POLITÉCNICA

UNIVERSIDAD POLITÉCNICA DE MADRID

DOCTORAL THESIS

A method for the very early detection of rotor-casing rub in aeroderivative gas turbines

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Doctorate in Mechanical Engineering

Written in the

Department of Applied Mathematics (DMAII)

and the

Mechanical Engineering Department (DIM)

at the Escuela Técnica Superior de Ingenieros Industriales, UPM, Spain

2020

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A Thesis presented and submitted in partial fulfillment of the requirements for the degree of Doctor in Mechanical Engineering with international recognition, at the School of Industrial Engineers, Universidad Politécnica de Madrid.

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“Valeu a pena? Tudo vale a pena quando a alma não é pequena.”

Fernando Pessoa – Mar Português

“She’s giving me the excitations, I’m pickin’ up good vibrations.”

The Beach Boys – Good Vibrations

Acknowledgements

Para el Prof. Alejandro Zarzo Altarejos, que ha sido como un padre para mí y al que debo todos mis logros académicos y profesionales hasta el momento. Porque nunca tiraste la toalla cuando yo tiraba la mía.

Para el Prof. Juan Manuel Muñoz Guijosa, la persona más competente con la que nunca he trabajado, siempre dispuesto a aportar sus conocimientos y experiencia al servicio de los demás.

Para todos mis compañeros del Departamento de Matemática Aplicada de la ETSII, por acogerme en su departamento y dejar que comience mi carrera académica a su lado. Y a mis alumnos de Álgebra; creedme, yo he aprendido tanto con vosotros como vosotros conmigo, espero.

Para mis compañeros de la División de Ingeniería de Máquinas de la ETSII, especialmente a Mar, Paco, Álex y Guli. Gracias por la ayuda prestada y por hacer mis largas jornadas en el laboratorio mucho más amenas.

Para los alumnos de trabajos fin de máster que han trabajado para este proyecto de investigación: Jacobo Manuel Machuca González, Jair León Olivares y Antonio Squicciarini. También vosotros habéis hecho posible esta tesis.

Para Francesco Miniello. No sólo he contado con tu asistencia técnica, sino con tu amistad, que es aún más valiosa.

Per il Prof. Paolo Pennacchi, il mio padrone durante la permanenza di ricerca a Milano. Grazie alla sua enorme fiducia in me ho partecipato in un interessantissimo progetto di ricerca ad un centro di grande prestigio mentre passavo i tre mesi più gioiosi della mia vita, vivendo dei momenti indimenticabili in mezzo ad una nuova cultura e lingua. Questo lavoro di ricerca venne sviluppato con il valioso aiuto ed assistenza dal Dr. Gabriele Cazzulani, dal Prof. Steven Chatterton e dal Prof. Francesco Castelli-Dezza. Tutti e quattro avete fatto tanto per la mia giovane carriera accademica.

Anche da Milano, non posso dimenticarmi del carissimo Dr. Nicola Toscani, mio compagno di ufficio, che mi ha accolto dal minuto uno con le braccia aperte e mi ha fatto sentire sempre come a casa. Neanche mi potrei mai dimenticare di Dr. Maria Stefania Carmeli, del Dr. Marco Mauri, del Ing. Davide Enrico Quadrelli e del Ing. Emanuele Riva, fra tanti altri. Finalmente, non posso mettere a parte i cari ragazzi del gruppo studentesco 'Teatro delle Biglie' con cui ho condiviso tantissime risate, e qualunque birra nel bar.

Naturalmente, es imposible llevar hasta el fin una empresa de tanto esfuerzo como una tesis doctoral sin el apoyo de mis seres más cercanos y queridos, así que las últimas líneas de estos agradecimientos son para ellos: gracias por convertirme en la persona que ahora soy, por no dejar de estar a mi lado tanto en los buenos como en los malos momentos. Por no dejar que faltara nunca un plato de comida en la mesa. Por animarme a apuntar alto en la vida y aprovechar las oportunidades académicas que vosotros nunca pudisteis tener. Pelo afeito e compreensão, por manter-me sempre com os pés na terra e per não deixar nunca de acreditar que passo a passo e com esforço podemos chegar tão longe como desejarmos.

Gracias. Obrigado. Grazie.

List of Abbreviations

NASA	N ational A eronautics and S pace A dministration
CBM	C ondition- B ased M aintenance
GE	G eneral E lectric
ADRE	A utomated D iagnostics for R otating E quipment
FEM	F inite E lement M ethod
DFT	D iscrete F ourier T ransform
FFT	F ast F ourier T ransform
SODE	S ystem of O rdinary D ifferential E quations
PDE	P artial D ifferential E quations
CWT	C ontinuous W avelet T ransform
WSST	W avelet S ynchro- S queezing T ransform
SNR	S ignal-to- N oise R atio
AISI	A merican I ron and S teel I nstitute
PLA	P oly L actic A cid

List of Symbols

$\mathbf{x}$	Vector of samples
$w(i)$	Discrete windowing function
x_k	k-th sample of input vector $\mathbf{x}$
$\hat{x}_i$	i-th element of the discrete Fourier transform of $\mathbf{x}$
$\exp(x)$	Exponential function in the x domain
j	Imaginary unit
W_N	N -th complex root of unity
$\hat{\mathbf{x}}$	Discrete Fourier transform of $\mathbf{x}$
t_s	Sampling period
Ω	Rotation speed
$\mathbf{FS}_{forward}$	Vector of forward components of the full spectrum of $\mathbf{x}$
$\mathbf{FS}_{reverse}$	Vector of reverse components of the full spectrum of $\mathbf{x}$
$\psi_{\hat{x}_{x,i}}, \psi_{\hat{x}_{y,i}}$	Phase angles of the i-th element of the spectra of $\hat{x}_{x,i}$ and $\hat{x}_{y,i}$
c	Cepstrum of $x(t)$
$\mathcal{F}(x(t))$	Fourier transform of $x(t)$
$\mathcal{F}^{-1}(x(t))$	Inverse Fourier transform of $x(t)$
t	time
ϵ_{zz}	Axial strain
σ_{zz}	Axial stress
R_i	Disk inner radius
R_e	Disk outer radius
$H(x)$	Heaviside function
φ_x, φ_y	Bending angular displacements
h	Shell thickness
F_R	Normal contact force
ϵ	Clearance between rotor and obstacle
d_R	Relative displacement between rotor and obstacle
d'_R	Relative velocity between rotor and obstacle
k_R	Rub stiffness coefficients
$\mathbf{u}$	Vector of rotor-casing system displacements
$\mathbf{u}'$	Vector of rotor-casing system velocities
μ	Rub friction coefficient
v_R	Rotor-casing relative velocity
c_R	Rub damping coefficients
$\mathbf{f}_R$	Vector of rub forces
$\mathbf{M}$	Mass matrix

D	Damping matrix
K	Stiffness matrix
f	Forces vector
D_{ij}	Damping matrix element at the i-th row and j-th column
K_{ij}	Stiffness matrix element at the i-th row and j-th column
T	Temperature
η	Fluid viscosity
f_U	Vector of unbalance forces
$M_{r,x}, M_{r,y}$	Bending moments of beam
A	Beam section area
F_z	Axial internal forces of beam
$\kappa(z)$	Beam curvature at axial coordinate z
E	Young modulus
u_x, u_y	Radial displacements of beam
I_x, I_y	Second moment of area of beam around the x and y axis
F_x, F_y	External radial loads
$V_{r,x}, V_{r,y}$	Internal shear forces in the x and y directions
ρ	Material density
D	Space-time domain of the solutions of the Euler-Bernoulli equations
$\tilde{\mathbf{u}}(z)$	Vector of approximate displacement solutions as a function of the axis coordinate z
$\tilde{u}_x(z), \tilde{u}_y(z)$	Approximate radial displacements in x and y as a function of the axis coordinate z
N_i	Trial function of the i-th system node
$\tilde{\mathbf{u}}_i(t)$	Vector of degrees-of-freedom displacements at the i-th node in the time domain
$u_{x,i}(t), u_{y,i}(t)$	Radial x and y displacements of the i-th node in the time domain
$u_{\varphi_{x,i}}(t), u_{\varphi_{y,i}}(t)$	Bending angular displacements around x and y of the i-th node in the time domain
$L_{e,j}$	Length of the j-th shaft element
$R_{s,j}$	Radius of the j-th shaft element
$\mathbf{r}(z, t)$	Residual vector of the approximate solution
$\mathbf{r}_{e,i}(z, t)$	Residual vector of the degrees-of-freedom displacement of the i-th node
L	Linear differential operator
$u(x)$	Unknown function in x
V	Function domain containing the solution subspace of the linear differential equation $L(u(x)) = f(x)$
$f(x)$	Forcing function of the linear differential equation $L(u(x)) = f(x)$
$\tilde{u}(x)$	Trial function of $L(u(x)) = f(x)$
r	Residual of the equation $L(\tilde{u}(x)) = f(x)$

α_j	Coefficients of a linear combination of the basis functions
$\phi_j(x)$	j-th basis function of the space of approximate solutions of $L(u(x)) = f(x)$
$\mathbf{M}_{e,i}$	Mass matrix of the i-th shaft element
$\mathbf{K}_{e,i}$	Stiffness matrix of the i-th shaft element
$\mathbf{V}_{r_{e,i}}$	Vector of internal shear forces at the boundary nodes of the i-th shaft element
$\mathbf{M}_{r_{e,i}}$	Vector of internal bending moments at the boundary nodes of the i-th shaft element
$\mathbf{M}_s$	Shaft mass matrix
$\mathbf{K}_s$	Shaft stiffness matrix
$\mathbf{f}_s$	Shaft force vector
$\mathbf{G}_{e,j}$	Gyroscopic effects matrix of the j-th shaft element
(x, y, z)	Global coordinate system
(x', y', z')	Local coordinate system on the disk's center of gravity
M_d	Disk mass
F_x, F_y	Radial forces on the disk
L_x, L_y	Disk angular momenta in x and y
I_T	Disk transverse moment of inertia
I_P	Disk polar moment of inertia
M_x, M_y	External radial torques on the disk
$\mathbf{C}$	Global viscous damping matrix
$\mathbf{G}$	Global matrix of gyroscopic effects
$\mathbf{G}_D$	Disk gyroscopic matrix
$\mathbf{M}_{S+D}$	Mass matrix of the assembled rotor
$\mathbf{G}_{S+D}$	Gyroscopic matrix of the assembled rotor
R_c	Radius between the casing mid-surface and the rotor centerline
θ_e, θ_i	Transversal boundary angles of a casing element
z_e, z_i	Axial boundary coordinates of a casing element
(x_i, y_i, z_i)	Vector of global coordinates of the i-th casing node
(u_i, v_i, w_i)	Vector of global displacements of the i-th casing node
ξ, η, ζ	Curvilinear coordinates of a point in a casing element
$\mathbf{v}_{2,i}$	Unitary normal vector of the i-th casing node
$N_i(\xi, \eta)$	Shape function of the i-th casing node
$\mathbf{v}_{1,i}, \mathbf{v}_{3,i}$	Tangential unitary vectors of the i-th casing node
α_i, β_i	Rotations of the casing mid-surface at the i-th node around the $\mathbf{v}_{1,i}$ and $\mathbf{v}_{3,i}$ directions
(x^*, y^*, z^*)	Local coordinate system at a point of the casing mid-surface
$\mathbf{P}_i$	Global Cartesian coordinates of the i-th casing node

$\mathbf{D}^*$	Local constitutive matrix
$\boldsymbol{\sigma}^*$	Vector of stresses in the local frame
$\boldsymbol{\epsilon}^*$	Vector of strains in the local frame
$\boldsymbol{\epsilon}_0^*$	Vector of initial strains in the local frame
ν	Poisson's ratio
κ	Ratio of relevant strain energies
$\sigma_{x^*}, \sigma_{z^*}$	Local normal stresses
$\tau_{x^*z^*}, \tau_{x^*y^*}, \tau_{y^*z^*}$	Local tangential stresses
$\epsilon_{x^*}, \epsilon_{z^*}$	Local linear strains
$\gamma_{x^*z^*}, \gamma_{x^*y^*}, \gamma_{y^*z^*}$	Local angular strains
$\mathbf{B}_e$	Matrix relating global nodal displacements and the global strain field
$\mathbf{B}_e^*$	Matrix relating global nodal displacements and the strain field in the local frame
$\mathbf{J}(\xi, \eta, \zeta)$	Jacobian matrix of a casing element point
$\mathbf{S}^*$	Matrix of differential operators in local coordinates
$\mathbf{u}^*$	Local displacement field vector of a point of the shell element
$\mathbf{u}_e$	Displacements of the degrees of freedom of a casing element
$\mathbf{u}_{e,i}$	Displacements of the degrees of freedom of the i-th element node
$\mathbf{S}$	Matrix of differential operators in global coordinates
$\mathbf{B}_{e,i}$	i-th column of matrix $\mathbf{B}_e$
$\boldsymbol{\Theta}$	Rotation matrix
V_e	Volume of a casing element in the spatial domain
$\mathbf{M}_C$	Casing mass matrix
$\mathbf{K}_C$	Casing stiffness matrix
$\mathbf{M}_{CU}, \mathbf{M}_{CL}$	Mass matrices of the upper and lower casing halves
$\mathbf{K}_{CU}, \mathbf{K}_{CL}$	Stiffness matrices of the upper and lower casing halves
$\mathbf{K}_b$	Bearing stiffness matrix
$K_{x,x}, K_{x,y}, K_{y,x}, K_{y,y}, K_{\theta_x}, K_{\theta_y}$	Bearing stiffness coefficients
$\mathbf{C}_b$	Bearing viscous damping matrix
$C_{x,x}, C_{x,y}, C_{y,x}, C_{y,y}, C_{\theta_x}, C_{\theta_y}$	Bearing viscous damping matrix
$\mathbf{K}_{bc}$	Stiffness matrix of the unions between casing and bearing
K	Stiffness parameter of the unions between casing and bearing
$\mathbf{C}_{bc}$	Viscous damping matrix of the unions between casing and bearing
C	Viscous damping parameter of the unions between casing and bearing
θ	Beam angle with respect to the x axis

$\mathbf{C}'_{bc}$	Rotated viscous damping matrix of the unions between casing and bearing
$\mathbf{K}'_{bc}$	Rotated stiffness matrix of the unions between casing and bearing
$\mathbf{K}_c$	Coupling stiffness matrix
$K_x, K_y, K_{\theta_x}, K_{\theta_y}$	Coupling stiffness coefficients
$\mathbf{C}_c$	Coupling viscous damping matrix
$C_x, C_y, C_{\theta_x}, C_{\theta_y}$	Coupling viscous damping coefficients
$\mathbf{K}_f$	Stiffness matrix of the casing supports
$K_x, K_y, K_\alpha, K_\beta$	Stiffness coefficients of the casing supports
$\mathbf{C}_f$	Viscous damping matrix of the casing supports
$C_x, C_y, C_\alpha, C_\beta$	Viscous damping coefficients of the casing supports
$\mathbf{K}_{cc}$	Stiffness matrix of the unions between the casing halves
$K_{x,x}, K_{x,y}, K_{x,\alpha}, K_{x,\beta} \dots K_{\beta,\beta}$	Stiffness coefficients of the unions between the casing halves
$\mathbf{C}_{cc}$	Viscous damping matrix of the unions between the casing halves
$C_{x,x}, C_{x,y}, C_{x,\alpha}, C_{x,\beta} \dots C_{\beta,\beta}$	Viscous damping coefficients of the unions between the casing halves
$F_{U_{x,i}}, F_{U_{y,i}}$	Projections of the unbalance force at the i-th casing node over the global Cartesian axes
$M_{U,i}$	Unbalance mass at the i-th shaft node
r_i	Distance between the unbalance mass at the i-th shaft node and the rotation axis
$\psi_{U,i}$	Phase angle of the unbalance mass at the i-th node
$\mathbf{f}_U$	Vector of global unbalance forces
d_R	Relative displacement between the rotor and the stator obstacle
d'_R	Relative velocity between the rotor and the stator obstacle
$\langle \cdot, \cdot \rangle$	Inner or dot product
$H(x)$	Heaviside function
$R_{r,j}$	Rotor radius at the j-th shaft node
$\mathbf{f}_{R,i}$	Rub force vector at the i-th system mode
γ	Angle of the rub obstacle with respect of the x axis
$\mathbf{M}_B$	Sub-matrix of $\mathbf{M}$ for the bearing nodes
$\mathbf{G}_{S+D}$	Sub-matrix of $\mathbf{G}$ for the rotor nodes
$\mathbf{C}_{S+D}, \mathbf{C}_{S+D,B}, \mathbf{C}_C, \mathbf{C}_{C,B}$	Sub-matrices of the global viscous damping matrix
$\mathbf{C}_{S+D,B}, \mathbf{C}_{C,B}$	
$\mathbf{K}_{S+D}, \mathbf{K}_{S+D,B}, \mathbf{K}_{C,B}, \mathbf{K}_{S+D,B}$	Sub-matrices of the global stiffness matrix
$\mathbf{K}_{C,B}, \mathbf{K}_B$	
$\mathbf{u}_{S+D}, \mathbf{u}_C, \mathbf{u}_B$	Sub-vectors of the global displacements vector with the rotor, casing and bearing degrees of freedom
C_M, C_K	Parameters of the Rayleigh damping matrix

$\zeta_{d,i}$	Damping factor of the i-th system mode
$\omega_{N,i}$	Natural frequency of the i-th system mode
$\mathbf{u}_B, \mathbf{u}_I$	Displacement vectors boundary and internal degrees of freedom
$\mathbf{f}_B, \mathbf{f}_I$	Force vectors boundary and internal degrees of freedom
$\mathbf{M}_{BB}, \mathbf{M}_{BI}, \mathbf{M}_{IB}, \mathbf{M}_{II}$	Boundary and internal sub-matrices of the global mass matrix
$\mathbf{G}_{BB}, \mathbf{G}_{BI}, \mathbf{G}_{IB}, \mathbf{G}_{II}$	Boundary and internal sub-matrices of the global gyroscopic matrix
$\mathbf{C}_{BB}, \mathbf{C}_{BI}, \mathbf{C}_{IB}, \mathbf{C}_{II}$	Boundary and internal sub-matrices of the global viscous damping matrix
$\mathbf{K}_{BB}, \mathbf{K}_{BI}, \mathbf{K}_{IB}, \mathbf{K}_{II}$	Boundary and internal sub-matrices of the global stiffness matrix
Φ_{CB}	Craig-Bampton reduction matrix
$\mathbf{u}_{CB}$	Vector of reduced nodal displacements of the system degrees of freedom
ϕ_I	Modal degrees of freedom of the reduced displacements vector
$\mathbf{I}$	Identity matrix
Φ_C	Sub-matrix of constraint modes
Φ_N	Sub-matrix of matrix modes
Λ	Diagonal matrix of eigenvalues
$\mathbf{M}_{CB}, \mathbf{G}_{CB}, \mathbf{C}_{CB}, \mathbf{K}_{CB}$	Reduced global matrices
$\mathbf{f}_{U,CB}, \mathbf{f}_{R,CB}$	Reduced unbalance and rub force vectors
Δt	Integration time step
β, γ	Newmark-beta parameters
ω	Frequency
$\psi(t)$	Mother wavelet function in the time domain
$\psi^*(t)$	Conjugated mother wavelet function in the time domain
a	Translation factor
s	Scaling factor
ψ	Vector of sampled values of the Mother wavelet in the time domain
$W(a, s)$	Wavelet coefficient of $x(t)$ with factors a and s
$\hat{\psi}(\omega)$	Mother wavelet function in the frequency domain
N_a, N_s	Number of translation and scaling factors of the Wavelet scalograms
$\mathbf{w}(s)$	Row of sampled Wavelet coefficients with scaling factor s
$\hat{\psi}^*$	Conjugated vector of sampled values of the Mother wavelet in the frequency domain
ω	Vector of sampled frequencies
$\omega'(a, s)$	Reassignment frequency for factors a and s
$f'(a, s)$	Reassignment frequency in Hertz for factors a and s
$\mathbf{S}$	Wavelet scalogram matrix

$\mathbf{S}'$	Reassigned wavelet scalogram matrix
$\mathbf{a}_R$	Resampled vector or acceleration samples from the model under rub and unbalance
$\mathbf{a}_U$	Resampled vector or acceleration samples from the model under unbalance
$\mathbf{a}_N$	Vector of white noise
$\mathbf{a}_{R+N}$	Resampled vector or noisy acceleration samples from the model under rub and unbalance
$\mathbf{a}_{U+N}$	Resampled vector or noisy acceleration samples from the model under unbalance
σ_N	Standard deviation
$\mathcal{N}(0, \sigma_N)$	Normally-distributed random variable with zero mean and σ_N standard deviation
$P(\mathbf{x})$	Power of the discrete signal $\mathbf{x}$
$\hat{\mathbf{a}}_{U+N}$	Fourier spectrum of $\mathbf{a}_{U+N}$
$\hat{\mathbf{a}}_{R+N}$	Fourier spectrum of $\mathbf{a}_{R+N}$
$\mathbf{S}_{U+N}$	Wavelet scalogram of $\mathbf{a}_{U+N}$
$\mathbf{S}_{R+N}$	Wavelet scalogram of $\mathbf{a}_{R+N}$
$\mathcal{W}$	Wavelet transform
ϵ_{SNR}	Sensitivity of the rub detection under Gaussian white noise
$\hat{\mathbf{u}}(\omega)$	Spectrum of the displacements
$\hat{\mathbf{f}}(\omega)$	Spectrum of the excitations
$\hat{\mathbf{a}}(\omega)$	Spectrum of the accelerations
$\mathbf{H}_u(\omega)$	Transfer function of the system displacements
$\mathbf{H}_a(\omega)$	Transfer function of the system accelerations
$\hat{\mathbf{a}}(\omega)$	Spectrum of the accelerations
$\mathbf{M}'_C$	Modified global mass matrix
K	Multiplicative factor of the mass matrix
Re	Real part of a complex number

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Abstract

The flawless operation of rotating machines such as turbomachines is key in many branches of Industry. During the last decades there was an increase in the net efficiency and performance of these machines, attained through a minimization of component mass, size and dimensional tolerances, rotor-to-stator clearances; the use of cutting-edge materials with top-class mechanical and thermal properties and the introduction of the newest cooling and lubrication technologies. However, higher performance came inevitably with a trade-off in reliability, since new designs and technologies made the turbomachines more vulnerable to malfunction. A breakdown during operation can carry over unassumingly high economical or even human losses, which justifies the convenience of preventive maintenance programs which allow for early malfunction detection before it could cause over time forced shutdown or even catastrophic failure.

One of the most common failures in rotating machinery is rotor-casing rub, which is caused by a combination of excessive vibration with abnormal rotor-stator clearances due to thermal expansion and assembly or maintenance errors. This phenomenon causes wear and overheating and, when unnoticed, may be highly destructive. In most gas and steam turbines, rub is easily detected with displacement sensors mounted between rotor and stator. However, in machines such as aeroderivative gas turbines, acceleration sensors on their casing are the often only means available for condition monitoring due to lack of space inside the machine and the very high stiffness of the bearings. Few published works deal with the detection of rub in turbomachines with acceleration sensors, and nothing has been found on early fault detection of rub with accelerometers. Since aeroderivative turbines are increasingly relevant to sectors such as energy generation and chemical processing this problem offers a fertile ground for active research with important industrial outcomes.

The aim of this Doctoral Thesis is the development of a signal-processing methodology for the very early detection of rotor-casing single-point rub in aeroderivative gas turbines with accelerometers on the casing, most often the only means available for their monitoring. This methodology has been devised with the assistance of a mathematical model of an aeroderivative gas turbine. When integrating the model equations—derived with the Finite-Element method—the accelerations of selected casing nodes under either rub or no-rub are available for post-processing and a range of detection methods can be tested and compared. The most optimal procedure must detect very weak rotor-stator contact at very small time scales and be implementable in real time. This novel methodology was then validated in an experimental rotor rig with flexible casing monitored with both accelerometers and proximity sensors. The research found that the Wavelet Synchrosqueezed Transform—WSST, a Continuous Wavelet Transform with energy reassignment—complied with all the requirements. Its efficiency is even more outstanding when acceleration signals are contaminated with noise.

Chapter 1

INTRODUCTION

1.1 The historical context of the gas turbine and its evolution

Since the beginnings of human civilization, the human species has striven for the generation of power and energy for the fulfillment of its survival needs. Even before the Neolithic Revolution, humans depended on fire to render meat digestible and to keep their bodies warm during the Last Glacial Period as it hit across Eurasia. Then about 12000 years ago, at the ice melted down and global temperatures arose, when the populations occupying what would be called *the Fertile Crescent*—encompassing the vast Tigris and Euphrates floodplains and the Levant region on the Middle East—learned to cultivate their food and settle down, their energy needs rose up. The business of conditioning, irrigating, sewing and reaping the fields turned out to be physically very exhausting, even if the food surplus that enjoyed the pioneering peasants—something unheard of across the Paleolithic—was perceived as advantageous: local differences in the amounts of hoarded food were exploited for social hierarchization and the world population would soar up, unbounded by food availability in the wildness.

Our forerunners learned that animals could be domesticated and then relied on for the most exhausting tasks. Cows, oxen, donkeys, horses and the like became the greatest source of power for agriculture, machine operation and transportation until 1712, the year of the invention of the steam engine. The Industrial Revolution was soon under way, powered up by coal and boiling water and enabled by the fast-developing sciences of Physics and Chemistry and the dawning of consumer-based Capitalism.

Energy consumption levels were again in the rise, and the increase would be exponential during the XIX and XX centuries with the advent of Globalization, the discovery and exploitation of new energy sources and the invention of appliances and gadgets bound to make our lives easier and safer in many ways—See Figure 1.1. The hyper-connected world we thrive in nowadays demands huge amounts of energy to keep on spinning. This continuous exploitation of natural resources constitutes a threat to the environment that may drag us all along if we do nothing about it. The future generations ought not to pay the price for the comfort and well-being of the present ones. So, the idea of *sustainability* has become a very sound buzzword in today's political and economical discourse.

