

w -cosmological singularities

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In this paper we characterize barotropic index singularities of homogeneous isotropic cosmological models [M. P. Dąbrowski and T. Denkiewicz, Phys. Rev. D **79**, 063521 (2009).]. They are shown to appear in cosmologies for which the scale factor is analytical with a Taylor series in which the linear and quadratic terms are absent. Though the barotropic index of the perfect fluid is singular, the singularities are weak, as it happens for other models for which the density and the pressure are regular.

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I. INTRODUCTION

The observational evidence from different sources [1–5] for the present stage of accelerated expansion of our Universe has driven the quest for theoretical explanations of such feature. Assuming the validity of the theory of gravity, one attempt of explanation is the existence of a disregarded, but dominant at present time, ingredient of the energy content of the Universe, known as dark energy [6–8], with unusual physical properties. The other possibility is modifying the general theory of relativity at large scales [9–11].

Both approaches have contributed to change our view of the final state of the Universe. Before the discovery of the accelerated expansion of the Universe, only two possibilities were considered. Either our Universe would expand forever or the matter content would force a contraction and recollapse of the Universe in a final big crunch.

Observations compatible with a barotropic index $w = p/\rho$ lower than -1 pointed out a final singularity in the form of an infinite scale factor of the Universe, named the big rip [12]. Other models were postulated and the family of candidates increased. The price to pay was violation of one or several energy conditions and hence these possibilities were not considered in classical theorems of singularities [13]. Among these we may find:

- (i) Sudden singularities: Finite-time singularities for which the weak and strong energy conditions hold, but the pressure of the cosmological fluid blows up whereas the density remains finite [14]. If the second derivative of the scale factor is positive, they are called big boost singularities [15]. Related to braneworld models for which the embedding of the brane in the bulk is singular at some point they have also been named quiescent singularities [16]. However, the name quiescent appeared originally in a different context in [17] related to nonoscillatory singularities.
- (ii) Generalized sudden singularities: These are finite-time singularities with finite density and pressure

[18] instead of diverging pressure. Again in the braneworld context they have also been called quiescent [19], though this name had already been assigned to sudden singularities.

- (iii) Big brake: These singularities originally arose in tachyonic models and are characterized by a negative infinite second derivative of the scale factor whereas the first derivative vanishes and the scale factor remains finite [20]. They are consequently a subcase of sudden singularities.
- (iv) Big freeze: These singularities were detected in generalized Chaplygin models and are characterized by a finite scale factor and an infinite density [21].
- (v) Inaccessible singularities: These singularities appear in cosmological models with toral spatial sections, due to infinite winding of trajectories around the tori. For instance, compactifying spatial sections of the de Sitter model to cubic tori. However, these singularities cannot be reached by physically well-defined observers. This fact suggests the name of inaccessible singularities [22].
- (vi) Directional singularities: Curvature scalars vanish at the singularity but there are causal geodesics along which the curvature components diverge [23]. That is, the singularity is encountered just for some observers. In a general framework they were dubbed p,p curvature singularities (curvature singularities with respect to a parallelly propagated basis) in [13].

Most of them are compiled in a classification due to Nojiri, Odintsov, and Tsujikawa in terms of which physical quantities blow up [24]:

- (i) Big bang/crunch: Zero a , divergent H , density and pressure.
- (ii) Type I: “Big rip”: Divergent a , density and pressure.
- (iii) Type II: “Sudden”: Finite a , H , density, divergent \dot{H} , and pressure. They enclose the big brake and most of quiescent singularities.
- (iv) Type III: “Big freeze” or “finite scale factor singularities”: finite a , divergent H , density and pressure.

- (v) Type IV: “Generalized sudden”: Finite a , H , \dot{H} , density, pressure, divergent higher derivatives. They comprise the subcase of quiescent singularities with finite pressure.

This classification is refined further in [25,26].

Since inaccessible and directional singularities are not in principle related to divergences in curvature scalars, they would fall out of this scheme.

Some of these cannot be taken as the end of the Universe, since the spacetime can be extended continuously beyond the singularity [27–29]. The case of a string surviving a sudden singularity is proven in [30].

