ARTICLE TYPE

A dichotomy result about Hessenberg matrices associated to measures in the unit circle.

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Summary

We characterize Hessenberg matrices $\mathbf{D}$ associated to measures in the unit circle ν which are matrix representations of compact, and actually Hilbert Schmidt, perturbations of the forward shift operator as those with recursion coefficients $\{\alpha_n\}_{n=0}^\infty$ verifying $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$, i.e. associated to measures verifying Szegő condition. As a consequence we obtain the following dichotomy result for Hessenberg matrices associated to measures in the unit circle: either $\mathbf{D} = \mathbf{S}_R + \mathbf{K}_2$ with $\mathbf{K}_2$ a Hilbert Schmidt matrix or there exists a unitary matrix $\mathbf{U}$ and a diagonal matrix Λ such that $\mathbf{D} = \mathbf{U}^* \Lambda \mathbf{U} + \mathbf{K}_2$ with $\mathbf{K}_2$ a Hilbert Schmidt matrix. Moreover, we prove that for $1 \leq p \leq 2$, if $\sum_{n=0}^\infty |\alpha_n|^p < \infty$ then $\mathbf{D} = \mathbf{S}_R + \mathbf{K}_p$ with $\mathbf{K}_p$ an absolutely p -summable matrix inducing an operator in the p -Schatten class. Some applications are given to classify measures on the unit circle.

KEYWORDS:

Hermitian positive matrices, compact perturbation, Hessenberg matrices, orthogonal polynomial

1 | INTRODUCTION

One of the most interesting topics in the theory of orthogonal polynomials on the unit circle (OPUC), see among many classic references¹, is its connections with other areas of current interest, like Khrushchev theory, Quantum walks, CMV matrices, Asymptotic quantum algorithm for the Toeplitz systems, see for example^{2,3} and references therein.

The Hessenberg matrix is the representation of the multiplication operator by z in terms of the orthonormal polynomials sequence, that is known in the literature as GGT matrix, and CMV matrix from Laurent orthogonal polynomials. The properties of these matrices can be used to obtain properties of the associated measure, for example matrix methods have been used to obtain quadrature formulas on the unit circle in⁴ and in⁵ properties of the perturbed measure was obtained.

Our goal is to analyze Hessenberg matrices associated to measures in the unit circle taking into account the asymptotic behaviour of the Hessenberg matrix. In this sense we will also use matrix methods and asymptotic linear algebra to classify measures on the unit circle taking into account this asymptotic behaviour.

First of all, we will provide a basic framework of the theory of orthogonal polynomials.

Let $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$ be an infinite Hermitian matrix, i.e., $c_{i,j} = \overline{c_{j,i}}$ for all i, j non-negative integers. Following⁶ we say that $\mathbf{M}$ is positive definite if $|\mathbf{M}_n| > 0$ for all $n \geq 0$, where $\mathbf{M}_n$ is the truncated matrix of size $(n+1) \times (n+1)$ of $\mathbf{M}$. An infinite Hermitian positive definite matrix (in short, an HPD matrix) defines an inner product $\langle \cdot, \cdot \rangle$ in the space $\mathbb{P}[z]$ of all polynomials with complex coefficients in the following way: if $p(z) = \sum_{k=0}^n v_k z^k$ y $q(z) = \sum_{k=0}^m w_k z^k$ then

$$\langle p(z), q(z) \rangle = v \mathbf{M} w^*$$

being $v = (v_0, \dots, v_n, 0, 0, \dots)$, $w = (w_0, \dots, w_m, 0, 0, \dots) \in c_{00}$ where c_{00} is the space of all complex sequences with only finitely many non-zero entries.

The most interesting class of HPD matrices are those which are moment matrices with respect to a measure μ , i.e., HPD matrices $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$, such that there exists a representing measure μ with infinite support on $\mathbb{C}$ and finite moments for all $i, j \geq 0$,

$$c_{i,j} = \int z^i \overline{z}^j d\mu.$$

In this case we denote by $\mathbf{M} = \mathbf{M}(\mu)$. For more information concerning the characterization of HPD matrices which are moment matrices with respect to a certain measure μ with support on $\mathbb{C}$ see among others^{7, 8} and⁹.

An HPD matrix $\mathbf{M} = (c_{i,j})_{i,j=0}^{\infty}$ is the Gram matrix of the inner product in $\mathbb{P}[z]$ with respect to $\{z^n\}_{n=0}^{\infty}$. Thorough the paper the following notation is used: Let $\{\varphi_n(z)\}_{n=0}^{\infty}$ denote the unique sequence of orthonormal polynomials induced by $\mathbf{M}$ with positive leading coefficient denoted by κ_n , let $\Phi_n(z) = \kappa_n^{-1} \varphi_n(z)$ denote the monic orthogonal polynomial of degree n and let $P^2(\mathbf{M})$ denote the completion of the vector space $\mathbb{P}[z]$ with the Hilbertian norm induced by $\mathbf{M}$.

Let Λ_z be the linear mapping from $\mathbb{P}[z]$ into $\mathbb{P}[z]$ defined by $\Lambda_z(p(z)) := zp(z)$, with Λ_z not necessarily continuous, and let $\mathbf{D} = (d_{i,j})_{i,j=0}^{\infty}$ denote the matrix representation of Λ_z with respect to $\{\varphi_n(z)\}_{n=0}^{\infty}$. Clearly this matrix is a Hessenberg matrix, i.e. $d_{i,j} = 0$ if $i - j > 1$, and it has positive subdiagonal $d_{i,i+1} > 0$ for all $i \geq 0$.

