at $(\nu, f) = (1.116, 1.099)$, where a homoclinic bifurcation set arising from one Bogdanov-Takens point collides with the upper branch of saddle-node bifurcations, initiating a set of saddle-node homoclinic bifurcations before quickly detaching and moving towards another Bogdanov-Takens point on the lower saddle-node branch; see the inset of Fig. 3.9 for the detail of this scenario, which is not visible on the scale of Fig. 3.17.

The main difference between this case and that of $\phi = \pi$ is the transition from primary Hopf to a pair of pitchfork bifurcations for $|\nu| \gtrsim 1$. These transitions, the details of which are shown in Fig. 3.18, are associated with an additional three Bogdanov-Takens points, two with $\mathbf{Z}_2$ symmetry and one without. The leftmost transition is organised by a Bogdanov-Takens bifurcation with $\mathbf{Z}_2$ symmetry (the simplest “ $s = 1$ ” case of Kuznetsov, 2004, Sec. 9.5.3, and of Peng and Li, 2014; see also Knobloch, 1986) that, in addition to the Hopf and pitchfork bifurcations, generates a heteroclinic bifurcation set. This heteroclinic bifurcation set soon collides with a curve of saddle-node bifurcations originating very near the same Bogdanov-Takens point (similar to the scenario seen in Tuckerman, 2012).

FIGURE 3.16: Primary Hopf (H) and pitchfork (P) bifurcation sets for (a) $\mu = 0.2i$, (b) $\mu = 0.04 + 0.2i$, (c) $\mu = 0.2$, and (d) $\mu = 0.2 + 0.04i$. $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points (not marked) lie on the intersection.

FIGURE 3.17: (a) Bifurcation sets for $\phi = \pi/2$, $a = 0$, $\mu = 0.2i$: Hopf (H: dashed curves), saddle-node of periodic orbit (SN Per: dashed curve), saddle-node (SN), heteroclinic (Het), saddle-node heteroclinic (SN Het), homoclinic (Hom), Bogdanov-Takens points (BT). Some bifurcation sets (SN Per, homoclinic, and gluing bifurcations, for example) are hard to see on the scale of the plot and are not labelled. Panel (b) shows a close-up of the complex transitions for positive detuning. In the larger regions, the types of solutions that exist there are indicated in parentheses.

The transition from Hopf to pitchfork bifurcation near $\nu = 1$ (see lower panel of Fig. 3.18) is again organised by a Bogdanov-Takens bifurcation with $\mathbb{Z}_2$ symmetry, but according to the more complex case (the “ $s = -1$ ” case of Kuznetsov, 2004, Section 9.5.3, and of Peng and Li, 2014), which generates an additional set of asymmetric Hopf bifurcations, producing asymmetric periodic orbits (P_a), and a set of homoclinic gluing bifurcations — a set of saddle-node bifurcations on the $\mathbb{Z}_2$ -symmetric periodic orbits is not shown.

FIGURE 3.18: Close-up of the (a) left and (b) right transition between pitchfork and Hopf bifurcation of Fig. 3.17 showing bifurcation sets: pitchfork (P), Hopf (H: dashed curves), saddle-node (SN), saddle-node heteroclinic (SN Het), heteroclinic (Het), homoclinic gluing (Hom Gluing), and Bogdanov-Takens points (BT). Some bifurcation sets (homoclinic and heteroclinic, near the lower, asymmetric BT point in the lower panel) are not visible on the scale of the plot and are not labelled. In the larger regions, the types of solutions that exist there are indicated in parentheses.

The asymmetric Hopf and gluing bifurcation sets remain close as f is decreased until the lower pitchfork bifurcation curve is crossed; at this point a gluing bifurcation is forbidden by the stability of the trivial (Z_2 -symmetric) state and it splits into separate homoclinic and heteroclinic bifurcations (as occurs for the single oscillator case of Fig. 3.3). The homoclinic bifurcation set terminates in another (asymmetric) Bogdanov-Takens point located on the set of saddle-node bifurcations originating from the lower pitchfork bifurcation near this crossing. The heteroclinic bifurcation set merges with the saddle-node bifurcation set near this same point, creating a set of saddle-node heteroclinic bifurcations.

FIGURE 3.19: Bifurcation diagrams for $\phi = \pi/2$, $a = 0$, $\mu = 0.2i$ showing the norm $|A| = |(A, B)|$ of steady states (S), Z_2 -symmetric periodic orbits (P_{Z_2}), and asymmetric periodic orbits (P_a): (a) $\nu = -1.5$, (b) $\nu = 0$, (c) $\nu = 1$, (d) $\nu = 2$. For $\nu = 1$ there is a secondary supercritical Hopf bifurcation on the isola of steady states that generates stable asymmetric periodic orbits, shown in the inset. Stable (unstable) solutions are shown with solid (dashed) lines.

Four representative bifurcation diagrams are shown in Fig. 3.19 for this set of parameters, and should be compared with those of Fig. 3.10.

For $\nu = -1.5$ there is no Hopf bifurcation, but a pair of pitchfork bifurcations at $f = 1.714$ and

$f = 1.888$, respectively. These are followed by a saddle-node bifurcation at $f = 2.275$. Note, in comparison with Fig. 3.10, the splitting of the stable and unstable steady state branches (for large f) into two distinct pairs.

For $\nu = 0$ the primary instability at $f = 1.01$ is a Hopf bifurcation, which produces a stable $\mathbf{Z}_2$ -symmetric periodic orbit that persists until a saddle-node heteroclinic bifurcation at $f = 1.15$. This is followed by a second saddle-node bifurcation at $f = 1.389$.

For $\nu = 1$ there is again a Hopf bifurcation, now at $f = 1.428$, but the unstable $\mathbf{Z}_2$ -symmetric periodic orbit created here is soon destroyed in a saddle-node heteroclinic bifurcation at $f = 1.374$ that forms the right boundary of an isola of steady states. The upper branch of this isola experiences a Hopf bifurcation (see the inset of Fig. 3.19(c)) at $f = 1.076$ leading to a branch of stable asymmetric periodic orbits that is quickly destroyed by homoclinic collision with the lower branch at $f = 1.0773$. A heteroclinic bifurcation near this point, at $f = 1.0848$, generates an unstable $\mathbf{Z}_2$ -symmetric periodic orbit that undergoes a saddle-node bifurcation at $f = 1.08461$, quickly followed by symmetry breaking bifurcations at $f = 1.08463$ and $f = 1.08467$; the stabilised periodic orbit is eventually destroyed in another heteroclinic bifurcation at $f = 1.121$ involving a larger amplitude steady state branch created via saddle-node bifurcation at $f = 1.112$. The final saddle-node bifurcation occurs at $f = 1.249$.

For $\nu = 2$ the primary instability is again steady, with subcritical pitchfork bifurcations at $f = 2.124$ and $f = 2.343$. The first branch is stabilised in a saddle-node bifurcation at $f = 1.01$, then destabilised again at $f = 1.567$ before merging with the second branch in a third saddle-node bifurcation at $f = 1.145$. Two more saddle-node bifurcations at $f = 1.104$ and $f = 1.22$ create the steady branches that persist for large f .

Phase portraits are shown in Fig. 3.20 for the case corresponding to Fig. 3.19(c) at $f = 1.374$, which is where the $\mathbf{Z}_2$ -symmetric periodic orbit created in the primary Hopf bifurcation terminates in a saddle-node heteroclinic bifurcation. At this forcing value there are two stable steady state branches (each representing two solutions related by the reflection R of Eq. (3.20), and three unstable steady state branches. The smallest (pair) of these solutions is colliding with the periodic orbit. Note that the projections onto (A_r, A_i) and (B_r, B_i) , which focus on the dynamics of one oscillator, are qualitatively similar to those of Fig. 3.11 (modulo a rotation for B due to the forcing phase). The projections onto (A_r, B_i) and (A_i, B_r) , however, which reflect the effects of coupling and symmetry, are quite different. The C_4 symmetry of Fig. 3.11(c,d) is absent in Fig. 3.20(c,d).

3.5 Comparison with experiment

Subharmonic surface waves in horizontally vibrated containers (see, e.g., Garrett (1970), Jones (1984), Miles (1988), Miles and Henderson (1990), and Underhill, Lichter, and Bernoff (1991)) are an important example of a real experimental system that can be modelled as a pair of coupled parametrically forced oscillators with $\phi = \pi$. The case considered in detail in Sec. 3.4.3 will be

FIGURE 3.20: Phase space portraits corresponding to the case of Fig. 3.19(c) with $f = 1.374$, at the saddle-node heteroclinic bifurcation. There are two stable steady states (upper branch denoted by solid dots, and the lower by asterisks) and two (larger amplitude) unstable steady states (upper branch denoted by open circles, and the lower by open squares). The $\mathbb{Z}_2$ -symmetric periodic orbit, which encloses the origin (trivial state not shown) is colliding with two smaller amplitude unstable steady states (open triangles). Projections onto (a) (A_r, A_i) , (b) (B_r, B_i) , (c) (A_r, B_i) , (d) (A_i, B_r) .

relevant if the subharmonic waves are dominant (i.e., the harmonic wave field can be neglected) and when the container is long enough (or forcing frequency high enough) that the subharmonic waves can be thought of as two weakly interacting (cross-wave) patterns generated by opposing endwalls. This is the case in several recent experiments (Porter et al., 2012; Tinao et al., 2014).

The results of Sec. 3.4.3 agree with recent theoretical work on cross-waves (Pérez-Gracia et al., 2014; Perez-Gracia et al., 2014; Porter et al., 2013; Tinao et al., 2014) in predicting a primary Hopf bifurcation to modulated (quasiperiodic) solutions. The results of that section go further in, among other things, predicting the destruction of these modulated solutions in a saddle-node heteroclinic bifurcation (less often, in a standard heteroclinic bifurcation). Both the region of existence of these modulated solutions and their frequency of modulation decrease as the strength of the coupling decreases.

Although modulated solutions have been observed before in cross-wave experiments (Ayanle, Bernoff, and Lichter, 1990; Lichter and Underhill, 1987; Shemer and Lichter, 1987; Underhill, Lichter, and Bernoff, 1991), these were likely due to either nonlinear effects or mode interactions, while the modulated solutions investigated here arise in a primary Hopf bifurcation and are due to interaction between the out-of-phase (by $\pi/2$) wave fields generated at opposite sides of the container. These solutions, shown in Fig. 3.21, are associated with a slow cycling between balanced and one-sided

FIGURE 3.21: Modulated (quasiperiodic) solution found experimentally in a 20 cm square container with 5 cSt oil driven at 13.95 Hz. The crosswise mode number is $N = 11$ and the modulation period is about 45.6 s. The snapshots (a)-(d) are separated by approximately $1/8$ of this period, or 5.7 s. Reproduced from Tínao et al. (2014).

excitation and, to the best of our knowledge, have not been observed prior to the recent experiments of Tínao et al. (2014).

Possible reasons for this include the complicating effects of mode interactions mentioned above and the difficulty of controlling boundary conditions (Porter et al., 2012; Tínao et al., 2014), but a more basic issue is simply selecting the right range of forcing frequencies for a given container. If the forcing frequency is too low, then cross-waves will rapidly invade the entire surface and the interaction will be too strong (i.e., a weakly coupled oscillator model will not apply) and, among other difficulties, the distinction between one-sided and two-sided excitation is lost. If the forcing frequency is too high, then, although a weakly coupled oscillator approximation is valid, the modulation frequency will be too low to be measured accurately. Furthermore, the range of existence of these quasiperiodic solutions may be so narrow that it is washed out by various experimental errors.

Modulated solutions that cycle between one-sided and balanced excitation were located in the experiments of Tínao et al. (2014) using a 20 cm square experimental container and a forcing frequency of 13-14 Hz. These solutions, located at onset, are an important confirmation of the modulated solutions predicted by theory. Additional details of Fig. 3.9 and Fig. 3.12, unfortunately, are impossible to verify due to the limitations of the measurement; there are substantial sources of

FIGURE 3.22: Modulation (Hopf) frequency (in Hz) of the solution shown in Fig. 3.21 over an interval of forcing frequencies ω/π (in Hz) around the resonance frequency of 13.52 Hz. The experimentally measured Hopf frequency Ω is compared with that predicted by the model obtained in Tinao et al. (2014) with the derived boundary conditions (solid curve) and with Dirichlet boundary conditions (dashed curve); remaining parameters for these calculations are: viscosity 5 cSt, surface tension 19.7 dyne/cm, density 0.913 g/cm³, container depth 5 cm, and width 9 cm. Reproduced from Tinao et al. (2014).

experimental error, particularly in maintaining constant boundary conditions.

3.5.1 Dependence on detuning

A significant issue in comparing experiment with theory is the sensitivity to the coupling coefficient μ . In the above analysis, this was treated as constant while forcing and detuning were varied, but in a real experiment it will not be so. Coupling strength depends on the overlapping wave fields and, thus, on all experimental parameters (container size, damping, boundary conditions, etc.). In particular, it varies strongly with detuning. Negative detuning leads to evanescent wave fields that decay quickly away from the endwall (Tinao et al., 2014) (weak coupling) while positive detuning leads to more oscillatory wave fields that extend further into the interior (stronger coupling). Furthermore, the phase of the subharmonic waves in the overlap (coupling) region is important — some phases will lead to constructive interference and others to destructive interference.

We thus expect that, for cross-waves excited in horizontally vibrated containers, the coupling coefficient μ should grow with ν and oscillate as well, at least for $\nu > 0$ when the wave fields are oscillatory in the interior. The oscillation with ν is clearly seen in the experimental measurements of the Hopf frequency, as well as theoretical models (Perez-Gracia et al., 2014; Tinao et al., 2014); see Fig. 3.22.

As a first step to a more realistic dependence of the coupling coefficient $\mu(\nu)$, we use the following model:

$$\mu(\nu) = \mu_0 e^{\sigma\nu} e^{i[\theta_0 + H(\nu)\Omega_\nu\nu]}, \quad (3.48)$$

where the exponent σ captures the growth rate of the coupling due to increased wave penetration and Ω_ν captures the oscillation due to varying overlap (phase). This oscillation is applied only for

FIGURE 3.23: Primary Hopf bifurcation for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (3.48) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$. (a) Critical forcing $f(\nu)$ (solid curve) compared to the pitchfork bifurcation of a single oscillator (dashed curve). (b) Hopf frequency $\Omega(\nu)$; this should be compared with Fig. 3.22.

positive ν via a Heaviside step function $H(\nu)$.

The primary Hopf bifurcation and the corresponding Hopf frequency, obtained using Eq. (3.48), are shown in Fig. 3.23, which should be compared with Fig. 3.22 (reproduced from Tinao et al., 2014). The particular parameters in Fig. 3.23 were chosen to qualitatively reproduce the experimental data (and theoretical curves) for the observed modulation frequency; Fig. 3.23 can be directly compared with Fig. 3.22 despite the different axes because of the rescaling (3.4) with γ that is done here and not in Tinao et al. (2014) (for that experiment $\gamma \sim 0.008$). This choice of coupling function captures the oscillations of the modulation frequency Ω for positive ν seen in the experiment. Note also that, since the coupling $\mu(\nu)$ is small except for large detuning, the location of the Hopf bifurcation is only shifted with respect to the (pitchfork) instability of an uncoupled oscillator (dashed curve in Fig. 3.23(a)) by a small amount, most visible when ν is large and positive.

Additional bifurcation sets can also be calculated using the ansatz of Eq. (3.48) and compared with the experimental results. In particular, the experiments (Tinao et al., 2014) show that *stable* modulated solutions exist well into the positive detuning regime. It is easy to show that this is not consistent with a conservative nonlinearity $a = 0$ because the transition from supercritical to subcritical Hopf bifurcation occurs near $\nu = 0$, and while stable periodic orbits can be found for larger ν they are soon destroyed by heteroclinic or saddle-node heteroclinic bifurcations; in Figs. 3.9 and 3.12 this occurs for $\nu \sim 1 - 1.5$, while experiments suggest stable modulated solutions are easily located for $\nu \sim 4$.

FIGURE 3.24: Selected bifurcation sets for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (3.48) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$: Hopf (H: dashed curve), saddle-node heteroclinic (SN Het), saddle-node of periodic orbit (SN Per, solid curves for $\nu \gtrsim 2.15$). Dotted vertical lines mark ν values where the Hopf frequency vanishes. Additional bifurcation sets near these double zero points, which may include saddle-node, pitchfork, asymmetric Hopf, homoclinic, and heteroclinic bifurcations (Marqués et al., 2013), are not shown in detail.

Figure 3.24 shows selected bifurcation sets calculated with $a = 2$ and remaining parameters as in Fig. 3.23. Since $a = 2$, the Hopf bifurcation is guaranteed to be supercritical for $\nu \leq 2$ (see Fig. 3.14) while for larger ν it can switch between supercritical and subcritical depending on the value of $\mu_r(\nu)$. The first two of these transitions with the parameters of Fig. 3.24 occur for $\nu \simeq 2.15$ (supercritical to subcritical) and $\nu \simeq 4.48$ (subcritical to supercritical); only the first of these is shown in the figure. Regardless of these transitions for $\nu > 2$, there are substantial regions of stable modulated solutions, bounded above by a saddle-node heteroclinic bifurcation, for nearly all values of detuning, consistent with experiment.

3.6 Comparison with simulations

Given the difficulties inherent in the experiments (Tinao et al., 2014), it is not possible to accurately verify additional details of the dynamics beyond the initial Hopf bifurcation and the existence of a stable modulated solution, although some progress may be possible in future experiments. Here, we take an alternative approach and compare with the results of direct numerical simulations of the Navier-Stokes equations². These are done in a two-dimensional domain to make the calculations more feasible and, thus, cannot replicate the patterns observed in experiments, whose orientation is nearly crosswise. The simulations only permit one (downstream) orientation for the wavevector. These types of parallel subharmonic waves are known to exist (Perez-Gracia et al., 2014; Porter et al., 2013; Varas and Vega, 2007), even if the instability is typically masked by the earlier onset of (nearly) crosswise oriented waves in the full three-dimensional problem.

²Performed with the simulation software COMSOL Multiphysics (<https://www.comsol.com/>).

In any case, the orientation of the subharmonic waves is incidental to the main idea examined here: that weakly coupled subharmonic patterns (more generally, oscillatory systems) driven by antisymmetric forcing, but otherwise identical, will reflect much of the dynamics contained in Eqs. (3.32). Although the values of the parameters in these equations are not known, we expect behaviour similar to that shown in Fig. 3.10(a) or (b) for most frequencies, since this is the typical scenario for small detuning when the nonlinearity is dissipative, $a > 0$.

The simulations described in this section were performed using fluid parameters similar to common silicone oils: viscosity of 5 cSt, surface tension of 20 dyne/cm, and (for simplicity) density of 1 g/cm³. The fluid domain was taken to be 7 cm long and 2.5 cm deep, and forcing frequencies (in the neighbourhood) of 30 Hz, 40 Hz, 50 Hz, and 60 Hz were used. A contact angle of 0° was imposed on the lateral walls.

3.6.1 Modulated solutions

Figure 3.25 shows the filtered amplitude of the harmonic and subharmonic components taken from the times series of the height $h(t)$ of the left edge of the surface. Initial transient behaviour is removed and the results are plotted versus forcing amplitude; the forcing was increased at a rate of 0.004 g 's per second over the relevant part of the range (a faster rate was used over the initial portion safely below the subharmonic instability). Observe that for applied forcing at 30 Hz, 40 Hz, and 50 Hz the subharmonic instability appears to be subcritical, while the case of 60 Hz forcing is less clear. Hysteresis is also seen using a slower rate of change, with both increasing and decreasing forcing (see Fig. 3.27). However, much of this observed hysteresis may be due to the delay effect of the (linear) drift in forcing (Baer, Erneux, and Rinzel, 1989; Mandel and Erneux, 1987), making it hard to distinguish the subcritical from the supercritical case.

With applied forcing at 30 Hz, 40 Hz, and 50 Hz, the subharmonic instability is oscillatory and modulated solutions are found immediately after it. In each of these cases the period of modulation increases with forcing until the modulations abruptly disappear — evidence of a global bifurcation. For 30 Hz and 40 Hz forcing, there are additional transitions over the range of the simulation. With 30 Hz forcing there is a second, more irregular, modulated state appearing at about 0.95 g 's, while with 40 Hz forcing there is an apparent transition to a second unmodulated branch of higher amplitude at about 1.49 g 's. The lack of an unambiguous region of modulated solutions with 60 Hz forcing suggests this case may be close to a double zero point where the Hopf frequency vanishes; see Fig. 3.24. This is discussed further below.

An instantaneous surface profile is shown in Fig. 3.26 for the case of 50 Hz forcing and a forcing of 0.9088 g 's, which is above the onset of subharmonic waves. Note that the subharmonic waves dominate over the harmonic waves (in this snapshot, harmonic waves can be seen clearly only near the left endwall) and that the waves extend into the interior of the domain. A relatively short domain of 7 cm was intentionally chosen so that the modulated solutions would have a relatively large frequency and region of existence. A similar requirement was needed for the experimental observations (Tinao et al., 2014). It is perhaps surprising that a relatively strong interaction such

as that in Fig. 3.26 can still be described by a model of weakly coupled parametrically forced oscillators.

FIGURE 3.25: Harmonic (black curve) and subharmonic (grey curve) components as forcing (in g 's) is increased for frequencies of (a) 30 Hz, (b) 40 Hz, (c) 50 Hz, and (d) 60 Hz. A transient has been removed from each simulation. Note the modulation of the subharmonic waves in all but (d) the 60 Hz case.

FIGURE 3.26: Snapshot of the change in fluid surface height (in cm) for the 50 Hz simulation at $0.9088 g's$. Above onset the subharmonic waves are dominant throughout the domain.

FIGURE 3.27: (a) Close-up of the Hopf bifurcation to modulated subharmonic waves in the case of 50 Hz in Fig. 3.25. Peak modulation amplitude (maximum over one modulation period) of the filtered subharmonic signal is shown for increasing forcing (open circles, up arrow) and decreasing forcing (open squares, down arrow). (b) Linear fit of the growth rate (in s^{-1}) of perturbations at fixed values of the forcing predicts an onset of $f = 0.804 g's$.

3.6.2 Character of the bifurcations

We now examine the 50 Hz case in more detail. Simulation results shown in Fig. 3.27(a) are obtained with a slower rate of change of applied acceleration ($0.0008 g's$ per second), and in both

increasing and decreasing directions. The Hopf transition still appears to be hysteretic, although a measurement of the growth rate of perturbations at fixed forcing values suggests that the real location of the Hopf bifurcation is near $f = 0.804 g$'s and thus, that much of this hysteresis is due to the linear drift in forcing (Baer, Erneux, and Rinzel, 1989; Mandel and Erneux, 1987).

As forcing amplitude is increased beyond the Hopf bifurcation, the modulated (quasiperiodic) solutions show an increase in modulation period and irregular (non-sinusoidal) temporal behaviour indicative of global bifurcation. This progression can be seen in Fig. 3.28.

The increase in modulation period (T_m) is shown quantitatively in Fig. 3.29 and compared with the inverse square root scaling expected of a saddle-node heteroclinic bifurcation. The excellent agreement argues that this is the bifurcation predicted in Eqs. (3.32) and not, for example, a standard heteroclinic bifurcation, which would show logarithmic divergence of T_m (a much poorer fit to the data).

It should be acknowledged that the fit shown in Fig. 3.29, while convincing, is not a completely rigorous confirmation of the saddle-node heteroclinic bifurcation. In particular, if the system is “close” to a double zero point (vanishing Hopf frequency), there may be additional complications, even in the reduced model (3.32), including small regions of (standard) heteroclinic bifurcations (or pitchfork bifurcations) that replace the saddle-node heteroclinic bifurcation (Marqués et al., 2013).

These issues are further blurred by the delay associated with the slow drift in forcing. Regarding these concerns, we note that all simulations with fixed parameters are consistent with the picture suggested above. Steady (unmodulated) subharmonic solutions were only found above the global (saddle-node heteroclinic) bifurcation, while the modulated subharmonic solution appears to be the only (nontrivial) attractor below it. In other words, the stable unmodulated branch seems to disappear below the assumed saddle-node heteroclinic bifurcation, as expected. Furthermore, the delay associated with the drift in forcing is relatively minor in this case: reducing the rate of change of the forcing from $0.004 g$'s per second to $0.0005 g$'s per second, and considering both forward and backward sweeps, only lowers the estimate for the bifurcation point by 1-2 %. For these reasons (as well as technical numerical issues) we did not attempt in the present paper a more rigorous bifurcation analysis using the Navier-Stokes simulations, following, e.g., the noninvasive control approach of Barton and Sieber (2013). This is left for future work.

3.6.3 Dependence on frequency

Significant dependence on frequency (detuning) is anticipated in these simulations for the reasons given in connection with Eq. (3.48). As the forcing frequency increases, the wavenumber selected by the dispersion relation increases too. Due to boundary constraints and the spatial dependence of the forcing (Porter et al., 2013; Tínao et al., 2014; Varas and Vega, 2007), this wavenumber (gradient of the phase) is not uniform in space and cannot simply follow the dispersion relation as it would in the unbounded (vertical) Faraday wave problem. Nonetheless, we expect it to increase

FIGURE 3.28: Time series showing (signed) modulation amplitude of time series for points equidistant from the left (solid curves) and right (dashed curves) boundaries for the case of 50 Hz forcing. The transition from sinusoidal modulation near the Hopf bifurcation to slower irregular modulation near the global bifurcation can be seen with increasing forcing values (in g 's): (a) 1, (b) 1.2, (c) 1.25, (d) 1.27. A 30 s transient has been removed from each simulation.

with frequency, at least locally near the endwalls, and thus for the phase of the interaction in the interior region (i.e., the phase of the coefficient μ in Eqs. (3.32)) to vary with detuning.

FIGURE 3.29: The increase in modulation period (T_m) as the global bifurcation is approached. The data from the simulation (open circles) fit well with the scaling expected for a saddle-node heteroclinic bifurcation: the solid line shows the fitting function: $0.059 + 0.835(1.296 - f)^{-1/2}$.

We can estimate the period of this variation using the gravity-capillary wave dispersion relation

$$\omega^2 = gk + \frac{\Gamma}{\rho}k^3, \quad (3.49)$$

where 2ω is the forcing frequency (in rad/s), Γ is the surface tension, and ρ is the density. The wavenumber k selected in the limit of small damping and forcing should satisfy $k_n \simeq n\pi/L$, where L is the length of the container. These resonant modes are separated by several Hz for the parameters of the simulations (approximately 2.5 Hz near 40 Hz forcing and 3 Hz near 60 Hz forcing, e.g.). Thus, we expect the behaviour of the fluid in the simulations to vary with frequency over a range of several Hz.

First consider the case of 60 Hz, where no clear region of modulated solutions is seen in Fig. 3.25. Figure 3.30 shows that this case is atypical; it just happens to be close to a double zero point where the region of existence of modulated solutions is very small (the dotted vertical lines in Fig. 3.24). With a forcing frequency of 60.4 Hz, for instance, we observe modulated solutions terminating in a saddle-node heteroclinic bifurcation, as with 30 Hz, 40 Hz, and 50 Hz forcing.

The simulation has been repeated, using the same linear increase in forcing amplitude of $0.004 g$'s per second, over a range of frequencies near 50 Hz. The onset of subharmonic waves was estimated from the data, along with the initial modulation frequency Ω , and the location of the saddle-node heteroclinic bifurcation where the modulations cease. It must be noted again that, due to the linear change (drift) in forcing amplitude and the finite time of the simulation, the threshold values will be somewhat overestimated. The results, which are shown in Fig. 3.31, verify the oscillations predicted above (with a period of about 3 Hz) and are consistent with the picture provided in Fig. 3.24, where a sequence of double zero points marks the boundaries of bubble-like regions where modulated solutions are found. Complex dynamics are expected in a neighbourhood of these double

FIGURE 3.30: Harmonic (black curve) and subharmonic (gray curve) components as forcing (in g 's) is increased with 60.4 Hz vibration.

FIGURE 3.31: (a) Estimated location of onset (triangles) and saddle-node heteroclinic bifurcation (circles); the parameters shown are applied forcing f (in g 's) and applied frequency ω/π (in Hz). (b) Estimated modulation frequency Ω (in Hz) near onset.

zero points (Marqués et al., 2013). The simulation results are further complicated by the mode transitions mentioned above. This adds a type of discontinuous behaviour not captured by the weakly coupled parametric oscillator model analysed in Sec. 3.4. Figure 3.31 suggests that these mode transitions occur just above the double zero point (about 0.5 Hz higher in forcing frequency). The proximity of these two critical points and their associated complexity explains the irregular nature of the curves near the double zero points.

3.7 Conclusions

In this paper, we considered the dynamics of weakly coupled parametrically forced oscillators in a neighbourhood of the dominant subharmonic instability. It was seen how different choices of coupling topology and individual forcing phases lead to distinct symmetry groups characterising the equivariant bifurcation problem describing the primary instability. The residual permutation symmetry that survives in the coupled problem is critical in determining invariant subspaces and associated solution classes. It plays a key role as well in determining whether the primary bifurcation itself is steady or oscillatory.

Detailed results were obtained in the case of two coupled oscillators because of the relevance to recent experiments on horizontally vibrated fluids (Porter et al., 2012; Tínao et al., 2014). In horizontally forced fluid experiments there is a direct harmonic (synchronous) response including surface waves and localised evanescent modes. These evanescent modes (also, oscillatory bulk flow; Varas and Vega, 2007) drive a variable pressure gradient that acts as a localised parametric forcing mechanism. The localised nature of the forcing and the resulting subharmonic modes means that, if the container is large compared to the spatial decay length of the waves, the subharmonic patterns generated at opposing endwalls can be treated as weakly interacting subharmonic oscillators.

The bifurcation sets and related bifurcation diagrams provided in Sec. 3.4.3 demonstrate the importance of the forcing phase in determining both primary and secondary instabilities. If the phase of the parametric forcing is identical for both oscillators, then the residual symmetry is $\mathbf{Z}_2 \times \mathbf{Z}_2$ and there are two nontrivial fixed point subspaces containing synchronous (even) and antisynchronous (odd) oscillations. This structure, in particular, rules out a primary Hopf bifurcation.

If the parametric forcing is antisymmetric, then the residual symmetry is $\mathbf{C}_4$ and there are no nontrivial fixed point subspaces. The primary bifurcation is oscillatory and generates a $\mathbf{C}_4$ -symmetric modulated (quasiperiodic) solution where the excitation cycles slowly between one oscillator and the other. These modulated solutions are destroyed via global bifurcation, which typically takes the form of a saddle-node heteroclinic connection, when the forcing exceeds the critical value by an $\mathcal{O}(|\mu|)$ amount. A particularly complex interaction of saddle-node, Hopf, heteroclinic, and homoclinic bifurcations is found for positive detuning and associated with a pair of Bogdanov-Takens points.

If the phase difference between the parametric forcing at each oscillator is not 0 or π , then there is no residual permutation (exchange) symmetry, only the basic subharmonic symmetry $\mathbf{Z}_2$. This case leads to intermediate behaviour organised by (generally two) $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points separating regions of primary Hopf and pitchfork bifurcations. As with antisymmetric forcing, a complex scenario is found for positive detuning and associated with additional (asymmetric) Bogdanov-Takens points.

The case of antisymmetric forcing was compared with the experimental results of Tínao et al.

(2014) in Sec. 3.5 and found to qualitatively capture the observed oscillations in the modulation frequency if a physically motivated dependence of coupling on detuning is assumed. Regions of stable modulated solutions exist for a wide range of detuning if the nonlinearity is dissipative ($a > 0$). Further comparison with direct numerical simulations were presented in Sec. 3.6. For most parameters, the predicted modulated (quasiperiodic) solutions were found, and these were seen to terminate in a global bifurcation with increasing forcing. Furthermore, this was shown to be a saddle-node heteroclinic bifurcation, as expected from the coupled oscillator results of Sec. 3.4.3. Oscillations with forcing frequency were expected in these two-dimensional simulations, both due to mode transitions and to the phase of the coupling (interaction). The results were consistent with the scenario discussed in Sec. 3.5.1, where bubbles of stable modulated solutions are separated by double zero points.

In summary, we have shown that coupled parametrically forced oscillators can have very interesting dynamics in the weakly nonlinear regime, and that this dynamics is especially influenced by the residual permutation symmetry, if any, that remains. The fact that the underlying oscillators are parametric, with (at least) two preferred phases, also plays a key role in determining the relevant symmetries. Although simple models like the one considered here have obvious limitations, they allow a much more complete analysis and can shed light on complex behaviour that would otherwise resist explanation. In this case, the relevance to cross-wave experiments was demonstrated by the symmetry-based explanation for the primary Hopf bifurcation, the presence of stable quasiperiodic solutions, the dependence of the modulation frequency on detuning, and the observation of the predicted saddle-node heteroclinic bifurcation in direct numerical simulations of the Navier-Stokes equations. It is hoped that future experiments will allow more of the predictions of Sec. 3.4.3 to be tested.

Acknowledgment. This work was supported by the Spanish Ministry of Science and Innovation under project AYA2010-19081 and the Spanish Ministry of Economy and Competitiveness under projects ESP2013-45432-P and ESP2015-70458-P.

Chapter 4

The CFVib experiment: Control of Fluids in microgravity with Vibrations

Fernández, J., Salgado Sánchez, P., Tíñao, I., Porter, J. and Ezquerro, J. M.

Microgravity Science and Technology **29**, 351–364 (2017).

<https://doi.org/10.1007/s12217-017-9556-7>

The Control of Fluids in Microgravity with Vibrations (CFVib) experiment was selected for the 2016 Fly Your Thesis! programme as part of the 65th ESA Parabolic Flight Campaign. The aim of the project is to observe the potentially complex behaviour of vibrated liquids in weightless environments and to investigate the extent to which small-amplitude vibrations can be used to influence and control this behaviour. Piezoelectric materials are used to generate high-frequency vibrations to drive surface waves and large-scale reorientation of the interface. The theory of vibroequilibria, which treats the quasi-stationary surface configurations achieved by this reorientation, was used to predict interesting parameter regimes and interpret fluid behaviour. Here we describe the scientific motivation, objectives, and design of the experiment.

4.1 Introduction

By its nature fluid moves easily, adjusting its shape and position in response to external forces. On Earth, the most prominent large-scale external force is gravity and fluid configurations generally evolve so as to minimise their gravitational potential energy. The effects of this are familiar to everyone. An undisturbed body of liquid, for example, whether part of the natural landscape or held in a container, will minimise its height by pooling in the lowest configuration possible, maintaining a (nearly) flat horizontal interface with the surrounding air (or other lighter fluid). Although this picture can be complicated by surface tension, which is important on small lengthscales, and various other physical processes like buoyancy, diffusion and evaporation, the basic flat interface selected at leading order in common situations is remarkably simple.

Periodic vibrations represent an interesting class of external forcing that can perturb or disrupt this initial (static) surface configuration. Vibrations have the potential to both induce instability (Faraday, 1831; Miles and Henderson, 1990; Vega, Knobloch, and Martel, 2001) and suppress it (Gresho and Sani, 1970; Wolf, 1969, 1970), and to affect important physical processes like mixing and mass transfer. In the laboratory, periodic vibrations can be introduced in a deliberate and controlled way and the effects of amplitude, phase, and frequency content systematically investigated.

The effects of vibration on a mass of fluid can be roughly divided according to the frequency and the orientation of the forcing. Low frequency vibrations (provided they are not purely orthogonal to the surface) tend to excite back-and-forth motion along the axis of the vibration, or sloshing modes (Feng, 1997; Funakoshi and Inoue, 1988; Miles, 1984). Higher frequencies excite surface (or interfacial) waves of shorter wavelength, which may be either harmonic or subharmonic (Barnard and Pritchard, 1972; Miles and Henderson, 1990). The smaller lengthscale of these waves leads to the possibility of pattern selection and associated dynamics, which has been a subject of much interest (Arbell and Fineberg, 2002; Edwards and Fauve, 1994; Kudrolli, Pier, and Gollub, 1998).

The orientation of the forcing is a key factor in the response of the fluid. If forcing is applied perpendicular to a large flat surface then its initial effect is homogeneous and isotropic. The flat surface will persist, but undergoes a parametric instability (Benjamin and Ursell, 1954) at a certain critical forcing amplitude to (typically) subharmonic surface waves, which appear with uniform amplitude across the surface. Horizontal (more generally, non-vertical) forcing has more complex effects, producing harmonic (synchronous) waves and, in many situations, subharmonic waves such as cross-waves (Garrett, 1970; Jones, 1984; Porter et al., 2012). In contrast to the case of vertical (perpendicular) motion, horizontal forcing typically produces nonuniform wave fields concentrated near the moving boundaries or wavemakers. An interesting exception to this is the appearance at a critical forcing amplitude of a Kelvin-Helmholtz type instability in heterogeneous systems, such as a stably stratified layer of immiscible fluids, that initially produces quasi-stationary frozen waves (Gaponenko et al., 2015b; Gaponenko and Shevtsova, 2016; Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007; Wolf, 1969). Sufficiently far from the boundaries these frozen waves are of uniform amplitude.

In addition to surface waves, which oscillate on the fast timescale set by the forcing, periodic vibrations can drive a slower reorientation of the fluid towards a new quasi-steady equilibrium distinct from the hydrostatic one. This phenomenon, observed by Faraday (1831) in the flattening of fluid drops suspended beneath a vibrating plate, is known as the vibroequilibria effect. Like frozen waves, vibroequilibria are supported by an oscillatory velocity field generated by the applied vibrations. It is the spatial inhomogeneity of this velocity field, and the corresponding oscillatory pressure gradient, that leads to deformation of the surface. Recent analysis (Fernández et al., 2017b), using both numerical and theoretical techniques, found that, depending on fluid volume, vibroequilibria in horizontally vibrated rectangular containers can suffer a saddle-node instability that initiates a disordered transition to one of many possible asymmetric states. In low-viscosity fluids, surface wave dynamics can also promote the destruction of vibroequilibria states, with a strong correlation found between modulated subharmonic surface waves and the (potentially unbounded) growth of the first odd sloshing mode.

In microgravity environments, fluid behaves quite differently than it does on Earth. Surface tension effects are no longer suppressed by gravity and there is no preference for flat surfaces (interfaces). A free-floating mass of liquid will, instead, minimise its surface energy by taking a spherical shape, while contact forces may cause a confined liquid to fully cover (wet) the solid boundaries. The distinction between vertical and horizontal forcing is of minimal use since most surfaces will be curved and, thus, the forcing mechanisms driving surface instabilities will generally be inhomogeneous. Among other things, this means that the distinction between Faraday waves, which are associated with uniform vertical forcing, and cross-waves, which arise in horizontally forced systems, is blurred. Other phenomena, like frozen waves, are altered as well, as shown by recent experiments (Gandikota et al., 2014b) in near-critical hydrogen that found the frozen wave amplitude increases with decreasing gravity level, eventually approaching the lengthscale of the container (domain) in microgravity conditions.

The absence of a constant gravitational force has a marked effect on the vibroequilibria phenomenon, which is no longer suppressed by the high gravitational cost of changing surface shape (height). In microgravity, vibrational energy primarily competes with surface and contact energy, and these may be comparable in magnitude for small to moderate forcing amplitudes. The ability of fluid masses to significantly adjust their shape in response to vibrations, combined with the selection of curved hydrostatic surfaces due to surface tension, means that fluid management in microgravity is considerably more complicated than it is on Earth.

Although vibrations are frequently unwanted in a microgravity environment, they may also be put to good use. They are under consideration to help reduce muscle and bone loss in astronauts, for example, and can mimic the effects of gravity in other ways too (Beysens, 2006), inducing convection patterns similar to Rayleigh-Bénard (buoyancy driven) convection in a fluid confined between two plates of different temperature, for instance. Vibrations generate mean flows that affect mixing, heat transfer, and phase separation, among other important physical processes. Further understanding of the various effects of vibrations on fluids may make it possible to use these effects for advantage

in microgravity — not only to avoid undesired phenomena, but to deliberately manipulate and manage fluids, a critical problem in space travel. The vibroequilibria effect, in particular, suggests artificial gravity since, as forcing amplitude increases, the interface typically evolves towards a more perpendicular alignment with respect to the axis of vibration, thereby restoring a preference for flat interfaces.

The Control of Fluid with Vibrations in Microgravity (CFVib) experiment examines the dynamic behaviour of representative liquid-gas and liquid-liquid combinations confined in moderately small transparent acrylic containers (several centimetres across) and subjected to periodic vibrations. Relatively high frequencies (from about 50 Hz to 12 kHz) are used to drive the reorientation of the fluid interface and explore the potential for controlling it. The use of piezoelectric materials to transmit vibrations to the fluid requires less space and power than an electromagnetic shaker, making it more conducive to potential microgravity applications. On the other hand, there are challenges for theory and simulations since the excited motion within the container is complicated, nonuniform, and sensitive to resonance conditions.

The period of weightlessness (more than 20 s) offered by a parabolic flight (Pletser et al., 2016) is well-suited to the scientific requirements of the experiment. Furthermore, the g-level cycles resulting from repeated parabolas can be treated as a natural and variable resetting of the liquid, in keeping with the idea of investigating a robust method for manipulating fluids in microgravity.

In this paper we first present, in Sec. 4.2, an overview of the most relevant microgravity experiments and theory related to the vibroequilibria effect in microgravity. In Sec. 4.3, the scientific objectives and experiment requirements are described, while the experiment design is presented in Sec. 4.4; this includes the container-shaker assembly, data acquisition system, and electrical subsystems, as well as the operational concept and experimental procedure. A final review of the CFVib experiment design is given in Sec. 4.5, preliminary results in Sec. 4.6, and conclusions in Sec. 4.7.

4.2 Prior results and scientific motivation

There have been relatively few experimental investigations of vibrated fluids in microgravity and none, that we are aware of, which look directly at the vibroequilibria effect or which used piezoelectric actuators to deliver the vibrations.

The most direct antecedent of the CFVib experiment is arguably the parabolic flight experiment of Ganiev, Lakiza, and Tsapenko (1977), which investigated the dynamic behaviour of water, oil, and air mixtures in partially filled vibrating cylindrical cavities. The interface was studied using both low-frequency and high-frequency vibrations applied with an electrodynamic shaker. Relatively disordered behaviour, involving multiple modes and oscillating bubbles, was observed at low frequencies (8 Hz, 12 Hz) while high-frequency vibrations (600–800 Hz) led to the formation of deep funnels or craters, which penetrated nearly to the bottom of the container. This phenomenon, which would now be recognised as a type of vibroequilibrium, was strongly resonant in character.

Experiments using the ALICE-2 instrument on the MIR space station (Garrabos et al., 2007; Zyuzgin et al., 2001) examined the influence of low-frequency vibrations (0.2 Hz, 1.6 Hz) on heat propagation in supercritical fluids and observed a strong effect.

Parabolic flight experiments (Mialdun et al., 2008; Shevtsova et al., 2010a,b) demonstrated vibration-induced convection in nonuniformly heated fluid in reduced gravity using excitation frequencies of 1–12 Hz. An extension of these efforts, the Influence of Vibrations on Diffusion in Liquids (IVIDL) experiments (Shevtsova, 2010; Shevtsova et al., 2011, 2015b) on the International Space Station (ISS) investigated the effects of vibration on diffusion and convection. Diffusion and thermodiffusion coefficients were measured for two binary mixtures and it was found that vibrations of constant frequency and amplitude can generate mean flows that affect both processes. Two methods of mass transfer were observed, depending on the vibrational forcing. With low forcing, the mass transport is controlled by the Soret effect. When the forcing is high, mass transfer is convective and the Soret effect is localised in the boundaries generating the heat flux. The importance of g-jitter for diffusion experiments was also evaluated and found to be largely negligible; only large transient events like docking or orbital corrections have a notable impact on such experiments.

The Vibrational Phenomena in Liquids (VIPIL) project focuses on vibrational phenomena in miscible and immiscible fluids, including heat and mass transfer. Associated parabolic flight experiments (Shevtsova et al., 2016) have found large-amplitude frozen waves that evolve into columnar structures stretching across the width of the container. The column interfaces, which are largely perpendicular to the axis of the forcing, can develop subharmonic surface waves (i.e., Faraday waves). This subharmonic instability occurs with constant gravity as well, although the average structure of the large-amplitude frozen waves then resembles a triangle wave. Finally, we note the Capillary Flow Experiment (CFE) (Jenson et al., 2010; Weislogel et al., 2009), which demonstrated the importance of contact line boundary conditions in selecting the preferred static equilibrium of perturbed fluids and in the response of the fluid to manually imparted perturbations of various types.

On the theoretical side, there has been a considerable amount of work on vibrated fluids, including the phenomenon of vibroequilibria (Beyer et al., 2001; Beyer, Günther, and Timokha, 2004; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov, Lyubimova, and Cherepanov, 2003; Lyubimov et al., 1997). The approach to vibroequilibria theory is based on a time-averaged variational method that assumes an inviscid fluid and irrotational flow. The forcing frequency is taken to be much larger than the frequency of the primary sloshing mode, while incompressibility, if assumed, further requires that this forcing frequency is below the frequency of the first acoustic resonance. The Lagrangian (Hamiltonian) variational method then leads to an energy functional accounting for surface energy, vibrational energy (Kapitza, 1951) and, if present, gravitational energy. The surface configurations associated with critical points of this functional are the vibroequilibria solutions.

The predictions of vibroequilibria theory for horizontally vibrated containers were tested against direct numerical simulations (Fernández et al., 2017b), where, with large enough damping, very good agreement was found. Low viscosity fluids support more prominent surface waves, which

can destabilise the underlying vibroequilibria state through the excitation of sloshing modes. This instability is correlated with the presence of modulated (quasiperiodic) subharmonic surface waves, which are typical at onset (Perez-Gracia et al., 2014; Porter et al., 2013; Salgado Sánchez et al., 2016; Tinao et al., 2014; Varas and Vega, 2007). Even without the complications of surface wave dynamics, an interesting dependence on depth was found for the symmetric connected vibroequilibria state, which develops a large central trough (crater) as forcing is increased and the fluid is pushed towards the vibrating endwalls. For shallow layers of fluid, this central dip gradually reaches the container bottom, but for sufficiently deep layers, the symmetric vibroequilibria state disappears through a saddle-node bifurcation, after which this central trough plunges rapidly towards the bottom in a more violent and unpredictable fashion. The transition to asymmetric vibroequilibria states found at still higher forcing is inherently hysteretic.

The various results summarised above highlight both the interesting fluid phenomena that can be observed with vibrations in microgravity environments and some of the challenges for fluid management in such situations. The aim of the CFVib experiment is to extend the level of scientific understanding in this area and, specifically, to explore the potential of small-amplitude vibrations to control liquid-air or immiscible liquid-liquid configurations.

4.3 Scientific objectives and requirements

The main scientific objectives and experiment requirements of the CFVib experiment are described here, including the parameters and constraints of the parabolic flight campaign.

4.3.1 Primary scientific objectives

As stated above, the principal objective of the CFVib experiment is to investigate the complex behaviour of vibrated liquids in microgravity and to evaluate the potential for small-amplitude vibrations in controlling this behaviour. More specifically, the main scientific objectives are:

- (i) Help evaluate the feasibility of managing fluids using small-amplitude vibrations in microgravity. In this regard, two different frequency regimes are considered: acoustic (high frequency) and hydrodynamic (low frequency).
- (ii) Allow a comparison between different representative fluid configurations having distinct surface energies and wetting characteristics, including water-air, oil-air, and liquid-liquid. Comparison is also made with unforced samples to differentiate the effect of the applied vibrations from that of residual ambient accelerations (like g-jitter).
- (iii) Provide experience in the design and execution of a microgravity experiment centred on the vibroequilibria effect. Strengths and weaknesses of the design can suggest improvements for future microgravity experiments.

4.3.2 Scientific requirements

The main scientific requirements of the experiment are outlined below. These relate to fluid properties, time constraints of the parabolic flight, sensitivity to g-level cycles and residual accelerations, forcing requirements, and observation criteria.

Experimental fluids

The experiment considers different liquid-gas and liquid-liquid configurations, which, in a qualitative sense, are representative of common space applications. Configurations include the case of a wetting fluid like silicone oil and a non-wetting fluid like water. The experiment fluids satisfy the safety requirements of the parabolic flight environment and are contained in relatively small, closed, transparent acrylic containers.

Relaxation time

There are two potentially distinct processes whose timescales must be smaller than the available microgravity time:

- (i) The decay time needed to reach the initial "static" configuration at the beginning of the microgravity phase. This time depends primarily on the damping experienced by the excited sloshing modes.
- (ii) The vibroequilibria timescale needed for the vibrated fluid to reach a new quasi-static equilibrium.

Calculations (Weislogel and Ross, 1990) using a cylindrical container comparable to the size of those in the CFVib experiment predict that the unforced sloshing mode decay time is on the order of 1–4 s, depending on the fluid viscosity, contact angle, surface tension and cylinder radius. In a rectangular container, numerical simulations of the full Navier-Stokes equations, like that shown in Fig. 4.1, give similar timescales (on the order of 2 s for a fluid with a viscosity of 50 cSt).

Once the vibrational forcing begins, the time needed for the fluid to approach the vibroequilibria state is, according to numerical simulations like that of Fig. 4.1, similar to the unforced relaxation time (on the order of several seconds). Although the dynamics are somewhat different with the CFVib configuration using piezoelectric actuators, and depend as well on the resonance mode selected, it is likely that the streaming flow responsible for driving the fluid to its vibroequilibria state is established quite rapidly and that the relaxation process is limited by sloshing motion in both the forced and unforced cases. The sloshing frequency and effective damping clearly depend on the asymptotic state about which the oscillations occur, and thus on the vibrational forcing, but in most cases, this dependence is relatively weak.

Since the characteristic relaxation time is on the order of several seconds, both to settle into an initial unforced microgravity configuration and to evolve towards the vibroequilibria state when the

FIGURE 4.1: Relaxation of a 2D fluid in a 4 cm wide container (fluid area/volume 6 cm²) subjected to horizontal vibrations at 100 Hz in the absence of gravity; forcing starts at $t = 2$ s, viscosity is 50 cSt, surface tension 20 dyne/cm, and contact angle 60°. (a) Surface profile $h(x)$ at $t = 0$ (initial flat surface; dashed line), at $t = 2$ s (hydrostatic surface; dash-dotted curve), and for large time (vibroequilibria surface; solid curve). (b) Heights $h(t)$ of the two contact points (upper curves) and midpoint (lower curve) showing relaxation to the hydrostatic solution ($t < 2$ s) and to the vibroequilibria solution ($t > 2$ s). The inset shows the out-of-phase harmonic waves at the two walls (one grey, one black).

forcing is turned on, this constitutes the basic time requirement of the CFVib experiment. Parabolic flights provide a period of microgravity of more than 20 s during each parabola, which is sufficient to observe the asymptotic behaviour.

Airplane disturbances: g-level cycles and g-jitter

The changing g-levels during a parabolic flight affect the position of the fluids in the experimental containers. Consequently, the configuration prior to each experimental run has a degree of unpredictability, as does the dynamics of the relaxation (sloshing) phase during the first few seconds of microgravity. One important scientific requirement is that g-level cycles and ambient g-jitter effects not mask or otherwise adversely affect the important experimental measurements.

Figure 4.2 shows a typical acceleration profile during the microgravity portion of a parabolic flight. The small-amplitude ambient accelerations that are present (amplitudes on the order of 0.05 g 's) at low frequencies like 1 Hz are particularly worrying as a potential source of fluid motion and experimental errors since these frequencies are comparable to the relevant sloshing frequencies. Higher-frequency accelerations, known as “quickly changing acceleration forces” (QCAFs) (Schmidt, 2004), are not captured in Fig. 4.2, but are not likely to adversely affect the scientific output of the experiment.

The g-level cycles, including high-g portions of the parabolic flight and small periods of negative g, are not detrimental to the success of the experiment since the initial conditions for the microgravity phase of the flight do not need to be known precisely. Although both g-level cycles and g-jitter may complicate precise quantitative comparison of different runs, this variation is compatible with

FIGURE 4.2: Acceleration (in g 's) during the microgravity portion of the parabolic flight in a 3-axis reference frame attached to the airplane showing the z -axis (“vertical” direction, g_z , dark grey), y -axis (“lateral” direction, g_y , medium grey), x -axis (“longitudinal” direction, g_x , light grey) and total magnitude ($|g|$, black).

the scientific objective of the CFVib experiment to investigate a general robust method for moving fluids in microgravity. The presence of unforced fluids in otherwise (nearly) identical experimental cells allows the importance of low-frequency noise and g -jitter to be directly evaluated.

Frequency regimes

The CFVib experiment uses high-frequency vibrations in the range of 50 Hz to 12 kHz. There are thus two distinct excitation regimes:

- (i) The hydrodynamic regime where compressibility can be neglected.
- (ii) The compressible regime where acoustic waves are essential in determining the velocity field.

The hydrodynamic regime pertains when the fluid is driven at frequencies up to several kHz while the acoustic regime is associated with higher frequencies (tens of kHz or more). The manner in which the vibrational energy is delivered to the fluid in these two regimes is different, and it is of interest to compare the vibroequilibria effect in each case.

Fluid velocity

Sufficient fluid velocity is required to clearly observe vibroequilibria effects. This means that the vibrational energy associated with changes in the rapidly oscillating velocity field be comparable (or greater), at least locally, to the characteristic variation in surface and contact energy terms. An induced velocity of approximately 10 cm/s or more is needed. Although fluid velocity is often

FIGURE 4.3: Vibroequilibria solution with 88 Hz forcing (near the lowest resonance) and 60° contact angle for a liquid with properties like water. The vibroequilibria surface is shown after 20 s while it takes about 4 s, starting from the hydrostatic equilibrium, for transients to decay.

measured using PIV techniques [see, e.g., (Hasegawa et al., 2009), where streaming flow excited by an acoustic levitator was observed], this is beyond the scope of the CFVib experiment; PIV measurements may be included in future extensions.

Interface observation

The experiment requires a series of images showing the deformation of the surface. Movement on the order of 1 mm can be detected by the camera. Recording two different views of each sample allows for more possibilities in post-processing of the images and (potential) reconstruction of the fluid interface. An example of the type of interface expected (obtained from numerical simulation) is shown in Fig. 4.3.

4.3.3 Design requirements and considerations

Low-power forcing device with a wide frequency envelope

Piezoelectric devices are used to induce a vibrational velocity field in the experiment fluids. To avoid having to operate the piezoelectric material at its own (high) resonance frequency, a bender beam configuration is used instead of attaching the piezoelectric ceramics directly to the external wall of the recipients. With this configuration (see subsequent sections for details), the resonant frequencies for prominent flexural (beam) and container modes range over several Hz to kHz, as desired. Furthermore, forcing can be supplied by a cost-effective low-power amplifier.

Interaction between different experimental cells

The effects of coupling between different experimental cells, which are mounted on the same base plate, are required to be no greater than those of the airplane disturbances (g-jitter; see Fig. 4.2). Some degree of mechanical coupling between distinct experimental cells is unavoidable, and the importance of this effect, which is quite sensitive to resonances, is difficult to predict. Direct observations are made after the experiment is assembled.

Real-time data acquisition and visualisation

Since the excitation is (often) started after the capillary equilibrium is reached, it is an advantage for the experiment operator to monitor the fluid configuration directly in (nearly) real-time.

4.3.4 Novespace requirements

Requirements like those summarised below help ensure safe experiment operation and compatibility with the airplane systems.

- Safety is strictly enforced during the parabolic flight and all potential hazards must be classified and analysed.
- Electronic systems must ensure protection for operating personnel, aircraft interfaces, and the experiment itself. This is achieved through proper grounding, isolation of high voltage electrical connections, and the use of fuses in case of short circuits.
- Liquids, powders and particles are required to be safe from accidental release into the airplane cabin; this contamination risk is reduced by a double containment policy.
- Structural hardware must withstand emergency-level forcing loads.

4.4 Experiment design

The experimental set-up, sketched in Fig. 4.4 for a single experimental cell, includes the following main items:

- (i) Four experiment fluids: distilled water, 5 cSt silicone oil (Dow Corning), 50 cSt silicone oil (Dow Corning), and vacuum oil (Krytox 1506).
- (ii) Twelve experimental containers: six rectangular cuboids and six cylinders, half excited by vibration and half passive.
- (iii) Twelve piezoelectric devices: two for each of the six excited containers, attached on opposite faces of the vibrating (bender) beam.

FIGURE 4.4: Sketch of CFVib experiment showing just one cuboidal cell. Rack #1 holds the experiment samples and main elements of the acquisition system. Rack #2 holds the control and electronic components.

	ρ (g/cm ³)	Γ (dyne/cm)	γ (cSt)
Water	1.0	72	1
5 cSt silicone oil	0.91	19.7	5
50 cSt silicone oil	0.96	20.8	50
Krytox 1506	1.9	19	250

TABLE 4.1: Characteristic physical properties of the experiment fluids at room temperature: density (ρ), surface tension (Γ), and kinematic viscosity (γ).

- (iv) Four cameras, twelve mirrors and an LED light strip for video acquisition.
- (v) Six accelerometers measuring the vibration of the excited cells.
- (vi) Three (dual channel) wave generators, six acoustic amplifiers and an electronics board (including a DC Power Supply) driving the piezoelectric devices.
- (vii) One laptop computer controlling the experiment.

4.4.1 Experiment fluids

As mentioned above, one goal of the CFVib experiment is to evaluate the effect of surface tension and contact angle in the selection of the vibroequilibria solution. In the CFE experiment (Jenson et al., 2010; Weislogel et al., 2009) the importance of the boundary condition in selecting the preferred hydrostatic configuration was shown. The CFVib experiment takes this into account, using different representative fluids with distinct surface and contact properties. The experiment uses water, two silicone oils of different viscosity, and a mixture of silicone oil and Krytox 1506 vacuum oil (which

FIGURE 4.5: Prototypes for (a) rectangular cuboidal container and (b) cylindrical container.

is about twice as dense as silicone oil). Characteristic physical properties of each experiment fluid are given in Table 4.1.

4.4.2 Experiment containers

Two different experiment containers are used; a brief description of each is provided below.

Cuboidal containers

Six rectangular cuboids made of methacrylate are used as containers for the experiment fluids. To protect against leaking, they are built using only two pieces: a body and a lid. The interface between the lid and the body is sealed using an O-ring and pressure is maintained by two aluminium plates connected by four long bolts; see Fig. 4.5(a). These containers have an interior dimension of $3\text{ cm} \times 3\text{ cm} \times 4\text{ cm}$ and a wall thickness of 0.5 cm . They are approximately half-filled in the case of a single liquid, and full (no air) in case of the two-liquid mixture. The total volume of fluid is about 70 ml .

Cylindrical containers

Six right circular cylinders made of 0.5 cm thick methacrylate are used as containers for the experimental fluids. The inner diameter is 3 cm and the height is 6 cm . As with the cuboids, they are built using two pieces (body and lid) that are sealed with an O-ring and pressed together by two aluminium plates connected by long bolts; see Fig. 4.5(b).

The six cylindrical containers have the same dimensions, as do the six regular cuboidal containers, respectively. Three of each type have a piezoelectric shaking device attached, while the other three are passive, providing an unforced reference configuration. The cuboidal cells are arranged at two different levels to optimise space on the base plate, while the cylindrical cells are at the same level. One container from each set of three has water, one (sometimes two) has silicone oil, and one (sometimes none) has a combination of silicone oil and Krytox vacuum oil.

<i>Direct attachment</i>	<i>Bender beam configuration</i>
– Expensive amplifier – Separate transducer – Thicker PZT ceramic (more expensive) – Simpler mechanical assembly – Complex numerical analysis – No analytical solutions – High resonance frequencies	– Acoustic amplifier – Bender acts as transducer – Thinner PZT ceramic (less expensive) – More complex mechanical assembly – Complex numerical analysis – Some analytical solutions – Some resonances in the range of interest

TABLE 4.2: Comparison of relevant characteristics for alternative piezoelectric actuator set-ups.

4.4.3 Piezoelectric actuators

Piezoelectric devices are used to induce the oscillatory velocity field in the experiment fluids. A bender configuration is selected for the reasons outlined in Table 4.2, including the need to shift the resonances from those of the piezoelectric ceramic itself.

The shaking device used in the CFVib experiment is constructed from a pair of piezoelectric ceramic materials and a metal plate or strip, referred to as a bender, that is attached to one of the walls of the container. The ceramics are placed on opposite horizontal faces of the bender in order to induce flexural motion that (in the primary mode) moves the recipient up and down. This configuration requires antisymmetric excitation of the two piezoelectric devices (achieved via opposite polarisation of the AC signal inputs). The longitudinal deformation created in the ceramics is transmitted to the beam as shear stress with opposite sign on opposing sides of the beam. This situation generates a linear shear stress distribution that, as predicted by Euler-Bernoulli theory in the limit of a slender (linear) beam, produces a purely flexural deformation (the integral related to longitudinal motion vanishes).

The bender, which is driven by the piezoelectric ceramics near one end, is attached to one of the walls of the experimental container. This attachment is made by gluing a piece of methacrylate to the container and fixing this with a bolt to the cantilever plate. The bender is also clamped to the base plate with another bolted aluminium piece. Since in this set-up the individual experimental cells are not mechanically isolated and some degree of coupling between different bender-container assemblies occurs, a heavy base plate is used to reduce unwanted interaction.

4.4.4 Data acquisition: cameras, mirrors, illumination, accelerometers

The data acquisition system includes the configuration of cameras, mirrors, illumination and accelerometers; each is described in more detail below. The cameras are powered via USB connection to the laptop. Videos showing the evolution of the surface are taken from two viewpoints for each

	Cuboids	Cylinders
Format	1/2 "	1/2 "
FOV	69.4	69.4
Object distance (★)	200 mm	250 mm
Focal length	6 "	6 "
Horizontal pixels/mm (★)	9.24	7.39
Vertical pixels/mm (★)	7.392	5.91

TABLE 4.3: Camera, lens and sensor characteristics. (★) Approximate numbers that vary somewhat in final configuration.

cell. Accelerometers, controlled by Analog Discovery 2 devices (see Sec. 4.4.5), measure the effective vertical acceleration of the excited containers. Each set of three containers has a dedicated video camera (four video cameras in total) while illumination is provided by an LED strip.

Video cameras

The video camera selected for the CFVib experiment is the ThorLabs DCC1545m, a CMOS camera that has a maximum frame rate of 25 frames per seconds at the maximum resolution of 1280×1024 pixels. The MVL6WA camera lenses from ThorLabs are used with the camera. A high-speed camera is not required since the mean (averaged) position of the interfaces is observed, not the rapid dynamics occurring at the forcing frequency. A camera and lens pair with the same format (1/2 ") were selected to avoid cropping and vignetting of the images and to have the camera field of view (FOV) at the maximum allowed by the lens. The lens has a depth of field (DOF) of around 50 mm, which allows the interior of the experimental cells to be observed. A summary of the main camera and lens characteristics is given in Table 4.3.

Illumination

A commercial LED array is used for illumination. The model EGLO LED DECO is a 3 m LED strip which includes a 12 V power supply and has a maximum power consumption of 13.5 W with 250 lm.

Accelerometers

Six Analog Devices ADXL001-500BEZ accelerometers are used to measure the vibration induced in the excited experimental cells. Each accelerometer is connected to one of the channels of an Analog Discovery 2 device, receiving the required 5 V and providing the measured signal. Two accelerometer measurements from the experiment are shown in Fig. 4.6.

4.4.5 Electronics and control systems

The control and electrical systems, sketched in Fig. 4.4 and described in more detail below, include:

- (i) Laptop computer

FIGURE 4.6: Accelerometer measurements taken from the cuboidal cell with water during the third flight in parabolas 5 and 15, using excitation frequencies of 79.8 Hz (first resonance) and 846.5 Hz (third resonance). (a) Time series of the accelerometer signals a (in g 's) for forcing at 79.8 Hz (upper panel) and 846.5 Hz (lower panel). Unfiltered (grey curves) and filtered signals (black curves) are shown. (b) Absolute value $|A|$ of the fast Fourier transform of both signals (lower frequency signal rendered with a thicker line) showing peaks at the corresponding excitation frequencies.

(ii) Amplifiers

(iii) Signal generators

(iv) Power supplies

(v) Other hardware: cabling, glands, etc.

Depending on the weight of the item, Velcro strips or mechanical straps are used to secure its position.

Laptop

A commercial high-performance MSI laptop computer is used to control the experiment and the piezoelectric actuators.

High voltage amplifiers

Six independent amplifiers are required. The amplifier selected is the PDm200B from PiezoDrive, which has the following features:

- Overload protection for current (at 0.5 A) and temperature (at 70 °C).
- Maximum AC output voltage of 200 V and current of 300 mA.

- Weight: 50 g.

The amplifiers are placed in well-ventilated electrical boxes to allow effective cooling.

Function generator and digital oscilloscope

Three Analog Discovery 2 devices are used to drive the piezoelectric ceramics. Because each one has two channels for arbitrary function generation and two oscilloscope channels for analog-to-digital conversion, it is used simultaneously to drive two piezoelectric actuators and to collect data from two accelerometers.

Power supply

One TENMA 72-10505 DC Power Supply is used to provide the required voltage (± 17 V) to the amplifiers. In order to drive the six amplifiers, a power distribution box is used to obtain six lines. The piezoelectric devices receive a sinusoidal electrical signal at the desired frequency produced by the Analog Discovery 2 function generators. The PDm200B amplifiers boost this to the required power level. Both systems are controlled and synchronised in time with the computer clock.

4.4.6 Operational concept: experimental procedure, software and control

The experiment procedure is organised in accord with the sequence of separate microgravity periods presented by the parabolic flight plan. Each microgravity period is used to measure one experimental data point. The natural variables of the experiment include the container geometry (shape, size, aspect ratio), the fluid properties (viscosity, surface tension, density) and the excitation parameters (frequencies, amplitudes). The first two of these are constant between runs — different geometries (cuboids and cylinders) and different fluid configurations are considered simultaneously during the experiment.

The control variables of the experiment are the excitation parameters: frequency f , and peak-to-peak voltage, V_{pp} . The frequency selects the vibrational mode of the experimental assembly and, thus, the (qualitative shape of the) vibroequilibria solution reached, while, in the absence of instabilities or transitions, the voltage selects its amplitude. The movement of the interface is recorded with a video camera.

The basic procedure for each run is as follows:

1. Video acquisition is started several seconds before injection (i.e., several seconds prior to microgravity).
2. A wait of 5 s (maximum) may be implemented after the start of the microgravity phase while the static equilibrium is reached.
3. Forcing is applied (with fixed f and V_{pp}) and accelerometer measurements are recorded for

FIGURE 4.7: Resonance behaviour in the cuboidal container with 50 cSt silicone oil. Here V_{pp} (applied to the piezoelectrics) is held at 40 V while the frequency f is swept between 70 Hz and 12 kHz.

at least 25 s; a minimum of 15 s of data is obtained during the microgravity period. The behaviour of the fluids in hypergravity may also be observed.

4. The forcing and data acquisition (video and accelerometers) are stopped.

The applied vibration is different for different sets of runs in order to explore distinct modes of excitation occurring at different (resonance) frequencies. These resonances can be seen in Fig. 4.7 for the case of 50 cSt silicone oil in a cuboidal container. Since the resonance frequencies vary a bit depending on the assembly procedure (bolt tightness, e.g., being one variable) the forcing frequencies are determined on site (i.e., after the experiment is fully assembled). More than seven frequencies are explored over the first sixty parabolas. Roughly, these frequencies may be classified as follows:

- (i) 50–100 Hz: near the first resonance.
- (ii) 150–250 Hz: near the second resonance.
- (iii) 500–1000 Hz: near the third resonance.
- (iv) 2–5 kHz: near the fourth resonance.
- (v) ~ 8 kHz: contact line displacement (deformation) observed here, even with normal gravity.

FIGURE 4.8: Vertical acceleration a (absolute value of harmonic component) measured during ground tests at the midpoint of the external rear wall for increasing (peak-to-peak) piezoelectric excitation voltages V_{pp} , at the first three resonance frequencies: 81 Hz (circles), 217 Hz (squares) and 841 Hz (triangles) of the cuboidal container with water. A linear fit is shown in each case.

(vi) 10–12 kHz: strongly acoustic regime.

Except for the first two resonances, where surface waves were seen during preliminary ground experiments and scientific interest is assured, the remaining frequencies are selected after the first flight (30 parabolas) — this is the only flight with a fully predefined experimental procedure. This flight is divided into two sets of ten runs and five sets of two runs. Over the two first sets of ten runs, the excitation amplitude (voltage) is increased linearly from an initially small value at a fixed frequency. These runs coincide with the first two resonances of the experimental assembly. For the remaining five sets of two runs each, various resonances over the range of interest are tested using intermediate V_{pp} values; depending on the results, they are selected (or not) for further study on subsequent flights. Acceleration values obtained during ground tests with different applied voltages for the first three resonance modes are shown in Fig. 4.8.

The final scientific data is contained in a compilation of four high quality videos (no compression) per parabola, six time series for the vertical acceleration (one per excited cell) and the accelerometer records of the airplane. This data is processed in a preliminary fashion between flights in order to decide on the selection of frequencies for subsequent flights. This processing includes:

- (i) Analysis of the videos for these experimental runs, evaluating the degree of surface motion (compared with those of the corresponding unforced experimental cell).
- (ii) Analysis of the accelerometer data to estimate the fluid velocities induced.

Additionally, minor changes in the experimental procedure can be implemented by adjusting the vibration start/stop time, for example, or the set of runs (number of V_{pp} amplitudes measured for each frequency). Among other things, such possibilities are allowed for because the location of the resonances can shift due to the capillary driven relocation of the fluid and starting the vibration before injection (microgravity phase), when the interface is still relatively flat due to hypergravity, may help mitigate this effect.

4.5 Review of final configuration

The final configuration of the CFVib experiment follows from integrating the subsystems described above, as well as additional hardware and material related to safety requirements. The heart of the experiment is a set of six cuboidal and six cylindrical methacrylate cells. Half of the experimental cells are vibrated using a PZT piezoelectric material attached via an aluminium bender. The other half of the cells are for reference and are not forced; although there is no piezoelectric actuator, the rest of the mechanical assembly is (nearly) identical to that of the forced cells. Each specific fluid configuration used, such as water/air for example, is found in a total of four experimental cells: two forced cells of different shape (cuboidal and cylindrical) and two corresponding unforced cells.

Because each excited fluid cell may have (slightly) different resonance frequencies when fully assembled, each piezoelectric actuator is operated independently. Specially designed software allows for independent frequency control while switching all piezoelectric devices on (and off) simultaneously. The experimental containers (both excited and reference) are observed using four cameras (controlled by the software), twelve mirrors and an LED illumination system.

The experiment hardware described, as well as the cabling, USB hubs, and the structures supporting the mirrors, is placed inside a Zarges box and attached to a base plate in the airplane. This Zarges box, referred to as Rack #1, contains the experimental cells, cameras, mirrors, and related hardware. Rack #1 is connected by a series of cables to the electronics and control assembly, placed on Rack #2. Six power cables transmit the forcing signal, three Ethernet cables connect to the accelerometers (two for each cable), USB cables allow communication with the four cameras, and an additional power cable supplies the LED lighting. Each cable is passed through one of four waterproof glands and attached to the interior base plate to avoid inadvertent contact with the experimental hardware.

The base of Rack #2 is a large aluminium plate where the different items of equipment are attached; this plate is directly joined to the airplane. Rack #2 includes the Laptop, two electrical boxes containing the six amplifiers, a power distribution box, three Analog Discovery 2 devices, one Power Supply Unit, one USB Hub, the LED lighting remote control, and the electrical power strip.

4.6 Preliminary results

The experiment flew in the 65th ESA Parabolic Flight Campaign as part of the 2016 Fly Your Thesis! programme (Callens et al., 2011; Callens, Ha, and Galeone, 2016). Although not all of the

FIGURE 4.9: Four snapshots from the experiment with 50 cSt silicone oil vibrated at 59.3 Hz in a cylindrical container: (a) At the beginning of microgravity, (b) Just before forcing is applied, (c) About 2 s after forcing of amplitude 18.5 cm/s is initiated, (d) About 4 s after forcing is initiated.

fluid configurations behaved as expected, the experiment was successful in many ways. A systematic analysis of the experimental results, which requires, among other things, additional vibroequilibria simulations, will be presented in a subsequent paper. Here, we summarise the main preliminary conclusions, enumerated below:

- (i) The power required to clearly observe the vibroequilibria effect was underestimated somewhat, perhaps due to the details of the boundary layer dynamics and an attenuation of the velocity transmitted to the fluid bulk. In some cases, the required power was roughly twice that originally estimated.
- (ii) No clear vibroequilibria effects were observed in the liquid-liquid mixtures, perhaps due to the greater load of these heavier experimental cells. This configuration was abandoned on the third and final flight.
- (iii) The length of the cables introduced additional noise in the signals and reduced the operational envelope of the amplifiers for the high-frequency forcing cases. This noise affected the accelerometer measurements as well, so filtering is required during post-processing.
- (iv) For silicone oil, a larger vibroequilibria effect was observed in the cylindrical containers compared to the cuboidal ones; see Fig. 4.9. In contrast, with water the vibroequilibria effect was more pronounced with the cuboidal containers.
- (v) The piezoelectric actuators were generally effective, allowing controlled excitation over a wide range of frequencies. A variety of modes can be selected (even more if additional bender configurations are used) and, thus, a variety of vibrational motion and fluid response investigated.