There are reasons other than environmental that push for more efficient energy consumption and power generation. In our free global markets, the Industry demands for lower production costs and the consumers demand for lower prices. All this poses economic incentives for the development of technologies that can achieve more—power output—for less—resource consumption. The most concerning primary energy sources are oil and coal, whose use take the greatest toll on

FIGURE 1.1: World energy consumption by source since the 1800s.
Source: [World Energy Consumption Since 1820 in Charts](#).

the Environment. As an alternative, renewable energies such as wind, solar, tidal and geothermic are mature technologies with lesser environmental costs whose implementation is actively endorsed by the Western governments with generous subsidies. Unfortunately, underdeveloped countries—and post-Communist authoritarian regimes such as China and Russia—cannot, or do not want to, afford the costs required by the deployment of these energy sources.

Alternatively, it is possible to raise the efficiency of the consumption of oil- and coal-based fuels for power generation and to minimize the polluting emissions. New heat engines would offer the potential for this well-awaited improvement. The steam engine inspired the later inventions of the Otto- and Diesel-cycle piston engines in the late 1800s. Then, between the XIX and the XX centuries came the steam and gas turbines. The steam turbine is dependent on an external heat source for the boiling of the pressurized gas expanding across the machine, as well as a condenser acting as a *cold source* for the cooling-off of the steam. The gas turbine, however, incorporates all the necessary components for the fulfillment of the Brayton thermodynamic cycle: fuel burners and working fluid—air, in open-cycle turbines—compressors and a turbine for gas expansion. Gas turbines may be deemed black boxes that take in fuel and atmospheric gas, supply either mechanical rotating power or thrust and expel lightly-heated exhaust fumes: the residuals of fuel combustion.

All the aforementioned heat engines have nowadays a myriad of industrial and domestic applications, wherever mechanical power is necessitated and a chemical source of combustion energy is readily available, there will be one of these. For a more comprehensive description and comparison we refer the curious reader to Ghojel, 2020. In the following, we will focus exclusively on open-cycle gas turbines.

Gas turbines contain these components that are necessary for the fulfillment of the Brayton cycle: a *turbocompressor*—comprised of a gas compressor and a turbine mechanically coupled through a common shaft; a *combustor chamber*—which supplies the thermal energy for mechanical power generation; and, in gas turbines for ground-based applications, a *power expansion turbine* that converts the gas entropy into mechanical energy in the *power shaft*—1.2. The compressor and turbine stages of the turbocompressor are constituted of a common *shaft*—or several independent

1.1. The historical context of the gas turbine and its evolution

shafts linked by an epicycloidal gearboxes—to which circumferential arrangements of *blades* called *disks* are attached. This assembly is known as the *rotor* of the machine and is the reason why these machines are known as a type of *rotating machine*. The rotor is supported on the machine *stator* or *housing* by means of *bearings* and is meant to rotate freely on them. The stator is the fixed part of the gas turbine assembly. The working fluid—air in gas turbines—is sucked into the turbine by the effect of the rotating blades and compressed to very high pressures—up to 40 times the atmospheric pressure in the most modern gas turbines. Then it is heated in the combustion chamber to temperatures up to 1500 degrees Celsius. This gas then releases its energy throughout the turbine disks, expanding and cooling off in the process. For aviation-purpose gas turbines—turbofans, turbojets and turboprops—a substantial fraction of the thermal energy contained by the hot, pressurized gas is converted into kinetic energy in an outlet nozzle which provides mechanical thrust. In ground-based gas turbines, however, most of the thermal energy—as much as possible—must be converted into mechanical energy in the turbine shaft for its use in electrical power generation, automotive traction or fluid pumping in chemical plants, among other uses. Figure 1.3 shows the inside of an aeronautical gas turbine.

FIGURE 1.2: (A) A turbojet—for aviation purposes; and (B) a ground-based gas turbine. Images owned by ipieca.org —Ipieca, 2014.

FIGURE 1.3: A cutaway of a gas turbine. Image with Creative Commons Rights. From wikipedia.org 2019

1.2 The issue of rub in rotating machinery

The very tiny clearances—of the order of hundredths of a millimeter—required to achieve the outputs and efficiencies of today’s top-range gas turbines have a serious drawback: they increase the possibility of physical contact, also called *rub*, between the mobile and the static parts of the machine during its operation. The probability of contact will depend on the intensities of the dynamical forces which the machine is subject to, on the occurrence thermal expansions and contractions during start-up and shut-down that alter the clearances and on possible human maintenance, assembly and operation errors, among other factors. It is, therefore, a secondary effect of another primary cause that triggers rotor-stator contact. Some events that may result in rub are excessive unbalance—Shen et al., 2007, misalignment—Lu et al., 2017, journal bearing instability—Hu et al., 2016—and sudden blade loss—Wang et al., 2016a. Thermal expansion due to component heating is another potential cause of rub since its effect is to decrease the clearances between components.

Rubbing-related failure is one of the most common malfunction in rotating machinery. According to a 1989 NASA report—Muszynska et al., 1989—between 1962 and 1976 a 10.2% of all rotor failures during commercial flights were caused by blade rub. The lightest and most common type imaginable of rub is *partial rub*—also called *single-point* or *fixed-point rub*, represented in Figure 1.5. The occurrence of rubbing during a prolonged period of time, even when it is of the lightest type, leads to machine heating and wear which in return causes physical damage, noise and energy consumption and increased vibration in a positive feedback loop. These effects may contribute to a further increase of the intensity of the rub, constituting a positive feedback loop. Figure 1.6 sums up the dynamic and physical effects of rub between rotor and stator in a rotating machine. Notice the presence of feedback loops in the diagram.

FIGURE 1.5: Single-point rotor-casing rub. One rotor spot impacts one point on the casing at a time. During the contact, normal and friction rub forces (light blue) change the rotor orbit direction (dark blue) and inflict vibration in the system.

The direct effects of rub are friction, rotor stiffening—a transient increase in the system stiffness during the contact between the rotor and the casing, see Chu et al., 2007, rotor-bearing-stator mechanical coupling and disruption of the fluidic and thermal balance. The friction and impacting cause wear, noise, parasitic vibration and increased energy consumption. Rotor stiffening and mechanical coupling change the dynamic characteristics of the system. Finally, rub friction leads to heating, which

FIGURE 1.6: The effects of mechanical rub between rotor and stator.
From Muszynska, 2005a.

leads to rotor deformation or *bow*—Goldman et al., 2000—in the direction of the rubbing spot. The deformed rotor has an extra mass unbalance in the direction of the rubbing spot which in turn raises the intensity of the rub. If unchecked, the process would result in either a permanently bowed rotor or in the thermal or mechanical destruction of the machine.

A particularly dangerous self-excited rub failure mode is *dry whip*—Muszynska, 2005b. It is a particularly dangerous sort of *full angular* or *continuous* rub. It involves a circumferential contact between rotor and stator in which the rotor rolls on the housing surface while spinning in the direction opposite to the rolling. This backward precession of the shaft causes very high stresses in the shaft material and extremely high normal and friction forces on the contacting surfaces, and can lead to the destruction of the machine in a few seconds.

A typical rub-related failure is the partial or total destruction of the labyrinth seals between the rotor and its housing due to wear and thermal deformation induced by the continuous contact between the fixed and mobile parts. Figure 1.7 shows an example of this.

Given the potential risk of rotor-casing rub, it is highly desirable to detect its presence inside the machine as soon as possible so as to prevent further damage, when its intensity is still too low to cause any effect visible to the naked eye. We will call this detection paradigm *early fault diagnosis*.

FIGURE 1.7: Destructive tear-off damage in the bearing labyrinth seal of a compressor caused by rotor-stator rub. From Bently et al., [2003a](#).

1.3 Conventional detection of rub in gas turbines

The detection and diagnosis of rub and other malfunctions in rotating machinery is of paramount importance to their operators for the prevention of failures that can often be catastrophic, bringing about large economic—and even human—losses. Thanks to developments in high-precision instrumentation and in post-processing techniques for the output data, machinery condition can be monitored during operation without having either to interrupt operation or disassemble it, thus saving lots of money on overhaul costs. This advancement has enabled the deployment of *condition-based maintenance*, or *CBM*, methodologies—Gillespie, [2015](#). CBM programs rely on monitoring systems that sense the state of the equipment under surveillance and on knowledge and past experience for an accurate interpretation for the data. If the gathered data indicates a high probability of reaching a state of failure—where the equipment is no longer useful—early corrective action can be taken, thus saving lots of maintenance costs and preventing catastrophic failure. The objective is to optimize operation costs and to minimize overhaul costs and downtime times—Figures [1.8](#) and [1.9](#).

Turning back to the question of rub, nowadays most gas turbines are equipped with monitoring systems that allow for the early detection of rub between rotor and stator before it could worsen the equipment condition beyond the operational limit, with the effects mentioned above in Section [1.2](#). Displacement sensors, also called *proximitors*—Figure [1.10](#)—are transducers that measure relative displacement between the rotor and the stator. They are most often mounted next to the shaft bearings. If two sensors are assembled on a common radial plane with an angular separation of 90° it is possible to have a two-dimensional picture of the rotor relative vibration known as the rotor *orbit*—Section [2.1.1](#).

FIGURE 1.8: The bathtub curve: when the machine approaches its life expectancy the probability of failure increases as the machine condition worsens. CBM seeks to estimate the probability of failure by means of condition monitoring. Image owned by ludeca.com —Lambert, 2018.

FIGURE 1.9: The *performance-failure curve*—or *P-F curve*. The idea of CBM is to measure how machine health condition changes in time so that corrective action is taken before ever reaching a Functional Failure (F) event. The deterioration of the machine becomes visible in the instrumentation at the Potential Failure (P) point. According to the bathtub curve, somewhere in the PF interval the probability of failure should be too high and the operation must be stopped. From Brkovic et al., 2017.

Displacement sensors provide a very precise picture of rotor vibration with very mild signal noise distortion. The visualization of rub on the orbit diagrams is very straightforward when looking over the orbit shapes and the vibration harmonics that rotor-casing contact excites—Muszynska, 2005b; Bently et al., 2003b. Rotor orbit becomes notably sharper at the spot of impact, and, if the machine operates at a speed multiple of any of its lateral critical speeds the strikes will excite these rotor resonances. In this situation, *subharmonics* (vibration harmonics whose frequencies are a fraction of the rotation speed) can appear in the rotor lateral vibration, resulting in an eight-number orbit shape with one impact each n rotor rotations—see Figure 1.11.

1.4. The problem of early rub detection in aeroderivative turbines

FIGURE 1.10: (A) Proximity probe, or *proximitor*, for the measuring of relative displacements of rotors, with the control electronics box. Image property of GE-Bently Nevada.

FIGURE 1.11: Rubbing at a rotor speed of more than twice the first lateral critical speed triggers a $1/2\times$ harmonic, resulting in a rotor-stator impact once every two rotations and an eight-number shaped orbit. From Muszynska, 2005b.

Unfortunately, not all gas turbines can be equipped with displacement sensors for condition monitoring. In the so-called *aeroderivative gas turbines*, their installation is highly complicated by design issues, with important implications whose discussion is the object of the next section.

1.4 The problem of early rub detection in aeroderivative turbines

Aeroderivative gas turbines—Figure 1.12—are a class of rotating machines which result from the modification of aeronautical jet engines—Figure 1.2a—for their application in ground-based purposes, mainly with the removal of the turbofan and the nozzle and the addition of a power turbine and a power output shaft—Figure 1.2b. These pieces of industrial equipment have many advantages over conventional heavy-frame gas turbines in the fields of power generation and in the oil & gas industry: they are lighter and more responsive to changes in load and power demand, they have smaller and modular—making assembly and maintenance much easier, reducing overhaul costs—and have higher efficiencies. These advantages make these equipment ideal for medium-power peaking and cycling power generation for offshore applications or isolated power systems—i.e. islands, for pumping in chemical processing plants and for heat and power generation in combined cycle power plants, where these turbines can reach up to an 80% of efficiency—Doom, 2011.

In 2012, it was estimated that aeroderivative gas turbines represented a 10-20% of the deployed gas turbines worldwide and yearly orders of these machines totaled about 6000-8000 MW of power.

On the other hand, the larger and more robust industrial, heavy-frame gas turbines offer much higher output power, have greater thermal and mechanical inertia and much larger change-out times—the times of change, which affect maintenance. They have lower thermal efficiencies but are better at polluting emissions thanks to the lower pressure rates and gas temperatures inside the combustion chamber. All these features suit base-load power applications more. A thoughtful comparison of aeroderivative and heavy-frame gas turbines can be found in integratechnical.com, 2018.

FIGURE 1.12: A schematic of the General Electric LM6000-PF+ aeroderivative gas turbine. Image property of GE.

In relation to condition-based maintenance, there is a key design difference between the two machines that gravely hinders the monitoring of aeroderivative gas turbines: unlike heavy-frame gas turbines, whose rotor is supported in either journal oil bearings or squeeze film dampers, aeroderivative gas turbines rely on much simpler, but stiffer ball bearings, which make the rotor too rigid near its supports to provide meaningful rotor vibration measurements. Another factor is the compactness of aeroderivative gas turbines, which means that there is not much room available for the installation of proximitors between rotor and stator for vibration monitoring. Instead, the vibration measurements for condition monitoring must be taken with accelerometers assembled across the machine outer casing—Figure 1.13a. Acceleration signals, due to their very high sensitivity to high-frequency vibration and their closeness to sources of fluid-related high-frequency excitation, tend to be much contaminated with signal noise—see Figure 1.13b—than proximitor signals, and are sensitive even to coupled vibrations from auxiliary or near-by industrial equipment. Moreover, these sensors provide only an indirect measure of the rotor vibration, the most important and well-documented source of information about the machine condition. Consequently, the detection of many malfunctions—such as rotor-casing rub—is much more difficult in aeroderivative gas turbines than in heavy-frame gas turbines. As we will remark when discussing the state of the art—Section 2—very little attention has been paid to the detection of rub with accelerometer signals, despite the great technical and industrial relevance of this problem.

1.5. Hypothesis, Thesis goals and scope

(A)

(B)

FIGURE 1.13: (A) Acceleration sensor, or *accelerometer*, model 330400, of Bently Nevada—image property of GE-Bently Nevada; and (B) an accelerometer output representing acceleration as a function of time, in the ADRE monitoring system interface.

1.5 Hypothesis, Thesis goals and scope

As an starting point, and considering all the above motivations and the knowledge already available by the state of the art, our hypothesis is that **time-frequency analysis methods provide the ideal tool for the extraction and the representation of the rub indicators or features that accelerometer signals may contain when there is a punctual rub in the monitored gas turbine**. Such signal features will be visible even in very short data samples and intermixed with signal noise. All in all to enable the very early detection of the fault.

With the motivations and background already stated in the former sections, the Thesis goal can be established as follows:

The conception, development and validation of an procedure for the detection of single-point rub in aeroderivative gas turbines from acceleration signals in its very early stages, that is, under very low intensities, based on time-frequency domain decomposition methods with robustness against noise. The fault features indicating rub must be visible and detectable the soonest possible to prevent potential damage in the machine.

The secondary goals of this Thesis work, whose resolution leads to the main objective stated above, are these:

- The model-assisted evaluation of a number of signal processing methods so as to find the one that suits early fix-point rub detection with accelerometer signals.
- The model-assisted quantification of the effect of post-processing parameters on the efficacy and precocity of the detection method.
- The model-assisted implementation of the methodology, with the post-processing tools and parameters selected for the case at hand.
- The experimental validation of the proposed methodology in a setting able to reproduce very weak single-point rub intensities.

The Thesis work was always carried out in the *Escuela Técnica Superior de Ingenieros Industriales* at the *Universidad Politécnica de Madrid*—UPM—in Spain, with a three-month-long hiatus in between in which the doctoral candidate enjoyed a stay at the *Politecnico di Milano* in Milan, Italy. The research work developed during the stay gravitated around the Thesis thematic: it was the experimental evaluation of the abilities of a methodology based on *Optical Backscatter Reflectometry* for condition monitoring of rotor blades.

1.6 Industrial relevance, assumptions and simplifications for this research work

The coronavirus pandemic and the oil trade wars between Russia and Saudi Arabia have severely disrupted the world market of gas turbines, but not in the same manner neither for heavy and light gas turbines nor for the electric power generation and the oil & gas sectors, as turbomachinerymag.com, 2020 explains. According to the article, The small gas turbines market may recover much faster from the present demand slump due to their highest suitability to the current needs of the power generation sector and because of the demand of these machines for processes in the oil & gas industry. As the power generation industry moves from carbon-based production to renewable sources, the variability of the electric power available in the grid increases. The resulting demand gap must be fulfilled with highly flexible and variable production systems. This is where aeroderivative gas turbines, for all their qualities mentioned in Section 1.4 prove themselves the perfect choice to fill the gap between production and demand. At the same time, many opportunities are arising in the oil & gas field, such as the compression and transportation of liquefied natural gas—LNG—, an industry in growth—Figure 1.14—for being more environmentally friendly than the production and consumption of traditional fossil fuels, mainly diesel gasoline and fuel oil.

FIGURE 1.14: Evolution of the LNG liquefaction capacity during the last years. From compressortech2.com, 2018

1.6. Industrial relevance, assumptions and simplifications for this research work

The previously cited article also points out that the reduced energy demand caused by restrictions due to the pandemic opens up an opportunity to increase expenditure and investment on maintenance operations. Since fewer new units are demanded and produced during this period, the gas turbine operators are constrained to extend the useful life of their oldest equipment; and maintenance tasks that have to be postponed during the operation peaks can be now taken up on the underutilized gas turbines. All in all, the fields of condition-based maintenance and malfunction detection in aeroderivative gas turbines provide a very fruitful research niche for the upcoming years from the industrial point of view.

The study of rub detection in aeroderivative gas turbines have required a series of assumptions that would bind the implications of the study to the Thesis goals enumerated in Section 1.5, thus ruling out the possibility of redundancies with other studies in the Literature or the expenditure of excessive resources on the reproduction of phenomena with little influence on the acceleration signal features due to rotor-stator or rotor-casing contact. These assumptions are:

- The rotor-casing system under study will be considered as an isolated whole with no external inputs or excitation but those induced by the motion of the modeled machine.
- The cause of the rub fault will be a combination of excessive unbalance of the rotor with a small rotor-to-stator clearance. There will be no other relevant cause of rub in the system under study.
- During the experimental study of rotor-casing rub only these two malfunctions will be under the control of the experimenter: rub and rotor unbalance. Imperfections such as misalignment, bearing-induced vibration, etc. must be minimized to an admissible range during the setup and the preparation of the experimental setting before the collection of test data so that their influence over the rub dynamics were residual.
- Following the two previous points, the computational model-based study of rub will considered only the malfunctions of rub and unbalance, with the rub being caused by excessive rotor unbalance.
- Since the interest is on the detection of rub in its early stages, only the weak single-point rub will be under the scope of this research. More severe forms of rotor-stator or rotor-casing contact, such as full annular rub, are out of our scope.
- In both the experimental and computational models, the original, complex industrial aeroderivative gas turbine will be reduced to its most basic form: a rotor with several solid cylindrical disks supported by two linear bearings and coupled to a source of mechanical power, and the casing as one, or an assembly of, cylindrical shells with homogeneous thickness and isotropy.
- Based on the previous point, blade or vane vibrations and their coupling with casing vibrations, fluid-dynamic effects derived from the rotation of the blade disks and other fluid-induced effects will be neglected in the study.

1.7 Research methodology and work structure

This work relies upon two pillars that determine the Thesis structure and the organization of the work to be done. The first of those pillars is the computational modelization of an aeroderivative gas turbine. This model must synthesize its most fundamental dynamics, including the possibility of rotor-stator fixed point rub. This model constitutes the basis for the design of the detection methodology. The second pillar is the validation of both the model and the methodology on an experimental model of an aeroderivative gas turbine. This model was designed and built at the *Mechanical Engineering Department—DIM—of the Escuela Técnica Superior de Ingenieros Industriales*. The computational and experimental modelization of an aeroderivative gas turbine with the possibility of single-point rub and the validation of the computational model have been the most time-consuming tasks of the Thesis research work.

In Chapter 1 we introduced the motivation for this Thesis research and its potential impact in the Industry. Then we presented the Thesis goals and scope.

In Chapter 2 the most relevant state of the art on fault detection in rotating machines is presented, beginning with the description of the traditional, standard data processing methods presently employed by the Industry for condition-based maintenance with vibration sensors. We dedicated some of the exposition to the state of the art of early detection of rub in rotating machinery, which unfortunately is very scarce; few authors paid any attention to this question.

Chapter 3 describes the evaluation of distinct rub detection techniques with the assistance of a model of an aeroderivative gas turbine, presented during the first half of the chapter. The presentation of this model includes model elements, hypothesis, the derivation of its highly-nonlinear dynamic equations with the Finite Element Method—FEM—and its numerical integration.

In Chapter 4 we present the experimental validation of the proposed methodology. First, parameters of the computational model described in Chapter 3 were adjusted to an experimental model of the same rotating machine. Then, post-processing parameters of the proposed method were tested on the adjusted model in order to strike a balance between computation speed and the efficacy of the detection. Given data from the adjusted model and the experimental measurements, the superiority of our methodology is confirmed over them with the same evaluation metrics defined in Chapter 3.

Finally, the conclusions of the Thesis research and possible future research lines are presented in Chapter 5.

Chapter 2

LITERATURE REVIEW

2.1 Traditional monitoring and data processing methods for rotating machines

This chapter begins with a description of the staple signal analysis techniques that make up the present industrial standard for the detection of malfunctions in rotating machinery since decades past, either with proximitors—Figure 1.10—between rotor and stator or with accelerometers—Figure 1.13a.

2.1.1 The orbit plot

The representation of time versus relative rotor vibration amplitude—measured with proximitors—in one Cartesian direction is done with the so-called *time-base plots*—Bently et al., 2003a, such as the one shown in Figure 2.1. The blue spot represents, in the Bently-Nevada notation standard, the instant in which a Keyphasor event occurs and is called Keyphasor mark. The interval between two consecutive Keyphasor marks determines one shaft cycle of rotation.

FIGURE 2.1: Time-base plot of the displacement of a rotor system degree of freedom. From Bently et al., 2003a.

Rotor displacement orbits in the radial plane where it rubs the casing are also of interest because in conventional turbomachines rotor orbits are the preferred monitorization tool. Rotor orbits bring us confirmation of the existence of a rub in the simulated rotor-flexible casing model—Bently et al., 2003b.

If the displacement data from two co-planar, perpendicular transducers over a section of the rotor shaft is plotted on a two-dimensional graphic, the result is called the *orbit plot*. Each point represented on the orbit plot corresponds to the position of the shaft centerline at a certain instant of time, therefore orbit plots represent the path taken by the rotor centerline in its vibrational motion on a transversal plane—Figure 2.2. Analogously, the simulation data from the x and y displacements of a shaft node can be taken to generate an orbit plot of the rotor system model.

FIGURE 2.2: Derivation of orbit plots from data from shaft displacement probes. From Bently et al., 2003a.

2.1.2 The Discrete Fourier Transform

A vector $\mathbf{x}$ of periodical discrete signal samples with length N can be decomposed into a Fourier sum series with sinusoidal terms called *harmonics*. The N coefficients $\hat{\mathbf{x}}(i)$ of the decomposition form the signal *spectrum* and are calculated with the *Discrete Fourier Transform*—DFT, see Vetterli et al. (2014). The sampling vector $\mathbf{x}$ may contain either signals of rotor displacement from proximitors or radial acceleration of a casing node, both measured with respect to time with T_s as its sampling period.

Before the calculation of the spectrum, the signal is periodized with the application of a *windowing transform* on the data in order to prevent spectral *leakage*: the smearing of the energy of the signal harmonics around their actual frequencies in the spectrum.—Lyon, 2009. The windowing transform of the sample vector $\mathbf{x}$ is performed multiplying each sample x_i , $i = 0, 1 \dots N - 1$ by a windowing function $w(i)$. Two popular windowing functions that can be applied to increase the degree of symmetry of the sample vector—therefore preventing leakage—are the Hanning and the Hamming windows, both being periodic cosine-sum window functions—Smith, 1997.

2.1. Traditional monitoring and data processing methods for rotating machines

Let $x_0, x_1, x_2 \dots x_{N-1}$ be an array of N discrete displacement values from one of the rotor system degrees of freedom, equally spaced in the domain of time. For each i -th value of a Discrete Fourier Transform output array $\hat{x}$ the next equation will be evaluated:

$$\hat{x}_i = \sum_{k=0}^{N-1} \exp\left(\frac{-2\pi j}{N} \cdot i \cdot k\right) \cdot x_k = \sum_{k=0}^{N-1} W_N^{-i \cdot k} \cdot x_k, \quad i = 0 \dots N-1, \quad (2.1)$$

where N is the number of samples, j is the imaginary unit and W_N is the N -th complex root of unity. The physical meaning of the discrete and continuous Fourier Transform is the change of a function domain from time to frequency. When applying the DFT, a set of function values evaluated in the domain of time in equally-spaced intervals is converted into a series of complex values that represent the amplitudes and phases of sinusoidal harmonics with a frequency spacing equal to f_s/N , where f_s is the *sampling frequency* of the vector of samples, equal to the inverse of the time interval between samples or *sampling period*: t_s .