In¹⁰, for finite dimensional matrices, the connections of normal Hessenberg matrices with subdiagonal positive with moment matrices associated to measures with finite support were studied, to conclude that there are few normal finite dimensional Hessenberg matrices. We here always consider *infinite* dimensional matrices. In¹¹ a one to one correspondence was established

between the set of all Hessenberg matrices with positive subdiagonal and the normalized HPD matrices. In this context, normal Hessenberg matrices are those verifying that the corresponding HPD matrix is indeed a moment matrix with respect to a compactly supported measure μ in the complex plane verifying that polynomials are dense in $L^2(\mu)$.

Most of the results concerning Hessenberg matrices are obtained for Toeplitz positive definite matrices $\mathbf{T}$. In this case it is well known the existence of a measure ν supported in the unit circle $\mathbb{T}$ such that $\mathbf{T}$ is indeed the associated moment matrix. The matrix elements of $\mathbf{D}$ are expressible in terms of the Verblunsky coefficients, a fact rediscovered many times (see e.g.¹²). In¹ this matrix is called the *GGT representation* after Geronimus¹³, Gragg¹⁴, and Teplyaev¹⁵. We point out that in¹⁴ and¹¹ this matrix description is obtained using only an algebraical approach and without using the fact of the existence of an associated measure. We use this algebraical description in order to study the asymptotic behaviour and properties of summability of the entries of the Hessenberg matrix $\mathbf{D}$ associated to an HPD Toeplitz matrix $\mathbf{T}$ in terms of the asymptotic behaviour and summability properties of the associated Verblunsky coefficients. In this direction, in¹⁶, the authors introduce the notion of *asymptotic Toeplitz* matrix, very closely related to the notion of asymptotic Toeplitz operator that appears in¹⁷. We first show that property $\lim_{n \rightarrow \infty} \Phi_n(0) = 0$ can be characterized with $\mathbf{D}$ being asymptotic Toeplitz to the unilateral shift right matrix $\mathbf{S}_R$. At this point, one can wonder if $\mathbf{D} - \mathbf{S}_R$ is the matrix representation of a compact, or more precisely, Hilbert Schmidt operator. We will prove that this is not true in general. Indeed, we prove that this only happens when $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$.

The paper is organized as follows. In the first section we prove that if $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ then the infinite matrix $\mathbf{D} - \mathbf{S}_R$ is a Hilbert-Schmidt matrix, that is, with absolutely square summable entries. For $p \geq 1$ we introduce matrices which are absolutely p -summable and we generalize this result to the case $1 \leq p < \infty$ relating it to the p -Schatten class of operators. Finally, using the well known Weyl-von-Neumann theorem (see¹⁸), which establishes that every normal operator is a diagonal operator plus a Hilbert-Schmidt operator, and using our previous results we conclude with a dichotomy result concerning CGT Hessenberg matrices associated to HPD Toeplitz matrices $\mathbf{T}$: either $\mathbf{D} - \mathbf{S}_R$ is Hilbert Schmidt matrix or there exists an unitary matrix $\mathbf{U}$ and a diagonal matrix Δ such that $\mathbf{D} - \mathbf{U}^* \Delta \mathbf{U}$ is a Hilbert Schmidt matrix. In the case of $1 \leq p \leq 2$ we prove that if $\sum_{n=0}^{\infty} |\alpha_n|^p < \infty$ then $\mathbf{D} = \mathbf{S}_R + \mathbf{K}_p$ with $\mathbf{K}_p$ an absolutely p -summable matrix inducing an operator in the p -Schatten class.

In the second section we summarize the well known results concerning different types of measures in the unit circle in order to classify such measures in terms of their associated Hessenberg matrices as asymptotic limit or perturbations by Hilbert-Schmidt matrices of the shift right matrix.

First of all, we will have in mind the following comments. We are mainly concerned with infinite matrices and, in order to avoid confusion, we distinguish infinite matrices and operators using different notation: an infinite matrix $\mathbf{A}$ defines a linear operator from the sequence space c_{00} to c_{00} if for every $v = (v_i)_{i=0}^{\infty} \in c_{00}$ the formal matrix product $\mathbf{A}v$ is defined and belongs to c_{00} . In our context, we will work in the vector space of the polynomials $\mathbb{P}[z]$ via the identification $p(z) = \sum_{k=0}^n a_k z^k$ with $(a_0, a_1, \dots, a_n, 0, \dots)$. Thus, every Hessenberg matrix $\mathbf{D}$ defines a linear operator, not necessary bounded, on $\mathbb{P}[z]$. If $\mathbf{D}$ induces

a bounded operator $D : \mathbb{P}[z] \rightarrow \mathbb{P}[z]$ with the norm induced by an HPD matrix $\mathbf{M}$ it can be extended to an operator D in the Hilbert space $P^2(\mathbf{M})$. Thorough all the paper our underlying Hilbert space will be $P^2(\mathbf{M})$ with the orthonormal basis $\{\varphi_n(z)\}_{n=0}^\infty$. On the other hand, for every bounded operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with an orthonormal basis, say $\{v_i\}_{i=0}^\infty$, the matrix representation of this operator with respect such basis is by constructing the infinite matrix with entries $a_{i,j} = \langle \mathcal{A}v_j, v_i \rangle$, i.e., $\mathbf{A} = (a_{i,j})_{i,j=0}^\infty$. The forward shift operator in a Hilbert space with basis $\{e_n\}_{n=0}^\infty$ denoted by S is the operator such that $S(e_n) = e_{n+1}$ which representing matrix with respect to such basis is the unilateral shift right matrix $\mathbf{S}_R$.