Signal generation and amplification worked well over this large range of frequencies.

4.7 Conclusions

In this paper, we described the scientific objectives and design of the Control of Fluids in Microgravity with Vibrations experiment, which investigates the complicated behaviour of fluids subjected to vibrations in a weightless environment. The experiment focuses on the vibroequilibria phenomenon, whereby an inhomogeneous oscillatory velocity field causes the fluid to move and adjust its average shape. The fact that the selected vibroequilibria surfaces are increasingly flattened (perpendicular to the axis of vibration) by the applied periodic forcing means that this effect can be considered as a type of artificial gravity in some situations, and as a potential means to control and manipulate fluids in microgravity.

As discussed in Sec. 4.2, although there has been substantial theoretical investigation of vibrated fluids, including the vibroequilibria effect, there are relatively few microgravity experiments examining this phenomenon (those of Ganiev, Lakiza, and Tsapenko (1977) being a notable exception). It is of interest to test the predictions of vibroequilibria theory directly and to look at potential instabilities of these quasi-stationary states, as has recently been done (Fernández et al., 2017b) using numerical simulations.

The experiment was proposed to serve as a first step in a more systematic investigation of vibroequilibria phenomena in microgravity. The experimental set-up is designed to record the behaviour of several different fluid configurations in six cuboidal and six cylindrical cells, only half of which are deliberately excited by vibrations. The forcing is delivered through a cantilever beam driven by a pair of piezoelectric PZT ceramics subjected to a counter-phase electrical signal. Locating the resonances of this piezoelectric actuator (where the most energy can be transferred to the fluid) is a critical part of the preparation of the experiment. Changes in vibrational energy associated with different interface configurations must be comparable to those due to contact and surface tension terms. The piezoelectric actuator used in the experiment allows for a wide range of frequencies to be investigated and the corresponding modes and vibroequilibria solutions include both hydrodynamic and acoustic cases. Different behaviour such as flattening or cratering can occur depending on the orientation of the dominant vibrational motion with respect to the initial surface.

Overall, the experiment was successful, providing a lot of useful data to compare with the predictions of vibroequilibria theory. As a full analysis of the results is completed, more specific modifications and improvements of both theoretical (numerical) models and microgravity experiment design will be suggested.

Acknowledgment. This work was supported by the ESA Education Office in connection with the Fly Your Thesis! programme and the 65th ESA Parabolic Flight Campaign, and by the Ministerio de Economía y Competitividad under Projects No. ESP2013-45432-P and No. ESP2015-70458-P. We thank the European Low Gravity Research Association (ELGRA) for their valuable scientific

support and Novespace for all their technical advice and contributions during the project. We thank Trenz Electronic for their Analog Discovery device, Analog Devices (ADI) for providing the accelerometers, and Krytox^{TM} Performance Lubricants, a business of The Chemours Company, for supplying the vacuum oil. Finally, we thank the Spanish User Support and Operations Centre (E-USOC) and the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio at the Universidad Politécnica de Madrid, which have supported this project from the beginning.

Chapter 5

Vibroequilibria in microgravity: comparison of experiments and theory

Salgado Sánchez, P., Fernández, J., Tíñao, I. and Porter, J.

Physical Review E **100**, 063103 (2019).

<https://doi.org/10.1103/PhysRevE.100.063103>

Experiments on vibrated fluids confined in cylindrical and cuboidal containers were performed under the reduced gravity conditions of a parabolic flight. The results constitute the first systematic quantitative investigation of the vibroequilibria effect, which refers to the reorientation of vibrated fluids in response to the inhomogeneous oscillatory velocity field and the accompanying dynamic pressure. This effect is amplified in microgravity, where the restoring force of gravity is small or absent. Here, the vibrations are transmitted via a pair of piezoelectric ceramics and a cantilever beam, excited in a resonant mode. The first and second resonances exhibit different types of motion and lead to different types of vibroequilibria surfaces, one with a dip or crater in the interior and the other flattened compared to the unforced reference experiment. The general tendency for interfaces to orient more perpendicular, on average, to the vibrational axis is confirmed. In the case of water in a cuboidal container, a quantitative comparison is made with vibroequilibria theory and with direct simulations of the Navier-Stokes equations. The good agreement confirms the predictions of vibroequilibria theory and suggests the capacity of this phenomenon to manipulate and position fluids in space environments through the choice of frequency and resonant mode.

5.1 Introduction

By nature, fluids move easily in response to external forces and the ability to manage and control this motion is a crucial aspect of wide-ranging applications in physics, chemistry, biology, medicine and engineering. While small volumes of fluid may often be effectively manipulated using capillary effects, electromagnetic fields, or acoustic forces, this becomes increasingly difficult for larger volumes of liquid, whose distribution is dominated by gravity. Density stratification to minimise gravitational energy is a near universal feature of fluid configurations larger than the capillary length scale and, among other things, explains the familiar sight of flat liquid-air interfaces on Earth.

In microgravity environments, on the other hand, fluid behaviour is different. Surface tension and contact forces come to the fore and largely determine the shape and location of any free interfaces. A floating mass of liquid will minimise its surface energy by taking a spherical shape, while adhesive forces may cause it to wet a solid boundary upon contact. The increased ease with which fluids can move (vertically) in microgravity, and the preference for curved hydrostatic interfaces rather than flat ones, means that fluid management is substantially more complicated than it is on Earth. At the same time, large-scale manipulation of fluids may become more feasible when gravity is no longer the limiting factor.

Vibrations are an inherent feature of most real environments, with their importance for a given physical process depending on amplitude, frequency and orientation. Even in the simple case of purely periodic motion, they can affect a range of important fluid phenomena such as mixing (Ito and Komori, 2006) and mass transfer.

If there is an interface separating fluids of different density, vibrations will generally excite waves. The vibrating motion of a solid support or boundary generates harmonic (synchronous) waves that decay, due to viscous effects, as they travel inwards. At a critical amplitude, subharmonic waves (having twice the period of the vibrational forcing) may be excited via a parametric forcing mechanism arising from the oscillating pressure gradient (Miles and Henderson, 1990; Varas and Vega, 2007). If the source of this parametric forcing is localised, as it is near a side wall in a horizontally vibrating open container of liquid, the subharmonic waves are similarly localised, and are commonly known as cross-waves (Barnard and Pritchard, 1972; Garrett, 1970; Jones, 1984; Tínao et al., 2014). On the other hand, if the motion is vertical, perpendicular to the unperturbed flat interface, then the forcing is homogeneous and produces subharmonic Faraday waves (Benjamin and Ursell, 1954; Faraday, 1831; Kumar and Tuckerman, 1994) across the entire domain.

In addition to standing or travelling waves, vibrations parallel to an interface separating fluids of different density can drive a Kelvin-Helmholtz type instability producing quasisteady, or “frozen” waves (Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007; Wolf, 1969). These frozen waves are initially sinusoidal in form and grow as the amplitude of vibrations increases. As with oscillatory disturbances like Faraday waves and cross-waves, they may be considered, in the usual supercritical case near onset, as perturbations of the initial flat interface. This is no longer so without

the restoring force of gravity. In weightlessness, the frozen waves rapidly develop into large-scale columnar structures with alternating (nearly) vertical interfaces (Beysens et al., 2009; Lyubimova et al., 2017). Recent microgravity experiments (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) have shown that Faraday waves can subsequently be excited on these columnar interfaces to create a complex structure with two characteristic wavelengths. The pattern selection process is mainly controlled by vibrational velocity, but is affected by viscosity (Talib, Jalikop, and Juel, 2007) and finite-size effects (Salgado Sánchez et al., 2019c).

The example of frozen waves illustrates the dramatic effect that even small-amplitude vibrations can have in microgravity. In contrast to the situation on ground, they can lead to a large-scale redistribution of the liquids involved and, in so doing, facilitate a secondary Faraday wave instability that would not otherwise occur. It is not necessary to have the frozen wave (Kelvin-Helmholtz) instability for such a redistribution to happen either. The inhomogeneous oscillatory velocity field is naturally accompanied by an inhomogeneous dynamic pressure that drives a reorientation of the fluid. This phenomenon, known as the vibroequilibria effect, was observed by Faraday (1831) in the flattening of drops beneath a vibrating plate. Ordinary gravity masks the vibroequilibria effect for most common fluid configurations (unless very large forcing is applied), while in microgravity environments, vibrations compete only with surface tension and contact forces so their effect is greatly enhanced.

The usual theoretical approach to vibroequilibria (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997) assumes a separation of timescales between the high-frequency forcing and the slower viscous diffusion of momentum. If the flow is further assumed to be potential, then a variational method can be used to determine the quasisteady equilibrium of the surface as the critical point of a Lagrangian averaged over the fast timescale.

A recent investigation (Fernández et al., 2017b) compared vibroequilibria theory with direct numerical simulations and found good agreement over a range of parameters. It was further demonstrated that, depending on fluid depth, the symmetric vibroequilibria solution in a horizontally vibrated rectangular container can disappear in a saddle-node bifurcation, after which the surface dips violently towards the lower boundary and undergoes a disordered transition to one of many possible asymmetric states. In low-viscosity fluids, surface waves can interact with the underlying vibroequilibria state and may even destabilise it through excitation of the first odd sloshing mode; this is likely related to the presence of temporal modulations resulting from the interaction between subharmonic wave fields emanating from opposite sides of the container (Perez-Gracia et al., 2014; Salgado Sánchez et al., 2016; Varas and Vega, 2007).

The first experimental observation of the vibroequilibria effect in confined fluids was by Wolf (1969), who applied large-amplitude vibrations (936 mm) to a 60 mm long cylindrical container of diameter 28 mm holding an aqueous solution of potassium iodide and a lighter oil (SAE 140). In microgravity, the only prior vibroequilibria experiment that we are aware of was conducted by Ganiev, Lakiza, and Tsapenko (1977), who investigated combinations of water, oil, and air in the reduced gravity

of a parabolic flight. Partially filled cylinders of length 100 mm and diameter 30 mm were vibrated at low frequencies (8 Hz, 12 Hz), producing disordered dynamics, and at high frequencies (600–800 Hz), where deep funnels or craters were seen for the air-water interface near a resonance frequency. This behaviour, which is now recognised as indicative of the vibroequilibria effect in horizontally vibrated containers (Fernández et al., 2017b), was only later shown (Gavrilyuk, Lukovsky, and Timokha, 2004) to be qualitatively consistent with vibroequilibria theory, and was not quantitatively compared with either theory or simulations.

A more systematic experimental investigation of vibroequilibria solutions and their importance in microgravity is called for, not only for the potential observation of novel states like those mentioned above, but for their ability to manipulate fluids in weightless environments. Indeed, vibrations are known to mimic the effect of gravity in certain aspects of convection and phase transitions (Beysens, 2006) and vibroequilibria can be regarded in this light too. The general tendency, with increasing forcing, is for the vibroequilibria states to develop so that large portions of the surface are nearly perpendicular to the axis of vibration, reminiscent of how fluid surfaces align perpendicular to gravity.

With this motivation, the behaviour of fluids in microgravity and the potential for controlling them through the vibroequilibria effect were investigated in the Control of Fluids in microgravity with Vibrations (CFVib) experiment (Fernández et al., 2017a), which looked at representative liquid-gas and two-liquid combinations in rectangular cuboidal and cylindrical containers excited by periodic vibrations and was performed during the 65th ESA Parabolic Flight campaign. The vibroequilibria effect was measured as a function of forcing amplitude at different resonance frequencies having different characteristic modes of vibration.

In this paper, we present the main scientific results obtained during the CFVib experiment and compare them to theoretical predictions. The experimental setup is described in Sec. 5.2. The mathematical formulation used in the numerical simulations is given in Sec. 5.3, together with a basic review of vibroequilibria theory. The most relevant results of the experiment are presented in Secs. 5.4 and 5.5, while the case of water in a cuboidal container is quantitatively compared both to direct numerical simulations and to vibroequilibria theory in Sec. 5.5.3. Conclusions are provided in Sec. 5.6.

5.2 The experiment

The experiment is designed to compare pairs of cuboidal and cylindrical cells (see Fig. 5.1) containing a liquid-air or immiscible two-liquid combination (the latter is not described here since the experimental results were inconclusive). One of the cells is subjected to vibrations at various amplitudes and frequencies, while the reference cell is unforced and responds only to the residual acceleration present on the parabolic flight. These two simple geometries, which represent common fluid containers, can be expected to lead to different interfacial dynamics. In addition, the use of cuboidal cells has the advantage that (quasi) two-dimensional behaviour can be assumed in some

FIGURE 5.1: Pictures of the experimental cells and the bender beam: (a) cuboid and (b) cylinder. The cells are held closed by four bolts, which compress the PMMA body and cover against an O-ring. The aluminium bender beam has two piezoelectric ceramics attached to opposite horizontal faces. Cables associated with electrical excitation of the ceramics and the clamping bolt can also be seen.

	ρ (kg/m ³)	Γ (N/m)	ν (m ² /s)
Water	1000	7.20×10^{-2}	1×10^{-6}
50 cSt silicone oil	960	2.08×10^{-2}	50×10^{-6}

TABLE 5.1: Characteristic physical properties of the experimental liquids at room temperature: density ρ , surface tension Γ , and kinematic viscosity ν . These values are used in the mathematical simulations using the formulation of Sec. 5.3.

cases, while the cylindrical cells can be compared with the previous observations of Ganiev, Lakiza, and Tsapenko (1977).

The containers are made of transparent polymethyl methacrylate (PMMA) to allow for visualisation of the liquid interface. The cuboids are $30 \times 40 \text{ mm} \times 30 \text{ mm}$ (length $\times$ width $\times$ height), while the cylinders have an interior radius of 15 mm and a height of 60 mm. Each cell is composed of two PMMA pieces that are held against each other using bolted aluminium plates and sealed with an O-ring; part of this can be seen in Fig. 5.1. The results described here are obtained with water and 50 cSt silicone oil, whose relevant properties are given in Table 5.1.

Each experimental cell is fixed to an aluminium beam (cantilever) that, in the case of the excited cells, has two piezoelectric ceramics (PZTs), attached on opposite horizontal faces. An alternating counterphase voltage is applied to each PZT after being boosted by an amplifier. The counterphase mechanical stress induced in the ceramics creates a torque that drives periodic flexural motion of the beam and shakes the cell.

Both the amplitude V (with a maximum determined by the amplifier) and frequency f of the applied voltage can be varied to control the magnitude and characteristics of the vibrations. The experiments use voltages up to 100 V, while the frequencies are selected to match the most relevant resonant modes of the assembly. The first and second resonances (at approximately 80 Hz and 215 Hz, respectively, for the cuboids) were found to be associated with the largest displacements and, therefore, the most pronounced vibroequilibria effects. Accelerometers directly measure the (vertical) acceleration on each vibrated container.

Although the motion of the container induced by the PZTs in a bender-beam configuration is not purely linear (see Sec. 5.3 for more details), combining translational motion with some degree of rotation, it has the advantage of small size and low power (compared to an electromagnetic shaker, for example). The selected transducer yields significantly higher amplitudes for a given voltage than alternative configurations (such as a biclamped beam), at least for the resonant modes excited here.

The optical system that monitors the position of the fluid includes a complementary metal-oxide-semiconductor (CMOS) camera, a mirror and source of illumination. For the vibrated cells, the CMOS sensor samples at (approximately) 20 frames per second, providing excellent resolution of the slow timescale dynamics of the surface; the reference cells were recorded at (approximately) 10 frames per seconds. The behaviour of the liquid surface was observed along the axis of the beam, and after reflection by a mirror. Depending on the cell geometry, this mirror provides either a top view (for the cuboidal cell) or a lateral view (for the cylindrical cell). Both experimental setups are sketched in Fig. 5.2, which shows the container, bender beam, piezoelectric ceramics (PZTs), CMOS camera and mirror. Selected optical paths are drawn to illustrate the direct and reflected (top or lateral) views. Included illustrations indicate the dominant type of motion for the first resonant mode [see the inset in Fig. 5.2(b)] and the second one [Fig. 5.2(a)].

Experiments were performed under the reduced gravity provided by a parabolic flight (Pletser et al., 2016), which lasts approximately 20 s. The microgravity level (residual acceleration) during the parabolic manoeuvres was measured by the onboard accelerometers and provided each day by the airplane operators¹. This microgravity level satisfied

$$|g_{\tilde{z}}| \lesssim 0.05 g_0, \quad |g_{\tilde{x}, \tilde{y}}| \lesssim 0.01 g_0, \quad (5.1)$$

where $g_0 = 9.81 \text{ m/s}^2$. The components are expressed in the airplane reference frame (Pletser et al., 2016), defined by axes $\tilde{x}$, oriented aft to front, $\tilde{y}$, oriented along the wings, and $\tilde{z}$, perpendicular to the floor and identical to the vertical z axis of the container (see Fig. 5.3). The cuboidal containers are aligned, approximately, with the airplane axes so that $(x, y) \simeq (-\tilde{y}, \tilde{x})$, while the cylindrical containers are rotated by approximately 45° (clockwise) with respect to the longitudinal axis of the plane.

In this paper, we highlight the results observed in microgravity with the cylindrical cells when

---

¹Novespace, <https://www.airzerog.com/novespace-and-avico/>

FIGURE 5.2: Sketch of experimental setup with: (a) cuboid and (b) cylinder. Shown are the (partially filled) experimental cell, bender beam, piezoelectric ceramics, mirror and CMOS camera. Selected optical paths indicate the direct and reflected images referred to in later sections. The inset in (b) shows a lateral view of the setup for a cylindrical cell. Shaded curves illustrate the excited motion of the beam for the first (b, inset) and second (a) resonant modes.

vibrated in the first resonant mode, and the cuboidal cells when vibrated in the first and second resonant modes. For further details about the experiment, see Fernández et al. (2017a), where a review of the design, implementation and execution is given.

5.3 Mathematical formulation

We consider a finite volume of fluid Ω (see Fig. 5.3) with a free surface held in a container in the presence of a gravitational field and subjected to vibrations of frequency ω . The fluid satisfies the incompressible Navier-Stokes momentum and mass continuity equations (Landau and Lifshitz, 1987)

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \mathbf{G}(\mathbf{x}, \mathbf{u}, t), \quad (5.2a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (5.2b)$$

where $\mathbf{u}$ is the velocity, ρ is the density, ν is the kinematic viscosity and p is the pressure.

FIGURE 5.3: (a) Midplane cut of the experimental setup for a cuboidal container showing the bender beam, piezoelectrics (PZTs), fluid volume Ω , free surface S and solid boundaries W . The container and lab reference frames are illustrated; since the displacement is very small compared to the container size, we assume hereafter that $(x, y, z) \simeq (x_{\text{lab}}, y_{\text{lab}}, z_{\text{lab}})$. This geometry is used for the numerical results shown in Sec. 5.5.3. (b,c) Illustrations of the motion produced in the first two resonant modes.

The inhomogeneous acceleration $\mathbf{G}(\mathbf{x}, \mathbf{u}, t)$, which is expressed in the accelerating frame of the container, can be written, to leading order, as

$$\mathbf{G}(\mathbf{x}, \mathbf{u}, t) = \mathbf{g}(t) + \omega \mathbf{v}_\eta \cos(\omega t) + 2\boldsymbol{\varpi} \times \mathbf{u} + \boldsymbol{\varpi} \times [\boldsymbol{\varpi} \times (\mathbf{x} - \mathbf{x}_\eta)] + \dot{\boldsymbol{\varpi}} \times (\mathbf{x} - \mathbf{x}_\eta), \quad (5.3)$$

where $\mathbf{g}(t)$ is the time-dependent residual gravity field in the parabolic flight environment, $\mathbf{v}_\eta$ is the (approximately) linear velocity of a reference point $\mathbf{x}_\eta$, taken here to be the centre of the unperturbed fluid (see Fig. 5.3), and $\boldsymbol{\varpi}(t)$ is the rotational velocity of the container about that reference point. Both the linear and rotational motion are assumed to lie in the (x, z) plane defined in Fig. 5.3. The magnitude of the rotational velocity $|\boldsymbol{\varpi}(t)| = \varpi_0 \sin(\omega t)$ depends on the mode of vibration and the forcing amplitude, but ϖ_0 is typically on the order of 1 rad/s.

Navier-slip boundary conditions (Salgado Sánchez et al., 2019c) are imposed on the solid boundaries W to preserve the contact angle β and allow for motion of the contact line. The balance between surface tension, pressure and viscous stresses holds at the free surface S defined by $f(\mathbf{x}, t) = 0$:

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \text{on } W, \quad (5.4a)$$

$$p \mathbf{n} - \left(\frac{\Gamma}{\rho} \right) \kappa \mathbf{n} - \nu (\nabla \mathbf{u}) \cdot \mathbf{n} = 0, \quad \text{on } S, \quad (5.4b)$$

$$f_t + \mathbf{u} \cdot \nabla f = 0, \quad \text{on } S. \quad (5.4c)$$

Here, κ is the mean curvature of the interface $f(\mathbf{x}, t) = 0$ defined as

$$2 \kappa(f) = \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right), \quad (5.5)$$

and $\mathbf{n} = \nabla f / |\nabla f|$ is a unit normal vector.

In Sec. 5.5.3, we compare experiments with the solution of Eqs. (5.2)–(5.5) in a two-dimensional domain. We also compare with established vibroequilibria theory, which is summarised below.

5.3.1 Vibroequilibria theory

The theory of vibroequilibria (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997) provides a method for calculating the quasisteady (average) equilibrium states of a fluid configuration with a free interface. It assumes a separation of timescales between the forcing at frequency ω and the slower (averaged) response of the fluid, characterised by the viscous timescale L^2/ν , and the period of the primary sloshing modes (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004). If incompressibility is assumed, this further requires the forcing frequency to be below the first acoustic resonance.

A variational approach using a time-averaged Lagrangian casts the problem as one of determining critical points for a functional that depends on the shape of the interface. For this step, it is further assumed that the leading order velocity field is potential, with $\mathbf{u} = \nabla \phi$, and varies harmonically at the frequency of the applied forcing:

$$\phi(\mathbf{x}, t) = v_\eta \tilde{\phi}(\mathbf{x}) \cos(\omega t) + \dots. \quad (5.6)$$

The free surface is split into oscillatory and average parts

$$f(\mathbf{x}, t) = \tilde{F}(\mathbf{x}) + \tilde{f}(\mathbf{x}) \cos(\omega t) + \dots. \quad (5.7)$$

Note that rotational motion of the container is not generally compatible with potential flow. Of the three rotational contributions in Eq. (5.3), it is typically the final term (the Euler acceleration) that is the largest for the resonant modes of interest. For these excitations, none of the rotational terms are as large as the linear acceleration ωv_η , which justifies neglecting them at leading order so that

the standard (variational) approach to vibroequilibria theory can be used; the error this introduces is discussed in Sec. 5.5.3.

If there is a constant gravitational acceleration g taken to be aligned with the vertical z -axis (see Fig. 5.3) then the vibroequilibria defined by the surface $\tilde{F}(\mathbf{x}) = 0$ is dynamically stable if it is a local minimum of the time-averaged functional (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004)

$$\mathcal{L} = \int_{\langle \Omega \rangle} \left(gz + \frac{v_\eta^2}{4} |\nabla \tilde{\phi}|^2 + P_0 \right) d\Omega + \frac{\Gamma}{\rho} \int_{\langle S \rangle} dS - \frac{\Gamma}{\rho} \sin \beta \int_{\langle W \rangle} dW, \quad (5.8)$$

where P_0 is a Lagrange multiplier for enforcing conservation of volume. The functional (5.8) should be minimised over permissible configurations of $\tilde{F}(\mathbf{x})$. Assuming the surface can be expressed as a single-valued function $z = F(x, y)$ and using the rescaling

$$\mathbf{x} \rightarrow L\mathbf{x}, \quad P_0 \rightarrow \frac{\Gamma}{\rho L} P_0, \quad \tilde{\phi} \rightarrow L\tilde{\phi}, \quad \mathcal{L} \rightarrow \frac{\Gamma L^2}{\rho} \mathcal{L}, \quad (5.9)$$

the functional can be rewritten (see Fernández et al. (2017b) for details) as

$$\mathcal{L} = P_0 \mathcal{V} + \int_S dS + \eta \int_S (\mathbf{e} \cdot \mathbf{x}) q(\mathbf{x}) dS - \sin \beta \int_W dW + \frac{\text{Bo}}{2} \int_S [F(x, y)^2 - d^2] dx dy, \quad (5.10)$$

where d is the liquid depth with a flat horizontal surface, and $q(\mathbf{x}) = \partial \tilde{\phi} / \partial n$.

The nondimensional parameter

$$\eta = \frac{\rho L}{4\Gamma} v_\eta^2, \quad (5.11)$$

characterises the ratio of vibrational energy to surface energy, while the Bond number

$$\text{Bo} = \frac{\rho L^2}{\Gamma} g, \quad (5.12)$$

characterises the relative importance of gravitational and surface energy.

We minimise Eq. (5.10) using the Boundary Element Method, as in Fernández et al. (2017b), to obtain the predicted vibroequilibria surface.

5.4 Vibroequilibria in a cylindrical container

Figure 5.4 shows the evolution of the fluid surface in three vibrated cylindrical containers approximately half-filled with silicone oil, to be compared with the unforced reference cell (top row). In each experiment, the free surface is captured from the front and from a lateral view, and is illustrated at three different times (labelled in the figure) measured with respect to the beginning of the excitation in microgravity at $t = 0$ s. These snapshots are selected to show the liquid surface at the

FIGURE 5.4: Snapshots (front and lateral views) showing the evolution of the surface of silicone oil in a cylindrical cell. The upper row of panels (a-c) shows the fluid behaviour in the reference cell (experiencing residual acceleration of the airplane) at the given times, paired with corresponding snapshots of a vibrated experiment. The vibrated experiments are excited at $f = 59.3$ Hz with vibrational velocities: (a-c) $v_z = 73.9$, (d-f) 84.6 , and (g-i) 94.8 mm/s, which correspond to applied (peak) voltages of 80, 90, and 100 V, respectively. To help visualise the free surface deformation, contact lines at “front” and “rear” halves of the lateral wall are highlighted with solid and dashed lines.

beginning of the excitation, at the time when the contact line reaches the upper lid of the container (or its maximum height), and at an intermediate point between these.

FIGURE 5.5: Evolution in time of the highest visible point of the contact line (solid dots) and the minimum visible point of the fluid surface (diamonds) in vibrated (black) and reference (grey) cells. Vibrations are delivered at $f = 59.3$ Hz and different applied vibrational velocities: (a) $v_z = 73.9$, (b) 84.6, and (c) 94.8 mm/s, which correspond to applied (peak) voltages of 80, 90, and 100 V, respectively. Measurements of the residual gravity level are shown below each panel, with the labelled components measured in the airplane reference frame. Vertical lines mark the times when the snapshots in Fig. 5.4 are taken [labelled as 4(a-i)].

The vibrations are delivered at a fixed frequency $f = 59.3$ Hz, which corresponds to the first resonant mode, and for three applied voltages of 80, 90, and 100 V; these give vertical vibrational velocities v_z (measured at the accelerometer) of 73.9 (a–c), 84.6 (d–f), and 94.8 mm/s (g–i), respectively. For the experiment shown in (a–c), each snapshot of the vibrated cell is shown below a corresponding snapshot of the reference cell. The front and rear halves of the contact line are highlighted in solid and dashed lines, respectively, as a visual reference. Recall that the lateral views are obtained after reflection, as illustrated in Fig. 5.2.

The experiments shown in Fig. 5.4 are quantified in Fig. 5.5, where the uppermost (solid dots) and lowermost (diamonds) visible points of the surface, which are located along the contact line and in the interior, respectively, are shown as a function of time. The vibrated surface (marked in black) and the reference experiment (marked in grey) can be compared. Each plot is shown above the corresponding residual gravity measurement, given in the airplane reference frame. The vertical vibrational velocities v_z measured by the accelerometer of 73.9 (a), 84.6 (b), and 94.8 mm/s (c) are indicated, as are the times (shown with vertical lines) of the snapshots included in Fig. 5.4.

Prior to the beginning of the excitation in microgravity, the equilibrium reflects a balance between residual gravity and contact forces. Since silicone oil easily wets most surfaces and has a low contact angle, the surface resembles a spherical cap; see Fig. 5.4(a,d,g). In the absence of applied vibrations, this spherical cap-like shape is maintained throughout the experiment, although it responds with some degree of motion to the changing residual gravity, as seen in the snapshots of the reference

cell in panels (a–c). Applied vibrations, in contrast, modify this interface shape significantly.

Immediately after the forcing is initiated, the fluid moves towards a new (quasisteady) equilibrium, evolving on a slow timescale compared to the forcing period. This vibroequilibria effect is quantified by recording the motion of the contact line with respect to the minimum height of the visible surface, which occurs in the interior. What is generally observed with increasing amplitude is the gradual upward motion of the “front” and “rear” halves of the contact line, as seen in Figs. 5.4 and 5.5. This movement of the surface, which is driven by the spatially inhomogeneous vibrational velocity field, is affected by the residual gravity level. In particular, there is an evident correlation with the vertical gravity component (black curve in the lower panels of Fig. 5.5), reflecting the suppression of vibroequilibria effects for a given vibrational forcing with increasing gravity level (Bond number). The interior of the free surface, on the other hand, tends to move downwards per conservation of mass; see Figs. 5.4 and 5.5. Depending on the forcing amplitude, the contact line may reach the upper lid of the container, as with the experiments at 84.6 and 94.8 mm/s [Fig. 5.5(b,c) and Fig. 5.4(f,i)] or stop prior to that, as with the experiment at 73.9 mm/s [Figs. 5.5(a) and 5.4(c)].

For the experiments where the contact line reaches the top of the container, the effect of increasing vibrational amplitude is reflected in the time required for the fluid to arrive at that boundary. This contact time, which decreases from 6.38 s for $v_z = 84.6$ mm/s to 2.44 s for $v_z = 94.8$ mm/s, is also affected by the residual gravity level. Generally, the effect of (vertical) gravity is to flatten the surface; a higher vertical gravity level thus requires greater applied forcing to achieve a similar vibroequilibria effect.

Although the vibrations induced by the first resonant mode of the beam are oblique [see Fig. 5.3(b)], the observed vibroequilibria is relatively symmetric. In this aspect, the vibroequilibria surface recalls the case of horizontally vibrated containers (Fernández et al., 2017b), suggesting that the horizontal component is dominant. These results are also qualitatively consistent with the experiments of Ganiev, Lakiza, and Tsapenko (1977), which found a similar cratering of the interior surface in vibrated cylindrical cavities.

A similar, but much smaller, vibroequilibria effect was found in the cylindrical cell that was half-filled with water. This is likely explained by the fact that water has a higher surface tension and contact angle than oil, so the barrier for vibrational energy to overcome surface and contact energy is greater; see Eq. (5.10). In addition, the reduced viscosity of water allows the earlier excitation of surface waves. At the highest voltages, these waves are of high amplitude and drive the ejection of drops (James, Smith, and Glezer, 2003; James et al., 2003). This drop ejection process has recently been analysed experimentally with vibrated immiscible liquids in microgravity (Salgado Sánchez et al., 2019d), where the interfacial energy was measured with increasing amplitude and found to be consistent with the scaling suggested by vibroequilibria theory via Eq. (5.10); the interfacial energy of the drops increases as the square of the dimensionless vibrational velocity, or η , measured relative to the drop ejection threshold.

FIGURE 5.6: Snapshots (top and front views) showing the evolution in time of experiments with silicone oil in a cuboidal container. The upper two rows show the reference cell, which is subjected to the airplane’s residual gravity, while the lower two rows show the cell excited at $f = 83.07$ Hz and $v_\eta = 161.6$ mm/s, corresponding to a peak voltage $V = 100$ V. When feasible, the contact lines along the front and rear walls (front view) are highlighted using solid and dashed lines, respectively, as is the contact line in the upper views.

5.5 Vibroequilibria in a cuboidal container

Here, we consider the experiments in the cuboidal container holding either silicone oil or water. For the latter case, the experimental results are quantitatively compared with simulations of the Navier-Stokes equations and with vibroequilibria theory.

5.5.1 Experiments with silicone oil

Figure 5.6 provides front and top views showing the evolution in time of silicone oil in a cuboidal container. The fluid behaviour in the reference cell (upper two rows of snapshots), which is affected by the residual gravity in the airplane, and the vibrated cell (lower two rows of snapshots) can be compared at the same times. The vibrated cell is driven in the first resonant mode of the beam at $f = 83.07$ Hz and $v_\eta = 161.6$ mm/s, which corresponds to an applied peak voltage of 100 V. When

they are clearly visible, the “front” and “rear” contact lines are highlighted using solid and dashed lines, respectively.

In contrast to the results of Sec. 5.4, when enough silicone oil is present, a bubble develops in the cuboidal container; see the upper row of Fig. 5.6. This bubble persists throughout the microgravity period, although its location and shape vary somewhat due to residual gravity. We attribute the appearance of this bubble in the cuboidal container (and its absence in the cylindrical one) both to geometry — the cuboidal cells have lateral corners, which facilitate wetting and capillary action, and they are half as high as the cylindrical cells, so the fluid can more easily reach the upper lid — and to the different roughness of the PMMA walls resulting from the distinct manufacturing procedures used. This roughness, which is assumed to be greater in the case of the cuboidal cells, can significantly modify the contact line dynamics.

In general, the contact line dynamics depend on surface properties and on forcing. Without forcing, meniscus effects extend, for a representative residual gravity of $g = 0.05 g_0$, over the length scale $l_c \sim \sqrt{\Gamma/(\rho g)} \simeq 12$ mm (water), 7 mm (silicone oil). The fact that the capillary length is comparable to the container size explains the curvature observed in the reference cells during the microgravity portion of the flight (see Sec. 5.5.2). With forcing, the vibrational capillary length can be estimated (Douady, Fauve, and Thual, 1989) as $l_c \sim \sqrt{\Gamma/(\rho A \omega^2)} \simeq 1$ mm, where A and ω are the vibrational amplitude and frequency, respectively. Meniscus effects are thus expected to have only a weak effect on the vibroequilibria solutions when the contact angle is relatively large, as with water. For silicone oil in cuboidal containers, however, strong capillary effects are observed even under applied vibrations, as is clear from the formation of the large air bubble in the interior.

The formation of a large bubble means that the effect of the applied vibrations is more subtle than in Sec. 5.4, but a vibroequilibria effect is clearly present, as demonstrated by the different location of the contact lines seen in the figure. The applied vibration changes the bubble from an approximately spherical to a more elliptical shape; compare the top view of the reference and vibrated cells in Fig. 5.6(c). To leading order, these vibrations flatten the surface and make it more perpendicular to the dominant vibrational axis, as vibroequilibria theory predicts (Fernández et al., 2017b). Similar behaviour was observed in recent parabolic flight experiments with immiscible fluids (Salgado Sánchez et al., 2019d), where the flattening of an initial drop-like configuration was seen for vibrations initiated in microgravity.

The initial condition of the fluid surface can clearly play an important role in selecting the observed vibroequilibria solution, with an approximately flat surface developing in quite a different manner than a large bubble. Notwithstanding this dependence on initial topology, we find that the smaller variations of the initial surface characteristic of the parabolic flight platform do not noticeably affect the development of vibroequilibria — residual gravity, blue as mentioned earlier, is a far more important factor. We note that other phenomena, like the frozen wave instability that arises in two-fluid systems vibrated parallel to the unperturbed interface, can be much more sensitive to initial conditions. It was recently demonstrated that in similar microgravity experiments (Salgado

Sánchez et al., 2020) the development of the frozen wave instability can be significantly affected, or suppressed, by a misalignment between the initial interface and the vibrational axis. In fact, to a large extent, this can be explained by the vibroequilibria effect and its ability to orient interfaces (density contours) perpendicular to the vibrational axis. Frozen waves do not affect the CFVib experimental results described here since the density of the upper fluid (air) is very low, making the frozen wave threshold (Lyubimova et al., 2017; Lyubimov and Cherepanov, 1986) much higher than the forcing amplitudes used.

Experiments were also performed using 5 cSt silicone oil confined in both cuboidal and cylindrical cells, with a volume ratio of liquid to air of approximately 1/3, rather than 1/2. As discussed, the lower viscosity allows the same vibrational amplitude to excite higher amplitude surface waves, which can trigger drop ejection, and this was observed with 5 cSt oil. Since there was less fluid volume than in the experiments with water, so less total mass, the vibrational amplitude (for a given applied voltage) was higher. This fact, along with the lower surface tension, led to a profusion of drops that, unfortunately, obscured the vibroequilibria effect. The final excited states were fully three-dimensional and chaotic (James, Smith, and Glezer, 2003; Salgado Sánchez et al., 2019d) and could not be effectively analysed.

Compared to the results that follow in Secs. 5.5.2 and 5.5.3, the experiments with silicone oil in a cuboidal container are more difficult to quantify and to compare directly with simulations or with theory. For one thing, the lateral motion of the bubble within the cuboidal cell makes the problem fully three-dimensional, just as in the cylindrical cell. Accurate simulations would also require the treatment of topological changes in the fluid boundaries. Such a comparison between experiment and simulations is not attempted here.

5.5.2 Experiments with water

Experiments performed in rectangular cuboidal containers with water provided good results both for the first and the second resonant modes. Results are discussed separately for each of these modes since the character of the vibrational motion is different, leading to distinct vibroequilibria.

First resonant mode

Figure 5.7(a–c) shows the vibroequilibria observed in three experiments at different vibrational velocities v_η of 70.7 (a), 122.6 (b), and 161.6 mm/s (c), corresponding to applied voltages of 50, 80 and 100 V, respectively. Snapshots of the reference cells at identical times are included for comparison.

The reference cells show (nearly) identical solutions, with the surface curved due to its contact angle and displaying a degree of leftward tilt due to the effect of residual gravity, particularly its component along the y axis (see Fig. 5.3), which approximately coincides with the $\tilde{x}$ axis of the airplane; this scenario is similar to that of other recent parabolic flight experiments (Salgado Sánchez et al., 2019d).

FIGURE 5.7: (a–c) Snapshots of vibroequilibria in a cuboidal container vibrated at 79.8 Hz in the first resonant mode. Vibrational velocities are: (a) 70.7, (b) 122.6, and (c) 161.6 mm/s, corresponding to peak voltages of 50, 80 and 100 V, respectively. The location of the “front” and “rear” contact lines is outlined with solid and dashed curves, respectively. Images of the reference cell at identical times are included for comparison. (d) Illustration of the measurement points used for quantitative comparison with the theory of Sec. 5.5.3.

The vibroequilibria effect is revealed in the comparison between the excited and reference cells, and can be quantified in terms of the contact line motion. Increasing the vibrational velocity (i.e., the applied voltage), causes the “front” contact line to move slowly downwards, while the “rear” contact line climbs more rapidly upwards. Consistent with conservation of volume, the interior portion of the surface dips downwards at the same time, which makes this vibroequilibria qualitatively similar to previous solutions observed in the cylindrical container. In this case, however, the cratering is less pronounced, which suggests an effective vibrational axis that is, on average, more oblique.

As discussed above, the low viscosity of water facilitates drop ejection (James, Smith, and Glezer, 2003; Salgado Sánchez et al., 2019d), which occurs when the forcing is large enough to excite high amplitude waves that rupture the surface; see Fig. 5.7(b,c). Generally, we observe drops of increased size in microgravity compared to hypergravity, consistent with the work of James, Smith, and Glezer (2003). For a given vibration amplitude above threshold, this drop ejection is observed regardless of the initial conditions, as in Li and Umemura (2014).

Second resonant mode

Figure 5.8 shows two snapshots of the water surface in a cuboidal container and allows the excited cell (lower row) to be compared with the unexcited reference cell (upper row) at the same moment. The vibrated cell is excited in the second resonant mode of the beam at $f = 215.4$ Hz using a 100 V signal that provides a vibrational velocity $v_\eta = 49.14$ mm/s.

The second resonant mode [see Fig. 5.3(c)] has, on average, a more vertical orientation of the vibrational axis. This is consistent with the observed vibroequilibria, which possess a flatter surface compared to the reference experiment. We note that the velocities obtained are approximately half those achieved for the first resonance at the same applied voltage (Fernández et al., 2017a), so the vibroequilibria effect is smaller.

FIGURE 5.8: Snapshots (top and front views) showing the time evolution of experiments with water in a cuboidal container when vibrated in the second resonant mode. The upper rows show the reference cell, while the lower rows show the results of vibrating at $f = 215.4$ Hz and $v_\eta = 49.19$ mm/s, corresponding to an applied voltage of 100 V.

5.5.3 Comparison with theory

Using the solutions shown in Fig. 5.7 (a–c), the vibroequilibria effect is characterised by the “front” and “rear” contact points in the midplane (see Fig. 5.3) parallel to the beam direction (x axis). These points, h_{rear} and h_{front} , are located by manually processing the video frames, with an estimated accuracy of ± 1 pixel. With a spatial resolution of 7.4 pixels/mm in the vertical direction (Fernández et al., 2017a), this gives an uncertainty in h of approximately ± 0.14 mm. The image contrast is enhanced via local histogram equalisation over the regions of interest.

The dimensionless difference between these contact points is calculated as

$$\Delta\mathcal{H} = \frac{h_{\text{rear}} - h_{\text{front}}}{L}, \quad (5.13)$$

where $L = 30$ mm is the container size, and shown in Fig. 5.9 versus η , which is proportional to the square of applied velocity; see Eq. (5.11). Error bars show the estimated uncertainty in the dimensionless difference $\Delta\mathcal{H}$ of 0.01. This figure confirms that the vibroequilibria effect is proportional to η at leading order, as predicted theoretically in Sec. 5.3.1; the curvature can be attributed primarily to the slight variation of the axis of linear velocity $\mathbf{v}_\eta$ with forcing amplitude, which occurs for this mode.

The experimental results are also compared in Fig. 5.9 with the theoretical models of Sec. 5.3, including the direct numerical simulation of the vibrated problem (open grey circles) and the prediction from the energy functional formulation of vibroequilibria theory (solid black curve). In both cases, the forcing frequency $f = 79.8$ Hz is that of the first resonant mode, the contact angle β between water and PMMA is taken to be 50° (Zhao et al., 2010), the residual gravity is set to

FIGURE 5.9: Comparison of experimental results (solid black dots) with direct numerical simulation of Eqs. (5.2)–(5.5) (open grey circles with solid fitting curve), with analogous simulations that ignore the rotational motion (open grey squares with dotted fitting curve), and with the predictions of vibroequilibria theory based on the minimisation of Eq. (5.10) (solid black curve). The contact angle used in these calculations is 50° , while the residual gravity is set to a representative value of $g_z = 0.05 g_0$.

the representative value of $g_z = 0.05 g_0$ (giving $Bo = 6.1$), and the remaining parameters are as in Table 5.1 except for the viscosity, which is increased to 200 cSt to discourage large amplitude surface waves and drop ejection. The influence of these last two parameters can be inferred from Fig. 5.10, which shows the result of several numerical simulations (including rotational motion) with different values of (a) viscosity and (b) gravity level.

The vibroequilibria effect does not depend on viscosity at leading order and this is confirmed in Fig. 5.10(a). As seen from these simulations with $\nu = 10$ cSt, 50 cSt, and 200 cSt (see also Fernández et al., 2017b), viscosity does not appreciably affect the final (average) positions of the contact points, but does have a clear effect on the relaxation time and on the amplitude of secondary surface waves. The value of 200 cSt was chosen to safely avoid the numerically problematic drop ejection that was frequently observed in the experiments; see Figs. 5.7 and 5.8.

The residual gravity level, on the other hand, is clearly an important parameter that broadly acts to suppress the vibroequilibria effect. It varies both during and across experimental runs; see Fig. 5.5. We selected $g_z = 0.05 g_0$ for the simulations partly because this is the level achieved in typical parabolic flights. More importantly, better agreement was found for $g_z = 0.05 g_0$ than for lower values because the vibroequilibria were more nearly symmetric (in y) when $|g_z| > |g_y|$, making the two-dimensional assumption more tenable; the average microgravity level for the data used in Fig. 5.9 is close to $g_z = 0.05 g_0$. We note that, as shown in Fig. 5.10(b), the front contact point is nearly independent of the gravity level, recalling its weak dependence on forcing, and residual gravity, in the experiment; see Fig. 5.7.

The Navier-Stokes simulations include the full (x, z) motion (acceleration) of the container as measured on ground by an accelerometer and gyroscope attached to the side of the container.

FIGURE 5.10: Influence of viscosity and gravity level for a fixed voltage $V = 50$ V, $\eta = 0.57$ on the predicted contact point separation used to quantify the vibroequilibria effect. (a) Simulations of the Navier-Stokes equations with $g_z = 0.01 g_0$ and $\nu = 10$ cSt (light grey curves), 50 cSt (medium grey curves), and 200 cSt (black curves). (b) Simulations using $\nu = 10$ cSt with $g_z = 0$ (light grey curves) and $g_z = 0.05 g_0$ (black curve). The remaining parameters are as in Fig. 5.9.

These measurements are shown in Fig. 5.11. The commercial software COMSOL Multiphysics is used to solve the governing equations in two dimensions, corresponding to the midplane $y = 0$. The initial conditions $\mathbf{u} = 0$ at $t = 0$ are evolved, using a maximum time step of $1/(10f)$, until a steady state is reached.

The procedure prescribed by vibroequilibria theory is implemented in a two-dimensional rectangular domain, as in Fernández et al. (2017b). We use MATLAB to minimise Eq. (5.10) over permitted surface configurations using the Boundary Element Method and expand the surface $z = F(x)$ in terms of basis functions $P_n(x)$ that are taken to be Legendre polynomials:

$$F(x) = \sum_{n=0}^N a_n P_n(x). \quad (5.14)$$

This yields a functional $\mathcal{L}\{a_n\}$ that can be minimised over a finite set of coefficients.

FIGURE 5.11: Measurements (performed on ground) of the maximum linear and angular velocities at the centre of the side wall for the first resonant mode. (a) Horizontal velocity $|v_x|$ (solid dots) and vertical velocity $|v_z|$ (solid squares). (b) Angular velocity ϖ_0 .

Since the vibroequilibria theory neglects rotational terms, we use the linear acceleration of the point $\mathbf{x}_\eta$ in the middle of the unperturbed liquid (assuming it occupies the bottom half of the container; see the inset of Fig. 5.11) for this calculation. For consistency, the estimated acceleration of this point is also used to calculate η for the experimental data points and the Navier-Stokes simulations. The absence of rotational acceleration in the vibroequilibria calculation leads to an error since the real motion [see Fig. 5.3(b)] is not purely linear, particularly in the case of the first resonant mode. From Fig. 5.11, this error is expected to be on the order of 20–30%; that estimate follows from multiplying ϖ_0 by 20 mm, which is half the length, and comparing with $|v|$. To confirm that this is the principal source of error, the Navier-Stokes simulations were repeated without rotational terms (open grey squares in Fig. 5.9) and found to be in excellent accord with the vibroequilibria calculations.

Overall, the experimental results provide a clear confirmation of the vibroequilibria effect in microgravity. There is convincing agreement between the predicted and the observed position of the fluid over the considered forcing range $\eta \lesssim 3$ despite the simplifying assumptions made in the calculations, which include neglect of the lateral (y) dimension to justify a two-dimensional model, the use of constant residual gravity and, in the case of vibroequilibria theory, the approximation of linear container motion. This latter error is noticeable for the excitation used here, but could be accounted for with an alternative separation of timescale approach that does not assume potential flow.

Although further comparison with vibroequilibria theory may be possible, the above issues, as well as the variation of the effective Bond number during the parabolic flight, make this difficult. The available vibrational amplitude was an additional limitation. For example, according to the results of Fernández et al. (2017b), the liquid was deep enough ($d/L \simeq 0.5 > 0.36$) to undergo a sudden transition from a symmetric to an asymmetric solution. However, not only were the vibroequilibria solutions in the experiment not symmetric in x , since even the principal component of the acceleration was oblique, but the predicted amplitude ($\eta \sim 15$) for this (saddle-node) transition was not reached.

5.6 Conclusions

Results from microgravity experiments investigating the vibroequilibria effect in cylindrical and cuboidal containers partially filled with water or silicone oil have been presented and discussed. The experiments utilised different resonance modes of a cantilever beam excited by a pair of piezoelectric ceramics controlled by an applied AC voltage. The resulting motion was generally oblique to the initial fluid surface and “more vertical” for the second resonant mode compared to the first. Each excited cell was compared to an unforced reference cell with the same liquid configuration that experienced only residual gravity. The experimental results confirm the predicted “gravity-like” tendency of vibroequilibria solutions to move large portions of the surface into a more perpendicular orientation with respect to the (average) vibrational axis. For the first resonance mode, this led to the formation of a dip or crater in the interior, with the rear (and sometimes front) contact line climbing upwards. For the second resonance mode, this led to a flattening of the surface compared to the unforced reference cell.

These experimental results provide clear evidence of the vibroequilibria effect in both containers and with both liquids. Although observed long ago on ground (Faraday, 1831; Wolf, 1969) and even in microgravity (Ganiev, Lakiza, and Tsapenko, 1977), this is the first experiment that the authors are aware of to quantify the vibroequilibria effect and to compare the observed amplitude of solutions directly with numerical simulations of the Navier-Stokes equations and with calculations using the more recently developed vibroequilibria theory, which relies on a separation of time scales (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997). An analogous experiment on ground would be hampered by the large amplitudes required and, with that, the simultaneous excitation of other dynamical modes.

In the CFVib experiment, in fact, drop ejection from large amplitude surface waves was often observed in the case of water and 5 cSt silicone oil, and excessive drop ejection sometimes prevented the vibroequilibria effect from being clearly measured. However, despite the presence of a small number of drops, the cuboidal containers with water provided good results that allowed for a clear quantitative comparison with direct numerical simulations and with theory. The agreement demonstrates, experimentally, the validity of the vibroequilibria theory and shows that it can be used to accurately predict the position of fluids in microgravity.

With appropriate container design and choice of excitation (here determined by the resonance modes of the beam), it should be possible to use the vibroequilibria effect for control and management of fluids in microgravity environments. Aside from this consideration, the amplification of the vibroequilibria effect in weightlessness, where the reorientation is countered only by surface tension and contact forces, means that it should be taken into consideration whenever sufficient vibrations, known as g -jitter (Nelson, 1994), are present.

Acknowledgment. This work was supported by the ESA Education Office Fly Your Thesis! programme, the 65th ESA Parabolic Flight Campaign, and the Ministerio de Economía y Competitividad under Projects ESP2013-45432-P and ESP2015-70458-P. We also acknowledge the essential contributions of ELGRA, Novespace, E-USOC and the ETS de Ingeniería Aeronáutica y del Espacio (UPM) during the project. Finally, we thank Trenz Electronic for their Analog Discovery device, Analog Devices (ADI) for providing the accelerometers, and Krytox Performance Lubricants (The Chemours Company) for supplying the vacuum oil.

Chapter 6

Simulating vibrated fluids and vibroequilibria in the CFVib microgravity experiment

Salgado Sánchez, P., Fernández, J., Tíñao, I., Porter, J. and Bello, A.

Proceedings of the 8th European Conference for Aeronautics and Aerospace Science, 0224 (2019).

<https://doi.org/10.13009/EUCASS2019-224>

The CFVib experiment investigated the use of piezoelectric actuators to manipulate fluids via the vibroequilibria effect. A model is developed to simulate fluid behaviour in this experiment, including interaction with the container and piezoelectrics. Since vibroequilibria evolve slowly compared to the applied forcing, a separation of timescales is used to obtain a linear problem on the fast timescale whose solution provides a quasisteady source term driving slow vibroequilibria dynamics. We find that the surface adopts different vibroequilibria depending on the selected mode of vibration. Simulations show good agreement with experiments and suggest a potential strategy for manipulating fluids in weightlessness.

6.1 Introduction

By their nature, fluids move easily and adjust their shape and position in response to external forces. In the presence of a gravitational field, a large body of liquid will minimise its height by pooling in the lowest configuration possible, with the interface generally flat and horizontal. External forcing can change this situation, driving the fluid to a new equilibrium and/or supporting excitations on the surface or in the bulk. Periodic forcing, for example, is relatively common and easy to apply, affects important phenomena like mixing and mass transfer, and has broad significance in engineering and biological systems. Periodic vibrations have the potential to destabilise the flat surface equilibrium, as with the well-known phenomenon of Faraday waves (Faraday, 1831), or to stabilise an otherwise unstable configuration, as with the Rayleigh-Taylor instability (Wolf, 1970).

The effects of vibrations on such systems can be roughly divided according to the forcing frequency and orientation. Low-frequency vibrations (provided they are not purely perpendicular to the surface) tend to excite back-and-forth motion along the axis of vibration, or sloshing modes (Miles, 1984). Higher frequencies can excite interfacial waves, which may be harmonic, with the same period as the forcing, or subharmonic (Miles and Henderson, 1990), with twice that period. The associated patterns, whose wavelength is often smaller than the system length scale, can display complex dynamics and transition scenarios, a subject of much interest in recent decades (Arbell and Fineberg, 2002).

The orientation is another key factor in determining the fluid response. If the forcing is applied perpendicular to a large flat surface, the flat solution persists and undergoes a parametric instability (Benjamin and Ursell, 1954) to (typically) subharmonic surface waves beyond a certain forcing threshold (Kumar and Tuckerman, 1994). These waves extend over the entire surface since, aside from boundary effects, the forcing can be considered homogeneous throughout the domain. Horizontal (more generally, non-vertical) forcing directly produces harmonic waves and, in many situations, subharmonic waves such as cross-waves (Garrett, 1970; Porter et al., 2012). In contrast to the case of vertical motion, these wave fields are concentrated near the rigid boundaries, which act as wavemakers. In a horizontally vibrated rectangular container the interaction between wave fields extending from opposite endwalls can induce temporal modulations (Varas and Vega, 2007) and a variety of secondary bifurcations (Salgado Sánchez et al., 2016), depending on the extent to which they penetrate into the interior of the container.

In addition to surface waves and other phenomena that evolve on the fast timescale of the forcing, the fluid may respond to average effects on a slower timescale, as with the Kelvin-Helmholtz instability in density-stratified systems. Vibrations parallel to density contours can excite, at a critical forcing amplitude, quasisteady waves known as frozen waves (Lyubimova et al., 2017; Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007). If the frozen wave instability is observed in a two-fluid system in weightlessness, the typical sinusoidal pattern at onset develops rapidly into a larger columnar structure, eventually forming a series of vertical interfaces separating alternating volumes

of the different liquids (Beysens et al., 2009; Gaponenko et al., 2015a). Recent experiments (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) have shown that, after this frozen wave column growth, Faraday waves can then be excited along the nearly vertical interfaces, thus producing an interesting two-scale wave pattern. In this complex scenario, the pattern selection problem is not only controlled by the vibrational velocity, but is strongly affected by viscosity (Talib, Jalikop, and Juel, 2007) and finite-size effects (Gandikota et al., 2014b; Salgado Sánchez et al., 2019c).

Another slow timescale phenomenon driven by periodic vibrations concerns the reorientation of the fluid towards a new quasisteady equilibrium, which may differ considerably from the hydrostatic one. This behaviour, first observed by Faraday (1831) in the flattening of fluid drops suspended beneath a vibrating plate, is known as the vibroequilibria effect. Like frozen waves, vibroequilibria are supported by the oscillatory velocity field generated by the applied vibrations. It is the spatial inhomogeneity of this field, and the corresponding oscillatory pressure gradient, that leads to deformation of the surface. Recent analysis (Fernández et al., 2017b), using both numerical and theoretical techniques, found that, for deep enough layers of fluid, symmetric vibroequilibria in horizontally vibrated rectangular containers can suffer a disordered transition to one of many possible asymmetric solutions. In low-viscosity fluids, surface wave dynamics may also lead to the destruction of vibroequilibria states due to nonlinear forcing of the first (odd) sloshing mode and its potentially unbounded growth.

In reduced gravity environments, fluids behave differently than on ground. Surface tension is no longer dominated by gravity, so there is no preference for flat surfaces. A floating volume of liquid will minimise its surface energy by taking a spherical shape, while capillary forces may cause a confined liquid to fully wet the boundaries. The curved surfaces promoted by the forces of cohesion and adhesion mean that the distinction between vertical and horizontal forcing is of limited value. Any applied external force will be inhomogeneous over such a surface. Nonetheless, it can be useful to identify effective local forcing orientations over particular portions of the surface and thus, to understand the behaviour and surface dynamics as a combination of the more familiar phenomena observed on Earth and their interaction (Salgado Sánchez et al., 2019d). The absence of a strong gravitational force has a marked effect on vibroequilibria as well, which deviate much more from the unforced equilibrium since, in weightlessness, the vibrational energy that drives this movement only needs to balance surface and contact energies and becomes comparable for small to moderate forcing amplitudes.

The facility that fluids have to adjust their shape when forced, combined with the preference for curved hydrostatic surfaces, means that fluid management in microgravity is different and often more complicated than it is on Earth. Although vibrations are frequently unwanted in a space environment, they may also, potentially, be put to good use. For example, vibrations are under consideration to help reducing muscle and bone loss in astronauts, and can mimic the effects of gravity in other ways too (Beysens, 2006). This includes the potential to induce convection, which has critical implications in physical processes like mixing, heat transfer, and phase separation.

Further understanding of the various effects of vibrations on fluids may make it possible to use them for advantage in space systems to deliberately manipulate and control fluids, which is a critical problem in space travel. The vibroequilibria effect, in particular, suggests artificial gravity since, as the forcing amplitude increases, the interface tends to evolve towards a more perpendicular alignment with respect to the axis of vibration, restoring some degree of the familiar preference for flat interfaces. The Control of Fluids in microgravity with Vibrations (CFVib) experiment (Fernández et al., 2017a) examined the dynamic behaviour of representative fluids combinations subjected to vibrations. The data obtained contribute to understanding the vibroequilibria effect, and help demonstrate the potential of vibrations for manipulating fluids in microgravity.

In this paper, we present the numerical model used to simulate the vibroequilibria effect in the CFVib experiment. In Sec. 6.2, the experimental setup is briefly described. In Sec. 6.3, the mathematical formulation of the vibrated fluid problem, based on separation of timescales, is described. This formulation includes a model of the interaction between the fluid, structure, and piezoelectric actuators. In Sec. 6.4, the algorithm and numerical scheme are presented. General predictions of the model are presented in Sec. 6.5 and compared to experiments on ground and in microgravity in Sec. 6.6. Conclusions are given in Sec. 6.7.

6.2 The CFVib experiment

The main objective of the experiment is to observe the vibroequilibria effect in small fluid volumes in microgravity. Under reduced gravity conditions, vibroequilibria theory predicts the general tendency of fluid interfaces to orientate more perpendicular to the axis of applied vibrations (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004).

The heart of the experiment is a small recipient (several centimetres wide) that is subjected to high frequency vibrations and either half-filled with water or silicone oil, or filled completely with a combination of silicone oil and vacuum oil. We focus here on the first two liquid-air configurations, which yielded better results than the two-liquid experiment. The recipient, which is cuboidal or cylindrical, is attached to an aluminium beam in a cantilever configuration with two piezoelectric ceramics attached on opposite horizontal faces near the clamped end. Figure 6.1 shows (a) a cuboidal container holding the oil-oil combination and (b) a cylindrical container holding water.

The pair of piezoelectric ceramics are electrically excited in counterphase to induce a linear distribution of tensile stress along the beam width. This stress generates a bending moment that drives flexural vibrations without longitudinal motion. The electrical signals driving the vibrations are passed through an acoustic amplifier to boost the power transmitted. The two main control parameters are voltage amplitude and frequency, which select the character of the applied vibrations. The amplitude of vibrations can be varied in a (nearly) continuous manner. The driving frequency, on the other hand, is selected to coincide with one of the many structural resonances, where larger displacements can be achieved. The resonances explored include both flexural beam modes and container modes, located at particular values over the range of 50 Hz to 12 kHz. Dedicated cabling

FIGURE 6.1: Images of the experimental cells: (a) cuboid and (b) cylinder. The piezoelectric ceramic attached on the upper side of the beam (the other is attached underneath) is visible in each assembly along with the associated cabling.

is attached to each piezoelectric to transmit the electrical signals, and part of this can be seen in Fig. 6.1. The fluid dynamics is monitored by means of a complementary metal–oxide–semiconductor (CMOS) camera using an acquisition rate of (approximately) 20 frames per seconds, which is enough to capture the average effects of the applied vibrations.

The microgravity experiments are thus defined by the selected fluid(s), container geometry and applied voltage (peak-to-peak amplitude and frequency). They were conducted on board an aircraft executing parabolic flight manoeuvres to provide repeated periods of approximately 20 s of reduced gravity. Since the different fluids have different surface tension and contact angles, a comparison reveals something of their influence on the vibrational phenomena. To observe clear vibroequilibria effects, the energy associated with the oscillatory velocity field should be (at least locally) greater or comparable to the characteristic variation in contact and surface energies. For experiments with silicone oil, which has a low contact angle (with most solids) and wets easily, this can be expected to be somewhat easier to achieve because contact angle, and thus contact energy, will be nearly constant. The experiments with water, in contrast, have more resistance to contact line motion.

In order to separate the effects of the forcing from accelerations caused by the aircraft (residual gravity or g-jitter), an unforced cell was included for each configuration. These passive cells have no piezoelectrics to excite the bender beam but are otherwise identical (to the extent possible). Further details of the experiment setup can be found in Fernández et al. (2017a).

6.3 Mathematical formulation

The experiments described in Sec. 6.2 depend on the complex interaction of different physical systems. The mathematical formulation presented below includes a treatment of fluid dynamics under a vibrational field (detailed in Sec. 6.3.1), the electromechanical coupling of piezoelectric materials and the structural assemblies that governs the conversion of electrical power into mechanical motion (detailed in Sec. 6.3.2), and the fluid-structure interaction, which determines how structural

FIGURE 6.2: Sketches showing a midplane cut of the cuboid assembly. The domains of the mathematical formulation for (a) fluid dynamics and (b) fluid-structure-piezoelectric interaction are included. Throughout this paper, simulations assume the midplane reflection symmetry illustrated here.

displacements generate fluid motion. Altogether, the mathematical model describes the transfer of electrical energy into the fluid and, in particular, to surface deformation.

6.3.1 Fluid dynamics

We consider a volume Ω of fluid held in a vibrated container in the presence of a gravitational field; see Fig. 6.2(a). The fluid satisfies the incompressible Navier-Stokes momentum and mass continuity equations (Landau and Lifshitz, 1987)

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \mu \Delta \mathbf{u} + \rho \mathbf{g}(t), \quad (6.1a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (6.1b)$$

where $\mathbf{u}$ is the velocity, ρ the density, μ the dynamic viscosity, p the pressure, and $\mathbf{g}(t)$ the gravitational acceleration.

No-slip boundary conditions apply on the solid boundaries $W(\mathbf{x}, t)$, which oscillate with frequency ω in the laboratory reference frame. On the free surface $F(\mathbf{x}, t) = 0$, there is a balance between surface tension, pressure and viscous stress.

$$\mathbf{u} = \mathbf{A}\omega \sin(\omega t), \quad \text{on } W(\mathbf{x}, t); \quad (6.2a)$$

$$p \mathbf{n} - (\Gamma \kappa) \mathbf{n} - \mu (\nabla \mathbf{u}) \cdot \mathbf{n} = 0, \quad \text{on } F(\mathbf{x}, t); \quad (6.2b)$$

$$F_t + \mathbf{u} \cdot \nabla F = 0, \quad \text{on } F(\mathbf{x}, t). \quad (6.2c)$$

Here, κ is the mean curvature of the interface defined as

$$2\kappa(F) = \nabla \cdot \left(\frac{\nabla F}{|\nabla F|} \right), \quad (6.3)$$

where $\mathbf{n} = \nabla F / |\nabla F|$ is a unit normal vector and $\mathbf{A}(\mathbf{x})$ describes the spatial distribution of the boundary displacement.

For the experiments of Sec. 6.2, the applied vibrations are of small amplitude, $|\mathbf{A}| \ll L$, and high frequency $\omega \gg \mu / (\rho L^2)$, where L is the characteristic length of the container. This allows a separation of timescales (Lyubimov and Cherepanov, 1986) to be applied, with different descriptions for phenomena occurring on the slow (vibroequilibria) timescale $t_v = t$ and the fast (harmonic) timescale $t_h = \omega t$. Fluid velocity and pressure are expressed as the sum of contributions from each of these timescales,

$$\mathbf{u} = \mathbf{u}_h(\mathbf{x}, t_h) + \omega^{-1} \mathbf{u}_v(\mathbf{x}, t_v) + \dots, \quad (6.4a)$$

$$p = \omega p_h(\mathbf{x}, t_h) + p_v(\mathbf{x}, t_v) + \dots, \quad (6.4b)$$

$$F = F(\mathbf{x}, t) + \dots, \quad (6.4c)$$

while temporal derivatives are written as

$$(\cdot)_t = \omega (\cdot)_{t_h} + (\cdot)_{t_v}. \quad (6.5)$$

The (imperfect) gravity level of the parabolic flight typically satisfies $|\mathbf{g}| < 0.05 g_0$ (Pletser et al., 2016), where $g_0 = 9.81 \text{ m/s}^2$. In situ measurements demonstrate that the dominant microgravity variations occur on the slow timescale, so that we may write $\mathbf{g} = \mathbf{g}(t_v)$. The appropriate fluid equations in this limit are derived below, where it is also assumed, for simplicity, that $\mathbf{g} = 0$ (i.e., no residual gravity). Relaxing this final condition does not change the mathematical formulation.

Fast timescale dynamics: linear harmonic field

The dynamics on the fast (harmonic) timescale, which occur in response to the motion of the walls W , satisfy

$$\rho \mathbf{u}_{h,t_h} = -\nabla p_h + \mu \Delta \mathbf{u}_h \quad \text{in } \Omega, \quad (6.6a)$$

$$\nabla \cdot \mathbf{u}_h = 0 \quad \text{in } \Omega, \quad (6.6b)$$

$$\mathbf{u}_h = \mathbf{A} \omega \sin(\omega t) \quad \text{on } W, \quad (6.6c)$$

$$p_h = 0 \quad \text{on } F, \quad (6.6d)$$

$$F_{t_h} = 0 \quad \text{on } F, \quad (6.6e)$$

where the viscous term is retained in the momentum equation to preserve no-slip boundary conditions on the solid boundaries. The free surface does not evolve on the fast timescale [Eq. (6.6e)] and the force balance reduces to a constant (zero) pressure condition [Eq. (6.6d)].

Slow timescale dynamics: mean flow field

The dynamics of the mean flow occurring on the slow timescale are driven by gravity (neglected here) and the convective term arising from the fast (harmonic) velocity field. As per the separation of timescales, this latter source term is averaged over one period (Lyubimov and Cherepanov, 1986), denoted by $\langle \cdot \rangle$, to obtain the governing equations

$$\rho [\mathbf{u}_v, t_v + \langle (\mathbf{u}_h \cdot \nabla) \mathbf{u}_h \rangle] = -\nabla p_v + \mu \Delta \mathbf{u}_v \quad \text{in } \Omega, \quad (6.7a)$$

$$\nabla \cdot \mathbf{u}_v = 0 \quad \text{in } \Omega, \quad (6.7b)$$

$$\mathbf{u}_v = 0 \quad \text{on } W, \quad (6.7c)$$

$$p_v \mathbf{n} - (\Gamma \kappa) \mathbf{n} - \mu (\nabla \mathbf{u}_v) \cdot \mathbf{n} = 0 \quad \text{on } F, \quad (6.7d)$$

$$F_{t_v} + \mathbf{u}_v \cdot \nabla F = 0 \quad \text{on } F. \quad (6.7e)$$

Again, viscous terms are retained in order to have no-slip boundary conditions on the solid walls.

Note that Eqs. (6.6) are coupled to Eqs. (6.7) through the evolution of the free surface F , which defines the fluid domain Ω . It is thus necessary to solve both the fast and slow timescale equations in an iterative manner. Further simplifications of this approach can be made in some cases, as described in subsequent sections.

6.3.2 Piezoelectricity and structural mechanics

The general formulation of piezoelectricity is in terms of differential equations describing the balance of momentum and energy conservation (Martin, 1972)

$$\rho_m \mathbf{d}_{tt} - \nabla \cdot [S] = \mathbf{p}, \quad (6.8a)$$

$$\nabla \cdot \mathbf{D} = \rho_v, \quad (6.8b)$$

where ρ_m is the density of the material in question, ρ_v is the charge density, $\mathbf{d}$ is a vector of displacements, $[S]$ is the (symmetric) mechanical stress tensor, $\mathbf{p}$ contains the volume forces acting on the piezoelectric material (or structure) and $\mathbf{D}$ is the electrical displacement. This electrical displacement is produced by charge distribution gradients in the element due to deformation in the molecular structure (Catarino et al., 2011).

Equations (6.8) are closed with the classical constitutive law

$$[\epsilon] = \frac{1}{2} \left(\nabla \mathbf{d} + (\nabla \mathbf{d})^T \right), \quad (6.9)$$

where $[\epsilon]$ is the (symmetric) mechanical strain tensor, the Maxwell equation relating the electric field $\mathbf{E}$ and voltage V

$$\mathbf{E} = -\nabla V, \quad (6.10)$$

and the material constitutive relationships

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{22} \\ \epsilon_{23} \\ \epsilon_{33} \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} s_{11}^E & s_{12}^E & s_{13}^E & s_{14}^E & s_{15}^E & s_{16}^E & d_{11}^M & d_{12}^M & d_{13}^M \\ s_{21}^E & s_{22}^E & s_{23}^E & s_{24}^E & s_{25}^E & s_{26}^E & d_{21}^M & d_{22}^M & d_{23}^M \\ s_{31}^E & s_{32}^E & s_{33}^E & s_{34}^E & s_{35}^E & s_{36}^E & d_{31}^M & d_{32}^M & d_{33}^M \\ s_{41}^E & s_{42}^E & s_{43}^E & s_{44}^E & s_{45}^E & s_{46}^E & d_{41}^M & d_{42}^M & d_{43}^M \\ s_{51}^E & s_{52}^E & s_{53}^E & s_{54}^E & s_{55}^E & s_{56}^E & d_{51}^M & d_{52}^M & d_{53}^M \\ s_{61}^E & s_{62}^E & s_{63}^E & s_{64}^E & s_{65}^E & s_{66}^E & d_{61}^M & d_{62}^M & d_{63}^M \\ d_{11}^E & d_{12}^E & d_{13}^E & d_{14}^E & d_{15}^E & d_{16}^E & \epsilon_{11}^M & \epsilon_{12}^M & \epsilon_{13}^M \\ d_{21}^E & d_{22}^E & d_{23}^E & d_{24}^E & d_{25}^E & d_{26}^E & \epsilon_{21}^M & \epsilon_{22}^M & \epsilon_{23}^M \\ d_{31}^E & d_{32}^E & d_{33}^E & d_{34}^E & d_{35}^E & d_{36}^E & \epsilon_{31}^M & \epsilon_{32}^M & \epsilon_{33}^M \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{22} \\ S_{23} \\ S_{33} \\ E_1 \\ E_2 \\ E_3 \end{pmatrix}. \quad (6.11)$$

Here, s_{ij}^E are the elastic coefficients, d_{ij}^M and d_{ij}^E are the piezoelectric coefficients and ϵ_{ij}^M is the dielectric permittivity (Sherrit and Mukherjee, 2007). For classical structural elements that do not couple to the electric field, this same formulation holds but with $d_{ij}^M = d_{ij}^E = \epsilon_{ij}^M = 0$. The formulation may be extended to include damping, as described in later sections.