The DFT algorithm takes a number of steps of N^2 —multiplies and adds. However, if the number of samples is a power of two, different symmetries that arise during the application of the DFT algorithm can be exploited, and as a result the number of steps needed for the calculation of the signal spectrum can drop to $N \log_2 N$ —multiplies and adds. This variety of the DFT algorithm is known as *Fast Fourier Transform*—FFT. There are many variants of the FFT algorithm, the oldest of them dating back to 1965, when J. W. Cooley and J. W. Tukey discovered the method—Cooley et al., 1965—even though it seems that the algorithm they published was already known to Carl Friedrich Gauss more than 150 years ago—Heideman et al., 1985.

Once the time-domain samples vector x has been multiplied by the windowing function we can apply the DFT/FFT algorithm on the resulting data. The result is a array of complex values $\hat{x}$ of length N whose modules can be represented on frequency-amplitude or frequency-energy spectral plots often known as *periodograms* or alternatively, in the Bently-Nevada nomenclature of monitorization in rotor-dynamics, *half spectra*—Bently et al., 2003a. Figure 2.3 shows examples of half spectra from proximator signals. Note that for every two cycles of rotation—two Keyphasor marks—the shape of the motion curve repeats itself, hence the period of vibration of the rotor node is twice the shaft rotation speed. Due to this periodicity, there is a $1/2 \times$ component in the spectra of both orthogonal directions.

Each spectral line that conforms the half spectrum of a displacement signal represents an harmonic of the rotor vibration. Figure 2.3 shows the harmonics contained in the shaft displacements measured in two orthogonal directions. Harmonics are labeled with their corresponding ratios of sinusoidal motion frequency to shaft rotation speed. The harmonic $1/2 \times$ has a frequency equal to half the shaft rotating speed. Since its frequency is lower than the rotating speed this is a *sub-synchronous* harmonic. The second spectral line represents amplitude of the $1 \times$ harmonic, also known as the *synchronous* harmonic, since its frequency equals the rotating speed. Higher harmonics— $3/2 \times$, $2 \times$, $3 \times \dots$ —are known as *super-synchronous* harmonics. This is equally valid in the case the spectral analysis is carried out on simulation data as it is with data from real machinery.

FIGURE 2.3: Half spectra of the radial displacements of a shaft node, derived from their time-base plots. From Bently et al., 2003a.

2.1.3 The full spectrum

In the field of Rotor-dynamics there exist a very powerful tool of analysis that is based on the spectral decomposition of the shaft orbits of vibration in a sum of circumferences around the rotation axis. Each of these circumferences that conforms the rotor orbit represents a circular motion of precession with a period that is a multiple of the sampling period, t_s , and their sum would result in the original rotor orbit. This spectral tool is known as *full spectrum* or as the *spectrum of an orbit*.

The harmonics of the full spectrum are of two classes: the *forward* precession is a motion whose direction matches the shaft rotation Ω . The *reverse* precession, on the other hand, opposes the rotor direction of rotation. It is the linear combination of these two sets of harmonics which yields the original rotor orbit. Figure 2.4 shows how an elliptical orbit can be split into two circular motions, one forward and the other backward, with synchronous rotation frequencies. At each instant of time, the position vector of the rotor centerline is the sum of the rotating vectors that give shape to each circumference. More intricate orbit shapes would be the combination of sub-synchronous, synchronous and super-synchronous forward and backward orbits with rotation frequencies equal or different to the shaft rotation speed.

FIGURE 2.4: Decomposition of an elliptical shaft orbit in two synchronous—precession frequencies equal to rotation speed—rotating circular motions, a *forward* motion and a *reverse* motion. From Muszynska, 2005a.

Calculation of the full spectra begins with the complex Fourier Transforms of the displacements of a shaft node in two orthogonal directions: $\hat{x}_x$ and $\hat{x}_y$ —see Figure 2.5. The result is a pair of arrays, $\mathbf{FS}_{forward}$ and $\mathbf{FS}_{backward}$, that contain the radii of the circular precession motions that compose the rotor orbit around the shaft node. Each of their values is calculated as

$$\begin{aligned} FS_{forward,i} &= \sqrt{|\hat{x}_{x,i}|^2 + |\hat{x}_{y,i}|^2 + 2 \cdot |\hat{x}_{x,i}| |\hat{x}_{y,i}| \cdot \sin(\psi_{\hat{x}_{x,i}} - \psi_{\hat{x}_{y,i}})}, \\ FS_{reverse,i} &= \sqrt{|\hat{x}_{x,i}|^2 + |\hat{x}_{y,i}|^2 - 2 \cdot |\hat{x}_{x,i}| |\hat{x}_{y,i}| \cdot \sin(\psi_{\hat{x}_{x,i}} - \psi_{\hat{x}_{y,i}})}, \end{aligned} \quad (2.2)$$

$$i = 0 \dots N - 1.$$

In Equation 2.2, $|\hat{x}_{x,i}|$ and $|\hat{x}_{y,i}|$ are the modules of the i -th spectral component of $\hat{x}_x$ and $\hat{x}_y$. $\psi_{\hat{x}_{x,i}}$ and $\psi_{\hat{x}_{y,i}}$ are their respective phase angles on the complex plane.

The full spectrum is graphically represented with the amplitudes of the forward harmonics on the right side of the horizontal axis, and those of the reverse harmonics on the left side of the horizontal axis—Figure 2.6. The forward spectral components are labeled with positive rational numbers representing the motion frequency-to-rotation speed ratio of each harmonic—as in the half spectra. Reverse components are labeled with the same rationals although with the minus sign.

2.1.4 The cepstrum

The *cepstrum*—Randall, 2017, also known as the *spectrum of the spectrum*, is a popular diagnosis tool in machinery condition monitoring. Generally, it is the inverse Fourier Transform of the logarithm of a spectrum. In the case of the *real cepstrum*, its mathematical definition is

$$c(t) = \mathcal{F}^{-1}(\log |\mathcal{F}(x(t))|), \quad (2.3)$$

with $\mathcal{F}$ representing the Fourier Transform, and $\mathcal{F}^{-1}$ its inverse.

FIGURE 2.5: Calculation of the full spectrum of a shaft node radial displacement. From Bently et al., 2003a.

FIGURE 2.6: Full spectrum of the orbit of a shaft node that combines forward and reverse synchronous and sub-synchronous harmonics. From Bently et al., 2003a.

2.2. Early fault detection in rotating machinery

First, the complex phase information of the signal $x(t)$ is removed with the calculation of the module of its spectrum, $|\mathcal{F}(x(t))|$. Then, the logarithm evens out the amplitude of any sidebands or impulse trains contained in the spectrum with the amplitudes of the main spectral components. After the application of the inverse Fourier Transform, impulse trains and amplitude or frequency-modulated components contained in the original signal are highlighted in the resulting cepstrum and shown as periodical pulses called *rahmonics* with a time-dimensional periodicity called *quefrequency*, measured in seconds.

2.2 Early fault detection in rotating machinery

During the course of this research, there was a surge of interest in the Literature in the field of early fault detection in rotating machines. However, the vast majority of this new knowledge focuses on the diagnosis of either bearing or gear fault with acceleration signals measured with accelerometers on the bearing or gearing housing. Wei et al., 2019 offers a comprehensive review of early fault detection approaches in rotating machinery which includes bearing fault, gear fault and cracked rotor, but with no reference to rubbing.

An enumeration of major recent advances on early bearing fault diagnosis shows a huge diversity of methods proposed for detection. Li et al., 2016 proposed early fault detection and identification with Symbolic Dynamic Filtering—SDF, Gupta et al., 2007—and Intrinsic Characteristic-scale Decomposition—ICD, Li et al., 2015—respectively. Brkovic et al., 2017 applied Wavelet Transform over acceleration signals for early detection of bearing faults in induction motors. After Wavelet decomposition the authors extracted the standard deviation and the logarithmic energy entropy of each wavelet sub-band, and then reduced the parameter space to two degrees of freedom. The values in the two-dimensional reduced space showed very good separation among the diverse bearing fault classes and the fault/no fault state—Figure 2.7.

Gao et al., 2018 developed an improved ADMM—Alternating Direction Method of Multipliers, Boyd et al., 2011—coupled with Minimum Entropy Deconvolution—MEC, Wiggins, 1978—for early bearing fault detection. It was tested over data from the Case Western Reserve University—CWRU—Bearing Data Center—Smith et al., 2015, which has become the standard reference in the field of fault detection in bearings. The data contained by this repository come from the simulation of a number of bearing fault types—Figure 2.8a—in the experimental rig shown in Figure 2.8b.

FIGURE 2.7: (A) Experimental setup in Brkovic et al., 2017 with bearing and sensor positions; and (B) Faulty—right—and normal—left—sample points in the reduced space, and the separation curve between the fault/no fault subsets achieved by Brkovic et al., 2017.

(A)

2.2. Early fault detection in rotating machinery

(B)

FIGURE 2.8: (A) Bearing malfunction types depending on the location of the fault—from Smith et al., 2015; and (B) The CWRU bearing test rig described in Smith et al., 2015.

Glowacz et al., 2018 analysed acoustic emission as an early indicator of bearing fault for induction motors. Li et al., 2018b proposed the combination of Hierarchical Symbol Dynamic Entropy or HSDE with Binary Tree Support Vector Machine—BT-SVM, Fei et al., 2006. Ma et al., 2019 presented early detection of bearing faults with Adaptive Variational Mode Decomposition—AVMD, Lian et al., 2018, then, optimal intrinsic mode functions—OIMFs—were picked up with a criterion based on envelope entropy. Then, Local Tangent Space Alignment—LTSA, Zhang et al., 2007—was used for the denoising of the OIMFs. Albezzawy et al., 2019 proposed the customization of Morlet Wavelet functions based on the maximization of the Gini index of the squared envelop of the signal for bearing fault detection. Haidong et al., 2019 explored the application of Enhanced Deep Gated Recurrent Unit and Complex Wavelet Packet Energy Moment Entropy for early detection. Hoseinzadeh et al., 2019 considered a modification of the Hilbert-Huang transform that considers entropy features in the signals, enabling early detection of bearing faults. Wan et al., 2019 presented Adaptive Asymmetric Real Laplace Wavelet—ARLW—and Envelope Analysis. Wang et al., 2020 proposed early detection of bearing faults with the Adaptive Piecewise Hybrid Stochastic Resonance method—APHSR, Li et al., 2019.

In the other hand, on early gear fault detection, or joint bearing-gear fault detection, we highlight these: Soleimani et al., 2015 explored phase reconstruction as a method to spot chaotic vibration that characterize certain failures even at its early stages. Three quantifiers are used to quantify the chaotic behavior and detect and classify the faults: Correlation Dimension—CD, Largest Lyapunov Exponent—LLE—and Approximate Entropy—AE. The method was tested over acceleration data from the CWRU bearing testing rig in Figure 2.8 for bearing fault detection, and from a experimental machine designed by the authors that can simulate gear faults—Figure 2.9a. Good separation between fault/no fault and fault type separation was achieved in the chaotic feature space—Figure 2.9.

FIGURE 2.9: (A) Experimental rig for simulation of gear faults, showing worn tooth, broken tooth and accelerometer location; (B) Fault identification of diverse bearing fault types; and (C) Fault identification of diverse gear fault types. Samples from the machine under normal operation are marked in either yellow or green. From Soleimani et al., 2015.

Li et al., 2017 studied early fault detection in gears and bearings within acceleration signals with a spline-based Local Mean Decomposition—LMD, Cheng et al., 2012. The product functions with most information were selected with the Kullback-Leibler divergence—Van Erven et al., 2014. Guo et al., 2018 combined Wavelet Packet Energy—WPE—and Modulation Signal Bispectrum—MSB—over accelerometer signals for early detection in planetary gearboxes. Li et al., 2018a developed a post-processing method that can extract impulsive features from acceleration signals even under the presence of strong signal noise. It is a combination of the Alternating Direction Method of Multipliers—ADMMs, Boyd et al., 2011—with the Improved Non-Convex Regularization Algorithm—I-NCRA. This method was tested on data from an experimental rig with bearing fault—Figure 2.10a—and another with gear wear fault—Figure 2.10b. In Figure 2.10 we show transient impulses extracted from samples from the experimental setups under bearing fault, gear fault and healthy state respectively. Notice that under healthy state the signals should contain no remarkable impulsive features and therefore the I-NCRA output is the original signal—Figure 2.10e.

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FIGURE 2.10: (A) Experimental setup for bearing fault; (B) Experimental setup for gear fault; (C) Periodic transient impulses demodulated using I-NCRA for bearing fault; (D) Periodic transient impulses demodulated using I-NCRA for gear fault; and (E) Periodic transient impulses demodulated using I-NCRA for the healthy state; From Li et al., 2018a.

Deng et al., 2020 proposed a sparse representation approach based on Optimized Fault Impulse Matching Algorithm—FIMA—for the early matching of the impulse features that characterize certain faults such as gearbox and bearing faults. After matching those features they can be isolated in a similar fashion to the Li et al., 2018a method described above. Figure 2.11 shows the results with experimental bearing fault data, very satisfactory despite the high levels of signal noise. Yao et al., 2020 achieved early fault feature extraction with Double-Window Spectrum Fusion Enhancement—DWSFE—which suits signals containing strong noise. It was tested for gear fault detection in a wind turbine gearbox.

Yet, by digging deep in the state of the art we still found the pioneering work Niaoqing et al., 2003 showing the application of Stochastic Resonance theory—SR, Qiao et al., 2019—for early rub diagnosis. Unfortunately, the study is constrained to the detection within signals from displacement sensors.

FIGURE 2.11: At the left column: (A) Time-domain signal under gear fault; (B) Frequency spectrum; and (C) Envelope spectrum. At the right column: (A) The gear fault signal features isolated with the proposed FIMA; and (B) Envelope spectrum of those features. From Deng et al., 2020.

2.3 Rub detection with accelerometers in rotating machinery

Rub dynamics, monitoring, detection and diagnosis of rub and other malfunctions have been research topics in Academia for the last decades and the Industry has already a large experience on the matter, as proved by benchmark monographs such as Muszynska, 2005a and Bently et al., 2003a. However, as we mentioned in Section 1.3, this mainstream knowledge is constrained to detection with the measurement of relative rotor-stator vibrations measure with displacement sensors. Only a handful of papers have been published on the subject of the detection of rub with acceleration sensors on casing, despite the Industrial relevance of the problem and the several handicaps associated to the use of casing acceleration measurements for fault detection, as discussed in Section 1.4. Here we sum up the existing state of the art on rub detection with accelerometers up to the realization of this Thesis research.

The authors of Chen, 2015a; Chen, 2015b and Yu et al., 2015 show the results of a series of experimental tests on an experimental aero-engine with a bladed disk and a casing that encloses it. Blades and casing rub with each other when the latter is squeezed onto one of the rotating disk by turning one or several screws—Figure 2.12a. This rub is continuous over some or all of the disc blades in one or several fixed points of the casing. Obviously, if all disk blades rub, the number of impacts per cycle of rotation in each rub point is equal to the number of blades. Casing acceleration is measured at a set of points with accelerometers and spectral analysis is conducted on the data—Figure 2.12b. Raw acceleration data—Figures 2.13a, 2.13b and 2.13c—and its spectra reveal periodic rotor-casing impacts taking place with frequency equal to the product of the rotation speed—equal to 24.8 Hz—with the number of blades—thirty-two. Again, these impacts have an amplitude-modulated component with frequency equal to rotation speed. Amplitude modulations are mainly caused by residual rotor unbalance and eccentricity, which vary impact force sinusoidally as the rotor turns one cycle. These dynamics are well visible in both

2.3. Rub detection with accelerometers in rotating machinery

the spectra—Figure 2.13d—and the Cepstrum—Figure 2.13f—of the acceleration signals. Notice the harmonic corresponding to the blade-pass frequency—794 Hz—being excited by the impacting, and its sidebands reflecting rotation-frequency modulation—Figure 2.13e. The amplitude modulation appears very clearly on the Cepstrum.

FIGURE 2.12: (A) and (B) Rubbing screws and accelerometer positions in the experimental rig of Chen, 2015a and Chen, 2015b; (C) and (D) Model schemes for the blade-casing rub simulations in Chen, 2015c and Chen, 2016.

Then, Chen, 2016 contrasted these experimental results with a simulation in a rotor model described in Chen, 2015c—Figures 2.12c and 2.12d: the shaft is discretized into beam elements; the disk blades are solid, undeformable, straight beams; and the casing is modelled as a two-dimensional curved non-rotating beam with two displacements in the radial plane per node.

Over the same experimental setup, Yu et al., 2018 proposes a combination of autocorrelation, Hilbert envelope spectrum and cyclo-stationary theory to analyse the amplitude-modulated casing accelerometer signals in order to extract the features characterizing the rub.

Wang et al., 2016b used another experimental rig to reproduce the same type of rub than in the previous papers—Figure 2.14: rub is inflicted by means of a nylon block on the casing that impacts with a number of disc blades as the shaft rotates. The data spectra also highlight the impact-frequency peak and the impact modulation, as well as a rotation-frequency amplitude peak and its super-synchronous multiples, clearly seen in the low-frequency spectrum band. Later on, their authors proposed in Wang et al., 2017a envelope demodulation for the diagnosis of rotor rub.

FIGURE 2.13: Accelerometer data plots from single-point rub experiments in Chen, 2015b. (A) Time series, no rub; (B) Time series, rub; (C) Detail of time series, rub; (D) Spectrum, rub; (E) Detail of spectrum, rub; and (F) Cepstrum, rub.

The results in Wang et al., 2016b are very comparable to those of the Chen articles above. The rub pattern is clear on the time domain plot—Figure 2.15a: there is an impact every time a blade crosses the rubbing block—420 times per second at a rotation speed of 35 Hz—with an amplitude modulation of 35 Hz caused by unbalance, eccentricity, etc. The referred modulation sidebands appear around the blade impact harmonic—Figure 2.15c. Apart from the rotation frequency—Figure 2.15d—and the blade-pass frequency harmonics—Figure 2.15c—the spectrum—Figure 2.15b—is dominated by natural frequencies of the casing and other system components, excited by the rub impacts.

FIGURE 2.15: Accelerometer data plots from single-point rub experiments in Wang et al., 2016b. (A) Time series, rub; (B) Spectrum, rub; (C) Detail one of spectrum, rub; (D) Detail two of spectrum, rub; (E) Cepstrum, rub; and (F) Time-frequency spectrums—STFT, rub.

2.4 Time-frequency analysis and post-processing of data from rotating machines under rub

Time-frequency decomposition methods are still unfamiliar to most industrial practitioners. However, they have been studied for fault detection in rotating machines during the last twenty years at least—Feng et al., 2013. Time-frequency representations of data can capture local frequency variations with very high resolution in both time and frequency domains, therefore, they are ideal for signals whose instantaneous frequencies vary very rapidly in time. Instead, Traditional frequency analysis methods such as Fourier Transform—Bracewell, 1963—require the data to be periodic or stationary, and when they are not so, anomalies such as *leaking* or *smearing* occur that blur the frequency representations—Smith, 1997, Tran, 2006.

Again, a myriad of time-frequency methods have been proposed for single-point rub detection over the years. These are some recent applications of time-frequency methods to the diagnosis of rotor-casing or rotor-stator rub: Wang et al., 2014 proposed Matching Demodulation Transform—Chen et al., 2017a—to extract the instantaneous frequency oscillations that characterize rotor-stator impact within displacement signals. Chandra et al., 2016 gave a comparison of three time-frequency techniques—the Continuous Wavelet Transform or CWT, the Short-Time Fourier Transform or STFT, and the Hilbert-Huang Transform or HHT—for detection of rotor faults such as punctual rub within proximator signals under Gaussian noise. Figures 2.16 and 2.17 show results from simulated run-ups in a rotor in healthy state and under light rub over the first critical speed, respectively. STFT fails altogether to detect the transient rub; while the HHT and the CWT can indeed detect it. The paper proves that even though the CWT is slower than the HHT it is better under signal noise than the other signal decomposition methods.

Yang et al., 2016a and Wang et al., 2017b presented instead a Nonlinear Squeezing Time-Frequency Transform—NSquTFT—for the visualization of the high oscillations of the instantaneous frequencies induced by rub, a key trait for its detection with displacement sensors. Zheng et al., 2017 developed rub detection with an adaptive Empirical Wavelet Transform—EWT, Gilles, 2013—with displacement sensors. Braut et al., 2018 applied the Teager-Huang Transform—Cexus et al.,

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FIGURE 2.16: Run-up response of a healthy rotor. From Chandra et al., 2016.

2006—for partial rotor-stator rubbing detection with displacement sensors. Islam et al., 2018 combined Ensemble Empirical Mode Decomposition—EEMD, Rilling et al., 2003 and Wu et al., 2009—with selection of Intrinsic Mode Functions—IMD—with Genetic Algorithms for the extraction of the rub features in displacement signals. Prosvirin et al., 2018 and Prosvirin et al., 2019 combined EMD with feature selection based on the Kullback-Leibler Divergence—Van Erven et al., 2014—and the so-called Ratio of Degree-of-Presence—DPR. Chen et al., 2019 used Adaptive Chirp Mode Decomposition to extract frequency modulation characteristics induced by rotor-stator rub on rotor displacement. Hu et al., 2019 analyzed the intra-wave frequency modulation induced by crack and single-point rub in shaft displacement signals. As stated above, these frequency modulations are very clear indicators of single-point rub in rotor displacement signals—Figure 2.18. Note that all these studies employ signals from displacement sensors instead of accelerometers as the indicator of failure.

Of all the existing time-frequency methods for the post-processing of signals, Wavelet Analysis techniques—Mallat, 1999—stand out for their omnipresence in so many diverse fields across Science. Applications include Noise Reduction—Patil, 2015, Data Compression—Boix et al., 2010, Image Processing—Xizhi, 2008, Speech Recognition—Farouk, 2014, Radar detection—Cao, 2014, Medicine—Singh et al., 2015, Biology, Geoscience, Astrophysics—Baleanu, 2012—and of course, maintenance and fault detection in rotating machinery. Two review papers discussing advancements on the fault detection field are Yan et al., 2014 and Chen et al., 2016. More recent advances on rotating machinery are enumerated as follows:

Peng et al., 2005 studied the use of a reassigned scalogram with the Morlet Wavelet Transform for rotor rub diagnosis although in displacement signals. Halim et al., 2007 illustrates the usefulness of Discrete Wavelet Transform—DWT—for

FIGURE 2.17: Run up response of a rotor under light rub. From Chandra et al., 2016.

rub detection with displacement sensors. Al-Badour et al., 2011 proposed the detection of blade-casing rub with gauge sensors mounted on the blades. Signals were decomposed and analysed with a combination of Wavelet-based tools: Continuous Wavelet Transform—CWT, Discrete Wavelet Transform—DWT—and Wavelet Packet Transform—WPT. Wavelet methods proved sensitive to the impulsive excitation of the blade natural frequencies during blade-to-casing contact, as we can see in Figure 2.19. The authors also noted the usefulness of splitting long series of data into smaller samples for a faster detection and lower use of computer resources.

Chen et al., 2017b validated the applicability of the Wavelet Transform-based algorithms as a fault detection tool for rotating machinery, particularly rub impact in a gearbox, with accelerometers on the bearing supports. Very recently, Tong et al., 2019 and Yu et al., 2019 applied two variants of the Synchrosqueezed Continuous Wavelet Transform—WSST, pioneered by Ingrid Daubechies, Jianfeng Lu and Hau-Tieng Wu—Daubechies et al., 2011; the former implements the Nonlinear WSST, and the latter chooses a Second-Order Multisqueezed WT—SMSST. These algorithms were tested on the rotor rig in Figure 2.20a. Rotor deflection was measured with proximitors—Figure 2.20b—and signals were processed with the two proposed methods. Both achieve high resolution for weak rub features appearing in the proximitors signals as high oscillations of the instantaneous frequencies. Notice the similarity between the instantaneous frequencies extracted with the SMSST, shown in Figure 2.20c, and those in Figure 2.18b.

It is worth mentioning that, to the best authors' knowledge none of the previous works on rub detection have ever focused on early fault detection: the diagnosis of the fault at the earliest stage and with the lowest fault severity that can be reproducible under an experimental setting and measurable with the most precise instrumentation.

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FIGURE 2.18: Instantaneous frequency of displacement signals from an experimental rotor (A) in normal condition; and (B) under period-two rub. From Hu et al., 2019.

FIGURE 2.19: CWT scalograms of strain gauge signals from the blades of an experimental rotor (A) in normal condition; and (B) under blade-to-casing rub. From Al-Badour et al., 2011.

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FIGURE 2.20: (A) The rotor test rig; (B) Time waveform measured at a rotation speed of 85 Hz under rub impact, with its Fourier spectrum; and (C) Its SMSST scalogram. From Yu et al., 2019.

Chapter 3

MODEL-BASED EVALUATION OF ADVANCED SIGNAL PROCESSING TECHNIQUES FOR THE EARLY FAULT DETECTION OF SINGLE-POINT RUB IN AERODERIVATIVE TURBINES

3.1 A non-linear model for the simulation of single-point rub

In this chapter we present the evaluation of a series of signal processing techniques that may be valid for the early detection of rub in an industrial aeroderivative gas turbines. This preliminary evaluation is fully computational: it relies on simulations of an ad-hoc mathematical model of such machine. The model conception, features and hypothesis, the derivation of its highly-nonlinear dynamic equations with the Finite Element Method—FEM—and their numerical integration are all presented below.