Following Barria and Halmos¹⁷ a bounded operator $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ is a *weakly asymptotic Toeplitz operator* if the sequence $S^{*n} \mathcal{A} S^n$ is strongly convergent. In this direction, in¹⁶ they introduce the following notion: an infinite matrix $\mathbf{A} = (a_{i,j})_{i,j=0}^\infty$ is an *asymptotic Toeplitz matrix* if for every $k \in \mathbb{Z}$ there exists $\lim_{n \rightarrow \infty} a_{n,n+k} = \tilde{a}_k$. In such a case it is denoted by $\text{Lim}(\mathbf{A}) = (\tilde{a}_{i-j})_{i,j=0}^\infty$, which obviously is a Toeplitz matrix. It is obvious that matrix representations of weakly asymptotic Toeplitz operators on Hilbert spaces are asymptotic Toeplitz matrices.

2 | ASYMPTOTICS OF HESSENBERG MATRICES ASSOCIATED TO TOEPLITZ MATRICES: COMPACT PERTURBATIONS.

From an HPD matrix $\mathbf{M}$ with $c_{00} = 1$ there is an unique Hessenberg matrix with positive subdiagonal $\mathbf{D}$ which is the matrix representation of the multiplication by z linear mapping with respect to the orthonormal basis induced in $\mathbb{P}[z]$ by $\mathbf{M}$. There is a reciprocal procedure to obtain from an infinite Hessenberg matrix with subdiagonal positive the unique suitable HPD Hermitian matrix as we establish in the following proposition. For the proof see e.g.¹¹:

Proposition 1. Let $\mathbf{D} = (d_{i,j})_{i,j=0}^\infty$ be an upper Hessenberg matrix with positive subdiagonal, i.e. $d_{i+1,i} > 0$ for every $i \in \mathbb{N}_0$. Then, there exists an unique HPD matrix $\mathbf{M}$ with $c_{00} = 1$ such that $\mathbf{D}$ is the matrix representation of the multiplication by z operator Λ_z defined in $\mathbb{P}[z]$ with respect to the orthonormal basis induced by the inner product given by $\mathbf{M}$.

Using Proposition 1, a correspondence can be established between the subset of normalized HPD matrices, say $\mathcal{M}_0 = \{\mathbf{M} : \mathbf{M} \text{ is HPD and } c_{00} = 1\}$, and the subset of Hessenberg matrices with positive subdiagonal, say $\mathcal{H}^+ = \{\mathbf{D} = (d_{i,j})_{i,j=0}^\infty : \mathbf{D} \text{ is Hessenberg with } d_{i+1,i} > 0, i \in \mathbb{N}_0\}$. More precisely, there is a one to one correspondence $\mathcal{B} : \mathcal{M}_0 \rightarrow \mathcal{H}^+$ in such a way that for every $\mathbf{M} \in \mathcal{M}_0$ the Hessenberg matrix $\mathcal{B}(\mathbf{M})$ is the matrix representation of the multiplication operator $\Lambda_z : \mathbb{P}[z] \rightarrow \mathbb{P}[z]$ with respect to $\{v_i\}_{i=0}^\infty$, induced by $\mathbf{M}$.

Recall that a bounded operator D on a Hilbert space $\mathcal{H}$ is a *normal* operator if $D^*D = DD^*$, and *unitary* if $D^*D = DD^* = \mathbf{I}$, where D^* is the adjoint operator. It is a *subnormal* operator if it is the restriction of a normal operator on an invariant subspace. From the results in^{7, 20, 21}

Corollary 1. Let $\mathbf{M} \in \mathcal{M}_0$,

1. There is a representing probability measure μ with bounded support on $\mathbb{C}$ such that $\mathbf{M} = \mathbf{M}(\mu)$ if and only if $B(\mathbf{M}) = \mathbf{D}$ induces a *subnormal* bounded operator on $P^2(\mathbf{M})$.
2. There is a representing probability measure μ with bounded support on $\mathbb{C}$ such that $\mathbf{M} = \mathbf{M}(\mu)$ and $P^2(\mu) = L^2(\mu)$ if and only if $B(\mathbf{M}) = \mathbf{D}$ induces a bounded *normal* operator on $P^2(\mathbf{M})$.

Remark 1. From the above result it follows that, for compactly supported measures μ , density of polynomials in $L^2(\mu)$ can be characterized with Hessenberg matrices inducing normal operators. Nevertheless, we do not know a characterization of such property of completeness via the moment matrix $\mathbf{M}(\mu)$. In this direction, we point out that in⁶ a necessary, but not sufficient, condition is given in terms of the behaviour of the smallest eigenvalues of the truncated matrices $\mathbf{M}_n$ of size $(n+1) \times (n+1)$ of the matrix $\mathbf{M}(\mu)$ denoted by $\lambda_{\min}(\mathbf{M}_n)$. More precisely, it is proved that if $P^2(\mu) = L^2(\mu)$ then $\lim_{n \rightarrow \infty} \lambda_{\min}(\mathbf{M}_n) = 0$. At this point, we introduce the following definition: a measure μ is *separating* if $\lim_{n \rightarrow \infty} \lambda_{\min}(\mathbf{M}_n) > 0$.

The description of the matrix $\mathbf{D}$ associated to an infinite HPD Toeplitz matrix is very well known (see e.g.¹⁴) and it can be obtained from a linear algebra point of view, without using orthogonal polynomials theory. For more details it can be seen also¹¹.