In [31] a cosmological model with just a singular barotropic index at $t = t_s$ is described,

$$a(t) = \frac{a_s}{1 - \frac{3\gamma}{2} \left(\frac{n-1}{n-3\gamma}\right)^{n-1}} + \frac{1 - \frac{2}{3\gamma}}{n - \frac{2}{3\gamma}} \frac{na_s}{1 - \frac{2}{3\gamma} \left(\frac{n-2}{n-1}\right)^{n-1}} \left(\frac{t}{t_s}\right)^{2/3\gamma} + \frac{a_s}{\frac{3\gamma}{2} \left(\frac{n-1}{n-3\gamma}\right)^{n-1} - 1} \left(1 - \frac{1 - \frac{2}{3\gamma}}{n - \frac{2}{3\gamma}} \frac{t}{t_s}\right)^n, \quad (1)$$

where $\gamma > 0$, in order to prevent the model from becoming phantom, and $n \neq 1$. The constant $\gamma = w - 1$ is related to the barotropic index w near the big bang at $t = 0$. We shall use the subindex s throughout the paper to refer to quantities calculated at the time of the singularity t_s .

The scale factor, $a(t_s) = a_s$ is regular and the density and the pressure vanish at t_s . Furthermore, if n is natural, the derivatives of the Hubble parameter are regular either. However, the effective barotropic index w is infinite at t_s .

In this paper we would like to characterize these w -singularities in Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models.

In the next section we obtain the cases for which the barotropic index is singular and check which of them have vanishing fluid density and pressure at the singularity. Finally, the cases with singularities in higher derivatives of the scale factor are removed. A final section of conclusions is included.

II. SINGULARITIES IN BAROTROPIC INDEX

The total content of a FLRW spacetime is described as a perfect fluid of density ρ and pressure p . Since both of them are functions of just the time coordinate, the fluid has at least locally an equation of state $p = p(\rho)$. The quotient of both is the barotropic index, $w = p/\rho$, which is also a function of time. Focusing on flat cosmologies,

$$ds^2 = -dt^2 + a^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \quad (2)$$

the barotropic index is constant just for power-law flat, $w \neq -1$, and de Sitter models, $w = -1$.

From Friedmann equations for the effective pressure and energy density

$$H = \frac{\dot{a}}{a}, \quad 3H^2 = \rho, \quad \dot{\rho} + 3H(\rho + p) = 0, \quad (3)$$

we get the expression for the barotropic index w ,

$$w = -\frac{1}{3} - \frac{2}{3} \frac{a\ddot{a}}{\dot{a}^2}, \quad (4)$$

in terms of the derivatives of the scale factor $a(t)$.

Assuming that the scale factor admits a generalized power expansion [28,32] of the form

$$a(t) = c_0(t_s - t)^{\eta_0} + c_1(t_s - t)^{\eta_1} + \dots, \quad (5)$$

$$\eta_0 < \eta_1 < \dots, \quad c_0 > 0,$$

around a value t_s , with real exponents, we may expand the barotropic index accordingly:

- (i) If $\eta_0 \neq 0$,

$$w = -\frac{1}{3} - \frac{2}{3} \left(\frac{\eta_0 - 1}{\eta_0} + \frac{c_1 \eta_1 (\eta_1 - 1)}{c_0 \eta_0^2} \right) \times (t_s - t)^{\eta_1 - \eta_0} + \dots$$

$$\times \left(1 + \frac{c_1}{c_0} (t_s - t)^{\eta_1 - \eta_0} + \dots \right)$$

$$\times \left(1 - \frac{c_1 \eta_1}{c_0 \eta_0} (t_s - t)^{\eta_1 - \eta_0} + \dots \right)^2$$

$$\simeq -\frac{1}{3} - \frac{2}{3\eta_0} (\eta_0 - 1) - \frac{2c_1}{3c_0 \eta_0}$$

$$\times \left(\eta_0 - 1 + \frac{\eta_1 (\eta_1 - 2\eta_0 + 1)}{\eta_0} \right) (t_s - t)^{\eta_1 - \eta_0},$$

the result is obviously consistent at $t = t_s$ with a linear barotropic perfect fluid, for which $\eta_0 = 2/3(1 + w_s)$ with finite w_s . In the limit of large η_0 de Sitter-like models would appear.