3.1.1 Model schematic

Our rotor-flexible casing model—Figure 3.1—comprises a cylindrical shaft— S , blue color—with some cylindrical rigid solid disks— D_1 and D_2 , red color—attached to it symmetrical positions that guarantee an homogeneous weight distribution between the bearings. Together, they form the machine rotor, which rotates at a constant speed Ω . The rotor is supported on bearings— B_1 and B_2 , violet—that are radially bound to the cylindrical shell that forms the machine casing— C , black. In an aeroderivative gas turbine, the ends of a casing are attached to solid elements called *frames*, which also house the rotor bearings. These bearing-casing joints are modeled as very stiff bar trusses between each bearing and some homogeneously-distributed points in the casing— BC_1 , not represented for simplicity; and BC_2 , yellow). All components are modeled under the hypotheses of linear elasticity—with the exception of the rigid solid disks), homogeneity and small displacements and deformations. Any of the two shaft extremes may incorporate a linear elastic coupling attached to the ground at the other side—*coupl*, dark blue. The casing rests on the ground held up by linearly elastic supports attached to several points of it— CF_4 , not represented; and CF_1 , CF_2 and CF_3 , black). The z axis of the global Cartesian reference system—black—follows the rotor axial direction and the x and y axes are contained in a rotor radial plane at one of the shaft ends. A positive rotor rotation speed follows the shortest path from x to y .

This system may vibrate due to the action of unbalance and rub forces that will be described further below. The motions caused by those forces are measured by one or several accelerometers on the casing—green—which output casing acceleration in time. The rotor radial motion—called the rotor *orbit*—is also of great interest because these rotor orbits give us a wider picture of the dynamics of the model under simulation, and because it is these rotor radial deflections what is monitored in conventional turbomachines as the indicator of malfunction. In fact, when we analyze rub in the experimental model—in Section 4—we will need measurements of rotor lateral deflection via proximitors along with the acceleration measurements in order to confirm the existence—or non-existence—of rub at each instant of time.

FIGURE 3.1: The rotor-flexible casing model: (a) perspective view; and (b) radial view of the model end, with the rotor displaced from the bearing node B_2 . K_B represents bearing stiffness.

This model can be adapted to any set of material properties, dimensions and numbers of disks, bearings and supports, as long as the model hypotheses hold. As shown in Figure 3.1, our model was set up with two disks, two rotor bearing supports at both casing sides, a coupling at one of the shaft extremes and four casing-ground supports attached to the casing ends in a plane that splits the casing in two identical halves. Shaft and disks are made of steel. The casing's mechanical properties correspond to carbon fiber. Bearing-casing joints—in yellow—are modeled as very rigid elements so that most of the bearing-casing union elasticity rests on the bearings themselves.

3.1.2 Model hypothesis

A simulation of this model requires the integration of a dynamical system of ordinary differential equations—SODE—. In this work, it was derived with the Finite-Element Method following the model hypotheses. A thin, slender shaft has been discretized into two-node Euler-Bernoulli beam elements with five degrees of freedom per node: three translations and two bending rotations. Mass, gyroscopic and stiffness local matrices of a two-node Euler beam element were taken from Jones, 2005, with the neglect of the torsional degrees of freedom. The rigid solid disks—modeled as in Muszynska, 2005c—share these five degrees of freedom with the shaft. The casing was split into a grid of equally-sized Mindlin-Reissner shell elements with four nodes at the element corners and two additional nodes at the radial boundaries to allow the isoparametric modeling of a curved cylindrical shell. The formulation we followed to derive the local equations of casing elements is in Ahmad et al., 1970 and is based on treating each shell element as a degenerate 3D element under shell hypotheses, with nodes only in the mid-plane. Each casing node has five degrees of freedom: the three Cartesian translations and two bending rotations around the in-plane local axes, as usual in shell elements. The bearing nodes and those of the stiff bars that bind them to the casing only have as degrees of freedom the three translations in space. As bearing hypotheses, there are no cross-coupled reaction forces between bearings and shaft and the shaft is supposed to rotate freely in the bearing, with no reaction forces caused by relative bearing-shaft angular deformations. This hypotheses are applicable when the shaft is supported on ball or roller bearings, as it is in most aeroderivative turbines.

Below we describe in detail, element by element, the hypotheses of our rotor-casing model under unbalance and rub.

Shaft hypotheses

The rotor shaft is modeled as a isotropic, homogeneous and linearly elastic beam with distributed mass and constant circular section. The forces and moments that act on the shaft are constraint to the rotor radial planes, with no axial projection over the axis z , and therefore there no axial displacements nor torques. According to the hypothesis of small deformations, any shaft deformations are sufficiently small so as to remain linear with respect to shell loading.

The Euler-Bernoulli Beam Theory is the oldest and most simple beam theory developed. However, it is accurate enough on long and thin beams with small deformations and an homogeneous, isotropic material. The Euler-Bernoulli hypothesis are these:

- Beam cross-sections remain planar after deformation.
- Beam cross-sections have their shape unchanged with deformation.
- Beam cross-sections remain normal to the beam neutral axis after deformation.

The *cross-sections* are defined as planar sections normal to the beam neutral axis, and the *neutral axis* is an axis along the beam cross-sections which undergoes no longitudinal strains or stresses—Figure 3.2.

FIGURE 3.2: Beam cross-section and neutral axis. Source: wikipedia.org. The shaft section, unlike the beam shown in the image, is circular in our model, therefore its stiffness is constant in any radial direction.

One of the consequences of these hypotheses for homogeneous, thin beams is that the only considerable strains and stresses in the beam are the normal strains and stresses in the axial direction: ϵ_{zz} and σ_{zz} . These strain and stress fields are zero in the neutral axis and linearly proportional at each beam point to its shortest distance to the neutral axis—Figure 3.3.

FIGURE 3.3: Normal axial stresses along the Euler-Bernoulli beam. These stresses are caused by bending internal moments on the beam cross-sections. Source: wikipedia.org.

Disks hypotheses

The disks are cylindrical, homogeneous rigid solids. They are fitted to the shaft through circular holes of radii R_i around their center. Each disk has an external radius R_e and a thickness H —Figure 3.4.

FIGURE 3.4: Dimensions of a rotor disk.

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The equations of angular motion are derived with the consideration of the following assumptions:

- The transverse moment of inertia I_T is the same in all radial directions due to the disk lateral symmetry.
- Small angular displacements: $\varphi_x \approx 0$, $\varphi_y \approx 0$. Consequently, the disk angular motions whose coupling causes the *gyroscopic* effect—the coupling between the angular motions in the two orthogonal radial directions, see Jones, 2005—will be linear.
- The axial rotation speed Ω remains constant, and no axial—external or inertial—torques exist on the disk.

Further down in Section 3.1.3 we will calculate the motion equations of the disks. Their calculation will be a simplified adaptation of the demonstration described by the reference Muszynska, 2005a.

Casing hypotheses

The most elementary casing model—Figure 3.5a—consists of a cylindrical, axisymmetric shell with distributed mass built concentrically around the rotor shaft and disks with a common centerline. This shell has a constant thickness h around the whole shell and the radius R_c between its mid-surface—a plane whose points are equidistant to the upper and lower shell surfaces—and the rotor centerline remains constant in the axial direction. The casing is made of an homogeneous, isotropic and linearly elastic material. The casing is subject to forces and moments that are contained in transversal xy rotor planes. Those external loads can have projections that are either normal or tangential to the shell surfaces. As with the shaft, the hypothesis of small deformations is taken under consideration.

However, in a real aeroderivative gas turbine the casing components are split in half for assembly purposes, and therefore, composed of upper and lower halves. For this reason, we may propose the more realistic casing model in Figure 3.5b. We only have to model identical casing halves and to couple them elastically at the mid-plane and to the foundation.

FIGURE 3.5: Sketches of (A) a model of a cylindrical casing; and (B) a model of a semi-cylindrical casing with flaps on the symmetry plane for attachment.

Since the casing dimensions are much larger in the two in-plane directions than in the direction normal to the surfaces, Plate Theory provides the model hypothesis and presumptions from which the casing dynamical equations will be derived. The two classical Plate Theories are the Kirchhoff-Love theory and the Mindlin-Reissner theory. The latter is an extension of Kirchhoff-Love theory applicable to thick shells and plates—in which shell in-plane dimensions are less than ten times larger than the thickness dimension—for more accuracy.

The Mindlin-Reissner hypotheses are these:

- Straight lines normal to the plate surfaces remain straight after deformation.
- The plate thickness remains unchanged with deformation.

The resulting shell model and deformation equations could be greatly simplified with the following additional assumption:

- Straight normal lines remain normal to the plate surfaces after deformation.

The three hypotheses put together form the Kirchhoff-Love theory. It is the first ever formulated plate theory and provides great accuracy if shells and plates have sufficiently small thickness-to-length ratio, as proven by Labuschagne et al., 2009. Notice that Kirchhoff-Love theory is a two-dimensional extension of the Euler-Bernoulli theory for beams described in Section 3.1.2.

Figure 3.6a shows linear normal-to-plane and angular deformations of a plate or shell differential element. If the plate is modeled as a Kirchhoff-Love plate, the blue normal line remains perpendicular to the plate surfaces—upper, lower and mid-surface—and its angular deformation in any in-plane direction φ equals the derivative of the mid-surface normal displacement w in that direction as in a Euler-Bernoulli beam. If it is modeled as a Mindlin-Reissner plate then normal shear deformations are significant and the presumption of perpendicularity is no longer valid.

As a result of the Kirchhoff-Love hypotheses, the plates and shells can sustain tangential—or in-plane—normal strains and stresses due to internal bending moments—Figure 3.6b—and torque-related strains and stresses—Figure 3.6c. In-plane shear strains and stresses due to in-plane torques, however, can cause noticeable effects only in the Mindlin-Reissner shell, that is, within thick plates and shells—Figure 3.6d. The Kirchhoff-Love theory instead neglects deformations by shear forces.

(A)

FIGURE 3.6: Deformations and stresses in a plate. (A) Normal and angular displacements; (B) Bending moments and normal in-plane stresses; (C) Torque internal forces and stresses; (D) Shear resultant and shear stresses. Source: wikipedia.org.

Hypotheses of the coupling, bearings and casing supports

We will suppose that the coupling—*Coupl* in Figure 4.17—the connections of the shaft to bearings B_1 and B_2 and the casing supports to the foundation— CF_1 to CF_4 —we suppose that all these elements behave as linear visco-elastic elements with no mass of their own.

If the casing was composed of several parts, these could be linearly and visco-elastically coupled to each other.

The rub

The rubbing forces form the core of this study. These forces arise when there is contact between rotor and casing. As a result of this contact, an elastic force acts on both rotor and casing on the point of contact, and a friction force, proportional to the elastic force by a factor, acts on a tangential direction to the contact surfaces. Regardless of the linearity or non-linearity of the elastic contact force, the rubbing forces are inherently piece-wise and therefore nonlinear—see Figure 3.7.

FIGURE 3.7: The rubbing forces as a function of relative rotor-casing displacement. Rubbing forces are non-zero only when the relative deformation is larger than a clearance ϵ between rotor and casing.

If there is energy dissipation during rubbing contact these rubbing forces are also dependent on the relative rotor-casing velocities of deformation. Other parameters that influence the rubbing forces are the contact stiffness function $k_R(\mathbf{u}, \mathbf{u}')$, which is often considered constant— k_R ; a friction coefficient that can vary with the rotor-casing relative velocity, $\mu(v_R)$, also usually taken as a constant μ ; and a contact damping function $c_R(\mathbf{u}, \mathbf{u}')$, or simply the constant c_R . All in all, a vector of rubbing forces in a rotor-casing model can be expressed as a function $\mathbf{f}_R(\epsilon, \mathbf{u}, \mathbf{u}', k_R(\mathbf{u}, \mathbf{u}'), \mu(v_R), c_R(\mathbf{u}, \mathbf{u}'))$, with ϵ being the rotor-casing clearance, and $\mathbf{u}$ and $\mathbf{u}'$ the system deformations and velocities.

3.1.3 Derivation and assembly of the model dynamic equations with the Finite-Element method

As in any mechanical system, the most general equations that will define the rotor-casing model dynamics can be written as

$$\mathbf{M}\mathbf{u}'' + \mathbf{D}\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}, \quad (3.1)$$

where $\mathbf{M}$ is a mass matrix, $\mathbf{D}$ is a damping matrix, $\mathbf{K}$ is a stiffness matrix, $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ are the system displacements field vector and its time derivatives—velocity and acceleration—and $\mathbf{f}$ is a vector of external forces on the model that may also contain non-linear forces, if there were any.

The components of the mass matrix represent the amount and distribution of the system inertia and are constant if the system mass remains constant with deformation and time. The physical meanings of the damping and stiffness matrix components are, respectively, the structural stiffness and damping of the flexible components—shaft and casing—that form the rotor system. The components of the damping and stiffness matrices— D_{ij} and K_{ij} —usually depend on variables that may or may not be coupled with either displacements or time, such as the rotating speed Ω , the temperature T and fluid viscosity η . Under such dependencies, damping and stiffness matrix components are functions $D_{ij} = D_{ij}(T, \Omega, \eta)$ and $K_{ij} = K_{ij}(T, \Omega, \eta)$.

The last component of the generalized equation, $\mathbf{f}$, is a vector that contains the external forces that act on the rotor system. In our rotor-casing model it will be the sum of two external forces: the shaft mass unbalance forces and the rubbing forces:

$$\mathbf{f} = \mathbf{f}_U + \mathbf{f}_R. \quad (3.2)$$

Derivation of the shaft equations

Remember the Kirchhoff-Love hypotheses—Section 3.1.2—and the distribution of the normal stresses under beam bending across each transversal cross-section—Figure 3.3. We will define a local coordinate system with the z axis tangent to the neutral axis and the x and y axis contained in the shaft cross-sections. Integration of the normal axial stresses along the cross-section A results in these internal bending moments:

$$\begin{aligned} M_{r,x}(z) &= \int_A y \sigma_{zz}(x, y, z) dA, \\ M_{r,y}(z) &= \int_A x \sigma_{zz}(x, y, z) dA. \end{aligned} \quad (3.3)$$

If there are no axial internal forces coupled with bending, the sum of the axial internal forces must be zero on the beam cross-section:

$$F_z = \int_A \sigma_{zz} dA = 0. \quad (3.4)$$

As the axial stresses—Figure 3.6b, the axial strains are zero in the neutral axis and proportional to the distance to it. For the following calculations, we can consider only bending deformations in the direction of the x axis and then generalize for the other transversal direction y by linear combination. For the beam strains due to bending moments in the x axis we have

$$\epsilon_{zz}(y, z) = \kappa(z)y, \quad (3.5)$$

where $\kappa(z)$ is the beam curvature in the axial coordinate z .

By the Hooke's law of linear elastic materials, if there are only normal stresses and strains the value of σ_{zz} in a point of the beam with coordinates y and z can be evaluated:

$$\sigma_{zz} = E \cdot \epsilon_{zz}(y, z) = E \cdot \kappa y, \quad (3.6)$$

where E is the Young's modulus. κ here is a function of the axial coordinate z .

Substituting Equation 3.6 in Equation 3.3 we obtain the bending moments at each coordinate z :

$$M_{r,x} = \int_A y \sigma_{zz} dA = E \cdot \int_A y \epsilon_{zz} dA = E \cdot \int_A \kappa y^2 dA. \quad (3.7)$$

The final step in the calculation of the bending moments is the evaluation of κ as a function of the radial displacement of the neutral axis, $u_y(z)$. As seen in Figure 3.8, on each beam cross-section with length dz , the infinitesimal of the axial displacement, $\epsilon_{zz} \cdot dz$, of a point on the cross-section with distance y to the neutral axis is a function of the infinitesimal of the beam rotation angle due to bending, $d\varphi_x$:

$$\frac{\epsilon_{zz} \cdot dz}{y} = \tan d\varphi_x. \quad (3.8)$$

FIGURE 3.8: Infinitesimal axial displacement and rotation angle of a beam cross-section.

By the hypothesis of small deformations the rotation angles can be approximated to their sines and cosines, and Equation 3.8 can be rewritten:

$$\varphi \approx 0 \implies \frac{\epsilon_{zz} \cdot dz}{y} \approx d\varphi_x. \quad (3.9)$$

The normal axial strain is now cleared:

$$\epsilon_{zz} = \frac{y \cdot d\varphi_x}{dz}, \quad (3.10)$$

and since $\kappa = \epsilon_{zz}/y$ —Equation 3.5—the beam curvature κ in each axial position z can be approximated to $d\varphi/dz$, which is the bending angular deformation of the beam cross-section in z :

$$\kappa = \frac{d\varphi_x}{dz}. \quad (3.11)$$

As shown in Figure 3.9, if we apply the hypothesis of small deformations we derive a relationship between the rotation angles and the derivatives of the radial displacements:

$$\frac{-du_y(z)}{dz} = \sin \varphi_x \approx \varphi_x. \quad (3.12)$$

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FIGURE 3.9: Relationship between the rotation angle and the infinitesimal radial displacement in a beam cross-section with length dz .

The axial derivation of the two sides of Equation 3.12 yields the following expression:

$$\frac{d\varphi_x}{dz} = \frac{-d^2u_y(z)}{dz^2}. \quad (3.13)$$

Now, κ can be expressed as a function of the radial displacements:

$$\kappa = \frac{-d^2u_y(z)}{dz^2}, \quad (3.14)$$

and substituting Equation 3.14 into Equation 3.7 we have

$$M_{r,x}(z) = E \cdot \int_A -y^2 \frac{d^2u_y(z)}{dz^2} dA = -E \cdot \frac{d^2u_y(z)}{dz^2} \int_A y^2 dA = -EI_x \cdot \frac{d^2u_y(z)}{dz^2}, \quad (3.15)$$

where $\int_A y^2 dA = I_x$ is the beam second moment of area around the x axis.

If this formulation were repeated for bending deformation around the y axis, the resultant internal bending moment would be

$$M_{r,y}(z) = E \cdot \frac{d^2u_x(z)}{dz^2} \int_A y^2 dA = EI_y \cdot \frac{d^2u_x(z)}{dz^2}, \quad (3.16)$$

where, analogously, $\int_A x^2 dA = I_y$ is the second moment of area around the y axis.

If the cross-section is circular, the second moment of area would be the same with respect to any radial direction in the cross-sections: $I_x = I_y = I$.

The final equations that relate radial displacements to external radial loads $F_x(z)$ and $F_y(z)$ on the beam are obtained from the balances of forces and moments on the infinitesimal cross-section—Figure 3.10.

FIGURE 3.10: Balance of forces and moments on an infinitesimal Euler-Bernoulli beam cross section.

We will only describe the balance of forces and moments on the yz plane. The balance on the xz plane is equivalent.

From the balance of moments we get

$$-V_{r,y} \cdot dz + (M_{r,x} + \frac{dM_{r,x}}{dz} \cdot dz) - M_{r,x} = -(F_y(z)dz) \cdot \frac{dz}{2} \implies V_{r,y} = \frac{-dM_{r,x}}{dz}, \quad (3.17)$$

and from the balance of shear forces we have

$$(V_{r,y} + \frac{dV_{r,y}}{dz} \cdot dz) - V_{r,y} = F_y(z) \cdot dz \implies F_y(z) = \frac{dV_{r,y}}{dz}. \quad (3.18)$$

Substituting Equation 3.17 into Equation 3.18 we obtain

$$EI \cdot \frac{d^4 u_y(z)}{dz^4} = F_y(z), \quad (3.19)$$

and equally on the xz plane:

$$EI \cdot \frac{d^4 u_x(z)}{dz^4} = F_x(z). \quad (3.20)$$

Equations 3.19 and 3.20 are the differential equations of the static Euler-Bernoulli beam.

The final dynamical equations require the addition of the time variable to the model mathematical domain. By Newton's second law, inertial forces on the beam

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infinitesimal sections with length dz are proportional to the mass of the infinitesimal section multiplied by its radial accelerations: $d^2u_x(z, t)/dt^2$ and $d^2u_y(z, t)/dt^2$. In the end, the following inertial terms must be added to the static sections:

$$\begin{aligned} \rho A \cdot \frac{d^2u_y(z, t)}{dt^2}, \\ \rho A \cdot \frac{d^2u_x(z, t)}{dt^2}. \end{aligned} \quad (3.21)$$

Hereafter, we use a simpler notation for the partial derivatives of the radial displacements $u_x(z, t)$ and $u_y(z, t)$:

$$\begin{aligned} \frac{d^2u_x(z, t)}{dt^2} &= u_{x,tt}, \\ \frac{d^2u_y(z, t)}{dt^2} &= u_{y,tt}, \\ \frac{d^4u_x(z, t)}{dz^4} &= u_{x,zzzz}, \\ \frac{d^4u_y(z, t)}{dz^4} &= u_{y,zzzz}. \end{aligned} \quad (3.22)$$

The definitive partial differential equations of the radial displacements of the dynamic Euler-Bernoulli beam are

$$\begin{aligned} \rho A \cdot u_{x,tt} + EI \cdot u_{x,zzzz} &= F_x(z, t), \\ \rho A \cdot u_{y,tt} + EI \cdot u_{y,zzzz} &= F_y(z, t). \end{aligned} \quad (3.23)$$

Solving this boundary value problem in the domain $D \ni (z, t)$ requires, for each of the two equations, two time conditions with the form $u(z, t^*)$ or $u_t(z, t^*)$ —where t^* is a fixed value of t —and four Dirichlet— $u(z^*, t)$ —or Newmann—over $u_z(z^*, t)$, $u_{zz}(z^*, t)$ or $u_{zzz}(z^*, t)$ —boundary conditions, where z^* is axial position where the boundary values are fitted.

Once solutions $u_{x,z}$ and $u_{y,z}$ are known, the bending rotation angles, internal bending moments and internal shear forces can be obtained from the axial derivatives of the radial displacements according to Equations 3.12, 3.16 and 3.17:

$$\begin{aligned} \varphi_x &= -u_{y,z}, \quad \varphi_y = u_{x,z}, \\ M_{r,x} &= -EI \cdot u_{y,zz}, \quad M_{r,y} = EI \cdot u_{x,zz}, \\ V_{r,x} &= -EI \cdot u_{y,zzz}, \quad V_{r,y} = EI \cdot u_{x,zzz}. \end{aligned} \quad (3.24)$$

Given a set of boundary condition and distributed forces along the beam, it is very difficult—if not impossible—to find an exact analytical solution of Equation 3.23. Instead, an approximate solution will be found with the application of a *weighted residual method* such as the *Finite Element Method*—Taylor et al., 2000a; Taylor et al., 2000b.

First, for simplicity, Equation 3.23 can be rewritten as a single system by expressing the displacements in both radial directions as a column vector. $\mathbf{u}(z, t)$ will be a vector of exact displacement solutions, and $\tilde{\mathbf{u}}(z, t)$ will be a vector of approximate displacement solutions:

$$\mathbf{u}(z, t) = \begin{pmatrix} u_x(z, t) \\ u_y(z, t) \end{pmatrix} \approx \tilde{\mathbf{u}}(z, t) = \begin{pmatrix} \tilde{u}_x(z, t) \\ \tilde{u}_y(z, t) \end{pmatrix}. \quad (3.25)$$

Then, the two approximate solutions contained in the vector $\tilde{\mathbf{u}}(z, t)$ are expressed as a linear combination of some so-called *nodal displacements* $\tilde{\mathbf{u}}_1(t)$, $\tilde{\mathbf{u}}_2(t)$, $\dots$ $\tilde{\mathbf{u}}_N(t)$ which represent the displacements of a set of N points in the shaft called *nodes* with axial coordinates $z_1, z_2, \dots, z_N$. The contributions of these nodal displacements to the bending deformation of a shaft point with axial coordinate z are weighted by the *trial functions* $\mathbf{N}_1(z)$, $\mathbf{N}_2(z)$, $\dots$ $\mathbf{N}_N(z)$.

Each shaft node has four displacements in four directions called *degrees of freedom*: two radial translations $u_{x,i}(t)$ and $u_{y,i}(t)$, and two bending rotations $u_{\phi_{x,i}}(t)$ and $u_{\phi_{y,i}}(t)$ —Figure 3.11:

$$\tilde{\mathbf{u}}(z, t) = \sum_{i=1}^N \mathbf{N}_i(z) \cdot \tilde{\mathbf{u}}_i(t), \quad \tilde{\mathbf{u}}_i(t) = \begin{pmatrix} u_{x,i}(t) \\ u_{y,i}(t) \\ u_{\phi_{x,i}}(t) \\ u_{\phi_{y,i}}(t) \end{pmatrix}. \quad (3.26)$$

FIGURE 3.11: Nodal displacements and degrees of freedom of a shaft node.

What makes the Finite Element Method—or FEM—different from other approximation methods to solve PDEs—*partial differential equations*—is that the solution domain of a boundary value problem is split into both nodes and *elements*. So, the bending deformation of a shaft point contained in a certain shaft element will depend only on the nodal deformations of the nodes which belong to such element. The resulting formulation is simple and provides a smooth solution. Therefore, when applying the FEM we intend to find an approximate interpolated solution function $\tilde{\mathbf{u}}_{e,i}(z, t)$ for each i -th element of the solution domain.

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The spatial domain of our problem—the neutral axis z —will be broken into a number of nodes and elements. In our particular model, we choose to discretize the shaft into Euler-Bernoulli beam elements with two nodes per element at its extremes—Figure 3.12. The nodes at the element extremes are shared with any contiguous elements attached at the sides.

FIGURE 3.12: Shaft 1D beam element with nodes, length and radius.

In the end, the solution can be approximated in each element as

$$\tilde{\mathbf{u}}_{e,i}(z, t) = \mathbf{N}_{e,i}(z) \cdot \begin{pmatrix} \tilde{\mathbf{u}}_i(t) \\ \tilde{\mathbf{u}}_{i+1}(t) \end{pmatrix}, \quad (3.27)$$

where $\tilde{\mathbf{u}}_i(t)$ and $\tilde{\mathbf{u}}_{i+1}(t)$ are the nodal displacements of the two element nodes i and $i + 1$ at instant t , and $\mathbf{N}_{e,i}(z)$ is the *interpolation matrix* of the element i .

The interpolation matrix is a 2×8 matrix whose elements are *shape functions* that define the shape of the beam under unitary normal displacements of each degree of freedom. Namely, each shape function $N_{e,i}^{j,k}(z)$ in the i -th row and j -th column is the radial displacement function in the i -th direction— x for the upper row elements and y for the elements of the lower row—when the displacement of the j -th degree of freedom is one, and the remaining nodal displacements are zero. These shape functions only depend on z and are calculated integrating the static Euler-Bernoulli equations—Equations 3.19 and 3.20—with the appropriate boundary conditions. The result of each integration is a third degree polynomial in z .