$$\mathbf{D} = \begin{pmatrix} -\Phi_1 \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_1} \Phi_2 \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_2} \Phi_3 \bar{\Phi}_0 & \dots & -\frac{\kappa_0}{\kappa_{n-2}} \Phi_{n-1} \bar{\Phi}_0 & -\frac{\kappa_0}{\kappa_{n-1}} \Phi_n \bar{\Phi}_0 & \dots \\ \frac{\kappa_0}{\kappa_1} & -\Phi_2 \bar{\Phi}_1 & -\frac{\kappa_1}{\kappa_2} \Phi_3 \bar{\Phi}_1 & \dots & -\frac{\kappa_1}{\kappa_{n-2}} \Phi_{n-1} \bar{\Phi}_1 & -\frac{\kappa_1}{\kappa_{n-1}} \Phi_n \bar{\Phi}_1 & \dots \\ 0 & \frac{\kappa_1}{\kappa_2} & -\Phi_3 \bar{\Phi}_2 & \dots & -\frac{\kappa_2}{\kappa_{n-2}} \Phi_{n-1} \bar{\Phi}_2 & -\frac{\kappa_2}{\kappa_{n-1}} \Phi_n \bar{\Phi}_2 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \dots & -\Phi_{n-1} \bar{\Phi}_{n-2} & -\frac{\kappa_{n-2}}{\kappa_{n-1}} \Phi_n \bar{\Phi}_{n-2} & \dots \\ 0 & 0 & 0 & \dots & \frac{\kappa_{n-2}}{\kappa_{n-1}} & -\Phi_n \bar{\Phi}_{n-1} & \dots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\Phi_i := \Phi_i(0)$ and $\kappa_i/\kappa_{i+1} = \sqrt{1 - |\Phi_i(0)|^2}$ for all $i \geq 0$.

In the sequel, in order to simplify the notation, we will denote by $\alpha_n = \Phi_n(0)$. More generally, for every sequence $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = 1$ and $|\alpha_n| < 1$ for all $n \in \mathbb{N}$ we may construct an associated Hessenberg $\mathbf{D}(\alpha_n)$ matrix replacing $\Phi_n(0)$ by α_n and defining $\kappa_n = \kappa_{n-1}/\sqrt{1 - |\alpha_n|^2}$, for $n \in \mathbb{N}$ and $\kappa_0 = 1$.

In the following result we characterize Hessenberg matrices $\mathbf{D}(\alpha_n)$ which are asymptotically Toeplitz to $\mathbf{S}_R$ in terms of the asymptotic behaviour of the sequence (α_n) :

Lemma 1. Let $\{\alpha_n\}_{n=0}^\infty$ with $\alpha_0 = 1$ and $|\alpha_n| < 1$ for all $n \in \mathbb{N}$ and the associated Hessenberg matrix $\mathbf{D}(\alpha_n)$. The following statements are equivalent

1. $\lim_{n \rightarrow \infty} \alpha_n = 0$.

2. The matrix $\mathbf{D}(\alpha_n)$ is asymptotic Toeplitz and $\text{Lim}(\mathbf{D}(\alpha_n)) = \mathbf{S}_R$.

Proof. Assume that $\lim_{n \rightarrow \infty} \alpha_n = 0$; since $\kappa_{n-1}/\kappa_n = \sqrt{1 - |\alpha_n|^2}$, it follows that the subdiagonal of the matrix $\mathbf{D}(\alpha_n)$ converges to 1. On the other hand, for every $k \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \alpha_n \overline{\alpha_{n+k}} = 0.$$

and since $\kappa_{n-1}/\kappa_n \leq 1$ all the diagonals converge to zero. The reciprocal result is obvious. $\square$

One can wonder whether in the case that $\lim_{n \rightarrow \infty} \alpha_n = 0$ the bounded operator $D(\alpha_n)$ on $P^2(\mathbf{M})$ induced by $\mathbf{D}(\alpha_n)$ is a compact perturbation of the forward shift operator. The main result in this section is to prove that this only happens when $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$; moreover, if such is the case, we can prove that $D(\alpha_n)$ is even a perturbation by a Hilbert-Schmidt operator of S . In order to prove it we show the following technical lemma concerning infinite matrices:

Lemma 2. Let $p \geq 1$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence with $\alpha_0 = 1$ and $|\alpha_n| < 1$ for all $n \in \mathbb{N}$ and the associated Hessenberg matrix $\mathbf{D} = \mathbf{D}(\alpha_n)$.

1. Assume $\sum_{n=0}^{\infty} |\alpha_n|^p < \infty$, then $\sum_{i,n=0}^{\infty} |(\mathbf{S}_R - \mathbf{D})[i, n]|^p < \infty$.
2. Assume $\sum_{i,n=0}^{\infty} |(\mathbf{S}_R - \mathbf{D})[i, n]|^p < \infty$, then $\sum_{n=0}^{\infty} |\alpha_n|^{2p} < \infty$.

Proof. By taking in mind the matrix representation of $\mathbf{S}_R - \mathbf{D}$ and denoting by $A = \sum_{n=0}^{\infty} |\alpha_n|^p$, consider the n -th column and since $\{\kappa_n\}_{n=1}^{\infty}$ is a non decreasing sequence we have:

$$\sum_{i=0}^{n-2} \left| \frac{\kappa_i}{\kappa_{n-1}} \alpha_n \overline{\alpha_i} \right|^p + |\alpha_{n-1} \alpha_n|^p + \left(1 - \frac{\kappa_{n-1}}{\kappa_n}\right)^p \leq A |\alpha_n|^p + \left(1 - \frac{\kappa_{n-1}}{\kappa_n}\right)^p.$$

Next, we show the following fact: there exist constants $c, C > 0$ such that for every $n \in \mathbb{N}$,

$$c |\alpha_n|^2 \leq 1 - \frac{\kappa_{n-1}}{\kappa_n} \leq C |\alpha_n|^2.$$

Indeed, note first that if $\alpha_n = 0$ the result is obvious. In other case, we apply the equivalence $\frac{x^2}{2} \sim 1 - \sqrt{1 - x^2}$ whenever $x \sim 0$ and since $\lim_{n \rightarrow \infty} |\alpha_n| = 0$ by taking $c, C > 0$ enough large we may assume as we required that for every $n \in \mathbb{N}$,

$$c |\alpha_n|^2 \leq 1 - \frac{\kappa_{n-1}}{\kappa_n} \leq C |\alpha_n|^2.$$

Therefore, since $p \geq 1$ and $\{\kappa_n\}_{n=0}^{\infty}$ is a non decreasing sequence, for every $n \in \mathbb{N}$

$$\left(1 - \frac{\kappa_{n-1}}{\kappa_n}\right)^p \leq \left(1 - \frac{\kappa_{n-1}}{\kappa_n}\right)^{p/2} \leq C |\alpha_n|^p.$$