Different boundary conditions are applied for the piezoelectric material compared to ordinary materials:

- (i) The electrical problem in the piezoelectric ceramics is solved by imposing ground voltage on the faces adjacent to the beam and the periodic voltage $V_0 = (V_{pp}/2) \sin(\omega t)$ of frequency ω on the opposite horizontal sides. Zero charge is enforced on the remaining boundaries.
- (ii) For ordinary material elements, zero displacement and stress are imposed at the fixed and free boundaries, respectively. Interfaces separating structural elements are considered to fully transmit stresses and strains.

Since the structure vibrates harmonically at the frequency ω of the applied voltage, the fluid-structure interaction on W is made consistent with the solution of Eqs. (6.8)–(6.11) by writing

$$\mathbf{d}_t = \mathbf{A} \omega \sin(\omega t). \quad (6.12)$$

Equations (6.6)–(6.12) are solved in dimensional variables with COMSOL Multiphysics 5.3 using the Finite Element Method. The numerical scheme is discussed below.

6.4 Numerical scheme

6.4.1 Algorithm

We solve the problem formulated in Sec. 6.3 with an iterative algorithm. Following the work of Koster (2006), the dynamics on the fast timescale are solved first, including the fluid-structure interaction. The resulting linear field is then averaged to determine its contribution to the slow timescale, and the corresponding mean flow equations (6.7) are evolved through one or more forcing periods. After that, the free surface location is updated and the process is repeated. This algorithm is illustrated in Fig. 6.3(a) using solid lines (arrows).

We can distinguish several processes and characteristic timescales for the CFVib experiment:

- (i) The timescale of the fast (harmonic) dynamics is $\omega^{-1} \sim 10^{-4} - 10^{-2}$ s, where frequencies up to tens of kHz are considered (Fernández et al., 2017a).
- (ii) The mean flow (vibroequilibria) timescale is on the order of seconds (Fernández et al., 2017b), $t_v \sim 1 - 10$ s.
- (iii) The timescale of sound waves propagating through the fluid is $L a^{-1} \sim 10^{-5}$ s, where $a = 1500$ m/s is the speed of sound in water and $L = 3$ cm is the characteristic length of the container.
- (iv) The typical transient time to relax to an unforced hydrostatic equilibrium, as with vibroequilibria, is on the order of seconds (Abramson, 1966).

Since the mean flow evolves over a much longer timescale than the harmonic field and the propagation of its changes via sound waves, transient behaviour in the harmonic solution may be neglected, treating it as (quasi)steady. This allows the harmonic solution to be expressed in the frequency domain, which reduces the computational cost. Moreover, since the speed of wave propagation (i.e., sound) in solids is even larger than in fluids, this assumption can naturally be extended to the structural mechanics part of the problem. Transient behaviour is thus only considered as part of the mean flow problem, along with surface reorientation.

The solution of a linear problem gives the harmonic velocity field $\mathbf{u}_h$ in the fluid for a fixed geometry as discussed above. In the case of interest here, where the wavelength of sound (on the order of meters) is much larger than the characteristic length of the container (several centimetres), the fluid dynamics may be treated as incompressible. Furthermore, if the deformation of the surface (vibroequilibria) with respect to the static equilibrium is not large, then the same harmonic field can be used repeatedly for the source term in the mean flow equations without a large error, allowing a simplification of the basic algorithm and lower computing cost. In the simplest case, the effect of the surface deviation on the harmonic field is ignored completely and the same source term is used throughout the calculation.

FIGURE 6.3: (a) Algorithm using the timescales separation approach, showing the complete iterative scheme (solid lines) and the simplified scheme (dashed lines) followed here. Here T is the forcing period and T_f the stopping time. (b) The complete (symmetric) geometry used in the model. The relevant physical properties of water (at room temperature) and of structural materials, reproduced from the COMSOL Library, are included: density ρ , surface tension Γ , dynamic viscosity μ , Young's modulus E and Poisson ratio ν , when applicable.

The simplified numerical scheme, which does not update the harmonic fields $\mathbf{u}_h$ and p_h , is illustrated in Fig. 6.3(a) by dashed lines (arrows). Three steps are followed: (1) solve the linear harmonic field in the frequency domain, and use the frequency dependence of the solution to locate resonances where more vigorous motion and a correspondingly enhanced vibroequilibria effect is expected, (2) calculate the (averaged) source term $\langle (\nabla \mathbf{u}_h) \mathbf{u}_h \rangle$ for the selected frequency, and (3) simulate the slow transient evolution towards the vibroequilibria solution at this frequency.

The linear harmonic field, resolved in the frequency domain, may be written as

$$\mathbf{u}_h = \frac{1}{2} (\mathbf{u}_h e^{i\omega t} + \text{c.c.}), \quad (6.13)$$

where $\mathbf{u}_h(\mathbf{x})$ are complex functions, and "c.c." denotes the complex conjugate. The convective source term can then be written as

$$\langle (\nabla \mathbf{u}_h) \mathbf{u}_h \rangle = \frac{1}{4} (\nabla \mathbf{u}_h \bar{\mathbf{u}}_h + \nabla \bar{\mathbf{u}}_h \mathbf{u}_h), \quad (6.14)$$

where integrals of the form $\int_t^{t+2\pi/\omega} (\cdot) e^{\pm 2i\omega t} dt$ vanish, since $(\cdot)$ only depends on spatial coordinates. Note that this expression is assumed constant during the transient analysis, as discussed above, and is applied at the corresponding nodes in both the initial and deformed meshes.

6.4.2 COMSOL model of the cuboid-water experiment

In the remainder of this paper, we apply the formulation of Sec. 6.3 to experiments in the cuboidal container that is half-filled with water. Figure 6.3(b) illustrates the geometry of the model, following the real container assembly as closely as possible. The vibrational dynamics of the system are principally affected by elements with relevant inertia or stiffness, so only these main elements are modelled. For example, neither the O-ring nor the screw-bolt unions (see Fig. 6.1), which are part of the real assembly, are retained in the model. Including these static compression loads does not modify the vibrational response in any significant way.

To model the experiment, we consider a rectangular cuboid with interior dimension (length $\times$ height $\times$ width) of 3 cm $\times$ 3 cm $\times$ 4 cm and solid walls of thickness of 0.5 cm. The cell is (approximately) half-filled with water. The dynamics of the contact line are simplified to impose a steady contact angle β while permitting contact line motion; this is crucial to reproducing the experimental results. We remark that this numerical scheme does not consider the complications associated with contact line hysteresis, which may play a role in the experiments.

The container is made of PMMA (acrylic) and is closed by two aluminium plates to ensure one of the two levels of containment required in parabolic flights. We model the attachment to the bender beam using a PMMA piece and a steel nut, which act together as a support for the cell. The real mounting includes a bolt that is fastened to the PMMA block and later fastened to the beam. The PMMA block is chemically soldered to the container using chloroform. Finally, the beam geometry includes the two piezoelectric ceramics and the metal support piece on the clamped end, whose dimensions and location match the real experiment assembly.

The physical properties of water and the relevant material parts, which determine the constants and matrices involved in the governing Eqs. (6.6)–(6.12), are listed in Fig. 6.3(b). In particular, in the equations of piezoelectricity, many elements of s_{ij}^E , d_{ij}^M , ϵ_{ij}^M and d_{ij}^E are either zero or determined by symmetries of the ceramic pieces. The experimental determination of these tensors has been undertaken by several investigations (see, e.g., Sherrit and Mukherjee, 2007). We do not attempt this precise characterisation here, using instead the typical piezoelectric properties provided for Lead Zirconate Titanate (PZT-2) ceramic in the COMSOL Library.

The unknown value of damping, which will be discussed more in subsequent sections, is taken to be small and equal for all structural elements: $\eta_s = 0.01, 0.02$ in the isotropic loss factor formulation (see Sec. 6.6.1 for details). We thus anticipate certain differences with the real experiments, especially regarding the predicted displacements in each mode. Further refinement of the model is done by determining the structural damping that best fits the experimental data at the resonance frequencies and including this frequency dependence.

For numerical convenience, a reflection symmetry about the midplane is assumed, which bars some types of dynamics (e.g., the first torsional mode of the beam is not permitted). We nonetheless

Mesh #	DoFs	f_1 (Hz)	$ \Delta_1 $ (%)	f_2 (Hz)	$ \Delta_2 $ (%)	f_3 (Hz)	$ \Delta_3 $ (%)	$ \Delta $ (%)
#1	19460	88	8.6	246	13.4	857	1.9	3.88
#2	37168	86	6.2	240	10.6	846	0.5	2.76
#3	56598	83	2.5	228	6.0	843	0.4	1.30
#4	118346	81	0.0	220	2.8	842	0.1	0.36
Experiment	—	81	—	217	—	841	—	—

TABLE 6.1: Results of the mesh convergence test against experimental measurements at the first three resonance frequencies. The selected mesh #4 is highlighted in bold.

expect to find good agreement with the experiments that are conducted near (non-torsional) resonances. For simulations of the free surface evolution, a BDF (backward differentiation formula) integration scheme is used with a *streamline* stabilisation technique.

Convergence of the numerical model

Convergence is examined for different meshes. The defining characteristics, results and numerical error compared to the experimental determination of the resonant modes, are included in Table 6.1. The simulations presented throughout this paper are performed using mesh #4, which gives acceptable accuracy with reasonable computing cost.

6.5 Some numerical predictions

Following Fernández et al. (2017b), we use the dimensionless velocity

$$(u_h, v_h, w_h) \rightarrow (\widehat{u}_h, \widehat{v}_h, \widehat{w}_h) \left(\frac{\Gamma}{\rho \sqrt[3]{\mathcal{V}}} \right)^{0.5}, \quad (6.15)$$

where $\mathcal{V} = 18 \text{ cm}^3$ is the volume of water used in the experiments. In Fig. 6.4, the absolute value of the dimensionless vertical velocity $|\widehat{w}_h|$ is calculated over the frequency range of interest for two selected damping values $\eta_s = 0.01$ (solid) and 0.02 (dashed lines) when the driving voltage has a peak-to-peak value $V_{pp} = 100 \text{ V}$. Both the velocity at the midpoint of the fluid surface (grey) and at the right lateral PMMA wall (black line) where the accelerometer is placed in experiments, are shown. The importance of a resonance is evaluated in terms of the vertical velocity $\widehat{w}_h$ predicted at the free-surface; accordingly, any mode that does not induce substantial motion of the fluid is not considered.

Damping modifies the solution by limiting the maximum velocity (equivalently, displacement). As mentioned above, a wide variety of resonances, including a combination of flexural modes of the beam and modes of the cuboid structure, may be found. Special attention is paid to resonances at low frequency (less than 1 kHz) since these were found to be the most effective excitations in the experiment and can, also, be modelled more accurately because they are well-characterised by a single accelerometer measurement and require fewer mesh points.

FIGURE 6.4: Magnitude of nondimensional harmonic velocity $|\hat{w}_h|$ at the midpoint of the free surface (grey) and the midpoint of the right lateral wall (black) for different damping values $\eta_s = 0.01$ (solid) and 0.02 (dashed) with forcing $V_{pp} = 100$ V. A close-up view of the first resonance in the inset shows the effect of increased damping. The three first resonances near 81 Hz, 220 Hz and 842 Hz are illustrated at the instant of maximum deformation (displacement scale not preserved between illustrations).

The first resonance mode, near 81 Hz, involves the primary bending (rotational) motion of the cantilever beam that induces both vertical displacement and an in-phase rotation of the container, as sketched in the upper-left illustration of Fig. 6.4. This movement is complicated, but resembles a mixture of vertical (up-and-down motion at the midsection of the beam) and horizontal (due to rotation) excitations, which corresponds an oblique local vibrational axis. This interaction plays a key role in determining the vibroequilibria solution.

The second mode occurs near 220 Hz, and can be roughly understood as a combination of the primary bending (rotational) motion of the beam and a counterphase rotation of the container about the PMMA support. This mode provides an effective vibration (upper middle illustration) that induces vertical motion of roughly the same magnitude in the interior of the fluid as at the lateral wall.

Increasing the frequency to 842 Hz locates the third resonance, which involves the second flexural mode of the cantilever beam and a small rotation around the PMMA piece. Again, the relative displacements are shown in the corresponding inset in Fig. 6.4.

At a frequency near 1940 Hz, there is a structural mode with less practical interest since very little motion is transmitted to the interior of the fluid. For still higher frequencies, the attenuation increases and the numerical model is less likely to provide an accurate picture of the real response.

FIGURE 6.5: (a,b) Vibroequilibria calculated numerically for the (a) first and (b) second resonances after 20 s of applied vibrations at 200 V. The parameters used are those of water, with the contact angle set to $\beta = 80^\circ$. (c) Nondimensional height of the front (grey squares) and rear (black squares) contact points at the midplane for increasing vertical vibrational velocity $\widehat{w}_h$ in the first resonance mode. Note the approximately linear dependence on the squared velocity for $|\widehat{w}_h|^2 \lesssim 3$.

These high frequency resonances involve a combination of higher beam modes and different structural modes of the container, including both lateral wall motion resembling the breathing of a membrane and similar movement of the aluminium covering plates.

Vibroequilibria states found at the first and second resonances are illustrated in Fig. 6.5(a,b), respectively, assuming a contact angle of 80° . As described above, the vibroequilibria state depends on the motion of the container and the spatial dependence of the harmonic velocity field that it induces. For the first mode, the fluid tends to climb the right lateral wall due to the more vigorous motion of this boundary, thereby (locally) orienting the surface more perpendicular to the effective oblique vibrational axis. A relatively high voltage of about 100 V or more is needed to observe a significant vibroequilibria effect. The second (counterphase) mode, on the other hand, selects a vibroequilibria state that tends to climb the left wall, which, in this case, vibrates more than the right wall. Again, this can be thought of as a mixture of vertical and horizontal excitations, modified by other types of motion.

Numerical simulations suggest that different modes of the beam, container and fluid can be used to excite various velocity distributions in the fluid. By vibrating the system near one of the resonances, different vibroequilibria solutions can be selected, depending on the type and relative importance of beam and container motion. If up-and-down motion is dominant, for instance, the surface will tend to flatten. In contrast, if the lateral vibration is stronger, a reorientation that tends to locally align the surface with the lateral walls is expected. Between these limiting cases, a range of more complicated mixed solutions can be found. If the dominant vibrations of the system can be predicted, the vibroequilibria effect can be anticipated as well, meaning that the choice of frequency (resonance) in this system is the key factor that selects the shape of the final fluid surface. To this

extent, deliberate movement, or control, of the fluid is possible.

Numerical investigations of vibroequilibria solutions were conducted by Fernández et al. (2017b) and Gavriluk, Lukovsky, and Timokha (2004). A general feature of the vibroequilibria effect is its scaling with the square of vibrational (harmonic) velocity or, equivalently, with the dimensionless parameter

$$\xi = |\widehat{\mathbf{u}}_h|^2/4. \quad (6.16)$$

In Fig. 6.5(c), we measure the vibroequilibria effect via the height of the rear and front contact points in the midplane and the difference is nearly linear in $|\widehat{w}_h|^2$, until a value of about 3. As discussed in Sec. 6.4, the simplified scheme we use ignores the effect of the free surface deviation on the fast timescale by keeping the same source term in the mean flow equations. Beyond $|\widehat{w}_h| \sim 2$, which corresponds to $V_{pp} = 150$ V, the displacement of the surface is significant (comparable to the fluid depth) so this assumption begins to fail. The figure suggests that the effect of this error is to overestimate the vibroequilibria effect, and that an update of the harmonic velocity field is required to go further in amplitude.

6.6 Comparison with experiments

6.6.1 Ground experiments: damping estimate

In Sec. 6.5, two different damping values were used to calculate fluid velocities, illustrating the need for a good estimate of the attenuation. An appropriate damping can be determined from tests performed near each resonance by looking at peak amplitude and bandwidth. Both Rayleigh damping $[\mathcal{C}]$ and the isotropic loss factor η_s are considered by introducing the modified matrices

$$[\mathcal{C}] = \alpha_M[\mathcal{M}] + \beta_K[\mathcal{K}], \quad (6.17a)$$

$$([\mathcal{M}], [\mathcal{K}]) \rightarrow (1 + j\eta_s) ([\mathcal{M}], [\mathcal{K}]), \quad (6.17b)$$

in the Finite Element Method formulation. Here, $\mathcal{C}$ is the damping matrix, $\mathcal{M}$ and $\mathcal{K}$ are the effective mass and rigidity of the system, respectively, while α_M and β_K are the Rayleigh damping coefficients and η_s represents the isotropic loss factor.

Experiments indicate that the first three resonances, all at frequencies less than 1 kHz, are of greatest interest since a clear vibroequilibria effect is observed. This means that estimates of the damping coefficients η_s (or α_M , β_K) are needed only over frequencies up to 1 kHz, where attention is focused hereafter. Resonances at higher frequencies appear to experience greater damping, which reduces the vibroequilibria effect (i.e., comparable deformation require higher forcing).

FIGURE 6.6: (a) Measured magnitude of the vertical harmonic velocity $|\widehat{w}_h|$ at the midpoint of the right endwall for different excitation voltages V_{pp} , at the three first natural frequencies: 81 Hz (solid circles), 220 Hz (solid squares) and 842 Hz (solid triangles); with associated linear fittings (grey lines). Values calculated numerically using $\eta_s(f)$ as in Eq. (6.20) are shown with grey open markers. (b) Magnitude of the vertical harmonic velocity $|w_h|$ for frequencies up to 1 kHz with a driving voltage of 200 V: experimental measurements (black curve) and numerical calculations (grey curves). Two damping models are used: $\eta_s(f)$ [grey solid line, Eq. (6.20)] and with Rayleigh coefficients [grey dashed line, Eq. (6.22)]. Close-up views of the three resonances are provided in the insets.

As shown in Fig. 6.6(a), the dependence of measured velocity on applied voltage is nearly linear for the first three modes, with respective linear fits

$$|\widehat{w}_h^{(1)}| = (1.21V_{pp} + 5.91) \times 10^{-2}, \quad (6.18a)$$

$$|\widehat{w}_h^{(2)}| = (0.54V_{pp} - 3.67) \times 10^{-2}, \quad (6.18b)$$

$$|\widehat{w}_h^{(3)}| = (0.56V_{pp} - 6.68) \times 10^{-2}, \quad (6.18c)$$

Here, $\widehat{w}_h^{(i)}$ refers to the vertical velocity measured for the i th resonant mode, measured at the midpoint of the rear wall of the assembly where the accelerometer is placed.

Neglecting the constant term at $V_{pp} = 0$, which is only finite because of experimental error and filtering effects, the slope of each expression gives the (dimensional) transfer function $|w_h|/V_{pp}$. Based on these values, damping expressions that capture each resonance are determined as functions of frequency. Furthermore, the results of a frequency sweep experiment are shown in Fig. 6.6(b) (black line) where the bandwidth of each resonance can be estimated.

Isotropic loss factor: $\eta_s(f)$

The damping coefficient η_s is chosen to match the predicted velocities at each resonance with those measured from experiments. We use a least squares fitting to determine the coefficients $\mathcal{P}_i$ of a

polynomial in frequency

$$\eta_s(f) = \sum_{i=0}^N \mathcal{P}_i f^i, \quad (6.19)$$

and find that a second order fitting polynomial ($N = 2$),

$$\eta_s(f) = -2.573 \times 10^{-7} f^2 + 2.337 \times 10^{-4} f - 1.093 \times 10^{-2}, \quad (6.20)$$

is sufficient to accurately reproduce the measured results over the frequency range of interest. The vertical velocity distribution obtained after introducing $\eta_s(f)$ into the FEM model is shown in Fig. 6.6(b) (grey solid curve). The agreement is good, both for the resonance locations and for the maximum velocities.

Despite the success of this approximation over the frequency range of interest, the expression for $\eta_s(f)$ has a negative quadratic coefficient, which gives negative (non-physical) values of the damping at higher frequencies. Extending the model would require a higher order polynomial and is not attempted here because the higher frequency resonance modes involve the cuboid container and cannot be measured well with a single accelerometer.

Rayleigh damping coefficients: α_M, β_K

In an analogous fashion, the Rayleigh damping coefficients can be estimated as a function of the frequency by fitting the resulting values at each resonance. Writing

$$\eta_s = \left(\frac{\alpha_M}{2\pi f} + 2\pi\beta_K f \right), \quad (6.21)$$

and using the frequencies of the first two resonances, one finds both coefficients:

$$\alpha_M = -2.2594 \text{ s}^{-1}, \quad \beta_K = 2.1165 \times 10^{-5} \text{ s}. \quad (6.22)$$

After introducing α_M and β_K into the FEM model, the vertical velocity distribution is calculated and shown in Fig. 6.6(b) (grey dashed curve). As expected, the velocity at the first two resonances is reproduced quite well, while neither the location nor maximum velocity is captured very accurately for the third resonance. As with the isotropic loss factor, this model does not agree with experimental measurements if the frequency range is extended; a generalisation with more frequency-dependent terms would be needed, writing, for example, $\alpha_M(f)$ and $\beta_K(f)$.

6.6.2 Microgravity experiments: vibroequilibria in the CFVib experiment

The predicted vibroequilibria at the first two resonance modes were illustrated in Fig. 6.5(a,b) of Sec. 6.5 for an arbitrary choice of η_s . Damping does not qualitatively affect the character of these resonance modes, but does affect their width and limit the maximum displacement (equivalently, velocity or acceleration). The attenuation thus affects the voltage required to generate a certain vibroequilibria amplitude (deformation), but has little influence on the shape of the free surface.

FIGURE 6.7: Simultaneous images of *Vibrated* and unforced *Reference* microgravity experiments in cuboidal containers partially filled with water: (a) vibroequilibria effect when vibrated at $f \simeq 81$ Hz (the first resonant mode) and $V_{pp} = 200$ V, (b) reference experiment, experiencing only the effect of airplane disturbances.

Here, we qualitatively compare the numerical predictions with experimental results obtained during the 65th ESA Parabolic Flight Campaign. Some discrepancy is expected owing to flight disturbances and g -jitter. Among other sources of error, the midplane symmetry assumed in the model for simplicity and feasibility does not hold in the real experiments.

Reasonable agreement is found between experiment and simulations for the first resonance. Images taken during the microgravity portion of a parabola reveal the tendency of the fluid to climb the rear wall, as seen in Fig 6.7(a); this behaviour was predicted in Fig. 6.5(a). Experiments also display surface waves and, sometimes, drop ejection when the waves are of large amplitude (James, Smith, and Glezer, 2003; Salgado Sánchez et al., 2019d). The effects of residual gravity can be seen by comparing the vibroequilibria with the unforced equilibria present in the corresponding passive cell shown in Fig. 6.7(b). This preliminary analysis makes it clear that flight disturbances are relevant for the results of the experiment, particularly for low to medium applied voltages (up to approximately 100 V) where they can partially mask the vibroequilibria effect.

6.7 Conclusions

We have described the mathematical and numerical formulation used to model the Control of Fluids with Vibrations (CFVib) microgravity experiment, which flew on the 65th ESA Parabolic Flight campaign. Predictions of the model were compared to experiments on ground and in microgravity.

When a fluid is periodically forced, there are interfacial instabilities like surface waves that occur on this same (fast) timescale and, also, slower (average) processes like the vibroequilibria effect, which describes the reorientation of the fluid interface toward a new quasiequilibrium. In the limit of high amplitude (less in reduced gravity), a free surface or other density contour will tend to position itself (largely) perpendicular to the effective vibrational axis. The physical reason for this is the inhomogeneous dynamic pressure generated by the rapid oscillatory velocity field. This pressure distribution deforms the surface until an appropriate balance with surface tension (and gravity, if

present) is achieved.

The CFVib microgravity experiment was designed to study the vibroequilibria effect in small volumes of fluid (dimensions of several centimetres) confined in closed, partially-filled acrylic containers, using vibrations applied via piezoelectric actuators, to deliberately modify the position of the fluid. Cuboidal and cylindrical containers were used with water, silicone oils of different viscosity, and a combination of silicone and vacuum oils. The effect of vibrations was examined under the microgravity conditions of a parabolic flight, in these two geometries and with the selected fluid combinations, and compared to the behaviour of unforced reference cells.

The numerical model developed here was used to understand and complement the experimental results. The piezoelectric ceramics, which are used to deliver vibrations, as well as the structure of the experiment container are treated using classical mechanics theory, while the fluid dynamics is analysed using a separation of timescales. The governing Navier-Stokes equations are expanded in a perturbation series to obtain equations acting on the fast timescale, which provides a linear harmonic velocity field, and on the slow timescale. Compressibility is neglected since the container size is much smaller than the wavelength of sound over the frequency range of interest (up to 1 kHz).

The separation between the period of the harmonic field oscillating at frequency ω and the longer time needed for relaxing to either a static equilibrium or a vibroequilibria state justifies the neglect of transient behaviour for the harmonic field and its resolution in the frequency domain. The linear problem on the fast timescale includes the fluid-structure interaction along the container walls and provides the (nonlinear) source term for the equations on the slow timescale. The numerical scheme can be further simplified if the surface deformation is assumed small enough that it does not significantly affect the harmonic velocity field, an approximation that fails at high enough forcing. Whether the harmonic velocity field is updated or not, the evolution towards the vibroequilibria is completed after the surface reaches a steady (on the slow timescale) position. As anticipated, the time for this relaxation is on the order of a few seconds, several orders of magnitude larger than the period of piezoelectric vibrations.

Experiments and simulations are consistent and suggest that one can take advantage of the different resonant modes of the experimental assembly to excite distinct oscillatory velocity fields in the fluid, and thereby achieve different vibroequilibria solutions. If, for example, the dominant motion of the container is up-and-down, the surface will tend to flatten “horizontally”. If the dominant vibration is along a lateral axis, the fluid will tend to climb the lateral walls of the container (into a more “vertical” orientation). A wide variety of intermediate scenarios can occur, generating vibroequilibria of a more complicated (mixed) type. The results presented here thereby offer support for the idea that vibrations can be used to deliberately reorient fluid interfaces, while the numerical model allows us to predict the vibroequilibria effect associated with the main resonant modes of the system. A more detailed analysis of the experimental results should allow the model to be further refined and improved.

Acknowledgment. This work was supported by the ESA Education Office in connection with the Fly Your Thesis! programme and the 65th ESA Parabolic Flight Campaign, and by the Ministerio de Economía y Competitividad under Projects No. ESP2013-45432-P and No. ESP2015-70458-P. We thank the European Low Gravity Research Association (ELGRA) for their valuable scientific support and Novespace for their technical advice and contributions during the project. We thank Trenz Electronic for their Analog Discovery device, Analog Devices (ADI) for providing the accelerometers, and Krytox^{TM} Performance Lubricants, a business of The Chemours Company, for supplying the vacuum oil. Finally, we thank the Spanish User Support and Operations Centre (E-USOC) and the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio at the Universidad Politécnica de Madrid, which have supported this project from the beginning.

Chapter 7

Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity

Salgado Sánchez, P., Yasnou, V., Gaponenko, Y., Mialdun, A., Porter, J. and Shevtsova, V.

Journal of Fluid Mechanics **865**, 850–883 (2019).

<https://doi.org/10.1017/jfm.2019.88>

We consider the response to periodic forcing between 5 Hz and 50 Hz of an interface separating immiscible fluids under the microgravity conditions of a parabolic flight. Two pairs of liquids with viscosity ratios differing by one order of magnitude are investigated. By combining experimental data with numerical simulations, we describe a variety of dynamics including harmonic and subharmonic (Faraday) waves, frozen waves, and drop ejection, determining their thresholds and scaling properties when possible. Interaction between these various modes is facilitated in microgravity by the relative ease with which the interface can move, altering its orientation with respect to the forcing axis. The effects of key factors controlling pattern selection are analysed, including vibrational forcing, viscosity ratio, finite-size effects, and residual gravity. Complex behaviour often arises with features on several spatial scales, such as Faraday waves excited on the interface of a larger columnar structure that develops due to the frozen wave instability — this type of state was previously seen in miscible fluid experiments but is described for the first time here in the immiscible case.

7.1 Introduction

The behaviour of the interface between two fluids is of fundamental interest in fluid dynamics, with significance in biological systems, in engineering applications such as liquid storage, solvent extraction, oil recovery, and in mixing — microfluidic geometries, in particular. Even in the relatively simple case of periodic excitation, a variety of distinct instabilities can occur in two-layer liquid systems depending on frequency, amplitude, and the direction of vibration.

Vibrations perpendicular to the interface (i.e., vertical) can be treated, in the co-moving frame, as a parametric modulation of the effective gravity. They can stabilise the corresponding Kapitza pendula (Krieger, 2017) and suppress the Rayleigh–Taylor instability (Wolf, 1970). At sufficient amplitude, this type of excitation leads to the well-known phenomenon of Faraday waves, which are standing waves oscillating at half the excitation frequency (Faraday, 1831). In the inviscid case, the Faraday instability can be described by the well-known Mathieu equation (Benjamin and Ursell, 1954) while viscosity was taken into account by Kumar and Tuckerman (1994), who used a Floquet analysis of the linearised Navier-Stokes equations to determine the threshold of the Faraday instability in immiscible fluid layers. A wide variety of patterns can be observed above onset (Arbell and Fineberg, 2002; Kumar and Tuckerman, 1994; Miles and Henderson, 1990; Prinnet, Juric, and Tuckerman, 2009), especially with multi-frequency forcing, and many of these can be understood theoretically in the weakly-nonlinear regime (see, e.g. Skeldon and Rucklidge, 2015).

Vibrations parallel to the interface (i.e., horizontal) immediately induce motion near the walls or other solid boundaries, which act as wavemakers to produce harmonic surface waves. Subharmonic waves, called cross-waves, can also be excited via a localised parametric forcing mechanism (Garrett, 1970; Jones, 1984; Miles and Henderson, 1990; Tinao et al., 2014) and this type of inhomogeneous forcing leads to interesting modulations and dynamics in the weakly-nonlinear regime (Perez-Gracia et al., 2014; Varas and Vega, 2007). With stratified layers of liquids of different densities, the harmonic motion of the interface near the walls (one fluid penetrating into the other), which is driven by an oscillating pressure gradient, is associated, as per volume conservation, with a counterflow in the more quiescent interior. In the case of a rectangular container, this shear counterflow causes a Kelvin-Helmholtz type instability at a critical forcing amplitude that generates a spatially periodic wave pattern on the interface. Since this pattern is stationary in the co-moving reference frame, it is referred to as a “frozen wave”. The effect of horizontal vibrations on two superposed immiscible liquids has been examined both experimentally (Jalilikop and Juel, 2009; Talib, Jalilikop, and Juel, 2007; Yoshikawa and Wesfreid, 2011) and theoretically (Khenner et al., 1999; Lyubimov and Cherepanov, 1986; Lyubimov, Khenner, and Shotz, 1998). Recent experiments demonstrated that this interfacial instability also exists for miscible liquids of similar (but non-identical) viscosities and densities (Gaponenko et al., 2015a,b). The possibility of stabilising the Rayleigh-Taylor instability in miscible liquids has been discussed by Wolf (2018).

Even in the absence of instabilities, vibrations affect the quasi-steady (time-average) interface between fluids, as has been known since the work of Wolf (1969, 1970). The theory of vibroequilibria

(Beyer et al., 2001; Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997) predicts the global reorientation of the average interface based on minimising a Lagrangian that considers vibrational, gravitational, interfacial and contact energies. The general tendency, in the limit of high forcing, is for the interface to orient so as to be more perpendicular to the axis of vibration. The vibroequilibria effect scales with the square of vibrational velocity (Fernández et al., 2017b). With high Bond numbers $Bo \gg 1$ (i.e., strong gravity) vibroequilibria effects are only significant at large forcing amplitudes.

Since interfacial (surface) waves are prominent in nearly all vibrated fluid systems, especially when forcing exceeds the threshold of subharmonic waves, nonlinear effects become increasingly significant as vibrational amplitude increases. At a certain point, drop formation may occur, a process wherein the liquid layer absorbs energy from the inertial force and channels it into the creation of drops that leave the interface. There have been a number of studies of vibration-induced drop ejection from the surface of a liquid layer (see, e.g., Goodridge et al., 1997) or an initially large liquid drop on a vertically vibrating substrate (James, Smith, and Glezer, 2003). Li and Umemura (2014) showed that in a deep liquid layer, the threshold condition for the formation of a spray of droplets is determined only by the forcing strength, and is independent of the initial conditions of the liquid surface.

Under reduced gravity conditions, fluid behaves differently than on Earth. The absence of a large scale force to keep density contours flat favours curved interfaces and blurs the distinction between vertical and horizontal excitations. Nonetheless, depending on frequency, amplitude, and orientation, vibrated fluid systems may exhibit a variety of behaviours that can be associated with the phenomena mentioned above and their interaction.

Periodic stripe patterns perpendicular to the axis of vibration were found by Beysens et al. (2009) and Beysens (2014) in supercritical CO_2 on a sounding rocket provided by European Space Agency. These were the first experiments on such patterns in weightlessness, and many of the investigations were continued on ground in near-critical fluid using magnetic compensation of gravity, with the most recent among them aiming to study either the Faraday wave (Gandikota et al., 2014a) or frozen wave (Gandikota et al., 2014b) instability. These experiments are complemented by a significant amount of numerical and theoretical work (Amiroudine and Beysens, 2008; Jounet et al., 2000; Lyubimova et al., 2017; Sharma et al., 2019). It should be noted that the supercritical state in a liquid-vapour fluid combination is achieved by taking small steps in temperature on the order of several millikelvins. In near-critical fluids, several characteristic properties like the liquid-vapour density difference and the interfacial tension tend to zero, a situation analogous to that of miscible liquids.

Recent experiments using ordinary miscible liquids (combinations of water and isopropanol) carried out by Gaponenko et al. (2015b) showed that in the low gravity environment of a parabolic flight, a saw-tooth wave structure related to the frozen wave instability is initially formed on the interface, as it is on ground. The frozen wave amplitude grows over time, with the angle of the wave crests

continually increasing until the interface reaches the upper and lower walls. The final long-lived pattern consists of a series of vertical columns of alternating liquids. The increase of the diffusive interface width promotes the selection of a smaller wavenumber for column formation (Gaponenko et al., 2018). A further increase of vibrational forcing leads to Faraday waves being excited as a secondary instability on the primary columnar pattern (Shevtsova et al., 2016). These subharmonic waves appear at lower frequency when the driving amplitude is comparable with the height of the liquid layer.

Even more recently, experiments on parabolic flights and on ground were carried out for a pair of immiscible liquids subjected to vertical vibrations with the aim of studying the influence of capillarity and gravity on the Faraday wave instability (Diwakar et al., 2018). The working liquids (FC-72 and 1 cSt silicone oil) become miscible at an upper critical consolute point (Ancherbak et al., 2018). The low-gravity experiments, conducted with frequencies between 3 Hz and 8 Hz at several temperatures approaching the consolute point showed that the critical amplitude increases with temperature, i.e., with decreasing interfacial tension value. However, this effect can be partly explained by the reduction in density difference between the layers with temperature. These experiments also revealed the existence of a crossover frequency, predicted by linear stability analysis (Batson, Zoueshtiagh, and Narayanan, 2013), where the interface in low gravity switches from being less to more stable than the analogous system on ground.

In reduced gravity, one expects more prominent vibroequilibria effects in moderate or large systems because vibrational energy competes only with surface and contact energy. Few experiments have directly studied vibroequilibria (see, e.g., Ganiev, Lakiza, and Tsapenko, 1977). The experiments of Fernández et al. (2017a), which were performed in a parabolic flight campaign, delivered vibrations using piezoelectric devices to liquid-gas and liquid-liquid configurations confined in rectangular and cylindrical containers. This permitted a wide range of frequencies to be investigated and the corresponding modes and vibroequilibria solutions included both hydrodynamic and acoustic cases. Different behaviour such as flattening or cratering were observed, depending on the orientation of the dominant vibrational motion with respect to the initial surface. Results showed that relatively large vibrational velocity is required to clearly observe vibroequilibria effects, due mainly to the masking effects of residual gravity and low frequency parabolic flight disturbances.

The phenomenon of drop ejection, which can occur with high-amplitude surface waves, is of much practical interest. Previous studies focused on threshold conditions, and generally did not take gravitational effects into account, although they were validated by ground experiments. In microgravity, experiments with capillary tubes showed that drops are much larger than with gravity (Wollman et al., 2016). The numerical results of James, Smith, and Glezer (2003) also examined the dependence of drop volume on Bond number. To the best of our knowledge there are no microgravity experiments related to the threshold of drop ejection. The experimental data presented here allows for an estimate of the drop ejection threshold in microgravity, which is compared with the scaling suggested by Goodridge et al. (1997), and for the energy dependence of the drop ejection process.

In this paper, we look at the response of the interface separating two immiscible fluids under the microgravity conditions of a parabolic flight. We find that much of the interface dynamics can be understood as a combination of phenomena typical of both vertically and horizontally vibrated experiments on ground. Interaction between these different modes is facilitated by the ease with which the interfaces can move in microgravity, altering the effective orientation of the forcing with respect to (a portion of) the interface. Complex behaviour is seen, including wave patterns on two spatial scales, with Faraday waves excited on the surface of a larger (frozen wave) columnar structure, as previously seen in miscible fluid experiments (Shevtsova et al., 2016). Drop ejection is found as well.

One consequence of this complexity is that comparisons with theory are more difficult, with only qualitative agreement expected in most cases. The theory of Faraday waves, for example, generally assumes that both the forcing and background state are uniform, while weakly-nonlinear results are most applicable in the limit of large-aspect-ratio containers and weak damping. In the experiments described here, the Faraday waves appear on a nonuniform columnar background state that is not completely perpendicular to the forcing, become strongly nonlinear very quickly, and are affected by the finite (small) size of the cell, which also limits the applicability of a long-wave treatment. For these reasons, only the onset of Faraday waves is compared with theory, with no further appeal to weakly-nonlinear results.

A better understanding of interfacial instabilities and their interaction in microgravity conditions is important for all operations involving liquids onboard space vehicles, since, if nothing else, such systems are subjected to various gravitational fluctuations, known as g-jitter; these can be primarily oscillatory, as from structural resonances, quasi-steady, as from low-frequency orbital variations or spacecraft positioning, or transient, as from manoeuvres, docking, and crew activity. A previous analysis of g-jitter effects was done with miscible mixtures (Shevtsova et al., 2015b).

The paper is organised as follows. The experimental set-up, conditions and measurement procedures are introduced in Sec. 7.2. The mathematical model used for accompanying simulations of an immiscible two-phase system is defined in Sec. 7.3, including a discussion of numerical convergence criteria. The initial conditions appropriate for a variable gravity experiment and the relevant quasiequilibria are discussed in Sec. 7.4. Fast timescale phenomena including Faraday waves and drop ejection are analysed in Sec. 7.5. Slow timescale phenomena, such as the frozen wave instability under different levels of gravity are examined in Sec. 7.6. An overview of the interface dynamics under periodic excitation is given in Sec. 7.7. Final conclusions and remarks are offered in Sec. 7.8.

7.2 The experiment

The heart of the experimental set-up consists of two identical fluid cells with interior dimensions $L \times 2H \times D = 15 \text{ mm} \times 7.5 \text{ mm} \times 5 \text{ mm}$ [see Fig. 7.1(a)] subjected to linear vibrations. Two side walls of the cell are made of transparent quartz glass (Hellma), while the other four walls are custom made from aluminium alloy. Filling or draining the cell is done through 0.5 mm holes. Safety

FIGURE 7.1: (a) General view of the experiment cell assembly, consisting of two identical cells filled with immiscible fluids of different viscosity and density ratios. The lower liquid layers in both cells are FC-40, while the upper layers are 2 cSt silicone oil (top cell) and 20 cSt silicone oil (bottom cell). (b) Behaviour of the interface in microgravity in the absence of vibrational forcing. Raw (cropped) image of fluid cells: heavier FC-40 (lighter region); lighter silicone oils (darker region).

requirements include double containment of the cells, and part of this can be seen in Fig. 7.1(a).

Interfacial dynamics are captured by means of a CCD camera. The lighter liquid is coloured, so images show two different grey scales corresponding to each fluid, as in Fig. 7.1(b). The camera acquisition rate is fixed at 80 frames per second, providing excellent sampling of the interface motion for low to intermediate frequencies. Below about 20 Hz, it is fairly easy to track the evolution of interfacial waves at a given phase, while less accurate phase resolution is achieved at the higher vibration frequencies. Nonetheless, since the oscillations of the interfacial waves are either harmonic or subharmonic, stroboscopic effects do not cause any real difficulties in identifying the mode of the instability for the frequencies considered. At the highest forcing frequency where oscillatory interfacial waves were recorded, 40 Hz, there are enough points (two and four, respectively) to distinguish harmonic from subharmonic waves, even if the phase is not known precisely.

The experiment cells and the camera are attached to a linear motor, which drives controlled translational oscillations along the x -axis (see Fig. 7.3) of the cell at a given frequency and amplitude. The linear motor is mechanically based on a single-axis bearing positioning stage, which by specification has a maximum off-axis displacement of $25\text{ }\mu\text{m}/300\text{ mm}$. This provides an angular uncertainty in the direction of the vibrations on the order of 0.01° . To isolate the effect of vibrations on the environment, a special 12.5 mm thick anti-vibration cork/elastomer pad is mounted between the linear motor and the rack structure. The oscillatory motion is monitored from position-time records obtained with an encoder, which permits a comparison between the nominal and real vibrational parameters.

The experimental fluid properties are detailed in Table 7.1. Each cell holds an equal volume of FC-40 and silicone oil (either 2 cSt or 20 cSt, the upper and lower cells in Fig. 7.1, respectively). A

	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)	σ (10 ⁻³ N/m)	δ_ν (10 ⁻³ m)	β_A (rad)	β_G (rad)
FC-40	1855	2.2	16.63	0.118 – 0.374		
2 cSt silicone oil	873	2	18.63	0.113 – 0.357		
20 cSt silicone oil	950	20	20.85	0.357 – 1.100		
Interfacial properties:						
FC-40 and 2 cSt			3.927		$\pi/4$	0.426π
FC-40 and 20 cSt			6.021		$\pi/4$	0.426π

TABLE 7.1: Physical properties of FC-40 (Flourinert Liquids^{TM}), 2 cSt silicone oil, 20 cSt silicone oil [from Someya and Munakata (2005)], and the pairings used here. Provided are: density ρ , kinematic viscosity ν and surface tension σ at 25 °C; thickness of the vibrational viscous boundary layers $\delta_\nu = \sqrt{\nu/\omega}$ for the range of parameters used; contact angle with the aluminium alloy endwalls β_A (used for numerical simulations) and with the quartz glass β_G . Both contact angles are estimated from image processing.

comparison of these two fluid combinations reveals something about how viscosity and interfacial tension influence vibrational phenomena.

The (average) contact angle with the aluminium endwalls β_A is estimated from the image processing of experimental results. The interface between FC-40 and silicone oils exhibits a finite contact angle β_G with the quartz glass of the container, which can be estimated by assuming a circular arc for the cross section:

$$\frac{\pi}{2} - \beta_G = \tan^{-1} \left[\frac{D \zeta}{\sqrt{4\zeta^4 + \left(\frac{D}{2}\right)^4}} \right] \simeq 0.074 \pi. \quad (7.1)$$

Here, ζ is the apparent interface thickness, found to be approximately equal in both cells, and D is the cell depth. The corresponding meniscus means that the interface profile in the recorded images is not as sharp as with miscible liquids in a similar experimental set-up (Gaponenko et al., 2015b). Applied vibrations in weightlessness generally induce capillary phenomena on the lengthscale $l \sim \sqrt{\sigma/(\rho A \omega^2)}$ (Douady, 1990), where A and ω are the driving amplitude and frequency, respectively. Variations in effective gravity and possible contact line motion drive harmonic meniscus waves. In the case of two superimposed liquids, the interfacial tension may be roughly estimated as the difference between the two corresponding liquid-air surface tensions. In Table 7.1 the interfacial tension is relatively small and the associated meniscus effects were not noticeable in any of the experiments or simulations; for the parameter range used here $l \lesssim 1$ mm. We have assumed (for simplicity) steady contact angles in the mathematical formulation of Sec. 7.3.

The experiments were carried out onboard an aircraft executing parabolic manoeuvres to provide repeated periods of approximately 20s of reduced gravity, as illustrated in Fig. 7.2. These microgravity periods are preceded by a pull-up phase and followed by a pull-out phase, each associated with approximately 20s of hypergravity (up to $1.8 g_0$, where $g_0 = 9.81$ m/s²), and separated by several minutes of normal gravity $g \sim 1 g_0$. The microgravity level during the parabolas satisfied $|g_{x,z}| \leq 0.01 g_0$, $|g_y| \leq 0.05 g_0$, with the direction of the axis relative to the experiment container

FIGURE 7.2: Gravity profiles from one of the parabolic flights showing $g_x(t)/g_0$, $g_y(t)/g_0$ and $g_z(t)/g_0$ (red, black and grey curves, respectively); see Fig. 7.3 for axes. (a) Gravity profiles together with the experiment intervals, both when forcing begins in hypergravity (yellow) and in microgravity (blue); the overlap region is shown in light green. (b) Close-up of the microgravity interval.

shown in Fig. 7.3.

Each experiment, of 15 s duration, was conducted in (nearly-)isothermal conditions. The temperature of the cell was measured by a negative temperature coefficient (NTC) thermistor directly fixed to the cell plate. Over the entire experimental campaign, which lasted about three hours, the temperature rose slightly from 22°C to 24°C, still very far from the mixing point of this class of liquids (Ancherbak et al., 2018). Surface tension variation over this range is less than 4 % (Someya and Munakata, 2005), which justifies the assumption of constant surface tension over the experimental campaign.

Particular experiments were defined by a combination of fluid properties and applied forcing: frequency, amplitude, and profile. The profile refers to when the forcing was initiated, i.e., whether it began during the hypergravity or microgravity portion of a particular parabolic flight. The experiment runs explored frequency and amplitude intervals between 5–50 Hz and 1–5.3 mm. Typically, experiments were repeated in different parabolas, which provided an idea of the statistical spread of the results, albeit limited by the lack of repeatability between parabolic manoeuvres.

Experiment results, although interesting and varied, were constrained by the nature of the parabolic flight: a limited number of data points (available parabolas) to explore parameter space, and the perturbed (imperfect) microgravity level. In order to investigate more frequencies and amplitudes, and to gain insight, for instance, into the importance of flight perturbations and initial conditions for the experiment outcome, a numerical model that describes immiscible fluid behaviour in the presence of vibrations is developed. The formulation and implementation are detailed in Sec. 7.3

FIGURE 7.3: Sketch illustrating the (two-dimensional) numerical model.

and Sec. 7.4, while numerical results are discussed throughout the paper.

7.3 Mathematical formulation

We consider two superposed horizontal layers of immiscible liquids, FC-40 and silicone oil, of equal volume and confined in a rectangular container with dimensions $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$, as sketched in Fig. 7.3. The cell is subjected to vibrations of amplitude A and frequency ω oriented in the x -direction. The heavier fluid, FC-40, is at the bottom so the system is gravitationally stable (in normal gravity). The physical properties of each cell are given in Table 7.1, where the interfacial tension values have been taken from Someya and Munakata (2005).

A level-set approach (Olsson and Kreiss, 2005), which tracks the interface by defining a (level-set) function ϕ , is implemented to simulate the dynamics of the immiscible interface. Equation (7.2) governs its evolution, assuring conservation of interface thickness,

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right), \quad (7.2)$$

where γ (units of velocity) is a tuning parameter that scales with the velocity of the interface, and ε is a small diffusive term added for numerical stability. The proper selection of these parameters is crucial for numerical convergence and accuracy. We set $\gamma = A\omega$ for solving the vibrated problem, while ε is taken to be on the order of the maximum element mesh size upon convergence.

Fluid properties can be written as functions of the level-set variable ϕ via the introduction of a Heaviside function $\mathcal{H}$ (smoothed, for numerical stability),

$$\mathcal{H}(\phi) = \begin{cases} 0, 1 & \phi < -\varepsilon, \phi > \varepsilon \\ \frac{1}{2} + \frac{\phi}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\phi}{\varepsilon}\right) & |\phi| \leq \varepsilon \end{cases} \quad (7.3)$$

where ε refers to half the interface thickness.

Density and viscosity may thus be written as

$$\rho = \rho_1 + (\rho_2 - \rho_1)\mathcal{H}, \quad (7.4a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1)\mathcal{H}, \quad (7.4b)$$

where the subscripts 1 and 2 denote the two fluids (FC-40, silicone oil) and correspond to $\phi = 0, 1$, respectively.

Equations (7.2) and (7.4) are solved jointly with the continuous incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991)

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}] = -\nabla p + \mu \Delta \mathbf{u} + \delta \mathbf{f}_\sigma + \rho \mathbf{G}(t), \quad (7.5a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (7.5b)$$

where the effect of interfacial tension σ is included via $\mathbf{f}_\sigma$ in the momentum equation, and is given by

$$\mathbf{f}_\sigma = (\sigma \kappa) \mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}. \quad (7.6)$$

Here, κ is the interface curvature and $\mathbf{n}$ the normal vector. Interfacial effects are localised by the introduction of δ , which may be approximated in terms of the level-set function as

$$\delta = |\nabla \phi|. \quad (7.7)$$

Acceleration is expressed as

$$\mathbf{G}(t) = \mathbf{g}(t) + A\omega^2 \cos(\omega t) \mathbf{i}, \quad (7.8)$$

where $\mathbf{g}(t)$ refers to the (generally) time-dependent gravity field and $\mathbf{i}$ is a unit vector along the forcing axis.

Navier-slip boundary conditions, which introduce an artificial frictional force $\mathbf{f}_\nu = (\mu/\delta_\nu)\mathbf{u}$ to preserve contact angle, are applied on the walls

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u} \cdot [\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta_A)\delta] = 0, \quad (7.9)$$

where $\mathbf{n}_w$ is the corresponding unit normal vector, β_A is the contact angle and $\delta_\nu = \sqrt{\nu/\omega}$, is the vibrational viscous boundary layer thickness. Crucially, these boundary conditions permit motion of the contact line, which is needed to match the experiments.

If the following scalings are used for time, length, density and viscosity,

$$(\hat{t}, \hat{\mathbf{x}}, \hat{\rho}, \hat{\mu}) = (\omega t, \mathbf{x}/L, \rho/\rho_1, \mu/\mu_1), \quad (7.10)$$

	$\hat{\rho}$	$\hat{\mu}$	$\hat{\sigma} (10^{-7})$	Γ
FC-40 and 2 cSt	0.470	0.427	0.014 – 1.410	0.25
FC-40 and 20 cSt	0.512	4.655	0.021 – 2.192	0.25

TABLE 7.2: Nondimensional parameters corresponding to the experiments of Sec. 7.2.

where a hat denotes dimensionless quantities, then the system dynamics (7.2)–(7.9) depend on the dimensionless frequency and amplitude

$$\hat{\Omega} = \frac{L^2 \omega}{\nu_1}, \quad \hat{A} = \frac{A}{L}, \quad (7.11)$$

and on the other nondimensional parameters,

$$\hat{\rho} = \frac{\rho_2}{\rho_1}, \quad \hat{\mu} = \frac{\mu_2}{\mu_1}, \quad \hat{\sigma} = \frac{\sigma}{\rho_1 \omega^2 L^3}, \quad \mathcal{G} = \frac{A \omega^2}{g_0}, \quad \Gamma = \frac{H}{L}, \quad (7.12)$$

where H is the initial height of each fluid layer (see Fig. 7.3). For the experiments of Sec. 7.2, the nondimensional parameters are shown in Table 7.2, while frequency and amplitude are in the range of

$$\hat{\Omega} = [3855, 32130] \gg 1, \quad \hat{A} = [0.0740, 0.222] < 1. \quad (7.13)$$

Finally, different initial conditions may be considered for the vibrated problem, depending on the gravity profile $\mathbf{g}(t)$ and the initialisation time of the forcing. In order to perform a consistent study based on experiments, we use the gravity profiles defined in Sec. 7.4 for numerical simulations.

The commercial software COMSOL Multiphysics 5.3 is used for solving the governing Eqs. (7.2)–(7.9) in two dimensions, written using dimensional variables, with the finite element method. Some details of the numerical method, including criteria used to select the interface thickness and meshing are discussed in Sec. 7.3.1.

The assumption of essentially two-dimensional phenomena is supported by previous modelling of the analogous vibrational problem for a similar geometry in the case of miscible fluids (Gaponenko et al., 2015b). We expect the reduced numerical model to capture the essential features of the fluid behaviour relating to both frozen waves and Faraday waves, which are the dominant phenomena observed over the relevant parameter range. The lower computational cost permits a more thorough parameter study, which is a big advantage here, although some phenomena cannot be captured. Three-dimensional irregular behaviour (with many drops) is sometimes observed experimentally with strong vibration and this is not modelled. Drop ejection, in general, is not accurately captured. In the applicable parameter regime, droplets are generated from the breaking of a fluid jet (see Sec. 7.5.2 for details) and the characteristic length l_B of the breaking region vanishes as the drop is ejected. The level-set formulation used here, which has a characteristic interface thickness ε , cannot

Mesh #	Max. size [$\times(H/12.5)$]	Mesh elements	Mesh DoF	Simulation time (s)	Max. (v_x, v_y) (mm/s)	Deviation (%)
#1	4	485	2868	165	(23.0, -10.4)	28.8
#2	2	1249	7636	358	(31.4, -13.5)	6.9
#3	1	3818	23920	601	(33.1, -14.9)	2.2
#4	0.75	6253	39511	905	(32.3, -14.5)	—

TABLE 7.3: Results of a mesh convergence test for numerical simulations of the FC-40 and 20 cSt silicone oil cell with $A = 2$ mm and $f = 20$ Hz (see Sec. 7.6.1 for the relevance of this experiment). The selected mesh #3 is in bold, with deviations measured with respect to the maximum velocity for mesh #4 over the selected profiles: $v_x(y)$ at $x = 4$ mm and $v_y(x)$ at $y = 3.75$ mm. The interface thickness is $\varepsilon = 1.5(H/12.5)$.

describe the dynamics for $l_B \ll \varepsilon$. Drop ejection, however, is observed mainly with the FC-40 and 2 cSt silicone oil interface and represents a relatively small portion of the phenomena described below.

7.3.1 Convergence of the numerical solutions

The convergence of the numerical solutions was examined for different meshes. The defining characteristics, associated numerical error and solution results are included in Table 7.3 and illustrated in Fig. 7.4. Simulations were begun from a hydrostatic solution, with vibrations initiated during the microgravity portion of the parabola and maintained for 1.25 s. In this regime, a two-scale wave pattern develops due to a secondary Faraday instability, which occurs after the formation of a (primary) frozen wave structure.

The interface thickness, which controls the scale of density and viscosity gradients across the interface ($\phi = 0.5$) was first varied with a fixed mesh size. The results were examined after the development of the two-scale wave pattern, and an interface thickness of $\varepsilon = 1.5(H/12.5)$ was selected.

The mesh was then modified by varying the maximum element size, defined as a fraction of $(H/12.5)$, and different features were recorded in Table 7.3: the number of mesh elements, the degrees of freedom (DoFs), the simulation cost (in time) and the solution accuracy. We selected the maximum velocity of the profiles $v_x(y)$ at $x = 4$ mm and $v_y(x)$ at $y = 3.75$ mm as representative solution values, shown in Fig. 7.4(b,c) and measured the deviation from the most accurate solution. Mass conservation is checked in Fig. 7.4(a) by showing the level-set function ϕ , averaged over the fluid volume. We found mesh #3 to be a good compromise between computational cost and numerical accuracy. The level of numerical error with this choice should be well below that of the primary experimental uncertainties discussed in Sec. 7.2.

The simulations presented throughout this paper are thus performed using a maximum element size of $(H/12.5)$ and $\varepsilon = 1.5(H/12.5)$, which allows acceptable accuracy over a wide range of frequencies and amplitudes without excessive computational cost — an extended parameter study would otherwise be unfeasible. A Generalised- α (Erlicher, Bonaventura, and Bursi, 2002) integration

FIGURE 7.4: Mesh convergence test for numerical simulations. Results for meshes #1–4 are shown in blue, red, green and black, respectively. (a) Volume average of the level-set function $\langle \phi \rangle_V$ during the vibrated part of the simulation. (b) Horizontal velocity $v_x(y)$ at $x = 4$ mm. (c) Vertical velocity $v_y(x)$ at $y = 3.75$ mm.

scheme is used, with a maximum time step manually set at $1/(10f)$; this step size may be reduced (automatically), as required for stability. Both *streamline* (Harari and Hughes, 1992) and *crosswind* (Codina, 1993) stabilisation techniques are applied.

7.4 Gravity profile, forcing start time and initial condition for the vibrated problem

The lack of repeatability of microgravity levels in parabolic flights (Pletser et al., 2016) makes comparing experiments and simulations more difficult. It is hard to reproduce a specific experiment with very much precision, and such an effort would lack generality. Two of the purposes of the numerical model are to allow the exploration of a wider range of amplitudes and frequencies, where no experimental results are available, and confirm experimental trends by means of a more systematic analysis.

Based on in situ accelerometer measurements, like those illustrated in Fig. 7.2, we define three idealised gravity profiles to be used in simulations. This permits, in some cases, a direct comparison of flight disturbances between simulations and experiments, and allows us to extend results of the model in a straightforward manner to parameters not covered by experimental data points.

The three different gravity profiles used for simulations are illustrated in Fig. 7.5 by dashed curves. These profiles are mainly distinguished according to the initial and final gravity levels: *ideal parabola* [profile I, (a,d)] when gravity only has a vertical component g_y , *quality parabola* [profile II, (b,e)] when the small lateral gravity g_x decreases from $0.05 g_0$ to $0.001 g_0$ before the microgravity period and *generic parabola* [profile III, (c,f)] when lateral gravity g_x decreases from $0.1 g_0$ to $0.01 g_0$ before the microgravity period. We use profiles I and III extensively for the simulations presented below,

FIGURE 7.5: Idealised two-dimensional gravity profiles $|g_x(t)|/g_0$ and $|g_y(t)|/g_0$ (dotted and dashed curves, respectively) during the initial period of a parabolic manoeuvre. Vibrational forcing (shown as solid curves, normalised to unity) may be started during microgravity (a–c) or hypergravity (d–f): (a,d) profile I - ideal parabola with $g_x = 0$, (b,e) profile II - quality parabola with final $g_x = 0.001 g_0$ and (c,f) profile III - generic parabola with final $g_x = 0.01 g_0$. In all profiles g_y goes from $1.5 g_0$ to $0.05 g_0$. Shaded regions in (a) and (d) show the transition (blue, duration 1.5 s) from hypergravity to microgravity and the forcing initialisation (green, duration 0.25 s), both smoothed in the manner of (7.3).

while profile II is used to have a more accurate model of a particular experimental result (shown later in Fig. 7.15).

Forcing (i.e., experiments) may be initiated during microgravity or during hypergravity, as illustrated in Figs. 7.2 (blue and yellow regions) and 7.5 [panels (a–c) and (d–f), respectively]. The transient time required to achieve a given vibrational amplitude is 0.25 s for the computations presented in this paper. Its role in selecting the experiment outcome, possibly important in weightlessness, has not been analysed here and will be investigated elsewhere.

7.4.1 Quasiequilibria driven by residual gravity

When forcing is initiated after the start of the microgravity phase of the parabolic flight, the fluids prior response to residual accelerations in the aeroplane is also captured. This allows some results concerning the (isolated) effect of flight disturbances on the shape and location of the immiscible interfaces. These states are referred to as quasiequilibria, since the characteristic timescale of their evolution (on the order of the residual gravity variation) is significantly slower than the timescale of vibrational phenomena, especially for the medium to high frequencies used. In Figs. 7.1(b) and

FIGURE 7.6: Interface between FC-40 and 20 cSt silicone oil (the lower experiment cell) when subjected to residual gravity. (a–c) Snapshots showing the temporal evolution of the interface when subjected solely to residual gravity during an experiment. (d–f) Simulations when subjected to the gravity profiles of Fig. 7.5 prior to initialising forcing in microgravity: (d) initial ($t = 0.5$ s, dashed-dotted line), intermediate ($t = 1.5$ s, dashed) and final ($t = 2.25$ s, solid) interface locations for profile III [see Fig. 7.5(c)], and (e,f) final quasi-steady equilibria selected with gravity profiles I and III, respectively.

7.6(c), some quasiequilibria captured during the first part of a generic parabola (i.e., during the initial seconds of microgravity) are shown. The results of simulations are qualitatively consistent with the observed evolution of the interface in experiments due to varying residual gravity — illustrated in Fig. 7.6(a–c).

Some computed quasiequilibria states using gravity profiles I and III are shown in Fig. 7.6(d–f). The evolution of the interface for profile III after 0.5 s, 1.5 s, and 2.25 s [see Fig. 7.5(c)] is shown in (d), while (e) and (f) show the final (at $t = 2.25$ s) unforced quasiequilibria for profiles I and III, respectively. These computations are qualitatively consistent with experiments. During the hypergravity period, interfaces are nearly flat except near the endwalls, where meniscus effects are dominant. The average interface [Fig. 7.6(a) and dashed-dotted line in panel (d)] is slightly tilted with respect to the vibrational axis, with an angle on the order of few degrees, due to the combined effect of the x and y components of the aeroplane acceleration. At the initiation of the microgravity phase, interface curvature develops in both cells [Fig. 7.6(b) and dashed line in panel (d)]. The heavier FC-40 fluid (nearly transparent in the image) eventually forms an asymmetric drop on the left side of the container [Fig. 7.6(c) and solid line in panel (d)], forced by the residual acceleration in the (negative) x -direction. The lighter silicone oil (darker shading in the image) surrounds the heavier one. Thus, the interface position is selected by a combination of residual acceleration and

interfacial tension, which are comparable in effect over most of the fluid domain. The final quasi-steady states are used as initial conditions for further numerical simulations with applied vibrations. Subsequent results should thus reflect the real experiments performed in this manner.

The vibrational phenomena described in the remainder of the paper are associated with noticeable differences with respect to these unforced (quasi-steady) states. Although the full picture is complex with many potential interactions, we find it useful to separate the various interfacial phenomena into those tied to fast modes like surface waves, which oscillate on the time scale of the forcing frequency, and slow modes, like vibroequilibria and frozen waves, which are related to mean, time-averaged effects.

7.5 Fast timescale phenomena: from surface waves to drop ejection

This section considers interfacial phenomena that are associated with the forcing frequency, including both harmonic and subharmonic surface waves and the drop ejection instability that can occur with sufficient amplitude of these waves.

7.5.1 Interfacial waves: harmonic waves and subharmonic Faraday mode

Both harmonic and subharmonic surface waves are excited by vibrations under weightless conditions. Depending on the vibrational velocity, these waves may appear superposed on top of a frozen wave pattern, as observed in miscible fluids by Shevtsova et al. (2016), or a drop-like configuration evolving on a slower timescale. In Fig. 7.7, a few different interfacial modes are shown; in each frame two different out-of-phase snapshots are superimposed and the interfaces in each snapshot are outlined for clarity.

Harmonic waves are not investigated in detail here since, except at low vibrational amplitude, they are masked by more energetic phenomena like subharmonic waves. Many of the conclusions are thus based on this higher-amplitude regime. The dynamical modes observed here typically show some degree of symmetry (see, e.g., Fig. 7.7) with respect to midplane reflection, although asymmetry due to finite-size effects and residual vertical acceleration, for example, is often clearly present as well, leading to a rotation of the (average) interface position. In addition, numerical and experimental results suggest that, in many cases, surface waves form on nearly vertical portions of the interface and are of nearly constant amplitude, so that the forcing mechanism can be considered roughly homogeneous and experiments with subharmonic waves, for instance, can be compared with the ideal case of a vertically vibrated container (Kumar and Tuckerman, 1994).

Subharmonic Faraday waves exist over a wide range of parameters and can be seen experimentally in Fig. 7.7(b,c) and, below, in Figs. 7.8 and 7.15(d,e). Just past onset, both experiments and simulations show supercritical small-amplitude waves whose threshold may be compared with the case of a vertically vibrated container (Faraday, 1831; Kumar and Tuckerman, 1994; Miles and Henderson, 1990).

FIGURE 7.7: Superimposed out-of-phase images for different surface wave modes with FC-40 and 2 cSt silicone oil: (a) harmonic mode at $f = 40$ Hz and $A = 1.06$ mm, (b) subharmonic mode at $f = 5$ Hz and $A = 4.1$ mm (approximately odd with respect to midplane reflection), and (c) subharmonic mode at $f = 18$ Hz and $A = 1.43$ mm (approximately even). The time between frames is labelled under each picture while solid and dashed lines outline the interfaces in each image.

In Shevtsova et al. (2016), the Faraday wave instability in a miscible binary mixture was analysed. The threshold of subharmonic waves was estimated based on the damped Mathieu equation (Kumar and Tuckerman, 1994) by minimising

$$A\omega = \frac{2}{\omega k} \left(\frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right) \sqrt{\left(\frac{\omega^2}{4} - \frac{\sigma}{\rho_1 + \rho_2} k^3 \right)^2 + 4k^4 \omega^2 \left(\frac{\mu_1 + \mu_2}{\rho_1 + \rho_2} \right)^2}, \quad (7.14)$$

with $\sigma = 0$ for the miscible fluid case.

The threshold obtained by minimising equation (7.14) over wavenumber provides a reference value that is later shown in Fig. 7.18 (dashed blue line) of Sec. 7.7 for the experiments of Sec. 7.2. There is qualitative, albeit limited, agreement of this theoretical prediction with experiment, which both display $f^{-1/2}$ scaling at high frequencies, although separated by a clear gap. This discrepancy is not surprising in light of the differences between theoretical assumptions and the experimental setup, including the inhomogeneous base state (the columnar frozen wave structure) and finite-size effects. The latter become increasingly important at low frequencies, which delays onset and modifies the scaling to f^{-1} .

The numerical simulations (dark blue solid line and blue open markers in Fig. 7.18 of Sec. 7.7), on the other hand, match somewhat better with the experimental results. Numerical thresholds have been located using a similar approach as in Gandikota et al. (2014a,b), using more precision in frequency and amplitude near threshold, so that the estimated uncertainty is on the order of ± 1.25 Hz and ± 0.0125 mm. For low frequency vibrations, residual gravity can couple with the forcing to slightly modify the instability onset. This effect is observed in both cells.

Figure 7.8 shows a series of experimental snapshots separated by approximately one half-period of the $f = 22$ Hz driving force. Depending on the phase, standing waves can be seen (mainly) on the left or right side of the underlying columnar structure. The explanation for this phase-shift

FIGURE 7.8: Successive snapshots showing the evolution of subharmonic Faraday waves on opposite sides of the frozen wave columnar structure for $f = 22$ Hz and $A = 1.76$ mm. Times are selected to best show the $\pi/2$ phase difference between these two subharmonic wave fields (1/4 period of these 11 Hz subharmonic waves is approximately 0.0227 s).

is the sign change in the effective vibrational acceleration at each interface due to the sign of the density jump. Since the Faraday waves are subharmonic, the π shift in forcing on opposite sides of the columnar structure implies a $\pi/2$ phase shift for the waves. The same type of behaviour was previously observed in miscible liquids (Shevtsova et al., 2016).

The importance of the phase difference for weakly-coupled subharmonic wave fields, like these two regions of interfacial Faraday waves, was investigated in Salgado Sánchez et al. (2016). It was found that the character of the instability not only changes from a supercritical to subcritical depending on detuning (most relevant here at low frequencies where the container dimensions most constrain the wavenumber), but can change from a steady-state pitchfork bifurcation to a Hopf bifurcation due to coupling. Hopf bifurcations have not been observed here, possibly due to the coupling being too weak and, thus, the modulation frequency too small, to be noticed on the timescale of the experiments and simulations.

By combining numerical with experimental results, we are able to construct the instability diagram for the subharmonic Faraday wave mode in each experiment cell. In Fig. 7.9, results for FC-40 and 20 cSt silicone oil are organised by the wavenumber k measured in the centre of the pattern and vibrational velocity under gravity profiles I [panel (a)] and III [panel (b)]. Four modes are distinguished according to the average number of wavelengths in the vertical direction, and corresponding curves are drawn for visual reference, suggesting boundaries within the main subharmonic tongue. Since the simulations of panel (a) are performed using the ideal gravity profile (no g_x component) they are not compared directly with experiments, but provide a reference for evaluating the effect of residual gravity seen in panel (b), which shows results obtained using a more generic gravity profile that may be better compared to experiments. We note that numerical simulations near $k = 1$ were not able to reproduce subharmonic waves as seen in experiments, perhaps due to the increased importance of residual gravity at those low forcing amplitudes in the experiments or to increased damping from the boundaries in the simulations. Over the limited region of overlap the experimental results are consistent with the simulations, both in terms of wavenumber selection and the qualitative structure of the surface.

FIGURE 7.9: Instability diagram of the subharmonic mode for FC-40 and 20 cSt silicone oil under different gravity profiles: (a) profiles I and (b) profile III. Experimental (filled circles) and numerical (open circles) results exhibit various dynamical modes, marked with alternating blue tones. The solid curves are drawn for visual reference to distinguish different average vertical wavenumbers. The dashed curve indicates that all these modes belong to the main subharmonic instability tongue. Illustrative snapshots in the upper-left corner show the initial condition, while the smaller images show the interface after $t \simeq 2$ s of applied vibration in microgravity.

As discussed in Batson, Zoueshtiagh, and Narayanan (2013), gravity plays a dual role in shaping the instability diagram, modifying not only the instability threshold but the location and profile of the mode boundaries. The effect of gravity can be seen in Fig. 7.9 by comparing the response of the interface to the two different microgravity profiles. When a small disturbance is introduced along the x -axis [panel (b), profile III], the mode boundaries shift a bit towards lower wavenumbers (frequencies), while the threshold is modified only slightly. Note that the conditions differ only in the gravity level along the x -direction, which is $0.01 g_0$ for profile III. Some differences with respect to the predictions of Batson, Zoueshtiagh, and Narayanan (2013) are expected. Also note that, compared

FIGURE 7.10: Instability diagram for the subharmonic mode with FC-40 and 2 cSt silicone oil under gravity profile I. Experimental data (filled circles) and numerical simulations (open circles) reveal various dynamical modes, marked with alternating blue tones. The solid curves are drawn for visual reference to distinguish different average vertical wavenumbers. The dashed curve indicates that all these modes belong to the main subharmonic instability tongue. Illustrative snapshots in the upper-left corner show the initial condition, while the smaller images show the interface after $t \simeq 2$ s of applied vibration in microgravity.

to the analysis of Batson, Zoueshtiagh, and Narayanan (2013) and Kumar and Tuckerman (1994), we only locate the subharmonic instability tongue, which is normally dominant in Faraday wave systems, and this is shown with the (dashed black) curve in Fig. 7.9. Other instability tongues (harmonic, for example) are masked by the primary subharmonic mode and difficult to observe.

Viscous and interfacial tension effects are revealed in a comparison of Figs. 7.9(a) and 7.10. Lower viscosity clearly reduces the threshold for Faraday waves. At the same time, the reduction of interfacial tension with 2 cSt silicone oil generally increases the observed wavenumber of the subharmonic mode (for a given forcing frequency). This is consistent with the dispersion relation, and is investigated experimentally, for example, in Gandikota et al. (2014a), where the wavenumber of the Faraday instability in near-critical hydrogen is observed to increase with approach to the critical point where interfacial tension vanishes.

As vibrational velocity increases, the subharmonic Faraday waves become increasingly nonlinear and, at sufficient velocity, interfacial integrity breaks down, initiating a process of drop ejection. This process is important, since it places an upper bound on the wave amplitudes that can be investigated experimentally and numerically.

FIGURE 7.11: Representative snapshots in time (values provided) showing the different drop ejection mechanisms observed in the FC-40 and 2 cSt silicone oil cell during microgravity: (a–c) mechanism A (near onset), (d–f) mechanism A (far from onset), (g–i) mechanism B.

7.5.2 Drop ejection instability of the subharmonic mode

At large enough amplitude, the increasingly nonlinear Faraday waves may become unstable to drop ejection (Goodridge, Tao Shi, and Lathrop, 1996; Goodridge et al., 1997; James et al., 2003). Visual observations indicate that the resulting interfacial pattern often has a three-dimensional structure, which makes the two-dimensional simulations less applicable. Despite the complexity of this process, especially when many drops are present, the experimental results allow us to distinguish two qualitatively different instability mechanisms that are associated with distinct velocity thresholds:

1. A – *Heavier-to-lighter liquid ejection* when a drop of FC-40 is ejected into the bulk of the silicone oil.
2. B – *Lighter-to-heavier liquid ejection* when a drop of silicone oil is ejected into the bulk of the FC-40 fluid.