From the boundary value problem defined by Equation 3.23 we can find this FEM approximate solution:

$$\rho A \cdot \mathbf{u}_{tt} + EI \cdot \mathbf{u}_{zzzz} - \mathbf{f}(z, t) = 0 \implies \rho A \cdot \tilde{\mathbf{u}}_{tt} + EI \cdot \tilde{\mathbf{u}}_{zzzz} - \mathbf{f}(z, t) = \mathbf{r}(z, t), \quad (3.28)$$

with the *residual function* $\mathbf{r}(z, t)$ representing the solution approximation error.

After substituting Equation 3.27 into Equation 3.23, we can evaluate the residual vector $\mathbf{r}_{e,i}$ across the solution domain (z, t) of the i -th element:

$$\begin{aligned} \rho A \cdot \mathbf{N}_{e,i}(z) \left(\frac{d^2 \tilde{\mathbf{u}}_i(t)}{dt^2} \right) + EI \cdot \frac{d^4 \mathbf{N}_{e,i}(z)}{dz^4} \left(\tilde{\mathbf{u}}_{i+1}(t) \right) - \mathbf{f}_{e,i}(z, t) &= \mathbf{r}_{e,i}(z, t) \implies \\ \implies \rho A \cdot \mathbf{N}_{e,i} \left(\tilde{\mathbf{u}}_i'' \right) + EI \cdot \frac{d^4 \mathbf{N}_{e,i}}{dz^4} \left(\tilde{\mathbf{u}}_{i+1} \right) - \mathbf{f}_{e,i}(z, t) &= \mathbf{r}_{e,i}(z, t). \end{aligned} \quad (3.29)$$

The challenge is to find a combination of nodal displacements in the nodes i and $i + 1$ that minimizes the norm of the residual vector in the spatial and temporal element domains.

The Galerkin method

There are many residual minimization techniques, each of them resulting in a different approximation. In the Finite Element Method, the minimization of the residual is accomplished when it is orthogonal to each one of the subspace of approximate solutions spanned by the trial functions. In other words, to find the nodal displacements that give an optimal approximation to the real solution of Equation 3.23 we force to zero the inner product of the residual and each trial function. This method is known as the *Galerkin Method* in honor of its discoverer.

Suppose L is a linear differential operator, $u(x)$ is the unknown solution and a domain $x \in V$, and $f(x)$ is a known forcing function in the same domain. Given the differential equation $L(u(x)) = f(x)$ we can approximate its solution by a certain *trial function* $\tilde{u}(x)$ belonging to a certain subspace of functions with domain V . This approximation results in a residual r :

$$L(u(x)) = f(x) \implies L(\tilde{u}(x)) - f(x) = r(\tilde{u}(x)). \quad (3.30)$$

with certain boundary conditions that the trial function $\tilde{u}(x)$ must hold.

The trial functions belong to a function subspace generated by a set of *basis functions* $\phi_j(x)$. Any linear combination of basis functions with coefficients α_j produces a candidate trial solution.

$$\tilde{u}(x) = \sum_{j=1}^{\infty} \alpha_j \cdot \phi_j(x). \quad (3.31)$$

Simplifying the notation and substituting Equation 3.31 into Equation 3.30 we get this expression for the residual:

$$L(\tilde{u}) - f = L\left(\sum_{j=1}^N \alpha_j \cdot \phi_j\right) - f = r(\tilde{u}). \quad (3.32)$$

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As reasoned above, the Galerkin method is based on the orthogonalization of the residual function with the trial functions ϕ_k , forcing their inner product to zero. Applying the usual definition of the inner product for functions, we expand the residual:

$$\forall k, \langle r, \phi_k \rangle = \int_L r \cdot \phi_k dL = \int_L L \left(\sum_{j=1}^{\infty} \alpha_j \cdot \phi_j \right) - \mathbf{f} \cdot \phi_k dL = 0. \quad (3.33)$$

Applying the Galerkin formulation given by Equation 3.33 to Equation 3.29 we derive the following expression:

$$\begin{aligned} \rho A \cdot \int_{L_{e,i}} \mathbf{N}_{e,i}^T \mathbf{N}_{e,i} dz \left(\frac{\tilde{\mathbf{u}}_i''}{\tilde{\mathbf{u}}_{i+1}''} \right) + EI \cdot \int_{L_{e,i}} \mathbf{N}_{e,i}^T \frac{d^4 \mathbf{N}_{e,i}}{dz^4} dz \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) - \\ - \int_{L_{e,i}} \mathbf{N}_{e,i}^T \mathbf{f}_{e,i}(z, t) dz = \mathbf{0}, \end{aligned} \quad (3.34)$$

where the trial functions are the shape functions contained in the interpolation matrices, and the domain of integration is the element length $L_{e,i}$ from node to node.

The first integral term, which is multiplied by the vector of second time derivatives of the nodal displacements, is the *mass matrix* of the i-th element, $\mathbf{M}_{e,i}$:

$$\rho A \cdot \int_{L_{e,i}} \mathbf{N}_{e,i}^T \mathbf{N}_{e,i} dz = \mathbf{M}_{e,i}. \quad (3.35)$$

The second integral term of the expression has a relation with the *stiffness matrix* of the i-th element that we will define further ahead:

$$EI \cdot \int_{L_{e,i}} \mathbf{N}_{e,i}^T \frac{d^4 \mathbf{N}_{e,i}}{dz^4} dz. \quad (3.36)$$

The shape functions that integrate the interpolation matrix are third degree polynomials, therefore differentiating this matrix four times with respect the axial coordinate would annihilate the term 3.36. We can get around this problem by applying integration by parts twice on the integral:

$$\begin{aligned} EI \cdot \int_{L_{e,i}} \mathbf{N}_{e,i}^T \frac{d^4 \mathbf{N}_{e,i}}{dz^4} dz \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) = \\ = EI \cdot \mathbf{N}_{e,i}^T \frac{d^3 \mathbf{N}_{e,i}}{dz^3} \Big|_{\partial L_{e,i}} \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) - EI \cdot \int_{L_{e,i}} \frac{d \mathbf{N}_{e,i}}{dz} \frac{d^3 \mathbf{N}_{e,i}}{dz^3} dz \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) = \\ = EI \cdot \mathbf{N}_{e,i}^T \frac{d^3 \mathbf{N}_{e,i}}{dz^3} \Big|_{\partial L_{e,i}} \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) - EI \cdot \frac{d \mathbf{N}_{e,i}}{dz} \frac{d^2 \mathbf{N}_{e,i}}{dz^2} \Big|_{\partial L_{e,i}} \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right) + \\ + EI \cdot \int_{L_{e,i}} \frac{d^2 \mathbf{N}_{e,i}}{dz^2} \frac{d^2 \mathbf{N}_{e,i}}{dz^2} dz \cdot \left(\frac{\tilde{\mathbf{u}}_i(t)}{\tilde{\mathbf{u}}_{i+1}(t)} \right). \end{aligned} \quad (3.37)$$

This formulation of the Galerkin method is known as the *weak formulation*, and is used to solve many weighted residual problems because of several advantages. One of them is that it allows to incorporate boundary conditions on $\partial L_{e,i}$ very easily before solving the integral.

Now, we can properly define the stiffness matrix of the i -th element:

$$EI \cdot \int_{L_{e,i}} \frac{d^2 \mathbf{N}_{e,i}}{dz^2}^T \frac{d^2 \mathbf{N}_{e,i}}{dz^2} dz = \mathbf{K}_{e,i}. \quad (3.38)$$

Two boundary value vectors result from the application of the weak form formulation. Namely, a vector $\mathbf{V}_{\mathbf{r}_{e,i}}$ of internal shear resultant forces on the element boundaries—the nodes i and $i + 1$ —and a vector of internal bending moments $\mathbf{M}_{\mathbf{r}_{e,i}}$ on the same element nodes:

$$\begin{aligned} EI \cdot \mathbf{N}_{e,i}^T \frac{d^3 \mathbf{N}_{e,i}}{dz^3} \Big|_{\partial L_{e,i}} \cdot \begin{pmatrix} \tilde{\mathbf{u}}_i(t) \\ \tilde{\mathbf{u}}_{i+1}(t) \end{pmatrix} &= \mathbf{V}_{\mathbf{r}_{e,i}} \\ -EI \cdot \frac{d \mathbf{N}_{e,i}}{dz}^T \frac{d^2 \mathbf{N}_{e,i}}{dz^2} \Big|_{\partial L_{e,i}} \cdot \begin{pmatrix} \tilde{\mathbf{u}}_i(t) \\ \tilde{\mathbf{u}}_{i+1}(t) \end{pmatrix} &= \mathbf{M}_{\mathbf{r}_{e,i}} \end{aligned} \quad (3.39)$$

Element matrices $\mathbf{M}_{e,i}$ and $\mathbf{K}_{e,i}$ can be assembled to build the global mass and stiffness matrices of the shaft: $\mathbf{M}_s$ and $\mathbf{K}_s$. When the internal boundary forces vectors $\mathbf{V}_{\mathbf{r}_{e,i}}$ and $\mathbf{M}_{\mathbf{r}_{e,i}}$ of each i -th shaft element are added to a force vector of the whole shaft, the result is a vector of external forces and moments applied on the shaft nodes, $\mathbf{f}_s$, that is integrated to $\mathbf{f}$ along with the external forces that act on the other system components. That is because the internal forces and moments of contiguous elements on the shared node cancel each other out when added, leaving the external loads on the nodes as a remainder.

Due to the shaft assumptions and hypotheses—Section 3.1.2, the coefficients of the resulting mass and stiffness matrix are constants with no linear or non-linear dependencies on either the model degrees of freedom or time.

Remember that we modeled each shaft element as an Euler-Bernoulli beam with two extreme nodes with four degrees of freedom each—radial and bending displacements. Node displacements are measured in relation to an Cartesian orthogonal reference frame where the axis x e y have radial direction, and the axis z has the rotor axial direction—same direction than rotation axis. Positive linear and angular displacements and their derivatives must be congruent with the right hand rule.

Since the shaft rotates around its axial centerline, when it undergoes bending deformation in one transversal direction gyroscopic coupling causes coupling moments that act in the other orthogonal transversal direction. Jones, 2005 describes the derivation of a shaft gyroscopic matrix under the same Euler-Bernoulli model assumptions. This reference also formulates the mass and stiffness matrices of the Euler-Bernoulli beam element, thus saving us from the need of doing the analytical calculations to derive them. We only changed locations and signs of matrix elements to suit our reference frame. The three matrices are shown below:

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FIGURE 3.13: Nodal displacements and degrees of freedom of a shaft node.

$$\mathbf{M}_{e,j} = \frac{\rho_s A_{s,j} L_{e,j}}{420} \cdot \begin{pmatrix} 156 & 0 & 22L_{e,j} & 0 & 54 & 0 & -13L_{e,j} & 0 \\ 0 & 156 & 0 & 22L_{e,j} & 0 & 54 & 0 & -13L_{e,j} \\ 22L_{e,j} & 0 & 4L_{e,j}^2 & 0 & 13L_{e,j} & 0 & -3L_{e,j}^2 & 0 \\ 0 & 22L_{e,j} & 0 & 4L_{e,j}^2 & 0 & 13L_{e,j} & 0 & -3L_{e,j}^2 \\ 54 & 0 & 13L_{e,j} & 0 & 156 & 0 & -22L_{e,j} & 0 \\ 0 & 54 & 0 & 13L_{e,j} & 0 & 156 & 0 & -22L_{e,j} \\ -13L_{e,j} & 0 & -3L_{e,j}^2 & 0 & -22L_{e,j} & 0 & 4L_{e,j}^2 & 0 \\ 0 & -13L_{e,j} & 0 & -3L_{e,j}^2 & 0 & -22L_{e,j} & 0 & 4L_{e,j}^2 \end{pmatrix} \quad (3.40)$$

$$\mathbf{G}_{e,j} = \frac{\rho_s A_{s,j} R_{s,j}^2}{60L_{e,j}} \cdot \begin{pmatrix} 0 & 36 & -3L_{e,j} & 0 & 0 & -36 & -3L_{e,j} & 0 \\ -36 & 0 & 0 & -3L_{e,j} & 36 & 0 & 0 & -3L_{e,j} \\ 3L_{e,j} & 0 & 0 & 4L_{e,j}^2 & -3L_{e,j} & 0 & 0 & -L_{e,j}^2 \\ 0 & 3L_{e,j} & -4L_{e,j}^2 & 0 & 0 & -3L_{e,j} & L_{e,j}^2 & 0 \\ 0 & -36 & 3L_{e,j} & 0 & 0 & 36 & 3L_{e,j} & 0 \\ 36 & 0 & 0 & 3L_{e,j} & -36 & 0 & 0 & 3L_{e,j} \\ 3L_{e,j} & 0 & 0 & -L_{e,j}^2 & -3L_{e,j} & 0 & 0 & 4L_{e,j}^2 \\ 0 & 3L_{e,j} & L_{e,j}^2 & 0 & 0 & -3L_{e,j} & -4L_{e,j}^2 & 0 \end{pmatrix} \quad (3.41)$$

$$\mathbf{K}_{e,j} = \frac{EI}{L_{e,j}^3} \cdot \begin{pmatrix} 12 & 0 & 6L_{e,j} & 0 & -12 & 0 & 6L_{e,j} & 0 \\ 0 & 12 & 0 & 6L_{e,j} & 0 & -12 & 0 & 6L_{e,j} \\ 6L_{e,j} & 0 & 4L_{e,j}^2 & 0 & -6L_{e,j} & 0 & 2L_{e,j}^2 & 0 \\ 0 & 6L_{e,j} & 0 & 4L_{e,j}^2 & 0 & -6L_{e,j} & 0 & 2L_{e,j}^2 \\ -12 & 0 & -6L_{e,j} & 0 & 12 & 0 & -6L_{e,j} & 0 \\ 0 & -12 & 0 & -6L_{e,j} & 0 & 12 & 0 & -6L_{e,j} \\ 6L_{e,j} & 0 & 2L_{e,j}^2 & 0 & -6L_{e,j} & 0 & 4L_{e,j}^2 & 0 \\ 0 & 6L_{e,j} & 0 & 2L_{e,j}^2 & 0 & -6L_{e,j} & 0 & 4L_{e,j}^2 \end{pmatrix} \quad (3.42)$$

Assembly of the shaft stiffness and mass matrices Finally, we need to assemble the shaft mass and stiffness matrices— $\mathbf{M}_S$ and $\mathbf{K}_S$ —by adding up the mass and stiffness matrices of each shaft element. Notice that the dimensions of each shaft element matrix is 8×8 , since each of the two shaft nodes has four degrees of freedom.

In Section 3.1.3 shaft mass and stiffness matrices will be assembled to the global rotor-casing system matrices.

Derivation of the disks equations

Our starting point for the derivation of the disks equations will be the disk hypotheses enumerated in Section 3.1.3.

A local system of coordinates is fixed on the disk center of mass with axes z' —which follows the disk centerline—and radial orthogonal axes x' and y' . These coordinate system must coincide with the global frame (x, y, z) when the disk is assembled on any of the nodes of the undeformed shaft. The disk has, as each of the shaft nodes, four degrees of freedom: three translational displacements in the Cartesian directions and two rotations around the radial axis—Figure 3.14. The disk rotates around the axis z' with constant speed Ω , with the sign criteria already described at the beginning of this section.

FIGURE 3.14: Degrees of freedom of a rotor disk.

The equations of translational motion are obtained from the Newton's second law of motion in the global frame:

$$\begin{aligned} M_d \cdot u_x'' &= F_x, \\ M_d \cdot u_y'' &= F_y, \end{aligned} \tag{3.43}$$

where M_d is the disk mass, u_x'' and u_y'' are the disk accelerations in the two radial directions, and F_x and F_y are the radial forces on the disk, all of them expressed in the global frame.

For the derivation of the equations of angular motion, we must bear in mind the disk hypotheses, which we remind below:

- The transverse moment of inertia I_T is the same in all radial directions due to the disk lateral symmetry.
- Small angular displacements: $\varphi_x \approx 0$, $\varphi_y \approx 0$. Consequently, the angular motion effects will be linear.
- The axial rotation speed Ω remains constant, and no axial—external or inertial—torques exist on the disk.

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The diagram in Figure 3.15 shows a disk that rotates around the radial directions on its center of gravity. The polar angular momentum $I_P\Omega$ is projected—considering small deformations—on the global radial axis, and these projections are then added to the transverse angular momenta to yield the total angular momenta in x and y :

$$\begin{aligned} L_x &= I_T \cdot u'_{\phi_x} + \Omega I_P \cdot u_{\phi_y}, \\ L_y &= I_T \cdot u'_{\phi_y} - \Omega I_P \cdot u_{\phi_x}, \end{aligned} \quad (3.44)$$

FIGURE 3.15: Angular momenta of a pivoting disk.

with I_P being the disk's polar moment of inertia around the Ω axis. By the Newton's second law of motion, the derivatives of the angular momenta are equal to the sum of the external radial torques applied on the disk, M_x and M_y . The result is the following set of equations of radial angular motion:

$$\begin{aligned} \frac{L_x}{dt} &= I_T \cdot u''_{\phi_x} + \Omega I_P \cdot u'_{\phi_y} = M_x, \\ \frac{L_y}{dt} &= I_T \cdot u''_{\phi_y} - \Omega I_P \cdot u'_{\phi_x} = M_y, \end{aligned} \quad (3.45)$$

with u'_{ϕ_x} and u'_{ϕ_y} being the velocities of the disk radial rotations.

The four disk equations of motion—the two translational equations and the two angular equations—can be expressed in matrix form:

$$\begin{pmatrix} M_d & & & \\ & M_d & & \\ & & I_T & \\ & & & I_T \end{pmatrix} \begin{pmatrix} u''_x \\ u''_y \\ u''_{\phi_x} \\ u''_{\phi_y} \end{pmatrix} + \Omega \cdot \begin{pmatrix} & & & \\ & & & \\ & & I_P & \\ -I_P & & & \end{pmatrix} \begin{pmatrix} u'_x \\ u'_y \\ u'_{\phi_x} \\ u'_{\phi_y} \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \\ M_x \\ M_y \end{pmatrix}. \quad (3.46)$$

The first matrix is called the disk *mass matrix*, and the second is a *matrix of gyroscopic effects*. The gyroscopic effects acting on a disk are not different to the gyroscopic effects on the shaft—Equation 3.41. It is, in short, a coupling between the angular motions in the two orthogonal radial directions whose intensity is proportional to the magnitude of I_T . For more on gyroscopic effects the reader can look down in Muszynska, 2005a. The global gyroscopic effects term of the global rotor-casing model equations will contain contributions from the inertias of disks and rotor shaft and will be proportional to the rotation speed Ω and the node displacement velocities.

As a result, the damping component of the global rotor system—Equation 3.1—can be split into two terms:

$$\mathbf{D}\mathbf{u}' = (\mathbf{C} + \Omega \cdot \mathbf{G})\mathbf{u}', \quad (3.47)$$

where $\mathbf{C}$ is the component of the damping matrix that represents viscous, dissipative damping from shaft—structural damping, bearings and supports; and $\mathbf{G}$ is the global matrix of gyroscopic effects.

A more detail description of the calculation of the angular motion equations of the disks can be found in Muszynska, 2005a.

The mass and gyroscopic matrices of each disk, which can be represented as $\mathbf{M}_D$ and $\mathbf{G}_D$, must be added to the corresponding positions of the shaft matrices $\mathbf{M}_S$ and $\mathbf{G}_S$, resulting in a pair of full rotor matrices $\mathbf{M}_{S+D}$ and $\mathbf{G}_{S+D}$ representing the rotor assembly with shaft and disks.

Derivation for the casing equations

Element dimensions, coordinates and shape functions As explained in Section 3.1.2, the casing was modeled as a cylindrical—or semi-cylindrical, axisymmetric shell mounted around the rotor shaft. This shell has a constant thickness h , with a mid-surface that is equidistant to the upper and lower surfaces. The casing radius between its mid-surface and the rotor centerline is R_c . Applying the Finite Element Method, the cylindrical shell can be axisymmetrically split into a set of

FIGURE 3.16: Nodes 1 to 6, dimensions and boundary coordinates of the mid-surface of a shell casing element.

FIGURE 3.17: Degenerate 3D casing element, with thickness directions given by normal unitary vectors at each node of the element mid-surface.

six-node elements whose mid-surface has the dimensions shown in Figure 3.16 in cylindrical coordinates: transversal boundary angles θ_e and θ_i , radius R and axial boundary coordinates z_e and z_i . This element has six nodes, all on the mid-surface boundary, and shared with any surrounding elements. There are four nodes on each one of the mid-surface corners—numerated 1, 3, 4 and 6—and two additional nodes—numerated 2 and 5—that enable isoparametric modeling in the curved transversal direction.

We derive for each shell element discrete stiffness and mass matrices with the Finite Element Method. A method for the analysis of curved shells with curved Finite Elements using isoparametric elements is described in Ahmad et al., 1970 and Taylor et al., 2000b. We will adapt that method in this study for the modeling and discretization of the casing with our particular element choice. This method models each casing shell element as a degenerate 3D element with no deformation in the direction normal to the mid-surface and with the consideration of the Mindlin-Reissner assumptions—Section 3.1.2. The advantage of this methodology is that it allows the analysis of curved or flat shells that are too thin to be numerically analyzed as 3D structures without risking the numerical imprecisions caused by ill-conditioned element matrices—since the stiffness in the planar direction is significantly greater than in the normal direction.

Figure 3.17 shows the degenerate 3D casing element. For each point on the mid-surface there is a thickness direction normal to such surface. Thickness h is constant at every mid-surface point. The nodal coordinates and displacements of any point in the three global Cartesian directions— (x_i, y_i, z_i) and (u_i, v_i, w_i) , respectively—can be estimated by interpolating of displacements and deformations at the nodes, expressed in curvilinear coordinates (ξ, η, ζ) .

The Cartesian global coordinates of any point of the element are given by the expression

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sum_{\forall i} N_i(\xi, \eta) \cdot \left[\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + \zeta \cdot \frac{h}{2} \cdot \mathbf{v}_{2,i} \right], \quad (3.48)$$

where $N_i(\xi, \eta)$ is the shape function of the i -th node. These shape functions depend on curvilinear coordinates tangent to the element mid-surface, ξ and η . A third curvilinear coordinate, ζ , determines the position of the element point along its thickness direction, defined by an unitary normal vector $\mathbf{v}_{2,i}$ on each i -th node. h is the shell thickness.

The global Cartesian displacement field of a point of the shell element—under small deflections—is defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{\forall i} N_i(\xi, \eta) \cdot \left[\begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} + \xi \cdot \frac{h}{2} \cdot (\mathbf{v}_{3,i} - \mathbf{v}_{1,i}) \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \right], \quad (3.49)$$

where u_i , v_i , w_i , α_i and β_i are the nodal displacements of the five degrees of freedom of each node i on the mid-surface: three translational displacements, u_i , v_i and w_i , expressed in global frame coordinates; and two rotations, α_i and β_i , of the node normal unitary vector around orthogonal directions tangential to the mid-surface. $\mathbf{v}_{3,i}$ and $\mathbf{v}_{1,i}$ are tangential unitary vectors that define the rotation directions. This definition of the displacement field is congruent with the shell theory assumption that the surface normal lines remain straight and undeformed after shell deformation.

Local coordinates and unitary vectors on the shell mid-surface, as well as the numeration of the element nodes, are shown in Figure 3.19. The origin of the local frame is fixed on the mid-surface. the y^* axis is normal to the mid-surface, and the x^* and z^* axes are tangential to it. The z^* axis remains parallel to the axial global axis z . Positive directions of bending rotations around the tangential unitary vectors are also shown. Figure 3.18 shows the curvilinear coordinate values at each node on the mid-surface, those on the element faces, and the coordinate value ξ on the upper and lower surfaces. It is assumed that the curvilinear coordinate ranges between -1 and 1 from element face to face.

From the curvilinear coordinates values of the nodes in Figure 3.18 we define over the element mid-surface the following shape functions for each node. Since the six functions are quadratic, the mid-surface that results when interpolating the six nodes with Equation 3.48 is parabolic, and after the thickness interpolation the resulting element is parabolic in both radial and transversal directions.

$$\begin{aligned} N_1 &= \frac{1}{4} \cdot (\xi + 1)(\eta + 1)\eta, & N_2 &= \frac{1}{2} \cdot (\xi + 1)(1 - \eta)(1 + \eta), \\ N_3 &= \frac{1}{4} \cdot (\xi + 1)(\eta - 1)\eta, & N_4 &= \frac{1}{4} \cdot (1 - \xi)(\eta + 1)\eta, \\ N_5 &= \frac{1}{2} \cdot (1 - \xi)(1 - \eta)(1 + \eta), & N_6 &= \frac{1}{4} \cdot (1 - \xi)(\eta - 1)\eta. \end{aligned} \quad (3.50)$$

FIGURE 3.18: Curvilinear coordinates of the element mid-surface nodes.

FIGURE 3.19: Local coordinate frame at a point of the the element mid-surface.