Using these facts,

$$\sum_{i=0}^{\infty} |(\mathbf{S}_R - \mathbf{D})[i, n]|^p \leq (A + C) |\alpha_n|^p$$

Then,

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} |(\mathbf{S}_R - \mathbf{D})[i, n]|^p \leq (A + C) \sum_{n=0}^{\infty} |\alpha_n|^p < \infty$$

To prove (2) note that if $\sum_{i,n=0}^{\infty} |(\mathbf{S}_R - \mathbf{D})[i, n]|^p < \infty$ in particular for the subdiagonal

$$\sum_{n=1}^{\infty} \left(1 - \frac{\kappa_{n-1}}{\kappa_n}\right)^p < \infty,$$

and using basic results of series $\sum_{n=0}^{\infty} |\alpha_n|^{2p} < \infty$. □

Definition: For $p \geq 1$, we will say that an infinite matrix $\mathbf{A} = (a_{i,j})_{i,j=0}^{\infty}$ is *absolutely p -summable* if $\sum_{i,j=0}^{\infty} |a_{i,j}|^p < \infty$. In the particular case of $p = 2$ we say that $\mathbf{A}$ is a Hilbert-Schmidt matrix (in short H-S matrix).

As a consequence of the above proposition we have:

Corollary 2. Let $p \geq 1$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence with $\alpha_0 = 1$ and $|\alpha_n| < 1$ for all $n \in \mathbb{N}$ and the associated Hessenberg matrix $\mathbf{D} = \mathbf{D}(\alpha_n)$. If $\sum_{n=0}^{\infty} |\alpha_n|^p < \infty$ then $\mathbf{D} = \mathbf{S}_R + \mathbf{K}_p$ with $\mathbf{K}_p$ an absolutely p -summable matrix.

Recall that given $1 \leq p < \infty$ the *Schatten p -class* of a Hilbert space $\mathfrak{S}_p$ of $\mathcal{H}$ is defined as the Banach space of all compact operators on $\mathcal{H}$ with its singular values $s_k(\mathcal{K})$ verifying $\sum_{k=0}^{\infty} |s_k(\mathcal{K})|^p < \infty$ (see e.g.²²). The two most interesting classes are for $p = 1$, the trace class of operators $\mathfrak{S}_1$ and for $p = 2$ the Hilbert-Schmidt class $\mathfrak{S}_2$. Concerning the connections of absolutely p -summable matrices with the Schatten class operators (see e.g.²²) it is well known that the matrix representation of Hilbert Schmidt operators in separable Hilbert spaces with respect to orthonormal basis are Hilbert Schmidt matrices; on the other hand, every H-S matrix induces a H-S operator. More generally, for $1 \leq p \leq 2$ it can be also proved that if $\mathbf{A}$ is an absolutely p -summable matrix and $\mathcal{H}$ and $\{e_n\}_{n=0}^{\infty}$ are prefixed then $\mathbf{A}$ induces an operator in the p -Schatten class. We prove it for the sake of completeness. We use that :

Proposition 2. Let $1 \leq p \leq 2$ and let $\mathcal{A}$ be a bounded operator on a separable Hilbert space $\mathcal{H}$. Denote by $\mathbf{A}$ its matrix representation with respect to a certain orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Assume that $\mathbf{A}$ is absolutely p -summable, then $\mathcal{A}$ belongs to the p -Schatten class $\mathfrak{S}_p$.

Proof. In²³ it is proved that for $1 \leq p \leq 2$ a compact operator $\mathcal{K}$ belongs to $\mathfrak{S}_p$ if and only if $\sum_{n=0}^{\infty} \|\mathcal{K}(e_n)\|^p < \infty$ for some orthonormal basis in a Hilbert space $\mathcal{H}$. Thus, we only need to prove that $\sum_{n=0}^{\infty} \|\mathcal{A}(e_n)\|^p < \infty$. Fix an $N \in \mathbb{N}$ and the $(N+1) \times (N+1)$ matrix $\mathbf{A}_N$ and let $c_j = \left(\sum_{i=0}^N |a_{i,j}|^2\right)^{1/2}$ be the Euclidean length of the j -th column of $\mathbf{A}_N$. Following the same arguments as in²⁴ when $1 \leq p \leq 2$, by using the convexity of the function $g(t) = -t^{p/2}$ on $[0, \infty)$ it follows the inequality

$$\sum_{j=0}^N c_j^p \leq \sum_{i,j=0}^N |a_{i,j}|^p.$$

Therefore, for every $N \in \mathbb{N}$,

$$\sum_{j=0}^N \left(\sum_{i=0}^N |a_{i,j}|^2 \right)^{p/2} \leq \sum_{i,j=0}^{\infty} |a_{i,j}|^p,$$

and $\sum_{n=0}^{\infty} \|\mathcal{A}(e_n)\|^p < \infty$ as we required. $\square$

As a consequence of the proposition 2 can be reformulated in terms of operators in the following way:

Proposition 3. Let $1 \leq p \leq 2$ and let $\mathcal{D}$ be the Hessenberg operator associated to an HPD Toeplitz matrix $\mathbf{T}$ in the Hilbert space $P^2(\mathbf{T})$ with orthonormal basis $\{\varphi(z)_n\}_{n=0}^{\infty}$. Assume that $\sum_{n=0}^{\infty} |\Phi_n(0)|^p < \infty$, then there exists a compact operator $\mathcal{K}_p$ belonging to the Schatten class $\mathfrak{S}_p$ such that $\mathcal{D} = S + \mathcal{K}_p$, where S is the forward shift operator such that $S(\varphi_n) = \varphi_{n+1}$.