The dynamics of mechanism A is shown in Fig. 7.11(a–c) for $A = 1.57$ mm and $f = 20$ Hz. In this process, drops are created in the lighter fluid and ejected away from the interface. A variation of mechanism A can occur with larger vibrational velocities, as in Fig. 7.11(d–f) with $A = 4.29$ mm and $f = 8$ Hz, where drops are still created in the lighter fluid but, instead of moving away, tend to remain trapped near the interface. Mechanism B occurs for larger amplitudes, sometimes

FIGURE 7.12: Drop ejection instability thresholds for FC-40 and 2 cSt silicone oil as a function of frequency f and (a) amplitude A , (b) vibrational velocity $A\omega$. Circles mark experimental data points, coloured in red if drop ejection is found. Fits using different scaling for the drop ejection threshold [mechanism A, dashed black line] are given in colour: $A \propto f^{-1}$ (red), $A \propto f^{-2/3}$ (green) and $A \propto f^{-1/2}$ (blue). The threshold curve for mechanism B (solid black line) is estimated from experiments (no scaling assumptions).

concurrently with mechanism A. The dynamics of the process are as shown in Fig. 7.11(g-i) for $A = 4.37$ mm and $f = 9$ Hz.

Drop ejection is associated with a jet of fluid emerging from a large-amplitude surface wave that then collapses to create a drop of fluid and, often, other smaller drops, which are known as satellites (Castrejón-Pita et al., 2015; James, Smith, and Glezer, 2003). As forcing amplitude is increased, especially if it becomes comparable to (or larger than) the thickness of the liquid layer $A > H$, the scenario can be very complex. The dynamics is often fully three-dimensional and disordered in this regime (James, Smith, and Glezer, 2003) and the drops interact with the strongly nonlinear subharmonic waves and the associated mean flow, which can lead to the drop being recaptured by the interface. We note that the highly nonlinear behaviour of the system for $A > H$ was also observed in the case of miscible liquids, which led either to secondary Faraday waves over frozen waves (Shevtsova et al., 2016) or to the formation of a fish-spine pattern (Shevtsova et al., 2015a).

The thresholds of the two droplet ejection mechanisms identified above are shown in Fig. 7.12 for FC-40 and 2 cSt silicone oil in terms of amplitude A [panel (a)] and vibrational velocity $A\omega$ [panel (b)] versus frequency f . The estimated thresholds shown are based on the available experimental data points (filled circles). The dashed black curve indicates the lower bound of the droplet ejection regime, while the solid black curve indicates the onset of mechanism B. For the majority of low-gravity experiments considered here, performed at $f < 15$ Hz, wave breaking with drop ejection is observed at $A\omega|_c \gtrsim 0.17$ m/s, as illustrated in Fig. 7.12(b).

We can compare the experimental results giving the (estimated) onset of drop ejection with the scaling suggested by Goodridge et al. (1997) for the cases of a system dominated by surface tension (with, e.g., low viscosity fluids like ethanol): $A\omega \sim c_1 F_1(\sigma/\rho)\omega^{1/3}$ and for more viscous fluids like water or glycerine: $A\omega \sim c_2 F_2(\nu)\omega^{1/2}$. We also consider a constant velocity threshold: $A\omega \sim c_3 (= 0.17 \text{ m/s})$. Here $c_j, j = 1, 2, 3$ are constants while $F_1(\sigma/\rho) = (\sigma/\rho)^{1/3}$ and $F_2(\nu) = \sqrt{\nu}$ for a single fluid system. The corresponding prefactors are obtained by fitting and the results shown in Fig. 7.12(b).

The results, although limited by the small number of experimental data points and the variable conditions onboard the parabolic flight (e.g., residual gravity effects), do suggest a transition from (nearly) constant critical velocity at low frequencies to an increase of critical velocity (most) consistent with the scaling proposed by Goodridge et al. (1997) for viscous fluids. Note that more experimental data points would be needed to more clearly determine the scaling definitively and that, in any case, there are substantial differences between the experiments of Sec. 7.2 and the system considered by Goodridge et al. (1997), which would affect the c_i and F_i factors.

In the limit of high frequency, equation (7.14) gives a reference value for subharmonic Faraday waves, while experiments suggest a roughly constant gap between the onset of Faraday waves and the drop ejection instability (both scale approximately as $f^{-1/2}$ for higher frequencies). The regularity of this gap can be seen later in Fig. 7.18 of Sec. 7.7 with the solid blue and red lines. Ignoring the details associated with both instabilities (see, e.g., Goodridge et al., 1997), the results suggest that the energy gap between Faraday waves and drop ejection depends only weakly on the frequency of the forcing, and mainly on the physical properties of the fluid. Since the less viscous combination of FC-40 and 2 cSt silicone oil allows more data points in the drop ejection regime, conclusions are drawn mainly from this case.

7.5.3 Energy estimation for drops and interfacial phenomena

For the experiments described in Sec. 7.2, drop ejection generally occurs after the interface has evolved towards a columnar or bubble-like structure, and then undergone an instability to Faraday waves. This means that the average interface is constrained by the system size — reaching from top to bottom and oriented, to a large extent, perpendicular to the forcing. Both contact energy, which is relevant when $\beta \neq \pi/2$, and the average interfacial energy are largely constrained (bounded) in this situation. An increase in interfacial energy must manifest through the dynamics of the interface via higher amplitude waves or drop ejection.

We estimate the interfacial energy of the main interface and of the drops after ejection begins ($A\omega > A\omega|_c \simeq 0.17 \text{ m/s}$ for $f < 15 \text{ Hz}$) by measuring the visible interface length in selected frames of each experiment run. Consistent with the two-dimensional approximation (near onset, at least, the subharmonic waves do appear to be quasi-two-dimensional), we assume the contribution from the perpendicular dimension D is constant for different interface positions and for the initial drops. With this simplifying approximation, the interfacial energy depends on its visible perimeter, which can be measured in a given snapshot simply by summing pixels: $E_S \sim \sigma \mathcal{P}_\sigma$, where $\mathcal{P}_\sigma$ is the pixel

FIGURE 7.13: Interfacial energies, measured by $\mathcal{P}_\sigma$ (in pixels), obtained for FC-40 and 2 cSt silicone oil, as a function of the $\delta_c = A\omega - A\omega|_c$, where $A\omega|_c$ is the drop ejection threshold found to be 0.1743 m/s from the fit. The interfacial energy of the drops (red circles) and the main interface (green squares) are included. The curves are fitted to quadratic and constant functions, respectively. The number of observed drops is marked in parentheses, while outlier data points marked with crosses are excluded from the fit.

count of the perimeter. In Fig. 7.13, the measured interfacial energies (sum of visible lengths) of the main average interface (green squares), neglecting surface waves, and the drops (red circles) are given as a function of

$$\delta_c = A\omega - A\omega|_c, \quad (7.15)$$

where $A\omega|_c$ is the drop ejection threshold.

As expected, we find that the main interfacial energy term is (nearly) constant since either the frozen wave instability has led to columnar structures, or the vibroequilibria effect has flattened an initially bubble-like configuration. In both cases, the main interfacial energy is bounded by the system size. Drop energy, in contrast, increases (roughly) as δ_c^2 , as shown by a quadratic fitting to the experimental points.

We note several uncertainties of the experiments, which were not designed to study drop ejection, and of the fit in Fig. 7.13. First, drop number and size varied even near onset and the significant interfacial energy of a single drop means that $\mathcal{P}_\sigma$ does not go to zero at onset, but jumps to a finite value instead. Also, some runs were excluded as outliers (marked by crosses in Fig. 7.13). Upon checking the corresponding data sets, we attribute these exceptions to the lack of repeatability in the residual gravity profile between parabolas, and to the potential hysteresis that occurs if drops survive between parabolas. The former modifies the contribution of gravity (neglected in the analysis) and may eventually interact with the drops (James, Smith, and Glezer, 2003) if a change

in gravity couples with the ejection process, for example. Hysteresis is problematic too, with the experiment starting from a metastable initial condition, which changes the evolution (and energies) of the phenomena involved.

One can try to understand the results of Fig. 7.13 by recalling the energy balance suggested by the theory of vibroequilibria (Beyer et al., 2001; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997), where selection of the time-average (quasi-static) interface shape is formulated as a minimisation problem using a Lagrangian method that considers vibrational and surface energy contributions, among others (like gravitational); surface waves, which do not affect the average interface (to leading order) are not accounted for.

Assuming constant interfacial tension, the system energy E may be expressed [see, for example, Fernández et al. (2017b)] in a comoving frame as

$$E = (\rho_1 - \rho_2)\varepsilon(S)(A\omega)^2 + \sigma \int_S \partial S + \dots, \quad (7.16)$$

where $\varepsilon(S)$ represents a nondimensional factor obtained by integrating over the fluid domain (determined by the walls and interface S) and suppressed terms (signified by dots) include contact energy and gravitational energy.

In the absence of gravity and other relevant forces, the principal balance is between vibrational energy, which increases as $(A\omega)^2$, and interfacial energy. At a minimum of the Lagrangian (the vibroequilibria solution), these two energy terms have equal and opposite derivatives with respect to any (small continuous) variation. A finite increase in forcing will directly increase the vibrational energy of the system and this will, in turn, cause the interface to reorient, increasing interfacial energy but decreasing vibrational energy until a new balance (minimum) is achieved. Neglecting the initial interfacial contribution (of a flat, unforced interface, for instance) the interfacial energy of a vibroequilibria solution is expected to scale as $(A\omega)^2$.

It is interesting to note that the balance suggested by vibroequilibria theory, with increases in vibrational energy being partly transferred into (average) interfacial energy, seems to be replaced by an analogous process for drop ejection. Following a nonlinear process in which the surface wave amplitude increases until breaking occurs, subsequent increases in vibrational energy are partly channelled into the creation of ejected drops, with the $(A\omega)^2$ scaling carried over as well when the shifted value $\delta_c = A\omega - A\omega|_c$ is used.

7.6 Slow timescale phenomena: the frozen wave instability and vibroequilibria

When horizontal vibrations are applied to immiscible layers of fluids with a flat interface, Lyubimov and Cherepanov (1986) predicted, in the inviscid limit, the appearance of standing (sinusoidal) waves, known as frozen waves, at a certain velocity threshold. In deep layers the critical wavelength

is the capillary length. The absence of a strong (gravitational) restoring force in weightlessness means that these initially sinusoidal patterns can develop into large columnar structures. The lengthscale of these structures is bounded by the size of the system and reflects the wavelength of the fastest growing perturbation for a given forcing (Lyubimova et al., 2017). Here, we show experimental evidence of the frozen wave instability in microgravity and hypergravity, and present numerical results, particularly in connection with the wavenumber selected in the instability.

The vibroequilibria effect is also a potentially important slow timescale phenomena in microgravity. Although we observe some evidence of vibroequilibria in the experiments of Sec. 7.2, such as a flattening of the drop-like configurations, the presence of residual gravity perturbations and the strong effect of the frozen wave instability for higher vibrational velocities makes it difficult to isolate their effects or to quantify them in a precise manner. We expect that vibroequilibria effects, insofar as they tend to cause the interface to orient more perpendicular to the axis of vibration, have a stabilising effect on the frozen wave instability, whose (Kelvin-Helmholtz type) mechanism supposes an interface aligned with the axis of vibration.

7.6.1 Frozen waves and the two-scale wave pattern

The series of snapshots in Figs. 7.14 and 7.15 show the evolution of the interface between immiscible liquids subjected to horizontal vibrations under microgravity conditions. Each figure contains experimental observations for $f = 22$ Hz that confirms the results of simulations for $f = 20$ Hz. It is clear that the amplitude, which is below the threshold for frozen waves in Fig. 7.14 and above this threshold in Fig. 7.15, has a big effect. Increasing from $A = 0.87$ mm to $A = 1.76$ mm in the experiments (from $A = 1.5$ mm to $A = 2.0$ mm in simulations) pushes the system beyond the critical value for the frozen wave instability. Recall that this instability is caused by a shear-driven mechanism similar to the Kelvin-Helmholtz instability.

Below the frozen wave threshold (Fig. 7.14), residual gravity drives the formation of a squeezed drop-like structure located near the left side of the cell. Subsequently, either harmonic surface waves (a,b,d,e) or Faraday waves (f) may be excited over the interface; see Sec. 7.5.1 for further details.

Above the frozen wave threshold (Fig. 7.15), the interface forms a pattern with $\lambda \sim L$, which then develops into a columnar-like structure. Subharmonic Faraday waves (see Sec. 7.5.1) can appear on both sides of a column [Fig. 7.15(d,e)]. Analogous two-scale patterns have been observed in miscible fluid experiments (Shevtsova et al., 2016).

The (numerical) onsets of Faraday (dark blue circles and dark blue solid line in Fig. 7.18) and frozen waves (green squares and green solid line in Fig. 7.18) cross at a frequency $f^* \simeq 35$ Hz for 2 cSt silicone oil and $f^* \simeq 17.5$ Hz for 20 cSt silicone oil. If we imagine an idealised scenario with $\beta = \pi/2$ and an initially flat interface in microgravity, then Faraday waves cannot immediately appear with horizontal excitation (although harmonic and cross-waves can), even if their threshold is surpassed. It is only after frozen waves have been excited that columnar structures develop and portions of the

FIGURE 7.14: Snapshots showing the temporal evolution of the FC-40 and 20 cSt silicone oil interface below the frozen wave threshold for both experiments and simulations. (a–c) Experiment response at $f = 22$ Hz and $A = 0.87$ mm. (d–f) Simulation performed at $f = 20$ Hz and $A = 1.5$ mm under gravity profile III. A comparison of experiment and simulations in time cannot be done quantitatively due to the definition of $\mathbf{g}(t)$ (see Sec. 7.4) so times values are not labelled.

interface are sufficiently perpendicular to the forcing axis to permit the excitation of Faraday waves. Thus, for $f < f^*$ one expects, with increasing amplitude, to first see the frozen wave instability, quickly followed (at sufficient wave amplitude) by Faraday waves — the frozen wave instability acts here, among other things, to facilitate or “trigger” Faraday waves by reorienting the interface. For $f > f^*$, on the other hand, the frozen waves develop free of Faraday waves, whether forming columnar structures or not, until the Faraday wave threshold is crossed. In a real parabolic flight experiment, however, plane manoeuvres and residual gravity do not permit a purely horizontal flat interface, and the situation is more complex. Faraday waves may appear any time a portion of the interface moves towards a (nearly) perpendicular orientation with respect to the vibrational axis and the Faraday wave threshold is exceeded.

Although some experiments were performed above the frozen wave threshold (see Fig. 7.18), the irregular microgravity conditions in those runs do not permit a satisfactory analysis of the instability. In addition, with the 2 cSt silicone oil, the experiments performed above the frozen wave threshold are complicated by high-amplitude Faraday waves or drop ejection, making the results hard to process. Here, we perform a simulation-based analysis of frozen waves and compare it, when possible, with experimental results and theoretical predictions. Unlike the experiments of Sec. 7.2, we analyse the instability under idealised microgravity conditions (mainly, profile I) in the numerical model.

7.6.2 The frozen wave instability in microgravity

Columnar-like structures developing from the frozen wave instability have been seen in previous experiments in microgravity either in ordinary miscible liquids (Gaponenko et al., 2015b) or in near-critical fluids where interfacial tension vanishes (Gandikota et al., 2014b). Here we report, for

FIGURE 7.15: Snapshots showing the temporal evolution of the FC-40 and 20 cSt silicone oil interface above the frozen wave threshold for both experiments and simulations. (a–e) Experiment response at $f = 22$ Hz and $A = 1.76$ mm (above frozen wave onset) showing the growth of the instability (a,c) and the secondary Faraday waves excited on the lateral boundaries of the columns. The snapshots (d) and (e) separated by $\Delta t = 0.05$ s $\simeq 1/(22$ Hz), showing the (nearly) out-of-phase subharmonic behaviour; waves on the right column are shifted in phase by $\pi/2$. (f–h) Simulation performed at $f = 20$ Hz and $A = 2$ mm under gravity profile II. A comparison of experiment and simulations in time cannot be done quantitatively due to the definition of $\mathbf{g}(t)$ (see Sec. 7.4) so the times values are not labelled.

the first time, columnar-like structures in microgravity for a pair of liquids with sizeable interfacial tension.

Previous treatments and investigations of frozen waves (Goldobin et al., 2015; Lyubimov et al., 2016; Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007) considered a sufficiently large container or the ideal case of a infinite domain so that finite-size effects could be neglected. In this limit, a linear stability analysis provides the velocity threshold

$$(A\omega)^2 = \frac{(\rho_1 + \rho_2)^3}{\rho_1 \rho_2 (\rho_1 - \rho_2)} \sqrt{\frac{\sigma g_y}{\rho_1 - \rho_2}}, \quad (7.17)$$

for the inviscid case. The most unstable wavenumber $k = \sqrt{(\rho_1 - \rho_2)g_y/\sigma}$ is related to the capillary length. When vibrated beyond the threshold (7.17), which is very small in microgravity, wavenumber selection depends on the vibrational velocity, which excites a band of wavenumbers. In Burnysheva,

Lyubimov, and Lyubimova (2011), it was assumed that the wavelength selection in this problem was defined by the perturbation with maximum growth rate. A continuous and monotonic relationship between vibrational velocity and selected wavenumber was found.

In order to compare experiments and simulations with this theoretical prediction, Fig. 7.18 includes Eq. (7.17) with $g_y = 0.05 g_0$ (green dashed lines) for each experimental cell. Theoretical predictions give

$$A\omega|_{\text{theory}} = 0.134 \text{ m/s (2 cSt oil)}, \quad 0.1581 \text{ m/s (20 cSt oil)}, \quad (7.18)$$

while the numerical results (green solid lines) yield

$$A\omega|_{\text{experiment}} = 0.157 \text{ m/s (2 cSt oil)}, \quad 0.207 \text{ m/s (20 cSt oil)}. \quad (7.19)$$

Our results are in line with findings by Talib, Jalikop, and Juel (2007) that inviscid predictions underestimate the true onset value, which is affected by viscosity and, due to the geometry of the experiment cells, finite-size effects. The results here are obtained in relatively small containers where finite-size effects are expected to be important. To investigate this, the wavenumber of the instability is shown in Fig. 7.16(a) alongside the vibrational velocity with gravity profile I ($g_x = 0$) and for each fluid combination.

For small amplitudes, a constant wavenumber $k_1 \sim 0.4 \text{ mm}^{-1}$ ($\lambda_1 \sim 15.7 \text{ mm} \simeq L$) is observed in both cells. When compared to the inviscid theory, it is evident that the wavenumber selection is driven by the container length. This wavenumber is nearly constant with increasing velocities until a certain value $A\omega \sim 0.3 \text{ m/s}$ is reached, at which point the wavenumber doubles to $k_2 \sim 0.8 \text{ mm}^{-1}$. For this second mode, the results exhibit a certain amount of dispersion in the less viscous combination (green squares), while the wavenumber is nearly constant for the more viscous case (red squares). This mode persists until a velocity of $A\omega \simeq 1 \text{ m/s}$, where the wavenumber appears to double again.

7.6.3 Residual gravity effects

The effect of residual gravity is analysed here and compared to the available experimental data point. In Fig. 7.16(b), the wavenumber of the frozen wave instability under different gravity conditions (i.e., gravity profiles) and forcing initiation times is shown for FC-40 and 20 cSt silicone oil.

First, note the differences arising from the initiation time of the forcing. This is shown for gravity profile I, both for initiation during the hypergravity period (pink open squares) and during microgravity (red open squares). Despite the fact that the final gravity level is identical for both experiments, the wavenumber at onset is distinct. For hypergravity initialisation, when g_y is reduced after the vibration begins, the frozen wave threshold, which depends on g_y via Eq. (7.17), may be surpassed before the final microgravity level is reached. Since the vertical gravity at this

FIGURE 7.16: (a) Frozen wave instability diagram showing wavenumber for both combinations (less viscous, green open squares) under gravity profile I, when vibrations are applied in microgravity. (b) Frozen wave instability diagram for FC-40 and 20 cSt silicone oil under different gravity profiles and forcing initiation times: profile I initiation in microgravity (red open squares), profile I initiation in hypergravity (pink open squares), profile III initiation in hypergravity (cyan open squares). An experimental measurement (black solid square) is also shown. Illustrations are included to help visualise the features of the instability.

instant is larger, an increased wavenumber is expected. This wavenumber is nearly constant before jumping at a certain velocity, as in the previous case.

With increased vibrational velocity in this hypergravity case, we observe greater dispersion in the wavenumber that may be explained in the same way. The variation of the gravity level results in varying (most dangerous) wavenumbers and so, depending on the growth rate of each unstable mode and its interaction with the changing gravity level, different final states may be found. In particular, a new mode is observed with $k \simeq 1.2 \text{ mm}^{-1}$.

FIGURE 7.17: Three snapshots, separated by 0.0125 s in time for FC-40 and 2 cSt silicone oil (a–c) and FC-40 and 20 cSt silicone oil (d–f) subjected to vibrations at $f = 50$ Hz and $A = 3.9$ mm. (a,d) Initial high-amplitude surface waves in hypergravity, (b,e) transient frozen wave pattern (in hypergravity), (c,f) transition from vibrated to residual gravity equilibria.

In the case of gravity profile III (cyan open squares), when there is a perturbation in the x -direction, a further increase in k is observed. In addition to that, the critical velocities for transition to different modes are notably increased as well; this limits the maximum expected wavenumber in experiments. We were able to process just one experimental point (black solid square), where the microgravity conditions of the parabola allowed the appropriate development of frozen waves.

To the best of our knowledge, the results presented here are the first to examine the effect of residual gravity and strong finite-size constraints on wavenumber selection for the frozen wave instability.

7.6.4 The frozen wave instability in hypergravity

Only one experiment run considered high frequency and high amplitude vibrations. Despite the scarcity of experimental data points in this regime, which was further aggravated by the fact that the run was quickly aborted for safety considerations, there is clear evidence of the frozen wave instability in both experimental cells. In Fig. 7.17, three snapshots of each cell after initiation of the forcing in hypergravity are shown; the excitation is with $f = 50$ Hz and $A = 3.9$ mm.

As with lower vibrational velocities, there may be a harmonic surface wave mode, typical of horizontally vibrated containers (see, e.g., Porter et al., 2012; Salgado Sánchez et al., 2016), directly excited on the interface during hypergravity [panels (a,d)]. These surface waves, which are strongly nonlinear here, seem to propagate inwards from the ends of the container, but quickly evolve towards a different fluid configuration over a characteristic time $\tau < 0.0125$ s during the hypergravity period, as illustrated in Fig. 7.17(b,e). Here, the weightlessness condition is less significant since the vibrational amplitude is so large, effectively dominating over gravity and allowing the formation of a nearly columnar pattern. The ratio of the estimated frozen wave instability wavenumber for the two experiments (2 cSt oil, Fig. 7.17(b), denoted by subscript 1; 20 cSt oil, Fig. 7.17(e), denoted by

subscript 2) is measured as

$$\left. \frac{k_1}{k_2} \right|_{\text{experiment}} = \frac{\lambda_2}{\lambda_1} \simeq 1.26, \quad (7.20)$$

which matches well with the predictions of Lyubimov and Cherepanov (1986)

$$\left. \frac{k_1}{k_2} \right|_{\text{theory}} = \sqrt{\frac{\sigma_2 (\rho_1 - \rho_2)_1}{\sigma_1 (\rho_1 - \rho_2)_2}} \simeq 1.29. \quad (7.21)$$

With FC-40 and 2 cSt silicone oil, the frozen wave pattern coexists with numerous drops near the interface. In the case of the higher viscosity 20 cSt silicone oil, however, the Faraday waves do not reach high enough amplitude to cause drop ejection in less than 0.0125 s (one could speculate that, had the experiment lasted longer, they would have). In Fig. 7.18(b), this experimental point is included for the sake of completeness, but marked in blue to emphasise the discrepancy with expected phenomena in this regime.

7.7 Overview of the interface dynamics under periodic excitation

Figure 7.18 summarises, in terms of the vibrational frequency f and amplitude A , the ways in which the interface was seen to respond to forced vibrations in microgravity. For interfacial modes associated with the fast timescale, low-gravity experimental results (marked with filled circles) can be separated into three different regimes depending on the dominant behaviour: harmonic waves (yellow), subharmonic Faraday waves (light blue) and drop ejection (red). Included in the figure are numerical results (open markers) estimating the threshold for Faraday waves (blue open circles) and frozen waves (green open squares). These onsets are fitted to obtain the dark blue and solid green curves, respectively. The Faraday instability undergoes a change in its scaling from $f^{-1/2}$ to f^{-1} as the frequency is reduced (below about 20 Hz) and finite-size effects become more important. The drop ejection threshold discussed in Sec. 7.5.2 is shown in panel (a), and is extended for visual reference to higher frequencies (the dashed portion). Panel (b) also includes a presumed curve (dashed red line) for drop ejection, although only one experimental data point is available to locate it.

For low vibrational velocity (i.e., low amplitude and/or frequency), residual gravity leads to a drop-like configuration similar to Fig. 7.1(b) or Fig. 7.6(a,d). At the same time, harmonic waves are (directly) excited on the interface, which experiences a dominant vertical excitation. As the vibrational velocity is increased, subharmonic Faraday waves are excited. Near onset, these waves oscillate (predominantly) with the frequency $f/2$ and they tend to grow quickly in size, overshadowing the pre-existing harmonic waves and dominating the dynamics of the interface. A combination of frequency and amplitude acts to select the pattern wavelength.

FIGURE 7.18: Response of the fluid interface to vibrations in microgravity as a function of frequency f and amplitude A : (a) FC-40 and 2 cSt silicone oil, (b) FC-40 and 20 cSt silicone oil. Solid circles denote experimental points in three regimes: harmonic waves (yellow), subharmonic Faraday waves (blue) and drop ejection (red). Open markers denote numerical results for the Faraday wave instability (blue open circles) and the frozen wave instability (green open squares). These thresholds are fitted to obtain the dark blue and green solid lines, respectively. The Faraday instability scales as $f^{-1/2}$ for high frequencies ($\gtrsim 20$ Hz) and as f^{-1} at lower frequencies where finite-size effects are important. Theoretical onset curves are shown for Faraday waves [Eq. (7.14), dashed blue lines] and frozen waves [Eq. (7.17) using $g_y = 0.05 g_0$, dashed green line]. The drop ejection threshold discussed in Sec. 7.5.2 is shown in panel (a), and is extended (dashed curve) for visual reference to higher frequencies. Panel (b) includes a presumed curve (dashed red line) for drop ejection, although supported by just one experimental data point.

Drop ejection arises from the destabilisation of the subharmonic surface wave mode in the limit of high vibrational velocity (solid red line in the figure). Far from onset, subharmonic waves acquire high amplitude and are very nonlinear, and the oscillating interface becomes unstable to the ejection of drops. If such ejection is vigorous, the interfacial dynamics becomes disordered and fully

three-dimensional, which makes a comparison with two-dimensional simulations difficult. Based on experimental evidence, it is possible to classify different instability mechanisms, depending on the applied forcing.

Evidence for the existence of frozen waves, combined with vibroequilibria effects, are found for both fluid combinations. This phenomena, which occurs on the slow timescale, is critically affected by aeroplane disturbances.

Finally, theoretical threshold predictions are included in Fig. 7.18 for subharmonic Faraday waves [Eq. (7.14), blue dashed line] and frozen waves [Eq. (7.17), green dashed line]. Note that each of these predictions is based on an idealised configuration: in the case of Faraday waves, an infinite flat interface with purely “vertical” homogeneous forcing, low damping, and deep layers, and, in the case of frozen waves, an infinite flat interface with purely “horizontal” high-frequency forcing and inviscid flow. Discrepancies are expected, especially for Eq. (7.14), due to both finite-size effects and the nontrivial base state on which the Faraday waves form. The less viscous case [panel (a)] allows for a wider variety of results since higher (effective) forcing values can be achieved experimentally.

7.8 Conclusions

Experiments with two immiscible fluids subjected to vibrations have been performed in a parabolic flight campaign covering frequency and amplitude ranges of 5–50 Hz and 1–5.3 mm. A wide variety of dynamics was observed, depending on the vibrational velocity. At low vibrational velocity, residual gravity promotes drop-like configurations with harmonic waves directly excited on the interface. For higher vibrational velocities, subharmonic Faraday waves appear at half the forcing frequency. Frozen waves are observed beyond a critical onset value. A pattern with features on two length scales, analogous that reported in Shevtsova et al. (2016) with miscible fluids, was described for the first time with immiscible fluids. Disordered behaviour, associated with drop ejection, is observed for high vibrational velocity when the subharmonic surface waves become large enough to drive jets and ruptures of the interface.

To compliment and extend the range of experimental results and provide insight, for example, into the effect of residual gravity on the observed phenomena, we implemented a level-set-based model to predict the dynamics of the interface, and defined idealised microgravity profiles in order to systematically study the influence of parabolic flight disturbances. Good qualitative agreement is found between experiments and simulations, particularly below the drop ejection onset; the two-dimensional model used here does not capture the details of drop creation, which features the breaking of a fluid jet, and cannot describe the three-dimensional nature of this behaviour when vibrated strongly.

Simulations have been extensively used to investigate the subharmonic Faraday wave instability, and were able to suggest details within the primary subharmonic instability tongue. These simulations

were compared with experiments and with theoretical predictions (based on an idealised configuration), demonstrating good qualitative agreement, particularly in the limit of high frequency. The drop ejection instability, by contrast, was analysed solely on the basis of experimental data. Two different instability mechanisms for drop ejection were distinguished. Quantitative results were obtained from the experimental data of Sec. 7.2 that suggested, in the two-dimensional approximation, quadratic dependence of drop interfacial energy on the relative vibrational velocity.

The frozen wave instability was also analysed to determine its onset and results were shown for wavenumber selection in microgravity and for the effect of parabolic flight disturbances on the observed behaviour; these were compared to the predictions coming from the theory of an idealised inviscid system. Again, reasonably good agreement was found over the frequency and amplitude intervals of interest. Finally, evidence of the frozen wave instability in hypergravity was presented.

The results presented here, which include a quantitative analysis when possible, should contribute to understanding of vibrated fluid systems in microgravity. However, additional numerical simulations and future experiments are needed to extend these investigations. The very different interfacial phenomena, such as frozen waves, Faraday waves, and vibroequilibria, which can occur simultaneously in microgravity, lead to a complex set of interactions and transitions, with interesting but complicated dynamics and pattern selection scenarios.

Acknowledgment. The work of V. Yasnou, Y. Gaponenko, A. Mialdun, and V. Shevtsova was supported by the PRODEX programme of the Belgian Federal Science Policy Office and by the European Space Agency through its 62nd and 63rd Parabolic Flight Campaigns. The work of P. Salgado Sánchez and J. Porter was supported by the Ministerio de Economía y Competitividad under Project No. ESP2015-70458-P. We also thank the Departamento de Aeronaves y Vehículos Espaciales in the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio, Universidad Politécnica de Madrid for supporting the stay of P. Salgado Sánchez at the Microgravity Research Centre, Université Libre de Bruxelles.

Chapter 8

Finite-size effects on pattern selection in immiscible fluids subjected to horizontal vibrations in weightlessness

Salgado Sánchez, P., Gaponenko, Y., Porter, J., and Shevtsova, V.

Physical Review E **99**, 042803 (2019).

<https://doi.org/10.1103/PhysRevE.99.042803>

A detailed numerical investigation of frozen wave pattern selection in finite rectangular containers under microgravity conditions is presented. The column growth cycle controlling mode transitions is described and the instability diagram showing selected wavenumber as function of the vibrational velocity is obtained. In contrast to the continuous monotonic dependence of pattern wavenumber on forcing predicted by linear inviscid theory in the limit of infinitely long containers, the pattern selection process in finite domains is characterised by solution branches that persist over discrete forcing intervals. We describe how properties of the selected frozen wave pattern, including the finite critical forcing value, depend on container length. In long containers, a transition from nearly symmetric to asymmetric columnar development is found at sufficiently high forcing values, with the loss of (approximate) reflection symmetry evident in the mean flow associated with the transient growth of the pattern. The effect of container height is considered separately, in the limit of both shallow and deep layers. Shallow layers suppress the development of long-wave perturbations leading to higher wavenumber patterns whose growth may be associated with an asymmetric series of collisions between developing columns and container boundaries. In thick layers, lower pattern wavenumbers are observed and, although the column growth is on average more regular, the final state is often asymmetric. Finally, we compare our results with the dependence on container aspect ratio $\Gamma = L/2H$ predicted by inviscid theory. For $\Gamma \gtrsim 2$, finite-size effects are reasonably weak and the numerically obtained thresholds are similar to but slightly higher than the theoretical values, most likely due to viscous effects. For $\Gamma \lesssim 2$, finite-size effects come to the fore and onset is significantly delayed.

8.1 Introduction

The dynamic response of an interface separating distinct fluids in the presence of external forces is of considerable interest in fluid dynamics, with a range of applications in engineering, biology, and physics. In the simple case where the excitation can be approximated as monochromatic, a number of distinct instabilities may arise, depending on the frequency, amplitude, and direction of the forcing. On Earth, where gravity acts to flatten interfaces larger than the capillary lengthscale, the forcing is usually taken either to be vertical (perpendicular to the initial surface) or horizontal (parallel to it).

The case of vertical excitation can be treated as a modulation of the effective gravity in a co-moving reference frame, and the isotropy and homogeneity of this parametric forcing mechanism lead to extended patterns characterised by a finite threshold. The subharmonic Faraday waves (Faraday, 1831) that appear on gravitationally stable interfaces are well-known. An enormous range of interesting patterns can be observed in such systems (Arbell and Fineberg, 2002; Kumar and Tuckerman, 1994; Miles and Henderson, 1990; Prinet, Juric, and Tuckerman, 2009), especially if the monochromatic assumption is relaxed and multi-frequency forcing is used. In the case of gravitationally unstable configurations, on the other hand, vertical vibration can act to suppress the Rayleigh-Taylor instability (Wolf, 1970).

The case of horizontal excitation is somewhat more complicated to treat theoretically due to the lack of isotropy and the stronger effect of container boundaries. Since the fluids, with different densities, are accelerated towards or away from the solid (lateral) boundaries containing the contact line, they will move up and down near this boundary due to the difference in pressure. Such motion immediately induces (harmonic) waves at the forcing frequency, which disrupt the initially flat interface (Gaponenko et al., 2015a). The oscillatory pressure gradient can also excite subharmonic waves, known as cross-waves (Garrett, 1970), through a parametric forcing mechanism analogous to that of Faraday waves, but lacking the spatial homogeneity of the vertical case (Tinao et al., 2014; Varas and Vega, 2007). Depending on the extent to which these localised cross-waves penetrate inwards from the lateral walls, their interaction can induce temporal modulations (Perez-Gracia et al., 2014; Varas and Vega, 2007) and a range of secondary bifurcations (Salgado Sánchez et al., 2016).

The counterflow associated with the mass transfer occurring at the lateral walls in horizontally forced layers provides the mechanism for an instability of Kelvin-Helmholtz type (Khenner et al., 1999; Lyubimov and Cherepanov, 1986). The resulting interfacial waves, which are stationary in a co-moving frame, are commonly referred to as frozen waves (Wunenburger et al., 1999). Theoretical studies in the large-aspect-ratio limit (Lyubimov and Cherepanov, 1986; Lyubimov et al., 1998), ignoring lateral boundaries and taking the initial interface to be flat, have determined the frozen wave threshold and the critical wavenumber. The instability may be supercritical or subcritical, depending on the density ratio, and associated with finite or vanishing wavenumber, depending on the layer depth. The predictions of the inviscid theory were critically examined in the work of Talib,

Jalikop, and Juel (2007), among others (Jalikop and Juel, 2009; Yoshikawa and Wesfreid, 2011), where it was shown numerically and experimentally that the threshold is underestimated for fluids of similar viscosity, and overestimated for fluids of high viscosity contrast.

Frozen waves are of increased importance in weightless environments, where the predicted threshold vanishes in the large-aspect-ratio limit (Lyubimova et al., 2017; Lyubimov and Cherepanov, 1986) for long-wave perturbations. The periodic stripe patterns found in supercritical CO₂ in the experiments of Beysens et al. (2009) were the first observation of the frozen wave instability in weightlessness; these studies of near-critical fluids were continued on ground using magnetic compensation (Gandikota et al., 2014a,b). More recent work (Sharma et al., 2019) on supercritical fluids in microgravity examined thermovibrational instabilities, including the influence of finite-size effects.

Experiments conducted by Gaponenko et al. (2015b) showed that the frozen wave interfacial instability also exists for the diffusive boundary of miscible liquids with different densities and viscosities. Under the microgravity conditions of a parabolic flight, Gaponenko et al. (2015b) and Shevtsova et al. (2016) showed that the saw-tooth wave structure that first develops on the diffusive interface, and which occurs on ground as well, evolves in a similar manner as in experiments using immiscible liquids. The amplitude of the pattern grows until it reaches the upper and lower walls and the resulting structure can be described as a series of vertical columns of alternating liquids. Increasing the forcing still further can excite Faraday waves on the diffusive columnar interfaces, and thus produce an interfacial pattern with two distinct length scales (Shevtsova et al., 2016). The pattern selection process in this system has been further analysed by Gaponenko et al. (2018), who revealed a transition from columnar to vortical structures as the width of the initial diffusive interface increases (i.e., when density and viscosity gradients across the interface decrease).

Frozen wave experiments have also been performed in microgravity using immiscible liquids, and the recent work of Salgado Sánchez et al. (2019d) demonstrates that the interaction of various interfacial phenomena occurs in them as well. In particular, the appearance of interfacial patterns with two length scales, where Faraday waves are excited on the sides of a columnar structure that develops subsequent to the frozen wave instability, was confirmed in an immiscible two-fluid system for the first time [see Fig. 8.4(c)]. The interpretation of the results was affected by the imperfect and variable gravity level of the parabolic flight, the finite contact angle and associated dynamics, and other experimental limitations. In particular, the constraints of the container size clearly affected the results.

Given the importance of finite-size effects for understanding frozen wave patterns and related interfacial instabilities in microgravity, we undertake a systematic numerical investigation of these patterns in immiscible liquids for various container sizes and a range of forcing amplitudes (and thus selected wavenumbers). Simulations are performed using parameters similar to those used by Salgado Sánchez et al. (2019d) to understand parabolic flight experiments, but differing, for convenience, in the viscosity ratio and contact angle. In contrast to the experiments of that paper, the

FIGURE 8.1: Sketch illustrating the numerical model: two horizontal superposed layers of immiscible liquids confined in a $L \times 2H$ rectangular container. The cell is subjected to vibrations of frequency ω and amplitude A parallel to the initial flat interface (dashed line). Fluid behaviour is analysed for a pair of fluids similar to recent parabolic flight experiments (Salgado Sánchez et al., 2019d).

frozen wave instability is analysed here under ideal weightless conditions, with no residual gravity and with a perfectly flat initial interface oriented parallel to the vibrational axis.

The article is organised as follows. In Sec. 8.2, we discuss the mathematical formulation and numerical implementation of the problem. In Sec. 8.3, the development of the instability in weightlessness is detailed in finite containers, and the cycle of column growth is described and summarised in a stability diagram. In Sec. 8.4 and Sec. 8.5, the effects of length and height on different features of the instability, such as symmetry, are considered separately. In Sec. 8.6, we compare our numerically obtained thresholds with the predictions of linear inviscid theory. Conclusions are given in Sec. 8.7.

8.2 Mathematical formulation

The basic problem considered here is the application of horizontal vibrations to a rectangular container holding two superposed layers of immiscible liquids of equal volume and separated by an initially flat horizontal interface. The container has interior length L and height $2H$, as sketched in Fig. 8.1, and vibrates harmonically with amplitude A and frequency ω in the x -direction.

We implement a level-set method (Olsson and Kreiss, 2005; Salgado Sánchez et al., 2019d; Sethian, 1999) for tracking the movement of the interface, which is represented by the level-set variable ϕ obeying

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right), \quad (8.1)$$

where $\gamma = A\omega$ is selected as the vibrational velocity, and ε is a small diffusive term added for numerical stability and taken to be on the order of the maximum mesh element size after convergence. This addition and that of the compression flux term $\phi(1 - \phi)\nabla \phi/|\nabla \phi|$ are described in Olsson and Kreiss (2005) and implemented by default in COMSOL Multiphysics.

Fluid properties are defined as functions of ϕ via the introduction of a (smoothed) Heaviside function $\mathcal{H}$ (as in Salgado Sánchez et al., 2019d) by writing

$$\rho = \rho_1 + (\rho_2 - \rho_1)\mathcal{H}, \quad (8.2a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1)\mathcal{H}, \quad (8.2b)$$

where

$$\mathcal{H}(\phi) = \begin{cases} 0, 1 & \phi < -\varepsilon, \phi > \varepsilon \\ \frac{1}{2} + \frac{\phi}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\phi}{\varepsilon}\right) & |\phi| \leq \varepsilon. \end{cases} \quad (8.3)$$

The subscripts 1 and 2 denote the two fluids and correspond to $\phi = 0, 1$, respectively, with the interface located at $\phi = 0.5$.

We solve Eqs. (8.1) and (8.2) along with the continuous incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991)

$$\rho[\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}] = -\nabla p + \mu\Delta\mathbf{u} + \delta\mathbf{f}_\sigma + \rho\mathbf{G}(t), \quad (8.4a)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (8.4b)$$

The effect of interfacial tension at the immiscible interface is included in the momentum equation via $\mathbf{f}_\sigma$, defined by

$$\mathbf{f}_\sigma = (\sigma\kappa)\mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}, \quad (8.5)$$

where κ is the interface curvature, $\mathbf{n}$ is the unit normal vector and $\delta = |\nabla\phi|$ restricts the associated effects to the vicinity of the interface.

The applied acceleration is written as

$$\mathbf{G}(t) = A\omega^2 \cos(\omega t) \mathbf{i}, \quad (8.6)$$

where ω is the forcing frequency, $A\omega$ the vibrational velocity, and $\mathbf{i}$ the unit vector in the x direction.

Navier-slip boundary conditions, applied on all solid walls, allow for fixing the contact angle and for motion of the contact line, which is observed in the relevant experiments (Salgado Sánchez et al., 2019d). This formulation, which includes an artificial frictional force $\mathbf{f}_\nu = (\mu/\delta_\nu)\mathbf{u}$ that scales with the viscous vibrational boundary layer δ_ν is

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u}[\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta)\delta] = 0, \quad (8.7)$$

where $\mathbf{n}_w$ is the corresponding normal vector.

At $t = 0$, the liquids are assumed to be motionless ($\mathbf{u} = 0$) and separated by a flat interface. The vibrational forcing is initialised using the same type of smoothed Heaviside function, $\mathcal{H}$, with the transient time selected as 0.25 s for its relevance to the recent experiments of Salgado Sánchez et al.

	ρ (kg/m ³)	ν (m ² /s)
20 cSt silicone oil	950	20e-6
FC-40	1855	5e-6 ¹
	σ (N/m)	β (rad)
FC-40 and silicone oil	6.021e-3	$\pi/2$

TABLE 8.1: Physical properties of 20 cSt silicone oil, FC-40, and the interface between them: density ρ , kinematic viscosity ν , surface tension σ (reproduced from Someya and Munakata, 2005), and contact angle β (selected to achieve an initial flat interface).

¹The real kinematic viscosity is 2.2e-6 m²/s. It is increased to delay the appearance of subharmonic Faraday waves at high frequency.

(2019d). Similarly, fluid properties, which are listed in Table 8.1, are similar to those of the FC-40 and 20 cSt silicone oil combination used in that reference. The values here differ from the actual experiment in that the viscosity of FC-40 is increased (by roughly a factor of 2) in order to delay the appearance of the secondary subharmonic surface wave mode (Salgado Sánchez et al., 2019d), and the ideal contact angle of $\beta = \pi/2$ is used to achieve a perfectly horizontal initial interface.

We use the following scalings for time, length, density and viscosity,

$$\begin{pmatrix} t \\ \mathbf{x} \\ \rho \\ \nu \end{pmatrix} \rightarrow \begin{pmatrix} \hat{t} \sqrt{\frac{(\rho_1 - \rho_2)L^3}{\sigma}} \\ \hat{\mathbf{x}} L \\ \hat{\rho} (\rho_1 - \rho_2) \\ \hat{\nu} \sqrt{\frac{\sigma L}{(\rho_1 - \rho_2)}} \end{pmatrix}. \quad (8.8)$$

The dynamics contained in Eqs. (8.1)–(8.7) depend on the dimensionless vibrational velocity parameter $\mathcal{F}$ and frequency Ω ,

$$\mathcal{F} = A\omega \sqrt{\frac{(\rho_1 - \rho_2)L}{\sigma}}, \quad (8.9)$$

$$\Omega = \omega L \sqrt{\frac{(\rho_1 - \rho_2)L}{\sigma}}, \quad (8.10)$$

as well as the density and viscosity ratios, $\hat{\rho}$ and $\hat{\nu}$, the capillary number Ca , and the aspect ratio Γ :

$$\hat{\rho} = \frac{\rho_2}{\rho_1}, \quad \hat{\nu} = \frac{\nu_2}{\nu_1}, \quad \text{Ca} = \nu_1 \sqrt{\frac{(\rho_1 - \rho_2)}{\sigma L}}, \quad \Gamma = \frac{L}{2H}. \quad (8.11)$$

Recall that H is the initial height of each fluid layer (see Fig. 8.1).

The commercial software COMSOL Multiphysics 5.3 is used for solving the two-dimensional governing Eqs. (8.1)–(8.7) with the finite element method. Numerical convergence was previously tested for a similar configuration (Salgado Sánchez et al., 2019d) and we make use of this to select the number of mesh elements and the interface thickness. Simulations throughout this paper are computed using a maximum element size $\mathcal{S}$ of

$$\mathcal{S} = \min\left(\frac{2H}{25}, \frac{7.5}{25}\right) \text{ mm} \quad (8.12)$$

and

$$\varepsilon = \left(\frac{3}{2}\right) \mathcal{S}. \quad (8.13)$$

A Generalised- α (Erlicher, Bonaventura, and Bursi, 2002) integration scheme, in combination with both *streamline* (Harari and Hughes, 1992) and *crosswind* (Codina, 1993) techniques is used. The maximum time step is set at $1/(10f)$, with stepsize reduced automatically when required for stability. Further details of the numerical scheme can be found in Salgado Sánchez et al. (2019d).

8.3 Development of the frozen wave instability in weightlessness

When two immiscible layers of fluid separated by a flat interface are subjected to horizontal vibrations, quasisteady undulations known as frozen waves appear at a critical forcing value. If lateral boundaries are ignored (i.e., the container is taken to be infinitely long) and the fluids are assumed to be deep and inviscid, then this threshold is given by Lyubimov and Cherepanov (1986)

$$(A\omega)^2 = \frac{(\rho_1 + \rho_2)^3}{\rho_1 \rho_2 (\rho_1 - \rho_2)} \sqrt{\frac{\sigma g}{\rho_1 - \rho_2}}, \quad (8.14)$$

and the frozen wave has a wavelength λ set by the capillary lengthscale:

$$\lambda = 2\pi \sqrt{\frac{\sigma}{(\rho_1 - \rho_2)g}}. \quad (8.15)$$

In microgravity conditions, this critical wavelength becomes very large, and diverges in the limit of $g = 0$ where the threshold also vanishes. Beyond lowering the threshold, the absence of the usual gravitational restoring force facilitates the growth of the initially sinusoidal patterns, which can rapidly develop into large columnar structures (Gandikota et al., 2014b; Gaponenko et al., 2015a). The lengthscale of these structures is determined by the fastest growing perturbation at the given forcing (Burnysheva, Lyubimov, and Lyubimova, 2011; Lyubimova et al., 2017) and the size constraints of the system (container).

We describe in Sec. 8.3.1 how the frozen wave instability develops in finite containers subjected to vibrations in weightless environments, and how a cycle of column growth is key to the pattern selection process. The associated instability diagram is presented in Sec. 8.3.2.

8.3.1 Column growth cycle and mode transitions

In Fig. 8.2, two series of snapshots illustrate the time evolution of the instability in a container of size $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$. The amplitude of the excitation is fixed at $A = 1.5 \text{ mm}$, while two different frequencies of (a) $f = 45 \text{ Hz}$ and (b) 60 Hz are used, corresponding to $\mathcal{F} = 28.48$ and 37.97 , respectively. These figures focus on the evolution of the mean interface and velocity field (shown with scaled arrows) by averaging over one period of excitation. The harmonic surface waves, which are present near the lateral boundaries, are therefore filtered out and not visible.

Immediately after the forcing is initiated, the mean vertical velocity is especially large near the vertical walls, where the two liquids move up and down, and there is motion of the contact line as permitted by the Navier-slip boundary conditions; see Eqs. (8.7). The horizontal velocity on the interface is, on average, directed inwards (Dore, 1973; Shevtsova et al., 2015a).

The net result of this initial agitation near the lateral walls is to provide a perturbation that activates column growth. These first columnar seeds can be seen near the lateral boundaries at $t = 0.2 \text{ s}$. As time passes, the frozen wave (column) growth moves inwards towards the centre of the container, as seen by the associated mean velocity; see the snapshots at $t = 0.3, 0.4 \text{ s}$. The interfacial perturbations grow differently depending on their amplitude and the amount of liquid they involve. While the larger lateral columnar seeds grow until reaching the top and bottom walls, surface tension limits the growth of the smaller perturbation located in the centre, which does not always reach the (upper and lower) container walls; see Fig. 8.2(a) at $t = 0.9 \text{ s}$. This can also be understood as a kind of wavenumber frustration, where the more rapidly growing columns near the lateral walls, with wavelengths closer to the prediction of linear theory, induce a mismatch or defect in the centre of the container.

The incomplete central column [Fig. 8.2(b) at $t = 0.5, 0.9 \text{ s}$] is observed over a certain interval of the vibrational parameter. When it first appears, the associated fluid volume is small and it oscillates noticeably at the forcing frequency. This oscillatory motion, which is asymmetric, generates a mean flow that pushes it laterally (left or right) until it collides and merges with an adjacent column. In this scenario, the central column is transient. As forcing increases, the wavelength of the lateral columns decreases and the central column grows in size. With more inertia, the harmonic oscillations diminish and the mean flow is more symmetric. In this scenario, the central column remains and can be located in the final structure.

As the forcing velocity is increased, the tip of the smaller column crosses the horizontal midline at $y = 0$ (the location of the initial unperturbed interface) and we take this forcing value to characterise a mode transition. The wavelength of the pattern at this point, where the emerging column surpasses the horizontal midline, is also used to indicate the evolution of the pattern towards regularly spaced columns as this “local” wavenumber decreases by approximately a factor of two; see the snapshots at $t = 0.9 \text{ s}$ in Fig. 8.2.

FIGURE 8.2: Time evolution, averaged over one excitation period, of the frozen wave instability under weightlessness in a container of size $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$ with $A = 1.5 \text{ mm}$ and two different frequencies: (a) $f = 45 \text{ Hz}$ and (b) $f = 60 \text{ Hz}$, corresponding to $\mathcal{F} \approx 28.48$ and $\mathcal{F} \approx 37.97$, respectively. The time is shown between the columns. The mean velocity field is shown using scaled arrows (scale not preserved between snapshots). The representative wavelength λ in the defect region of each pattern is defined using the final snapshot at $t = 0.9 \text{ s}$. The appearance of subharmonic Faraday waves (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) on the lateral boundaries of the columnar structure is also seen at that time in column (b).

The size of the incomplete column (defect) grows with increasing forcing. At a certain value it reaches the upper wall and, beyond that, its wavelength increases (i.e., the column width widens) with vibrational velocity. At some point its wavelength is too large and a new column seed appears, this time enclosing the other fluid and growing downwards from the top. The repetition of this process defines the column growth cycle, which marks the transition from one mode to another and is driven by the increase in vibrational forcing and the associated selection of shorter wavelengths. This scenario characterises the majority of results presented in this paper.

It is also notable for the forcing parameters shown in Fig. 8.2 that the column growth is approximately symmetric with respect to horizontal reflection about the middle of the container. This is evident in the mean velocity field, which is included in all except the last ($t = 0.9$ s) snapshots; when the selected steady state is reached the mean flow is null.

The final columnar structures are in qualitative agreement with the parabolic flight experiments of Salgado Sánchez et al. (2019d) in immiscible liquids [see Fig. 8.4(c)], with the experiments of Gandikota et al. (2014a,b) in supercritical fluid under magnetic compensation, and with the observations in microgravity of Gaponenko et al. (2015b) and Shevtsova et al. (2016) using miscible liquids; similar behaviour is found in that case when the diffusive interface is thin and the density and viscosity gradients produce an effective interfacial tension (Gaponenko et al., 2015b).

8.3.2 Pattern selection and instability diagram

The wavelength selection process in weightlessness was considered by Lyubimova et al. (2017) for the inviscid, infinitely extended problem, where the wavelength of the fastest growing perturbation was shown to increase continuously and monotonically with the vibrational parameter. When finite (longitudinal) size effects are relevant, however, numerical simulations show that wavenumber selection can differ significantly from these predictions.

To characterise the discontinuous selection process in finite length containers, we construct a mode instability diagram by measuring two relevant (dimensionless) wavenumbers. One wavelength λ is measured locally at the location (usually the centre) of the emerging column (or defect) as illustrated in Fig. 8.2, with the dimensionless wavenumber defined as

$$\mathcal{K} = \frac{L}{\lambda}. \quad (8.16)$$

The other is an average wavenumber $\langle \mathcal{K} \rangle$, equal to the number of paired columns (identifying lateral boundaries) in the columnar structure. This averaged wavenumber is used to label the mode of instability in the results that follow. Both quantities are calculated over a wide set of vibrational parameters $\mathcal{F}$ for a 30 mm $\times$ 7.5 mm domain in Fig. 8.3. Snapshots are included showing selected final states for each mode of the instability.

At the onset of the frozen wave instability, the interface develops towards a single “pair” of columns, with the upper fluid (silicone oil) located in the centre and the lower fluid (FC-40) divided nearly equally and aligned against the lateral walls; this is the $\langle \mathcal{K} \rangle = 1$ mode. As the vibrational velocity is increased, the cycle of column growth described in Sec. 8.3.1 is observed. Mode transitions are considered to occur when the small central column (defect) grows large enough to pass the horizontal midline $y = 0$. As this happens, the surrounding column widens and splits apart.

For the range of vibrational velocities considered, there are five distinct modes, which are delimited by horizontal dashed lines in Fig. 8.3. Each mode is found over a certain interval of vibrational velocities, which widens as the wavenumber grows, indicating the increased cost of adding new

FIGURE 8.3: Instability diagram in terms of the vibrational parameter $\mathcal{F}$ for a $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$ container showing the nondimensional averaged wavenumber $\langle \mathcal{K} \rangle$ (solid vertical lines), equal to the number of column pairs along the container length, and the wavenumber $\mathcal{K}$ (solid dots) near the defect in the central part of the container. The frozen wave threshold (see Sec. 8.4.2) is distinguished by a circular gray marker. Insets illustrate the final pattern for selected points on each branch, while dashed lines provide a visual guide for the forcing interval associated with each mode.

columnar structures. In fact, in any real system, there is likely to be a point of wavenumber saturation, or a transition to new phenomena, as the vibrational velocity becomes very large and acoustic or viscous effects assume a dominant role.

As mentioned above, the column growth cycle is often nearly symmetric with respect to reflection about the middle of the container. This approximate symmetry is observed over a range of low to medium strength excitations, especially in moderately sized containers where λ remains on the order of the container length. Asymmetric behaviour is also found, as will be discussed below, and the size of the container appears to play a crucial role in its emergence. We begin investigating the effect of container size on the frozen wave instability by varying the container length.

8.4 Effect of container length

Here, we describe the dependence on container length of the instability diagram (Sec. 8.4.1) and the frozen wave threshold (Sec. 8.4.2). The appearance of symmetry-breaking in larger containers is discussed (Sec. 8.4.3) and, lastly, we suggest aspects of the transition from finite-sized containers to effectively infinite systems (Sec. 8.4.4).

8.4.1 Instability diagram with increasing length

Figure 8.4 shows the frozen wave instability diagram in terms of (a) the nondimensional average wavenumber $\langle \mathcal{K} \rangle$ and (b) the local wavenumber $\mathcal{K}$ measured in the region of the emerging column. Results for containers lengths of $L = 7.5 \text{ mm}$, 15 mm , 30 mm and 60 mm , all with a fixed height $2H = 7.5 \text{ mm}$, are displayed together, covering aspect ratios from $\Gamma = 1$ to $\Gamma = 8$. Container lengths are labelled in the figure for clarity.

FIGURE 8.4: (a,b) Instability diagrams in terms of the vibrational parameter $\mathcal{F}$ for fixed vertical dimension $2H = 7.5$ mm and different container lengths (labelled in each curve): $L = 7.5$ mm (red, upright triangles), 15 mm (green, inverted triangles), 30 mm (black, squares) and 60 mm (blue, circles). (a) Average wavenumber $\langle \mathcal{K} \rangle$. (b) Wavenumber $\mathcal{K}$ near the defect region. Mode transitions are shown with dashed lines for visual reference. (c) Experiment showing a frozen wave pattern in microgravity in a $L \times 2H = 15\text{mm} \times 7.5$ mm container of FC-40 and 20 cSt silicone oil subjected to horizontal vibrations at $f = 22$ Hz, $A = 1.755$ mm, $\mathcal{F} = 11.519$.

The case of the shortest length $L = 7.5$ mm with $\Gamma = 1$ differs from the scenario of column creation described previously (see Fig. 8.3 for comparison) since, over the range of forcing considered, the

pattern selected is consistently the first mode $\langle \mathcal{K} \rangle = 1$. There is just one branch of solutions over this range of the vibrational parameter because the development of more columns is prohibited by the (small) container size.

As expected, when the container length is increased, additional modes appear over the same range of the vibrational parameter. The interval over which each mode exists decreases with length, as does the absolute instability threshold itself, which is that of the largest permitted wavelength. In the largest container with $L = 60$ mm and $\Gamma = 8$, numerous modes, up to $\langle \mathcal{K} \rangle = 9$, can be excited. Furthermore, we observe a type of symmetry-breaking transition at $\mathcal{F} \simeq 40$ where $\lambda/L \lesssim 0.2$; this is described in Sec. 8.4.3.

To support the numerical results, Fig. 8.4(c) presents a snapshot from a parabolic flight experiment showing the frozen wave instability in microgravity in a $L \times 2H = 15$ mm $\times$ 7.5 mm container with FC-40 and 20 cSt silicone oil (see Table 8.1). Horizontal vibrations are applied at $f = 22$ Hz with $A = 1.755$ mm, corresponding to $\mathcal{F} = 11.519$. The observed pattern is that of the $\langle \mathcal{K} \rangle = 1$ mode, with a single pair of columns along the container. This is consistent with the simulations shown in Fig. 8.4(a,b) for $L = 15$ mm at that value of $\mathcal{F}$, despite the different contact angle and viscosity. Further comparison with available experiments is complicated by the irregularity of the microgravity level and the complex interaction between the frozen wave patterns, high-amplitude Faraday waves and drop ejection (Salgado Sánchez et al., 2019d).

8.4.2 Effect of length on frozen wave threshold

It was shown by Lyubimova et al. (2017) and Lyubimov and Cherepanov (1986) that in the absence of gravity, frozen waves arise at arbitrarily small vibrational forcing in deep and infinitely extended layers of inviscid fluids. Finite length domains, however, do not permit the associated long-wave perturbation and there is, in contrast, a finite threshold for the instability, even in the inviscid limit. This finite threshold results from, and is strongly affected by, the container length.

To get a more accurate estimate of the onset value, we measure the transient time τ for the interface amplitude at $x = L/2$ to reach the bottom of the container. The depth d of this growing column is shown as a function of time in Fig. 8.5(a) using a recipient of length $L = 15$ mm and height $2H = 7.5$ mm, and for seven different forcing amplitudes just above onset. At least three different regimes can be clearly distinguished. First, there is an initial stage reflecting the exponential growth of the linearised system as well as the timescale (0.25 s) of the smoothing function that modulates the forcing [see Eq. (8.3)]. After that, there is a transition to a nearly linear (constant growth) regime, which dominates the growth time of the column close to threshold. Finally, there is a more rapid (approximately exponential) acceleration towards the container ceiling.

We find that, despite the distinct growth stages involved, the inverse square of the total time needed for the column to reach the boundary τ^{-2} is proportional to $\mathcal{F} - \mathcal{F}_c$, as demonstrated in Fig. 8.5(b).

FIGURE 8.5: (a) Time evolution of the interface depth d (in mm) at the centre of a $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$ container for decreasing vibrational forcing (1–7) near threshold showing (at least) three distinct regions: initial quasi-exponential growth, a transition to nearly constant growth rate, and a transition to exponential growth when the column approaches the wall. (b) Inverse square of the frozen wave growth time, τ^{-2} (grey circles), for the smallest wavenumber $\langle K \rangle = 1$ as a function of the vibrational parameter $\mathcal{F}$. The linear dependence allows straightforward estimation of the instability threshold (black circle). Numbers refer to the different forcings used, as in (a). (c) Frozen wave instability threshold $\mathcal{F}_c$ as a function of the container length L for fixed height $2H = 7.5 \text{ mm}$. Fitting curves (dashed) assuming L^{-1} , $L^{-0.5}$ and $L^{-0.25}$ scalings are shown for comparison.

In that figure, the data points (open circles) are the same as in Fig. 8.5(a) and the threshold (filled black circle) is estimated by a simple linear regression. Note that this procedure assumes there is no significant interval of bounded (small-amplitude) columns, which were never observed.

The same method is used for different container lengths to construct Fig. 8.5(c), which also shows fitting curves (dashed) with scalings of $\mathcal{F}_c \propto L^{-1}$, $\mathcal{F}_c \propto L^{-0.5}$ and $\mathcal{F}_c \propto L^{-0.25}$. These fits are

shown mainly for the sake of comparison, but do suggest a slow scaling like $L^{-0.25}$ for the decrease of onset values as $L \rightarrow \infty$ where theory (Lyubimova et al., 2017; Lyubimov and Cherepanov, 1986) predicts a vanishing threshold. In the limit of small containers, on the other hand, the frozen wave instability is delayed by the need for a large (dimensional) wavenumber and, thus, high vibrational velocity. The simulations in Fig. 8.5(c) suggest that $\mathcal{F}_c$ diverges as L^{-1} in the limit $L \rightarrow 0$.

We note that threshold results for long large-aspect-ratio containers have a certain degree of uncertainty because, as discussed later in Sec. 8.5.1, the development of long wavelength modes $\langle \mathcal{K} \rangle = 1, 2, \dots$ is affected by the upper and lower boundaries. Near onset, the calculation of the growth time for the first mode $\langle \mathcal{K} \rangle = 1$ is more difficult, and sometimes not even possible. This problem, among other potential sources of numerical error, explains the lack of results for still larger L values in Fig. 8.5(c) and may affect the threshold estimation for $L = 75$ mm, which is slightly lower than the general trend would suggest.

8.4.3 Symmetry breaking in the column growth cycle

For sufficiently large containers, a transition from symmetric to asymmetric column growth is observed at moderate vibrational parameters. Figure 8.6 contains the instability diagram for $L = 60$ mm with insets showing selected final states along each branch.

Pattern selection at low vibrational velocities is characterised, in this domain, by cycles of nearly symmetric column growth, as described in Sec. 8.3. Emerging columns develop in the middle of the container and remain there. When the vibrational parameter $\mathcal{F}$ reaches a value of about 40, however, there is a transition to more asymmetric column growth. To illustrate this more clearly, we show the final steady density distribution along the horizontal midline in Fig. 8.7 as a function of the vibrational parameter.

As the pattern wavelength becomes smaller compared to the container length, lateral boundary effects become weaker; for $\mathcal{F} \simeq 40$ we find $\mathcal{K}^{-1} = \lambda/L \lesssim 0.2$. This can also be interpreted as a weakening of the bias associated with the initiation of the forcing, which tends to perturb the interface more strongly near the lateral walls, causing the columns to grow more rapidly there. As the container length increases, the initial growth becomes more uniform across the interior and wavenumber frustration can occur at various locations, not always at the centre.

The mechanism of column creation in the asymmetric regime is more variable and complicated than in the symmetric case, and often involves a type of fingering or branching during the transient stage as nascent columns of different size merge, either partly or completely. These fingers may branch off to the side of the main column at an angle only to be reabsorbed eventually, sometimes leaving drops behind, or they can reach the upper or lower boundary to form a new column. This process leads to the sudden appearance, in the final state, of a (pair of) new finite-width columns.

The location of the defect also varies, with a tendency to alternate from one side to the other, as seen in Fig. 8.7. The first asymmetric defect appears close to the centre of the container. These

FIGURE 8.6: Frozen wave instability diagram in terms of the vibrational parameter $\mathcal{F}$ for a $L \times 2H = 60$ mm $\times$ 7.5 mm container. The average wavenumber $\langle \mathcal{K} \rangle$ (vertical lines) gives the number of column pairs in the container, while the local wavenumber $\mathcal{K}$ (solid dots) near the defect region reflects the growth of this thin column (defect) into a regular column. Dashed lines are included for reference to delineate the mode transitions. A change from symmetric to asymmetric column growth (both regions are labelled) is observed near $\mathcal{F} \simeq 40$.

defect locations, although seemingly random, are remarkably repeatable — indeed, so is the final columnar pattern itself, despite the complicated nonlinear dynamics that precedes it. Figure 8.7, which shows mainly continuous motion of the column boundaries versus $\mathcal{F}$, is constructed from a series of simulations that all start from the same initial condition of motionless fluids and a flat interface; the darker horizontal lines in this figure show the actual $\mathcal{F}$ values used, with the boundaries obtained by interpolation. The consistent defect locations across successive simulations are likely determined, at least in part, by the form of the forcing function.

Note that the measurement of the local (defect) wavenumber in the asymmetric regime must adjust to the changing location. We try to maintain continuity by switching from the region of one emerging column to the next when the measured column widths at the horizontal midline are comparable, i.e., after the first column has widened and developed, and before the next one appears.

Figure 8.7 makes clear that there is generally a distribution of column widths in the final state. Although this is not surprising given the complicated nonlinear dynamics involved in column growth and their collisions with the container walls (and with each other), the local wavenumber variation seems to possess a degree of regularity as well. It can be seen in Figs. 8.6 and 8.7 that, while the energy barrier for adding new columns tends to increase with $\langle \mathcal{K} \rangle$, there is an apparent modulation

FIGURE 8.7: Density distribution (silicone oil shaded) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a rectangular container with $L \times 2H = 60\text{mm} \times 7.5\text{ mm}$. Darker lines show the resolution used, which is higher for $\mathcal{F} \lesssim 50$. The column formation cycle becomes asymmetric at $\mathcal{F} \simeq 40$.

of this trend when the column growth cycle switches from the symmetric (central) mechanism to the asymmetric one. That is, while the $\langle \mathcal{K} \rangle = 7$ mode exists over a relatively wide range ($42.5 \lesssim \mathcal{F} \lesssim 61.5$), the next mode, $\langle \mathcal{K} \rangle = 8$, is found over a noticeably smaller interval ($61.5 \lesssim \mathcal{F} \lesssim 75$). Following that, the $\langle \mathcal{K} \rangle = 9$ mode is again observed over a large interval ($75 \lesssim \mathcal{F} \lesssim 106$). This alternation, although inferred from limited data, suggests hysteresis in the asymmetric growth mechanism, and that it may be useful to consider asymmetric mode transitions in pairs; a more systematic investigation of this will be undertaken elsewhere.

When considering this symmetry-breaking transition, it is worth clarifying again that the symmetry referred to for the column growth cycle is an average, or approximate, one. The horizontal forcing is not symmetric on the fast timescale since the motion of the two lateral walls is out of phase by π , as are the harmonic waves generated by them. There is always a certain degree of asymmetry present in the columnar state and this is amplified for the lowest $\langle \mathcal{K} \rangle = 1$ mode near onset, as seen in Fig. 8.7 for $7 \lesssim \mathcal{F} \lesssim 10$. One reason for the greater asymmetry in the lowest mode is its interaction with the upper and lower walls, something that will be discussed more in Sec. 8.5.1.

8.4.4 Approaching the infinitely extended case

Considering the results described above for the frozen wave instability in microgravity, we can summarise the qualitative features suggested for the transition from finite to infinitely extended containers.

- (i) The forcing interval associated with each frozen wave mode decreases, approaching zero (Sec. 8.4.1);
- (ii) The absolute instability threshold decreases, approaching zero (Sec. 8.4.2);

- (iii) The cycle of column growth that characterises mode transitions becomes increasingly asymmetric, with defects appearing at various points along the container (Sec. 8.4.3).

The first two of these statements are natural consequences of approaching the extended limit where arbitrary wavenumbers are permitted and the threshold of long-wave perturbations is known to vanish for inviscid fluids (Lyubimov and Cherepanov, 1986). The third is more speculative, since the appearance (or not) of defects in the infinite system, where discrete mode transitions do not generally occur, is unknown. It may be that the asymmetric column growth cycle observed here at larger forcing values eventually fades, giving way to more regular and uniform patterns.

It is also worth noting that the scaling of $\mathcal{F}$ with $\mathcal{K}$ observed here, which is very roughly $\mathcal{F} \propto \mathcal{K}^2$ (see, e.g., Fig. 8.6) is different, at least over the range of parameters considered, than that predicted by Lyubimova et al. (2017) for extended domains and inviscid fluids: $\mathcal{F} \propto \mathcal{K}$ for small k and $\mathcal{F} \propto \sqrt{\mathcal{K}}$ for large k . This suggests [see Fig. 8 of Lyubimova et al. (2017)] that, besides finite-size effects, viscosity may be playing a crucial role.

8.5 Effect of container height

In addition to the effects that finite length has on wavenumber selection, mode transitions and defect growth, the development of the frozen wave instability is also significantly affected by the container height. We describe here how the instability diagram changes if the length L is fixed and the height $2H$ is varied. The development of small wavenumber patterns in thin containers is frustrated by collisions with the boundaries, as discussed in Sec. 8.5.1, while thick containers promote more asymmetry in the final frozen wave state, as discussed in Sec. 8.5.2.

Figure 8.8 shows the instability diagram in terms of the average wavenumber $\langle \mathcal{K} \rangle$ for fixed length $L = 30$ mm, and different container heights: $2H = 3.75$ mm, 7.5 mm and 15 mm. As the container height is increased, the interval of the vibrational parameter $\mathcal{F}$ over which each mode is observed tends to increase as well (the addition of a new column requires more energy). The pattern selection process is generally similar to that described in Sec. 8.4 with increasing $\mathcal{F}$, although there are differences when the height is much less, or much greater, than the wavelength. For $2H = 3.75$ mm, for instance, the thin container frustrates the development of the largest wavelength pattern, $\langle \mathcal{K} \rangle = 1$, which is found only over a very small range of $\mathcal{F}$. In this domain, the second mode, $\langle \mathcal{K} \rangle = 2$, exists over a much wider range; an explanation for this is given below.

8.5.1 Frozen wave instability in shallow containers: suppression of small wavenumbers

When the height of the container is sufficiently small, the patterns of lowest wavenumber, $\langle \mathcal{K} \rangle = 1, 2$, etc., are frustrated before they can fully develop. Low frequency vibrations, particularly if the forcing amplitude $A \sim H$, enhance the harmonic motion produced by the endwalls. This harmonic excitation, as mentioned, is out of phase at the left and right walls. As soon as the amplitude near one (or both) of the endwalls is large enough for the interface to contact the upper wall, the

FIGURE 8.8: Nondimensional wavenumber $\langle \mathcal{K} \rangle$ of the selected columnar pattern versus the vibrational parameter $\mathcal{F}$ for $L = 30$ mm and different container heights (labelled): $2H = 3.75$ mm (red, medium thickness), 7.5 mm (grey, thicker) and 15 mm (black, thinner). Container heights are labelled for reference.

dynamics is controlled in large part by surface and contact forces, and a column rapidly forms along the lateral wall. This process repeats, usually in alternating (asymmetric) fashion, inwards along the container and, taken together, acts to select a pattern of higher average wavenumber than the initial perturbation.