Considering a cylindrical or semi-cylindrical casing with axis z and the element dimensions in 3.16, The global Cartesian coordinates of the six nodes are

$$\begin{aligned} \mathbf{P}_1 &= \begin{pmatrix} R_c \cdot \cos \theta_i \\ R_c \cdot \sin \theta_i \\ z_i \end{pmatrix}, \quad \mathbf{P}_2 = \begin{pmatrix} R_c \cdot \cos \frac{\theta_e + \theta_i}{2} \\ R_c \cdot \sin \frac{\theta_e + \theta_i}{2} \\ z_i \end{pmatrix}, \quad \mathbf{P}_3 = \begin{pmatrix} R_c \cdot \cos \theta_e \\ R_c \cdot \sin \theta_e \\ z_i \end{pmatrix}, \\ \mathbf{P}_4 &= \begin{pmatrix} R_c \cdot \cos \theta_i \\ R_c \cdot \sin \theta_i \\ z_e \end{pmatrix}, \quad \mathbf{P}_5 = \begin{pmatrix} R_c \cdot \cos \frac{\theta_e + \theta_i}{2} \\ R_c \cdot \sin \frac{\theta_e + \theta_i}{2} \\ z_e \end{pmatrix}, \quad \mathbf{P}_6 = \begin{pmatrix} R_c \cdot \cos \theta_e \\ R_c \cdot \sin \theta_e \\ z_e \end{pmatrix}. \end{aligned} \quad (3.51)$$

The interpolation of the positions and displacements across the element necessitates the definition of unitary vectors at the nodes. $\mathbf{v}_{2,i}$ is the normal-to-mid-surface unitary vector of node i ; it follows the thickness direction and is calculated as the cross product of the partial derivatives of the mid-surface position function—Equation 3.48—with respect to ξ and η .

Then, the two remaining unitary vectors of the mid-surface can be easily calculated as follows:

$$\begin{aligned} \mathbf{V}_2 &= \begin{pmatrix} x_\xi \\ y_\xi \\ z_\xi \end{pmatrix} \times \begin{pmatrix} x_\eta \\ y_\eta \\ z_\eta \end{pmatrix} = \begin{pmatrix} y_\xi z_\eta - y_\eta z_\xi \\ z_\xi x_\eta - z_\eta x_\xi \\ x_\xi y_\eta - x_\eta y_\xi \end{pmatrix} \Rightarrow \mathbf{v}_2 = \frac{\mathbf{V}_2}{\|\mathbf{V}_2\|}, \\ \mathbf{V}_1 &= \mathbf{V}_2 \times \mathbf{k} = \mathbf{V}_2 \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \frac{\mathbf{V}_1}{\|\mathbf{V}_1\|}, \quad \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2. \end{aligned} \quad (3.52)$$

Constitutive equations, stiffness and mass matrices of a casing element
According to the Mindlin-Reissner plate theory, the differential shell element is subject to normal stresses in the local tangential—also called *in-plane*—directions, and shear stresses in the local normal and in-plane directions. Normal stresses in the local normal direction are negligible in shell and plate theory. The vectors of local stresses

and strains in a Mindlin-Reissner shell are related by a linear *constitutive equation*, with the following *constitutive matrix* $\mathbf{D}^*$ relating local stress and strains vectors $\boldsymbol{\sigma}^*$ and $\boldsymbol{\epsilon}^*$, both in the local coordinate system:

$$\boldsymbol{\sigma}^* = \begin{pmatrix} \sigma_{x^*}^* \\ \sigma_{z^*}^* \\ \tau_{x^*z^*}^* \\ \tau_{x^*y^*}^* \\ \tau_{y^*z^*}^* \end{pmatrix} = \mathbf{D}^* \cdot (\boldsymbol{\epsilon}^* - \boldsymbol{\epsilon}_0^*), \quad (3.53)$$

with $\boldsymbol{\epsilon}_0^*$ being the vector of initial strains due to factors such as thermal expansion or residual strains and stresses.

Considering that the linear elastic Hooke's law holds in the casing model and according to Mindlin-Reissner plate theory, the constitutive matrix $\mathbf{D}^*$ is defined as

$$\mathbf{D}^* = \frac{E}{1-\nu^2} \cdot \begin{pmatrix} 1 & \nu & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 \\ & & & \kappa(1-\nu) & 0 \\ & & & & \kappa(1-\nu) \end{pmatrix}, \quad (3.54)$$

where ν is the material Poisson's ratio and κ is the material ratio of relevant strain energies or *Timoshenko's shear coefficient*. Ahmad et al., 1970 proposes a value κ of 5/6 for an optimal shear distribution. $\mathbf{D}^*$ is a symmetrical matrix: elements under the matrix diagonal have been neglected for clarity.

If we considered the Kirchhoff-Love hypothesis of normal straights remaining normal to the surfaces after the deformation, normal shear strains and stresses— $\tau_{x^*y^*}$, $\tau_{y^*z^*}$, $\sigma_{x^*y^*}$ and $\sigma_{y^*z^*}$ —would be neglected. In the constitutive Equation 3.1.3, negligible shear deformations are attained by setting a zero value of κ .

Strains in a plate or shell expressed in the local coordinate system can be written as a function of the partial derivatives of the shell local displacement field, also in local coordinate frame:

$$\boldsymbol{\epsilon}^* = \begin{pmatrix} \epsilon_{x^*}^* \\ \epsilon_{z^*}^* \\ \gamma_{x^*z^*}^* \\ \gamma_{x^*y^*}^* \\ \gamma_{y^*z^*}^* \end{pmatrix} = \begin{pmatrix} \partial u^*/\partial x^* \\ \partial w^*/\partial z^* \\ 1/2 \cdot (\partial u^*/\partial z^* + \partial w^*/\partial x^*) \\ 1/2 \cdot (\partial u^*/\partial y^* + \partial v^*/\partial x^*) \\ 1/2 \cdot (\partial v^*/\partial z^* + \partial w^*/\partial y^*) \end{pmatrix}. \quad (3.55)$$

The shell element stiffness matrix is defined in the global reference frame as

$$\mathbf{K}_e = \int_{V_e} \mathbf{B}_e^T \mathbf{D} \mathbf{B}_e \, dx dy dz, \quad (3.56)$$

where $\mathbf{B}_e$ is a matrix that relates the vector of displacements of the element nodes with the strain field of any point of the shell element, both expressed in the global coordinate frame:

$$\boldsymbol{\epsilon} = \mathbf{B}_e(\xi, \eta, \zeta) \cdot \mathbf{u}_e. \quad (3.57)$$

Notice that the matrix $\mathbf{B}_e$ is a function of the three curvilinear coordinates.

3.1. A non-linear model for the simulation of single-point rub

Since the constitutive equation is defined in local coordinates, we need to convert the $\mathbf{B}_e$ and $\mathbf{D}$ matrices from global to the local frame at each mid-surface point to be able to calculate the stiffness and mass element matrices. The new stiffness matrix in local coordinates is

$$\mathbf{K}_e = \int_{V_e} \mathbf{B}_e^{*T} \mathbf{D}^* \mathbf{B}_e^* \cdot \det \mathbf{J}(\xi, \eta, \zeta) d\xi d\eta d\zeta, \quad (3.58)$$

where $\mathbf{J}(\xi, \eta, \zeta)$ is the Jacobian matrix of the operator relating global and curvilinear coordinates of any point in the element—Equation 3.48. This Jacobian is composed of the partial derivatives of Equation 3.48:

$$\mathbf{J}(\xi, \eta, \zeta) = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{pmatrix} = \begin{pmatrix} x_\xi & y_\xi & z_\xi \\ x_\eta & y_\eta & z_\eta \\ x_\zeta & y_\zeta & z_\zeta \end{pmatrix}. \quad (3.59)$$

One approach to calculate the stiffness matrix with Equation 3.1.3 is to carry out numerical piece-wise integration throughout the element volume in the three curvilinear coordinates. In that case, the Jacobian 3.59 must be recalculated at each step. However, notice that Equation 3.48 is easy to differentiate analytically and therefore the calculation of the Jacobian 3.59 is not quite time-consuming.

Calculation of the $\mathbf{B}_e$ matrix The $\mathbf{B}_e$ matrix must be calculated at every integration step of the stiffness matrix, but once again, Ahmad et al., 1970 offers the following analytical formulation for this task.

First, the shell local stress field is expressed as the product of a differential operator $\mathbf{S}$ —or $\mathbf{S}^*$ in local coordinates—and the displacement field vector:

$$\begin{aligned} \boldsymbol{\epsilon}^* &= \begin{pmatrix} \epsilon_{x^*} \\ \epsilon_{z^*} \\ \gamma_{x^*z^*} \\ \gamma_{x^*y^*} \\ \gamma_{y^*z^*} \end{pmatrix} = \begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial w^*}{\partial z^*} \\ 1/2 \cdot (\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}) \\ 1/2 \cdot (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}) \\ 1/2 \cdot (\frac{\partial v^*}{\partial z^*} + \frac{\partial w^*}{\partial y^*}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x^*} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z^*} \\ \frac{1}{2} \cdot \frac{\partial}{\partial z^*} & 0 & \frac{1}{2} \cdot \frac{\partial}{\partial x^*} \\ \frac{1}{2} \cdot \frac{\partial}{\partial y^*} & \frac{1}{2} \cdot \frac{\partial}{\partial x^*} & 0 \\ 0 & \frac{1}{2} \cdot \frac{\partial}{\partial z^*} & \frac{1}{2} \cdot \frac{\partial}{\partial y^*} \end{pmatrix} \cdot \begin{pmatrix} u^* \\ v^* \\ w^* \end{pmatrix} \\ &\implies \boldsymbol{\epsilon}^* = \mathbf{S}^* \cdot \mathbf{u}^*. \end{aligned} \quad (3.60)$$

Then, the global displacement field function—Equation 3.49—is written as the product of an element shape function matrix $\mathbf{N}_e$ and the vector of the nodal displacements of the six element nodes, $\mathbf{u}_e$:

$$\begin{aligned} \mathbf{N}_e \cdot \mathbf{u}_e &= \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{u}, \\ \mathbf{u}_e &= (\mathbf{u}_{e,1} \dots \mathbf{u}_{e,6})^T, \quad \mathbf{u}_{e,i} = (u_i, v_i, w_i, \alpha_i, \beta_i)^T. \end{aligned} \quad (3.61)$$

The shape function matrix $\mathbf{N}_e(\xi, \eta, \zeta)$ is an assembly of six column matrices $\mathbf{N}_{e,i} \in \mathbb{R}^{3 \times 5}$, one for each node i :

$$\mathbf{N}_e = [\mathbf{N}_{e,1} | \dots | \mathbf{N}_{e,i} | \dots | \mathbf{N}_{e,6}], \quad \mathbf{N}_{e,i} = N_i(\xi, \eta) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} \xi \cdot \frac{h}{2} \mathbf{v}_{3i} \\ -\zeta \cdot \frac{h}{2} \mathbf{v}_{1i} \end{vmatrix}. \quad (3.62)$$

Substituting Equation 3.61 into Equation 3.55 we can derive the analytical expression of $\mathbf{B}_e$:

$$\boldsymbol{\epsilon} = \mathbf{S} \cdot \mathbf{u} = \mathbf{S} \cdot \mathbf{N}_e \cdot \mathbf{u}_e = \mathbf{B}_e \cdot \mathbf{u}_e \implies \mathbf{B}_e = \mathbf{S} \cdot \mathbf{N}_e. \quad (3.63)$$

It turns out that each column of matrix $\mathbf{B}_e$ is the derivative of the shape function sub-matrix $\mathbf{N}_{e,i}$.

$$\begin{aligned} \mathbf{B}_e = \mathbf{S} \cdot \mathbf{N}_e &\implies [\mathbf{B}_{e,1} | \dots | \mathbf{B}_{e,i} | \dots | \mathbf{B}_{e,6}] = \mathbf{S} \cdot [\mathbf{N}_{e,1} | \dots | \mathbf{N}_{e,i} | \dots | \mathbf{N}_{e,6}] \implies \\ &\implies \mathbf{B}_{e,i} = \mathbf{S} \cdot \mathbf{N}_{e,i}, \quad \mathbf{B}_{e,i} \in \mathbb{R}^{3 \times 5}. \end{aligned} \quad (3.64)$$

However, in our formulation, we need the matrix $\mathbf{B}_e^*$ relating global nodal displacements and the local strain field of a point of the shell. This matrix must be evaluated for every element point with local coordinates (ξ, η, ζ) with the expression

$$\boldsymbol{\epsilon}^* = \mathbf{B}_e^* \cdot \mathbf{u}_e. \quad (3.65)$$

Notice from 3.1.3 that each of the columns of $\mathbf{B}^*(\xi, \eta, \zeta)$ is the local strain field caused at point (ξ, η, ζ) by an unitary deformation of each of the degrees of freedom in the vector $\mathbf{u}_e$ while the remaining degrees of freedom have displacement zero. Therefore, for each element point (ξ, η, ζ) we need to calculate the local strain fields for successive unitary displacements of the degrees of freedom in order to assemble the matrix $\mathbf{B}^*$.

The calculation of these local field strains requires the following operations: first, we use the partial derivatives of Equation 3.49 and the inverse of the Jacobian 3.59 to calculate a matrix of partial derivatives of the global displacements of the point (ξ, η, ζ) with respect to the global coordinate frame:

$$\begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} = \mathbf{J}(\xi, \eta, \zeta)^{-1} \cdot \begin{pmatrix} u_\xi & v_\xi & w_\xi \\ u_\eta & v_\eta & w_\eta \\ u_\zeta & v_\zeta & w_\zeta \end{pmatrix} \quad (3.66)$$

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We need partial derivatives of the local displacements (x, y, z) with respect to the local reference frame (x^*, y^*, z^*) . To obtain those we rotate the matrix of partial derivatives of the global displacements with respect to the global coordinate frame, using a rotation matrix Θ whose columns are the three local unitary vectors at point (ξ, η, ζ) :

$$\begin{pmatrix} u_{x^*}^* & v_{x^*}^* & w_{x^*}^* \\ u_{y^*}^* & v_{y^*}^* & w_{y^*}^* \\ u_{z^*}^* & v_{z^*}^* & w_{z^*}^* \end{pmatrix} = \Theta^T \cdot \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \cdot \Theta, \quad \Theta = [\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3]. \quad (3.67)$$

Finally, we can apply Equation 3.55 to calculate the local strain vector ϵ^* under unitary displacement of the i -th element degree of freedom, and therefore the i -th column of matrix $\mathbf{B}_e^*$. Integrating numerically throughout the element domain we can now calculate its stiffness matrix with Equation 3.1.3.

At last, we calculate the element mass matrix. The following expression in global coordinates of the mass matrix of a shell element $\mathbf{M}_e$ is given by reference Taylor et al., 2000a and coincides with the most general expression of the mass matrix of a mechanical system derived by Finite Elements:

$$\mathbf{M}_e = \rho \cdot \int_{V_e} \mathbf{N}_e^T \mathbf{N}_e \, dx dy dz \quad (3.68)$$

Since $\mathbf{N}_e$ is a function of the curvilinear coordinates of each element point and these coordinates are much comfortable to work with, we can render the calculation of the element mass matrix easier if we carry out its integration in local coordinates:

$$\mathbf{M}_e = \rho \cdot \int_{V_e} \mathbf{N}_e^T \mathbf{N}_e \cdot \det \mathbf{J}(\xi, \eta, \zeta) \, d\xi d\eta d\zeta, \quad (3.69)$$

where $\mathbf{N}_e$ is the element shape function matrix, and ρ is the shell density.

Assembly of the casing stiffness and mass matrices The final step in the application of the Finite Element Method to the casing is the assembly of the casing mass and stiffness matrices— $\mathbf{M}_C$ and $\mathbf{K}_C$ —from the mass and stiffness matrices of each casing element. Notice that the dimensions of each element matrix is 30×30 : six nodes per element, and five degrees of freedom per node.

If we modeled the casing as an union of two semi-cylindrical halves, the casing matrices would be each composed of two sub-matrices: one for the upper half, and another for the lower half:

$$\mathbf{M}_C = \begin{bmatrix} \mathbf{M}_{CU} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{CL} \end{bmatrix}, \quad \mathbf{K}_C = \begin{bmatrix} \mathbf{K}_{CU} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{CL} \end{bmatrix}. \quad (3.70)$$

In our model, there are neither forces nor deformations in the z direction, therefore we can reduce matrix side by dropping all rows and columns associated with the z degree of freedom in the casing matrices.

In Section 3.1.3 we will add the casing mass and stiffness matrices to the global rotor-casing system matrices.

Bearing stiffness and damping matrices

If we suppose linearity, visco-elasticity and stationarity, each model bearing—violet in Figure 3.1, marked as B_1 and B_2 — can be characterized by the following constant stiffness and damping matrices:

$$\mathbf{K}_b = \begin{pmatrix} K_{x,x} & K_{x,y} & 0 & 0 \\ K_{y,x} & K_{y,y} & 0 & 0 \\ 0 & 0 & K_{\theta_x} & 0 \\ 0 & 0 & 0 & K_{\theta_y} \end{pmatrix}, \quad \mathbf{C}_b = \begin{pmatrix} C_{x,x} & C_{x,y} & 0 & 0 \\ C_{y,x} & C_{y,y} & 0 & 0 \\ 0 & 0 & C_{\theta_x} & 0 \\ 0 & 0 & 0 & C_{\theta_y} \end{pmatrix}. \quad (3.71)$$

The indices of each matrix element are, respectively, the shaft node coordinate and the bearing node coordinate which the element couples. Bearings and shaft share only the x and y degrees of freedom, therefore elastic coupling between the two is possible only through these to radial translational degrees of freedom. Bearing bending stiffnesses K_{θ_x} , K_{θ_y} , C_{θ_x} and C_{θ_y} cannot be coupled with the motion of the bearing, and therefore with the stator. Cross-coupled terms $K_{x,y}$, $K_{y,x}$, $C_{x,y}$ and $C_{y,x}$ are in general equal and the matrices are symmetrical. The coupling between the radial and the angular—bending—deformations can be neglected due to short bearing hypothesis—Kirk et al., 1976. When assembling the global matrices—Section 3.1.3—each element of $\mathbf{K}_b$ and $\mathbf{C}_b$ must be added to them accordingly.

Stiffness and damping matrices of the bearing-casing joints

Figure 3.1b shows that the unions between each bearing node and the casing— BC_1 and BC_2 , yellow—are modeled as star-shaped trusses composed of beams that either bind their ends to two neighboring casing nodes or connect the bearing node to each of the casing nodes contained in the same radial plane than the bearing. These beams have two degrees of freedom at the extremes: two Cartesian displacements in x and y .

For simplicity, we can consider that beam stiffness and damping are equal in x and y and that there are no elastic couplings between the two axes. Therefore, their elementary stiffness and damping matrices are

$$\mathbf{K}_{bc} = \begin{pmatrix} K & 0 & -K & 0 \\ 0 & K & 0 & -K \\ -K & 0 & K & 0 \\ 0 & -K & 0 & K \end{pmatrix}, \quad \mathbf{C}_{bc} = \begin{pmatrix} C & 0 & -C & 0 \\ 0 & C & 0 & -C \\ -C & 0 & C & 0 \\ 0 & -C & 0 & C \end{pmatrix}, \quad (3.72)$$

where K and C are stiffness and damping coefficients.

Since the beams conforming the bearing-casing joints are usually rotated with respect to the axes x and y , these matrices must be transformed before their assembly in the global matrices. Let θ be the beam angle with respect to the x axis, measured in the counterclockwise direction. Then, we define a rotation matrix Θ as

$$\Theta = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}. \quad (3.73)$$

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The rotated stiffness and damping matrices are

$$\mathbf{K}'_{bc} = \mathbf{\Theta}^T \cdot \mathbf{K}_{bc} \cdot \mathbf{\Theta}, \quad \mathbf{C}'_{bc} = \mathbf{\Theta}^T \cdot \mathbf{C}_{bc} \cdot \mathbf{\Theta}. \quad (3.74)$$

$\mathbf{K}'_{bc}$ and $\mathbf{C}'_{bc}$ are then assembled in the global stiffness and damping matrices.

Stiffness and damping matrices of the shaft coupling

The linear stiffness and damping of the shaft coupling are introduced into the model equations by adding to the diagonals of the global stiffness and damping matrices the corresponding terms. These are contained in the following matrices:

$$\mathbf{K}_c = \begin{pmatrix} K_x & 0 & 0 & 0 \\ 0 & K_y & 0 & 0 \\ 0 & 0 & K_{\theta_x} & 0 \\ 0 & 0 & 0 & K_{\theta_y} \end{pmatrix}, \quad \mathbf{C}_c = \begin{pmatrix} C_x & 0 & 0 & 0 \\ 0 & C_y & 0 & 0 \\ 0 & 0 & C_{\theta_x} & 0 \\ 0 & 0 & 0 & C_{\theta_y} \end{pmatrix}, \quad (3.75)$$

with x, y, θ_x and θ_y being the shaft degrees of freedom.

Stiffness and damping matrices of the casing supports

If we drop the z degree of freedom from the casing matrices we have left with four degrees of freedom to describe the casing deformations: x, y, α and β . We can define, for each casing support— CF_1 to CF_4 in Figure 3.1—stiffness and damping coefficients in each of the four degrees of freedom. These are contained in the following matrices:

$$\mathbf{K}_f = \begin{pmatrix} K_x & 0 & 0 & 0 \\ 0 & K_y & 0 & 0 \\ 0 & 0 & K_\alpha & 0 \\ 0 & 0 & 0 & K_\beta \end{pmatrix}, \quad \mathbf{C}_f = \begin{pmatrix} C_x & 0 & 0 & 0 \\ 0 & C_y & 0 & 0 \\ 0 & 0 & C_\alpha & 0 \\ 0 & 0 & 0 & C_\beta \end{pmatrix}. \quad (3.76)$$

The coefficients must be added to the diagonal elements of the global matrices corresponding to the casing nodes which are attached to the ground.

Stiffness and damping matrices of the unions between the casing halves

The linear visco-elastic unions between the casing halves—when the casing is modeled as two symmetrical casing halves—involve the four degrees of freedom of each of the casing nodes. The couplings between the two parts are between one node of the upper half and another of the lower half.

The stiffness and damping coefficients of these couplings are represented in these matrices:

$$\mathbf{K}_{cc} = \begin{pmatrix} K_{x,x} & K_{x,y} & K_{x,\alpha} & K_{x,\beta} \\ K_{y,x} & K_{y,y} & K_{y,\alpha} & K_{y,\beta} \\ K_{\alpha,x} & K_{\alpha,y} & K_{\alpha,\alpha} & K_{\alpha,\beta} \\ K_{\beta,x} & K_{\beta,y} & K_{\beta,\alpha} & K_{\beta,\beta} \end{pmatrix}, \quad \mathbf{C}_{cc} = \begin{pmatrix} C_{x,x} & C_{x,y} & C_{x,\alpha} & C_{x,\beta} \\ C_{y,x} & C_{y,y} & C_{y,\alpha} & C_{y,\beta} \\ C_{\alpha,x} & C_{\alpha,y} & C_{\alpha,\alpha} & C_{\alpha,\beta} \\ C_{\beta,x} & C_{\beta,y} & C_{\beta,\alpha} & C_{\beta,\beta} \end{pmatrix}. \quad (3.77)$$

Each one of the two indices of the coefficients denote either the upper or the lower half. Coefficients must be added to the corresponding positions of the global matrices during their assembly.

The unbalance forces

The mass distribution of shaft and disks around the axis of rotation—the rotor centerline—is never completely symmetric, resulting in a center of gravity that is not aligned with it. The offset between the shaft and disks center of mass and their common centerline can be represented by an unbalance mass M_u on a point of the rotor system—or a set of unbalance masses on different points—at a distance r of the centerline. The result of the mass unbalance distribution is, under the hypothesis of constant rotation speed, a set of centrifugal forces with amplitudes $M_{U,i}r_i\Omega^2$ that act on the shaft along lines that contain the i -th unbalance mass point and a point of the centerline, always remaining perpendicular to it.

FIGURE 3.20: Unbalance mass on a rotor and the resulting unbalance force.

These unbalance centrifugal forces rotate around the centerline on rotor radial planes with constant speed Ω as seen in Figure 3.20 and their projections over the x and y axis are sinusoidal curves:

$$\begin{aligned} F_{U_{x,i}} &= M_{U,i}r_i\Omega^2 \cdot \cos(\Omega t + \psi_{U,i}), \\ F_{U_{y,i}} &= M_{U,i}r_i\Omega^2 \cdot \sin(\Omega t + \psi_{U,i}), \end{aligned} \quad (3.78)$$

where ψ_U is an unbalance phase angle, and i is the index of a point—or *node*—in the shaft.

Given the set of unbalance forces over each of the shaft nodes, a global vector of unbalance forces can be assembled and then added to the rotor-casing model SODE—Equation 3.1:

$$\mathbf{M}\mathbf{u}'' + (\mathbf{C} + \Omega \cdot \mathbf{G})\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}_U(t). \quad (3.79)$$

The rub forces

We can define in our model as many rotor-casing rubs as we desire, each between a casing node and a rotor node—Figure 3.21. An obstacle with stiffness and damping coefficients k_R and c_R is fitted at the casing node j . This obstacle may impact the rotor at node i . When the system is at rest, the clearance between rotor and obstacle is ϵ . If the rotor-casing relative deformation becomes larger than ϵ , two forces appear in the system: a normal, visco-elastic force— F_R —and a tangential friction force proportional to the normal rub force by a friction coefficient, μ . This friction coefficient can be

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considered constant if the contact surfaces never stick with each other and if changes in the relative rub velocity during contact are small.

Single-point rub—also called *fixed-point rub*—models such as this one have been studied in some publications: Ma et al., 2013 and Behzad et al., 2013 applied Lagrangian methods to model the point-to-point contact between an undeformable rotor surface and an elastic rod that disturbs the rotor motion. The model formulations define a gap function between the two surfaces with a master-slave approach and enforce a no-penetration condition. The models were simulated at sub-critical—below the first rotor critical speed—and supercritical—above the first rotor critical speed—rotation speeds. Following the same approach, Ma et al., 2014 then used the same formulation to study single-point rub with three-pin-three-rub spots and four-pin-four-rub spots stators. Mokhtar et al., 2017 added rotor torsional deformations to a model similar to those of the aforementioned references, concluding that the spectra of torsional vibration, with the presence of multiples of the rotation speed and the excitation of the torsional natural frequencies, may be indicative of rub.