One can wonder if proposition 3 holds for $p > 2$; this is not true at all. Moreover, in the case that $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = \infty$, even with $\lim_{n \rightarrow \infty} \Phi_n(0) = 0$, the associated operator $\mathcal{D}$ is not even a compact perturbation of the forward shift operator. At this point, in order to show it we need to use that any HPD Toeplitz matrix $\mathbf{T}$ is indeed the moment matrix associated to a measure ν with infinite support on $\mathbb{T}$. Note that in this case $P^2(\nu) = P^2(\mathbf{T})$ and $\mathcal{D}$ is the multiplication by z -operator on such space. We now prove our main result:

Theorem 1. Let $\mathbf{T}(\nu)$ be a moment matrix associated to a measure ν with infinite support on $\mathbb{T}$ and let $\mathcal{D}$ be the multiplication by z -operator on $P^2(\nu)$ and let S be the forward shift operator verifying $S(\varphi_n(z)) = \varphi_{n+1}(z)$. The following statements are equivalent:

1. $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$
2. $\mathcal{D} = S + \mathcal{K}$ with $\mathcal{K}$ being a compact operator.
3. $\mathcal{D} = S + \mathcal{K}_2$ with $\mathcal{K}_2$ being a Hilbert Schmidt operator.

Proof. The part (1) implies (3) is a consequence of proposition 3 for $p = 2$. (3) implies (2) is obvious. In order to prove (2) implies (1), assume (2) is true and nevertheless $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = \infty$. By classical results in the Szegő theory for orthogonal polynomials on the unit circle (see e.g.¹) $\mathcal{D}$ is an unitary operator. This is not possible since it is well known²⁵ that an unitary operator can not be a compact perturbation of the forward shift operator; this is because S is Fredholm of index -1 , and it would follow that $\mathcal{D}$ is likewise Fredholm of index -1 , contradicting the fact that an unitary Fredholm operator has index 0. $\square$

As an immediate consequence we obtain the following dichotomy result:

Theorem 2. (Dichotomy result for operators) Let ν be a measure with infinite support in $\mathbb{T}$ and let $\mathcal{D}$ be the associated Hessenberg operator. Then,

1. either $\mathcal{D} = S + \mathcal{K}_2$ with $\mathcal{K}_2$ a Hilbert Schmidt operator,

2. or $D = D_{\text{diag}} + \mathcal{K}_2$ with D_{diag} a diagonal operator and $\mathcal{K}_2$ a Hilbert Schmidt operator.

Proof. First, if $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 < \infty$ the result is consequence of Proposition 2. Otherwise, if $\sum_{n=0}^{\infty} |\Phi_n(0)|^2 = \infty$ the operator D is unitary and by Weyl-von-Newmann theorem¹⁸ D is a diagonal operator, say D_{diag} plus a Hilbert Schmidt operator $\mathcal{K}_2$. $\square$

This dichotomy result can be reformulated for infinite matrices in the following way:

Theorem 3. Let ν be a measure with infinite support in $\mathbb{T}$ and let $\mathbf{D}$ be the associated Hessenberg matrix. Then,

1. either $\mathbf{D} = \mathbf{S}_R + \mathbf{K}_2$ with $\mathbf{K}_2$ a Hilbert Schmidt matrix,
2. or $\mathbf{D} = \mathbf{U}^* \Lambda \mathbf{U} + \mathbf{K}_2$ with $\mathbf{K}_2$ being a Hilbert Schmidt matrix, Λ a diagonal matrix and $\mathbf{U}$ an unitary matrix.

Proof. It is enough to have in mind that a diagonal operator is an operator such that there exists an orthonormal basis, say $\{f_n\}_{n=0}^{\infty}$, for which the matrix representation is a diagonal matrix, say Λ . Consider now the unitary mapping on $P^2(\nu)$ such that $\mathcal{U}(f_n) = \varphi_n$ and consider $\mathbf{U}$ the matrix representation of $\mathcal{U}$ in the basis $\{\varphi(z)_n\}_{n=0}^{\infty}$. Therefore, the matrix representation of D with respect to $\{\varphi(z)_n\}_{n=0}^{\infty}$ is

$$\mathbf{D} = \mathbf{U}^* \Lambda \mathbf{U} + \mathbf{K}_2$$

with $\mathbf{K}_2$ being a Hilbert Schmidt matrix. $\square$

Remark 2. In Theorem above the most interesting case is when the Hessenberg matrix is asymptotically Toeplitz to $\mathbf{S}_R$ and nevertheless it is not a compact, or more generally Hilbert Schmidt, perturbation of $\mathbf{S}_R$. We provide an example in this situation.

Consider the Hessenberg matrix associated to $\alpha_n = (n+1)^{-1/2}$, since

$$\frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - |\alpha_n|^2 \Rightarrow \frac{\kappa_{n-1}^2}{\kappa_n^2} = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Therefore $\kappa_n = \sqrt{\frac{n+1}{n}} \kappa_{n-1}$ and assuming that $\kappa_0 = 1$, it follows that

$$\kappa_n = \sqrt{\frac{n+1}{n}} \sqrt{\frac{n}{n-1}} \sqrt{\frac{n-1}{n-2}} \cdots \sqrt{\frac{2}{1}} \kappa_0 = \sqrt{n+1}.$$