Figure 8.9 illustrates this process in a $L \times 2H = 30 \text{ mm} \times 1.875 \text{ mm}$ container vibrated at 12 Hz with $A = 1.5 \text{ mm}$ (note $A > H$). When the excitation has been applied for 5 s, a long wavelength perturbation is clearly visible on the interface. It grows in amplitude, especially near the endwalls, something that is likely promoted by the vibroequilibria effect, wherein the inhomogeneous oscillatory velocity field drives a reorientation of the interface that, for a horizontally vibrated interface, causes the heavier fluid to move up along the sides, leaving a trough in the centre (Fernández et al., 2017a,b; Ganiev, Lakiza, and Tsapenko, 1977). Harmonic waves (at the forcing frequency) are also excited, though these cannot be seen in the figure due to the averaging over one period. Eventually, one contact line traverses the upper corner of the cell and the interface dynamics is then locally driven by surface and contact forces acting to maintain the prescribed contact angle with the upper boundary. In this region undergoing rapid motion of the contact line, the average velocity field is considerably larger than in the rest of the container, as seen at $t = 10.325 \text{ s}$. Shortly after, the opposite (left) side undergoes the same process, while the first column (on the right) stabilises and the vigorous motion of the interface is transported inwards in preparation for another collision with the upper wall. In this manner, successive collisions of the interface with the upper and lower walls dynamically increase the wavenumber of the columnar pattern from $\langle \mathcal{K} \rangle \simeq 2$ at onset to $\langle \mathcal{K} \rangle = 3$ in the final state, seen at $t = 12.5 \text{ s}$. Given this strongly nonlinear and asymmetric mechanism of column formation, the final structure is surprisingly symmetric, although not completely so.

FIGURE 8.9: Evolution of the frozen wave instability (averaged over one excitation period) in weightlessness in a $L \times 2H = 30 \text{ mm} \times 1.875 \text{ mm}$ rectangular container vibrated with $f = 12 \text{ Hz}$ and $A = 1.5 \text{ mm}$, corresponding to $\mathcal{F} \approx 7.59$. Snapshots are labelled by time after initiation. The mean velocity field is shown in all but the final snapshot (scaling of arrow length is not constant between snapshots). Note the markedly asymmetric evolution of the interface, especially after the first (right) corner of the cell is covered by FC-40.

FIGURE 8.10: Density distribution (silicone oil shaded; darker lines show resolution used) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for $L = 30 \text{ mm}$ and container heights (a) $2H = 3.75 \text{ mm}$ and (b) $2H = 1.875 \text{ mm}$. The thinner layer not only displays more asymmetry, but frustrates the development of patterns with large wavelengths: modes with $\langle \mathcal{K} \rangle = 1, 2$ are not observed when $2H = 1.875 \text{ mm}$.

Figure 8.10 shows the final steady fluid density distributions along the midline $y = 0$ as function of the vibrational parameter in a container of length $L = 30 \text{ mm}$ for two different container heights: (a) $2H = 3.75 \text{ mm}$ and (b) 1.875 mm . In the thinner container, both of the first two modes, $\langle \mathcal{K} \rangle = 1, 2$, are prevented from developing into columnar structures by the process of repeated collisions with the upper and lower walls just described. The final columnar structure has evident asymmetry over most of the forcing range considered.

In the thicker container with $2H = 3.75 \text{ mm}$, the $\langle \mathcal{K} \rangle = 1$ mode is found only over a narrow range

FIGURE 8.11: Time evolution of the frozen wave instability in weightlessness in a $L \times 2H = 30 \text{ mm} \times 30 \text{ mm}$ rectangular container at $f = 60 \text{ Hz}$, $A = 1.5 \text{ mm}$, resulting in $\mathcal{F} \approx 37.97$. Snapshot times are labeled. The mean velocity field — averaged over one excitation period — is shown by scaled arrows (scaling not preserved between snapshots) for all but the final time. Asymmetric evolution driven by the harmonic motion of the lateral column crests is observed.

that depends on the complicated dynamics of the transient stage related with the movement of the contact line. The $\langle \mathcal{K} \rangle = 2$ mode is considerably easier to obtain.

8.5.2 Frozen wave instability in deep containers

We now consider containers whose horizontal and vertical dimensions are comparable so $\Gamma \sim \mathcal{O}(1)$. Compared to more shallow containers of the same length, the frozen wave columnar modes exist over a wider interval of vibrational parameters and greater asymmetry is observed in the final state, except very close to onset. The mechanisms behind this are described below.

The snapshots shown in Fig. 8.11 illustrate the time evolution of a frozen wave pattern in a square container, $L \times 2H = 30 \text{ mm} \times 30 \text{ mm}$, vibrated with $f = 45 \text{ Hz}$ and $A = 1.5 \text{ mm}$. When the forcing is first turned on, frozen waves grow in the same manner described in Sec. 8.3.1, with larger deformation near the lateral walls and smaller perturbations in the centre of the container (see the snapshots for $t = 0.1 \text{ s}$ and $t = 0.325 \text{ s}$). Compared to thinner containers of identical length (see Fig. 8.2) the development of the columnar structure takes somewhat longer. The longer columns

FIGURE 8.12: Density distribution (silicone oil shaded; darker lines show resolution used) along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a deep container with $L \times 2H = 30\text{mm} \times 30\text{ mm}$. Asymmetric behavior is observed quite generally for $\mathcal{F} \gtrsim 20$.

and increased transient time amplify the effect of the harmonic waves, which are out of phase on each side, as well as other sources of asymmetry produced by the forcing. The final columnar patterns reflect this enhanced asymmetry, as seen in the snapshot at $t = 0.6\text{ s}$, which shows one column on the left that has already reached the lower wall while its counterpart on the right has not. Substantial asymmetry persists in the final steady columnar structure, as seen for $t = 1.4\text{ s}$.

The final columnar state ($t = 1.4\text{ s}$) also has an asymmetric distribution of subharmonic interfacial waves, most prominent on the right side of the central column. These Faraday waves oscillate at half the forcing frequency (recall that the figures show an average over one excitation period) and introduce a second length scale to the frozen wave columnar structure. Solutions of this type were previously observed in microgravity experiments in horizontally vibrated containers with miscible liquids by Shevtsova et al. (2016) and with immiscible liquids by Salgado Sánchez et al. (2019d).

Figure 8.12 shows the fluid (density) distribution as a function of the vibrational parameter $\mathcal{F}$. Other than a small range near onset for the $\langle \mathcal{K} \rangle = 1$ mode, the increased asymmetry of the final structure in this deeper container is evident.

8.6 Dependence of the instability threshold on aspect ratio

Here, we focus on the onset of the first frozen wave mode with $\langle \mathcal{K} \rangle = 1$ and compare the thresholds obtained numerically with the theoretical prediction for inviscid fluids in infinitely extended layers. Since this data is obtained by varying the length L , it is convenient to rescale using H so that the nondimensional parameters characterising the problem are independent of L . In analogy with Eqs. (8.9) and (8.11), we define the dimensionless velocity parameter and capillary number scaled

FIGURE 8.13: Frozen wave threshold as a function of the inverse aspect ratio Γ^{-1} . Numerically obtained onset values (solid dots with dashed line for visual reference) were calculated for a capillary number $\text{Ca}_H = 0.0317$. The simulation values are slightly higher than the theoretical prediction (solid line) of Eq. (8.19) for $\Gamma^{-1} \lesssim 0.5$ but deviate substantially beyond that as finite size effects play a dominant role.

with H :

$$\mathcal{F}_H = A\omega \sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad \text{Ca}_H = \nu_1 \sqrt{\frac{(\rho_1 - \rho_2)}{\sigma H}}. \quad (8.17)$$

The threshold prediction (Lyubimova et al., 2017) for the inviscid system is

$$(A\omega)^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)^2} \sigma k \tanh(kH), \quad (8.18)$$

where k is the dimensional wavenumber. Writing this in terms of $\mathcal{F}_H$ and using $k = 2\pi/L$ for the first mode gives the critical curve

$$\mathcal{F}_H^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)} \left(\frac{\pi}{\Gamma}\right) \tanh\left(\frac{\pi}{\Gamma}\right), \quad (8.19)$$

in terms of the aspect ratio Γ .

Figure 8.13 provides a comparison of the threshold predicted by Eq. (8.19) (shown with a solid line) and the results of Sec. 8.4 (marked with solid dots). For $\Gamma^{-1} \lesssim 0.5$ the calculated onset values are about 10–20% higher, but reasonably close to the theoretical prediction, with the difference likely explained by viscosity (Talib, Jalikop, and Juel, 2007) and possibly also vibroequilibria effects (Fernández et al., 2017a,b; Ganiev, Lakiza, and Tsapenko, 1977). As Γ^{-1} becomes larger than 0.5 (short containers), the frozen wave instability is increasingly delayed by finite-size effects, diverging decisively from the theoretical prediction.

8.7 Conclusions

An extensive investigation of the columnar patterns triggered by the frozen wave instability in horizontally vibrated immiscible layers in weightless conditions was carried out numerically. The corresponding time-dependent problem was solved using a level-set formulation, as described in Sec. 8.2. Our principal interest was in determining how pattern selection is affected by the finite size of the container, particularly for experimental configurations and liquids similar to those considered in Salgado Sánchez et al. (2019d).

The evolution of the interface after the frozen wave instability was described in Sec. 8.3, including the column growth cycle defining mode transitions. An instability diagram showing the selected wavenumber for a given forcing was presented. In contrast to the continuous monotonic dependence of wavenumber on forcing predicted by linear inviscid theory neglecting lateral boundaries (Lyubimova et al., 2017), the finite domain selects distinct solution branches that persist over finite intervals of the forcing.

The consequences of varying container length for fixed height were considered in Sec. 8.4. In particular, the frozen wave threshold decreases with increasing length, consistent with the requirement of inviscid theory (Lyubimova et al., 2017; Lyubimov and Cherepanov, 1986) that it vanish in the infinitely extended case. The onset of frozen waves in short containers is significantly delayed. In longer containers, where higher wavenumbers can be studied, an interesting transition was found. The approximately symmetric (central) column growth cycle governing mode transitions at lower forcing is replaced at a certain point by a more complex, asymmetric column growth cycle, with the loss of average symmetry clearly visible in the mean velocity field.

The results of varying container height for fixed length were discussed in Sec. 8.5. In thin containers, the development of large wavelength patterns is frustrated by repeated collisions with the upper and lower boundaries, where surface and contact forces play a key role. This highly nonlinear and asymmetric process acts to increase the average wavenumber of the final columnar structure, which may be relatively symmetric nonetheless. In thick containers, the asymmetry of the forcing and the associated harmonic surface waves, which are out of phase at the left and right endwalls, is amplified by the longer transient growth time, leading to final columnar structures with more asymmetry.

The onset values obtained from numerical simulations for the lowest $\mathcal{K} = 1$ mode were quantitatively compared with the prediction for inviscid, extended layers of fluid (Lyubimova et al., 2017) for different aspect ratios Γ . Reasonable agreement was found for $\Gamma \gtrsim 2$, with a small delay likely due to viscosity (Talib, Jalikop, and Juel, 2007), but for $\Gamma \lesssim 2$, a more substantial increase in onset was seen, suggesting a limit for the applicability of theoretical results relying on the neglect of lateral boundaries.

Several aspects of the results described here suggest themselves for further investigation, an effort that will be undertaken separately. First, the vibroequilibria effect, which was mentioned briefly,

clearly has a role in selecting the final interfacial structure. Evidence of this can be found in the fact that the denser liquid (FC-40) *always* ends up along the lateral walls, despite the fact that the frozen wave instability does not select for this. The vibroequilibria effect, however, does imply that the denser fluid should move up these walls rather than down them (Fernández et al., 2017a,b; Ganiev, Lakiza, and Tsapenko, 1977). This effect also partly explains the difficulty of accurately measuring the frozen wave threshold in very long containers.

Finally, the manner in which the forcing is initiated is undoubtedly important. Here, motivated by parabolic flight experiments (Salgado Sánchez et al., 2019d), the forcing was turned on relatively quickly. This led to larger disturbances and more rapid growth of the frozen waves near the lateral walls, with the central column being the last to develop and the most likely to suffer defects due to wavenumber frustration, at least with forcing amplitudes below that of the asymmetric regime. In fact, the location of the column-generating defects in the more disordered asymmetric regime is, presumably, also related to the forcing function used and merits further investigation as well, particularly regarding the transition to the extended limit where discrete mode transitions are not anticipated.

Acknowledgment. The work of Y. A. Gaponenko and V. Shevtsova was supported by the PRODEX programme of the Belgian Federal Science Policy Office. The work of P. Salgado Sánchez and J. Porter was supported by the Ministerio de Economía y Competitividad under Project No. ESP2015-70458-P. We also thank the Departamento de Aeronaves y Vehículos Espaciales in the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio, Universidad Politécnica de Madrid for supporting the stay of P. Salgado Sánchez at the Microgravity Research Centre, Université Libre de Bruxelles.

Chapter 9

Effect of initial interface orientation on patterns produced by vibrational forcing in microgravity

Salgado Sánchez, P., Gaponenko, Y., Yasnou, V., Mialdun, A., Porter, J. and Shevtsova, V.
Journal of Fluid Mechanics **884**, A38-1 (2020).
<https://doi.org/10.1017/jfm.2019.955>

When a container with two distinct fluids is subjected to vibrations in microgravity, the interface may undergo a variety of instabilities and develop towards a complex structure, as seen in recent parabolic flight experiments using both miscible and immiscible liquids. Among other things, the selected pattern depends on the frequency and amplitude of the forcing and, crucially, on its orientation with respect to the initial interface. In a parabolic flight experiment, this initial orientation is largely determined by the stage of the parabolic manoeuvre when the forcing is started and the residual gravity level during the period of microgravity. It plays a key role in the appearance of defects and irregularities during the evolution of the interface triggered by the frozen wave instability. Using numerical simulations, we systematically investigate the effect of initial interface orientation on pattern selection in microgravity for both miscible and immiscible fluids, and compare to available experiments. When the interface and the forcing are nearly aligned, the frozen wave instability is dominant, leading to the development of approximately regular columnar patterns. As the initial angle becomes more oblique, the frozen wave growth becomes more irregular and asymmetric and may involve thin auxiliary columns. Sufficiently large angles suppress the frozen wave instability and, depending on the container aspect ratio, may result in a simple two-column final state.

9.1 Introduction

The instabilities suffered by free surfaces or interfaces are of long-standing interest in fluid dynamics, with relevance to physics, biology, and engineering applications such as solvent extraction, oil recovery and mixing. Gravitationally unstable configurations are susceptible to the Rayleigh-Taylor instability (Kull, 1991), which tends to generate fingers or plumes. Temperature differences can induce density gradients, leading to buoyant (gravitational) convection (Batchelor, 1954) in the bulk, but also surface tension gradients, leading to thermocapillary (Bénard-Marangoni) convection (Birikh, 1966). If there is a jump in velocity across the interface, the resulting shear force can lead to a Kelvin-Helmholtz instability (Miles, 1959), which is involved in familiar phenomena such as wind-driven water waves and billow (Kelvin-Helmholtz) clouds. Moving boundaries (or time-varying forcing) generally drive interfacial waves and, depending on the form and direction of the motion, such waves can be standing or travelling, localised or distributed.

The dynamics arising from these instabilities, and fluid behaviour in general, can be dramatically affected by the diminution or absence of gravity. An obvious effect of reducing gravity for moderate or large fluid systems is a weakening of the usual preference for flat surfaces perpendicular to the direction of gravity. Surface tension favours configurations of minimal area (perfectly spherical drops, for example) while contact forces may draw the fluid into covering (wetting) adjacent solid boundaries. Broadly speaking, in the absence of strong gravity, fluids are more apt to move and change shape and, among other things, this plays a critical role in the interaction and evolution of the interfacial instabilities that occur.

Some notable interfacial phenomena are significantly enhanced in low gravity environments including vibroequilibria (Beyer et al., 2001; Fernández et al., 2017b; Lyubimov et al., 1997), which involve the reorientation of the (average) fluid interface due to vibrations, and instabilities of Kelvin-Helmholtz type. For a flat interface separating two fluids and subjected to periodic horizontal forcing, this latter case leads to a quasistationary undulatory surface deformation referred to as a frozen wave (Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007; Wolf, 1969).

The critical amplitude for the frozen wave instability between two infinitely long layers of inviscid immiscible fluids of height H , densities ρ_1 and ρ_2 ($< \rho_1$), and interfacial tension σ , was found by Lyubimov and Cherepanov (1986) via a linear stability analysis:

$$(A\omega)^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)^2} \left[\sigma k + \frac{(\rho_1 - \rho_2)g}{k} \right] \tanh(kH), \quad (9.1)$$

where A is the amplitude of the vibrational displacement, ω is the frequency, g is the gravitational acceleration normal to the interface and k is the perturbation wavenumber. This predicted threshold was examined by Talib, Jalikop, and Juel (2007) experimentally and theoretically, where it was seen that the inviscid theory consistently underestimates the threshold for liquids of similar viscosity, while generally overestimating it for layers with large viscosity contrast. Despite the complicating effect of viscosity, it is clear from Eq. (9.1) that the wavelength of the first unstable mode is set by

the capillary length for sufficiently long and deep layers. In microgravity, on the other hand, long-wave modes are expected to be the most dangerous since the threshold without viscosity vanishes as $k \rightarrow 0$.

The pattern selection problem for the frozen wave instability in microgravity is thus more complicated than in ordinary gravity, and relatively few studies have been conducted. The work of Beysens et al. (2009) in sounding rockets was one of the first, and reported the existence of a stripe pattern in supercritical CO₂ that was later related to frozen waves. The investigations were continued on ground using magnetic compensation in supercritical fluid, and both frozen waves (Gandikota et al., 2014b) and Faraday waves (Gandikota et al., 2014a) were studied. The pattern selection process in periodically excited miscible liquids has recently been studied both under normal gravity and in microgravity conditions (Gaponenko et al., 2015a,b; Shevtsova et al., 2016). A theoretical analysis was conducted (Lyubimova et al., 2017) by assuming that the wavenumber at a given forcing amplitude is controlled by the fastest growing perturbation. In the infinitely extended limit, the dependence of this wavenumber on forcing is monotonic and continuous.

One of the more striking features of the frozen wave instability in microgravity is the rapid growth of large columnar structures (Gaponenko et al., 2018; Shevtsova et al., 2016). This is due both to the very low threshold (Lyubimova et al., 2017) and to the weak restoring force, especially in the case of miscible liquids where the effective interfacial tension is very small. Such columnar structures have the effect of rotating large portions of the interface into a nearly perpendicular orientation with respect to the forcing, which, in turn, allows Faraday waves to form when their threshold is surpassed. The interfacial structures resulting from this type of interaction are complex, with features on multiple length and time scales (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016). These two-scale interfacial patterns were first observed in parabolic flight experiments (Shevtsova et al., 2016) and, although the wavenumber is dependent on the vibrational velocity (Gaponenko et al., 2015a) in accordance with predictions of linear theory, there are a number of additional factors that play an important role in selecting the experimental patterns.

One of these factors is the level of residual gravity, whose effect on the frozen wave instability was considered by Gaponenko and Shevtsova (2016). It was found that the pattern wavenumber increases with gravity, and that the development of the waves can be divided into three different gravity regimes according to the amplitude and vertex angle of the triangular (nonlinear) waveform. The effect of the initial diffusion layer thickness in the case of miscible liquids was then examined by Gaponenko et al. (2018) through a series of experiments performed in consecutive parabolic manoeuvres without resetting the interface. Due to diffusion, the interface thickness increased with successive experiments, which reduced the shear force and promoted columnar structures of smaller wavenumber. Still further mixing revealed a transition to a vortical flow structure.

Interfacial patterns in immiscible liquids were studied by Salgado Sánchez et al. (2019d) under the microgravity conditions of a parabolic flight. Both experiments and simulations were considered, and a complex interaction was found between faster phenomena, like interfacial waves and drops, and

slower phenomena like frozen waves. The two-scale wave patterns combining a columnar structure with subharmonic Faraday waves were observed for the first time in immiscible liquids. Finite-size effects, whose influence was also apparent in this study, were systematically investigated by Salgado Sánchez et al. (2019c), where the process governing discrete mode transitions with increasing vibrational velocity was described.

The parabolic flight experiments showed a range of complex behaviour involving defects and non-reproducible columnar patterns that could not be explained by these previous studies. It is clear that, aside from fluid properties and the amplitude of the applied forcing, there are other important factors including the angle between the initial interface and the axis of the forcing, which is difficult to control precisely in the experiments and may take values within a certain range depending on the residual gravity level and, most critically, on the stage of the parabolic manoeuvre when the forcing begins. We investigate here the question of how the patterns that develop due to periodic forcing in microgravity depend on the (average) angle between the forcing axis and the initial unforced interface. The importance of this parameter has not been examined before.

The evolution of these patterns and, more generally, the various fluid phenomena that can be excited by vibrations in microgravity are of interest for a number of reasons. It is important, for example, to understand the effects of residual gravity (g-jitter) on other fluid experiments (looking at diffusion, container-less processing, liquid bridges, etc.) and on fluid containment systems. Vibrations in microgravity have more potential to disrupt and alter fluid configurations since they need only overcome surface tension and adhesive forces. This point is illustrated by the dramatic difference between the frozen wave instability on ground, where small-amplitude waves are (typically) observed above onset, and in microgravity, where the threshold for long-wave perturbations approaches zero and the rapid formation of large columnar structures is observed. Such a large-scale redistribution of fluid caused by (in principal) small-amplitude vibrational forcing clearly has implications for fluid management strategies in space environments.

The organisation of the paper is as follows. The relevant results from parabolic flight experiments are presented first in Sec. 9.2, while the equations and numerical schemes used for simulations of miscible and immiscible liquids are described in Sec. 9.3. The results of systematic numerical simulations, including interfacial dynamics and wavenumber selection, are then discussed in Sec. 9.4. Conclusions are given in Sec. 9.5.

9.2 Microgravity experiment

9.2.1 Experimental setup

The microgravity experiments are performed in cuboidal containers of interior dimension $L \times 2H \times D = 15 \text{ mm} \times 7.5 \text{ mm} \times 5 \text{ mm}$ [see Fig. 9.1(a)]. The two large faces with area $L \times 2H$ are made of transparent quartz glass (Hellma) to allow observation of the fluids in the interior. The remaining boundaries are made from aluminium alloy, with a hole drilled through each; the holes in the upper and lower walls are used to fill the container, while those in the lateral walls are for drainage.

(a) Schematic view of optical setup (left) and experimental cell (right)

(b) Experimental cells holding immiscible liquids

(c) Configuration of immiscible liquid combinations I_a (upper) and I_b (lower) in microgravity

(d) Experimental cell holding miscible liquids

(e) Frozen wave pattern in mixture M_a (upper) and M_b (lower)

FIGURE 9.1: (a) Schematic view of the optical setup (left) and the experimental cell (right). (b,d) Images of the experimental cells holding immiscible and miscible liquids, respectively. (c,e) Unprocessed cropped images showing the interface in reduced gravity: (c) unforced immiscible combinations (see Table 9.1) I_a (upper container) and I_b (lower container); the lighter regions of the image correspond to FC-40 and the darker regions to silicone oil, (e) miscible liquids under horizontal (x axis) forcing: mixture M_a (upper snapshot) and M_b (lower snapshot).

(a) Miscible combinations				
	$\tilde{c}_0$, water (mass fraction)	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)	D (10 ⁻¹⁰ m ² /s)
M _a (1):	0.9	980.9	1.44	1.76
M _a (2):	0.5	902.4	3.39	1.76
M _b (1):	0.67	941.0	2.88	2.85
M _b (2):	0.27	847.5	3.21	2.85
(b) Immiscible combinations				
	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)	σ (10 ⁻³ N/m)	
FC-40 (1)	1855	2.2	16.63	
2 cSt silicone oil (2)	873	2	18.63	
20 cSt silicone oil (2)	950	20	20.85	
Interfacial properties:				
I _a : FC-40 / 2 cSt oil	–	–	3.927	
I _b : FC-40 / 20 cSt oil	–	–	6.021	

TABLE 9.1: Physical properties of the experimental fluids at 25°C. (a) Miscible water–IPA combinations: density ρ , kinematic viscosity ν , and diffusion coefficient D [from Mialdun et al. (2012)]. (b) Immiscible combinations of FC-40 (Flourinert Liquids™) and silicone oil: density ρ , kinematic viscosity ν , and surface tension σ [from Someya and Munakata (2005)].

The experimental containers hold selected combinations of miscible or immiscible liquids, whose relevant physical properties are given in Table 9.1. In the miscible experiments, two layers of water-isopropanol (IPA) with different concentrations are used. The initial concentrations (mass fractions) of water in the two miscible layers were either 0.9 and 0.6 (mixture M_a in Table 9.1), or 0.67 and 0.27 (mixture M_b in Table 9.1). In the immiscible experiments, two different silicone oils (2 cSt and 20 cSt) are used in combination with FC-40, a denser fluorocarbon oil. These immiscible combinations (respectively, I_a and I_b in Table 9.1) have similar density ratios and interfacial tension, but differ by a factor of 10 in the viscosity ratio.

The position of the interface is monitored through (one of) the transparent sides of the container. In the immiscible case, the silicone oil is coloured with the additive Sudan IV so that its transparency is different from that of the FC-40 [see Fig. 9.1(b)]. The two liquids thus appear with different shading when illuminated from behind, as seen in the greyscale images of Fig. 9.1(c). In the miscible experiments, the interface is revealed by a shadowgraphy technique based on the variation of refractive index with concentration. Refraction caused by the large gradient across the interface makes it appear dark, as seen in the images of Fig. 9.1(e).

The image acquisition system uses light from a collimated LED source, which is directed towards the experimental cell by a series of three mirrors. The image of the liquids and their interface is redirected by a final mirror before passing through a set of lenses to adjust the field of view and

resolution (see Fig. 9.1(a) and Gaponenko et al. (2015a) for more details). The CCD camera is set to record images at 48 or 80 frames per second. The higher rate is used for all experiments with immiscible liquids and provides very good resolution of interfacial waves at low or moderate frequencies. Even at higher forcing frequencies, there is no practical difficulty with stroboscopic effects since the frequencies of the relevant interfacial modes are known. At the highest forcing frequency where oscillatory interfacial waves were recorded (40 Hz), there are enough points (two and four, respectively) to clearly distinguish harmonic from subharmonic waves.

The experimental cell, along with the lenses and camera, is mounted on a translation assembly attached to a linear motor that drives oscillations along the x -axis (the lengthwise dimension L) at the selected frequency $\omega = 2\pi f$ and amplitude A . The position of the motor is recorded by an encoder, which can be compared with the setting of the control software for calibration. The experiments using miscible liquids cover frequency and amplitude ranges of 2–24 Hz and 1.5–16 mm, respectively, while the experiments with immiscible liquids cover frequency and amplitude ranges of 5–50 Hz and 1–5.3 mm. Viewed from a reference frame attached to the container, the vibrated fluids are subjected to an effective gravity $\mathbf{G}(t)$ given by

$$\mathbf{G}(t) = \mathbf{g}(t) + A\omega^2 \cos(\omega t) \mathbf{i}, \quad (9.2)$$

where $\mathbf{i}$ is the unit vector along the x -axis. Note that, for convenience, we will continue to refer to the x direction as “horizontal” in microgravity, and to the two container boundaries parallel to the x axis as the “upper” and “lower” walls.

The microgravity experiments were conducted during the 59th and 62nd ESA Parabolic Flight campaigns, which provided repeated periods of reduced gravity lasting approximately 22 s (forcing was applied for 15 s). The residual gravity level during this period satisfies (Pletser et al., 2016)

$$|g_{x,z}| \leq 0.01 g_0, \quad |g_y| \leq 0.05 g_0, \quad (9.3)$$

where $g_0 = 9.81 \text{ m/s}^2$ and the x, y, z subscripts refer to the coordinate axes shown in Fig. 9.1(a). Conditions on the flight were nearly isothermal during the experimental campaign, as confirmed by a negative temperature coefficient (NTC) thermistor placed in the experimental cells. Over a full (daily) set of 31 parabolas, the temperature only increased by approximately 2°C. This justifies neglecting the variation of fluid properties with temperature, which is done in the simulations discussed below.

With miscible liquids, the diffusive interface naturally thickens during the experiments. Gaponenko et al. (2018) considered the effect of initial interface thickness on pattern selection after the frozen wave instability and found that a more diffuse interface promotes patterns of smaller wavenumber due to the reduced shear force. In the absence of convection, the mixing of the two liquids is slow and driven by diffusion. Vibrations strongly enhance mixing, to the extent that the interface thickens noticeably on a timescale of several seconds in the present experiments.

In order to have nearly constant initial conditions, a filling procedure is used to reinitialise the interface between experiments in the miscible case. Both liquids are injected simultaneously at an optimised flow rate by two identical pumps through the inlets on the upper and lower walls. Upon completion of the experiment, the liquids are drained through the lateral outlets. Flexible tubes [see Fig. 9.1(d)] connect the inlets with the filling system and the outlets with a waste bag; these do not constrain the mechanical vibration of the container. Valves are used to isolate the experimental cell and prevent interaction with the fluid in the tubes (Gaponenko et al., 2015a).

Experiments are characterised by the properties of the liquids used, by the forcing applied (frequency and amplitude) and by the start time of this forcing, i.e., whether vibrations begin during the hypergravity or microgravity portion of the given parabolic manoeuvre. If vibrations are initiated in hypergravity, the initial interface is relatively flat, although its orientation may not be completely parallel to the vibrational axis. If the vibrations are initiated during the microgravity period, then the interface has some time to move prior to forcing. With miscible liquids, the typical result of this is an increase in the average angle of the interface, which leads to a qualitative change in the interfacial pattern selected after forcing is applied.

The behaviour of immiscible liquids in microgravity is different due to interfacial tension, which causes the unforced interface to take a drop-like shape (partly attached to the walls); see Fig. 9.1(c). Such an interface cannot be well characterised by an average angle. The simulations of immiscible liquids presented in this paper are therefore not intended to reproduce the experiments initiated in microgravity, but rather to generalise and compare with those for miscible liquids and to complement results of Salgado Sánchez et al. (2019c), where pattern selection in finite containers was considered when starting from a perfectly flat (horizontal) interface.

Further details of the experimental setup and observed interfacial phenomena can be found in Gaponenko et al. (2015a) and Salgado Sánchez et al. (2019d).

9.2.2 Experimental results

The development of the frozen wave instability in miscible liquids is shown via four snapshots in Figs. 9.2 and 9.3, with mixture M_a (see Table 9.1) in panels (a–d) and mixture M_b in panels (e–h). For each mixture, the experiments in those two figures are performed with identical vibrational velocities but differ in whether the forcing starts during hypergravity (Fig. 9.2) or during microgravity (Fig. 9.3). The different orientations of the initial interface (just prior to forcing) that result from this choice have a marked effect on the final pattern shown in panels (d,h).

When forcing begins prior to the microgravity portion of a parabolic manoeuvre, as in Fig. 9.2, the initial interface [see panels (a,e)] is flat and only slightly tilted (approximately 5°). Depending on the specific profile of $\mathbf{g}(t)$, some features of the frozen wave growth under gravity may be visible. In Fig. 9.2(f), which shows the initial small-amplitude waves, the triangular waveform typical of the instability in gravity (Gaponenko and Shevtsova, 2016) can be seen since the transition to microgravity in this experiment has not yet occurred [it happens between panels (f) and (g) at

FIGURE 9.2: Snapshots (times labelled) showing the temporal evolution of the frozen wave instability in miscible layers subjected to horizontal vibrations starting from a slightly tilted interface. (a–d) Mixture M_a of Table 9.1 with $f = 14$ Hz, $A = 2.5$ mm ($A\omega = 219.9$ mm/s) and initial angle $\alpha \simeq 4.5^\circ$. (e–h) Mixture M_b of Table 9.1 with $f = 8$ Hz, $A = 11$ mm ($A\omega = 552.9$ mm/s) and initial angle $\alpha \simeq 5.0^\circ$. Microgravity begins at $t \simeq 0.75$ s and $t \simeq 1.2$ s, respectively, for these experiments.

$t \simeq 1.20$ s]. This contrasts with the small-amplitude waves in Fig. 9.2(c), which have more rounded crests since the microgravity period for this experiment begins, approximately, with panel (b) at $t \simeq 0.75$ s. Furthermore, a portion of the interface near the endwalls displays a “fish spine” type pattern, as described in (Shevtsova et al., 2015a), related to nonlinear effects when $A \sim H$.

The formation of columns spanning the height of the container is common to both mixtures, as seen in panels (d,h). The differences in the final (long-lived) patterns can be attributed mainly to the different applied vibrational velocity, which is the main control parameter of the frozen wave instability (Lyubimov and Cherepanov, 1986) and selects the wavenumber (number of columns) in the absence of gravity (Lyubimova et al., 2017); this can be extended to the miscible case if the diffusive interface is thin, behaving as if it had finite (small) interfacial tension (Lacaze et al., 2010).

After the formation of columns, large portions of the interface are nearly perpendicular to the applied forcing and subharmonic Faraday waves may be excited on them (Shevtsova et al., 2016). This secondary instability, which can be seen most clearly in panel (h), produces an interfacial structure with two length and time scales: quasisteady columns spanning the container height, originating in the frozen wave instability, and subharmonic standing waves on the column walls, resulting from the Faraday instability. These two-scale patterns were observed in the recent experiments of Shevtsova et al. (2016) and, in the case of immiscible liquids, by Salgado Sánchez et al. (2019d).

Faraday waves can occur even in the absence of both gravity and surface tension since during half of the forcing period the surface is susceptible to the Rayleigh-Taylor instability, while in the other half there is an inertial restoring force. The threshold (Shevtsova et al., 2016) depends on density and viscosity,

$$A = \frac{2}{\rho_1 - \rho_2} \sqrt{\frac{(\rho_1 + \rho_2)(\rho_1 \nu_1 + \rho_2 \nu_2)}{\omega}}. \quad (9.4)$$

Here, the subscripts 1 and 2 refer to the lower and upper fluids, respectively. Using the values from Table 9.1, Eq. (9.4) gives 8.6 mm for mixture M_a at 12 Hz and 10.3 mm for mixture M_b at 8 Hz. These estimates are consistent with Fig. 9.2 in that subharmonic waves are not seen for mixture M_a at $A = 2.5$ mm, while they are for mixture M_b at $A = 11$ mm.

When Faraday waves are excited on the sides of a column, they propel the interface back and forth during each oscillation period. This process gradually deforms the interface as waves interact with those on neighbouring column walls, or with the remnants of the interface (density gradients) left behind by previous wave cycles on the same column. This latter interaction tends to generate spike-like formations, which can be seen in Fig. 9.2(h) and, for numerical simulations, in Fig. 9.6(a,b) at $t = 3$ s. As this type of mixing proceeds, the waves gradually decay and disappear, leaving a more diffuse interface (compared to the case without surface waves). It is clear that, although parabolic flight experiments cannot properly investigate this slow process, the columnar structure itself will eventually be destroyed by diffusion as well.

When forcing begins in the microgravity period of a parabolic flight, as in Fig. 9.3, the unforced

FIGURE 9.3: Snapshots (times labelled) showing the temporal evolution of the frozen wave instability in miscible layers subjected to horizontal vibrations starting from a more tilted interface. (a–d) Mixture M_a of Table 9.1 with $f = 12$ Hz, $A = 2.91$ mm ($A\omega = 219.4$ mm/s) and initial angle $\alpha \simeq 27^\circ$. (e–h) Mixture M_b of Table 9.1 with $f = 8$ Hz, $A = 11$ mm ($A\omega = 552.9$ mm/s) and initial angle $\alpha \simeq 15^\circ$. These initial angles are measured in panels (b) and (e), respectively. The apparent breaks in the interface in (c,f) are optical effects.

interface may rotate towards a more oblique orientation, particularly due to the balance of residual gravity components in the x and y directions, as seen in panels (a,e). Even after forcing begins, the interface remains relatively flat at first [see panels (b,f)] without the clear frozen wave growth observed in Fig. 9.2. Instead, the vibration promotes continued rotation of the interface and an associated mean flow (see Sec. 9.4.1 for details). While the lateral portions of the interface bend along the upper and lower boundaries, the central portion continues to rotate towards a vertical orientation, sometimes overshooting somewhat [panels (c,g)] before settling into a long-lived final state with the two liquids basically occupying opposite sides of the container [panels (d,h)]. This final state is consistent with vibroequilibria theory (Fernández et al., 2017a,b; Gavriluk, Lukovsky, and Timokha, 2004), which, if gravity and surface tension are neglected, selects density contours perpendicular to the vibrational forcing.

Note that auxiliary frozen waves often develop during the latter stages of column development along the horizontal portions of the interface that lie close to the upper and lower walls. Their evolution is analogous to the frozen wave growth explained above, but with thinner columns (fingers) that may not reach the top/bottom.

Among other things, it is apparent from the experimental results discussed above that the initial surface orientation has a crucial influence on the dynamics of the interface and the structure that eventually develops. In the remainder of this paper, we use numerical simulations to systematically investigate the effect of initial orientation on pattern selection in both miscible and immiscible liquids.

9.3 Mathematical formulation and numerical models

This section summarises the equations and numerical methods used to simulate the vibrated liquids and follow the dynamics of the interface separating them. The simulations, for both miscible and immiscible liquids, are performed in a two-dimensional rectangular domain, a simplification that is justified by the following considerations.

The basic frozen wave instability produces two-dimensional patterns with a wavenumber aligned with the projection of the forcing axis along the interface. In the absence of certain secondary instabilities [see, e.g., Jalikop and Juel, 2012] there is no dependence on the crosswise direction. This was generally found to be true of the experimental patterns as well, with the main exception being the appearance of disordered three-dimensional patterns related to vigorous drop ejection at high forcing amplitude (Salgado Sánchez et al., 2019d), a situation that is not considered here. The two-dimensional numerical model used for the immiscible case was shown (Salgado Sánchez et al., 2019c,d) to reproduce the main features of the fluid behaviour related to frozen waves and Faraday waves, which are the dominant phenomena in the microgravity experiments of interest. For the case of miscible fluids, the two-dimensional nature of the interfacial patterns was confirmed by a direct comparison with three-dimensional simulations (Gaponenko et al., 2015a).

FIGURE 9.4: Miscible simulations are performed in a domain of length L and height $2H$. Vibrations of frequency ω and amplitude A are applied in the x direction. A solution is shown for illustration (the interface appears dark).

Curvature in the transverse direction and capillary waves are also generally negligible. In microgravity environments, applied vibrations induce (Douady, 1990) capillary phenomena on the lengthscale $l_c \sim \sqrt{\sigma/\rho(A\omega^2)}$. For the immiscible liquid layers of Table 9.1, the interfacial tension $\sigma \sim 10^{-3}$ N/m is relatively small, making $l_c \lesssim 1$ mm. Meniscus effects were inappreciable in the experiments, where the selected patterns were manifestly two-dimensional. For the experiments with miscible liquids, conducted on a timescale of tens of seconds, the gradients at the diffusive interface remain large enough to act as a type of effective interfacial tension. As discussed in Gaponenko et al. (2015a), the estimated interfacial tension of $\sigma \sim 10^{-6}$ N/m gives a very small l_c . Again, two-dimensional behaviour was clearly observed in the experiments, as well as the three-dimensional simulations (Gaponenko et al., 2015a).

9.3.1 Miscible case

The two miscible liquids are assumed to be incompressible and to occupy equal volumes in the rectangular domain of Fig. 9.4. The parameters of the liquids are those given for mixtures M_a and M_b in Table 9.1.

Let $\tilde{c}$ be the mass fraction of water and denote the initial concentrations in the i th layer as $\tilde{c}_{0i}$, $i = 1, 2$ (lower and upper layers, respectively). Regardless of the mixture used, we may use a rescaled relative concentration C ,

$$C = \frac{\tilde{c} - \tilde{c}_{02}}{\tilde{c}_{01} - \tilde{c}_{02}}, \quad 0 \leq C \leq 1, \quad (9.5)$$

which varies between 0 (initial upper layer) and 1 (initial lower layer). Both fluid density and viscosity depend on the concentration of the mixture. For the experiments of Sec. 9.2 using mixtures M_a and M_b (see Table 9.1), the density differences are not large, which justifies the use of a linear approximation for this dependence (Mialdun et al., 2012),

$$\rho = \rho_2 (1 - \beta_\rho C), \quad \beta_\rho = (\rho_2 - \rho_1)/\rho_2, \quad (9.6)$$

where $\beta_\rho = -0.0870$ for mixture M_a and -0.110 for mixture M_b . Viscosity μ , on the other hand, has a maximum at $\tilde{c}_0 = 0.5$ (Mialdun et al., 2012). For experiments using mixture M_a , we assume linear dependence

$$\mu = \mu_2 (1 - \beta_\mu C), \quad \beta_\mu = (\mu_2 - \mu_1)/\mu_2, \quad (9.7)$$

where $\beta_\mu = 0.538$, since the region of interest of concentration is $0.5 \leq \tilde{c}_0 \leq 0.9$. In experiments with mixture M_b , the viscosity is approximated (Mialdun et al., 2012) using the fourth order polynomial

$$\mu = (0.3719 C^4 + 1.0456 C^3 + 0.5679 C^2 + 1.2344 C + 2.7208) 10^{-3} \text{ Pa s}, \quad (9.8)$$

which provides the values for M_b (2) and M_b (1) given in Table 9.1 (using $\mu = \rho \nu$) when $C = 0, 1$, respectively.

The effective interfacial tension at the diffusive interface is several orders of magnitude smaller than that of most immiscible liquids (Gaponenko et al., 2015a). It is thus neglected at leading order and the system is treated as a single layer with a variable concentration; the interface is located by the contour $C = 0.5$, which is a region of large concentration gradient.

With no residual (constant) gravity, $\mathbf{g} = 0$ in Eq. (9.2), the force experienced by the liquid is

$$\mathbf{G}(t) = A\omega^2 \cos(\omega t) \mathbf{i}, \quad (9.9)$$

and momentum and mass transport equations read (Gaponenko et al., 2015a)

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \nabla (\mu \nabla \mathbf{u}) + \rho \mathbf{G}(t), \quad (9.10a)$$

$$C_t + \mathbf{u} \cdot \nabla C = D \nabla^2 C, \quad (9.10b)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (9.10c)$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, and D is the diffusion coefficient given in Table 9.1. No-slip and impermeability conditions apply on the container walls,

$$\mathbf{u} = 0, \quad \partial C / \partial n = 0, \quad (9.11)$$

where $\partial / \partial n$ denotes the normal derivative.

The initial flow is assumed to be quiescent at $t = 0$, with $\mathbf{u} = 0$, while the concentration begins as a step function (see Gaponenko et al., 2015a),

$$C_0(x, y) = \begin{cases} 0 & y - y_0(x) > 0; \\ 1 & y - y_0(x) < 0; \end{cases} \quad (9.12)$$

where y_0 is the interface location at $t = 0$. The angle between the vibrational axis and the initial interface, locally given by $\tan^{-1}(dy_0(x)/dx)$, is implicit in the choice of $y_0(x)$. We take a linear initial profile for $y_0(x)$ and vary this orientation between 0° (parallel to the vibrational axis) and

30°.

The commercial solver FLUENT v.17.2 is used for solving Eqs. (9.6)–(9.12) for mixtures M_a and M_b in the two-dimensional domain of Fig. 9.4. Time dependence is treated using a standard implementation of the Pressure-Implicit with Splitting of Operators (PISO) pressure-velocity coupling scheme, while spatial discretization is done using the Pressure Staggering Options (PRESTO) technique for the pressure equation and a third-order Monotone Upstream-Centred Scheme for Conservation Laws (MUSCL) technique for the momentum and mass transport equations. Numerical convergence was considered in Gaponenko et al. (2015a) and an orthogonal homogeneous discretisation of $N_x \times N_y = 400 \times 200$ was selected, with N_x and N_y the number of mesh elements along the x and y axes, respectively. The maximum timestep is set to 10^{-4} s to satisfy the Courant–Friedrichs–Lewy (CFL) condition at small mesh sizes.

9.3.2 Immiscible case

The two immiscible liquids are assumed to be incompressible and to occupy equal volumes of the rectangular domain shown in Fig. 9.5. The denser fluid is placed at the bottom so that the system is stable in the presence of gravity. Vibrations of amplitude A and frequency ω are again applied along the x -axis and the contact angle β is set to 90°. Note that the corners of this domain are rounded, with a radius of 0.25 mm. This is done to match the experimental container, which was machined with a drill bit of this radius, and has the advantage of eliminating the numerically singular behaviour that would otherwise occur as the interface moves across a corner.

A level-set approach (Olsson and Kreiss, 2005) is implemented for tracking the interface, with fluid properties defined as functions of the level-set variable ϕ , which obeys

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right). \quad (9.13)$$

Here, the parameter γ is set equal to the vibrational velocity $A\omega$, while ε is a small diffusive term added for numerical stability and taken to be on the order of the maximum mesh element size after convergence. A smoothed Heaviside step function $\mathcal{H}$ is defined (Salgado Sánchez et al., 2019d) by

$$\mathcal{H}(\phi) = \begin{cases} 0, 1 & \phi < -\varepsilon, \phi > \varepsilon \\ \frac{1}{2} + \frac{\phi}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\phi}{\varepsilon}\right) & |\phi| \leq \varepsilon, \end{cases} \quad (9.14)$$

and used to write the density ρ and viscosity μ as

$$\rho = \rho_1 + (\rho_2 - \rho_1)\mathcal{H}, \quad (9.15a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1)\mathcal{H}, \quad (9.15b)$$

where the subscripts 1 and 2 denote the lower and upper fluids, respectively, and correspond to $\phi = 0, 1$.

FIGURE 9.5: Sketch illustrating the immiscible numerical model: two superposed layers of immiscible liquids confined in a $L \times 2H$ rectangular container, displaying a finite contact angle β with the solid walls. The cell is subjected to vibrations of frequency ω and amplitude A in the x direction.

Equations (9.13)–(9.15) are solved together with the continuous incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991)

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \nabla (\mu \nabla \mathbf{u}) + \delta \mathbf{f}_\sigma + \rho \mathbf{G}(t), \quad (9.16a)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (9.16b)$$

The effect of surface tension at the interface is captured by the $\mathbf{f}_\sigma$ term in the momentum equation:

$$\mathbf{f}_\sigma = (\sigma \kappa) \mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \quad (9.17)$$

where κ is the curvature of the interface and $\mathbf{n}$ is the normal vector. These interfacial effects are localised by the prefactor δ , which can be approximated in terms of the level-set function by $\delta = |\nabla \phi|$.

Navier-slip boundary conditions, which introduce an artificial frictional force $\mathbf{f}_\nu = (\mu/\delta_\nu) \mathbf{u}$ to preserve contact angle β while permitting contact line motion, are applied on all solid walls,

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u} \cdot [\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta) \delta] = 0, \quad (9.18)$$

where $\mathbf{n}_w$ is the corresponding normal vector.

The applied acceleration is given by Eq. (9.2) while the initial velocity is zero throughout the domain. Because of numerical difficulties (related to contact angle) when starting from a prescribed initial interface, the different interface orientations are achieved by first allowing the unforced interface to relax with a fixed ratio of g_x/g_y . The residual gravity is then gradually reduced (while maintaining the same ratio of g_x/g_y) until reaching a magnitude below $0.01 g_0$. Since the corresponding capillary wavelength $\lambda = 2\pi\sqrt{\sigma/(\rho_1 - \rho_2)g} \geq 50 \text{ mm} > L$, this level of residual gravity ensures that the

critical wavelength is selected by the container size and the magnitude of the vibrational velocity (Salgado Sánchez et al., 2019d); it has a negligible effect on pattern selection when compared to the ideal zero-gravity case.

After the fluid reaches a hydrostatic equilibrium in this manner, the forcing is activated. A transient time of 0.25 s is used in place of the parameter ε in $\mathcal{H}(t)$ [see Eq. (9.14)] as a prefactor to smooth the onset of vibrations, as in Salgado Sánchez et al. (2019d). For the results presented below, we approximate the average interface orientation by measuring the slope in the centre of the container at the moment of forcing activation.

The commercial software COMSOL Multiphysics 5.3 was used for solving the governing two-dimensional equations. Convergence was tested previously in Salgado Sánchez et al. (2019d) for a similar configuration to select a maximum mesh size $H/12.5$. A Generalised- α (Erlicher, Bonaventura, and Bursi, 2002) integration scheme is used, which produces less numerical diffusion than the extended *Backward Differentiation Formulae* (BDF) schemes. The vibrated problem is initialized using a *Backward Euler* method for one step. The maximum timestep is set to $1/(10 f)$ and is automatically reduced to satisfy the CFL stability condition using *streamline* (Harari and Hughes, 1992) and *crosswind* (Codina, 1993) stabilisation techniques.

9.4 Numerical results

The results of a series of numerical simulations are presented in this section showing the effect of initial orientation on the dynamics of the interface as it evolves towards a quasisteady state. The final long-lived pattern is characterised by several complementary wavelength measurements.

9.4.1 Miscible liquids

A series of simulations using mixtures M_a and M_b were performed in the rectangular domain of Fig. 9.4 with $L \times 2H = 15 \text{ mm} \times 7.5 \text{ mm}$. Since the results were qualitatively similar, with the same type of dependence on initial inclination, only the (slightly clearer) results for mixture M_b are presented here.

Pattern development for different initial angles

The results of five numerical simulations are shown in Fig. 9.6, which can be compared with the experimental results shown in panels (e–h) of Figs. 9.2 and 9.3. The same horizontal vibration is applied, with $f = 8 \text{ Hz}$ and $A = 11 \text{ mm}$ ($A\omega = 552.9 \text{ mm/s}$), while different initial interface orientations of 0° , 5° , 10° , 20° and 30° are used in rows (a–e), respectively. The four snapshots in each row show the interface at $t = 0.2 \text{ s}$ (superimposed on the initial condition), 0.5 s , 1 s and 3 s . The mean velocity (averaged over two excitation periods) associated with the motion of the interface is also shown for the three earlier times, before the final quasisteady state is reached; note that the scale of the arrows is not preserved between frames.

FIGURE 9.6: Numerical simulations showing the temporal evolution of the frozen wave instability in mixture M_b with $f = 8$ Hz and $A = 11$ mm ($A\omega = 552.9$ mm/s) for increasing initial interface angles: (a) 0° , (b) 5° , (c) 10° , (d) 20° and (e) 30° . The columns show snapshots at $t = 0.2$ s, 0.5 s, 1 s and 3 s. The mean velocity field (averaged over two periods) is shown for the first three snapshots (arrow scale not maintained between frames); the solution at $t = 3$ s is quasisteady. The initial interface condition is shown in the leftmost panels at $t = 0.2$ s by a solid black line, while the interface is plotted in grey.

The evolution of the frozen wave instability from an initial horizontal interface, as illustrated in Fig. 9.6(a), was described by Gaponenko and Shevtsova (2016) for various gravity levels. Here, this case provides a reference to compare with the simulations initiated from a rotated interface. As soon as the forcing begins, approximately sinusoidal perturbations appear on the interface, most prominently near the lateral walls. These perturbations grow rapidly in the absence of strong gravity or surface tension to constrain them, and soon reach the upper and lower boundaries. The final long-lived structure takes the form of alternating columns. Note that the antisymmetric nature of the horizontal forcing — more precisely, the symmetry under horizontal reflection through the midline and translation by half a period of the forcing — is transmitted, in part, to the frozen wave structure. An investigation of similar structures in immiscible liquids (Salgado Sánchez et al., 2019c) found that the symmetry of the final state depends on the container size as well as the applied vibration. Although approximately symmetric patterns were found at moderate forcing, this symmetry can be lost as the ratio between the wavelength and the container length decreases.

When the initial interface is rotated by 5° , as in the simulations of Fig. 9.6(b), the evolution of the frozen wave instability is not substantially different from the horizontal case, although there are subtle changes. Comparing the snapshots at $t = 0.2$ s, one can see that the mean flow has more of a vertical component in Fig. 9.6(b), which indicates some rotation of this central portion of the interface. In this case, its effect is not strong enough to modify the final structure in a significant

way.

As the angle of the initial interface is increased further to 10° , as in the simulations of Fig. 9.6(c), the change in the mean flow is more obvious. In contrast to the horizontal and 5° cases, the mean flow in the centre of the cell is now stronger than the vortical flow near the lateral walls associated with the growth of the frozen waves. The rotation of the central portion of the interface during this initial period does not prevent the growth of frozen waves, which are visible on the inclined interface at $t = 0.5$ s. However, by $t = 1$ s, its continued rotation disrupts the regular growth of these interior frozen waves (columns), twisting them before they can extend across the height of the container. The final quasisteady state contains several anomalous thin columns in the central region.

When the angle of the initial interface is 20° , as in the simulations of Fig. 9.6(d), the rotation of the central portion of the interface is stronger and largely prevents the development of frozen waves into full columns. However, there are small perturbations along the horizontal portions of the interface, and two of these are noticeably enlarged at $t = 0.5$ s. These humps, or fingers, grow slowly into thin columns, although they do not fully reach the opposite boundary during the observation time. Since these thin fingers are qualitatively different from the “primary” frozen wave columns, they are considered auxiliary and are not included in the wavelength calculations presented below. Again, the pattern is approximately asymmetric with respect to horizontal reflection about the midline.

When the angle of the initial interface is 30° , as in the simulations of Fig. 9.6(e), the central region rapidly rotates towards a vertical orientation, first overshooting this somewhat [see the snapshot at $t = 0.5$ s and the experiment in Fig. 9.3(g)], before settling into a final quasisteady state with each of the liquids (almost, but not completely) occupying one half of the container.

Wavenumber selection for different initial angles

In Fig. 9.7, we represent the dependence of the final (long-lived) columnar structure on initial interface angle (α) with three dimensionless wavelengths (λ/L) measured along the midline $y = 0$: the minimum (blue), maximum (black) and average (red), which is the same as the inverse number of paired columns. The location of the interface is taken as $C = 0.5$. Snapshots showing the selected patterns for each angle are included, as is a sketch illustrating the calculation of minimum ($\lambda_{\min}$) and maximum ($\lambda_{\max}$) wavelengths.

For initial angles of 10° or less, the observed final states are qualitatively similar. Although the columns do change shape, the average wavenumber (half the number of abrupt concentration changes along the midline $y = 0$) does not change. The changing shape of the columns is captured by the maximum and minimum wavelengths, which separate from each other as the angle α increases. Qualitative transitions occur beyond $\alpha \simeq 10^\circ$, as the final pattern goes from one with three average wavelengths along the midline of the container, to one with two wavelengths at $\alpha \simeq 12.5^\circ$, to one with a single wavelength for $\alpha \gtrsim 17.5^\circ$ (the thin fingers seen with $\alpha = 20^\circ$, for example, are not included in the wavelength measurement).

FIGURE 9.7: Diagram showing three measured nondimensional wavelengths λ/L of the final quasisteady state versus initial interface angle α (in degrees): minimum (blue), average (red) and maximum (black). The vibrational velocity is held constant at $A\omega = 552.9$ mm/s ($f = 8$ Hz, $A = 11$ mm). Snapshots of the selected patterns are included, with the colourmap showing the concentration field (red for $C = 0$, blue for $C = 1$), as is a sketch illustrating how wavelengths are measured from the contour $C = 0.5$. Note that the snapshots are taken after a longer time of $t = 10$ s so are slightly different than those at $t = 3$ s in Fig. 9.6.

The type of pattern observed for inclinations of 20° and 30° is the limiting one for large angles. Increasing the initial angle does not change the number of (large) columns, which remains at two, but does lead to a more rapid approach to the final state with its central vertical interface and, correspondingly, less time for any type of frozen wave column to develop. Auxiliary fingers and irregularities along the central boundary both diminish with greater initial inclination.

9.4.2 Immiscible liquids

The simulations of immiscible liquids were performed in the rectangular domain (with rounded corners) shown in Fig. 9.5. As in Salgado Sánchez et al. (2019c), we do not use the real viscosity of FC-40 given in Table 9.1, but increase this to 5 cSt in order to delay the onset of secondary Faraday waves. Otherwise, the parameters are those of combination I_b .

Pattern development for different initial angles

Figure 9.8 shows the evolution of the frozen wave instability in a rectangular cell of size $L \times 2H = 30$ mm $\times$ 7.5 mm for four initial interface orientations: (a) 0° , (b) 7° , (c) 7.2° and (d) 14.15° . The amplitude and frequency are set at $A = 1.5$ mm and $f = 45$ Hz. The mean velocity field (averaged

FIGURE 9.8: Numerical simulations illustrating the development of the frozen wave instability in weightless in a $L \times 2H = 30 \text{ mm} \times 7.5 \text{ mm}$ container with $f = 45 \text{ Hz}$ and $A = 1.5 \text{ mm}$. The time of each snapshot is given under the corresponding column. The mean velocity (averaged over two excitation periods) is illustrated by scaled arrows (scale not preserved between panels) during the transient phase. The angle α of the initial interface is varied: (a) 0° , (b) 7° , (c) 7.2° and (d) 14.16° . The manner of calculating maximum ($\lambda_{\max}$) and minimum ($\lambda_{\min}$) wavelengths along the midline $y = 0$, used in Fig. 9.9, is illustrated for the final state of (c). The initial interface is shown (solid black curve) for comparison in the rightmost panels.

over two excitation periods) is also shown for the two earlier snapshots, while the final image includes the initial interface for comparison.

The development of the frozen wave instability from an initially flat horizontal interface, as illustrated in Fig. 9.8(a), was described in detail by Salgado Sánchez et al. (2019c). The pattern selection process was shown to be characterised by a cycle of column growth. The oscillatory vertical motion of the interface near the lateral boundaries creates a disturbance that tends to initiate the growth of frozen waves. Perturbations grow first near these walls, but propagate inwards over time, as revealed by the horizontal component of the mean velocity at $t = 0.2 \text{ s}$ [see also Dore (1973) and Shevtsova et al. (2015a)]. These frozen wave perturbations grow rapidly, approaching the upper and lower boundaries; see Fig. 9.8(b) at $t = 0.4 \text{ s}$. After a bit more time, they settle into quasisteady (nearly) vertical columns, as in the snapshot at $t = 1.5 \text{ s}$. Note the average left-right reflection symmetry of the solution in this case, both for the mean flow associated with the transient states and for the interface.

If the angle of the initial interface is increased slightly, the mean flow field changes substantially, as demonstrated in Fig. 9.8(b). Compared to the horizontal case ($\alpha = 0^\circ$), where the mean flow in the centre of the cell is very small, there is for $\alpha = 7^\circ$ a significant vortical flow, which indicates a rotation of the central portion of the interface towards a perpendicular orientation with respect

to the forcing axis, consistent with the theory of vibroequilibria. The rotation, however, is not fast enough to prevent the growth of some frozen waves into columns; see Fig. 9.8(b) at $t = 0.4$ s. The final steady pattern, seen at $t = 1.5$ s, contains two large columns and several smaller ones. Note that this selected interfacial structure has quite different symmetry compared to the horizontal case ($\alpha = 0^\circ$). In fact, it is approximately antisymmetric (or, symmetric under inversion through the centre of the container and exchange of liquids), as were all of the solutions in Fig. 9.6 with miscible liquids.

Larger angles for the initial interface have the same effect as they did in Fig. 9.6, increasing the rotation rate of the central portion of the interface, as seen in the stronger mean flow, and reducing the time needed to reach the final state. The development of the frozen waves is hampered by this rotation. With $\alpha = 7^\circ$, the two central perturbations are large enough to continue growing and form columns, but for $\alpha = 7.2^\circ$ they are reabsorbed by the larger columns before the final state is achieved. This marks a type of mode transition since the average wavenumber (i.e., number of paired columns) is reduced by one.

Figure 9.8(d) shows the evolution of the interface for $\alpha = 14.16^\circ$. This case exhibits even more rotation of the central part of the interface and reduced development of thinner columns. The lateral perturbation seen near the right wall in the previous three simulations at $t = 0.2$ s, for example, is nearly gone. Although some FC-40 is trapped near the bottom right corner, it is not enough to form a column, nor is the small mass of silicone oil in the upper left corner. Further increase in α will reduce the size of these regions even more, until α becomes large enough that the contact points of the initial interface already reach the corners of the container without forcing. When this occurs, the evolution of the forced interface is driven by contact conditions, with the final pattern emerging after a complex dynamical process involving collisions between the interface and the upper and lower boundaries (Salgado Sánchez et al., 2019d). In the simplest case, this process results in a final state with two columns, as with miscible liquids.

Wavenumber selection for different initial angles

The selection of the final columnar structure and its characteristic wavelength was considered for an initial horizontal interface by Salgado Sánchez et al. (2019c), where finite-size effects were investigated. In contrast to the continuous monotonic dependence of wavenumber on forcing predicted for the infinite inviscid system (Lyubimova et al., 2017), the boundaries permit only discrete modes, which exist over finite forcing intervals separated by transitions related to the creation or destruction of new columns. Analogous but different types of mode transitions are observed if the initial angle α is varied rather than the the forcing.

The mode diagram presented in Fig. 9.9 shows three relevant nondimensional wavenumbers measured along the midline $y = 0$ in the final steady structure: the minimum and maximum, and the averaged wavenumber, which gives the number of paired columns. These wavenumbers are shown over a range of initial interface angles between 0° and 16° in a $30 \text{ mm} \times 7.5 \text{ mm}$ container, i.e., with aspect ratio $\Gamma = L/(2H) = 4$, and with two different vibrational velocities of (a) $A\omega = 424$

FIGURE 9.9: Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ (measured at $y = 0$) for two applied vibrational velocities: (a) $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm), (b) $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). The domain measures 30 mm $\times$ 7.5 mm, with aspect ratio $\Gamma = 4$. Three different wavenumbers are shown: minimum (black), average (red) and maximum (blue). Snapshots of the final selected patterns are included for illustration. The simulations used for (a) include those shown in Fig. 9.8.

mm/s and (b) 565 mm/s. Illustrative snapshots showing the final patterns are included for selected angles.

When $\alpha = 0^\circ$, the final structure contains either three or four pairs of columns, depending on the applied forcing. Since these are not exactly the same length, the minimum and maximum wavelengths differ from the average value. For the higher vibrational velocity used in (b), the four-column pattern is observed only in the ideal $\alpha = 0^\circ$ case, with a transition to a three-column pattern observed when the initial inclination is increased to $\alpha = 0.1^\circ$.

As the initial inclination is increased, the difference between the maximum and minimum wavenumber grows until a mode transition [at $\alpha \simeq 7.15^\circ$ in (a) and $\alpha \simeq 8.60^\circ$ in (b)] reduces the average wavenumber by one as the thin central columns fail to fully develop and are absorbed by their neighbours. The resulting mode has two thick central columns and two thin outer columns.

With further increase in α , these two outer columns grow thinner and, at some point, fail to span the height of the container. When [at $\alpha \simeq 14.12^\circ$ in (a) and $\alpha \simeq 14.16^\circ$ in (b)] these lateral fluid masses no longer extend enough to cross the midline $y = 0$, the final mode transition to a single pair of columns is marked. Note that, although the location of the intermediate mode transition is delayed by higher vibrational velocity, the final mode transition is almost unaffected by the forcing amplitude.

FIGURE 9.10: Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ (measured at $y = 0$) for different container aspect ratios: (a) $\Gamma = 3$, (b) $\Gamma = 4$ and (c) $\Gamma = 6$. The minimum (black), average (red) and maximum (blue) wavenumbers are shown for each container at two different vibrational velocities: $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm) and $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). Snapshots of the final selected patterns are included for illustration.

To summarise, the orientation of the initial interface has a dramatic effect on the selected wavenumber (number of column pairs) of the final state. As in the case of miscible liquids, this results from the rotation of the central portion of the interface, consistent with vibroequilibria theory, and the corresponding reduction of the shear flow that drives the frozen wave (Kelvin-Helmholtz) instability.

Effect of the container aspect ratio

The orientation of the interface is a geometric feature that naturally relates to the aspect ratio of the container — since an interface inclined at a certain angle will cross the corners. This relationship is explored in Fig. 9.10, which shows the dependence on α of the pattern wavenumber for three different aspect ratios: (a) $\Gamma = 3$, (b) $\Gamma = 4$, and (c) $\Gamma = 6$. The same two vibrational velocities are applied as in Fig. 9.9: $A\omega = 424$ mm/s (upper row) and 565 mm/s (lower row).

For the container with $\Gamma = 3$, the mode diagram is relatively simple. The four-column (dimensionless wavenumber 2) pattern is consistently observed over most of the α values considered, although the details of its appearance change. An especially prominent change is observed at $\alpha \simeq 11.25^\circ$ (upper)

FIGURE 9.11: Angle $\alpha_{\max}$ of the final mode transition in the formation of columnar structures triggered by the frozen wave instability for different container aspect ratios $\Gamma = L/(2H)$. This angle is only weakly dependent on forcing, and the values plotted are the average of those found with $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm) and 565 mm/s ($f = 60$ Hz, $A = 1.5$ mm). Examples of patterns observed before and after the transition are shown in the insets. The predicted curve based on idealised geometry, $\tan^{-1}(\Gamma^{-1})$, is included (dashed line) for comparison.

and 12.75° (lower), where an asymmetric pattern transforms into one that is (nearly) antisymmetric under reflection in x (through the midline). The only mode transition, which produces a two-column pattern, occurs for an interface orientation approximately equal to that for which the contact points cross the corners even in the unforced hydrostatic case. The subsequent development of the pattern is driven largely by boundary conditions (contact angle).

For the longest container with $\Gamma = 6$, patterns of higher wavenumber are observed. As with $\Gamma = 4$, the amplitude of the vibrational velocity modifies the angles at which intermediate transitions occur. At a certain inclination, the interface immediately crosses the corners and the pattern formation process is driven mainly by contact conditions. As described in more detail in Salgado Sánchez et al. (2019d), there may be a series of collisions of the interface with the upper and lower boundaries to select the final steady solution, even in the absence of applied forcing. In this case, the final pattern is more complicated than the simple two-column state observed with $\Gamma = 3, 4$.

Besides modifying the location of intermediate transitions, the most prominent effect of aspect ratio is on the final angle $\alpha_{\max}$ where either the two-column state is found or, as described above, the pattern selection process is more complicated but driven by contact conditions and collisions. Figure 9.11 shows the variation of this maximum angle (solid dots) with container aspect ratio Γ . The observed values are then compared to the formula $\alpha_{\max} = \tan^{-1}(\Gamma^{-1})$ (dashed line) that predicts the corner crossing transition for a perfectly flat unforced interface.

FIGURE 9.12: Snapshots showing initial (left) and final (right) interfacial patterns in experiments (a–d) with combination I_b in a parabolic flight. Vibrations at 9 Hz with $A = 4.4$ mm begin during hypergravity in panels (a,b) and during microgravity in (c,d). Images showing the final states are averaged over (approximately) two excitation periods.

As seen in the figure, the observations lie very close to this curve. The contact angle of 90° induces curvature in the initial interface (see Fig. 9.8) and delays its contact with the corners. This is most clearly seen for smaller and larger aspect ratios where the measured $\alpha_{\max}$ is somewhat above the theoretical prediction. For intermediate aspect ratios, the measured values are closer; the final mode transition in this regime is more influenced by vibrational velocity. In any case, the strong dependence on aspect ratio of both the final mode transition point $\alpha_{\max}$ and the final selected wavenumber is clear. For the simulations performed here, the two-column final state is found for $\Gamma \lesssim 4$, while four and eight columns are observed for $\Gamma = 6$ and $\Gamma = 8$, respectively.

Comparison with experiment

This is some experimental evidence for the importance of the initial interface orientation in microgravity experiments with immiscible liquids, although it is not as clear as in the miscible case of Sec. 9.2.

Figure 9.12 shows initial and final states from two experiments with combination I_b at identical vibrational velocities. When vibrations are initiated during (the end of) the hypergravity period, with a flat initial interface [Fig. 9.12(a)], the final configuration is an approximately left-right symmetric state consistent with the frozen wave instability (with small angles of initial orientation); the denser FC-40 surrounds the silicone oil [Fig. 9.12(b)]. Subharmonic Faraday waves are also excited on the interface in this case, though they are not evident in the figure due to averaging

(over approximately two excitation periods).

When vibrations are initiated in microgravity, surface tension (and contact forces) lead to the formation of a drop-like configuration [Fig. 9.12(c)]. The applied vibrations act to squeeze and flatten this drop so that, as expected from vibroequilibria theory, large portions of the interface are nearly perpendicular to the forcing. The immiscible simulations presented in Sec. 9.4.2 do not apply to this case since the initial interface is far from an inclined (linear) profile.

9.5 Conclusions

Using careful simulations and available experiments, we demonstrated the importance of the initial interface orientation for pattern selection in miscible and immiscible liquids subjected to horizontal vibrations in microgravity. In a parabolic flight experiment, for example, the level of gravity varies from normal, to about twice that (hypergravity), to about 20 times less (microgravity), with each phase lasting for approximately 22 s. The initial condition for a fluid experiment with a free interface in this environment depends in a critical way on the gravity level that is present when the experiment begins.

Experimental evidence was first presented in the case of miscible liquids, where the evolution of the interface under applied vibrations was traced in four experiments using two distinct pairs of miscible liquids with different initial conditions. A decrease in the wavenumber characterising the final quasisteady state was observed as the angle of the initial interface with respect to the vibrational axis was increased.

Numerical models based on the solution of the Navier-Stokes equations allowed us to systematically investigate the dependence of interface dynamics on the initial angle. For miscible liquids, this involved an additional equation for the concentration field, while a level set approach was used to track the interface in the immiscible case. The slow (compared to the forcing) dynamics of the interface motion is reflected in the mean flow, and it was seen how this changes for different initial orientations. With a flat horizontal initial interface, the mean flow during the early transient period is largest near the lateral walls due to the growth and inward spread of the frozen waves, and there is negligible vertical flow in the interior; in the case of immiscible liquids, there is also an approximate left-right symmetry. For oblique initial interfaces, a rotating (vortical) mean flow is generated in the central part of the container that acts to rotate the interface towards a more vertical orientation. The strength of this central mean flow and the associated rotation increases with initial interface angle. It is this rotation of the central portion of the interface that frustrates and interferes with the development of the columnar structures that are characteristic of the frozen wave instability in microgravity.

The dependence of the final quasisteady state on initial inclination was summarised in mode diagrams showing three representative wavenumbers measured along the horizontal midline of the

container. Mode transitions to patterns of smaller wavenumber were observed with increasing initial angle. For sufficiently large angles, the selected structure is often one where each fluid primarily occupies one side of the container. The maximum angle characterising this limit in the immiscible case depends on aspect ratio and is related to the point where the unforced interface crosses the corners of the container. For aspect ratios of approximately 5 and larger, the final state exhibits multiple columns and depends on contact conditions and the dynamics of collisions between the interface and the upper and lower boundaries. Although intermediate mode transitions depend on vibrational velocity, with higher amplitudes favouring more columns, this maximum inclination angle is only weakly dependent on forcing.

Based on the consistency of the numerical simulations for both miscible and immiscible liquids, and on the qualitative agreement with available experimental results, it is clear that the development of interfacial instabilities in microgravity — in particular, the frozen wave instability and potential excitation of secondary Faraday waves — is strongly influenced by the initial position of the interface. This effect must be considered in addition to residual gravity, finite-size effects, and other experimental concerns.

Acknowledgment. The work of the Belgian team was supported by the PRODEX programme of the Belgian Federal Science Policy Office. The work of P. Salgado Sánchez and J. Porter was supported by the Ministerio de Economía y Competitividad under Project No. ESP2015-70458-P. We also thank the Departamento de Aeronaves y Vehículos Espaciales in the Escuela Técnica Superior de Ingeniería Aeronáutica y del Espacio, Universidad Politécnica de Madrid for supporting the stay of P. Salgado Sánchez at the Microgravity Research Centre, Université Libre de Bruxelles.

Chapter 10

Influence of gravity on the frozen wave instability in immiscible liquids

Gligor, D., Salgado Sánchez, P., Porter, J. and Shevtsova, V.

Physical Review Fluids **5**, 084001 (2020).

<https://doi.org/10.1103/PhysRevFluids.5.084001>

The influence of the gravity level on the frozen wave instability in finite containers of immiscible liquids is investigated numerically to shed light on the transformation from a supercritical pitchfork bifurcation in normal gravity, which produces stable finite-amplitude waves, to an apparently degenerate bifurcation in weightlessness, leading to large columnar patterns that collide with the container walls. The vibroequilibria effect grows in importance as gravity is reduced and has a symmetry-breaking effect on the bifurcation, selecting frozen waves with the heavier fluid displaced upward along the lateral walls. Three possible growth regimes are identified. With finite gravity, there is an initial gravity-capillary regime with the square-root growth predicted by weakly-nonlinear theory, associated with small-amplitude waves. This is followed by a linear growth regime that is dominated by gravity and associated with larger finger-like waves. Lastly, in some cases with reduced gravity, we observe a third regime characterized by the close proximity of the frozen wave crests to the upper wall and an interval of nearly saturated wave amplitude. The defining parameters of the bifurcation diagram are measured and presented as a function of Bond number and, when possible, compared with theoretical predictions. In particular, there is excellent agreement between theory and simulations for the weakly-nonlinear cubic (branching) coefficient and reasonable qualitative agreement for the threshold. Differences are attributed to finite-size effects, vibroequilibria, and viscosity.