- (ii) If $\eta_0 = 0$, the expansion becomes more involved,

$$w = -\frac{1}{3} - \frac{2}{3\eta_1} \left(\eta_1 - 1 + \frac{c_2 \eta_2 (\eta_2 - 1)}{c_1 \eta_1} \right) \times (t_s - t)^{\eta_2 - \eta_1} + \dots$$

$$\times \left(\frac{c_0}{c_1} (t_s - t)^{-\eta_1} + 1 + \frac{c_2}{c_1} (t_s - t)^{\eta_2 - \eta_1} + \dots \right)$$

$$\times \left(1 - \frac{c_2 \eta_2}{c_1 \eta_1} (t_s - t)^{\eta_2 - \eta_1} + \dots \right)^2$$

$$\simeq -\frac{1}{3} - \frac{2c_0}{3c_1 \eta_1} (\eta_1 - 1) (t_s - t)^{-\eta_1}$$

$$- \frac{2c_0 c_2 \eta_2}{3c_1^2 \eta_1^2} (\eta_2 - 2\eta_1 + 1) (t_s - t)^{\eta_2 - 2\eta_1}$$

$$- \frac{2(\eta_1 - 1)}{3\eta_1},$$

since several possibilities arise:

If $\eta_1 \neq 1$, the barotropic index diverges as a power $(t_s - t)^{-\eta_1}$.

If $\eta_1 = 1$, depending on the value of η_2 ,

$$w \simeq -\frac{1}{3} - \frac{2c_0c_2\eta_2(\eta_2 - 1)}{3c_1^2}(t_s - t)^{\eta_2 - 2},$$

we may have a singular barotropic index for $\eta_2 \in (1, 2)$ and a regular one for $\eta_2 > 2$. The subcase $\eta_2 = 2$,

$$w \simeq -\frac{1}{3} - \frac{4c_0c_2}{3c_1^2} + \left(\frac{16c_0c_2^2}{3c_1^3} - \frac{4c_2}{3c_1}\right)(t_s - t) - \frac{2c_0c_3\eta_3}{3c_1^2}(t_s - t)^{\eta_3 - 2},$$

produces also a regular w around t_s .

Models with scale factors admitting no generalized power series, typically models with $a(t) \sim e^{b/(t_s - t)^p}$, $p > 0$, produce finite barotropic indices of the form

$$w \sim -1 - \frac{2}{3} \frac{p + 1}{bp} (t_s - t)^p,$$

and are therefore no candidates for producing w -singularities.

A directional singularity of the type of [23] cannot be a w -singularity since the former has a finite barotropic index.

Therefore, the only chances for a diverging barotropic index arise for $\eta_0 = 0$, $\eta_1 \neq 1$ or $\eta_0 = 0$, $\eta_1 = 1$, $\eta_2 < 2$, as consigned in Table I.

In order to get a w -singularity, besides a diverging barotropic index, we need vanishing density and pressure,

$$\rho = 3\left(\frac{\dot{a}}{a}\right)^2, \quad p = -\left(\frac{\dot{a}}{a}\right)^2 - \frac{2\ddot{a}}{a}. \quad (6)$$

We check these conditions for both singular cases:

(1) $\eta_0 = 0$, $\eta_1 \neq 1$: $a(t) = c_0 + c_1(t_s - t)^{\eta_1} + \dots$.

$$\rho = \frac{3c_1^2\eta_1^2}{c_0^2}(t_s - t)^{2(\eta_1 - 1)} + \dots,$$

$$p = -\frac{2c_1\eta_1(\eta_1 - 1)}{c_0}(t_s - t)^{\eta_1 - 2} + \dots.$$

The expansions show that density tends to zero for $\eta_1 > 1$, whereas a vanishing pressure requires $\eta_1 > 2$.

TABLE I. Singularities in barotropic index.

η_0	η_1	η_2	w_s
$\neq 0$	(η_0, ∞)	(η_1, ∞)	Finite
0	(0, 1)	(η_1, ∞)	Infinite
	1	(1, 2)	Infinite
	1	[2, ∞)	Finite
	(1, ∞)	$[\eta_1, \infty)$	Infinite

We have then both vanishing density and pressure and divergent barotropic index for $\eta_0 = 0$, $\eta_1 > 2$.