Consequently the matrix $\mathbf{D}$ is

$$D = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2 \cdot 3}} & -\frac{1}{\sqrt{3 \cdot 4}} & -\frac{1}{\sqrt{4 \cdot 5}} & -\frac{1}{\sqrt{5 \cdot 6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ \frac{\sqrt{1}}{\sqrt{2}} & -\frac{1}{\sqrt{2 \cdot 3}} & -\frac{1}{\sqrt{3 \cdot 4}} & -\frac{1}{\sqrt{4 \cdot 5}} & -\frac{1}{\sqrt{5 \cdot 6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3 \cdot 4}} & -\frac{1}{\sqrt{4 \cdot 5}} & -\frac{1}{\sqrt{5 \cdot 6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{4}} & -\frac{1}{\sqrt{4 \cdot 5}} & -\frac{1}{\sqrt{5 \cdot 6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ 0 & 0 & 0 & \frac{\sqrt{4}}{\sqrt{5}} & -\frac{1}{\sqrt{5 \cdot 6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{\sqrt{6}} & \cdots & -\frac{1}{\sqrt{n(n+1)}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{n}}{\sqrt{n+1}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}.$$

Since $\sum_{n=0}^{\infty} |\alpha_n|^2 = \infty$ the matrix $\mathbf{D}$ is unitary (it can also be obtained in a direct way, without using Szegő's theory). Then, there exists a measure ν with infinite support in the unit circle such that the associated Hessenberg matrix is $\mathbf{D}$. Moreover, such matrix is asymptotic Toeplitz with $\text{Lim}(\mathbf{D}) = \mathbf{S}_R$, and consequently the matrix $\mathbf{S}_R - \mathbf{D}$ has all its diagonals convergent to zero and nevertheless by the dichotomy result it does not define a compact operator. On the other hand, taking into account that $\kappa_{n-1}/\kappa_n \rightarrow 1$, then $1/\sqrt[n]{\kappa_n} \rightarrow 1$ and from Ullman's results (see e.g.¹⁹) we obtain that $\text{Supp}(\mu) = \mathbb{T}$. This example is also interesting because it provides Hessenberg matrices with all its diagonals convergent to zero and which does not define a compact operator.

3 | SOME APPLICATIONS TO CLASSIFICATION OF MEASURES IN THE UNIT CIRCLE.

We now summarize several well known results concerning measures in unit circle, providing a classification in terms of the summability of their recurrence coefficients. Recall that for every positive measure ν on $\mathbb{T}$ we have that $d\nu = w(\theta)d\theta + d\nu_s$ after the Lebesgue-Radon-Nikodym decomposition theorem where $w(\theta) \in L^1[0, 2\pi]$ and ν_s is a singular measure. We recall the definition of certain kinds of measures on $\mathbb{T}$ (see e.g.²⁶). A measure ν with infinite support on $\mathbb{T}$ is in the *class of Nevai* if $\lim_{n \rightarrow \infty} |\Phi_n(0, \nu)| = 0$ and it is an Erdős measure if $w(\theta) = \frac{d\nu}{d\mathbf{m}} > 0$ almost everywhere on $[0, 2\pi]$; both classes do not coincide although every Erdős measure belongs to the Nevai class (for more information see e.g.²⁶). A measure belongs to the Szegő class if $\log(w(\theta)) \in L^1[0, 2\pi]$. On the other hand Baxter's theorem characterizes measures ν with $w(\theta) > 0$ continuous on $[0, 2\pi]$ and $\inf\{w(\theta) : \theta \in [0, 2\pi]\} > 0$ as those verifying $\sum_{n=0}^{\infty} |\Phi_n(0, \nu)| < \infty$. Our results provide a classification of such measures in terms of properties of perturbations of the associated Hessenberg operators $D(\nu)$ as follows:

Corollary 3. Let $\nu = w(\theta)d\theta + \nu_s$ be a measure with infinite support on $\mathbb{T}$ and $\mathbf{T}(\nu) = (c_{i-j})_{i,j=0}^\infty$ its moment matrix. Let $\mathbf{D}(\nu)$ be the associated Hessenberg matrix,

1. ν is in the Nevai class if and only if $\mathbf{D}(\nu)$ is asymptotic Toeplitz to $\mathbf{S}_R$.
2. If $w(\theta) > 0$ a.e. in $[0, 2\pi]$, i.e., ν is in the Erdős class, then $\mathbf{D}(\nu)$ is asymptotic Toeplitz to $\mathbf{S}_R$.
3. $\log(w(\theta)) \in L^1[0, 2\pi]$ if and only if, $\mathbf{D}(\nu) = \mathbf{S}_R + \mathbf{K}_2$ with $\mathbf{K}_2$ being a Hilbert Schmidt matrix.
4. If $w(\theta)$ is continuous and $\inf \{w(\theta), \theta \in [0, 2\pi]\} > 0$ and $\sum_{n=0}^\infty |c_n| < \infty$, then $\mathbf{D}(\nu) = \mathbf{S}_R + \mathbf{K}_1$ with $\mathbf{K}_1$ in the trace class.

Remark 3. Assume the case $\lim_{n \rightarrow \infty} \Phi_n(0)\Phi_{n+1}(0) = 0$ that happens when the associated measure ν belongs to the Maté-Nevai class. Note that, in this case, the asymptotic limit of the matrix $\mathbf{D}$ can not be exist. Nevertheless, all of the diagonals, except possibly the subdiagonal, are convergent to zero. That means that if we consider the upper triangular matrix $\tilde{\mathbf{D}}$ with $\tilde{d}_{i,j} = d_{i,j}$ if $i - j \neq 1$ and $\tilde{d}_{i,j} = 0$ otherwise it follows that $\text{Lim } \tilde{\mathbf{D}} = 0$.

Let $\nu = w(\theta)d\theta + \nu_s$ with $w(\theta) \in L^1[0, 2\pi]$ and ν_s singular. The following diagram is a classification of some types of measures supported in the unit circle which show different behaviours and examples (see^{1, 27} and⁶).