10.1 Introduction

Understanding the behavior of free surfaces or interfaces and the instabilities they undergo is of basic interest in fluid dynamics, with numerous applications in biological systems and engineering. There are many well-known examples with a range of features. Gravitationally unstable configurations — or a mass of lighter fluid accelerating into a denser one after an explosion or volcanic eruption, for instance — are susceptible to the Rayleigh-Taylor instability (Kull, 1991), which tends to generate fingers or plumes. Temperature differences arising from an inhomogeneous heat source generally induce density gradients, which lead to buoyant (gravitational) convection (Batchelor, 1954), and surface tension gradients, which lead to thermocapillary (Bénard-Marangoni) convection (Birikh, 1966). The shear force resulting from a velocity difference across a fluid interface can drive a Kelvin-Helmholtz instability, which is involved in familiar phenomena such as wind-driven water waves and billow clouds. Vibrations or moving boundaries generate interfacial waves, which may be either standing or traveling, localised or distributed, depending on the direction and frequency content of the forcing.

The dynamics resulting from these instabilities, and others, can vary considerably depending on fluid properties, the character and amplitude of the forcing mechanism, and the ratio between the system size and the lengthscale of the unstable mode. However, aside from small systems, gravity plays a consistently important role due to its strong preference for density stratification and the flat horizontal interfaces that characterize the static equilibrium configurations of most medium-to-large fluid masses on Earth. The familiar behavior of fluids can be dramatically altered in the absence of gravity, where surface tension favours minimum-surface configurations that are usually curved (perfectly spherical drops, for example) and contact forces can lead the fluid to fully cover any adjacent solid boundaries.

In addition to its decisive influence on equilibrium states, gravity provides a crucial restoring force for several instabilities that must be overcome at threshold (in the sense of the linear eigenvalue problem) by the driving mechanism, and which limits the subsequent growth of an unstable perturbation. As the gravity level is reduced, there is both a lowering of the threshold and more rapid growth of the excited mode. For example, small-amplitude vibrations (often called g-jitter) that would be inconsequential under normal gravity, can lead to significant disturbances in microgravity environments like the International Space Station (ISS).

The Kelvin-Helmholtz instability can be triggered in a container holding two liquids of different density by periodic horizontal vibration. Starting from a perfectly flat horizontal interface in the presence of gravity, this instability arises at a critical forcing amplitude through a supercritical pitchfork bifurcation and leads initially to a quasi-stationary, approximately sinusoidal surface deformation referred to as a frozen wave (Lyubimov and Cherepanov, 1986; Talib, Jalikop, and Juel, 2007; Wolf, 1969). The lengthscale of this frozen wave pattern, in sufficiently deep layers, is set by the capillary length. In microgravity, on the other hand, the threshold is drastically lowered, vanishing completely in the limit of inviscid fluids in unbounded domains. With the capillary length

effectively infinite, wavenumber selection in microgravity is controlled by a combination of growth rate (Lyubimova et al., 2017) and finite-size effects (Salgado Sánchez et al., 2019c). Moreover, the resulting patterns are not sinusoidal, or even of finite amplitude, but take the form of extended columnar structures limited only by the height of the container (Gaponenko et al., 2015b; Gaponenko et al., 2018; Shevtsova et al., 2016). Since these columnar structures effectively rotate large portions of the interface into a nearly perpendicular orientation with respect to the forcing, this, in turn, allows Faraday waves to form when their threshold is surpassed (Lyubimova et al., 2019; Salgado Sánchez et al., 2019d; Shevtsova et al., 2016). The solutions resulting from this interaction are complex, with features on multiple length and time scales (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016).

Another fluid phenomenon that is driven by vibrations and suppressed by gravity, is that of vibroequilibria (Beyer et al., 2001; Lyubimov et al., 1997; Wolf, 1969). This effect is due to the inhomogeneous velocity field produced by the vibrations and the corresponding oscillatory pressure gradient, which induces a slow deformation of the interface toward a quasi-steady equilibrium that balances forcing, gravity, surface tension, and contact energy. Faraday (1831) observed this effect in his experiments on the flattening of fluid drops suspended beneath a vibrating plate. Since the vibroequilibria effect is proportional to the square of applied velocity, there is no threshold. However, it is primarily in reduced gravity environments that it becomes important for moderate or large systems (i.e., those not dominated by surface tension). In such systems, the preexisting vibroequilibria states can be expected to modify the onset of threshold instabilities like frozen waves, as well as the nature and growth of the excited mode.

In this paper we investigate the changing character of the frozen wave instability as the gravity level is varied. This dependence is especially interesting given the dramatic difference between frozen waves in normal gravity and in microgravity. In the first case, as the forcing is increased beyond the critical value, the waves grow gradually in the manner of a supercritical pitchfork bifurcation. In the second case, any forcing above the very low threshold value leads to the rapid development of large columnar patterns; no small-amplitude frozen waves are observed. The transition between these strikingly different manifestations of the same instability is examined through a series of numerical simulations using parameters relevant to both ground and microgravity experiments (Jalikop and Juel, 2009; Salgado Sánchez et al., 2019d, 2020). The transformation is characterised through the defining parameters of the bifurcation diagram including the critical forcing value, the branching (cubic nonlinear) coefficient, and a symmetry-breaking parameter reflecting the magnitude of the vibroequilibria effect. These are compared to theoretical predictions when available.

The paper is organised as follows. The mathematical formulation and numerical approach are presented in Sec. 10.2. The frozen wave instability is described for the case of strong gravity in Sec. 10.3 and for reduced gravity in Sec. 10.4. Finite-size effects are discussed in each case. The complete results of the numerical simulations are summarised in Sec. 10.5 with the bifurcation parameters given as a function of Bond number and compared to theory. Final conclusions are offered in Sec. 10.6.

FIGURE 10.1: Sketch illustrating the two-layer system of immiscible liquids. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis and parallel to the initial flat interface (dashed line).

10.2 Mathematical formulation

We consider a rectangular container with interior dimensions $L \times 2H$ holding two layers of immiscible liquids, as sketched in Fig. 10.1. The corners of the container are rounded to more closely match the relevant experiments (Salgado Sánchez et al., 2019d, 2020) and avoid numerical singularities. The system is subjected to vibrations of amplitude A and frequency $\omega = 2\pi f$ oriented along the x -axis, which is also parallel to the initially flat (unperturbed) horizontal interface that separates the two liquids (dashed line in the figure). In the presence of vertical gravity, this initial configuration is stable, with the heavier fluid in the lower layer.

The dynamical response to applied vibrations is resolved using a level-set method (Olsson and Kreiss, 2005; Salgado Sánchez et al., 2019c,d, 2020) that follows the position of the interface via a function ϕ governed by the equation

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right). \quad (10.1)$$

The tuning parameter γ is set equal to $A\omega$ in this case, ε is a small diffusive term added for numerical stability and taken to be on the order of the maximum mesh element size, and the compression flux term $\phi(1 - \phi)\nabla \phi/|\nabla \phi|$ counteracts the tendency of the interface to widen from numerical diffusion. This flux term was first introduced by Olsson and Kreiss (2005) and is part of the standard implementation of the level-set method in the simulation environment of COMSOL Multiphysics.

The level-set function ϕ is used to continuously define density and viscosity across both layers after introducing the smoothed Heaviside function $\mathcal{H}$

$$\rho = \rho_1 + (\rho_2 - \rho_1) \mathcal{H}, \quad (10.2a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1) \mathcal{H}, \quad (10.2b)$$

where

$$\mathcal{H}(\tilde{\phi}) = \begin{cases} 0 & \tilde{\phi} < -\varepsilon, \\ \frac{1}{2} + \frac{\tilde{\phi}}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\tilde{\phi}}{\varepsilon}\right) & |\tilde{\phi}| \leq \varepsilon, \\ 1 & \tilde{\phi} > \varepsilon. \end{cases} \quad (10.3)$$

Here $\tilde{\phi} = \phi - 0.5$ with the values $\phi = 0, 1$ corresponding, respectively, to the heavier and lighter fluids denoted by subscripts 1 and 2 in Eqs. (10.2). The interface is associated with the contour $\phi = \mathcal{H} = 0.5$. The use of the function $\mathcal{H}$ instead of the usual Heaviside step function improves numerical stability.

In the reference frame of the container, Eqs. (10.1)–(10.3) are solved together with the continuous formulation of the incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991):

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \nabla(\mu \nabla \mathbf{u}) + \delta \mathbf{f}_\sigma + \rho \mathbf{G}(t), \quad (10.4a)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (10.4b)$$

Here, the effect of interfacial tension σ is included via $\mathbf{f}_\sigma$ with

$$\mathbf{f}_\sigma = (\sigma \kappa) \mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \quad (10.5)$$

where κ is the curvature of the interface and $\mathbf{n}$ the normal vector. The effect of interfacial tension is kept local by the parameter δ , which is defined in terms of the level-set function as

$$\delta = 6\phi(1 - \phi)|\nabla \phi|. \quad (10.6)$$

Gravity and applied periodic forcing are included in the $\mathbf{G}(t)$ term of Eq. (10.4a):

$$\mathbf{G}(t) = -g\mathbf{j} + A\omega^2 \cos \omega t \mathbf{i}, \quad (10.7)$$

where g is a steady vertical gravitational field and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y axes, respectively. Values of g are given below in terms of the usual terrestrial acceleration $g_0 = 9.81 \text{ m/s}^2$.

Navier-slip boundary conditions are applied on all solid walls in order to preserve the contact angle β while permitting contact line motion, as observed in the motivating experiments (Salgado Sánchez et al., 2019d, 2020). These conditions are written as

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u} \cdot [\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta)\delta] = 0, \quad (10.8)$$

	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)
FC-40	1855	5
100 cSt silicone oil	965	100
	σ (10 ⁻³ N/m)	β (rad)
FC-40 and silicone oil	6.021	$\pi/2$

TABLE 10.1: Physical and interfacial properties of FC-40 and 100 cSt silicone oil: density ρ , kinematic viscosity ν , interfacial tension σ and contact angle β , which has been selected to achieve an initially flat interface. According to Someya and Munakata (2005), interfacial tension measurements between silicone oils and FC-40 vary between 4 – 7 mN/m. Due to the lack of experimental measurements for 100 cSt silicone oils and FC-40, we use the same interfacial tension value as in Salgado Sánchez et al. (2019c,d, 2020). We also note that the real kinematic viscosity of FC-40 is 2.2 cSt but is increased here to 5 cSt in order to delay the secondary Faraday wave instability.

where $\mathbf{n}_w$ is a unit vector normal to the wall. They introduce an artificial frictional force $\mathbf{f}_v = (\rho\nu)/\delta_v \mathbf{u}$, where $\delta_v = \sqrt{\nu/\omega}$ is the thickness of the vibrational viscous boundary layer.

As initial conditions at $t = 0$, the liquids are assumed to be motionless with $\mathbf{u} = 0$ and to be separated by a perfectly flat interface (the dashed line in Fig. 10.1) with $\beta = \pi/2$. To improve numerical stability, the vibrational forcing is initiated smoothly in the manner of Eq. (10.3) over a 0.25 s transition time, as in Salgado Sánchez et al. (2019c,d, 2020).

The simulations of interfacial dynamics with Eqs. (10.1)–(10.8) are performed with parameters appropriate for a combination of FC-40 and 100 cSt silicone oil, as given in Table 10.1. This choice allows a comparison with several recent parabolic flight experiments (Salgado Sánchez et al., 2019d, 2020) and with the ground experiments of Jalikop and Juel (2009) that used silicone oil and Galden HT135, an immiscible combination with similar physical properties. Note, however, that the viscosity of FC-40 is increased in the simulations from 2.2 to 5 cSt in order to delay the appearance of secondary Faraday modes (Salgado Sánchez et al., 2019c,d), which can interact with and complicate the analysis of the primary frozen wave structure.

The system described by Eqs. (10.1)–(10.8) may be nondimensionalised by taking H for the unit of distance, $\rho_1 - \rho_2$ for density and $\sqrt{(\rho_1 - \rho_2)H^3/\sigma}$ for time. Behavior can then be characterised by the dimensionless forcing velocity $\mathcal{F}$ and frequency Ω :

$$\mathcal{F} = A\omega\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad \Omega = \omega H\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad (10.9)$$

as well as the density ratio $\hat{\rho}$, viscosity ratio $\hat{\nu}$, capillary number Ca, Bond number Bo and aspect ratio Γ :

$$\hat{\rho} = \frac{\rho_2}{\rho_1}, \quad \hat{\nu} = \frac{\nu_2}{\nu_1}, \quad \text{Ca} = \nu_1\sqrt{\frac{(\rho_1 - \rho_2)}{\sigma H}}, \quad \text{Bo} = \frac{(\rho_1 - \rho_2)H^2}{\sigma}g, \quad \Gamma = \frac{L}{2H}. \quad (10.10)$$

The simulation software COMSOL Multiphysics, which includes a standard implementation of the level-set method, is used for solving the two-dimensional governing equations (10.1)–(10.8) in dimensional variables with the finite element method. Numerical convergence was previously tested in Salgado Sánchez et al. (2019c,d) for a similar configuration, and we make use of those results for the selection of the mesh size and the interface thickness ε . The simulations described below are performed with a maximum element size $\mathcal{S} = (7.5/50)$ mm, while ε is set to $(3/4) \mathcal{S}$ for $g \geq g_0$, and reduced to $(3/5) \mathcal{S}$ in reduced gravity to avoid unstable numerical behavior. The evolution in time is implemented with a generalised- α integration scheme (Erlicher, Bonaventura, and Bursi, 2002) combined with streamline (Harari and Hughes, 1992) and crosswind Codina, 1993 stabilization techniques. The maximum time step is set at $1/(10f)$, with automatic reduction as required for stability. Further details of the numerical model can be found in Salgado Sánchez et al. (2019c,d, 2020).

10.3 Frozen wave instability in strong gravity

When two layers of immiscible fluids in a gravitationally stable configuration are subjected to horizontal vibrations, an initially flat interface becomes unstable to frozen waves only above a certain threshold. An expression for the critical forcing velocity $v = A\omega$ and its dependence on the perturbation wavenumber k was derived by Lyubimov and Cherepanov (1986) in the limiting case of an infinitely long container with inviscid fluids of equal heights H , densities ρ_1 and $\rho_2 < \rho_1$, and interfacial tension σ :

$$v^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)^2} \left[\sigma k + \frac{(\rho_1 - \rho_2)g}{k} \right] \tanh(kH), \quad (10.11)$$

which, in dimensionless form, becomes

$$\mathcal{F}^2 = \frac{(1 + \hat{\rho})^3}{2\hat{\rho}(1 - \hat{\rho})} \left[kH + \frac{\text{Bo}}{kH} \right] \tanh(kH). \quad (10.12)$$

For deep layers with $kH \gg 1$, Eq. (10.11) implies that the wavelength of the first unstable mode is set by the capillary length,

$$\lambda_c = 2\pi \sqrt{\frac{\sigma}{(\rho_1 - \rho_2)g}}, \quad (10.13)$$

and that its threshold v_c is given by

$$v_c^2 = \frac{(\rho_1 + \rho_2)^3}{\rho_1\rho_2(\rho_1 - \rho_2)} \sqrt{\frac{\sigma g}{\rho_1 - \rho_2}}. \quad (10.14)$$

These predictions were critically examined by Talib, Jalikop, and Juel (2007), who considered the effect of viscosity both experimentally and theoretically. It was found that the inviscid theory consistently underestimates the threshold for liquids of similar viscosity, while generally overestimating it for layers with a high viscosity contrast.

Lyubimov and Cherepanov (1986) also derived an expression for the (peak-to-peak) amplitude Δh of the frozen wave pattern (see also Gandikota et al., 2014b) from a weakly nonlinear analysis:

$$\Delta h^2 = \frac{16\rho_1(\rho_1 + \rho_2)}{\rho_2(42\rho_1\rho_2 - 11\rho_1^2 - 11\rho_2^2)} \sqrt{\frac{\sigma(\rho_1 - \rho_2)}{g^3}} (v^2 - v_c^2), \quad (10.15)$$

or, in dimensionless form,

$$\left(\frac{\Delta h}{H}\right)^2 = \frac{16(1 - \hat{\rho}^2)\text{Bo}^{-3/2}}{\hat{\rho}(42\hat{\rho} - 11 - 11\hat{\rho}^2)} (\mathcal{F}^2 - \mathcal{F}_c^2), \quad (10.16)$$

where the subscript c indicates the critical value. It follows from Eq. (10.15) that for $\hat{\rho} = \rho_2/\rho_1 > 0.2829$, which holds for most experimentally practical combinations of liquids, the instability appears via a supercritical pitchfork bifurcation. The supercritical nature of the instability was confirmed experimentally by Jalikop and Juel (2009) using a pair of liquids with density contrast $\hat{\rho} \simeq 0.548$. However, it was also observed that the square-root dependence of amplitude Δh on the distance above onset predicted by Eq. (10.15) changes at larger forcing values to a nearly linear one. This transition is accompanied by a change in the shape of the frozen waves, which grow from small sinusoidal waves to larger finger-like deformations.

In the remainder of this section, we analyze the frozen wave instability in finite-size containers under strong gravity. First, the behavior of the interface in terrestrial conditions at different applied forcing values is described and bifurcation diagrams showing $\Delta h(v)$ are constructed. Following this, the effects of hypergravity and finite container size, which were apparent in the experiments of Salgado Sánchez et al. (2019d), are discussed.

10.3.1 Dependence on applied forcing

Figure 10.2 shows the temporal evolution of the frozen wave instability with normal gravity ($g = g_0$, $\text{Bo} = 81.57$) in a $L \times 2H = 30 \times 15$ mm container ($\Gamma = 2$). The four series of snapshots use the same excitation frequency of $f = 55$ Hz while the amplitude A varies in (a–d) from 1.1 to 1.67 mm to reach $v = 0.380, 0.432, 0.546$ and 0.577 m/s, corresponding to dimensionless forcing values $\mathcal{F} = 12.65, 14.38, 15.19$ and 19.22 , respectively. Note that these forcing amplitudes satisfy $A \ll H$, as required for a valid comparison with linear theory (Lyubimov and Cherepanov, 1986). They are also below the threshold for secondary Faraday waves (Salgado Sánchez et al., 2019d).

The interface and mean velocity field are both obtained by averaging over one excitation period and are shown together during the transient growth of the pattern; in the final steady state the mean velocity is negligible and is not shown. This filters out the harmonic motion that is always present near the sidewalls due to surface waves, the evanescent oscillatory bulk flow (Tinao et al., 2014; Varas and Vega, 2007) and the associated motion of the contact point. It can be seen from the figure that these harmonic modes do interact nonlinearly, leaving a visible bump near the sidewalls even after averaging over time. Although not often observed with the parameters used here, subharmonic waves may also be present. Neither harmonic nor subharmonic waves are investigated in this work,

FIGURE 10.2: Time evolution (averaged over one excitation period) of the interface and mean velocity field in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$) under normal gravity ($g = g_0$, $Bo = 81.57$) with different forcing amplitudes: (a) $v = 0.380$ m/s, (b) 0.432 m/s, (c) 0.456 m/s and (d) 0.577 m/s, which correspond to $\mathcal{F}$ values of 12.65, 14.38, 15.19 and 19.22, respectively. The forcing frequency is fixed at $f = 55$ Hz while the amplitude A is increased (1.1, 1.25, 1.32 and 1.67 mm). The critical forcing is $v_c = 0.429$ m/s ($\mathcal{F}_c = 14.25$) as shown in Sec. 10.3.2. The velocity field during the transient portion of the simulation is shown with scaled arrows (scale differs between snapshots). The selected time of the intermediate snapshot in (d) is less than that of (a–c) due to the more rapid growth of that pattern. The final pattern in (d) also illustrates how the amplitude Δh is measured from the contour $\phi = 0.5$.

but we note that they are expected to play an increasingly important role with lower viscosity liquids. As shown by Salgado Sánchez et al. (2016), the interaction of subharmonic waves from opposing lateral walls can lead to a range of complex dynamics. This interaction may even drive sloshing modes (particularly in reduced gravity) that destabilize the underlying state (Fernández et al., 2017b). For some fluids and forcing parameters, the onset and development of frozen waves will be affected much more by harmonic and subharmonic surface waves than it is here.

The interface in Fig. 10.2 exhibits different dynamics depending on whether the applied forcing is below or above the frozen wave threshold (found in Sec. 10.3.2 to be $v_c = 0.429$ m/s, or $\mathcal{F}_c = 14.25$). Below this value, as in Fig. 10.2(a), gravity maintains a nearly flat interface across the interior portion of the container. There is, however, an evident upward movement of the contact points along the lateral walls, as expected from the theory of vibroequilibria (Fernández et al., 2017a,b; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997; Salgado Sánchez et al., 2019a). The vibroequilibria effect is proportional to the square of the induced vibrational velocity and is most prominent near the sides, where this inhomogeneous velocity field is strongest. It is observed across all finite applied forcing values since, unlike the frozen wave instability, it has no threshold.

When the applied forcing is just above the frozen wave threshold, two small-amplitude waves appear along the interior (flat) portion of the interface, as shown in Fig. 10.2(b) at $t = 1.5$ s. With further increase in forcing, these waves grow steadily in amplitude, as illustrated in Fig. 10.2(c), and their profile begins to take on features of an inverted trochoid. These relatively small-amplitude waves reflect the restoring forces of both gravity and interfacial tension and are in good agreement with the results of Jalikop and Juel (2009). We note that the frozen wave pattern develops over the interior portion of the interface and is inhibited near the lateral walls where the vibroequilibria effect is strongest. If the cell is vibrated with still larger amplitude, as in Fig. 10.2(d), the tips of the inverted trochoid grow into longer finger-like protrusions, also in agreement with the experiments of Jalikop and Juel (2009). This is discussed in more detail below.

In keeping with the predicted supercritical bifurcation of Eq. (10.15), Fig. 10.2 shows the continuous increase of wave amplitude with applied forcing. We take a similar approach as Jalikop and Juel (2009) in the analysis of these patterns and construct bifurcation diagrams from the measured wave amplitude.

10.3.2 Bifurcation diagram

The final (asymptotic) trough-to-crest amplitude Δh of the frozen wave pattern is measured in the manner sketched in Fig. 10.2(d) and bifurcation diagrams are constructed from its dependence on applied vibrational velocity v , as in Fig. 10.3. This figure reveals two distinct regimes, the first associated with the initial square root dependence, $\Delta h \propto \sqrt{v^2 - v_c^2}$, and the second with an approximately linear relationship, $\Delta h - \Delta h^* \propto v - v^*$, where the asterisk denotes the transition value. These regimes may be associated, as discussed below in Sec. 10.3.2, with a different balance between the vibrational forcing and the restoring forces of gravity and interfacial tension.

Character of the instability

Figure 10.3(a) shows that the frozen wave instability appears via a perturbed (asymmetric) supercritical pitchfork bifurcation (Strogatz, 1994). Although we do not directly test for hysteresis by gradually lowering the forcing — simulations are always begun from rest using the smoothed Heaviside function $\mathcal{H}$ to initiate vibrations — the supercritical nature of the pitchfork bifurcation is evident from the bifurcation diagram. It is also consistent with the ideal linear theory for this density ratio and with experiments (Jalikop and Juel, 2009).

The symmetry breaking (perturbation) of the pitchfork bifurcation comes mainly from the vibroequilibria effect, which is most visible near the lateral walls. While an infinitely long flat interface that is unstable to frozen waves may evolve toward any solution on the group orbit of equivalent patterns (i.e., solutions related by translation through some fraction of a wavelength), this is no longer possible in finite containers where the presence of the container walls modifies the vibrational velocity field. It is this same inhomogeneous velocity field that drives the average reorientation of the interface described by vibroequilibria theory. Since the effect induces the heavier fluid to climb up the lateral walls (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004), it acts to

FIGURE 10.3: Bifurcation diagram showing the trough-to-crest amplitude Δh of the frozen wave patterns versus the vibrational velocity v for $g = g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The numerically-obtained points are fit to (a) the perturbed pitchfork bifurcation of Eq. (10.17) or (b) the square-root dependence of Eq. (10.15), both shown as solid curves, and to the linear function (10.19), shown with a dotted line. These fits correspond to the two regimes identified by Jalikop and Juel (2009), with the transition between them marked by a vertical line. The points corresponding to the snapshots of Fig. 10.2(b–d) are distinguished by solid markers and labelled.

promote the compatible frozen wave solution and suppress the others. Put another way, in the presence of the vibroequilibria effect, one cannot obtain an equivalent frozen wave pattern by inverting the crests and troughs, even at small amplitude. Due to this symmetry-breaking effect, the selected branch of the perturbed supercritical pitchfork bifurcation does not truly appear at a critical value but, instead, grows smoothly as the forcing is varied, as observed in Fig. 10.3(a).

The two regimes of amplitude growth distinguished by Jalikop and Juel (2009) are also apparent in Fig. 10.3. Data points in the first regime are fit, for comparison, to both an ideal and a perturbed pitchfork branching equation, while those in the second regime are fit to a linear function. The branching equation for a perturbed supercritical pitchfork bifurcation can be written

$$\Delta h^3 - K^2(v^2 - v_c^2) \Delta h + \epsilon = 0, \quad (10.17)$$

which, with the corresponding value of K^2 , reduces to Eq. (10.15) when $\epsilon = 0$. With $g = g_0$, the values of v_c and K obtained from least squares fits with $\epsilon = 0$ and $\epsilon \neq 0$ are similar:

$$v_c = 0.427 \text{ m/s} \quad \text{for } \epsilon = 0, \quad (10.18a)$$

$$v_c = 0.429 \text{ m/s} \quad \text{for } \epsilon \neq 0. \quad (10.18b)$$

In the latter case, the fitting yields $\epsilon = 0.298 \cdot 10^{-9}$.

Gravity level	$\epsilon \times 10^9$	v_c	v^*	$v^* - v_c$	$K \times 10^3$	$\mathcal{M} \times 10^3$
(g_0)	(m^3)	(m/s)	(m/s)	(m/s)	(s)	(s)
1	0	0.427	0.466	0.039	16.19	35.43
1	0.298	0.429	0.466	0.037	16.39	35.43
2	0	0.529	0.587	0.058	10.17	20.32
2	$5e-12$	0.530	0.587	0.057	10.16	20.32
0.5	0	0.345	0.374	0.029	26.17	67.38
0.5	0.750	0.347	0.374	0.027	26.86	67.38

TABLE 10.2: Parameters characterising the bifurcation diagrams obtained at different gravity levels: symmetry-breaking parameter ϵ , critical vibrational velocity v_c , transition point v^* , width of square root regime $v_c - v^*$, branching coefficient K , linear fitting slope $\mathcal{M}$. Values are obtained from a least squares fit of the data to Eq. (10.17), in one case with $\epsilon = 0$, and to Eq. (10.19).

The linear regime is characterised by a least squares fit to the linear function

$$\Delta h - \Delta h^* = \mathcal{M}(v - v^*), \quad (10.19)$$

which provides the slope $\mathcal{M}$ as well as the forcing v^* and amplitude Δh^* characterising the transition. The points included in each fitting are selected from the root mean square error (RMSE) and the associated R^2 value. Successive data points, ordered by increasing v , are included in the first regime rather than the second only if the resulting RMSE decreases or the R^2 value increases. The results of these fittings for $g = g_0$ are summarised in the first row of Table 10.2.

Curvature and energy balance

Following Jalikop and Juel (2009), we quantify the curvature of the final patterns obtained by simulations. Figure 10.4(a) shows the curvature r_k measured at the crests of frozen wave patterns with $g = g_0$ as a function of vibrational velocity v . This radius of curvature is measured by averaging the $\phi = 0.5$ contour of the level-set function over one forcing period. The averaged contour is then linearly interpolated to obtain a representation $y = f(x)$ from which first and second derivatives, f' and f'' , are calculated using second order Lagrange polynomials. The radius of curvature is

$$r_k = \frac{\left[1 + (f')^2\right]^{\frac{3}{2}}}{|f''|}. \quad (10.20)$$

Figure 10.4(a) clearly shows how this curvature r_k decreases toward a value commensurate with the capillary length (shown by a dashed line):

$$l_c = \frac{\lambda_c}{2\pi} = \sqrt{\frac{\sigma}{(\rho_1 - \rho_2)g}}. \quad (10.21)$$

FIGURE 10.4: Radius of curvature r_k at the wave crests as a function of the applied velocity v for (a) $g = g_0$ and (b) $g = 0.5g_0$. Data points are fit with a solid curve for visual reference. Asymptotic behavior is seen at large v when r_k is comparable to the capillary length (shown with a dashed line) given by Eq. (10.21). Points corresponding to the snapshots of Fig. 10.2 are labelled and distinguished by solid markers in (a), while the same is done for (b) with regard to Fig. 10.6 (shown below).

This asymptotic behavior corresponds to the transition from a trochoid-like shape to a finger-like shape; compare panels (c) and (d) of Fig. 10.2. The transition occurs near the forcing value v^* (shown by a solid vertical line) defined by the change from square root to linear growth of the amplitude Δh (see Fig. 10.3). The limiting behavior reflects the fact that, beyond v^* , the shape of the frozen wave crests does not vary significantly with amplitude. This is in agreement with the experiments of Jalikop and Juel (2009), which exhibited the same trend. The presence of these two different supercritical regimes with their corresponding frozen wave shapes can be related to the balance between forcing, gravity, and interfacial tension that holds in each case, as discussed as well in Jalikop and Juel, 2009.

Energy balance in the gravity-capillary regime:

When the frozen waves are small and sinusoidal with $k\Delta h \ll 1$ [see Fig. 10.2(b)], the gravitational and interfacial energies, E_g and E_σ , of a single frozen wave can be approximated as

$$E_g = \frac{\pi(\rho_1 - \rho_2)g}{8k} \Delta h^2, \quad E_\sigma \simeq \frac{2\pi\sigma}{k} + \frac{\pi\sigma k}{8} \Delta h^2. \quad (10.22)$$

The expression for E_g follows from the fact that the centroid of the positive half of a sine wave is located at a height $\pi\Delta h/16$ while its area is $\Delta h/k$. The expression for E_σ follows from an expansion of the elliptic integral giving the arc length of a sine wave; note that the first term is just the energy of the unperturbed flat interface.

If forcing is increased, the amplitude Δh of the frozen wave will grow until it is matched by the restoring forces of gravity and interfacial tension,

$$\frac{\partial E_g}{\partial \Delta h} = \frac{\pi(\rho_1 - \rho_2)g}{4k} \Delta h, \quad \frac{\partial E_\sigma}{\partial \Delta h} \simeq \frac{\pi\sigma k}{4} \Delta h, \quad (10.23)$$

which are both proportional to the amplitude Δh and are equal in the ideal case of Eq. (10.13) with $k = 2\pi/\lambda_c = \sqrt{(\rho_1 - \rho_2)g/\sigma}$. For frozen waves near this selected wavenumber, both gravity and interfacial tension provide restoring forces of similar magnitude. This balance characterises the initial supercritical regime of Fig. 10.3 showing an approximate square-root dependence: $\Delta h \propto \sqrt{v^2 - v_c^2}$. As the frozen wave amplitude grows, however, the curvature of the crests and troughs reaches a limiting value (see Fig. 10.4) and, beyond this, interfacial energy increases slower than gravitational energy, which marks the gravity-dominated regime.

Energy balance in the gravity regime:

Far from threshold, when $k\Delta h \gtrsim 1$, the frozen waves develop into finger-like shapes with the tips maintaining approximately the same form (i.e., their curvature is nearly constant). As in the gravity-capillary regime, the gravitational energy of the finger-like waves is proportional to Δh^2 (coming from the product of centroid height and wave area). In contrast, the interfacial energy of these nonlinear waves is proportional to Δh (finger length):

$$E_g \simeq c_1(\rho_1 - \rho_2)g\Delta h^2, \quad E_\sigma \simeq E_{\text{tips}} + c_2\sigma\Delta h, \quad (10.24)$$

where c_1 and c_2 are approximately constant geometric factors depending on the shape and size of the tips and E_{tips} denotes their nearly constant interfacial energy. In this regime, the restoring forces can be estimated from

$$\frac{\partial E_g}{\partial \Delta h} \simeq 2c_1(\rho_1 - \rho_2)g\Delta h, \quad \frac{\partial E_\sigma}{\partial \Delta h} \simeq c_2\sigma, \quad (10.25)$$

which means that gravity dominates over interfacial tension for large Δh and is responsible for determining the amplitude of large finger-like frozen waves. This corresponds to the second, nearly linear regime of Fig. 10.3.

These basic energy arguments show how the evolution of frozen waves from small-amplitude sinusoidal profiles to nonlinear trochoid or finger-like shapes is associated with the transition from a gravity-capillary forcing balance to a gravity-dominated restoring force, which produces a noticeable change in the $\Delta h(v)$ curve. The same transition was analysed by Jalikop and Juel (2009) for layers of Galden HT135 and silicone oil of 100 or 200 cSt. We note that, although the growth of the frozen wave amplitude in the nonlinear finger-like regime is dominated by gravity, interfacial tension is essential in determining the shape of the tips and limiting their radius of curvature. In the absence of (strong) interfacial tension, as in the experiments with miscible liquids of Gaponenko et al. (2015a), the frozen waves adopt a more triangular profile, which is also affected by gravity (Gaponenko and Shevtsova, 2016).

FIGURE 10.5: (a) Bifurcation diagrams showing $\Delta h(v)$ with $g = 2g_0$ (Bo = 163.14, squares) and $g = g_0$ (Bo = 87.57, circles) in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$). The initial capillary-gravity regime of each diagram, with the interval given in Table 10.2 by $(v^* - v_c)$, is shaded. (b) Bifurcation diagrams under normal gravity, $g = g_0$, in containers of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$, circles) and $L \times 2H = 60 \times 15$ mm ($\Gamma = 4$, triangles). The insets show representative patterns, to be compared with Fig. 10.2, for (a) the hypergravity case and (b) the longer container with $\Gamma = 4$.

10.3.3 Effect of hypergravity

The linear results of Lyubimov and Cherepanov (1986) show how the combination of gravity and interfacial tension determines both the frozen wave threshold, given by Eq. (10.11), and the wavelength of the emerging pattern, given by Eq. (10.13) for $kH \gg 1$. With an increase in gravity level, greater forcing is required to excite frozen waves and their wavelength (set by the capillary length) is reduced. Recent results from parabolic flight experiments (Salgado Sánchez et al., 2019d) (see Sec. 6.4 of that reference) included some observations of frozen waves initiated in hypergravity, with the expected decrease in wavelength compared to the pure microgravity results.

These effects of increased gravity level are demonstrated in Fig. 10.5(a), which shows the bifurcation diagram for $g = 2g_0$ (Bo = 163.14) alongside the simulations of Fig. 10.3 performed with normal gravity. A typical frozen wave pattern is illustrated in the inset provided for the hypergravity case, showing a clear decrease in wavelength compared to the patterns of Fig. 10.2.

The threshold with $g = 2g_0$ is delayed with respect to normal gravity [see Table 10.2 and Eq. (10.18)]:

$$v_c|_{2g_0} = 0.529 \text{ m/s} \quad \text{for } \epsilon = 0 \quad (10.26a)$$

$$v_c|_{2g_0} = 0.530 \text{ m/s} \quad \text{for } \epsilon \neq 0. \quad (10.26b)$$

The relative increase in threshold is approximately 1.24 ($0.529/0.427$ for $\epsilon = 0$ and $0.530/0.429$ for $\epsilon \neq 0$) and in reasonable agreement with the value of $2^{1/4} \simeq 1.19$ predicted by Eq. (10.14). The

difference may be attributed to the effect of viscosity (Lyubimov et al., 2016; Talib, Jalikop, and Juel, 2007) and to the finite container size (Salgado Sánchez et al., 2019c), the influence of which is discussed below.

10.3.4 Effect of finite container length

Figure 10.5(b) presents bifurcation diagrams for two different container sizes: $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$, shown with circles) and $L \times 2H = 60 \times 15$ mm ($\Gamma = 4$, shown with triangles). A representative frozen wave pattern is shown in the inset for the $\Gamma = 4$ case and can be compared with Fig. 10.2.

Finite-size effects are expected to play a significant role whenever the number of frozen waves appearing in the container is not large. The lateral boundary conditions in that case are an important part of the wavenumber selection problem and can be expected, in general, to prevent frozen waves at the ideal wavenumber (the minimum of the neutral stability curve for an infinitely long container) from developing; this may be thought of as wavenumber frustration. Another, more subtle, effect is the modification of the base flow due to the boundaries. The ideal horizontal counter flow assumed by Lyubimov and Cherepanov (1986) cannot hold in finite containers. Since the oscillatory flow must turn up or down as it approaches the lateral walls, this can be expected to weaken the Kelvin-Helmholtz mechanism and increase the frozen wave threshold. Consistent with the diminishing of both effects in longer containers, a slight reduction of the threshold is observed with $L = 60$ mm in Fig. 10.5(b). The overall shape of the bifurcation diagram is not significantly affected, however.

Finite-size effects are much more prominent in microgravity conditions, as discussed by Salgado Sánchez et al. (2019c), since the most dangerous perturbations according to Eq. (10.13) are longwave modes that invariably interact with the boundaries. In hypergravity, on the other hand, the selection of a higher wavenumber reduces the importance of the boundaries.

10.4 Frozen wave instability in reduced gravity

In reduced gravity conditions, the patterns that emerge from the frozen wave instability can exhibit very different behavior compared to terrestrial experiments. Equation (10.13) shows that in the limiting case of $g = 0$, the capillary length becomes infinitely large. Furthermore, the threshold (10.11) vanishes for modes with $k \rightarrow 0$ so that some type of extended frozen wave mode is expected with minimal forcing. If a larger finite forcing value is given, the length scale of the instability can be estimated by determining the fastest growing perturbation within the band of unstable wavenumbers (Burnysheva, Lyubimov, and Lyubimova, 2011; Lyubimova et al., 2017).

In addition to the contrasting features of the linear problem, the nonlinear evolution of frozen waves above onset can also differ dramatically in the absence of the (usually dominant) gravitational restoring force. Rapid growth is typically observed, with the frozen waves rearranging into large columnar structures (Gaponenko et al., 2015b; Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) that are constrained only by the size of the container (Salgado Sánchez et al., 2019c). Moreover, the

FIGURE 10.6: Time evolution (averaged over one excitation period) of the interface and mean velocity field in a container of size $L \times 2H = 30 \times 15$ mm ($\Gamma = 2$) under reduced gravity $g = 0.5 g_0$ ($Bo = 40.79$). Increasing vibrational forcing is applied: (a) $v = 0.282$, (b) 0.344 , (c) 0.364 and (d) 0.410 m/s, corresponding to $\mathcal{F} = 9.41, 11.49, 12.14$ and 13.65 , respectively. The frequency is fixed at 45 Hz with the amplitude increased from 1 to 1.45 mm. The critical forcing is $v_c = 0.347$ m/s ($\mathcal{F}_c = 11.55$); see Sec. 10.4.2. The velocity field during the transient portion of the simulation is shown with scaled arrows (scale differs between snapshots). The selected times of series (a–c) are labelled at the top while, due to more rapid growth, the intermediate snapshot in (d) is taken at 0.7 s instead of 0.9 s.

absence of gravity promotes the vibroequilibria effect (Fernández et al., 2017a), which can interact with the frozen wave instability.

Given the fact that the frozen wave instability and subsequent pattern selection process in microgravity differ so dramatically from the case of normal gravity, significant qualitative changes in the bifurcation diagram must occur as gravity (Bond number) is reduced. An analysis of the frozen wave instability under various gravity levels, including the limiting case of $g = 0$, is presented below.

10.4.1 Dependence on applied forcing

Through four series of snapshots with increasing v between 0.282 and 0.410 m/s ($\mathcal{F}$ between 9.41 and 13.65), Fig. 10.6 illustrates the evolution of the instability under reduced gravity conditions ($g = 0.5 g_0$, $Bo = 40.79$) in the $L = 30$ mm container. The simulations are performed at a fixed frequency of $f = 45$ Hz with increasing amplitudes A from 1 to 1.45 mm, all satisfying $A \ll H$. The interface position is again averaged over one excitation period, as is the transient velocity field shown in the first two snapshots of each series.

As discussed in Sec. 10.3, the interface begins to take on a different shape as the forcing increases

beyond the frozen wave threshold, $v_c = 0.347$ m/s ($\mathcal{F}_c = 11.55$); the determination of this onset value is discussed in Sec. 10.4.2. Below this threshold, as in the snapshots of Fig. 10.6(a), the interface slowly reorients by moving upward in the vicinity of the lateral walls, which is the behavior expected from vibroequilibria theory (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004). The degree of deformation is comparable to that found in the simulations of Fig. 10.2(a) under normal gravity, but for a vibrational velocity approximately 26 % smaller. The behaviour shown in Fig. 10.6(c,d) for $v = 0.364$ and 0.410 m/s, respectively, reveals a more pronounced vibroequilibria effect with greater displacement of the contact points than in Fig. 10.2(a), which was made with an intermediate value of $v = 0.380$ m/s. This magnification of the vibroequilibria effect in reduced gravity is expected.

At forcing values near the threshold of $v_c = 0.347$ m/s, small-amplitude frozen waves coexist with the underlying vibroequilibria state, as seen in Fig. 10.6(b). The waves first appear in the interior (flat) portion of the interface since the Kelvin-Helmholtz mechanism is generally suppressed by proximity to the lateral walls and by the curvature of the vibroequilibria solution. For the frozen wave instability, this results in a reduction of the effective container length.

Further above threshold, as in Fig. 10.6(c), the frozen wave amplitude grows large enough to be clearly seen above the vibroequilibria effect. This solution can be compared with that of Fig. 10.2(c), where the frozen wave profile is composed of two waves spanning approximately half of the container length, in contrast to the single peak that develops here for the same container geometry. This increase in wavelength under the reduced gravity of $g = 0.5 g_0$ is consistent with the ideal linear theory of Eq. (10.13) and with the reduction of effective container length induced by the vibroequilibria effect. The qualitative shape of the frozen wave, however, is similarly trochoid-like in both cases.

Still further above threshold, as in Fig. 10.6(d), the frozen wave grows in amplitude and undergoes a qualitative change in its shape, similar to the transition described in Sec. 10.3. Figure 10.4(b) shows that the radius of curvature at the crests again reaches an asymptotic value, in this case larger than in Fig. 10.4(a) with $g = g_0$, that limits the sharpness of the finger-like tip. Once more, the asymptotic value of r_k is comparable to the capillary length, given by Eq. (10.21).

The mean velocity field behaves in a very similar fashion in Figs. 10.6 and 10.2. The development of a frozen wave profile is associated with a structure of large vortices that travels inward, deforming the interface. While qualitatively similar, these vortices driving the interface toward the final frozen wave state are larger here, consistent with the longer wavelength (a single frozen wave peak in the final pattern).

10.4.2 Bifurcation diagram

As in Sec. 10.3.2, we obtain the bifurcation diagram characterising the frozen wave instability with $g = 0.5 g_0$ by measuring the trough-to-crest distance Δh . These values are shown in Fig. 10.7 (using solid circles) as a function of applied velocity v . Three distinct regions can be identified from that

FIGURE 10.7: Bifurcation diagram showing the trough-to-crest distance Δh of the frozen wave patterns versus vibrational velocity v for $g = 0.5 g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The measurements in the gravity-capillary regime are fit to the branching equation (10.17) of a perturbed pitchfork (dashed curve) and of the unperturbed ($\epsilon = 0$) case (solid curve). The gravity regime is fit to the linear function (10.19) (dotted curve). The transitions between different regimes (labelled above) are highlighted by vertical lines, including the new finite-depth regime where the crest approaches the upper boundary. Points corresponding to the snapshots of Figs. 10.6(b–d) and Fig. 10.8(a,b) are labelled. The diagram of Fig. 10.3 for normal gravity is included for comparison (gray open circles).

figure: the initial capillary-gravity (square-root) regime, the subsequent gravity regime and, for even higher forcing, a new regime dominated by the effect of the finite container depth. The data points corresponding to the solutions of Fig. 10.6(b–d) are labelled and distinguished by open circles, as are the two simulations shown later in Fig. 10.8. For ease of comparison, the bifurcation diagram with normal gravity from Fig. 10.3 is included using open gray circles.

Figure 10.7 reconfirms that the frozen waves instability occurs via a perturbed (asymmetric) supercritical pitchfork (Strogatz, 1994). Recall that, in the ideal linear theory, the supercritical nature depends only on the density ratio $\hat{\rho}$, while the asymmetric perturbation is mainly due to the vibroequilibria effect. As in Sec. 10.3.2, the numerical points near onset are fit to the branching equation (10.17) of a pitchfork bifurcation both in the ideal ($\epsilon = 0$) and perturbed ($\epsilon \neq 0$) cases, which yields the critical amplitudes

$$v_c|_{0.5g_0} = 0.345 \text{ m/s} \quad \text{for } \epsilon = 0, \quad (10.27a)$$

$$v_c|_{0.5g_0} = 0.347 \text{ m/s} \quad \text{for } \epsilon \neq 0, \quad (10.27b)$$

and the coefficient K . While the frozen wave threshold increases in the hypergravity case of

Sec. 10.3.3, it is reduced with lower gravity levels, by a factor of approximately 0.81 for $g = 0.5 g_0$, which is consistent with the prediction of $2^{-1/4} \simeq 0.84$ from Eq. (10.14).

For higher values of vibrational velocity, the data is fit to Eq. (10.19) to obtain the transition value v^* and the linear growth rate $\mathcal{M}$. These fits use the same RMSE and R^2 error criteria as Sec. 10.3.2, and provide the values listed in Table 10.2. The coefficient K of the capillary-gravity regime is larger than with normal gravity, reflecting the more rapid growth of frozen waves with a reduced gravitational restoring force. This is also evident from direct comparison of the bifurcation diagrams for $g = 0.5 g_0$ and $g = g_0$ (gray markers) in Fig. 10.7.

The more rapid growth of the frozen waves extends as well, if not more so, into the gravity regime — less forcing is needed to achieve a given increase in amplitude compared to the normal gravity case. The approximately linear dependence, however, does not hold beyond $v \simeq 0.416$ m/s. At this point, there is an abrupt increase in Δh followed by a kind of saturation with $\Delta h \sim H$. This transition is explained by the finite depth of the container, which affects and limits the growth of large-amplitude waves, as described in the following section.

Effect of finite container depth

Figure 10.8 shows the evolution of the frozen wave pattern in the finite-depth regime for 45 Hz forcing with amplitudes of $v = 0.430$ m/s ($\mathcal{F} = 14.31$) and $v = 0.438$ m/s ($\mathcal{F} = 14.59$). The initial evolution is broadly similar in both cases to that previously illustrated in Fig. 10.6(d). Vortices appear near the lateral walls accompanying the motion of the contact points and the progression of the deformations (frozen waves) toward the interior of the container. As observed in Figs. 10.2 and 10.6, the wave profile is not reflection symmetric about the horizontal midline. In addition to vibroequilibria effects, the troughs are broader and shallower than the crest(s). If the forcing is strong enough, the central crest reaches the vicinity of the upper wall while the troughs remain a certain distance from the lower boundary (see the snapshots at $t = 1.3$ and 2 s). The increase in forcing from 0.430 to 0.438 m/s does not noticeably affect the vertical location of the crest but does broaden it slightly, while the troughs grow a bit deeper and more separated.

For the forcing values of Fig. 10.8, the tip of the finger-like wave flattens and avoids a true collision with the upper wall. Prior to this, the maximum curvature of the tip takes an asymptotic value near the capillary length, but there can be a significant increase in curvature in the finite-depth regime at the “corners” of the flattened tip. This suggests that interfacial energy is locally balanced here by strong fluid motion (through the narrow gap above the tip) rather than gravity. At sufficient forcing amplitude, the interface will collide with the upper boundary and split to create a columnar shape, as commonly observed following the frozen wave instability in weightlessness (Gaponenko et al., 2015b; Salgado Sánchez et al., 2019d; Shevtsova et al., 2016). Behavior similar to that of Fig. 10.8 was described by Salgado Sánchez et al. (2019c) for patterns resulting from the frozen wave instability in microgravity in finite containers.

FIGURE 10.8: Time evolution (averaged over one excitation period) of the interface and mean velocity field illustrating the frozen wave instability for $g = 0.5 g_0$ in the finite-depth regime in a container of size $L \times 2H = 30 \times 15$ mm vibrated at $f = 45$ Hz and amplitudes: (a) $A = 1.52$ mm ($v = 0.430$ m/s, $\mathcal{F} = 14.31$), (b) $A = 1.55$ mm ($v = 0.438$ m/s, $\mathcal{F} = 14.59$). The time of the snapshot is labelled between the columns. The velocity field is shown using scaled arrows (scale differs between snapshots). The Δh values obtained from these two simulations are labelled in Fig. 10.7.

Mode competition

As described in Salgado Sánchez et al. (2019c), the container dimensions play a crucial role in the pattern selection process in reduced gravity. The finite length L permits only certain wavenumbers and suppresses the others. Transitions between distinct modes occur as the forcing increases since the wavenumber of the fastest growing perturbation generally increases with forcing (Lyubimova et al., 2017). For the vibrational amplitudes of Fig. 10.6, the fastest growing perturbation is close to a mode with two large troughs and a single crest in the middle of the container, which is eventually

selected as the final pattern. Modes of higher wavenumber can be observed, over finite intervals, at higher forcing.

The pattern selection process in this system is complicated by the vibroequilibria effect, as mentioned Sec. 10.3. The heightened curvature of the average interface near the lateral walls, in addition to shifting the threshold, can be assumed to reduce the effective length of the container that is susceptible to Kelvin-Helmholtz instability, thereby directly affecting wavenumber selection. The vibroequilibria effect also promotes those solutions having the heavier fluid displaced upward near both lateral walls. In general, the complicated interaction between vibroequilibria and frozen waves continues into the nonlinear regime and can be expected to influence mode interactions like that seen in Fig. 10.8.

For convenience, we define the pattern wavenumber $\mathcal{K}$ as the number of troughs observed in the frozen wave profile. For example, the final patterns shown in Figs. 10.2(d) and 10.6(d) corresponds to the $\mathcal{K} = 3$ and $\mathcal{K} = 2$ modes, respectively. The snapshots of Fig. 10.8 at $t = 0.5$ s indicate that the initially dominant frozen wave perturbation is close to the $\mathcal{K} = 3$ mode but, by $t = 1.3$ s, the small-amplitude wave in the center of the container is absorbed into the faster-growing $\mathcal{K} = 2$ mode.

Vibrational forcing was increased in order to encourage the $\mathcal{K} = 3$ mode in Fig. 10.9, which shows snapshots of the interface development in the same container with $g = 0.5g_0$ vibrated at $f = 45$ Hz, but with $A = 1.6$ mm ($v = 0.452$ m/s). Up to $t \simeq 0.5$ s, the average interface is almost left-right symmetric, with both crests growing at similar rates. With more time, however, the solution becomes increasingly asymmetric as the crest on the right, but not the left, grows to reach the vicinity of the upper wall by $t \simeq 1.5$ s. The amplitude of the left crest then diminishes and disappears, completing a transition to the $\mathcal{K} = 2$ mode by $t \simeq 2.5$ s. It requires forcing exceeding $v = 0.452$ m/s to fully stabilize the $\mathcal{K} = 3$ mode.

10.4.3 Microgravity

As demonstrated above, a reduction in the gravity level has a clear effect on the frozen wave instability, lowering its threshold and allowing more rapid growth of large-amplitude patterns. The experiments of Shevtsova et al. (2016) with miscible liquids and those of Salgado Sánchez et al. (2019d) with immiscible liquids both showed that, in microgravity, the finite container depth is what limits wave growth in moderately sized containers. Initially sinusoidal frozen wave perturbations grow quickly into large columnar structures stretching between the lower and upper boundaries. The threshold for observing these patterns is extremely low in microgravity and vanishes in the ideal limit of an infinitely long container with $g = 0$, as seen from Eq. (10.11). It is the finite container length that determines the minimum wavenumber and, thus, the threshold for instability (Salgado Sánchez et al., 2019c).

This theoretical threshold, however, is modified by the vibroequilibria effect, which is magnified in microgravity and belies the assumption of an underlying flat interface. The effect is proportional to v^2 and has no threshold (Gavrilyuk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997) so, prior

FIGURE 10.9: Time evolution (averaged over one excitation period) of the frozen wave instability with $g = 0.5g_0$ in a container of size $L \times 2H = 30 \times 15$ mm vibrated at $f = 45$ Hz with $A = 1.6$ mm ($v = 0.452$ m/s, $\mathcal{F} = 15.06$). A mode transition between the $\mathcal{K} = 3$ and $\mathcal{K} = 2$ patterns is observed. The snapshot times are labelled and the mean velocity field during the transient period is shown using scaled arrows (scale differs between snapshots). For $t \lesssim 0.5$ s the evolution is similar to that of Fig. 10.8(b).

to the frozen wave instability, the average interface profile is that of the symmetric vibroequilibria state (Fernández et al., 2017b). The amplitude of this state is characterised in Fig. 10.10(a) by the difference Δh between the height of the contact points and the interface at the midpoint of the container. This figure confirms the predicted quadratic dependence: $\Delta h \propto v^2$. Furthermore, the vibroequilibria solutions are in good agreement with the theoretical results of Fernández et al. (2017b) and the microgravity experiments of Salgado Sánchez et al. (2019a).

Although the vibroequilibria state qualitatively resembles the $\mathcal{K} = 1$ frozen wave mode, one can still locate an instability threshold separating continuous vibroequilibria growth from the rapid formation of a columnar pattern like that shown in the inset of Fig. 10.10(b). As noted earlier, the preexisting vibroequilibria state strongly favours a frozen wave (columnar) solution with the heavier liquid along the sides and (for $\mathcal{K} = 1$) a doubly wide region of lighter liquid in the interior. We note that this final configuration can also develop directly from the symmetric vibroequilibria state if the amplitude reaches $\Delta h \simeq 2H$.

FIGURE 10.10: (a) Evolution of the symmetric vibroequilibria state in microgravity with increasing velocity v in a container of size $L \times 2H = 30 \times 15$ mm. The difference Δh between the height of the contact points and the midpoint of the interface is shown and fitted (dashed line) to a quadratic function as per the theoretical prediction. The inset illustrates a typical vibroequilibria state and the Δh measurement. (b) Inverse square of the frozen wave growth time, τ^{-2} (open squares), for the smallest wavenumber $\mathcal{K} = 1$ versus v . The observed linear dependence allows for an estimation of the instability threshold (black marker). The inset shows the final columnar pattern resulting from the $\mathcal{K} = 1$ mode.

To locate the frozen wave threshold for $g = 0$, we follow Salgado Sánchez et al. (2019c) and measure the transient time τ required for the interface at the central trough to reach the bottom of the container. For stable vibroequilibria states, τ is infinite, while it decreases beyond the frozen wave threshold. From the observation that τ^{-2} is nearly linear in v and the assumption that the frozen wave instability is either subcritical or nearly degenerate (i.e., there is no significant interval of bounded frozen waves with $\Delta h < 2H$), one can estimate the threshold v_c from the intercept of a simple linear fit. The measured (inverse square) growth times are shown in Fig. 10.10(b) (open squares) along with the linear fit (dashed line) and the estimated threshold (solid square) at $v_c \simeq 0.124$ m/s.

10.5 Effect of Bond number

Figure 10.11 summarises the results of a systematic series of numerical simulations with gravity levels (in units of g_0) of 0.25, 0.5, 0.75, 1, 1.5 and 2. The same analysis and fitting procedure is performed at each g value to construct Fig. 10.12, which shows the dependence on Bond number (Bo) of the critical amplitude $\mathcal{F}_c$, the transition value $\mathcal{F}^*$ that marks the beginning of the gravity regime, and the remaining fitting parameters expressed in dimensionless form (denoted by a hat),

$$(\hat{K}, \hat{\mathcal{M}}) = \sqrt{\frac{\sigma}{(\rho_1 - \rho_2)H^3}} (K, \mathcal{M}), \quad \hat{\epsilon} = \frac{\epsilon}{H^3}. \quad (10.28)$$

FIGURE 10.11: Bifurcation diagrams obtained for the $L \times 2H = 30 \times 15$ mm container with varying gravity levels (in units of g_0): 0.25, 0.5, 0.75, 1, 1.5, 2. The data sets are distinguished with different markers and labelled with the gravity level used. Note that for $g = 0.25 g_0$ and $0.5 g_0$, the finite-depth regime discussed in Sec. 10.4.2 is visible at large amplitude.

In Fig. 10.12(a,b), the fitting parameters obtained with $\hat{\epsilon} = 0$ are shown with open circles, while those obtained with the $\hat{\epsilon} \neq 0$ values of Fig. 10.12(d) are shown with crosses; the values often overlap.

From the results summarised in these figures, the following conclusions about the influence of gravity may be drawn.

- (i) The frozen wave threshold $\mathcal{F}_c$ decreases with decreasing Bo until reaching a minimum value determined by the longest wavelength allowed in the container (Salgado Sánchez et al., 2019c).
- (ii) The branching coefficient $\hat{K}$ of the capillary-gravity regime grows with decreasing Bo and appears to diverge as $\text{Bo} \rightarrow 0$.
- (iii) The slope $\hat{\mathcal{M}}$ of the approximately linear growth in the gravity regime similarly grows with decreasing Bo and appears to diverge as $\text{Bo} \rightarrow 0$.

The apparent divergence of $\hat{K}$ and $\hat{\mathcal{M}}$ with decreasing Bo reflects the crucial role of gravity as a restoring force for frozen waves. Gravity dominates for large-amplitude waves and is commensurate with interfacial tension near onset. Since the scale of the frozen wave instability in an infinitely deep and extended system is set by the capillary length l_c , both gravitational and interfacial restoring

FIGURE 10.12: Dimensionless fitting parameters characterising the frozen wave bifurcation diagram as a function of the Bond number Bo : (a) critical forcing $\mathcal{F}_c$ (open circles for $\hat{\epsilon} = 0$, crosses for $\hat{\epsilon} \neq 0$) and the point of transition to the gravity regime $\mathcal{F}^*$ (solid circles), (b) branching coefficient $\hat{K}$ of the gravity-capillary regime (open circles for $\hat{\epsilon} = 0$, crosses for $\hat{\epsilon} \neq 0$), (c) slope $\hat{\mathcal{M}}$ of the gravity regime, (d) symmetry-breaking parameter $\hat{\epsilon}$. The predicted value of Eq. (10.12) for the critical forcing is shown with a solid curve in (a) while the theoretical formula for $\hat{K}$ from Eq. (10.16) is shown in (b). The dashed curve in (c) shows a fit to Bo^{-1} while the dotted curve in (d) is intended only as a visual aid. The values of $\mathcal{F}^*$ and $\hat{\mathcal{M}}$ change imperceptibly between the $\hat{\epsilon} = 0$ and $\hat{\epsilon} \neq 0$ cases.

forces are in balance near onset and decrease together with decreasing Bo (see Sec. 10.3.2). For $Bo = 0$, there is no restoring force at all and the pitchfork bifurcation is degenerate ($\hat{K} = \infty$). This argument, however, does not hold for finite containers since there is a minimum allowed wavenumber, which arrests the declining influence of interfacial tension. For large enough σ in a container of finite

length, it should be possible to observe small-amplitude frozen waves even with $\text{Bo} = 0$. Such small-amplitude frozen waves were not seen with the parameters of Table 10.1 suggesting that, for realistic liquids in containers of the size used here, or larger, the (perturbed) pitchfork bifurcation to frozen waves with $g = 0$ is, effectively, degenerate. The same conclusion can be inferred directly from the tendency of the bifurcation diagrams in Fig. 10.11.

10.5.1 Comparison with theory

The expression (10.12) derived by Lyubimov and Cherepanov (1986) can be compared with the critical forcing $\mathcal{F}_c$ obtained from the fitting procedure. Since the wavenumber here is limited by the length L , we minimize Eq. (10.12) over wavenumbers between $k_{\min} = 2\pi/L$ and $k_{\max} \approx 7\pi/L$, which is the maximum observed in the simulations. This prediction is shown in Fig. 10.12(a) with a solid curve. Note that the measured threshold values are in good agreement with the dependence of the theoretical curve, except for a delay of approximately 20 %. This shift is likely due to a combination of the vibroequilibria effect, other finite-size effects, and viscosity. In fact, the viscosity contrast used here, $\hat{\nu} = 20$, lies in a range shown by Talib, Jalikop, and Juel (2007) to delay the onset of frozen waves compared to the prediction of the inviscid linear theory.

The branching coefficient $\hat{K}$ in the case of an infinitely deep extended system is that of Eq. (10.16):

$$\hat{K} = \sqrt{\frac{16(1 - \hat{\rho}^2)}{\hat{\rho}(42\hat{\rho} - 11 - 11\hat{\rho}^2)}} \text{Bo}^{-3/4} \simeq 1.688 \text{Bo}^{-3/4}. \quad (10.29)$$

This expression is evaluated for $\hat{\rho} = 0.52$, which is the density ratio for 100 cSt silicone oil and FC-40, and is shown as a solid curve in Fig. 10.12(b). The agreement of the fitting coefficient with Eq. (10.29) is excellent, despite the fact that wavenumber selection in the simulations is complicated by vibroequilibria and finite-size effects. The level of agreement further suggests that viscosity does not significantly affect the weakly nonlinear frozen wave growth.

10.5.2 Symmetry-breaking due to vibroequilibria

The symmetry-breaking effect of vibroequilibria is quantified by the parameter $\hat{\epsilon}$. Figure 10.12(d) shows the values obtained with Bond numbers corresponding to $g = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 2g_0$. As expected from the flattening effect of gravity on the interface, $\hat{\epsilon}$ decreases with increasing Bond number. Strong gravity masks the preference for solutions with heavier fluid along the lateral walls, and the bifurcation diagram more closely resembles an ideal supercritical pitchfork.

10.6 Conclusions

The influence of gravity on the frozen wave instability induced by periodic horizontal excitation was investigated with extensive numerical simulations modeling two immiscible fluids with properties characteristic of FC-40 and 100 cSt silicone oil, which are relevant to recent microgravity (Salgado Sánchez et al., 2019d, 2020) and ground (Jalikop and Juel, 2009) experiments. The simulations were begun from an initially flat interface separating liquids of equal volume, used gravity levels

from 0 to $2g_0$, and (most) were performed in a 30×15 mm rectangular domain. The dynamics were resolved using a standard level-set formulation by means of the finite element method, as described in Sec. 10.2.

The evolution of the interface following the frozen wave instability was considered under terrestrial and hypergravity conditions in Sec. 10.3 and characterised by the trough-to-crest distance for different applied velocities. The resulting diagram shows that frozen waves emerge via a perturbed (asymmetric) supercritical pitchfork bifurcation. The symmetry-breaking is attributed to the vibroequilibria effect, which promotes a fluid distribution with the heavier fluid displaced upward along the lateral walls. Near onset, wave amplitude varies as the square root of the applied velocity difference, consistent with the prediction of inviscid theory in infinitely deep, extended systems (Lyubimov and Cherepanov, 1986). The threshold and branching coefficients are determined from least squares fits, as described in Sec. 10.3.2. The interface often exhibits an inverted trochoid-like shape, in agreement with the experiments of Jalikop and Juel (2009).

As the forcing is raised further above onset, a transition is observed wherein the frozen waves adopt a more finger-like profile and the amplitude grows nearly linearly with applied velocity. This transition is also evident from the radius of curvature (Jalikop and Juel, 2009) at the crests, which near this point reaches an asymptotic value comparable to the capillary length. As the shape of the tips (crests) remains nearly constant, an increase in forcing (equivalently, wave amplitude) is limited primarily by gravity, with interfacial energy playing a diminishing role. This is in contrast to the situation near onset, where interfacial and gravitational forces are of similar magnitude.

The frozen wave instability in reduced gravity is considered in Sec. 10.4. As anticipated from the ideal linear theory (Lyubimov and Cherepanov, 1986), the threshold is lowered. Furthermore, beyond the transition from the initial gravity-capillary (square-root) regime to the gravity-dominated (linear) regime, a third type of behavior is found when the wave amplitude becomes comparable to the container depth. The growth of the crests is bounded in this regime by the upper wall of the container, while the troughs remain a certain distance from the bottom; this is reflected in the bifurcation diagram as a kind of amplitude saturation. In the limit of zero gravity, the frozen waves rapidly develop into a series of alternating columns spanning the container height. The threshold, although low, remains finite in containers of finite length, since there is a minimum allowed wavenumber, and is calculated following the method of Salgado Sánchez et al. (2019c).

The results of all simulations are summarised through the parameters defining the bifurcation diagram, shown as a function of Bond number (Bo) in Sec. 10.5. The threshold is compared with the prediction of the ideal linear theory and the branching coefficient with the prediction of the corresponding weakly-nonlinear theory (Lyubimov and Cherepanov, 1986). Qualitative agreement is obtained for the threshold, which is delayed due to viscosity (Talib, Jalikop, and Juel, 2007), vibroequilibria and finite-size effects (Salgado Sánchez et al., 2019c). Excellent agreement is obtained for the branching (nonlinear) coefficient despite the complicated wavenumber selection occurring in this system, suggesting as well that viscosity does not significantly affect the nonlinear frozen wave

growth, at least near onset. Since both coefficients relating wave amplitude to forcing appear to diverge as $Bo \rightarrow 0$, the (perturbed) pitchfork bifurcation can be considered to be effectively degenerate for $g = 0$. However, because the symmetry-breaking effect of vibroequilibria is magnified in this limit, there is no "true" bifurcation but, rather, a continuous transition from a vibroequilibria to a frozen wave solution. Despite this fact, the suddenness of the (nearly degenerate) transition allows a threshold to be located with reasonable accuracy. In the large Bond number limit, on the other hand, the vibroequilibria effect is reduced and the bifurcation approaches the predicted supercritical pitchfork.

Acknowledgment. The work of D.G, P.S.S and J.P was supported by the Ministerio de Economía y Competitividad under Project No. ESP2015-70458-P. The work of V.S. was supported by the PRODEX programme of the Belgian Federal Science Policy Office. We also thank A. Nepomnyaschy for helpful discussions.

Chapter 11

Symmetry breaking in large columnar frozen wave patterns in weightlessness

Troitiño, M., Salgado Sánchez, P., Porter, J. and Gligor, D.
Microgravity Science and Technology (2020).
<https://doi.org/10.1007/s12217-020-09812-0>

The large columnar patterns resulting from the frozen wave instability in zero gravity conditions are investigated numerically for finite containers of immiscible liquids. The mode selection process is affected by forcing amplitude, frequency, container size and the vibroequilibria effect. The final columnar states are often approximately left-right symmetric and a quantitative measure of their asymmetry is used to distinguish three different growth regimes depending on the applied vibration. Solutions at low frequencies are affected by the antiphase forcing at the lateral walls and by early collisions of the longwave modes with the upper and lower boundaries. At moderate frequencies, the development of the pattern is characterized by centrally located defects and nearly symmetric modes. At higher frequencies, there are off-center defects and asymmetric modes. The critical value of the symmetry-breaking transition separating these final two types of behavior is determined as a function of the container aspect ratio.

11.1 Introduction

Symmetry is an inherent feature of innumerable physical systems that may arise from the fundamental laws as expressed by the governing equations, from the domain geometry and boundary conditions and, during analysis, from any mathematical procedures that are used to reduce or simplify the original equations. Results based on group theory can be a powerful tool for organising the subspaces, solutions and bifurcations of such systems according to their symmetry properties (Hoyle, 2006). In the familiar example of a local symmetry-breaking bifurcation, an existing symmetric solution loses stability to perturbations of less symmetry, which drives the system to a new asymmetric solution.

Pattern forming systems almost invariably exhibit symmetry-breaking bifurcations of this type, at least under the simplifying assumptions that are commonly made. The Faraday instability (Faraday, 1831) in an open container of fluid, for example, is normally analysed after assuming a perfectly flat horizontal interface subjected to periodic vertical vibrations. If the container is large, so that boundaries may be ignored, then the unperturbed flat state has the symmetry of the plane (reflections, translations, rotations). The bifurcation producing Faraday waves breaks this symmetry in creating rolls, squares, hexagons, or more complicated patterns (see, e.g., Arbell and Fineberg, 2002).

The frozen wave instability (Ivanova, Kozlov, and Evesque, 2001; Lyubimov and Cherepanov, 1986; Wolf, 1969) that occurs in horizontally vibrated layers of distinct fluids due to the Kelvin-Helmholtz mechanism is another well-known pattern forming transition. As with the Faraday system, the unperturbed state is usually assumed to be flat and lateral boundaries are ignored so that the oscillatory counterflow is translationally invariant. The bifurcation producing quasisteady frozen waves breaks this symmetry, replacing it with one of discrete translation by one wavelength. The principal results (Lyubimov and Cherepanov, 1986) are also obtained under the assumptions of inviscid flow and a separation of timescales, which dictates that the phase of the (high frequency) forcing, and that of induced oscillatory phenomena like harmonic or subharmonic surface waves, is irrelevant to the slower evolution of the frozen waves.

The frozen wave bifurcation under ordinary gravity is supercritical for most practical combinations of liquids (Jalilop and Juel, 2009; Lyubimov and Cherepanov, 1986) and, thus, leads initially to stable small-amplitude waves. However, this picture changes dramatically in microgravity environments where the threshold is much lower and the frozen waves rapidly grow into large columnar structures (Beysens et al., 2009; Gaponenko et al., 2015b; Shevtsova et al., 2016) extending across the full height of the container. These columnar states are complex, strongly nonlinear, and constrained in amplitude only by the container size.

In microgravity, frozen waves are also strongly affected by the vibroequilibria phenomenon, which results from the spatial inhomogeneity of the underlying oscillatory flow and the consequent slow deformation of the average interface toward a new quasisteady configuration (Beyer et al., 2001;

Fernández et al., 2017b; Lyubimov et al., 1997; Wolf, 1969). In horizontally vibrated containers, the inhomogeneity of this oscillatory flow is strongest near the lateral boundaries and, if the denser fluid initially occupies the lower layer, the vibroequilibria effect causes it to move upward along them (Fernández et al., 2017b; Lyubimov et al., 2016; Salgado Sánchez et al., 2019a). This process clearly favors frozen wave patterns sharing that same feature (Gligor et al., 2020).

Here, we investigate the properties of large-amplitude columnar frozen wave patterns in weightless conditions through a series of numerical simulations. In addition to the vibroequilibria effect, the final patterns depend on forcing amplitude, container size, and the suitability of a separation of timescales. A function measuring left-right asymmetry is introduced and used to characterise the distinct regimes of column development. We find evidence for a type of spontaneous symmetry-breaking transition that separates central defect growth and symmetric modes from off-center defect growth and asymmetric modes.

The paper is organised as follows. The mathematical formulation and numerical approach are presented in Sec. 11.2. The features of the frozen wave instability in microgravity are discussed in Sec. 11.3, including the predictions of established theory and the parameters affecting mode selection. A quantitative measure of asymmetry is defined in Sec. 11.4 and used to distinguish three distinct columnar growth regimes as the forcing (frequency) is varied. The critical value of the symmetry-breaking transition separating the final two regimes is examined in Sec. 11.5, with final conclusions given in Sec. 11.6.

11.2 Mathematical formulation

We consider a rectangular container holding two layers of immiscible liquids, as sketched in Fig. 11.1. The liquids are of equal volume with an unperturbed height H for each layer. The interior dimensions of the container are $L \times 2H$, but with the corners rounded (with radius 0.25 mm) to avoid numerical difficulties and match the setup used in recent microgravity experiments (Salgado Sánchez et al., 2019d, 2020). The entire container is vibrated with amplitude A and frequency ω along the x (horizontal) direction parallel to the initial unperturbed interface, which is marked with a dashed line in Fig. 11.1. The heavier liquid begins in the lower layer so that the initial configuration is gravitationally stable.