If the obstacle contact surface is flat and parallel to the casing at the attachment node j , its normal unitary vector equals the casing unitary normal vector at j , $\mathbf{v}_{2,j}$.

FIGURE 3.21: Rub model and forces at shaft and casing nodes.

The relative displacement d_R between rotor and obstacle and the relative velocity d'_R can be expressed as

$$\begin{aligned} d_R &= \langle (u_{x,j} - u_{x,i}, u_{y,j} - u_{y,i}), \mathbf{v}_{2,j} \rangle - \epsilon \\ d'_R &= \langle (u'_{x,j} - u'_{x,i}, u'_{y,j} - u'_{y,i}), \mathbf{v}_{2,j} \rangle, \end{aligned} \quad (3.80)$$

being $u_{x,j}$, $u_{y,j}$, $u_{y,i}$ and $u_{x,i}$ the radial displacements of the casing and rotor nodes respectively. $u'_{x,j}$, $u'_{x,i}$, $u'_{y,j}$ and $u'_{y,i}$ are node velocities. $\langle \cdot, \cdot \rangle$ is the notation for the inner product of two vectors.

The normal rub force results from the product of relative displacements and velocities with stiffness and damping contact coefficients k_R and c_R and a Heaviside function that equals one when the relative deformation is larger than the clearance ϵ , and zero otherwise:

$$F_R(\mathbf{u}, \mathbf{u}') = (k_R d_R + c_R d'_R) \cdot H(d_R - \epsilon). \quad (3.81)$$

Since the rub spot stands at a distance $R_c - R_{r,i}$ of the casing node, $R_{r,i}$ being the rotor radius at shaft node i , the friction force at the rub spot inflicts, by balance of forces, a bending moment at the casing around the z axis. Normal and friction rub forces at rotor and casing are projected over the global coordinate axes, resulting in the following local rub vectors of rotor node i and casing node j :

$$\begin{aligned} \mathbf{f}_{R,i}(\mathbf{u}, \mathbf{u}') &= F_R \cdot \begin{pmatrix} -\cos \gamma \\ -\sin \gamma \\ 0 \\ 0 \end{pmatrix} + \mu F_R \cdot \begin{pmatrix} \sin \gamma \\ -\cos \gamma \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{f}_{R,j}(\mathbf{u}, \mathbf{u}') &= F_R \cdot \begin{pmatrix} \cos \gamma \\ \sin \gamma \\ 0 \\ 0 \end{pmatrix} + \mu F_R \cdot \begin{pmatrix} -\sin \gamma \\ \cos \gamma \\ 0 \\ -(R_c - R_{r,i}) \end{pmatrix}, \end{aligned} \quad (3.82)$$

where γ is the angle of $\mathbf{v}_{2,j}$ with respect to x .

As we do with the unbalance forces vector, we can add all rub forces present in the model to a global rub vector $\mathbf{f}_R(\mathbf{u}, \mathbf{u}')$.

Assembly of the global model equations

The definitive rotor-casing model equations are these:

$$\mathbf{M}\mathbf{u}'' + (\mathbf{C} + \Omega\mathbf{G})\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}_U(t) + \mathbf{f}_R(\mathbf{u}, \mathbf{u}'). \quad (3.83)$$

$\mathbf{M}$, $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{K}$ are the mass, viscous damping, gyroscopic and stiffness matrices, respectively. These global matrices are assemblies of the local matrices of shaft and disks, $S+D$; casing, C ; and bearings, B . The elastic couplings between shaft and bearings and between bearings and casing—the latter, through the bearing-casing joints marked in yellow in Figure 3.1—are accounted for with the addition of elements outside the main diagonal to the global viscous damping and stiffness matrices:

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} \mathbf{M}_{S+D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_B \end{bmatrix}, & \mathbf{G} &= \begin{bmatrix} \mathbf{G}_{S+D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ \mathbf{C} &= \begin{bmatrix} \mathbf{C}_{S+D} & \mathbf{0} & \mathbf{C}_{S+D,B} \\ \mathbf{0} & \mathbf{C}_C & \mathbf{C}_{C,B} \\ \mathbf{C}_{S+D,B} & \mathbf{C}_{C,B} & \mathbf{C}_B \end{bmatrix}, & \mathbf{K} &= \begin{bmatrix} \mathbf{K}_{S+D} & \mathbf{0} & \mathbf{K}_{S+D,B} \\ \mathbf{0} & \mathbf{K}_C & \mathbf{K}_{C,B} \\ \mathbf{K}_{S+D,B} & \mathbf{K}_{C,B} & \mathbf{K}_B \end{bmatrix}. \end{aligned} \quad (3.84)$$

$\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ are the vectors of node displacements and their time derivatives: velocity and acceleration. Analogously to the matrices, these vectors are assemblies of sub-vectors of shaft, casing and bearing nodes displacements, each with its own degrees of freedom:

$$\mathbf{u} = (\mathbf{u}_{S+D} | \mathbf{u}_C | \mathbf{u}_B)^T = (\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_N)^T, \quad (3.85)$$

where N is the total number of nodes of the global discretized system.

Rayleigh damping matrix

When defining a damping matrix $\mathbf{C}$ there are two options: one is manually setting damping coefficient values across elements, couplings and supports; another more comfortable option is to suppose proportional damping in our models, hence setting a damping matrix that is proportional to either the mass or the stiffness matrices, or even a linear combination of the two.

Works such as Chen, 2015b follow the latter alternative. In that study, the rotor-casing model had been hypothesized to have a Rayleigh damping: the global viscous damping matrix $\mathbf{C}$ as a linear combination of the mass and stiffness matrix:

$$\mathbf{C} = C_M \cdot \mathbf{M} + C_K \cdot \mathbf{K}. \quad (3.86)$$

The two constants C_M and C_K are the result of solving a two-variable system of equations given two system natural frequencies $\omega_{N,1}$ and $\omega_{N,2}$ with known damping factors $\xi_{d,1}$ and $\xi_{d,2}$:

$$\xi_{d,i} = \frac{1}{2} \cdot \left(\frac{C_M}{\omega_{N,i}} + C_K \omega_{N,i} \right), \quad i = 1, 2. \quad (3.87)$$

The drawback of proportional damping models is that, because the resulting damping matrices are constant, they cannot represent changes in the damping characteristics of the model with the vibration frequencies. Although there exist generalized formulations of proportional damping that are flexible enough to allow variations of the damping features with frequency, such as Adhikari, 2006.

Model reduction with the Craig-Bampton method

It is expected that the SODE 3.83 contains a very large number of degrees of freedom, of the order of tens of thousands. This would hinder the numerical calculation of the solutions of the rotor-casing model SODE, mostly in terms of time. For this reason, we are interested in reducing the size of the SODE by applying any feasible reduction method.

For this task we have selected the Craig-Bampton method—Craig Jr et al., 1968. The aim of this method is to reduce the number of degrees of freedom of the rotor-flexible casing system, represented by Equation 3.83. Craig-Bampton was especially design for its use during Finite-Element modeling of structures, particularly complex structural assemblies. Its defining feature is that this method splits the system degrees of freedom into two categories: *boundary* and *interior*. Only interior degrees of freedom are reduced, while the boundary degrees of freedom are left unaffected, which allows to either apply boundary conditions to those nodes, or to couple boundary degrees of freedom to adjacent structures, or to introduce external loads to them. In our case, we set as boundary degrees of freedom those on which we apply unbalance forces—the shaft nodes—or rub forces—casing nodes with rotor-casing contact—.

The first step of the Craig-Bampton method is to split the system degrees of freedom, contained in the modal displacements vector $\mathbf{u}$, into two categories: the *boundary* degrees of freedom—subscript B —and the *interior* degrees of freedom—subscript I . This division prompts a reordering of the nodal displacements

vector and its derivatives, the external force vectors and the system matrices as follows:

$$\begin{aligned} \mathbf{u} &= (\mathbf{u}_B | \mathbf{u}_I)^T, \quad \mathbf{f} = (\mathbf{f}_B | \mathbf{f}_I)^T = (\mathbf{f}_B | \mathbf{0})^T, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}_{BB} & \mathbf{M}_{BI} \\ \mathbf{M}_{IB} & \mathbf{M}_{II} \end{pmatrix}, \\ \mathbf{G} &= \begin{pmatrix} \mathbf{G}_{BB} & \mathbf{G}_{BI} \\ \mathbf{G}_{IB} & \mathbf{G}_{II} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{C}_{BB} & \mathbf{C}_{BI} \\ \mathbf{C}_{IB} & \mathbf{C}_{II} \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BI} \\ \mathbf{K}_{IB} & \mathbf{K}_{II} \end{pmatrix}. \end{aligned} \quad (3.88)$$

Note that the vector of external forces at the interior degrees of freedom is a zero vector, because by definition there are no external forces applied on them.

The core of the Craig-Bampton method is the orthogonal projection of the state space of the original system, represented by the reordered nodal displacements vector and its derivatives, over a reduced subspace with fewer degrees of freedom. Naming the Craig-Bampton reduction matrix Φ_{CB} , the projection of the nodal displacements vector is

$$\mathbf{u}_{CB} = \Phi_{CB}^T \cdot \mathbf{u} = (\mathbf{u}_B | \phi_I)^T. \quad (3.89)$$

The projection retains the boundary degrees of freedom in $\mathbf{u}_B$ and yields a set of *modal* degrees of freedom: ϕ_I . The Craig-Bampton reduction matrix is

$$\Phi_{CB} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \Phi_C & \Phi_N \end{bmatrix}. \quad (3.90)$$

$\mathbf{I}$ is an identity sub-matrix. Φ_C is the matrix of *constraint modes*, calculated as

$$\Phi_C = -\mathbf{K}_{II}^{-1} \cdot \mathbf{K}_{IB}. \quad (3.91)$$

Each i -th column of Φ_C represents the displacement of the interior degrees of freedom when a unitary displacement is set at the i -th boundary degree of freedom while the remaining boundary displacements are set to zero.

The final component of Φ_{CB} is the sub-matrix of *matrix modes*, Φ_N . Each of its columns is a modal shape of the system with the displacements of the boundary degrees of freedom constraint to zero. Those modal shapes are solutions of an eigenproblem formed by the interior stiffness and mass sub-matrices:

$$\mathbf{K}_{II} \cdot \Phi_N = \mathbf{M}_{II} \cdot \Phi_N \cdot \Lambda, \quad (3.92)$$

where Λ is the diagonal matrix with the eigenvalues. It is usual to normalize the columns of Φ_N so that the generalized modal mass equals one for every modal shape. This condition is expressed in matrix notation as

$$\Phi_N^T \cdot \mathbf{M}_{II} \cdot \Phi_N = \mathbf{I}. \quad (3.93)$$

The system reduction is attained when only a few of the system matrix modes are retained in the sub-matrix Φ_N .

3.1. A non-linear model for the simulation of single-point rub

The system global matrices are now projected over the Craig-Bampton subspace:

$$\begin{aligned}\mathbf{M}_{CB} &= \Phi_{CB}^T \cdot \mathbf{M} \cdot \Phi_{CB}, & \mathbf{G}_{CB} &= \Phi_{CB}^T \cdot \mathbf{G} \cdot \Phi_{CB}, \\ \mathbf{C}_{CB} &= \Phi_{CB}^T \cdot \mathbf{C} \cdot \Phi_{CB}, & \mathbf{K}_{CB} &= \Phi_{CB}^T \cdot \mathbf{K} \cdot \Phi_{CB}.\end{aligned}\quad (3.94)$$

The projection of the external force vectors results in vectors that conserve their values at the boundary degrees of freedom, while the forces at the interior degrees of freedom remain zero. Their product by Φ_{CB}^T simply shortens them as the projection subspace is smaller than the original state space:

$$\begin{aligned}\mathbf{f}_{U,CB} &= \Phi_{CB}^T \cdot \mathbf{f}_U = [F_{U,1} \dots F_{U,N}]^T, \\ \mathbf{f}_{R,CB} &= \Phi_{CB}^T \cdot \mathbf{f}_R = [F_{R,1} \dots F_{R,N}]^T,\end{aligned}\quad (3.95)$$

with N representing here the number of degrees of freedom, or dimension, of the Craig-Bampton subspace.

The reduced rotor-casing system of differential equations can now be assembled:

$$\mathbf{M}_{CB}\mathbf{u}_{CB}'' + (\mathbf{C}_{CB} + \Omega \cdot \mathbf{G}_{CB})\mathbf{u}_{CB}' + \mathbf{K}_{CB}\mathbf{u}_{CB} = \mathbf{f}_{U,CB}(t) + \mathbf{f}_{R,CB}(\mathbf{u}, \mathbf{u}'). \quad (3.96)$$

Numerical integration with the Newmark-Beta method

As the Craig-Bampton method for reduction, the Newmark- β method for numerical time integration—Newmark, 1959—was originally conceived for the resolution of mechanical and structural problems. This method uses the following implicit finite-difference expressions to calculate the velocities and displacements in the step $i + 1$ from the displacements, velocities and accelerations in the former step i and the accelerations in the present step $i + 1$, with an integration time step equal to Δt :

$$\begin{aligned}\mathbf{u}_{i+1}' &= \mathbf{u}_i' + \Delta t \cdot [(1 - \gamma)\mathbf{u}_i'' + \gamma\mathbf{u}_{i+1}''], \\ \mathbf{u}_{i+1} &= \mathbf{u}_i + \Delta t \cdot \mathbf{u}_i' + \Delta t^2 \cdot [(1/2 - \beta)\mathbf{u}_i'' + \beta\mathbf{u}_{i+1}'']. \end{aligned}\quad (3.97)$$

These expressions are a function of two parameters, β and γ , that determine how much weight the $i + 1$ step accelerations have in the calculation of the $i + 1$ velocities and displacements. If the parameter values is zero the method is explicit. Stability and convergence are functions of β and γ . Newmark—Newmark, 1959—recommends the values $\beta = 1/4$ and $\gamma = 1/2$ for unconditional stability.

Equation 3.83 with the displacements, velocities and accelerations in the step $i + 1$ has the form

$$\mathbf{M}\mathbf{u}_{i+1}'' + (\mathbf{C} + \Omega \cdot \mathbf{G})\mathbf{u}_{i+1}' + \mathbf{K}\mathbf{u}_{i+1} = \mathbf{f}_U(t_{i+1}) - \mathbf{f}_R(\mathbf{u}_{i+1}, \mathbf{u}_{i+1}'). \quad (3.98)$$

Substituting $\mathbf{u}_{i+1}$ and $\mathbf{u}_{i+1}'$ and clearing $\mathbf{u}_{i+1}''$ yields the accelerations in the present $i + 1$ step. Then, from Equation 3.97 the velocities and displacements in the $i + 1$ step can be calculated.

When there is a nonlinear term in the SODE that is a function of either displacements or velocities, the algebraic system formed by Equations 3.97 and 3.98 becomes nonlinear in $i + 1$ state variables. For solving this nonlinear problem, Newmark—Newmark, 1959—proposes an iterative scheme where some initial $\mathbf{u}_{i+1}'$ and $\mathbf{u}_{i+1}$ are tested in Equation 3.98 to obtain an $i + 1$ acceleration vector before

calculating new velocities and accelerations with Equation 3.97. The difference between the new and the initial $i + 1$ state vectors would determine the accuracy of the supposed solution. In a multi-degree-of-freedom system as ours that would mean a high degree of additional computational complexity. Instead, we adopted the following alternative: if the step size is small enough the nonlinear—rubbing—forces in the $i + 1$ state can be substituted with those of the previously calculated step i :

$$\mathbf{f}_R(\mathbf{u}_{i+1}, \mathbf{u}'_{i+1}) \approx \mathbf{f}_R(\mathbf{u}_i, \mathbf{u}'_i). \quad (3.99)$$

3.2 Model-based evaluation of potential strategies for rub feature detection

In this section, we will study some available alternatives for detection of single-point rub in gas turbines. This analysis will be entirely model-based, understanding that our rotor-casing system model represents accurately the dynamics of a gas turbine in general—when the casing is thicker and both casing and rotor and heavier—and of an aeroderivative gas turbine in particular.

Dimensions and mechanical properties of the model we set for the preliminary studies of this section are shown in Table 3.1. The values in Table 3.1 are within the range of those appearing in some reduced scale experimental devices typically employed in the literature for model validation—Chen, 2016; Yang et al., 2016b.

TABLE 3.1: Rotor-casing model dimensions and mechanical properties.

Shaft length:	0.6 m
Distance between shaft bearings:	0.4 m
Shaft diameter:	0.01 m
Disk mass:	1.5 kg
Disk thickness:	0.025 m
Disk diameter:	0.1 m
Casing mid-plane diameter:	0.134 m
Casing length:	0.4 m
Casing thickness:	0.003 m
Young modulus of shaft:	2×10^{11} Pa
Shaft density:	7850 kg/m^3
Casing density:	1600 kg/m^3
Young modulus of casing:	7×10^{10} Pa
Poisson's ratio of casing:	0.1
Bearing radial direct stiffness:	$1 \times 10^8 \text{ N/m}$
bearing-casing union stiffness:	$1 \times 10^{12} \text{ N/m}$
Bearing mass:	0.02 kg
Casing support horizontal stiffness:	$5 \times 10^5 \text{ N/m}$
Casing support vertical stiffness:	$5 \times 10^5 \text{ N/m}$

The simulated rotor-casing model comprises two disks symmetrically attached to the shaft. The second disk contains an unbalance mass M_U of $1.5 \times 10^{-3} \text{ kg}$ at a distance r_U of $3.75 \times 10^{-2} \text{ m}$ from the rotation axis with zero phase angle ψ_U . The viscous damping matrix has been estimated considering a damping factor of 0.04

3.2. Model-based evaluation of potential strategies for rub feature detection

and 0.06 at the critical speeds of 1500 and 5700 rpm, respectively. The casing was modeled as an homogeneous and solid cylindrical shell following Section 3.1.3. A rub condition between the rotor and an element attached to the casing affects a shaft node close to one of the disks and one of the casing nodes in the same radial plane, with $k_R = 2.25 \times 10^6$ N/m being the average stiffness of a seal strip tested as rub obstacle in Pennacchi et al., 2009, $c_R = 0$ representing no impact damping, with and $\mu = 0.2$ being the friction coefficient of steel-to-steel rub.

Equation 3.83 was reduced with Craig-Bampton and then integrated with a time step of $\Delta t = 2^{-17}$ s. Such a small time step is necessary to guarantee the stability of the numerical solutions, otherwise spurious oscillations would blur the high-frequency casing vibration harmonics. After the simulation, we extract two-second long sample vectors or *time records* of casing radial acceleration from a casing node close to one of the casing ends—i.e. next to one of the bearings—. The sample vectors have then 2^{18} samples, at a rate of one sample per integration step. Accelerometers on turbine casings measure radial accelerations of casing node j , u_j'' , therefore we need to project the accelerations in the x and y directions on the direction normal to the casing mid-surface at the measurement point: $u_j'' = \langle (u_{x,j}'', u_{y,j}''), \mathbf{v}_{2,j} \rangle$.

Our first step was the modal analysis of the free undamped system with $\Omega = 0$. The first rotor and casing modes of vibration and their corresponding undamped natural frequencies are shown in Figure 3.22. Notice that for this preliminary study the casing was modeled as a one-piece cylindrical shell. Note also that the bearing-casing unions—in Figure 3.1: BC_1 and BC_2 , yellow—were not drawn in these mode shape figures, although they are represented in the figures showing the mode shapes of the experimental rotor model—Figure 4.12.

The variation of natural frequencies with rotation speed in the damped free rotor-flexible casing due to the speed-variable damping term of Equation 3.83 is represented on the *Campbell diagram* shown in Figure 3.23. From the Campbell diagram we can obtain the lowest *resonance* or *critical* speeds of the rotor under unbalance by intersecting the $1 \times$ excitation line—in red—with the blue lines representing the bifurcations between natural modes in x and y caused by the rotor gyroscopic effects.

(A) Natural frequency: 24.5 Hz

(B) Natural frequency: 94.5 Hz

FIGURE 3.22: Lowest shaft and casing natural modes: (A) Rotor bending mode with one anti-node; (B) Rotor bending mode with two anti-nodes; (C) Casing rigid solid mode with shaft deflection; (D) Lowest casing longitudinal mode; and (E) and (F) Lowest casing flexural modes.

FIGURE 3.23: Campbell diagram of the two first rotor natural modes. The two bending modes split with increasing rotation speed due to the gyroscopic effect. The intersections of the $1\times$ line with the natural modes mark the rotor critical speeds. All represented modes are stable.

3.2. Model-based evaluation of potential strategies for rub feature detection

From the Campbell diagram—Figure 3.23, the first three rotor critical speeds are approximately 1500 rpm, corresponding to the rotor bending mode with one anti-node, and 5400 and 5900 rpm corresponding to the bending mode with two anti-nodes. The model was simulated at a speed Ω of 3600 rpm—60 Hz, between the first and the second critical speeds: the working range of a real aeroderivative machine. The purpose of selecting this rotation speed is to reproduce a rub condition that is observed very often in conventional turbomachines working at speeds above twice the first rotor critical speed: a single-point rub with one impact every two rotation cycles—Bently et al., 2003c. This period-two regime under supercritical rotation speeds is also observed in other works that deal with single-point—or single-point—rub Ma et al., 2013; Behzad et al., 2013; Mokhtar et al., 2017.

3.2.1 Signal processing with traditional time-domain methods

Here we apply the standard detection methods described in Section 2.1 to the results—node displacements, velocities and accelerations—of the simulations of the rotor-casing model in Figure 3.1 under unbalance and rub, and under only rotor unbalance.

The Discrete Fourier Transform

The Fourier spectra of the acceleration and velocity records simulated in the model under single-point rub through numerical time integration of Equation 3.83 are shown in Figure 3.24. It is common practice to integrate the accelerometer signals to obtain sampling vectors containing radial casing velocity, $\mathbf{u}'$, and then to monitor velocity instead of acceleration. The purpose of this integration is to attenuate the high-frequency noise contained in the accelerometer signal and to highlight the low-frequency band of the spectrum, which is the source of information that is most commonly used for protection proposes in rotating machinery—Littrell, 2013. Once a low-pass FIR filter was applied to the discrete sampling vectors to prevent aliasing, the acceleration data were down-sampled to 28 kHz to produce a 0-14 kHz spectrum, and the velocity data were down-sampled to 2 kHz for a 0-1000 Hz spectrum. Then, a Hamming windowing was applied to the new vectors of samples. The two sample records were two seconds long, hence the frequency resolution is 0.5 Hz. The two spectra are qualitatively very similar: each displays an impulse train with fundamental frequency equal to the rotor-casing impact frequency—half the rotation speed, 30 Hz. The harmonics with the highest amplitude, apart from the fundamental harmonic in the velocity spectrum, are found in the 90-240 Hz band and are related to the excitation of the 200 Hz casing rigid solid vibration mode—Figure 3.22c. This amplification phenomenon of the rub frequency harmonics near the casing natural frequencies has already been observed in works such as Tang et al., 2010.

The full spectrum

In Figures 3.25 and 3.26 we represent the rotor orbits, the full spectra of the rotor orbits—Bently et al., 2003d—and casing acceleration and velocity plots of a rotor-casing model in steady-state regime under two conditions: no rub and single-point rub. In both cases, the rotor rotates at a speed of 3600 rpm in the positive anticlockwise direction—from x to y . If there is no rub—Figure 3.25, the acceleration and velocity signals are dominated by a synchronous sine wave resulting

FIGURE 3.24: Periodograms of the steady-state casing signals with single-point rub and low-band details: (A) Casing acceleration; and (B) Casing velocity. For simplicity, only the low-band side of the spectra—between 0 and 600 Hz—is shown.

from the unbalance force excitation. If the clearance ϵ is small enough, a stable single-point rotor-casing sets off, its intensity being of such magnitude so that the unbalance-related sinusoidal vibration is masked by new signal features resulting from the rub condition. Black dotted lines in Figure 3.26 mark the boundaries of the leftmost rotor-casing contact.

Every time the rotor impacts the casing, two impulse peaks appear in the acceleration plot—Figure 3.25c: the first when rotor and casing come together, and the second when rotor and casing come apart. A transient that contains at most free high-frequency highly-damped vibration of rotor and casing follows the first impulse. The second impulse at the instant when rotor and casing split apart is followed by a damped transitory dominated by the 200 Hz casing rigid solid mode—Figure 3.22c. The existence of these two sharp impulses at the boundaries of rub contact and the damped high-frequency oscillations following can be confirmed by results found in the literature as the stator lateral response plots of Mokhtar et al., 2017. The number of rotor-casing impacts per second is half the rotation frequency: 30 Hz.

3.2. Model-based evaluation of potential strategies for rub feature detection

FIGURE 3.25: (A) Rotor orbit shaft displacement in the radial plane; (B) Full spectrum of the rotor orbit; and (C) and (D) Acceleration time plot and velocity time plot, respectively.

The cepstrum

The cepstrum analysis of the down-sampled and filtered acceleration and velocity signals—Figure 3.27—underlines the rotor-casing impact frequency of 15 Hz—represented in the cepstra as impulse trains of quefrency 0.033 s. Both cepstra are qualitatively and quantitatively very similar, although the velocity cepstrum looks coarser due to the lower sampling frequency of the down-sampled velocity data.

3.2.2 An advanced time-frequency domain alternative: the Synchrosqueezed Wavelet Transform

In Section 2.4 we enlisted the latest advances on rub detection with time-frequency methods, mostly over proximitor signals. It is in the scope of this Thesis research to pick up one of this algorithms and to compare it with the staple vibration-based detection methods described in Section 2.1 for early detection of rub in signals from casing accelerometers of aeroderivative gas turbines. Here we chose a variant of the *Continuous Wavelet Transform*—or CWT—called the *Synchrosqueezed Wavelet Transform*, which adds a *frequency reassignment* step to the calculation of the Wavelet coefficients in order to increase the resolution of the time-frequency representations. Below we describe this algorithm.