Remark 4. In the case of coefficients $\{\Phi_n(0)\}_{n=0}^\infty$ verifying $\Phi_n(0) = 1/a$, ($a > 1$), $n \geq 1$ and $\Phi_0 = 1$, the corresponding Hessenberg matrix is the following unitary matrix (see³⁰)

$$\mathbf{D} = \begin{pmatrix} -\frac{1}{a} & -\frac{(a^2-1)^{1/2}}{a^2} & -\frac{(a^2-1)^{2/2}}{a^3} & -\frac{(a^2-1)^{3/2}}{a^4} & -\frac{(a^2-1)^{4/2}}{a^5} & \dots \\ \frac{(a^2-1)^{1/2}}{a} & \frac{1}{(a^2-1)^{1/2}} & \frac{a^3}{(a^2-1)^{1/2}} & \frac{a^4}{(a^2-1)^{2/2}} & \frac{a^5}{(a^2-1)^{3/2}} & \dots \\ 0 & \frac{a^2}{(a^2-1)^{1/2}} & \frac{1}{a^3} & -\frac{(a^2-1)^{1/2}}{a^4} & \frac{(a^2-1)^{2/2}}{a^5} & \dots \\ 0 & 0 & \frac{(a^2-1)^{1/2}}{a^3} & -\frac{1}{a^4} & \frac{(a^2-1)^{1/2}}{a^5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

As it is proved in³⁰ the associated operator to $\mathbf{D}$ is a Hilbert-Schmidt perturbation of the Toeplitz operator given by $\text{Lim}(\mathbf{D}) = \mathbf{T}$. Following³¹ the measure associated to this matrix is supported on an arc of the unit circle; indeed, in the arc $\{e^{i\theta} : \beta \leq \theta \leq 2\pi - \beta\}$ where $\cos(\beta/2) = \sqrt{1-a^2}$ with $\beta \in (0, \pi)$.

Remark 5. We now provide an example of coefficients $\{\Phi_n(0)\}_{n=0}^\infty$ verifying $\lim_{n \rightarrow \infty} \Phi_n(0)$ does not exist, mainly, $\Phi_n = (-1)^{n+1} \frac{n}{n+1}$, $n > 0$. Then, the corresponding Hessenberg matrix is

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{15}}{8} & -\frac{\sqrt{105}}{30} & \frac{\sqrt{105}}{48} & \dots \\ \frac{\sqrt{3}}{2} & \frac{1}{3} & -\frac{\sqrt{5}}{8} & \frac{\sqrt{35}}{30} & -\frac{\sqrt{35}}{48} & \dots \\ 0 & \frac{\sqrt{5}}{3} & \frac{1}{2} & -\frac{\sqrt{28}}{15} & \frac{\sqrt{7}}{12} & \dots \\ 0 & 0 & \frac{\sqrt{7}}{4} & \frac{3}{5} & -\frac{\sqrt{9}}{8} & \dots \\ 0 & 0 & 0 & \frac{\sqrt{9}}{5} & \frac{2}{3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The matrix $\mathbf{D}$ is unitary and the associated measure fails Szegő condition. In this case, it is clear that $\kappa_{n-1}/\kappa_n \rightarrow 0$, and the main diagonal converges to 1. It is easy to prove that, in fact, the associated measure is a discrete measure with $z = 1$ being the unique cluster point of atoms.

Remark 6. In the following we connect our results with the results obtained for the CMV matrix in²⁷ to point out that our results can not be obtained as a consequence of such results. Recall that for a measure ν with infinite support on the unit circle $\mathbb{T}$, we may consider the moment matrix associated to the sequence $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$; the matrix obtained now is a Toeplitz matrix which defines an inner product in $L^2(\nu)$. By the procedure of orthogonalization applied to the sequence $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ we obtain an orthonormal basis in the space $L^2(\nu)$. The CMV matrix, denoted by $C(\nu)$, is the matrix representation of multiplication by z operator on $L^2(\nu)$; in³² is proved that this matrix representation is five-diagonal. One can consider another matrix representations of the multiplication operator with respect to other basis as in³³. In our framework, the Hessenberg matrix is the matrix representations with respect to the orthonormal polynomials $P^2(\nu)$, which only coincides with $L^2(\nu)$ when the Szegő condition, $\sum_{j=0}^\infty |\Phi_j(0)|^2 < \infty$, fails. Thus, the problem of whether $C(\nu)$ is normal does not make sense, since by construction

it is a normal operator. Therefore, it is a remarkable fact that the problem of whether a measure ν with compact support in $\mathbb{T}$ verifies $P^2(\nu) = L^2(\nu)$ can be studied in terms of certain perturbations of the associated Hessenberg matrix $\mathbf{D}(\nu)$.

4 | CONCLUSIONS

In this paper we consider the one-to-one correspondence between the set $\mathcal{H}^+$ of all upper Hessenberg matrices with positive subdiagonal and the infinite normalized HPD matrices $\mathcal{M}_0$. We characterize Hessenberg operators $\mathcal{D}$ associated to measures in the unit circle which are compact perturbations, and actually Hilbert Schmidt perturbations, of the forward shift operator. We have proven in Theorem 2 and Theorem 3 the interesting Dichotomy result: either the measure satisfies the Szegő condition, and in this case $\mathbf{S}_R - \mathbf{D}(\alpha_n)$ is Hilbert-Schmidt, or, if it is not satisfied, then $\mathbf{D}(\alpha_n)$ is unitary. Additionally we show a way to obtain many examples of Hessenberg matrices all of whose diagonals go to zero, although they do not define compact operator in ℓ^2 . For this purpose it is enough to consider measures whose support is equal to $\mathbb{T}$ and not satisfying the Szegő condition. Moreover we present a classification of some types of measures supported in the unit circle. We point out the different behaviour of $\Phi_n(0)$ related to the support of measure in the three cases shown in example 1 and remarks 3 and 4, where the Szegő condition is not satisfied.

5 | ACKNOWLEDGEMENTS

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