The dynamics of the interface in response to the applied vibrations is resolved with a level-set method (Olsson and Kreiss, 2005; Salgado Sánchez et al., 2019c,d, 2020; Sethian, 1999). The evolution of the level-set function ϕ is given by

$$\phi_t + \nabla \cdot (\mathbf{u}\phi) = \gamma \nabla \cdot \left(\varepsilon \nabla \phi - \phi(1 - \phi) \frac{\nabla \phi}{|\nabla \phi|} \right), \quad (11.1)$$

where γ is a tuning parameter that is set equal to $A\omega$ and ε is a small diffusive term, which is included for numerical stability and is of the order of the maximum mesh element size. The nonlinear compression flux term $\phi(1 - \phi)\nabla \phi/|\nabla \phi|$, which was first introduced by Olsson and Kreiss (2005)

FIGURE 11.1: Sketch illustrating the two-layer system of immiscible liquids, with the selection of FC-40 and silicone oil indicated. The rectangular container of size $L \times 2H$ is subjected to vibrations of frequency ω and amplitude A along the x axis and parallel to the initial flat interface (dashed line).

and is a standard part of the implementation of the level-set method of COMSOL Multiphysics, is included to counteract the effect of numerical diffusion on the width of the interface.

The level-set function ϕ is used to define continuous density and viscosity fields throughout the domain. The sudden change of these physical properties across the interface is modelled as

$$\rho = \rho_1 + (\rho_2 - \rho_1) \mathcal{H}, \quad (11.2a)$$

$$\mu = \mu_1 + (\mu_2 - \mu_1) \mathcal{H}, \quad (11.2b)$$

with a smoothed Heaviside function $\mathcal{H}$ to improve numerical stability:

$$\mathcal{H}(\tilde{\phi}) = \begin{cases} 0 & \tilde{\phi} < -\varepsilon, \\ \frac{1}{2} + \frac{\tilde{\phi}}{2\varepsilon} + \frac{1}{2\pi} \sin\left(\frac{\pi\tilde{\phi}}{\varepsilon}\right) & |\tilde{\phi}| \leq \varepsilon, \\ 1 & \tilde{\phi} > \varepsilon. \end{cases} \quad (11.3)$$

Here, $\tilde{\phi} = \phi - 0.5$, with the heavier and lighter liquids denoted by subscripts 1 and 2 in Eqs. (11.2) and associated with the values $\phi = 0, 1$, respectively. The contour $\phi = 0.5$ defines the interface.

Equations (11.1)–(11.3) are combined with the incompressible Navier-Stokes equations (Kothe, Mjolsness, and Torrey, 1991), written as

$$\rho [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] = -\nabla p + \nabla(\mu \nabla \mathbf{u}) + \delta \mathbf{f}_\sigma + \rho \mathbf{G}(t), \quad (11.4a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (11.4b)$$

with interfacial tension σ included through the force $\mathbf{f}_\sigma$:

$$\mathbf{f}_\sigma = (\sigma \kappa) \mathbf{n}, \quad \kappa = \nabla \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \quad (11.5)$$

	ρ (kg/m ³)	ν (10 ⁻⁶ m ² /s)
FC-40	1855	5
20 cSt silicone oil	950	20
	σ (10 ⁻³ N/m)	β (rad)
FC-40 and silicone oil	6.021	$\pi/2$

TABLE 11.1: Selected physical properties of FC-40, 20 cSt silicone oil, and their interface: density ρ , kinematic viscosity ν , surface tension σ (reproduced from Someya and Munakata, 2005), and contact angle β (chosen for an initial flat interface). We note that the real kinematic viscosity of FC-40 is 2.2 cSt but this is increased here to 5 cSt to delay the secondary Faraday wave instability.

where κ is the curvature of the interface and $\mathbf{n}$ is a unit normal vector. The parameter δ , expressed in terms of the level-set function as

$$\delta = 6\phi(1 - \phi)|\nabla\phi|, \quad (11.6)$$

keeps the effect of $\mathbf{f}_\sigma$ localised near the interface.

The oscillatory forcing is combined, in the noninertial frame of the container, with the steady vertical gravitational acceleration g (if present) to get the final term of Eq. (11.4a):

$$\mathbf{G}(t) = -g\mathbf{j} + A\omega^2 \cos\omega t \mathbf{i}, \quad (11.7)$$

with $(\mathbf{i}, \mathbf{j})$ the unit vectors along the x and y axes, respectively.

Navier-slip boundary conditions are applied on the solid walls in order to both preserve the contact angle β and allow for motion of the contact line, which is observed in experiments (Salgado Sánchez et al., 2019d, 2020):

$$\mathbf{u} \cdot \mathbf{n}_w = 0, \quad \mathbf{u} \cdot [\sigma(\mathbf{n}_w - \mathbf{n} \cos \beta)\delta] = 0, \quad (11.8)$$

where $\mathbf{n}_w$ is a unit vector normal to the wall. These boundary conditions invoke an artificial frictional force $\mathbf{f}_v = (\rho\nu)/\delta_v \mathbf{u}$, where ν is the kinematic viscosity and $\delta_v = \sqrt{\nu/\omega}$ is the width of the vibrational viscous boundary layer.

The initial conditions of the simulations impose motionless fluids ($\mathbf{u} = 0$) separated by a perfectly flat interface (dashed line of Fig. 11.1) with $\beta = \pi/2$. To increase numerical stability, the vibrational forcing is initiated in the manner of Eq. (11.3) with a transient time of 0.25 s as in Salgado Sánchez et al. (2019c,d, 2020).

The subsequent dynamics are obtained by solving Eqs. (11.1)–(11.8) with parameters characteristic of FC-40 and 20 cSt silicone oil, which are listed in Table 11.1. This selection is comparable to

that used in recent parabolic flight experiments (Salgado Sánchez et al., 2019d, 2020), where a complex interaction between columnar patterns developing from the frozen wave instability and secondary subharmonic Faraday waves was observed. The increase in viscosity of the FC-40 oil delays the appearance of these Faraday waves (Salgado Sánchez et al., 2019c,d) and thus simplifies the analysis of the primary frozen wave structure, which is the interest here. The combination of liquids in Table 11.1 is also similar to the silicone oil (100 cSt and 200 cSt) and Galden HT135 used in the ground experiments of Jalikop and Juel (2009).

If the time is scaled by $\sqrt{(\rho_1 - \rho_2)H^3/\sigma}$, length by H and density by $\rho_1 - \rho_2$, then the system is governed by the dimensionless forcing (velocity) and frequency:

$$\mathcal{F} = A\omega\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad \Omega = \omega H\sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad (11.9)$$

by the capillary and Bond numbers:

$$\text{Ca} = \nu_1\sqrt{\frac{(\rho_1 - \rho_2)}{\sigma H}}, \quad \text{Bo} = \frac{(\rho_1 - \rho_2)H^2}{\sigma}g, \quad (11.10)$$

and by the density, viscosity, and aspect ratios:

$$\hat{\rho} = \frac{\rho_2}{\rho_1}, \quad \hat{\nu} = \frac{\nu_2}{\nu_1}, \quad \Gamma = \frac{L}{2H}. \quad (11.11)$$

The simulations of frozen wave patterns in this work are conducted with $g = 0$ and a fixed height of $2H = 7.5$ mm, while the length L is varied. Together with the parameters of Table 11.1, this choice corresponds to $\text{Ca} = 0.032$, $\text{Bo} = 0$, $\hat{\rho} = 0.512$ and $\hat{\nu} = 4$. We also note that the radius of the rounded corners is fixed at $r = 0.25$ mm, for which the dimensionless ratio $r/(2H) \simeq 0.03 \ll 1$; this small deviation from a perfect rectangle helps avoid numerical difficulties at the corners but, otherwise, has a negligible effect on the results.

The scientific simulation software COMSOL Multiphysics, which includes a standard implementation of the level-set method, is used for solving the two-dimensional governing equations (11.1)–(11.8) in dimensional variables with the finite element method. Numerical convergence was previously tested in Salgado Sánchez et al. (2019c,d) for a similar configuration, and we make use of those results to select the number of mesh elements and the interface thickness ε . The simulations here are performed with a maximum element size of $\mathcal{S} = (7.5/25)$ mm, while ε is set $(3/2)\mathcal{S}$. Temporal evolution is implemented with a Generalised- α (Erlicher, Bonaventura, and Bursi, 2002) integration scheme in combination with streamline (Harari and Hughes, 1992) and crosswind (Codina, 1993) stabilization techniques. The maximum time step is set to $1/(10f)$, with smaller steps automatically taken as required for stability. Further details of this numerical model can be found in Salgado Sánchez et al. (2019c,d, 2020).

11.3 Frozen waves in microgravity

The frozen wave instability leads to dramatically different patterns in microgravity compared to ordinary gravity. Not only do frozen waves appear at much lower forcing values, but they grow rapidly into large columnar structures that interact both with the container length and with its height (Gaponenko et al., 2015b; Salgado Sánchez et al., 2019c). They develop in a nonuniform fashion that depends on the way in which the forcing is initiated, often moving from the lateral walls inward. Mode competition and defect formation are common. Moreover, since the final state is composed of a series of nearly vertical interfaces separating the columns, it is susceptible to secondary instabilities like that of Faraday waves (Lyubimova et al., 2019; Shevtsova et al., 2016) and, beyond that, drop ejection (Salgado Sánchez et al., 2019d).

Several of these complicating factors are anticipated in the results of classical frozen wave theory, which is summarised in the following section.

11.3.1 Theoretical predictions

Most analytical results on frozen waves follow Lyubimov and Cherepanov (1986) in assuming two infinitely extended layers of immiscible fluids in a gravitationally stable configuration subjected to sinusoidal horizontal vibrations. The initial interface is taken to be perfectly flat and the frequency ω of the vibrations much greater than the inverse of the characteristic viscous time ($\omega \gg \nu/H^2$). The assumption of high frequency allows a separation of timescales between the oscillating horizontal counterflow induced by the fast vibrations, and the slower growth of the frozen waves.

Viscosity does not affect the linear problem, which reveals an instability to steady perturbations of wavenumber k at a critical forcing velocity $v = A\omega$ given by

$$v^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)^2} \left[\sigma k + \frac{(\rho_1 - \rho_2)g}{k} \right] \tanh(kH), \quad (11.12)$$

which, in dimensionless form, becomes

$$\mathcal{F}^2 = \frac{(1 + \hat{\rho})^3}{2\hat{\rho}(1 - \hat{\rho})} \left[kH + \frac{\text{Bo}}{kH} \right] \tanh(kH). \quad (11.13)$$

For sufficiently deep layers with $H > H^* = [3\sigma/(\rho_1 - \rho_2)g]^{1/2}$, the minimum of Eq. (11.12) occurs at finite k and, for deep layers with $H \gg H^*$, this critical wavenumber is very close to the inverse of the capillary length l_c :

$$k_c = 1/l_c = \sqrt{\frac{(\rho_1 - \rho_2)g}{\sigma}}. \quad (11.14)$$

Upon neglecting the $\tanh(kH)$ term in Eq. (11.12), the critical forcing velocity v_c at $k = k_c$ is found from

$$v_c^2 = \frac{(\rho_1 + \rho_2)^3}{\rho_1 \rho_2 (\rho_1 - \rho_2)} \sqrt{\frac{\sigma g}{\rho_1 - \rho_2}}. \quad (11.15)$$

For thin layers with $H < H^*$, on the other hand, the minimum occurs at $k = 0$ and longwave perturbations are the most dangerous. These appear above a critical forcing velocity given by

$$v_c^2 = \frac{(\rho_1 + \rho_2)^3 g H}{2 \rho_1 \rho_2 (\rho_1 - \rho_2)}. \quad (11.16)$$

Equations (11.14)–(11.16) show that, regardless of the height H , a finite frozen wave threshold requires finite gravity. The ideal inviscid theory of infinitely extended systems predicts the vanishing of both v_c and k_c for $g = 0$.

Lyubimov and Cherepanov (1986) also performed a weakly-nonlinear analysis and obtained an expression for the (peak-to-peak) amplitude h of the frozen wave pattern (see also Gandikota et al., 2014b) in the case of deep layers with k_c given by Eq. (11.14):

$$h^2 = \frac{16 \rho_1 (\rho_1 + \rho_2) \sqrt{\sigma (\rho_1 - \rho_2)}}{\rho_2 (42 \rho_1 \rho_2 - 11 \rho_1^2 - 11 \rho_2^2)} g^{-3/2} (v^2 - v_c^2). \quad (11.17)$$

This equation predicts a supercritical bifurcation to stable small-amplitude waves when $\hat{\rho} = \rho_2/\rho_1 > 0.2829$, which holds for most practical combinations of experimental liquids. The bifurcation was confirmed experimentally by Jalikop and Juel (2009) and more recently, by the numerical work of Gligor et al. (2020). In the limit $g \rightarrow 0$, however, this bifurcation becomes degenerate, with no cubic order term in the weakly-nonlinear amplitude equation.

The ideal inviscid theory shows that gravity is essential to the standard picture of the frozen wave instability. It is needed to dampen perturbations at low forcing values and, thus, to set a threshold, and it is also required (in the weakly-nonlinear analysis) to limit the growth of unstable perturbations at higher forcing values. The fact that frozen waves in microgravity are excited easily and grow quickly is consistent with this.

11.3.2 Growth of columnar patterns

Figure 11.2 illustrates the development of a frozen wave pattern in microgravity through a series of snapshots. The mean flow (averaged over one forcing period) is also shown with scaled arrows to indicate where and how the interface is evolving at that time. Four stages of development can be distinguished as follows.

FIGURE 11.2: Evolution of the frozen wave instability in zero gravity in a container of length $L = 60$ mm vibrated with $\mathcal{F} = 8.95$ (40 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots). This column growth is characteristic of the medium frequency symmetric regime discussed in Sec. 11.4.2.

Initial stage

Immediately after the forcing is applied, the previously flat interface is disturbed more near the lateral walls than in the interior. Localised harmonic waves (filtered out in Fig. 11.2 by averaging)

are present. The vibroequilibria effect (Beyer et al., 2001; Lyubimov et al., 1997; Wolf, 1969) is also evident and causes the contact points to move up along the walls (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004), as seen at $t = 0.25$ s in Fig. 11.2.

Column formation

The frozen waves grow rapidly into large columns spanning the height of the container, with this process beginning near the lateral walls and moving inward. The column formation is accompanied by an alternating series of counter-rotating vortices that are strongest near the column formation "front" and subside as the corresponding column reaches the upper and lower boundaries and relaxes to its final position. When this chain of vortical flow reaches the middle of the container, the ensuing interaction determines the properties of the final central column (see the snapshots at $t = 0.625$ s and $t = 0.75$ s).

Defect evolution

When the column formation process reaches the center of the container, the final column must accommodate the interaction of both fronts. This typically results in a column of different size, or a defect. If the defect (central column) is large enough, it remains in the final state but, if not, it is eventually absorbed by a neighbouring column on the left or right (Salgado Sánchez et al., 2019c).

Final state

The final columnar state (shown at $t = 0.85$ s in the figure) is often approximately left-right symmetric and consists of a series of nearly vertical interfaces. For large enough forcing, these can sustain Faraday waves (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016).

Due to the vibroequilibria effect, the outermost columns always contain the heavier liquid. However, for many parameters, as in Fig. 11.2, the final state includes small drops of the lighter liquid in the upper left and right corners. These form when the points of maximum height of the outer columns are distinct from the contact points during their growth. Their collision with the upper boundary then traps a small volume of the lighter liquid in the corner. Because this mechanism is the same at both sides, the sizes of the drops are comparable. For other parameters, the lateral contact points reach the upper boundary before any other part of the outer columns and no drops are created.

11.3.3 Mode selection

The theoretical results discussed in Sec. 11.3.1 imply that the most dangerous perturbations in microgravity environments have $k \approx 0$ and that their threshold is extremely low. A real system of finite size, with viscosity, will still require a minimum forcing to excite frozen waves but there can be a large number of unstable modes at forcing values above this low threshold. The frozen wave mode selection problem in microgravity can thus be considerably more complicated than in ordinary gravity where sufficiently deep layers exhibit waves with $k = 1/l_c$.

Effect of forcing

Since the threshold of the longwave mode [see Eq. (11.16)] is so much lower in microgravity, a given forcing value may be enough to excite many nearby modes. In the infinitely extended inviscid case, there is an interval of unstable modes $0 \leq k \leq k_m$ for each forcing $\mathcal{F}$, whose upper limit is given by the solution of Eq. (11.13) with $\text{Bo} = 0$:

$$\mathcal{F}^2 = \frac{(1 + \hat{\rho})^3}{2\hat{\rho}(1 - \hat{\rho})} k_m H \tanh(k_m H). \quad (11.18)$$

Lyubimova et al. (2017) solved the mode selection problem in microgravity by maximising the growth rate over this interval and compared the prediction with numerical simulations and experiments. They found that, although viscosity can have a significant effect, there is a monotonic increase of observed wavenumber with forcing. This tendency was also confirmed by the numerical simulations of Salgado Sánchez et al. (2019c). A given forcing $\mathcal{F}$ may drive the growth of multiple frozen wave modes, but the one that grows fastest eventually dominates. This simple prediction of larger wavenumber for larger forcing is modified by finite-size effects, as discussed in the next section.

The way the forcing is initiated can also be important. In the simulations performed here, the forcing is turned on smoothly in the manner of Eq. (11.3) over a transition time $\tau = 0.25$ s. Specifically, the forcing amplitude A is multiplied by $\mathcal{H}(t - \tau/2)$ with $\varepsilon \rightarrow \tau/2$. This is done for numerical stability, but also affects the growth of frozen waves for $t < \tau$. Mode selection during this early period occurs for forcing values that are lower than the nominal one. Indeed, we have confirmed by varying τ that there is a strong correlation between frozen wave patterns with similar values of $\mathcal{F}/\tau$, which suggests that mode selection is occurring, to a significant extent, prior to reaching the final forcing amplitude (as at $t = 0.25$ s in Fig. 11.2, where the wavelength of the initial columns is already selected). Compared to a setup where vibrations are applied suddenly, the results in this paper are shifted somewhat toward lower wavenumbers.

Finite-size effects

There are several ways in which the finite size of the container affects the mode selection process. First, the finite length L places a lower bound on the wavenumber, $k_1 \approx 2\pi/L$, and permits only integer multiples of this satisfying $k_n \simeq 2\pi n/L$. This selection depends on the boundary conditions and is significantly complicated by the rapid and nonuniform growth described in Sec. 11.3.2. The advance of front-like regions with large columns on the outer side and a relatively flat interface on the inner side (see Fig. 11.2 between $t = 0.375$ s and $t = 0.5$ s) means that interior regions are effectively shielded from the boundaries when the instability develops there. The nonlinear interaction with other columns, through the associated vortices, is presumably more relevant over much of the domain than the lateral walls. The nonuniformity of this complicated process is reflected in the fact that, although the final states necessarily have an integer number of column pairs (neglecting drops and identifying left and right boundaries when the liquids are the same there), the individual column widths are not identical.

The finite size of the container also affects the base flow that exists prior to the instability. For infinitely extended inviscid systems, as in Sec. 11.3.1, this is a uniform counterflow (Lyubimov and Cherepanov, 1986). However, such a purely horizontal flow cannot extend to the vicinity of the lateral walls, where it must alternately turn up or down during each forcing cycle. Since the Kelvin-Helmholz mechanism arises from this counterflow, it cannot be uniform across the container either. Mode selection near the sides will be different from that in the interior, even in the case of ordinary gravity with small-amplitude frozen waves over a nearly flat interface.

The finite height of the container plays another important role since it limits the growth of the columns. Interaction with the upper and lower boundaries occurs continually during the column growth process (see Fig. 11.2), which affects mode selection as well. In a shallow container, for example, perturbations of low k are inhibited by the fact that they have little chance to grow before colliding with the boundaries. The very nonlinear dynamics of these collisions interrupts and controls their development, effectively replacing them with modes of higher wavenumber (Salgado Sánchez et al., 2019c).

The finite container size is also essential to the vibroequilibria effect, which is driven by inhomogeneities in the oscillatory flow. Because the variation in the oscillatory base flow is concentrated near the lateral walls, so is the vibroequilibria effect; it causes the heavier liquid to move upward along these walls. The selection of frozen wave modes that are consistent with this is discussed again in Sec. 11.4. Here, we remark that this provides the justification for ignoring the longwave mode with $k \simeq \pi/L$ (and odd multiples of that), as done above.

Results

Figure 11.3 shows the results of a series of numerical simulations in a container of length $L = 60$ mm, which can be compared with results from Salgado Sánchez et al. (2019c); note that length is scaled with L rather than H in that reference. The simulations here, and throughout the paper, are performed with a fixed value of $A = 1.5$ mm, which satisfies the small-amplitude criterion for vibrations, $A < H$, but with varying frequency ω . The figure shows the dimensionless forcing $\mathcal{F}$ and the average dimensionless wavenumber $\langle \mathcal{K} \rangle$ of the corresponding pattern, which is simply the number of column pairs measured along the horizontal midline $y = 0$.

The results confirm that the number of columns in the final state (illustrated with insets) increases with forcing $\mathcal{F}$. Due to finite-size effects, this increase is not continuous but occurs through discrete jumps at the mode transition points. The range of forcing values over which each mode is found tends to increase with $\langle \mathcal{K} \rangle$.

The mode transitions are associated with the growth of defects into larger columns as the forcing increases (Salgado Sánchez et al., 2019c). Over a wide range of $\mathcal{F}$, these defects first appear in the center of the container where the two chains of developing columns and the associated vortices meet. If, for a particular $\mathcal{F}$, the vortices match up well as these two chains reach the center, then there is a low level of “wavenumber frustration” and a central column is created. With increasing

FIGURE 11.3: Frozen wave instability diagram in terms of $\mathcal{F}$ for a container with $L = 60$ mm. The nondimensional average wavenumber $\langle \mathcal{K} \rangle$ (solid vertical lines) gives the number of column pairs along the midline $y = 0$. The insets illustrate the final pattern at selected points on each branch, while the dashed line marks the value $\mathcal{F} \simeq 10.7$ where the symmetry-breaking transition occurs.

$\mathcal{F}$, this central column widens until a small defect (incomplete column) appears in its midst. The transient development of this defect is associated with a small vortex, which grows in amplitude as $\mathcal{F}$ increases further.

For a sufficiently large value of $\mathcal{F}$, the defects that initiate wavenumber transitions begin appearing at off-center locations. This marks the symmetry-breaking transition discussed in the following section.

11.4 Symmetry breaking

The infinitely extended system that is commonly assumed in the analysis of the frozen wave instability (see Sec. 11.3.1) has a lot of symmetry. However, much of it depends on the mathematical simplifications employed and is, therefore, only approximate.

The neglect of lateral boundaries means that the ideal system has translational symmetry. Frozen wave solutions lie on group orbits of equivalent solutions related by a translation $\delta x \in [0, 2\pi/k]$ through some fraction of a period. The presence of boundaries clearly breaks this symmetry. Neumann boundary conditions with a length $L = 2\pi n/k$, for example, will select two of these solutions, related by reflection.

The left-right reflection symmetry of the ideal problem is an approximation as well, resulting from the separation of timescales procedure. The system treated in Sec. 11.2 has a symmetry that

combines left-right reflection with time translation through half a forcing period — it is not left-right reflection symmetric at a fixed time. We have confirmed this shift-reflection symmetry by changing the sign of the sinusoidal forcing term in Eq. (11.7), which results in a pattern whose asymmetries and defects are mirror images of the previous ones. It is only to the extent that the high frequency assumption is valid that frozen wave patterns are expected to be left-right symmetric. As the frequency is lowered, there is less separation between the fast timescale characterising the forcing as well as oscillatory (harmonic or subharmonic) surface waves and the slower timescale of the frozen wave dynamics. Moreover, the highly nonlinear nature of this development, as columns grow and collide with the boundaries, may serve to amplify the initial left-right asymmetry (Salgado Sánchez et al., 2019c).

The full two-layer liquid system is not symmetric under vertical reflection through the midplane. However, a small-amplitude sinusoidal frozen wave profile in an extended system does have this symmetry, which is equivalent to a translation by π/k in x . Translation symmetry depends on the uniformity of the frozen wave pattern, which in turn, depends on the homogeneity of the initial flat state. Vertical reflection symmetry is consistent with some boundary conditions in finite containers, as with the choice of $\beta = \pi/2$ in Table 11.1, but it is broken by the vibroequilibria effect (Gligor et al., 2020), which deforms the flat surface, particularly near the lateral walls.

The vibroequilibria effect, which is proportional to v^2 is amplified in microgravity (Salgado Sánchez et al., 2019a) and consistently selects columnar frozen wave patterns with the heavier liquid along both sides. Indeed, the left-right reflection symmetry of the vibroequilibria effect is likely a key factor behind the patterns observed here. If there is an asymmetric perturbation, as with the finite initial interface inclinations considered in Salgado Sánchez et al. (2020), then antisymmetric patterns can be selected instead.

The symmetries discussed above are inherently broken in the real frozen wave problem, since they stem from mathematical simplifications or the neglect of concurrent phenomena like vibroequilibria. The level of asymmetry associated with them is expected to vary continuously with parameters. Of course, it is also possible for symmetry to be broken *spontaneously* when a symmetric solution loses stability to asymmetric perturbations. Although it is difficult in any particular simulation or experiment to determine the source of observed asymmetries, we present evidence in Sec. 11.5 for the existence of a threshold transition to asymmetric solutions that is suggestive of spontaneous symmetry breaking.

FIGURE 11.4: Evolution of the frozen wave instability in zero gravity in a container of length $L = 60 \text{ mm}$ vibrated with $\mathcal{F} = 10.74$ (48 Hz, $A = 1.5 \text{ mm}$). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots). This column growth is characteristic of the high frequency asymmetric regime discussed in Sec. 11.4.2 with $\mathcal{F}$ close to the symmetry-breaking transition.

Figure 11.4 shows the evolution of the frozen wave instability above the forcing value where such a symmetry-breaking transition is observed. Note that the transient development of the columnar pattern is almost left-right symmetric until $t \simeq 0.5 \text{ s}$ when noticeable asymmetry appears during the growth of the final columns. The final state is approximately symmetric near the sides and clearly asymmetric in the center.

FIGURE 11.5: (a) Density distribution of the final steady state (lighter liquid shaded) along the horizontal midline $y = 0$ as a function of $\mathcal{F}$ for a container with $L = 60$ mm. (b) The asymmetry $\mathcal{A}$ of these same solutions. The column growth and defect locations become asymmetric at $\mathcal{F} \simeq 10.7$ (marked by the horizontal line), which is reflected in the increase of $\mathcal{A}$. For visual guidance, a moving average of $\mathcal{A}$ is shown in (b) with the dashed curve.

11.4.1 Quantifying pattern symmetry

In order to quantitatively study the left-right reflection symmetry of frozen wave patterns, we define a measure of asymmetry based on the level-set function ϕ , which is first given sharp boundaries by setting all values below (above) 0.5 to 0 (1). The asymmetry function $\mathcal{A}$ is calculated from

$$\mathcal{A} = \frac{100}{LH} \int_{-H}^H \int_0^{L/2} |\phi(x, y) - \phi(-x, y)| dx dy. \quad (11.19)$$

The integrand simply compares a point with its mirror image on the other side of the container and assigns a 0 if the liquids at those two points are the same and a 1 if they are not. The asymmetry measured by $\mathcal{A}$ is given as a percent value between 0 (completely left-right symmetric) and 100 (completely antisymmetric).

11.4.2 Characteristic regimes and transitions

Figure 11.5 illustrates the changing level of symmetry of columnar solutions in a container of length $L = 60$ mm as forcing is varied by increasing the frequency $f = \omega/2\pi$ from 7 Hz to 120 Hz. Figure 11.5(a) shows the distribution of the two liquids along the horizontal midline in the final state — recall that counting column pairs along this line gives $\langle \mathcal{K} \rangle$, and that all simulations are initiated from a perfectly flat initial interface. The accompanying plot in (b) provides the asymmetry

measure $\mathcal{A}$ from Eq. (11.19). Both plots point to a substantial increase in asymmetry near $\mathcal{F} \simeq 10.7$ (shown with a horizontal line). Beyond this value, the new columns (defects) appear off-center.

The profile of $\mathcal{A}$ in Fig. 11.5(b) is typical of other parameters too. With some variations, one can distinguish three principal regimes, with each exhibiting qualitatively different mechanisms of column development.

Low frequency asymmetric regime

At low frequencies (equivalently, low forcing, since A is fixed) there is often a significant level of asymmetry, which can be explained by a couple of factors.

First, since the separation of timescales is not well-justified, the difference in forcing phases at the left and right walls plays a role. The side wall that moves inward first will have an advantage in pushing the heavier liquid upward. This portion of the interface may then collide with the upper wall to create a column well before its counterpart, which initiates an alternating series of collisions (left, right, left, etc.) and, in general, leads to a final state with substantial asymmetry.

A related factor that increases the level of asymmetry is the fact that lower wavenumber patterns collide earlier (at smaller values of kH) with the upper and lower boundaries. The interruption of linear or weakly-nonlinear growth by the highly nonlinear collision process can be expected to amplify preexisting differences between the two sides.

This type of low-frequency column development is illustrated in Fig. 11.6 for 8 Hz forcing. Note that, with $\tau = 0.25$ s, the forcing period is comparable to the initialization envelope. The vortices in the mean flow are also quite different from those of Fig. 11.2. They do not always appear in counter-rotating pairs, but in a more asymmetric, alternating fashion.

Medium frequency symmetric regime

As the forcing frequency is raised, the separation of timescales has more validity and the wavenumber of the dominant mode increases, both of which reduce the effect of inherent (forcing phase-dependent) asymmetry. There is typically a very low level of asymmetry, $\mathcal{A} \lesssim 3$, although minor oscillations can appear due to defects and mode transitions. The width of this symmetric regime becomes smaller as the container length increases.

The column development process is as illustrated in Fig. 11.2 with $\mathcal{F} = 8.95$. Both the transient growth of the columns and their final distribution show a high degree of left-right symmetry. Throughout this regime, the new columns (defects) appear in the center of the container.

High frequency asymmetric regime

At a certain frequency (forcing), there is a transition to asymmetric column growth. This loss of symmetry cannot easily be attributed to the forcing phase since a separation of timescales is

FIGURE 11.6: Evolution of the frozen wave instability in zero gravity in the low frequency asymmetric regime. The container of length $L = 60$ mm is vibrated with $\mathcal{F} = 1.79$ (8 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots).

well-justified. The transition triggers a steady increase in $\mathcal{A}$ followed by approximate saturation; at $\mathcal{A} \simeq 9$ in Fig. 11.5(b).

Mode transitions throughout this regime occur asymmetrically, with defects (new columns) forming at various off-center points. Figure 11.7 illustrates the column growth process for $\mathcal{F} = 22.38$, which is not close to a mode transition. It is clear that the transient dynamics of the columns and associated mean flow vortices are asymmetric with numerous irregular regions.

For some parameters, the high frequency regime exhibits additional changes and different levels of asymmetry, often correlated with mode transitions. The value of $\mathcal{A}$ usually, but not always, increases during subsequent transitions. The approximate saturation of $\mathcal{A}$ (considering only frequencies below those that cause numerical difficulties) is also sometimes observed, typically with a value between 20 and 30; this type of limiting behaviour was also noted in Salgado Sánchez et al. (2019c).

FIGURE 11.7: Evolution of the frozen wave instability in zero gravity in the high frequency asymmetric regime (well above the symmetry-breaking transition). The container of length $L = 60$ mm is vibrated with $\mathcal{F} = 22.38$ (100 Hz, $A = 1.5$ mm). The snapshots are averaged over one forcing period, with the time provided underneath. The mean velocity field is shown using scaled arrows (scale differs between snapshots).

Asymmetry distribution

The asymmetric features of the final columnar patterns are not distributed uniformly. Since the vibroequilibria effect ensures that the heavier liquid always aligns along the lateral walls, the outer part of the solution tends to be quite similar on both sides. The center of the pattern, on the other hand, is the last to develop and is affected by the interaction of both column-forming fronts. It often exhibits more asymmetry than the rest of the domain.

Figure 11.8 shows the distribution of the asymmetry function $\mathcal{A}$ across the length of the container in the high frequency asymmetric regime. It is calculated from restricting $\mathcal{A}$ to windows of 3 mm, starting at the center ($x = 0$) and moving outward. This figure demonstrates that the region of maximum asymmetry does not remain in the center, but moves outward with increasing frequency (forcing). This process can be observed directly in Fig. 11.5(a) from the locations of successive defects. The level of asymmetry close to the lateral walls, however, remains small.

FIGURE 11.8: Distribution of asymmetry $\mathcal{A}$ along the x axis, calculated using 3 mm windows, for the final columnar states in a $L = 60$ mm container beyond the symmetry-breaking transition at $\mathcal{F} \simeq 10.7$. Simulations were performed with $\mathcal{F} = 11.19$ (50 Hz, circles with solid lines), $\mathcal{F} = 13.43$ (60 Hz, squares with dashed lines), $\mathcal{F} = 20.14$ (90 Hz, diamonds with dash-dotted lines) and $\mathcal{F} = 26.85$ (120 Hz, triangles with dotted lines). The maximum asymmetry is first found in the middle, then is displaced outward at higher forcing.

FIGURE 11.9: (a,b) Density distribution of the final steady state (lighter liquid shaded) along the horizontal midline $y = 0$ as a function of $\mathcal{F}$ for rectangular containers of lengths (a) $L = 90$ mm and (b) $L = 120$ mm. The column growth becomes asymmetric at $\mathcal{F} \simeq 7.4$ (solid line) and $\mathcal{F} \simeq 6.4$ (dashed line), respectively. (c) The asymmetry $\mathcal{A}$ of these solutions (black circles for $L = 90$ mm, grey squares for $L = 120$ mm). Note that the vertical scale is logarithmic.

11.5 Symmetry-breaking threshold

Figure 11.9 shows the density distribution along the horizontal midline and the corresponding asymmetry measure $\mathcal{A}$ for simulations with $L = 90$ mm and $L = 120$ mm. The transition from

FIGURE 11.10: Threshold of the symmetry-breaking transition as a function of container aspect ratio Γ . The relevant asymmetric mode is indicated under the corresponding portion of the curve. The onset value for frozen waves, obtained as in Salgado Sánchez et al. (2019c), is shown for reference.

the medium frequency symmetric regime to the high frequency asymmetric regime is marked. This symmetry-breaking tends to occur earlier for larger L . Note that the behaviour is similar to that of Fig. 11.5 but, in these longer containers, there appear to be secondary transitions (jumps in $\mathcal{A}$) occurring further into the high frequency regime.

A series of simulations were performed for various aspect ratios (taking L between 7.5 mm and 150 mm) with increasing $\mathcal{F}$. While no asymmetric behaviour was seen for $L < 35$ mm (presumably because the forcing was not large enough), the symmetry-breaking threshold for the remaining aspect ratios is shown in Fig. 11.10. These points were located through a combination of direct visual inspection of the final state for off-center defects and the behaviour of $\mathcal{A}(\mathcal{F})$. The average wavenumber $\langle \mathcal{K} \rangle$ of the emerging asymmetric mode is also labelled in the figure.

The symmetry-breaking threshold curves resemble neutral stability tongues despite the fact that they are far above the onset of frozen waves, which depends on the wavenumber $k \simeq 2\pi/L$ of the $\langle \mathcal{K} \rangle = 1$ mode and is shown in the figure as a solid black curve; see Salgado Sánchez et al. (2019c) for the determination of this curve. For example, if $\mathcal{F}$ is gradually increased from the frozen wave onset with $\Gamma = 8$ (the $L = 60$ mm case of Figs. 11.2–11.8), there is a succession of mode transitions (see Fig. 11.3) from $\langle \mathcal{K} \rangle = 1$ to 6 via the nearly symmetric growth of defects (new columns) in the center. The $\langle \mathcal{K} \rangle = 7$ mode, however, appears asymmetrically, as shown in Fig. 11.4.

The fact that the symmetry-breaking transition occurs with particular modes in such a regular way as Γ is varied strongly suggests that it is related to a *spontaneous* symmetry-breaking transition, that for modes beyond a certain $\langle \mathcal{K} \rangle$, the nearly left-right symmetric columnar state has lost stability to (one or more) asymmetric states. At the same time, it should be understood that all of the columnar states are relatively stable once they have collided with the upper and lower boundaries. There is no

bifurcation in the usual sense of a stable state losing stability to perturbations. Instead, the mode selection process takes place during a complicated and highly nonlinear transient evolution with multiple excited modes competing with each other. Given the complexity of this system and its solutions, it is rather surprising that a symmetry-breaking transition with such regular behaviour exists.

11.6 Conclusions

This paper investigated the symmetry properties of the large columnar frozen wave patterns that arise due to periodic horizontal vibrations in microgravity. A series of numerical simulations using a level-set method were performed in containers of various lengths for parameters corresponding to FC-40 and 20 cSt silicone oil.

The mode selection process in this system is determined mainly by the applied forcing level and by the container size, but is also clearly affected by the vibroequilibria phenomenon, which consistently selects solutions having the heavier liquid along the sides. Frozen waves develop rapidly and nonuniformly, moving inward as successive columns grow and collide with the upper and lower boundaries. This is accompanied by two opposing chains of vortices that collide and interact in the central region. For moderate forcing values, it is this central region where defects appear, to either be absorbed by neighbouring columns or grow, with more forcing, into new columns that increase the wavenumber.

Using the level-set function, an integral was proposed to quantify the left-right asymmetry. By combining this measure with direct visual inspection of the final pattern, three typical regimes were identified depending on frequency (forcing). The low frequency asymmetric regime is affected by the 180° phase shift in vibrational forcing at each lateral wall and, due to low wavenumbers, by early collisions with the upper and lower boundaries. The medium frequency regime is characterised by much more symmetric patterns with a series of mode transitions occurring through the appearance and growth of defects in the central region. The high frequency asymmetric regime is triggered by a symmetry-breaking transition after which the new columns (defects) arise at off-center locations.

The threshold of this symmetry-breaking transition was determined for a range of aspect ratios and was seen to be associated with particular modes in the manner of neutral stability tongues. The regularity strongly suggests that, despite the complexity of the mode selection process in this system, this transition corresponds to a type of *spontaneous* symmetry-breaking, in contrast to the inherent left-right asymmetry (of the fast timescale) that manifests in the low frequency regime.

The spatial distribution of asymmetry after the transition was also investigated. While the consistent presence of the heavier liquid in the outer columns ensures that these regions remain relatively symmetric, the central region may contain a high degree of asymmetry. With increasing frequency, the region of maximal asymmetry moves outward.

This work shows that the large columnar structures that develop following the frozen wave instability in microgravity possess many regular features, despite the complicated nonlinear dynamical process, including collisions and topological changes, that selects them. The final states may still be considered to vary continuously with parameters and to undergo regular transitions, several of which can be analysed by considering the level of left-right reflection symmetry.

Acknowledgment. This work was supported by the Ministerio de Economía y Competitividad under Project No. ESP2015-70458-P.

Part III

DISCUSSION AND CONCLUSIONS

Chapter 12

Discussion

In this Thesis, we have considered a variety of interfacial phenomena related to vibrated fluids, ranging from surface waves that inherit the (fast) timescale set by the applied (high-frequency) vibrations, to slower dynamics related to the associated mean flows. In microgravity, these very different phenomena can occur simultaneously, generating patterns that exhibit different time and length scales, and leading to a complex picture of interactions and transitions with interesting but complicated dynamics and pattern selection scenarios.

The investigations began with the dynamics of parametrically forced subharmonic waves, which is one of the predominant phenomena occurring on the fast timescale.

Dynamics of parametrically forced surface waves

Vibrated fluid experiments can produce a wide range of surface wave patterns (Arbell and Fineberg, 2002; Miles and Henderson, 1990), and some of the most studied are the subharmonic waves generated by parametric forcing mechanisms (Faraday, 1831; Jones, 1984). The case of cross-waves, which are subharmonic waves typical of horizontally forced experiments, is a representative example (Barnard and Pritchard, 1972; Lin and Howard, 1960; McGoldrick, 1968; Moisy et al., 2012; Taneda, 1994). Cross-waves are parametrically excited by the variable pressure gradient resulting from localised evanescent modes (also, oscillatory bulk flow; Varas and Vega, 2007). If these cross-waves arise in a rigid container that is relatively large compared to their decay length, then the surface wave patterns generated at opposite endwalls can be considered to be weakly interacting. In more general wavemaker experiments, with two or more wavemakers, a similarly weak interaction will occur when the different wave trains overlap only in the far-field (subcritically forced) region of the surface or interface.

Under these assumptions — parametric forcing and weak interaction — the dynamics of the surface waves can be effectively modelled by (a set of) coupled parametrically forced oscillators in the limit of weak coupling (Salgado Sánchez et al., 2016). In the simplest case of two (identical) wave patterns (i.e., two identical oscillators), the problem is described by a single coupling coefficient, and

the residual symmetry, if any, is entirely determined by the difference between the forcing phases of the wavemakers.

If these two wavemakers oscillate in phase, then the patterns they produce after the subharmonic instability may be approximated by weakly coupled oscillators with synchronised parametric forcing. The symmetry they exhibit is $\mathbf{Z}_2 \times \mathbf{Z}_2$, which reflects the synchronous (even) and antisynchronous (odd) oscillation of the system. There are two nontrivial fixed point subspaces associated with these two classes of solutions, which appear via primary steady (pitchfork) bifurcations. The presence of these two invariant subspaces does not permit a primary Hopf bifurcation.

If these same two wavemakers are driven with different phases, the coupled system behaves differently. This scenario occurs naturally in cross-waves experiments, where subharmonic waves are excited by the horizontal vibration of a rigid container. In this setup, while the left endwall is moving into the fluid, the right endwall is pulling away, which produces an antisymmetric upward and downward motion of the contact lines at the two sides. The dynamics in this scenario can be approximated by equations describing two weakly coupled parametrically forced oscillators with out-of-phase (antisymmetric) forcing,

$$\dot{A} = (-1 + i\nu)A + f\bar{A} - (a + i)|A|^2A + \mu B, \quad (12.1a)$$

$$\dot{B} = (-1 + i\nu)B - f\bar{B} - (a + i)|B|^2B + \mu A, \quad (12.1b)$$

which exhibit the cyclic symmetry $\mathbf{C}_4$. The primary instability here takes the form of a Hopf bifurcation and generates a $\mathbf{C}_4$ -symmetric modulated (quasiperiodic) solution, characterised by a slow cycle between activity concentrated at one oscillator and the other. These modulated solutions are typically destroyed via (global) saddle-node heteroclinic connection, when the forcing exceeds the critical value by an $\mathcal{O}(|\mu|)$ amount. The general picture of secondary bifurcations is organised around a pair of Bogdanov-Takens points, which generate, for positive detuning, a particularly complex interaction of saddle-node, Hopf, heteroclinic, and homoclinic bifurcations (Marqués et al., 2013; Salgado Sánchez et al., 2016).

If the wavemakers oscillate with phases different from the symmetric (0) or the antisymmetric (π) cases, the system of weakly coupled oscillators does not exhibit any residual exchange symmetry, but only the basic subharmonic symmetry $\mathbf{Z}_2$. This case leads to intermediate behaviour organised by (generally two) $\mathbf{Z}_2$ -symmetric Bogdanov-Takens points separating regions where the primary instability is oscillatory (Hopf) or steady (pitchfork).

The primary Hopf bifurcation predicted in the case of antisymmetric forcing was first observed in the cross-waves experiments of Tinao et al. (2014). Modulated solutions are caused by the interaction between the wave fields generated on opposite sides of the container and result in a slow cycling between balanced and one-sided excitation. These experiments reveal a strong dependence of the coupling on detuning. For negative detuning, the driving force remains concentrated near the walls and waves decay quickly in the interior. For positive detuning, the waves are more oscillatory and

FIGURE 12.1: Selected bifurcation sets for $\phi = \pi$ and $\mu(\nu)$ given by Eq. (12.2) with $\mu_0 = 0.06$, $\sigma = 0.4$, $\theta_0 = 7\pi/18$, $\Omega_\nu = 0.76\pi$: Hopf (H: dashed curve), saddle-node heteroclinic (SN Het), saddle-node of periodic orbit (SN Per, solid curves for $\nu \gtrsim 2.15$). Dotted vertical lines mark ν values where the Hopf frequency vanishes. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.

penetrate further, leading to a stronger interaction (coupling). Furthermore, Tinao et al. (2014) demonstrated that stable modulated solutions exist well into the positive detuning regime. These two features can be described in the weakly coupled oscillators model by using a modified coupling term capturing this type of dependence on detuning:

$$\mu(\nu) = \mu_0 e^{\sigma\nu} e^{i[\theta_0 + H(\nu)\Omega_\nu\nu]}, \quad (12.2)$$

where $H(\nu)$ is a Heaviside step function, and by selecting an adequate (positive) value of the cubic (dissipative) nonlinearity (Salgado Sánchez et al., 2016). With an appropriate choice of parameters, the general bifurcation picture is shown in Fig. 12.1.

It is worth noting that the observation of modulated cross-waves is experimentally challenging for various reasons (Tinao et al., 2014). For one thing, there is a need to select the right range of frequencies for a given container. If the frequency is too low, the resulting large waves, which cover the entire domain, violate the assumption of weak coupling. If the frequency is too high, the very weak coupling leads to a large modulation period that is difficult to measure accurately. This experimental limitation is further complicated by the effects of mode interactions and the difficulty of controlling boundary conditions in a precise manner. Therefore, it was not possible to verify additional details of the oscillator dynamics beyond the initial Hopf bifurcation and the existence of stable modulated solutions.

Given these experimental difficulties, we completed the analysis by comparing with numerical simulations of the Navier-Stokes equations. The simulations are performed in a rectangular two-dimensional domain of dimensions 70 mm (length) $\times$ 25 mm (depth), for parameters corresponding to 5 cSt silicone oil with a prescribed contact angle of 0° . The forcing is applied at a fixed frequency,

with exploration of the range between 30 Hz and 60 Hz, and with the amplitude slowly increased (ramped). Note that the two-dimensional nature of these simulations only capture one (downstream) orientation for the wavevector. These types of lengthwise subharmonic waves are known to exist (Perez-Gracia et al., 2014; Porter et al., 2013; Varas and Vega, 2007), and are legitimately described by the weakly-coupled oscillator model, even if their instability is typically masked by the earlier onset of cross-waves in the full three-dimensional problem.

In general, the numerical simulations show modulated waves immediately after the primary subharmonic instability, with a period that increases with the applied forcing (amplitude) until the modulations abruptly disappear — this is evidence of a global bifurcation. The increase in modulation period shows the inverse square root scaling characteristic of a saddle-node heteroclinic bifurcation, in agreement with the predictions of the oscillator model. Although convincing, it must be acknowledged that this is not a completely rigorous confirmation of the saddle-node heteroclinic bifurcation for all cases. In particular, if the system is in the vicinity of a Bogdanov-Takens (double zero) point (vanishing Hopf frequency), there may be additional complications including small regions of (standard) heteroclinic bifurcations (or pitchfork bifurcations) that replace the saddle-node heteroclinic connection (Marqués et al., 2013).

The onset of subharmonic waves, the initial modulation frequency, and the location of the saddle-node heteroclinic bifurcation where the modulations cease, are estimated from the data and shown in Fig. 12.2. The simulations verify the results of the oscillator model in predicting the existence of a sequence of double zero points that mark the boundaries of bubble-like regions where modulated solutions are found; see Fig. 12.1 for comparison. Complex dynamics are expected in the vicinity of these double zero points (Marqués et al., 2013), which are further complicated by mode transitions. This leads to a type of discontinuous behaviour not captured by the weakly coupled parametric oscillator model. Simulations suggest that these mode transitions occur just above the double zero point (about 0.5 Hz higher in forcing frequency). In the vicinity of these double zero points, the irregular nature of the curves can be explained by the proximity of these two critical points and their associated complexity.

In short, the relevance of weakly coupled parametrically forced oscillator models to the dynamics of subharmonic waves, especially to cross-waves (ground) experiments, is demonstrated by the symmetry-based explanation for the primary Hopf bifurcation, the presence of stable quasiperiodic solutions, the dependence of the modulation frequency on detuning, and the observation of the saddle-node heteroclinic bifurcation in direct numerical simulations of the Navier-Stokes equations.

Subharmonic waves, however, occur in other scenarios too. In microgravity, they often appear superimposed on deformed interfacial configurations. One interesting example is the development of Faraday waves over the vertical interfaces of a frozen wave columnar pattern (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016). The waves on opposite sides of a column oscillate at half the forcing frequency and exhibit a characteristic phase difference of $\pi/2$; we anticipate that this feature can be explained with the C_4 symmetry of Eqs. (12.1) (Salgado Sánchez et al., 2016, 2019d). Subharmonic

FIGURE 12.2: (a) Estimated location of onset (triangles) and saddle-node heteroclinic bifurcation (circles); the parameters shown are applied forcing f (in g 's) and applied frequency ω/π (in Hz). (b) Estimated modulation frequency Ω (in Hz) near onset. Reproduced from Salgado Sánchez et al. (2016), Chapter 3.

waves can also appear on a deformed interface selected by the vibroequilibria effect (Fernández et al., 2017b; Salgado Sánchez et al., 2019a). The work of Fernández et al. (2017b) suggests that, depending on fluid properties, these waves can interact with the underlying vibroequilibria state and may even destabilise it. In low viscosity fluids, the presence of (low-frequency) modulated subharmonic patterns is likely related to the unbounded growth of the first odd sloshing mode, and the destruction of the vibroequilibria solution.

These two examples illustrate the complex patterns that are often observed in vibrated fluid systems under reduced gravity conditions, where fast and slow timescale phenomena coexist and interact. To continue filling out this general picture, we discuss the results on the vibroequilibria effect.

The vibroequilibria effect

Applied vibrations in a fluid system generate an inhomogeneous oscillatory velocity field that is naturally accompanied by an inhomogeneous dynamic pressure. This inhomogeneous (distribution of) pressure is the driving mechanism for the reorientation of the fluid. This phenomenon, known as the vibroequilibria effect, was first observed by Faraday (1831) in the flattening of drops beneath a vibrating plate. On ground, gravity masks the vibroequilibria effect for most common fluid configurations unless a very large forcing is applied, while in microgravity environments, vibrations compete only with surface tension and contact forces so their effect is greatly enhanced.

The first experimental observations of vibroequilibria in confined fluids were made on ground by Wolf (1969), and in microgravity by Ganiev, Lakiza, and Tsapenko (1977). The experiments of Ganiev, Lakiza, and Tsapenko (1977), in particular, investigated combinations of water, oil, and air during the reduced gravity periods of a parabolic flight. Partially filled cylinders of length 100 mm and diameter 30 mm were subjected to vibrations at both low (8 Hz, 12 Hz) and high frequencies (600–800 Hz), where disordered dynamics and the creation of deep funnels or craters were observed, respectively. This latter behaviour, which is now recognised as indicative of the vibroequilibria effect, was only later shown by Gavriluk, Lukovsky, and Timokha (2004) to be qualitatively consistent with vibroequilibria theory; no quantitative comparison was made with either theory or simulations.

A systematic experimental investigation of the vibroequilibria effect and its relevance in reduced gravity was needed, not only for the expected observation of novel states like those mentioned above, but for its potential to manipulate fluids in weightless environments. Indeed, vibrations are known to mimic certain effects of gravity in convection and phase transitions (Beysens, 2006) and vibroequilibria can be regarded in this light too. The general tendency, with increasing forcing, is for the vibroequilibria solutions to develop so that large portions of fluid interfaces align nearly perpendicular to the axis of vibration. This can be understood as a restored preference for the familiar class of flat fluid interfaces (density contours) aligned perpendicular to gravity.

With this motivation, the Control of Fluids in microgravity with Vibrations (CFVib) experiment (Fernández et al., 2017a) investigated the vibroequilibria effect. The aim the project was to examine the complex behaviour of vibrated liquids in microgravity and to evaluate the potential of small-amplitude vibrations to control this behaviour. More specifically, the principal scientific objectives were: (i) to help evaluate the feasibility of managing fluids with the vibroequilibria effect using small-amplitude vibrations in microgravity, (ii) to allow a comparison between vibroequilibria states in different representative fluid configurations having distinct surface energies and wetting characteristics, and with unforced samples to distinguish the effect of vibrations from that of residual accelerations, and (iii) to obtain experience in the design and execution of a microgravity experiment centred on the vibroequilibria effect.

The experiment (Fernández et al., 2017a) incorporated a set of cuboidal and cylindrical cells, half of which were subjected to small-amplitude periodic forcing by a pair of PZT piezoelectric ceramics attached to opposite faces of an aluminium cantilever beam. The amplitude of vibrations was controlled by the applied oscillatory voltage amplitude V , while the frequency f determined its mode (vibrational characteristics). In practice, f was chosen to match one of the resonant modes of the assembly, with the first mode of particular importance since it yielded the largest displacements. This experimental setup relied on a complex energy conversion from the electrical energy delivered to the ceramics, to surface and contact energies of the test fluids (Salgado Sánchez et al., 2019b). Experiments were performed in parabolic flights, which provided a microgravity level $|\mathbf{g}| \lesssim 0.05 g_0$, where $g_0 = 9.81 \text{ m/s}^2$.

We focus on the results obtained with a cylinder holding silicone oil and with a cuboid holding

FIGURE 12.3: (a–c) Front and lateral view snapshots showing the position of silicone oil in a cylinder vibrated at $f = 59.3$ Hz and $V = 100$ V (measured vertical velocity v_z at the accelerometer of 94.8 mm/s). Observation times are labelled with respect to the initiation of the forcing in microgravity: (a) $t = 0$ s, (b) $t = 2.021$ s and (c) $t = 2.492$ s. The contact lines along the front and rear portions of the cylinder wall are outlined in solid and dashed lines, respectively, and the location of points $\mathcal{A}$ (front contact line maximum), $\mathcal{B}$ (interior minimum), $\mathcal{C}$ (rear contact line maximum) are indicated. (d) Time evolution of contact points $\mathcal{B}$ (solid diamonds; open diamonds for the reference cell) and $\mathcal{C}$ (solid dots; open dots for the reference cell) when vibrated at 80 V ($v_z = 73.9$ mm/s; grey) and 100 V ($v_z = 94.8$ mm/s; black). The times used in panels (b,c) are marked with vertical dashed lines. (e) The vertical gravity level of the corresponding experiments.

water at the primary resonance of each assembly; the associated frequencies f were 59.3 Hz and 79.8 Hz, respectively (Salgado Sánchez et al., 2019a). The relevant fluid properties are detailed in Table 2.1, Chapter 2.

Figure 12.3(a–c) provides front and lateral views showing the evolution of silicone oil in a cylinder after initiation of forcing at $V = 100$ V and $f = 59.3$ Hz (vertical velocity of 94.8 mm/s). Prior to the beginning of the excitation in microgravity, the interface position reflects the competition between residual gravity, surface tension, and contact forces. Since silicone oil has a low contact angle (i.e., it is a liquid that easily wets most materials), the interface shows clear curvature; see Fig. 12.3(a). After forcing begins, the fluid reorients slowly due to the vibroequilibria effect, with the “front” and “rear” contact lines moving upwards while the interior surface dips down. If the forcing is strong enough (also depending on the residual gravity) the front and rear contact lines may reach the top of the container; see Fig. 12.3(c). Although they behave similarly, the rear contact line climbs slightly faster than its front counterpart due to the asymmetry of the vibrational mode itself (Salgado Sánchez et al., 2019a) and the residual acceleration during the parabola.

To quantify this vibroequilibria effect, we show in Fig. 12.3(d) the time evolution of the interior

minimum (point $\mathcal{B}$, diamonds) and the rear maximum (point $\mathcal{C}$, circles) contact point for three experiments: two vibrated experiments (solid markers) differing in forcing amplitude, and one reference (unforced) experiment (open markers). With the lower forcing, the “final” vibroequilibria state does not quite reach the top of the container, but it does for higher forcing. The influence of (vertical) residual gravity, shown in Fig. 12.3(e), is also evident. The vibroequilibria timescale is slow compared to the period of the applied vibrations, but not compared to the primary oscillations of the residual gravity (approximate 0.25 Hz) which are clearly reflected in the evolution of the surface. One can infer from panels (d,e) that a gravity level of less than approximately $0.025 g_0$ is needed to obtain satisfactory results at these forcing amplitudes.

As introduced in Chapter 1, the established theoretical approach to vibroequilibria is based on minimising the averaged energy functional (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004; Lyubimov et al., 1997)

$$\mathcal{L} = \eta \int_{\mathcal{V}} |\nabla \phi|^2 d\mathcal{V} + \text{Bo} \int_{\mathcal{V}} z d\mathcal{V} + \int_S dS - \sin \beta \int_W dW + \dots, \quad (12.3)$$

where the terms describe vibrational, gravitational, surface and contact energies, respectively. The relative importance of each contribution to the energy is captured by the nondimensional parameters η and Bo (Salgado Sánchez et al., 2019a).

A detailed comparison between predictions from vibroequilibria theory and direct numerical simulations was made in the work of Fernández et al. (2017b). However, despite the early experimental observations (Faraday, 1831; Wolf, 1969) and theoretical progress, there was no direct comparison between experiments and theory prior to the work of Salgado Sánchez et al. (2019a). Figure 12.4 reproduces this comparison for the experiments with water in a cuboidal container. Four snapshots of the surface are included in Fig. 12.4(a–d) with applied voltages between 0 and 100 V, which correspond to dimensionless forcing values η between 0 and 2.72. The Bond number is $\text{Bo} \simeq 6.1$.

The increasing deformation of the surface with η is most evident from the contact line, which moves up in the rear and down in the front. The vibroequilibria effect can be quantified by measuring the scaled difference between the contact points in the middle of the front and rear walls,

$$\Delta \mathcal{H} = \frac{h_{\text{rear}} - h_{\text{front}}}{L}. \quad (12.4)$$

These measured values of $\Delta \mathcal{H}$ are shown versus vibrational forcing η (measured at the centre of the unperturbed liquid, Salgado Sánchez et al., 2019a) in Fig. 12.4(e). Since the surface is approximately symmetric about the midplane, we make the simplification of comparing with the two-dimensional version of Eq. (12.3), as in the work of Fernández et al. (2017b). Despite this, and the assumption of constant $\text{Bo} \simeq 6.1$, there is reasonably good agreement between the experimental measurements (solid dots with error bars) and the predictions of vibroequilibria theory (solid black line). In fact, the principal discrepancy, which amounts to an overestimate of approximately 20–30% in the calculations, is almost entirely due to the absence of rotational terms in the potential formulation of

FIGURE 12.4: (a–d) Water surface in a cuboidal container vibrated at (a) $V = 0$ V, $\eta = 0$; (b) $V = 50$ V, $\eta = 0.5201$; (c) $V = 80$ V, $\eta = 1.565$ and (d) $V = 100$ V, $\eta = 2.72$. The contact lines are highlighted (front, solid; rear, dashed). (e) Comparison of experimental results (solid dots) with vibroequilibria theory (solid black curve) and direct numerical simulations using $Bo = 6.1$. Simulations are performed including the complete motion of the cuboid (open grey circles with solid fitting curve) and ignoring the rotational motion (open grey squares with dashed fitting curve). Reproduced from Salgado Sánchez et al. (2019a), Chapter 5.

vibroequilibria theory (Fernández et al., 2017b) — the actual motion of the container is a combination of linear translation and rotation (Salgado Sánchez et al., 2019a). This is confirmed by comparing against direct numerical simulations of the Navier-Stokes equations both with rotational terms (open grey circles) and without rotational terms (open grey squares).

Overall, these results provide a clear confirmation of the vibroequilibria effect in microgravity. There is convincing agreement between the predicted and the observed position of the fluid over the considered forcing range $\eta < 3$, despite the simplifying hypotheses made in the calculations. Although further comparison with theory may be possible, the available vibrational amplitudes were a limitation. For instance, according to the results of Fernández et al. (2017b), the liquid volume was deep enough to undergo a sudden transition from a symmetric to an asymmetric vibroequilibria state. However, not only were these vibroequilibria states not symmetric along the container midplane in the experiment, but the predicted amplitude ($\eta = 15$) for this (saddle-node) transition was not reached.

Experiments provide obvious evidence of the vibroequilibria effect in both containers and with both liquids, and confirm the predicted “gravity-like” tendency of vibroequilibria solutions to move large portions of the surface into a more perpendicular orientation with respect to the vibrational axis. This further suggests that, with appropriate container design and choice of excitation, it should be possible to use the vibroequilibria effect for controlling and managing fluids in microgravity applications. Aside from this consideration, the enhancement of the vibroequilibria effect under weightless conditions means that it should be considered whenever sufficient vibrations, known as

g-jitter (Nelson, 1994), are present.

The example of vibroequilibria illustrates the dramatic effect that small-amplitude vibrations can have in microgravity. In contrast to the situation on ground, vibrations applied to a partially filled container of liquids can induce a large-scale redistribution of the fluids inside, supported primarily by the energy transfer from the applied vibrations (vibrational energy) into surface and contact energies (Salgado Sánchez et al., 2019b). A similar large-scale redistribution, however, can also happen via another mechanism. If this same container holds two fluids of different densities, and vibrations are applied parallel to the interface separating them, a Kelvin-Helmholtz type instability can occur and drive the formation of quasisteady waves, known as frozen waves.

The frozen wave instability

On Earth, these frozen waves are initially of small amplitude and sinusoidal and, in most common fluid systems, they appear through a supercritical pitchfork bifurcation (Jalilikop and Juel, 2009; Lyubimov and Cherepanov, 1986); they grow in amplitude as the square root of the vibrational forcing. This is no longer so in weightlessness, where frozen waves rapidly develop into large columnar structures with alternating (nearly) vertical interfaces (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016). This shows their increased importance and ability to comprehensively change the global distribution of fluids in microgravity, similar to what occurs with vibroequilibria.

The first theoretical analysis of frozen waves was by Lyubimov and Cherepanov (1986), who obtained the threshold for inviscid layers of infinite extension,

$$\mathcal{F}_H^2 = \frac{(1 + \hat{\rho})^3}{2\hat{\rho}(1 - \hat{\rho})} \left[kH + \frac{\text{Bo}}{kH} \right] \tanh(kH); \quad (12.5)$$

see Eq. (1.2) for its dimensional form. Here, $\hat{\rho} = \rho_2/\rho_1 (> 1)$ is the density ratio and $\mathcal{F}_H$ the vibrational forcing, defined as

$$\mathcal{F}_H = A\omega \sqrt{\frac{(\rho_1 - \rho_2)H}{\sigma}}, \quad (12.6)$$

where σ is the surface tension. It is clear from Eq. (12.5) that, in the limit of weightlessness ($\text{Bo} \rightarrow 0$), the threshold for the long-wave mode ($k \rightarrow 0$) vanishes $\mathcal{F} \rightarrow 0$.

The first experimental observation of the frozen wave instability in microgravity was by Beysens et al. (2009), who found periodic stripe patterns in supercritical CO₂. Following these seminal investigations, a relatively large number of experimental (see, e.g., Gandikota et al., 2014b; Gaponenko et al., 2015b) and theoretical (see e.g., Lyubimova et al., 2017) efforts have been undertaken to understand the pattern selection process in both miscible and immiscible liquids. The effect of viscosity, for example, was considered by Talib, Jalilikop, and Juel (2007). By combining theory and ground experiments, they showed that the inviscid prediction of Eq. (12.5) underestimates the onset of frozen waves in liquid layers of similar viscosity, while generally overestimating it in layers with

a large viscosity ratio. Regarding microgravity conditions, the recent analysis of Lyubimova et al. (2017) is especially relevant. In this theoretical work, the authors propose that the pattern selection process is controlled by the perturbation with maximum growth rate within the band of unstable wavenumbers $0 \leq k \leq k_{\max}$, where $k_{\max}$ is given by Eq. (12.5) for a given applied forcing. They found that, although viscosity plays a clear role in selecting the wavenumber, there is a continuous and monotonic dependence of the selected k on vibrational forcing.

Justified by a generally large difference between the pattern wavelength and the lengthwise dimension of the container, these investigations neglect the contribution of finite-size effects. Although such effects can be subtle in many of these systems, especially under ordinary gravity, finite-size effects are crucial in microgravity, where they largely control the process of pattern selection near onset. In order to analyse this systematically, we consider the problem of two immiscible layers subjected to vibrations of amplitude A and frequency ω in weightlessness, with the initial unperturbed interface parallel to the vibrational axis (Salgado Sánchez et al., 2019c). The liquids, with properties similar to those of relevant experiments (Salgado Sánchez et al., 2019d, 2020), are held in a finite rectangular container of dimensions $L \times 2H$, where L refers to the container length.

An extensive set of numerical simulations reveals that pattern selection in finite containers is controlled by a cycle of column growth, which may be described as follows. Immediately after forcing is initiated, the mean vertical velocity is largest near the lateral boundaries of the container, where the two liquids move up and down. This agitation provides the initial perturbation that activates the column growth. As time passes, the frozen wave (column) growth moves inward, transported by the horizontal component of the mean flow (Shevtsova et al., 2015a), towards the centre of the container. It is in this central region where the perturbations can grow differently depending on their amplitude and the amount of liquid they entrain. The larger perturbations near the lateral boundaries reach the top and bottom walls to form complete columns, but the smaller perturbation located in the interior may be limited by surface tension before that occurs. This can be understood as a kind of wavenumber frustration, where the more rapidly growing lateral columns, with wavelengths closer to the prediction of linear theory, induce a mismatch or defect in the centre of the container. With increasing forcing, the size of this incomplete column tends to grow. When, at a certain forcing value, it develops beyond half the container height (reference midline), we mark the occurrence of a mode transition. Higher forcing leads to the complete formation of the column and, beyond that, an increase of its local wavelength is observed. As soon as this wavelength is large enough, the growth of a new column begins, which initiates a new growth cycle.

Figure 12.5(a) shows selected modes in a 30 mm $\times$ 7.5 mm rectangular domain as a function of the (modified) dimensionless forcing $\mathcal{F} = A\omega\sqrt{(\rho_1 - \rho_2)L/\sigma}$, which is now scaled with the length of the container. The number of column pairs along the midline is recorded (black vertical lines, identifying lateral boundaries), as well as a local wavenumber $\mathcal{K} = L/\lambda$ measured using the wavelength λ at the location of the emerging column (grey dots).

At the onset of the frozen wave instability in microgravity, the interface develops towards a single

FIGURE 12.5: (a) Mode diagram in terms of the vibrational forcing $\mathcal{F}$ for a container of length $L = 30$ mm and depth $2H = 7.5$ mm showing the number of column pairs (solid vertical lines) and the wavenumber (solid dots) near the defect in the central part of the container. Small insets illustrate the final pattern for selected points on each branch. (b) Density distribution along the horizontal midline of the final steady state as a function of the vibrational parameter $\mathcal{F}$ for a rectangular container with $L \times 2H = 60$ mm $\times$ 7.5 mm. The column formation cycle becomes asymmetric at $\mathcal{F} \simeq 40$. Reproduced from Salgado Sánchez et al. (2019c), Chapter 8.

pair of columns with the lighter liquid located in the centre, and the denser one divided nearly equally and aligned against the lateral walls. This is the largest wavelength permitted by the container size, $\lambda = L$ ($\mathcal{K} = 1$). As the vibrational velocity is increased, a new column grows from the centre of the container, eventually splitting the central column into two. Each mode is found over a certain interval of vibrational velocities, which widens as the wavenumber grows, indicating the increased cost of adding new columnar structures. The column growth cycle is often nearly symmetric with respect to reflection about the vertical midline of the container, especially over a range of low to medium strength excitations when the pattern wavelength is on the order of the container length.

For sufficiently large containers, a transition from symmetric to asymmetric column growth is observed at moderate forcing values. Figure 12.5(b) shows the distribution of liquid (density distribution, lighter liquid in blue) along the horizontal midline for increasing forcing. As the pattern wavelength becomes smaller than the container length, lateral boundary effects become weaker, the initial growth is more uniform across the interior, and wavenumber frustration can occur at various locations, not always at the centre. The mechanism of column creation in the asymmetric regime is more complicated than in the symmetric case, and often involves a type of fingering or branching during the transient stage as nascent columns of different size merge, either partly or completely. The locations of the defects vary, with a tendency to alternate from one side to the other. These defect locations, although seemingly random, are repeatable and continuous, as is the final columnar pattern itself, despite the complicated nonlinear dynamics that precedes it.

FIGURE 12.6: (a) Frozen wave threshold as a function of the inverse aspect ratio Γ^{-1} . Numerically obtained onset values (solid dots with dashed line for visual reference) are slightly higher than the theoretical prediction (solid line) of Eq. (12.7) for $\Gamma^{-1} \lesssim 0.5$ but deviate substantially beyond that as finite size effects play a dominant role. Reproduced from Salgado Sánchez et al. (2019c), Chapter 8. (b) Threshold of the symmetry-breaking transition as a function of container aspect ratio Γ . The relevant asymmetric mode is indicated under the corresponding portion of the curve. The onset value for frozen waves, obtained as in Salgado Sánchez et al. (2019c), is shown for reference. Reproduced from Troitiño et al. (2020), Chapter 11.

The container depth has also an important effect. In thin containers, the development of large wavelength patterns is frustrated by repeated collisions with the upper and lower boundaries, where surface and contact forces play a central role. This highly nonlinear process effectively increases the average wavenumber of the final columnar structure, especially near onset. In thick containers, the asymmetry of the forcing and the associated harmonic surface waves, which are out-of-phase at opposite lateral endwalls, is amplified by the longer transient growth time, leading to final structures that inherit more of the asymmetric character of the forcing.

Aside from these distinct mechanisms of column growth, another feature that is critically affected by finite-size effects is the frozen wave threshold. The theoretical prediction of Lyubimova et al. (2017) in the inviscid case can be expressed, after taking the largest wavelength mode permitted in a finite container $k = 2\pi/L$, as

$$\mathcal{F}_H^2 = \frac{(\rho_1 + \rho_2)^3}{2\rho_1\rho_2(\rho_1 - \rho_2)} \left(\frac{\pi}{\Gamma}\right) \tanh\left(\frac{\pi}{\Gamma}\right), \quad (12.7)$$

where $\Gamma = L/(2H)$ is the container aspect ratio. Figure 12.6(a) provides a comparison of Eq. (12.7) (shown with a solid line) and the results of numerical simulations in containers of different aspect ratios (marked with solid dots). For $\Gamma^{-1} \lesssim 0.5$, the numerically obtained onset values are about 10–20% higher, but in reasonable agreement with the theoretical prediction. The observed difference can be attributed mainly to viscosity (Talib, Jalikop, and Juel, 2007) and, possibly also to the vibroequilibria effect (Fernández et al., 2017a,b; Ganiev, Lakiza, and Tsapenko, 1977). As Γ^{-1} becomes larger than 0.5 (short containers), the frozen wave instability is delayed by finite-size

effects, diverging decisively from the theoretical solution.

Analogously, a detailed analysis of the aforementioned symmetry-breaking shows a critical dependence on Γ . In Fig. 12.6(b), the critical forcing to observe such transition is given as a function of Γ . The symmetry-breaking threshold curves resemble neutral stability tongues despite being far above the onset of frozen waves. Furthermore, the fact that the symmetry-breaking transition occurs with particular modes in such a regular way strongly suggests that it is related to a *spontaneous* symmetry-breaking transition (Troitiño et al., 2020). At the same time, it should be understood that there is no bifurcation in the classical meaning of a stable state losing stability to perturbations. Instead, the mode selection process occurs during a complicated and strongly nonlinear evolution with a band excited modes competing with each other. Given the complexity of this system, it is rather surprising that a symmetry-breaking transition with such regular behaviour exists.

One aspect of these results that suggested itself for further investigation was the fact that the denser liquid *always* ends up along the lateral walls, although the pitchfork bifurcation characterising the appearance of the frozen wave instability does not select for this (Jalikop and Juel, 2009). The vibroequilibria effect, however, does imply that the denser fluid should move up these walls rather than down them (Salgado Sánchez et al., 2019a).

A detailed analysis of this has revealed that the vibroequilibria effect has a symmetry-breaking effect on the frozen wave bifurcation (Gligor et al., 2020). The final (asymptotic) trough-to-crest amplitude Δh of the frozen wave pattern was measured to construct bifurcation diagrams from its dependence on applied vibrational velocity v . Figure 12.7 shows that the frozen wave instability appears via a perturbed (asymmetric) supercritical pitchfork bifurcation (Strogatz, 1994). The symmetry breaking of the pitchfork bifurcation relies mainly from the vibroequilibria effect, which is most visible near the lateral walls. While an infinitely long flat interface that is unstable to frozen waves may evolve toward any solution on the group orbit of equivalent patterns (i.e., solutions related by translation through some fraction of a wavelength), this is no longer possible in finite containers where the presence of the container walls modifies the vibrational velocity field. It is this same inhomogeneous velocity field that drives the average reorientation of the interface described by vibroequilibria theory. Since the effect induces the heavier fluid to climb up the lateral walls (Fernández et al., 2017b; Gavriluk, Lukovsky, and Timokha, 2004), it acts to promote the compatible frozen wave solution and suppress the others. Due to this symmetry-breaking effect, the selected branch of the perturbed supercritical pitchfork bifurcation does not truly appear at a critical value but, instead, grows smoothly as the forcing is varied; see dashed line of Fig. 12.7.