(2) $\eta_0 = 0$, $\eta_1 = 1$, $\eta_2 < 2$: $a(t) = c_0 + c_1(t_s - t) + c_2(t_s - t)^{\eta_2} + \dots$.

$$\rho = \frac{3c_1^2}{c_0^2} + \frac{6\eta_2c_1c_2}{c_0^2}(t_s - t)^{\eta_2 - 1} - 6\left(\frac{c_1}{c_0}\right)^3(t_s - t) + \dots,$$

$$p = -\frac{2c_2\eta_2(\eta_2 - 1)}{c_0}(t_s - t)^{\eta_2 - 2} - \frac{c_1^2}{c_0^2} + \dots.$$

Since the density is finite and the pressure diverges in this case, it cannot be a w -singularity, but a sudden singularity.

For vanishing pressure and density and divergent barotropic index we are left just with the $\eta_0 = 0$, $\eta_1 > 2$ case:

A FLRW cosmological model has a singular barotropic index w with vanishing pressure and density at a finite time t_s if and only if the generalized power expansion of the scale factor $a(t)$ is of the form

$$a(t) = c_0 + c_1(t_s - t)^{\eta_1} + \dots, \quad (7)$$

with $\eta_1 > 2$.

If we allow finite pressure, the condition is relaxed to $\eta_1 > 1$.

Finally, since the scale factor does not vanish at t_s the only possibility for a singularity in higher derivatives of the Hubble factor is that a derivative of the scale factor (7) blows up. If η_1 is noninteger, there will be derivatives $a^{(p)}(t) \sim c_1(t_s - t)^{\eta_1 - p}$ which blow up for $p > \eta_1$.

The only way to prevent this is to require that η_1 be natural. But then the reasoning would be the same for η_2 and the subsequent exponents. Hence, the only possibility to avoid a diverging derivative of the scale factor is that every exponent η_i be natural. But in this case the series is no longer a generalized power series, but a Taylor series. Since $\eta_1 > 2$, the lowest power would be at least three:

A FLRW cosmological model has a w -singularity at a finite time t_s if and only if the scale factor $a(t)$ admits a Taylor series at t_s with vanishing linear and quadratic terms,

$$a(t) = c_0 + \sum_{n=3}^{\infty} c_n(t_s - t)^n. \quad (8)$$

If we allow finite pressure, then just the linear term is to vanish.

III. DISCUSSION

Cosmological models with generalized power expansions of the scale factor have been discussed in [28]. The exponents of the power expansion are related to the appearance of cosmological singularities, which can be strong or weak.

TABLE II. Singularities in cosmological models.

η_0	η_1	η_2	$\{\eta_i\}$	Tipler	Królak	Nojiri, Odintsov, and Tsujikawa
$(-\infty, 0)$	(η_0, ∞)	(η_1, ∞)	I	Strong	Strong	I
0	$(0, 1)$	(η_1, ∞)	S	Weak	Strong	III
	1	$(1, 2)$	S	Weak	Weak	II
		$[2, \infty)$	S	Weak	Weak	IV
	$(1, 2)$	(η_1, ∞)	S	Weak	Weak	II
	$[2, \infty)$	(η_1, ∞)	S	Weak	Weak	IV
	$[3, \infty)^a$	$[\eta_1 + 1, \infty)$	N	Weak	Weak	w
$(0, \infty)$	(η_0, ∞)	(η_1, ∞)	I	Strong	Strong	Crunch

^aIf we include finite pressure w -singularities, then $\eta_1 \in [2, \infty)$.

Weak singularities are not actual singularities in the sense that the spacetime can be extended continuously beyond the singularity. Or, put in another way, from the physical point of view, a finite object is not necessarily crushed on crossing a weak singularity. The classification of singularities [28] in terms of the exponents of the scale factor expansion is recorded in Table II.

The column $\{\eta_i\}$ stands for the properties of the exponents of the expansion: I means no additional condition on them, S means that at least one exponent must be non-natural in order to have a singularity in one of the derivatives and N means that every exponent is natural.

The difference between Tipler's [33] and Królak's [34] criterion for the strength of singularities is just that, whereas the former requires the volume of finites objects to tend to zero at a strong singularity, the latter just imposes the derivative of the volume with respect to proper time to be negative, which is a milder requirement. Conditions for

checking both criteria may be found in [35]. Another criterion is the one in [36].

All cosmological models with w -singularities therefore belong to the last but one line of the classification and hence we may conclude that w -singularities are weak singularities.

Therefore, the diverging barotropic index for w -singularities, which is not shared necessarily by type IV singularities, does not influence the weak character of both families of singularities.

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