FIGURE 3.26: (A) Rotor orbit, with the orbit segment where the rotor rubs the casing marked in red; (B) Full spectrum of the rotor orbit; (C) Acceleration time plot; and (D) Velocity time plot at 3600 rpm with single-point rub ($\epsilon = 2.5 \times 10^{-4}$ m).

As the Continuous Fourier Transform—Kovačević et al. (2013)—the CWT is defined as the integral correlation in the time domain of a function $x(t)$ with some basis function. In the Continuous Fourier Transform, that basis function is the complex unit circumference with frequency ω : $\exp(j\omega t)$. In the case of the CWT the chosen basis function is the dilated and translated version of a function ψ known as *mother wavelet*:

$$W(a, s) = \frac{1}{\sqrt{|s|}} \cdot \int_{-\infty}^{\infty} x(t) \cdot \psi^*\left(\frac{t-a}{s}\right) dt, \quad (3.100)$$

where a and s are, respectively, translation and scaling factors. If $x(t)$ is a discrete function—i.e. a vector of time samples—then Equation 3.100 becomes a discrete convolution with a wavelet function $\psi(t)$ sampled at the same time values as $x(t)$. In this case we can substitute $x(t)$ and $\psi(t)$ with $\mathbf{x}$ and $\boldsymbol{\psi}$ to emphasize that we are working with vectors. To simplify future calculations, we will normalize the sampling time of $\mathbf{x}$ to one second.

The coefficient $W(a, s)$ gives a measure of the similarity between the function and the dilated and translated mother wavelet. The big advantage of the CWT over the Fourier Analysis is its potential for feature localization in both time and frequency domains, provided that the selected mother wavelet has good properties, some of

3.2. Model-based evaluation of potential strategies for rub feature detection

FIGURE 3.27: Real cepstra of the accelerometer signals with single-point rub: (A) Cepstrum of the acceleration data; and (B) Cepstrum of the velocity data.

which are enumerated below—from Valens, 1999:

- Admissibility: This condition is given by the expression

$$\int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < +\infty, \quad (3.101)$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$. This condition guarantees that $\psi(t)$ can be used to de-construct and construct a signal without loss of information—Sheng, 1996. It implies, in one hand, that $\hat{\psi}(\omega)$ vanishes at zero frequency:

$$|\hat{\psi}(\omega)|^2|_{\omega=0} = 0, \quad (3.102)$$

and in the other, that the average value of $\psi(t)$ must be zero:

$$\int_{-\infty}^{+\infty} \psi(t) dt = 0. \quad (3.103)$$

- Regularity: it is a measure of the mother wavelet's degree of smoothness in both time and frequency domains. It is often quantified with the number of *vanishing moments* of $\psi(t)$, i.e. the highest approximation order achievable by

the wavelet transformation with a certain mother wavelet function—Burrus et al., 1998. The more vanishing moments that $\psi(t)$ has, the higher the order of the functions $x(t)$ that can be accurately de-constructed with $\psi(t)$.

When these features hold, the wavelet transform is ideal for signals with a high content in transient components and irregularities that are invisible to the Fourier decomposition methods.

Let N_a and N_s be the number of translation and scaling factors of the desired CWT scalograms. Suppose for simplicity a number of translation factors equal to N —the number of samples or elements of the vector $\mathbf{x}$ —and a normalized sampling time of 1 s, then this scalogram will be a complex matrix with N columns, N_s rows and $N_s \times N$ elements in total, with $s = 1, 1/2, 1/4 \dots 2/N$ —up to the normalized Nyquist frequency—as the scaling coefficients for each row. To obtain this scalogram, we would require $N_s \times N$ evaluations of Equation 3.100, which is often very costly in terms of time. Calculation of the scalograms are much faster if we applied the correlation theorem to Equation 3.100: this correlation in the time domain can be substituted by an element-wise product of the complex spectra of both the signal and the mother wavelet function for a fixed value of the scaling coefficient s . Then, the inverse Fourier Transform of these element-wise products yields each one of the rows of the spectrograms: $W(a, s)$, $a = 1 \dots N$.

$$\mathbf{w}(s) = \frac{1}{\sqrt{|s|}} \cdot \mathcal{F}^{-1}(\hat{\mathbf{x}} \odot \hat{\psi}^*(s\omega)), \quad (3.104)$$

where $\odot$ represents the element-wise product of vectors, $'^*$ represents the element-wise complex conjugate, $\hat{\mathbf{x}}$ is the Fourier spectrum of $x(t)$, $\hat{\psi}(s\omega)$ is the definition of the wavelet function in the frequency domain, and $\mathcal{F}^{-1}$ represents again the inverse Fourier Transform. The resulting vector, $\mathbf{w}(s) = (W(1, s), W(2, s) \dots W(N, s))$, would be one of the scalogram rows, with N elements.

To improve the resolution of the wavelet scalograms, we applied to the wavelet coefficients a frequency reassignment algorithm called *synchrosqueezing*. This post-processing algorithm, described in Daubechies et al., 2011, corrects the smearing of energy in the wavelet decomposition by concentrating the spectral energy closer to the signal instantaneous frequencies at each instant of time. The result is a less blurred representation of the frequency content of the signals across time. This reassignment step will require the derivatives of the wavelet coefficients calculated with Equation 3.104 with respect to frequency:

$$\frac{d\mathbf{w}(s)}{d\omega} = \frac{1}{\sqrt{|s|}} \cdot \mathcal{F}^{-1}(\hat{\mathbf{x}} \odot j\omega \odot \hat{\psi}^*(s\omega)), \quad (3.105)$$

where j is the imaginary unit, and $\omega = (0, 2\pi, 4\pi \dots N \cdot 2\pi)$ is the vector of sampled frequencies of the wavelet function spectrum $\hat{\psi}(s\omega)$ with the sampling time of $x(t)$ normalized to one. As with $\mathbf{w}(s)$, the vector $d\mathbf{w}(s)/d\omega = (dW(1, s)/d\omega, dW(2, s)/d\omega \dots dW(N, s)/d\omega)$ contains all the derivatives of the corresponding row of the scalogram.

Once we know all wavelet coefficients $W(a, s)$ of the scalograms and their derivatives with respect to frequency we can calculate the instantaneous frequencies $\omega'(a, s)$ to which we reassign each coefficient:

$$\omega'(a, s) = \frac{dW(a, s)/d\omega}{j \cdot W(a, s)}. \quad (3.106)$$

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ω' , as ω , are expressed in rad/s. It will be useful to express the instantaneous frequencies in hertz:

$$f'(a, s) = \frac{\omega'(a, s)}{2\pi}. \quad (3.107)$$

The last step of the synchrosqueezing algorithm is the reassignment step: Suppose that the wavelet coefficient $W(a, s)$ with translation and scaling coefficients a and b is located in the i -th row and the k -th column of the CWT scalogram matrix $\mathbf{S}$. In the reassigned scalogram, the coefficient $W(a, s)$ must be added to the $f'(a, s) + 1$ -th row in the same k -th column of the new reassigned scalogram $\mathbf{S}'$:

$$\mathbf{S}_{i,k} \longrightarrow \mathbf{S}'_{[f'(a,s)]+1,k} \quad (3.108)$$

where $[\cdot]$ represents the nearest integer operator.

The new scalogram $\mathbf{S}'$ represents the Synchrosqueezed Wavelet Transform of $x(t)$.

The algorithm that combines the CWT with the synchrosqueezing energy reallocation is generally called *Synchrosqueezed Wavelet Transform* or WSST.

In this study we applied two definitions of the mother wavelet function: the Analytic Morlet—Ashmead, 2012—and the Bump wavelet—Jiang et al., 2017. Their definitions in the complex domain are shown in Table 3.2. The parameters and scaling coefficient values used during the computation of the CWT scalograms are shown in Table 3.3. However, in this Section, for brevity, we show only results with the Analytic Morlet. This is a wavelet function that has been extensively used with success in the state of the art for the study of failure in rotating machinery, as shown by Lin et al., 2000, Zheng et al., 2002, Wu et al., 2006, Rafiee et al., 2010, Su et al., 2010, Tang et al., 2010 and Kankar et al., 2011.

TABLE 3.2: Definitions of the Analytic Morlet and the Bump wavelet mother functions. $\mathbb{1}$ is the indicator function.

Wavelet functions	Definitions in frequency domain
Analytic Morlet	$\hat{\psi}(s\omega) = \pi^{-1/4} \cdot \exp(-\frac{(s\omega - \omega_0)^2}{2}), \omega \geq 0,$ $\hat{\psi}(s\omega) = 0, \omega < 0.$
Bump	$\hat{\psi}(s\omega) = \exp(1 - \frac{1}{1 - (s\omega - \mu)^2/\sigma^2}) \cdot \mathbb{1}_{[(\omega - \mu)/\sigma, (\omega + \mu)/\sigma]}.$

TABLE 3.3: Parameter values of the Analytic Morlet and the Bump wavelet mother functions, and scaling coefficient values for the representation of the scalograms.

Wavelet functions	Parameter values	Scaling coefficients
Analytic Morlet	$\omega_0 = 6$	$s_i = \frac{1}{i},$ $i = 1, 2, \dots, \frac{N}{2}.$
Bump	$\mu = 6, \sigma = 1.2$	

The acceleration and velocity data were decomposed into wavelet coefficients with the Morlet wavelet function. First, the two second-long time records were passed

through a low-pass filter and resampled to 3.2 kHz, resulting in two sampling vectors with 8000 samples. Then, only the last 160 samples of both vectors—with a length of 0.1 s, equivalent to six shaft rotation cycles at 3600 rpm—were picked for the calculation of the wavelet coefficients, discarding the rest. As mentioned above, a synchrosqueezing algorithm was applied to the coefficients for improved resolution. The 0–800 Hz acceleration and velocity scalograms representing coefficient absolute values with a colored scale are shown in Figure 3.28.

FIGURE 3.28: WSST scalograms of the steady-state casing signals with the Morlet wavelet for single-point rub and the frequency axis in logarithmic scale: (A) Casing acceleration, color scale in g ; and (B) Casing velocity, color scale in mm/s. The boundaries of one of the rotor-casing rubs are marked with white dotted lines.

The two scalograms clearly highlight the fundamental rub frequency of 30 Hz and its lowest multiples, with the $3\times$ harmonic—90 Hz, near the second shaft bending mode in Figure 3.23—being the largest of them, and disclose the excitation of the 200 Hz casing vibration mode after the rub impact. The synchronous 60 Hz harmonic is very shady and therefore the scalograms hold no information about the rotor unbalance, which has been masked by the rub features. The localization of the rotor-casing contact event in time is possible because of these features: first, when

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rotor and rub obstacle touch, the 200 Hz band splits into two branches with different frequencies. This is a consequence of a change in the system modal characteristics when rotor and casing are mechanically coupled through their contact—see chapter two of Bently et al., 2003a. Secondly, when rotor and casing contact ends, those two branches get slightly blurry before merging back into one 200 Hz band. Finally, the acceleration scalogram in Figure 3.28a shows two faint tongues in the 400-600 Hz band right at the boundaries of rotor-casing contact. These three signal features characterize rotor-casing rub, allow to localize the rub event in time and are not visible to the Fourier spectra in Figure 3.24. Edge effects are negligible.

3.2.3 Synthesis of efficiency metrics for the comparison of detection strategies

In this section, we quantify the performance of both DFT/FFT and the WSST over the simulated accelerometer signals of Figures 3.25c and 3.26c. We employ the following two metrics as a measure of the performance of each post-processing method: estimated calculation times of DFT and wavelet coefficients and sensitivity—or *robustness*—to the presence of white Gaussian noise.

Detection times

Table 3.4 shows an estimation of the rub detection time with the Fourier spectrum and with WSST. The estimated time is the sum of the data acquisition time—equal to the time length of the vector of samples—plus the calculation time of the Fourier/wavelet coefficients during the data processing phase. These data processing times are the result of averaging the times of ten executions of the Matlab in-built functions *fft*—for the DFT/FFT—and *wsst*—for the Synchrosqueezed Wavelet Transform—with the samples inputs.

TABLE 3.4: Comparison of time performances for rub fault detection.

Time for Data Acquisition				
DFT/FFT	Acceleration data	2 s	Velocity data	2 s
CWT	Acceleration data	0.1 s	Velocity data	0.1 s
Time for Data Processing				
DFT/FFT	Acceleration data	0.001172 s	Velocity data	0.000132 s
CWT	Acceleration data	0.026428 s	Velocity data	0.009997 s
Total Time				
DFT/FFT	Acceleration data	2.00117 s	Velocity data	2.00013 s
CWT	Acceleration data	0.12643 s	Velocity data	0.11000 s

Even though the CWT input vectors, with 160 samples, are far shorter than the DFT input vectors, with 56,000—acceleration—and 4000 samples—velocity, the calculation of CWT coefficients is considerably slower than the calculation of the spectra. Nevertheless, since a more effective detection of rub can be carried out with much shorter sampling vectors, with lower acquisition times, the balance tips on

the side of the WSST as a preferable monitoring tool. Finally, we remark that if the WSST algorithms are optimized for speed and implemented in a fast and efficient data processing equipment, WSST-based real-time monitoring and detection become feasible.

Robustness to noise

The simulated measurements shown in Figures 3.25 and 3.26 are clean, while real-life sensor measurements are always contaminated with noise, particularly acceleration signals, since accelerometers are specially sensitive to random measurement errors which amplify in proportion to the square of the harmonic frequency. Therefore, a study of robustness against noise should be added for a rigorous and comprehensive comparison between post-processing methodologies for acceleration signals.

First of all, rub and no rub signals $\mathbf{a}_R$ and $\mathbf{a}_U$ are resampled as we did above for the calculation of the WSST scalogram—Figure 3.28. Then, a vector of white Gaussian noise $\mathbf{a}_N$ is randomized and added to both $\mathbf{a}_R$ and to the acceleration signals from the machine under unbalance but no rub, $\mathbf{a}_U$:

$$\mathbf{a}_N = \mathcal{N}(0, \sigma_N), \quad \mathbf{a}_{U+N} = \mathbf{a}_U + \mathbf{a}_N, \quad \mathbf{a}_{R+N} = \mathbf{a}_R + \mathbf{a}_N. \quad (3.109)$$

The standard deviation σ_N of the Gaussian random variable $\mathcal{N}(0, \sigma_N)$ depends on a certain signal-to-noise power ratio—SNR—value desired for the noisy signal that we want to create. Since we calculate the SNR as the quotient between the powers of the no-rub signal and of the Gaussian noise—equal to σ_N^2 —, the standard deviation of the Gaussian random variables of Equation 3.109 can be cleared as

$$SNR = 10 \cdot \log_{10} \left(\frac{P(\mathbf{a}_U)}{\sigma_N^2} \right) \implies \sigma_N = \sqrt{\frac{P(\mathbf{a}_U)}{10^{SNR/10}}}. \quad (3.110)$$

Once the noisy signals with and without rub, $\mathbf{a}_{R+N}$ and $\mathbf{a}_{U+N}$, are obtained for a certain SNR value, these time series are transformed into their one-dimensional complex Fourier spectra $\hat{\mathbf{a}}_{U+N}$ and $\hat{\mathbf{a}}_{R+N}$, and their bi-dimensional complex WSST scalograms $\mathbf{S}_{U+N}$ and $\mathbf{S}_{R+N}$:

$$\begin{aligned} \mathbf{a}_{U+N} &\longrightarrow \mathcal{F}(\mathbf{a}_{U+N}) = \hat{\mathbf{a}}_{U+N}, \quad \mathcal{W}(\mathbf{a}_{U+N}) = \mathbf{S}_{U+N}, \\ \mathbf{a}_{R+N} &\longrightarrow \mathcal{F}(\mathbf{a}_{R+N}) = \hat{\mathbf{a}}_{R+N}, \quad \mathcal{W}(\mathbf{a}_{R+N}) = \mathbf{S}_{R+N}. \end{aligned} \quad (3.111)$$

The absolute values of the DFT/FFT spectra and WSST scalograms of simulated acceleration signals of $1/2 \times$ rub are all depicted in Figure 3.29 for four values of the SNR—10, 0 and -10 dB. Notice that when the signal noise level is either 10 or 0 dB both post-processing methods can still capture the periodical impulses of rub hidden behind the noise for $1/2 \times$ rub. Nevertheless, as the noise level increases beyond 0 dB signal noise becomes more and more prominent as the impulsive rub features become less noticeable. Still, the diagrams shown in Figure 3.29 only display the complex amplitudes the Fourier and Wavelet coefficients from Equation 3.111. Coefficient phases could still retain some information about rub impulse features concealed by the noise in the acceleration signals. We need to define and apply a proper quantifying parameter for full characterization of the ability of post-processing methods to differentiate between the states of normal operation and failure.

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FIGURE 3.29: WSST scalograms of the simulated casing accelerations under $1/2$ rub in Figure 3.25c. Diverse white noise levels were added to the signals before post-processing.

As a quantitative measure of the ability of each post-processing technique to distinguish between the no-rub and the rub conditions, we propose the following relative sensitivities of the Fourier—and Wavelet—coefficients of the noisy signals under rub and under unbalance without rub:

$$\begin{aligned} \epsilon_{SNR}(\hat{\mathbf{a}}_{U+N}, \hat{\mathbf{a}}_{R+N}) &= \frac{\|\hat{\mathbf{a}}_{U+N} - \hat{\mathbf{a}}_{R+N}\|}{\|\hat{\mathbf{a}}_{U+N}\|}, \\ \epsilon_{SNR}(\mathbf{S}_{U+N}, \mathbf{S}_{R+N}) &= \frac{\|\mathbf{S}_{U+N} - \mathbf{S}_{R+N}\|}{\|\mathbf{S}_{U+N}\|}, \end{aligned} \quad (3.112)$$

where SNR is a value of the signal-to-noise ratio—Equation 3.110—of white noise with respect to the nominal no-rub state signals, and $\|\cdot\|$ represents the complex two-norm. This positive index provides a measure of similarity between the acceleration spectra—or scalograms—from the machine without rub and the signal

from the machine under rub, both signals containing the same Gaussian background noise. So, the smaller the ϵ_{SNR} , the more difficult the spectra—or scalograms—of the rub and no-rub state signals are to distinguish. Figure 3.30 represents relative sensitivities of DFT/FFT and WSST coefficients over a range of SNRs between 10 and -10 dB.

FIGURE 3.30: Relative sensitivities of complex DFT/FFT spectra and WSST scalograms of noisy acceleration signals from the model with parameters in Table 3.1 to the presence of $1/2 \times$ rub. White noise SNRs are with respect to the power of the no-rub signals.

As expected, the sensitivities of DFT/FFT spectra coefficients to the presence of rub signals drop to zero as signal noise level increases. For the simulated $1/2 \times$ rub signals in Figure 3.25c, if the white noise levels are low, traditional DFT order analysis results more sensitive to the presence of rub than the WSST algorithm. However, as the signals contain higher noise levels, the latter may become more sensitive to the presence of rub features in the acceleration data, particularly when the wavelet mother function selected for the decomposition is the Analytic Morlet. This means that a properly designed and adjusted WSST-based algorithm could be advantageous for detection within highly noisy signals.

3.2.4 The Synchrosqueezed Wavelet Transform as the best alternative for detection of early single-point rub in aeroderivative turbines

The model of an aeroderivative gas turbine presented in Section 3.1 has succeeded in reproducing a common type of rub encountered in conventional turbomachinery: the supercritical partial single-point rub with period-two—chapter two of Bently et al., 2003a. This is the first time that this type of rub has been studied in the context of aeroderivative gas turbines, with a model able to mimic either a compressor or turbine section. Also for the first time, the features of casing vibration have been extracted and analyzed as the available source of condition information and of diagnosis of this type of rub.

Single-point rub is reflected in the casing acceleration and velocity signals as impulse trains—Figure 3.26. Each impulse marks the beginning or the end of rotor-casing contact. Since each one of those sharp peaks supposes a perturbation of the casing, they are followed by transient oscillations containing a number of rotor and casing vibration modes. the most notorious of those transients is the rigid solid

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casing mode in Figure 3.22c: the lowest of all the casing natural modes, and, therefore, the least affected by damping.

The Fourier spectra of the signals are mainly composed of impulse trains with fundamental frequency equal to the impact frequency: half the rotor rotation speed—Figure 3.24. The highest impulse harmonics in the spectra are the fundamental frequency of 30 Hz and the peaks lying next to the 200 Hz casing natural mode. This vibration amplification near a casing natural frequency due to rotor-casing rub has been observed in Tang et al., 2010. The cepstrum analysis of these spectra produces trains of impulses with fundamental frequency equal to the rub frequency—Figure 3.27, evidencing the periodicity of the rotor-casing rub.

The Fourier spectra and the cepstra in Figures 3.24 and 3.27, due to their averaging nature and the small frequency resolution—0.5 Hz, fail to make visible the rotor and casing transient modes excited by the rub perturbation. This elements can instead be identified and precisely localized in time with the application of the Continuous Wavelet Transform on the signals—Figure 3.28. Because of the very large time resolution of the WSST at high frequencies, the analysis of a short vector of acceleration samples containing just a few of the rotor-casing impacts suffices to confirm the presence of those brief vibration transients and to localize them at the two ends of each rotor-casing contact, confirming the existence of rotor-casing rub. The WSST scalograms of the velocity data are better suited to visualize the low-frequency features of the rotor-casing rub, including the fundamental rub frequency band and some of its low-band multiples.

The visualization of those short-lived pieces of evidence of rotor-casing rub in a Fourier spectrum with high frequency resolution would require collecting very large sampling vectors with large time intervals between the first and the last sample. The acquisition of large data sets is time-consuming and comes with a loss of precocity in the diagnosis. On the other hand, the WSST has the potential to identify rotor-casing rub with the analysis of only a few impacts in a very short time span with no more than a small processing time penalty, as shown in Table 3.4. This enables very early detection of this malfunction.

Another advantage of the WSST is its suitability for the analysis of very noisy acceleration data. Figure 3.30 proves the higher robustness of the WSST to white noise in the accelerometer signals, specially under a proper choice of the wavelet mother function—when picking up the Analytic Morlet instead of the Bump function. All in all, the WSST seems to be an appropriate post-processing method for the early detection of rub in aeroderivative gas turbines with accelerometer signals. In the next chapter we will validate this choice experimentally and provide further proof of the superior performance of a time-frequency method such as the WSST for this purpose.

Chapter 4

EXPERIMENTAL VALIDATION AND PARAMETER ADJUSTMENT OF THE PROPOSED METHODOLOGY FOR EARLY RUB DETECTION IN AERODERIVATIVE GAS TURBINES

4.1 An experimental rig reproducing the dynamic behavior of a gas turbine with flexible casing and rotor-casing single-point rub

In this chapter, we describe the experimental validation of the detection methodology proposed in Section 3.2.2 and the testing of several post-processing parameters—sampling frequencies, time series periods, etc., providing guidelines for an efficient use of the Synchrosqueezed Wavelet Transform or WSST for single-point rub detection within accelerometer signals, in compliance with the demands of the very early detection paradigm—Section 1.4. We also demonstrate quantitatively the superior sensitivity of the WSST-based approach to rub detection in contrast with the classic Fourier Transforms over both the experimental signals and the simulated data from the model described in Section 3.1, this time adjusted to fit the dynamics of the experimental model.

All this necessitates the design and construction of a proper experimental rotor-casing model able to reproduce the dynamics of single-point rub in aeroderivative gas turbines. Such a model will be described in this section.

4.1.1 Design and construction of the experimental setting

We designed and built, at the composites and nanocomposites lab of the *División de Ingeniería de Máquinas*—Machinery Engineering Division—, a rotor rig with the purpose of reproducing and measuring unbalance and single-point rub between the rotor and a flexible casing—Figure 4.1. The data collected under diverse conditions will provide validation of the detection methodology designed in Section 1.4 and is modelizable with the model described in Section 3.1.

The foundation is an assembly of 35×35 mm aluminum profiles supporting the rotor and casing, a 0.55 kW asynchronous motor and a speed variator. The rotor is composed of an AISI 4140 steel shaft with a 10 mm diameter and two steel disks, and is flexibly coupled to the motor with a jaw coupling. The casing is made of two identical semi-cylindrical carbon fiber-epoxy composite shells with isotropic

draping. The lower casing half can be seen in Figure 4.1b while the upper half is visible in Figures 4.1a and 4.1d. These two casing halves are vertically clamped to the foundation at four points at the casing ends—two per side. The rotor is joined to the casing halves with two 3D-printed cylindrical bearing supports made of PLA, each containing an SKF 6000 ball bearing. The axial position of the two disks guarantee that the first rotor bending critical speed is the lowest possible in order to maintain the rotor speed intervals during testing in a safe, low vibration range. The most important weights, dimensions and mechanical properties are collected in Table 4.2. This system was designed and set up to operate between its second and third critical speeds at about 95 Hz—5700 rpm. The commercial denominations of the most fundamental components of the rotor rig are shown in Table 4.1. For more details on its design, features and setup parameters we refer the reader to Machuca González, 2019.

TABLE 4.1: Specifications of rotor rig components.

Foundation aluminum profiles	Fasten 35×35 mm with 10 mm guides.
Motor speed variator	Siemens Micromaster 420 6SE6420-2AB15-5AA1.
Carbon fiber sheets	Torayca T700SC with prepreg.
Electric motor	Siemens 1LA7073-2AA10.
Rub spindle	M12×1.25 brass bolt
Coupling	Rotex No. 001 Aluminum 19.
Bearings	SKF 6000 ball bearing.
Shaft	10x600 mm, AISI 4140 grade steel

(A)

(B)

(C)