The two regimes of amplitude growth distinguished by Jalikop and Juel (2009) are also apparent in Fig. 12.7. Data points in the first regime are fit, for comparison, to both an ideal and a perturbed pitchfork branching equation, while those in the second regime are fit to a linear function. The branching equation for a perturbed supercritical pitchfork bifurcation can be written

$$\Delta h^3 - K^2(v^2 - v_c^2) \Delta h + \epsilon = 0, \quad (12.8)$$

FIGURE 12.7: Bifurcation diagram showing the trough-to-crest distance Δh of the frozen wave patterns versus vibrational velocity v for $g = 0.5g_0$ in a container of size $L \times 2H = 30 \times 15$ mm. The measurements in the gravity-capillary regime are fit to the branching equation (12.8) of a perturbed pitchfork (dashed curve) and of the unperturbed ($\epsilon = 0$) case (solid curve). The gravity regime is fit to the linear function (dotted curve). The transitions between different regimes (labelled above) are highlighted by vertical lines, including the new finite-depth regime where the crest approaches the upper boundary. The diagram for normal gravity is included for comparison (gray open circles). Reproduced from Gligor et al. (2020), Chapter 10.

while the linear regime is characterised by a least squares fit to a linear function. In addition, a new finite-depth regime is shown in Fig. 12.7, where the patterns growth is strongly influenced by the height of the container.

Figure 12.8 summarises the results of a systematic series of numerical simulations to show the dependence on Bo of (a) the critical amplitude $\mathcal{F}_c$ and the transition value $\mathcal{F}^*$ that marks the beginning of the gravity regime, and (b) the branching coefficient $\hat{K}$ of the capillary-gravity regime (expressed in dimensionless form). In Fig. 12.8, the fitting parameters obtained with $\epsilon = 0$ are shown with open circles, while those obtained with the $\epsilon \neq 0$ are shown with crosses; the values often overlap.

From these results, the following conclusions about the influence of gravity may be drawn: (i) the frozen wave threshold $\mathcal{F}_c$ decreases with decreasing Bo until reaching a minimum value determined by the longest wavelength allowed in the container (Salgado Sánchez et al., 2019c); (ii) the branching coefficient $\hat{K}$ of the capillary-gravity regime grows with decreasing Bo and appears to diverge as $Bo \rightarrow 0$.

FIGURE 12.8: Dimensionless fitting parameters characterising the frozen wave bifurcation diagram as a function of the Bond number Bo : (a) critical forcing $\mathcal{F}_c$ (open circles for $\epsilon = 0$, crosses for $\epsilon \neq 0$) and the point of transition to the gravity regime $\mathcal{F}^*$ (solid circles), (b) branching coefficient $\hat{K}$ of the gravity-capillary regime (open circles for $\epsilon = 0$, crosses for $\epsilon \neq 0$). The predicted value of Eq. (12.5) for the critical forcing is shown with a solid curve in (a) while the theoretical formula for $\hat{K}$ from Eq. (12.9) is shown in (b). Reproduced from Gligor et al. (2020), Chapter 10.

The apparent divergence of $\hat{K}$ with decreasing Bo reflects the crucial role of gravity as a restoring force. Gravity dominates for large-amplitude waves and is commensurate with interfacial tension near onset. For $Bo = 0$, there is no restoring force at all and the pitchfork bifurcation is degenerate ($\hat{K} = \infty$).

The threshold derived by Lyubimov and Cherepanov (1986) [see Eq. 12.5] can be compared with the critical forcing $\mathcal{F}_c$ obtained from the fitting procedure. This prediction is shown in Fig. 12.8(a) with a solid curve. Note that the measured threshold values are in good agreement with the dependence of the theoretical curve, except for a delay of approximately 20%. This shift is likely due to a combination of the vibroequilibria effect, other finite-size effects, and viscosity (Talib, Jalikop, and Juel, 2007).

The branching coefficient $\hat{K}$ in the case of an infinitely deep extended system is:

$$\hat{K} = \sqrt{\frac{16(1 - \hat{\rho}^2)}{\hat{\rho}(42\hat{\rho} - 11 - 11\hat{\rho}^2)}} Bo^{-3/4} \simeq 1.688 Bo^{-3/4}, \quad (12.9)$$

evaluated for $\hat{\rho} = 0.52$, which is the density ratio for silicone oil and FC-40, and is shown as a solid curve in Fig. 12.8(b) (Gligor et al., 2020). The agreement of the fitting coefficient with the theoretical predictions is excellent.

FIGURE 12.9: Patterns generated by (horizontal) vibrational forcing in miscible water-isopropanol layers of different densities (i.e., different initial concentrations) in microgravity. (a) Typical frozen wave columnar pattern excited when the initial interface is (mostly) parallel to the forcing axis. (b) Frozen wave pattern with irregular shape, observed for a small misalignment of the initial interface. (c) Simple two-column state observed for a large inclined initial interface.

Furthermore, parabolic flight experiments have uncovered a range of complex behaviour associated with these columnar patterns, including defects and nonreproducible structures that were unexplained by previous studies. We have found that, aside from fluid properties and the characteristics of the applied forcing (A, ω), the angle between the initial interface and the forcing axis, which is used on ground to classify the observed phenomena according to the vertical or horizontal nature of the excitation, plays a crucial role in microgravity as well (Salgado Sánchez et al., 2020).

When the initial interface and the forcing are closely aligned, the frozen wave instability is dominant, leading to the development of nearly regular columns; see Fig. 12.9(a). As the initial angle becomes more oblique, the frozen wave growth becomes increasingly irregular and asymmetric, and may involve thin auxiliary columns; see Fig. 12.9(b). For large enough angles, the frozen wave instability is suppressed by the development of a simple two-column structure; see Fig. 12.9(c). This picture is consistent with the predictions of vibroequilibria theory (Salgado Sánchez et al., 2019a) and a preference for the perpendicular orientation of density contours with respect to the vibrational axis.

The experimental evidence is supported by numerical simulations in both miscible and immiscible liquids. Similar to the pattern selection process and wavenumber transitions explained above (Salgado Sánchez et al., 2019c; see Fig. 12.5), analogous but different types of mode transitions are observed when the interface aligns at an initial angle α with respect to the forcing axis. Calculated for immiscible layers, the mode diagrams presented in Fig. 12.10 show three relevant nondimensional wavenumbers for the final steady interfacial structure measured at half the container height (i.e., at the midline): the minimum and maximum, and the averaged wavenumber, which gives the number of paired columns. These are measured over a range of initial interface angles between 0° and 16° in a $30 \text{ mm} \times 7.5 \text{ mm}$ container, i.e., with aspect ratio $\Gamma = 4$, and with two different vibrational velocities of (a) $A\omega = 424 \text{ mm/s}$ and (b) 565 mm/s . Illustrative snapshots showing the final patterns are included for selected angles.

When $\alpha = 0^\circ$, the final structure contains either three or four pairs of columns, depending on the applied forcing. Since these are not exactly the same length, the minimum and maximum wavelengths differ from the average value. For the higher vibrational velocity used in (b), the

FIGURE 12.10: Diagrams showing the effect of initial interface angle α (in degrees) on dimensionless pattern wavenumber $kL/(2\pi)$ for two applied vibrational velocities: (a) $A\omega = 424$ mm/s ($f = 45$ Hz, $A = 1.5$ mm), (b) $A\omega = 565$ mm/s ($f = 60$ Hz, $A = 1.5$ mm). The domain measures 30 mm $\times$ 7.5 mm, with aspect ratio $\Gamma = 4$. Three different wavenumbers are shown: minimum (black), average (red) and maximum (blue). Snapshots of the final selected patterns are included for illustration. Reproduced from Salgado Sánchez et al. (2020), Chapter 9.

four-column pattern is observed only in the ideal $\alpha = 0^\circ$ case, with a transition to a three-column pattern observed when the initial inclination is increased to $\alpha = 0.1^\circ$.

As the initial inclination is increased, the difference between the maximum and minimum wavenumber grows until a mode transition — at $\alpha \simeq 7.15^\circ$ in (a) and $\alpha \simeq 8.60^\circ$ in (b) — reduces the average wavenumber by one as the thin central columns fail to fully develop and are absorbed by their neighbours. The resulting mode has two thick central columns and two thin outer columns. With further increase in α , these two outer columns grow thinner and, at some point, fail to span the height of the container. When these lateral fluid masses no longer extend enough to cross the midline, the final mode transition to a single pair of columns is marked — at $\alpha \simeq 14.12^\circ$ in (a) and $\alpha \simeq 14.16^\circ$ in (b). Note that, although the location of the intermediate mode transition is delayed by higher vibrational velocity, the final mode transition is almost unaffected by the forcing amplitude. The orientation of the initial interface thus has a dramatic effect on the selected wavenumber (number of column pairs) of the final state. This results from the rotation of the central portion of the interface, consistent with vibroequilibria theory, and the corresponding reduction of the shear flow that drives the frozen wave (Kelvin-Helmholtz) instability (Salgado Sánchez et al., 2020). In parabolic flight experiments, it is a fact that the initial orientation of the interface has certain degree of uncertainty, since it depends on the stage of the parabolic manoeuvre when the forcing is initiated and on the effective acceleration during that experiment, which may vary considerably.

FIGURE 12.11: Successive snapshots showing the evolution of subharmonic Faraday waves on opposite sides of the frozen wave columnar structure for $f = 22$ Hz and $A = 1.76$ mm. Times are selected to best show the $\pi/2$ phase difference between these two subharmonic wave fields (1/4 period of these 11 Hz subharmonic waves is approximately 0.0227 s). Reproduced from Salgado Sánchez et al. (2019d), Chapter 7.

Complex patterns in reduced gravity

The steady interfacial patterns observed in microgravity after the frozen wave instability or the vibroequilibria effect, or even the basic equilibrium shapes selected by surface and contact forces, can be further altered with vibrations. Even for the simplest long-wave columnar pattern mentioned above, the final average interface separating the liquids experiences a (nearly) perpendicular forcing (effectively vertical) and is susceptible to secondary instabilities. Depending on forcing amplitude, these secondary modes can be excited and add to the complexity of the interfacial structure.

A good example of this is the two-scale wave pattern discovered by Shevtsova et al. (2016) in vibrated miscible layers. It is created after the formation of Faraday waves, oscillating at half the forcing frequency, on the vertical interfaces of a columnar structure generated by the frozen wave instability. An analogous interfacial pattern was recently discovered with immiscible layers in the parabolic flight experiments of Salgado Sánchez et al. (2019d). In both cases, the resulting pattern inherits the length and time scales of both the primary (frozen wave) and secondary (Faraday wave) instabilities. Figure 12.11 illustrates the two-scale wave pattern in the FC-40 and 20 cSt silicone oil immiscible combination tested during the VIPIL experiment. A series of snapshots separated by approximately one half-period of the driving force at $f = 22$ Hz are shown.

Depending on the phase, standing waves can be seen (mainly) on the left or right side of the underlying columnar structure. The explanation for this phase shift is the sign change in the effective vibrational acceleration at each interface due to the sign of the density jump. Since the Faraday waves are subharmonic, the π shift in forcing on opposite sides of the columnar structure induces a $\pi/2$ phase shift for the waves. Note that this is another system where the dynamics of two parametrically forced oscillators as analysed above (Salgado Sánchez et al., 2016) is relevant, although the coupling mechanism, effected through the bulk flow in the column interior, is different.

In general, these Faraday waves form on nearly vertical portions of the interface and are of nearly constant amplitude. This suggests that the forcing mechanism can be considered roughly homogeneous and can be compared with the ideal case of a vertically vibrated container (Faraday, 1831;

Kumar and Tuckerman, 1994; Miles and Henderson, 1990). With this assumption, Shevtsova et al. (2016) analysed the Faraday wave instability in a miscible binary mixture and estimated the threshold of subharmonic waves by minimising (Kumar and Tuckerman, 1994)

$$A\omega = \frac{2}{\omega k} \left(\frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} \right) \sqrt{\left(\frac{\omega^2}{4} - \frac{\sigma}{\rho_1 + \rho_2} k^3 \right)^2 + 4k^4 \omega^2 \left(\frac{\mu_1 + \mu_2}{\rho_1 + \rho_2} \right)^2}, \quad (12.10)$$

with $\sigma = 0$ for the miscible fluid case.

With immiscible layers, the threshold obtained by minimising Eq. (12.10) over wavenumber provides a reference value that is later shown in Fig. 12.13 (dashed blue line) for the combinations of FC-40 with 2 cSt and 20 cSt silicone oils tested in the VIPIL experiments (Salgado Sánchez et al., 2019d). There is qualitative, albeit limited, agreement of this theoretical prediction with experiments, which both display $f^{-1/2}$ scaling at high frequencies, although separated by a clear gap. This discrepancy, however, is not surprising in light of the differences between theoretical assumptions and the experimental setup, including the inhomogeneous base state (the columnar frozen wave structure) and finite-size effects. The numerical simulations (dark blue solid line and blue open markers in Fig. 12.13) match somewhat better with the experiments.

As the vibrational velocity increases, the subharmonic Faraday waves become increasingly nonlinear and, at sufficient velocity, a process of drop ejection is initiated. Drop ejection is associated with a jet of fluid emerging from a large-amplitude surface wave that then collapses to create a drop of fluid and, often, other smaller drops, which are known as satellites (Castrejón-Pita et al., 2015; James, Smith, and Glezer, 2003). As forcing amplitude is increased, especially if it becomes comparable to (or larger than) the initial liquid layer depth $A > H$, the dynamics is often fully three-dimensional and disordered (James, Smith, and Glezer, 2003). In this regime, the drops interact with the strongly nonlinear subharmonic waves and the associated mean flow, which can lead to the drop being recaptured by the interface. It should be noted that the highly nonlinear behaviour for $A > H$ is also observed in the case of miscible liquids, which leads either to secondary Faraday waves (Shevtsova et al., 2016) or to the formation of a fish-spine pattern (Shevtsova et al., 2015a).

Experimental results with drop ejection can be compared with the scaling suggested by Goodridge et al. (1997) for the cases of a system dominated by surface tension $A\omega \propto \omega^{1/3}$, and for more viscous fluids $A\omega \propto \omega^{1/2}$, giving the (estimated) onset. We also consider a constant velocity threshold $A\omega \propto 1$. This comparison, although limited by the small number of experimental data points and the variable conditions onboard the parabolic flight, does suggest a transition from (nearly) constant critical velocity at low frequencies, to an increase of critical velocity (most) consistent with the scaling $A\omega \propto \omega^{1/2}$ for viscous fluids (Goodridge et al., 1997).

In addition, a simple analysis of the interfacial energy of the drops suggests an interesting balance analogous to that of vibroequilibria theory (Fernández et al., 2017b; Salgado Sánchez et al., 2019a). After the drop ejection threshold, an increase in vibrational energy is partly transferred into (average) interfacial energy, following a nonlinear process in which the surface wave amplitude grows

FIGURE 12.12: (a) Snapshot of an immiscible liquids combination (FC-40 and 2 cSt silicone oil) subjected to vibrations at $f = 50$ Hz and $A = 3.9$ mm: a complex interfacial pattern consisting of a frozen wave column, Faraday waves and drops is shown. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7. (b) Snapshots showing out-of-phase subharmonic waves oscillating over a vibroequilibria state in a half-filled cylinder holding 50 cSt silicone oil subjected to vibrations in microgravity.

until rupture occurs. In this process, that increase in vibrational energy is partly channelled into the creation of drops, with the $(A\omega)^2$ scaling of vibroequilibria carried over if the forcing is shifted with respect to the critical value of drop ejection. At high vibrational velocity, therefore, interfacial patterns reflect a complex superposition of different phenomena. Provided the initial interface and the applied forcing are (nearly) aligned, the frozen wave instability develops first. A large-scale fluid redistribution occurs, leading to a quasisteady columnar structure with a wavenumber (length scale) selected by the applied forcing and constrained by finite-size effects (Salgado Sánchez et al., 2019c). During the frozen wave growth, the threshold of Faraday waves may be surpassed when the forcing becomes effectively vertical as the interface aligns more perpendicularly to it. These waves inherit the fast timescale of the applied vibrations, pulsating at half the driving frequency, and exhibit a characteristic wavelength selected by the capillary dispersion relation (Kumar and Tuckerman, 1994). As time passes, the amplitude of the Faraday waves increases until the interface becomes unstable to the ejection of drops, which oscillate harmonically at the forcing frequency, not subharmonically. This introduces another frequency to the interfacial dynamics.

Figure 12.12(a) includes a snapshot from the rectangular container holding FC-40 and 2 cSt silicone oil when subjected to vibrations at $f = 50$ Hz and $A = 3.9$ mm. A complex interfacial pattern with frozen waves, Faraday waves and drops is seen. In the absence of the frozen wave instability, secondary Faraday waves can be also excited on an underlying vibroequilibria state, as anticipated above. This type of pattern is shown in Fig. 12.12(b) for a cylinder partially filled with 50 cSt silicone oil and subjected to vibrations in reduced gravity. Note that, in a liquid-air mixture, frozen waves cannot be observed in any practical situation since the threshold is extremely high; corresponding to $\rho_2 \simeq 0$ in Eq. (1.2) or $\hat{\rho} \simeq 0$ in Eq. (12.5).

General picture of interfacial phenomena in microgravity

To summarise, Fig. 12.13 includes the distinct ways in which the interface between two fluids can respond to forced vibrations in microgravity. Results are shown for the immiscible liquid combinations used in VIPIL and are organised in terms of the vibrational frequency f and amplitude A .

For interfacial modes associated with the (fast) timescale of the forcing frequency, results (filled circles) can be separated into three different regimes depending on the dominant behaviour: harmonic waves (yellow), subharmonic Faraday waves (light blue) and drop ejection (red). Numerical results (open markers) estimating the threshold for Faraday waves (blue open circles) and frozen waves (green open squares) are included. The Faraday instability undergoes a change in its scaling from $f^{-1/2}$ to f^{-1} as the frequency is reduced (below about 20 Hz) and finite-size effects become more important. The drop ejection threshold (red line) is shown in panel (a), and is extended for visual reference to higher frequencies (the dashed portion).

For low vibrational velocity (i.e., low amplitude and/or frequency), the combined effect of residual gravity and contact forces leads to a drop-like configuration that can show a certain amount of flattening with respect to the axis of vibration due to the vibroequilibria effect. At the same time, harmonic waves can be (directly) excited on the interface.

As the vibrational velocity is increased, subharmonic Faraday waves are excited. Near onset, these waves oscillate (predominantly) with the frequency $f/2$ and they tend to grow quickly in size, overshadowing the pre-existing harmonic waves and dominating the dynamics of the interface. Drop ejection arises from the destabilisation of the subharmonic surface waves in the limit of high vibrational velocity (solid red line in the figure) when they acquire high amplitude and are very nonlinear. If such ejection is vigorous, the interfacial dynamics becomes disordered and fully three-dimensional. In both fluid combinations, evidence for the existence of frozen waves is found.

For completeness, theoretical threshold predictions are included for subharmonic Faraday waves [Eq. (12.10), blue dashed line] and frozen waves [Eq. (1.2), green dashed line]. Note that each of these predictions is based on an idealised configuration: in the case of Faraday waves, an infinite flat interface with purely “vertical” homogeneous forcing, low damping, and deep layers, and, in the case of frozen waves, an infinite flat interface with purely “horizontal” high-frequency forcing and inviscid flow. Discrepancies are thus expected due to both finite-size effects and the nontrivial base state on which the Faraday waves form.

FIGURE 12.13: Response of the fluid interface to vibrations in microgravity as a function of frequency f and amplitude A : (a) FC-40 and 2 cSt silicone oil, (b) FC-40 and 20 cSt silicone oil. Solid circles denote experimental points in three regimes: harmonic waves (yellow), Faraday waves (blue) and drop ejection (red). Open markers denote numerical results for the Faraday wave instability (blue open circles) and the frozen wave instability (green open squares). The Faraday instability scales as $f^{-1/2}$ for high frequencies ($\gtrsim 20$ Hz) and as f^{-1} at lower frequencies where finite-size effects are important. Theoretical onset curves are shown for Faraday waves [Eq. (12.10), dashed blue lines] and frozen waves [Eq. (1.2), dashed green line]. The drop ejection threshold is shown in panel (a), and is extended (dashed curve) for visual reference to higher frequencies. Panel (b) includes a presumed curve (dashed red line) for drop ejection, although supported by just one experimental data point. Reproduced from Salgado Sánchez et al. (2019d), Chapter 7.

Chapter 13

Conclusions and future work

In this Thesis, an extensive theoretical and experimental investigation of vibrated fluids in microgravity has been conducted. In the absence of gravity, vibrational energy only needs to overcome interfacial and contact forces, which leads to much more complicated dynamics and pattern selection scenarios than in analogous fluid systems on ground, but also offers new possibilities for fluid control and management, which is a critical problem in space exploration.

We first analysed the dynamics of two weakly coupled parametrically forced oscillators in the vicinity of the primary subharmonic resonance in Chapter 3. It was shown how the nature of both the primary and secondary instabilities of the coupled problem, depends on the phase difference between the parametric forcing terms, which determines the residual permutation symmetries. The character of the primary instability, in particular, not only changes from supercritical to subcritical depending on parameters, but also changes from a steady-state pitchfork bifurcation to a Hopf bifurcation due primarily to this phase difference. Attention was drawn to the case of out-of-phase forcing for its relevance to recent experiments on modulated cross-waves (Tinao et al., 2014). The predictions of the model were compared with experiments and with direct numerical simulations of the Navier-Stokes equations, confirming both the initial Hopf bifurcation to modulated solutions, and their destruction via a saddle-node heteroclinic bifurcation.

Although the motivating experiments (Tinao et al., 2014) were performed on ground, the appearance of subharmonic waves in vibrated fluids does not depend on the presence of gravity. Subharmonic waves frequently appear in microgravity as well (Salgado Sánchez et al., 2019d; Shevtsova et al., 2016) where they interact with other phenomena. Indeed, the recent numerical work of Fernández et al. (2017b) on vibroequilibria in rectangular containers has suggested that the presence of modulated subharmonic waves may be related to the appearance of low-frequency sloshing motion. Depending on whether this sloshing motion decays, persists or grows, the underlying vibroequilibria state may survive, suffer variations, or be destroyed completely.

From the perspective of fluid management and control in microgravity, the degree of interfacial alignment achieved by the vibroequilibria effect is especially relevant, and suggestive of artificial gravity. A detailed investigation of vibroequilibria was presented in Chapters 4 to 6 through the

analysis and scientific review of the CFVib experiment.

The experimental results showed clear evidence of the vibroequilibria effect in both cuboidal and cylindrical geometries and for fluids having different interfacial and contact properties, i.e., water (non-wetting) and silicone oil (wetting). When the principal component of the vibrations was horizontal, the vibroequilibria states in both geometries were characterised by the upward motion of the (rear and front) contact lines and the downward motion of the liquid-air interface in the interior. In the contrasting case of dominant vertical motion, the vibroequilibria solutions were characterised by the flattening of the free surface compared to the (unforced) reference experiments. Overall, the general tendency for vibroequilibria states to reorient large portions of the interface to be more perpendicular to the vibrational axis was confirmed.

Experiments with water in the cuboidal geometry also allowed for a direct quantitative comparison with theory. Fairly good agreement was found with the (two-dimensional) predictions of vibroequilibria theory over the relevant interval of forcing, $\eta < 3$. The major differences (leading to a discrepancy of 20–30%) were shown to arise from the rotational motion of the experimental containers, which is not compatible with the hypothesis of potential flow assumed by vibroequilibria theory. This was confirmed by direct comparison with (two-dimensional) numerical simulations of the Navier-Stokes equations, both including and neglecting this rotational motion. Very good agreement with experiments was found when the rotational terms were included, while excellent agreement with vibroequilibria theory was obtained in the other case. It is clear that an extension of vibroequilibria theory to include rotational terms is a natural subject for future work.

Finally, detailed analyses of the frozen wave instability and other interfacial phenomena in microgravity were presented. First, the VIPIL experiments were carefully processed and complemented with numerical simulations in Chapter 7. Then, motivated by such experiments, two different geometrical features of the frozen wave pattern selection process were investigated separately: the effect of finite container size in Chapter 8, the effect of the initial angle between the vibrational axis and the unperturbed interface in Chapter 9, the influence of gravity in Chapter 10 and the symmetry-breaking transition in Chapter 11.

The critical role of finite-size effects for pattern selection was demonstrated (Gligor et al., 2020; Salgado Sánchez et al., 2019c; Troitiño et al., 2020). A finite container length does not permit the continuous wavenumber selection predicted by theory. Instead, pattern selection is characterised by selected wavenumbers that exist over discrete forcing intervals, with mode transitions occurring through repeated cycles of column growth. The finite container height, on the other hand, can suppress long-wave modes in the limit of shallow containers, and enhance the asymmetry of the final columnar patterns in the limit of deep containers. For containers of aspect ratio $\Gamma < 2$, the instability threshold is significantly delayed with respect to the theoretical prediction (Salgado Sánchez et al., 2019c). In addition, a finite container size enhances the effect of vibroequilibria, which acts perturbing the supercritical pitchfork bifurcation that characterises the appearance of frozen waves near onset and selecting patterns compatible with vibroequilibria states (Gligor et

al., 2020). Far from onset, on the other hand, there exists a symmetry-breaking transition whose threshold curves resemble neutral stability tongues. This symmetry-breaking occurs with particular modes in a surprisingly regular way, fact that strongly suggests that it is related to a *spontaneous* symmetry-breaking transition (Troitiño et al., 2020).

Furthermore, we found that the angle between the initial interface and the forcing axis also plays a crucial role in microgravity (Salgado Sánchez et al., 2020). When the initial interface and the forcing are nearly aligned, the frozen wave instability is dominant, leading to the development of nearly regular columns. As the initial angle becomes more oblique, the frozen wave growth is increasingly irregular and asymmetric, and may involve thin auxiliary columnar structures. For large enough angles, the frozen wave instability is suppressed by the development of a simple two-column structure.

Aside from the complexity of the pattern selection process for frozen waves, interfacial patterns driven by moderate to high forcing in microgravity are characterised by the coexistence and interplay of distinct instabilities. At high vibrational velocity, provided the initial interface and the applied forcing are (nearly) aligned, the frozen wave instability develops first. A large-scale fluid redistribution occurs as a quasisteady columnar structure forms, with a wavenumber (length scale) selected by the applied forcing and constrained by finite-size effects (Salgado Sánchez et al., 2019c). During the frozen wave growth, the threshold of Faraday waves may be surpassed as the forcing becomes effectively “vertical” with the increasing alignment of the interface perpendicular to it (Salgado Sánchez et al., 2019d). These waves oscillate on the fast timescale of the applied vibrations, pulsating at half the driving frequency, and exhibit a characteristic wavelength selected by the capillary dispersion relation (Kumar and Tuckerman, 1994). As time passes, the amplitude of the Faraday waves increases until the interface becomes unstable to the ejection of drops, which oscillate harmonically at the forcing frequency.

The results presented in this Thesis should contribute to the understanding of vibrated fluid systems in microgravity. The very different interfacial phenomena, such as frozen waves, Faraday waves, and vibroequilibria, which can arise simultaneously in microgravity, lead to complex interactions and transitions, with interesting but complicated dynamics and pattern selection scenarios. Altogether, these results strongly suggest that it should be possible to use vibrations to help control and manage fluids in microgravity environments. However, additional numerical simulations and future experiments are needed to extend these investigations. In particular, the research group is currently working on the effect of shear-dependent viscosity on the frozen wave pattern selection in both miscible and immiscible liquids. This specific objective is part of a more ambitious project that aims to understand the mechanisms of particle adsorption through mucus layers.

The analysis of this complex problem is currently ongoing, and some progress has been already achieved with the Complex Fluids under Periodic Excitation (COMFLU) experiment¹. The COMFLU experiment was motivated by the growing environmental crisis caused by microplastic waste.

---

¹<http://eea.spaceflight.esa.int/portal/exp/?id=9792>

FIGURE 13.1: Preliminary results of the COMFLU experiment: (a) ground test, (b) simulation.

Microplastics that are dumped or find their way into the oceans interact with marine creatures (Cole et al., 2013; Nelms et al., 2019) who can ingest or adsorb them, and are easily transported by waves and currents (van Sebille et al., 2015). In these processes of ingestion or adsorption, the primary contact is via a mucus layer.

The COMFLU experiment investigates the dynamics of the interface separating a Newtonian liquid (like water or brine) and a non-Newtonian layer (like mucus) when subjected to periodic excitation (emulating waves, swimming, breathing, etc.) and anticipates the effect of this on particle adsorption. The experiment was evaluated positively by ESA and the first phase was conducted during the 72nd ESA Parabolic Flight campaign in November 2019. This experimental campaign has already yielded interesting new results, which are being extended by numerical simulations. Plumes and finger-like patterns often appear. The length scale and evolution of these patterns depend on the periodic forcing, but also on the properties of the interface, whose thickness (and effective interfacial tension) varies in space and time, and on the viscoelastic character of the non-Newtonian fluid. In addition to this, intriguing new interfacial behaviour related to the frozen wave (Kelvin-Helmholtz) instability was observed, which is a natural extension of the work presented in this Thesis. Some preliminary results are shown in Fig. 13.1. We note that the experimental efforts of COMFLU are currently on hold and will continue during the 73rd or 74th Campaign (depending on adjustments related to the developing COVID-19 health situation).

These parabolic flight experiments, however, are limited by the shorter microgravity time of parabolic flights compared to the actual timescale of the absorption phenomenon in common marine situations (e.g., jelly fish swimming). In close connection to COMFLU, the experiment Microparticle Adsorption by Viscoelastic Fluid under Periodic Excitations (MAVFLU) was recently submitted to the Continuously Open Research Announcement from ESA, and is currently under evaluation to be executed in a Sounding Rocket, which provides longer microgravity times.

The main objective of the MAVFLU project is to investigate the effect of fluid motion of various frequencies and amplitudes on the inertial aspects of microparticle adsorption through a mucus layer. Specifically, we will observe dynamics and particle motion in the presence of a periodically forced flow generated by the controlled vibration of a solid container enclosing two fluids. Preliminary

FIGURE 13.2: Simulations of the MAVFLU experiment showing particle adsorption events on (a) ground and (b) microgravity.

results from two numerical simulations are illustrated in Fig. 13.2, showing particle transport across the interface separating the two (miscible) fluid layers (coloured). Such periodically forced flows are associated with vibrational convection, and with movement and instabilities of the interface like these presented in this Thesis. The resulting interfacial dynamics and the evolving fluid flow will be investigated in different viscoelastic fluids and under different frequencies and amplitudes.

Finally, progress is expected to be made regarding the application of coupled oscillator models to parametrically forced surface waves. Some research has already been conducted by extending the model presented in Chapter 3 to mixed (vertical and horizontal) forcing, and by comparing its dynamics with corresponding two-dimensional numerical simulations. From an experimental perspective, the research team is currently developing a setup with two wavemakers, made from vibrating plates, whose phase difference can be tuned to any value. We expect this simple experiment to confirm theoretical predictions in the case of synchronised forcing, as well as other forcing phases, and to allow a more complete analysis in the case of mixed excitation.

BIBLIOGRAPHY

- Abramson, H. N. (1966). *The Dynamic Behavior of Liquids in Moving Containers, with Applications to Space Vehicle Technology*. Tech. rep. NASA.
- Amiroudine, S. and Beysens, D. (2008). “Thermovibrational instability in supercritical fluids under weightlessness”. In: *Physical Review E* 78, p. 036325.
- Ancherbak, S., Yasnou, V., Mialdun, A., and Shevtsova, V. (2018). “Coexistence Curve, Density, and Viscosity for the Binary System of Perfluorohexane + Silicone Oil”. In: *Journal of Chemical and Engineering Data* 63, pp. 3008–3017.
- Arbell, H. and Fineberg, J. (2002). “Pattern formation in two-frequency forced parametric waves”. In: *Physical Review E* 65, p. 036224.
- Audier, P., Fénot, M., Bénard, N., and Moreau, E. (2016). “Flow control of an elongated jet in cross-flow: Film cooling effectiveness enhancement using surface dielectric barrier discharge plasma actuator”. In: *Applied Physics Letters* 108, p. 084103.
- Ayanle, H., Bernoff, A. J., and Lichter, S. (1990). “Spanwise modal competition in cross-waves”. In: *Physica D* 43, pp. 87–104.
- Baer, S. M., Erneux, T., and Rinzel, J. (1989). “The slow passage through a Hopf bifurcation: delay, memory effects, and resonance”. In: *SIAM Journal of Applied Mathematics*.
- Barton, D. A. W. and Sieber, J. (2013). “Systematic experimental exploration of bifurcations with noninvasive control”. In: *Physical Review E* 87, p. 052916.
- Barnard, B. J. S. and Pritchard, W. G. (1972). “Cross-waves. Part 2. Experiments”. In: *Journal of Fluid Mechanics* 55, pp. 245–255.
- Batson, W., Zoueshtiagh, F., and Narayanan, R. (2013). “Dual role of gravity on the Faraday threshold for immiscible viscous layers”. In: *Physical Review E* 88, p. 063002.
- Batchelor, G. K. (1954). “Heat convection and buoyancy effects in fluids”. In: *Quarterly Journal of the Royal Meteorological Society* 80, pp. 339–358.
- Benjamin, T. B. and Ursell, F. J. (1954). “The stability of the plane free surface of a liquid in vertical periodic motion”. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 225, pp. 505–515.
- Bena, I. and Van den Broeck, C. (1999). “Coupled parametric oscillators”. In: *Europhysics Letters* 48, pp. 498–504.
- Beyer, K., Gawriljuk, I., Gunther, M., Lukovsky, I., and Timokha, A. (2001). “Compressible Potential Flows with Free Boundaries. Part I: Vibrocapillary Equilibria”. In: *Journal of Applied Mathematics and Mechanics* 81, pp. 261–271.

- Beyer, K., Günther, M., and Timokha, A. (2004). “Variational and finite element analysis of vibroequilibria”. In: *Computational Methods in Applied Mathematics* 4, pp. 290–323.
- Beysens, D. (2006). “Vibrations in space as an artificial gravity?”. In: *Europhysics News* 37, pp. 22–25.
- Beysens, D., Garrabos, Y., Chatain, D., and Evesque, P. (2009). “Phase transition under forced vibrations in critical CO₂”. In: *Europhysics Letters* 86, p. 16003.
- Beysens, D. (2014). “Critical point in space: a quest for universality”. In: *Microgravity Science and Technology* 26, pp. 201–218.
- Birikh, R. V. (1966). “Thermocapillary convection in a horizontal layer of liquid”. In: *Journal of Applied Mechanics and Technical Physics* 7, pp. 43–44.
- Brunet, P., Eggers, J., and Deegan, R. D. (2007). “Vibration-Induced Climbing of Drops”. In: *Physical Review Letters* 99, p. 144501.
- Buks, E. and Roukes, M. L. (2002). “Electrically tunable collective response in a coupled micromechanical array”. In: *Journal of Microelectromechanical Systems* 11, pp. 802–807.
- Burnysheva, A. V., Lyubimov, D. V., and Lyubimova, T. P. (2011). “Disturbance spectrum of a plane fluid-fluid interface in the field of tangential high-frequency vibrations under weightlessness”. In: *Fluid Dynamics* 46, pp. 1000–1009.
- Callens, N., Ventura-Traveset, J., de Lophem, T. L., López de Echazarreta, C., Pletser, V., and van Loon, J. J. W. A. (2011). “ESA parabolic flights, drop tower and centrifuge opportunities for university students”. In: *Microgravity Science and Technology* 23, pp. 181–189.
- Callens, N., Ha, L., and Galeone, P. (2016). “Benefits of ESA Gravity-Related Hands-on Programmes for University Students’ Careers”. In: *Microgravity Science and Technology* 28, pp. 519–527.
- Catarino, S. O., Miranda, J. M., Lanceros-Mendez, S., and Minas, G. (2011). “Modeling and simulation of the mixing process of fluids in microchannels promoted by acoustic streaming”. In: *Proceedings of the 3rd Micro and Nano Flows Conference*.
- Codina, R. (1993). “A discontinuity-capturing crosswind-dissipation for the finite element solution of the convection-diffusion equation”. In: *Computer Methods in Applied Mechanics and Engineering* 110, pp. 325–342.
- Cole, M., Lindeque, P., Fileman, E., Halsband, C., Goodhead, R., Moger, J., and Galloway, T. S. (2013). “Microplastic Ingestion by Zooplankton”. In: *Environmental Science and Technology* 47, pp. 6646–6655.
- Copelli, M. and Lindenberg, K. (2001). “Phase-induced stability in a parametric dimer”. In: *Physical Review E* 63, p. 036605.
- Castrejón-Pita, J. R., Castrejón-Pita, A. A., Thete, S. S., Sambath, K., Hutchings, I. M., Hinch, J., Lister, J. R., and Basaran, O. A. (2015). “Plethora of transitions during breakup of liquid filaments”. In: *Proceedings of the National Academy of Sciences of the United States of America* 112, pp. 4582–4587.
- Danzl, P. and Moehlis, J. (2010). “Weakly coupled parametrically forced oscillator networks: existence, stability, and symmetry of solutions”. In: *Nonlinear Dynamics* 59, pp. 661–680.
- Diwakar, S. V., Jajoo, V., Amiroudine, S., Matsumoto, S., Narayanan, R., and Zoueshtiagh, F. (2018). “Influence of capillarity and gravity on confined Faraday waves”. In: *Physical Review Fluids* 3, p. 073902.

- Dore, B. D. (1973). “On mass transport induced by interfacial oscillations at a single frequency”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 74, pp. 333–347.
- Douady, S., Fauve, S., and Thual, O. (1989). “Oscillatory phase modulation of parametrically forced surface waves”. In: *Europhysics Letters* 10, pp. 309–315.
- Douady, S. (1990). “Experimental study of the Faraday instability”. In: *Journal of Fluid Mechanics* 221, pp. 383–409.
- Edwards, W. S. and Fauve, S. (1994). “Patterns and quasi-patterns in the Faraday experiment”. In: *Journal of Fluid Mechanics* 278, pp. 123–148.
- Erlicher, S., Bonaventura, L., and Bursi, O. S. (2002). “The analysis of the Generalized- α method for non-linear dynamic problems”. In: *Computational Mechanics* 28, pp. 83–104.
- Faltinsen, O. M., Rognebakke, O. F., and Timokha, A. N. (2006). “Transient and steady-state amplitudes of resonant three-dimensional sloshing in a square base tank with a finite fluid depth”. In: *Physics of Fluids* 18, p. 012103.
- Faraday, M. (1831). “On peculiar class of acoustical figures; and on certain forms assumed by groups of particles upon vibrating elastic surfaces”. In: *Philosophical Transactions of the Royal Society of London* 121, pp. 299–340.
- Feng, Z. C. (1997). “Transition to traveling waves from standing waves in a rectangular container subjected to horizontal excitations”. In: *Physical Review Letters* 79, pp. 415–418.
- Fernández, J., Salgado Sánchez, P., Tíñao, I., Porter, J., and Ezquerro, J. M. (2017a). “The CFVib experiment: Control of Fluids in microgravity with Vibrations”. In: *Microgravity Science and Technology* 29, pp. 351–364.
- Fernández, J., Tíñao, I., Porter, J., and Laverón-Simavilla, A. (2017b). “Instabilities of vibroequilibria in rectangular containers”. In: *Physics of Fluids* 29, p. 024108.
- Foresti, D., Nabavi, M., Klingauf, M., Ferrari, A., and Poulikakos, D. (2013). “Acoustophoretic contactless transport and handling of matter in air”. In: *Proceedings of the National Academy of Sciences* 110, pp. 12549–12554.
- Funakoshi, M. and Inoue, S. (1988). “Surface waves due to resonant horizontal oscillation”. In: *Journal of Fluid Mechanics* 192, pp. 219–247.
- Gandikota, G., Chatain, D., Amiroudine, S., Lyubimova, T., and Beysens, D. (2014a). “Faraday instability in a near-critical fluid under weightlessness”. In: *Physical Review E* 89, p. 013022.
- (2014b). “Frozen-wave instability in near-critical hydrogen subjected to horizontal vibration under various gravity fields”. In: *Physical Review E* 89, p. 012309.
- Ganiev, R. F., Lakiza, V. D., and Tsapenko, A. S. (1977). “Dynamic behavior of the free liquid surface subject to vibrations under conditions of near-zero gravity”. In: *Soviet Applied Mechanics* 13, pp. 499–503.
- Gaponenko, Y. A., Torregrosa, M., Yasnou, V., Mialdun, A., and Shevtsova, V. (2015a). “Dynamics of the interface between miscible liquids subjected to horizontal vibration”. In: *Journal of Fluid Mechanics* 784, pp. 342–372.
- Gaponenko, Y. A., Torregrosa, M. M., Yasnou, V., Mialdun, A., and Shevtsova, V. (2015b). “Interfacial pattern selection in miscible liquids under vibration”. In: *Soft Matter* 11, pp. 8221–8224.
- Gaponenko, Y. and Shevtsova, V. (2016). “Shape of diffusive interface under periodic excitations at different gravity levels”. In: *Microgravity Science and Technology* 28, pp. 431–439.

- Gaponenko, Y., Mialdun, A., and Shevtsova, V. (2018). “Pattern selection in miscible liquids under periodic excitation in microgravity: Effect of interface width”. In: *Physics of Fluids* 30, p. 062103.
- Garrabos, Y., Beysens, D., Lecoutre, C., Dejoan, A., Polezhaev, V., and Emelianov, V. (2007). “Thermoconvectonal phenomena induced by vibrations in supercritical SF₆ under weightlessness”. In: *Physical Review E* 75, p. 056317.
- Garrett, C. J. R. (1970). “On cross-waves”. In: *Journal of Fluid Mechanics* 41, pp. 837–849.
- Gavrilyuk, I., Lukovsky, I., and Timokha, A. (2004). “Two-dimensional variational vibroequilibria and Faraday’s drops”. In: *Journal of Applied Mathematics and Physics* 55, pp. 1015–1033.
- Gligor, D., Salgado Sánchez, P., Porter, J., and Shevtsova, V. (2020). “Influence of gravity on the frozen wave instability in immiscible liquids”. In: *Physical Review Fluids* 5, p. 084001.
- Goldobin, D. and Pikovsky, A. (2002). “Collective modes in parametrically excited oscillator arrays”. In: *Europhysics Letters* 59, pp. 193–198.
- Goldobin, D. S., Pimenova, A. V., Kovalevskaya, K. V., Lyubimov, D. V., and Lyubimova, T. P. (2015). “Running interfacial waves in a two-layer fluid system subject to longitudinal vibrations”. In: *Physical Review E* 91, p. 053010.
- Goodridge, C. L., Tao Shi, W., and Lathrop, D. P. (1996). “Threshold dynamics of singular gravity-capillary waves”. In: *Physical Review Letters* 76, pp. 1824–1827.
- Goodridge, C. L., Tao Shi, W., Hentschel, H. G. E., and Lathrop, D. P. (1997). “Viscous effects in droplet-ejecting capillary waves”. In: *Physical Review E* 56, pp. 472–475.
- Gresho, P. M. and Sani, R. L. (1970). “The effects of gravity modulation on the stability of a heated fluid layer”. In: *Journal of Fluid Mechanics* 40, pp. 783–806.
- Harari, I. and Hughes, T. J. R. (1992). “What are C and h?: Inequalities for the analysis and design of finite element methods”. In: *Computer Methods in Applied Mechanics and Engineering* 97, pp. 157–192.
- Hasegawa, K., Abe, Y., Kaneko, A., Yamamoto, Y., and Aoki, K. (2009). “Visualization measurement of streaming flows associated with a single-acoustic levitator”. In: *Microgravity Science and Technology* 21, S9–S14.
- Hoyle, R. (2006). *Pattern Formation: An Introduction to Methods*. Cambridge University Press.
- Huygens, C. (1669). “Instructions concerning the use of pendulum-watches, for finding the longitude at sea”. In: *Proceedings of the Royal Society of London, Transactions of the Royal Society* 4, pp. 937–976.
- Ito, Y. and Komori, S. (2006). “A vibration technique for promoting liquid mixing and reaction in a microchannel”. In: *American Institute of Chemical Engineers Journal* 52, pp. 3011–3017.
- Ivanova, A. A., Kozlov, V. G., and Evesque, P. (2001). “Interface Dynamics of Immiscible Fluids under Horizontal Vibration”. In: *Fluid Dynamics* 36, pp. 362–368.
- Jalikop, S. V. and Juel, A. (2009). “Steep capillary-gravity waves in oscillatory shear-driven flows”. In: *Journal of Fluid Mechanics* 640, pp. 131–150.
- (2012). “Oscillatory transverse instability of interfacial waves in horizontally oscillating flows”. In: *Physics of Fluids* 24, p. 044104.
- James, A. J., Smith, M. K., and Glezer, A. (2003). “Vibration-induced drop atomization and the numerical simulation of low-frequency single-droplet ejection”. In: *Journal of Fluid Mechanics* 476, pp. 29–62.

- James, A. J., Vukasinovic, B., Smith, M. K., and Glezer, A. (2003). “Vibration-induced drop atomization and bursting”. In: *Journal of Fluid Mechanics* 476, pp. 1–28.
- Jenson, R. M., Weislogel, M. M., Klatte, J., and Dreyer, M. E. (2010). “Dynamic Fluid Interface Experiments Aboard the International Space Station: Model Benchmarking Dataset”. In: *Journal of Spacecraft and Rockets* 47, pp. 670–679.
- Jones, A. F. (1984). “The generation of cross-waves in a long deep channel by parametric resonance”. In: *Journal of Fluid Mechanics* 138, pp. 53–74.
- Jounet, A., Mojtabi, A., Ouazzani, J., and Zappoli, B. (2000). “Low-frequency vibrations in a near-critical fluid”. In: *Physics of Fluids* 12, pp. 197–204.
- Kapitza, P. L. (1951). “Pendulum with a vibrating suspension”. In: *Physics-Uspekhi* 44, pp. 7–15.
- Kawai, R., Lindenberg, K., and Van den Broeck, C. (2002). “Parametrically modulated oscillator dimer: an analytic solution”. In: *Physica A: Statistical Mechanics and its Applications* 312, pp. 119–140.
- Khenner, M. V., Lyubimov, D. V., Belozerova, T. S., and Roux, B. (1999). “Stability of plane-parallel vibrational flow in a two-layer system”. In: *European Journal of Mechanics - B/Fluids* 18, pp. 1085–1101.
- Kitanov, P. M., William, W. F., and Willms, A. R. (2013). “Double zero bifurcation with Huygens symmetry”. In: *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* 20, pp. 197–226.
- Knobloch, E. (1986). “Normal forms for bifurcations at a double zero eigenvalue”. In: *Physics Letters A* 115, pp. 199–201.
- Koster, D. (2006). *Numerical Simulation of Acoustic Streaming on SAW-driven Biochips*.
- Kothe, D. B., Mjolsness, R. C., and Torrey, M. D. (1991). *RIPPLE: A Computer Program for Incompressible Flows with Free Surfaces*. Technical Report. Los Alamos National Laboratory.
- Krieger, M. S. (2017). “Interfacial fluid instabilities and Kapitza pendula”. In: *The European Physical Journal E* 40, p. 67.
- Kudrolli, A., Pier, B., and Gollub, J. P. (1998). “Superlattice patterns in surface waves”. In: *Physica D: Nonlinear Phenomena* 123, pp. 99–111.
- Kull, H. J. (1991). “Theory of the Rayleigh-Taylor instability”. In: *Physics Reports* 206, pp. 197–325.
- Kumar, K. and Tuckerman, L. S. (1994). “Parametric instability of the interface between two fluids”. In: *Journal of Fluid Mechanics* 279, pp. 49–68.
- Kuramoto, Y. (1984). *Chemical oscillations, waves, and turbulence*. Berlin: Springer-Verlag.
- Kuznetsov, Y. A. (2004). *Elements of applied bifurcation theory*. Ed. by J. E. Marsden and L. Sirovich. 3rd. Applied mathematical sciences. Springer.
- Lacaze, L., Guenoun, P., Beysens, D., Delsanti, M., Petitjeans, P., and Kurowski, P. (2010). “Transient surface tension in miscible liquids”. In: *Physical Review E* 82, p. 041606.
- Landau, L. D. and Lifshitz, E. M. (1987). *Fluid Mechanics*. Pergamon books Ltd.
- Li, Y. and Umemura, A. (2014). “Threshold condition for spray formation by Faraday instability”. In: *Journal of Fluid Mechanics* 759, pp. 73–103.
- Lichter, S. and Underhill, W. B. (1987). “Mode-number shifting of nonlinear cross-waves”. In: *Physical Review A* 35, pp. 5282–5284.

- Lifshitz, R. and Cross, M. C. (2003). “Response of parametrically driven nonlinear coupled oscillators with application to micromechanical and nanomechanical resonator arrays”. In: *Physical Review B* 67, p. 134302.
- Lin, J. D. and Howard, L. N. (1960). *Non-linear standing waves in a rectangular tank due to forced oscillation*. Hydrodynamics Laboratory Technical Report. Massachusetts Institute of Technology.
- Lyubimov, D. V., Lyubimova, T., and Cherepanov, A. A. (2003). *Dynamics of Interfaces in Vibrational Fields*. in Russian. Moscow: Fizmatlit.
- Lyubimov, D. V., Ivantsov, A. O., Lyubimova, T., and Khilko, G. L. (2016). “Numerical modeling of frozen wave instability in fluids with high viscosity contrast”. In: *Fluid Dynamics Research* 48, p. 061415.
- Lyubimova, T., Ivantsov, A., Garrabos, Y., Lecoutre, C., Gandikota, G., and Beysens, D. (2017). “Band instability in near-critical fluids subjected to vibration under weightlessness”. In: *Physical Review E* 95, p. 013105.
- Lyubimova, T. P., Ivantsov, A., Garrabos, Y., Lecoutre, C., and Beysens, D. (2019). “Faraday waves on band pattern under zero gravity conditions”. In: *Physical Review Fluids* 4, p. 064001.
- Lyubimov, D. V. and Cherepanov, A. A. (1986). “Development of a steady relief at the interface of fluids in a vibrational field”. In: *Fluid Dynamics* 21, pp. 849–854.
- Lyubimov, D. V., Cherepanov, A. A., Lyubimova, T. P., and Roux, B. (1997). “Interface orienting by vibration”. In: *Comptes Rendus de l’Académie des Sciences - Series IIB - Mechanics-Physics-Chemistry-Astronomy* 325, pp. 391–396.
- Lyubimov, D. V., Khenner, M. V., and Shotz, M. M. (1998). “Stability of a fluid interface under tangential vibrations”. In: *Fluid Dynamics* 33, pp. 318–323.
- Lyubimov, D. V., Lyubimova, T., Croell, A., Dold, P., Benz, K. W., and Roux, B. (1998). “Vibration-induced convective flows”. In: *Microgravity Science and Technology*, pp. 42–87.
- Mandel, P. and Erneux, T. (1987). “The slow passage through a steady bifurcation: Delay and memory effects”. In: *Journal of Statistical Physics* 48, pp. 1059–1070.
- Marqués, F., Meseguer, A., López, J. M., Pacheco, J. R., and Lopez, J. M. (2013). “Bifurcations with imperfect SO(2) symmetry and pinning of rotating waves”. In: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 469, p. 20120348.
- Martin, R. M. (1972). “Piezoelectricity”. In: *Physical Review B* 5, p. 1607.
- McGoldrick, L. F. (1968). *Faraday waves: the cross-wave resonant instability*. Technical Report. University of Chicago, Department of Geophysical Sciences.
- McLachlan, N. W. (1947). *Theory and application of Mathieu functions*. Clarendon Press, p. 401.
- Mialdun, A., Ryzhkov, I. I., Melnikov, D. E., and Shevtsova, V. (2008). “Experimental evidence of thermal vibrational convection in a nonuniformly heated fluid in a reduced gravity environment”. In: *Physical Review Letters* 101, p. 084501.
- Mialdun, A., Yasnou, V., Shevtsova, V., Königer, A., Köhler, W., Alonso De Mezquia, D., and Bou-Ali, M. M. (2012). “A comprehensive study of diffusion, thermodiffusion, and Soret coefficients of water-isopropanol mixtures”. In: *The Journal of Chemical Physics* 136, p. 244512.
- Millikan, R. A. (1913). “On the Elementary Electrical Charge and the Avogadro Constant”. In: *Physical Review* 2, pp. 109–143.

- Miles, J. W. (1959). “On the generation of surface waves by shear flows Part 3. Kelvin-Helmholtz instability”. In: *Journal of Fluid Mechanics* 6, pp. 583–598.
- (1984). “Resonantly forced surface waves in a circular cylinder”. In: *Journal of Fluid Mechanics* 149, pp. 15–31.
- Miles, J. (1988). “Parametrically excited, standing cross-waves”. In: *Journal of Fluid Mechanics* 186, pp. 119–127.
- Miles, J. and Henderson, D. (1990). “Parametrically forced surface waves”. In: *Annual Review of Fluid Mechanics* 22, pp. 143–165.
- Mirollo, R. E. and Strogatz, S. H. (1990). “Synchronization of pulse-coupled biological oscillators”. In: *SIAM Journal of Applied Mathematics* 50, pp. 1645–1662.
- Moisy, F., Michon, G. J., Rabaud, M., and Sultan, E. (2012). “Cross-waves induced by the vertical oscillation of a fully immersed vertical plate”. In: *Physics of Fluids* 24, p. 022110.
- Nelms, S., Barnett, J., Brownlow, A., Davison, N., Deaville, R., Galloway, T., Lindeque, P., Santillo, D., and Godley, B. (2019). “Microplastics in marine mammals stranded around the British coast: ubiquitous but transitory?” In: *Scientific reports* 9, p. 1075.
- Nelson, E. S. (1994). *An examination of anticipated g-jitter on space station and its effects on materials processes*. Tech. rep. 103775. NASA Lewis Research Center, pp. 1–119.
- Nobili, M., Ciliberto, S., Cocciaro, B., Faetti, S., and Fronzoni, L. (1988). “Time-dependent surface waves in a horizontally oscillating container”. In: *Europhysics Letters* 7, pp. 587–592.
- Olsson, E. and Kreiss, G. (2005). “A conservative level set method for two phase flow”. In: *Journal of Computational Physics* 210, pp. 225–246.
- Peng, G. and Li, C. (2014). “Computation of universal unfolding of the double zero bifurcation in the Z_2 -symmetric system”. In: *International Journal of Computer Mathematics* 91, pp. 461–479.
- Pérez-Gracia, J. M., Porter, J., Varas, F., and Vega, J. M. (2014). “Oblique cross-waves in horizontally vibrated containers”. In: *Fluid Dynamic Research* 46, p. 041410.
- Perez-Gracia, J. M., Porter, J., Varas, F., and Vega, J. M. (2014). “Subharmonic capillary-gravity waves in large containers subject to horizontal vibrations”. In: *Journal of Fluid Mechanics* 739, pp. 196–228.
- Phua, P. B., Lai, K. S., Wu, R. F., and Chong, T. C. (1998). “Coupled tandem optical parametric oscillator (OPO): an OPO within an OPO”. In: *Optical Letters* 23, pp. 1262–1264.
- Pletser, V., Rouquette, S., Friedrich, U., Clervoy, J. F., Gharib, T., Gai, F., and Mora, C. (2016). “The First European Parabolic Flight Campaign with the Airbus A310 ZERO-G”. In: *Microgravity Science and Technology* 28, pp. 587–601.
- Porter, J., Tínao, I., Laverón-Simavilla, A., and López, C. A. (2012). “Pattern selection in a horizontally vibrated container”. In: *Fluid Dynamics Research* 44, p. 065501.
- Porter, J., Tínao, I., Laverón-Simavilla, A., and Rodríguez, J. (2013). “Onset patterns in a simple model of localized parametric forcing”. In: *Physical Review E* 88, p. 042913.
- Prinet, N., Juric, D., and Tuckerman, L. S. (2009). “Numerical simulation of Faraday waves”. In: *Journal of Fluid Mechanics* 635, pp. 1–26.
- Salgado Sánchez, P., Porter, J., Tínao, I., and Laverón-Simavilla, A. (2016). “Dynamics of weakly coupled parametrically forced oscillators”. In: *Physical Review E* 94, p. 022216.

- Salgado Sánchez, P., Fernández, J., Tinao, I., and Porter, J. (2019a). “Vibroequilibria in microgravity: comparison of experiments and theory”. In: *Physical Review E* 100, p. 063103.
- Salgado Sánchez, P., Fernández, J., Tinao, I., Porter, J., and Bello, A. (2019b). “Simulating vibrated fluids and vibroequilibria in the CFVib microgravity experiment”. In: *Proceedings of the 8th European Conference for Aeronautics and Aerospace Science*, p. 0224.
- Salgado Sánchez, P., Gaponenko, Y. A., Porter, J., and Shevtsova, V. (2019c). “Finite-size effects on pattern selection in immiscible fluids subjected to horizontal vibrations in weightlessness”. In: *Physical Review E* 99, p. 042803.
- Salgado Sánchez, P., Yasnou, V., Gaponenko, Y., Mialdun, A., Porter, J., and Shevtsova, V. (2019d). “Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity”. In: *Journal of Fluid Mechanics* 865, pp. 850–883.
- Salgado Sánchez, P., Gaponenko, Y., Yasnou, V., Mialdun, A., Porter, J., and Shevtsova, V. (2020). “Effect of initial interface orientation on patterns produced by vibrational forcing in microgravity”. In: *Journal of Fluid Mechanics* 884, A38.
- Samiei, E., Tabrizian, M., and Hoorfar, M. (2016). “A review of digital microfluidics as portable platforms for lab-on a-chip applications”. In: *Lab on a Chip* 16, pp. 2376–2396.
- Scafetta, N. (2010). “Empirical evidence for a celestial origin of the climate oscillations and its implications”. In: *Journal of Atmospheric and Solar-Terrestrial Physics* 72, pp. 951–970.
- Schmidt, W. (2004). “Quickly changing acceleration forces (QCAFs) vibration analysis on the A300 ZERO-G”. In: *Microgravity Science and Technology* 15, pp. 42–48.
- Sethian, J. A. (1999). *Level Set Methods and Fast Marching Methods : Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Science*.
- Sharma, D., Erriguible, A., Gandikota, G., Beysens, D., and Amiroudine, S. (2019). “Vibration-induced thermal instabilities in supercritical fluids in the absence of gravity”. In: *Physical Review Fluids* 4, p. 033401.
- Sherrit, S. and Mukherjee, B. K. (2007). *Characterization of piezoelectric materials for transducers*. Tech. rep.
- Shevtsova, V. (2010). “IVIDIL experiment onboard the ISS”. In: *Advances in Space Research* 46, pp. 672–679.
- Shevtsova, V., Gaponenko, Y., Melnikov, D., Ryzhkov, I., and Mialdun, A. (2010a). “Study of thermoconvective flows induced by vibrations in reduced gravity”. In: *Acta Astronautica* 66, pp. 166–173.
- Shevtsova, V., Ryzhkov, I. I., Melnikov, D. E., Gaponenko, Y. A., and Mialdun, A. (2010b). “Experimental and theoretical study of vibration-induced thermal convection in low gravity”. In: *Journal of Fluid Mechanics* 648, pp. 53–82.
- Shevtsova, V., Mialdun, A., Melnikov, D., Ryzhkov, I., Gaponenko, Y., Saghir, Z., Lyubimova, T., and Legros, J. C. (2011). “The IVIDIL experiment onboard the ISS: Thermodiffusion in the presence of controlled vibrations”. In: *Comptes Rendus Mecanique* 339, pp. 310–317.
- Shevtsova, V., Gaponenko, Y., Yasnou, V., Mialdun, A., and Nepomnyashchy, A. (2015a). “Wall-Generated Pattern on a Periodically Excited Miscible Liquid/Liquid Interface”. In: *Langmuir* 31, pp. 5550–5553.

- Shevtsova, V., Gaponenko, Y. A., Sechenyh, V., Melnikov, D. E., Lyubimova, T., and Mialdun, A. (2015b). “Dynamics of a binary mixture subjected to a temperature gradient and oscillatory forcing”. In: *Journal of Fluid Mechanics* 767, pp. 290–322.
- Shevtsova, V., Gaponenko, Y. A., Yasnou, V., Mialdun, A., and Nepomnyashchy, A. (2016). “Two-scale wave patterns on a periodically excited miscible liquid-liquid interface”. In: *Journal of Fluid Mechanics* 795, pp. 409–422.
- Shemer, L. and Lichter, S. (1987). “Identification of cross-wave regimes in the vicinity of a cut-off frequency”. In: *Physics of Fluids* 30, pp. 3427–3433.
- Shilnikov, L. P. (1969). “On a new type of bifurcation of multidimensional dynamical systems”. In: *Doklady Mathematics* 10, pp. 1368–1371.
- Skeldon, A. C. and Rucklidge, A. M. (2015). “Can weakly nonlinear theory explain Faraday wave patterns near onset?” In: *Journal of Fluid Mechanics* 777, pp. 604–632.
- Sokolov, A., Goldstein, R. E., Feldchtein, F. I., and Aranson, I. S. (2009). “Enhanced mixing and spatial instability in concentrated bacterial suspensions”. In: *Physical Review E* 80, p. 031903.
- Someya, S. and Munakata, T. (2005). “Measurement of the interface tension of immiscible liquids interface”. In: *Journal of Crystal Growth* 275, e343–e348.
- Strogatz, S. (1994). *Nonlinear dynamics and Chaos: with applications to Physics, Biology, Chemistry, and Engineering*.
- Talib, E., Jalikop, S. V., and Juel, A. (2007). “The influence of viscosity on the frozen wave stability: theory and experiment”. In: *Journal of Fluid Mechanics* 584, pp. 45–68.
- Taneda, S. (1994). “Visual observations of the flow around a half-submerged oscillating circular cylinder”. In: *Fluid Dynamics Research* 13, pp. 119–151.
- Tinao, I., Porter, J., Laverón-Simavilla, A., and Fernández, J. (2014). “Cross-waves excited by distributed forcing in the gravity-capillary regime”. In: *Physics of Fluids* 26, p. 024111.
- Troitiño, M., Salgado Sánchez, P., Porter, J., and Gligor, D. (2020). “Symmetry breaking in large columnar frozen wave patterns in weightlessness”. In: *Microgravity Science and Technology*.
- Tuckerman, L. S. (2012). “Amplitudes from eigenvalues”. In: *Fluid Dynamics Research* 44, p. 031202.
- Turchin, P. (2003). *Complex population dynamics: A theoretical/empirical synthesis*. Princeton University Press.
- Underhill, W. B., Lichter, S., and Bernoff, A. J. (1991). “Modulated, frequency-locked, and chaotic cross-waves”. In: *Journal of Fluid Mechanics* 225, pp. 371–394.
- van Sebille, E., Wilcox, C., Lebreton, L., Maximenko, N., Hardesty, B. D., van Franeker, J. A., Eriksen, M., Siegel, D., Galgani, F., and Lavender Law, K. (2015). “A global inventory of small floating plastic debris”. In: *Environmental Research Letters* 10, p. 124006.
- Varas, F. and Vega, J. M. (2007). “Modulated surface waves in large-aspect-ratio horizontally vibrated containers”. In: *Journal of Fluid Mechanics* 579, pp. 271–304.
- Vega, J. M., Knobloch, E., and Martel, C. (2001). “Nearly inviscid Faraday waves in annular containers of moderately large aspect ratio”. In: *Physica D: Nonlinear Phenomena* 154, pp. 313–336.
- Weislogel, M. M., Jenson, R., Chen, Y., Collicott, S. H., Klatte, J., and Dreyer, M. (2009). “The capillary flow experiments aboard the International Space Station: Status”. In: *Acta Astronautica* 65, pp. 861–869.

- Weislogel, M. M. and Ross, H. D. (1990). *Surface settling in partially filled containers upon step reduction in gravity*. Tech. rep. NASA Technical Memorandum, pp. 1–19.
- Wollman, A., Weislogel, M., Wiles, B., Pettit, D., and Snyder, T. (2016). “More investigations in capillary fluidics using a drop tower”. In: *Experiments in Fluids*.
- Wolf, G. H. (2018). “Dynamic stabilization of the Rayleigh-Taylor instability of miscible liquids and the related "frozen waves"”. In: *Physics of Fluids* 30, p. 021701.
- (1969). “The dynamic stabilization of the Rayleigh-Taylor instability and the corresponding dynamic equilibrium”. In: *Journal for Physics A* 227, pp. 291–300.
- (1970). “Dynamic stabilization of the interchange instability of a liquid-gas interface”. In: *Physical Review Letters* 24, pp. 444–446.
- Wunenburger, R., Evesque, P., Chabot, C., Garrabos, Y., Fauve, S., and Beysens, D. (1999). “Frozen wave induced by high frequency horizontal vibrations on a liquid-gas interface near the critical point”. In: *Physical Review E* 59, pp. 5440–5445.
- Yoshikawa, H. N. and Wesfreid, J. E. (2011). “Oscillatory Kelvin–Helmholtz instability. Part 2. An experiment in fluids with a large viscosity contrast”. In: *Journal of Fluid Mechanics* 675, pp. 249–267.
- Zhao, X., Cheng, J., Chen, S., Zhang, J., and Wang, X. (2010). “Hydrophilic modification of poly(vinylidene fluoride) (PVDF) by in situ polymerization of methyl methacrylate (MMA) monomer”. In: *Colloid and Polymer Science* 288, pp. 1327–1332.
- Zyuzgin, A. V., Ivanov, A. I., Polezhaev, V. I., Putin, G. F., and Soboleva, E. B. (2001). “Convective motions in near-critical fluids under real zero-gravity conditions”. In: *Cosmic Research* 39, pp. 175–186.

LIST OF PUBLICATIONS

Salgado Sánchez, P., Porter, J., Tinao, I. and Laverón-Simavilla, A. Dynamics of weakly coupled parametrically forced oscillators. *Physical Review E* **94**, 022216 (2016). Editor's suggestion. <https://doi.org/10.1103/PhysRevE.94.022216>

Salgado Sánchez, P., López Negro, P. and García-Fogeda, P. Vibration-based method for damage detection at welded beams and rods. *Latin American Journal of Solids and Structures* **13**, 2336–2355 (2016). <http://dx.doi.org/10.1590/1679-78252881> (not included in the Thesis)

Fernández, J., Salgado Sánchez, P., Tinao, I., Porter, J. and Ezquerro, J. M. The CFVib experiment: Control of Fluids in Microgravity with Vibrations. *Microgravity Science and Technology* **29**, 351–364 (2017). <https://doi.org/10.1007/s12217-017-9556-7>

Salgado Sánchez, P., Yasnou, V., Gaponenko, Y., Mialdun, A., Porter, J. and Shevtsova, V. Interfacial phenomena in immiscible liquids subjected to vibrations in microgravity. *Journal of Fluid Mechanics* **865**, 850–883 (2019). <https://doi.org/10.1017/jfm.2019.88>

Salgado Sánchez, P., Gaponenko, Y., Porter, J. and Shevtsova, V. Finite-size effects on pattern selection in immiscible fluids subjected to horizontal vibrations in weightlessness. *Physical Review E* **99**, 042803 (2019). <https://doi.org/10.1103/PhysRevE.99.042803>

Ezquerro, J. M., Bello, A., Salgado Sánchez, P., Laverón-Simavilla, A. and Lapuerta, V. The Thermocapillary Effects in Phase Change Materials in Microgravity experiment: design, preparation and execution of a parabolic flight experiment. *Acta Astronautica* **162**, 185–196 (2019). <https://doi.org/10.1016/j.actaastro.2019.06.004> (not included in the Thesis)

Salgado Sánchez, P., Fernández, J., Tinao, I., Porter, J. and Bello, A. Simulating vibrated fluids and vibroequilibria in the CFVib microgravity experiment. *Proceedings of the 8th European Conference for Aeronautics and Aerospace Science*, 0224 (2019). <https://doi.org/10.13009/EUCASS2019-224>

Salehi Paniagua, K., Rafael de la Cruz, V., Salgado Sánchez, P. and García-Fogeda, P. Investigation of uncertain parameters for characterizing welded structures using Vibration-Based Methods. *Proceedings of the 8th European Conference for Aeronautics and Aerospace Science*, 0730 (2019). <https://doi.org/10.13009/EUCASS2019-730> (not included in the Thesis)

Rafael de la Cruz, V., Salehi Paniagua, K., Salgado Sánchez, P. and García-Fogeda, P. A Vibration-Based Method for assessing the integrity of welded structures. *Nondestructive Testing and Evaluation*, 1–21 (2019). <https://doi.org/10.1080/10589759.2019.1692011> (not included in the Thesis)

Salgado Sánchez, P., Fernández, J., Tinao, I. and Porter, J. Vibroequilibria in microgravity: Comparison of experiments and theory. *Physical Review E* **100**, 063103 (2019). <https://doi.org/10.1103/PhysRevE.100.063103>

Salgado Sánchez, P., Gaponenko, Y., Yasnou, V., Mialdun, A., Porter, J. and Shevtsova, V. Effect of initial interface orientation on patterns produced by vibrational forcing in microgravity. *Journal of Fluid Mechanics* **884**, A38–1 (2020). <https://doi.org/10.1017/jfm.2019.955>

Ezquerro, J. M., Salgado Sánchez, P., Bello, A., Rodríguez, J., Lapuerta, V. and Laverón-Simavilla, A. Experimental evidence of thermocapillarity in phase change materials in microgravity: Measuring the effect of Marangoni convection in solid/liquid phase transitions. *International Communications in Heat and Mass Transfer* **113**, 104529 (2020). <https://doi.org/10.1016/j.icheatmasstransfer.2020.104529> (not included in the Thesis)

López-Fernández, D., Salgado Sánchez, P., Fernández, J., Tinao, I. and Lapuerta, V. Challenge-Based Learning in Aerospace Engineering Education: The ESA Concurrent Engineering Challenge at the Technical University of Madrid. *Acta Astronautica* **171**, 369–377 (2020). <https://doi.org/10.1016/j.actaastro.2020.03.027> (not included in the Thesis)

Salgado Sánchez, P., Ezquerro, J. M., Porter, J., Fernández, J. and Tinao, I. Effect of thermocapillary convection on the melting of phase change materials in microgravity: experiments and simulations. *International Journal of Heat and Mass Transfer* **154**, 119717 (2020). <https://doi.org/10.1016/j.ijheatmasstransfer.2020.119717> (not included in the Thesis)

Troitiño, M., Salgado Sánchez, P., Porter, J. and Gligor, D. Symmetry breaking in large columnar frozen wave patterns in weightlessness. *Microgravity Science and Technology* (2020). <https://doi.org/10.1007/s12217-020-09812-0>

Gligor, D., Salgado Sánchez, P., Porter, J. and Shevtsova, V. Influence of gravity on the frozen wave instability in immiscible liquids. *Physical Review Fluids* **5**, 084001 (2020). <https://doi.org/10.1103/PhysRevFluids.5.084001>

Seta, B., Dubert, D., Massons, J., Salgado Sánchez, P., Porter, J., Gavaldá, J., Bou-Ali, M. M. and Ruiz, X. On the impact of gravity in low Prandtl liquid bridges. *Advanced Technologies, Systems, and Applications V* (2020), accepted.

(Publications under review)

Salgado Sánchez, P., Ezquerro, J. M., Fernández, J. and Rodríguez, J. Thermocapillary effects during the melting of phase change materials in microgravity: steady and oscillatory flow regimes. Submitted to *Journal of Fluid Mechanics*.

Salgado Sánchez, P., Ezquerro, J. M., Fernández, J. and Rodríguez, J. Thermocapillary effects during the melting of phase change materials in microgravity: heat transport enhancement. Submitted to *International Journal of Heat and Mass Transfer*.

Salgado Sánchez, P., Ezquerro, J. M., Porter, J., Fernández, J., Rodríguez, J., Tinao, I., Lapuerta, V., Laverón-Simavilla, A., Ruiz, X., Gavaldá, F., Bou-Ali, M. M. and Ortiz, J. The effect of thermocapillary convection on PCM melting in microgravity: results and expectations. Submitted to *Advances in Astronautics Science and Technology*.