Asymptotic values of entire meromorphic functions

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Definitions

Let $f : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ meromorphic, $a$ is an asymptotic value for $f$ if there exists a continuous curve $\gamma$ such that

$$\lim_{z \to \infty, \ z \in \gamma} f(z) = a \in \mathbb{C} \cup \{\infty\}$$

$\gamma$ is an asymptotic path for $a$. 
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$\gamma$ is an asymptotic path for $a$. $\text{As}(f)$ denotes the set of asymptotic values of $f$. \hfill \square
Overview and known results

Theorem (Mazurkiewicz)

\[ f : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \text{ meromorphic then the set of asymptotic values of } f \text{ is a (Suslin) analytic set of } \mathbb{C} \cup \{\infty\}. \]
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The continuous image of an analytic set is also analytic.
Analytic sets

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In a complete separable metric space an analytic set can be defined as

- a continuous image of $\mathbb{N}^\mathbb{N}$ (or equivalently, a continuous image of the irrational numbers in the unit interval).
- the kernel of the $A$-operation, i.e. $A$ analytic iff

$$A = \bigcup_{N^\mathbb{N}} \bigcap_{n_k} S_{n_1 \ldots n_k}$$

with $\{S_{n_1 \ldots n_k}\}$ a family of sets indexed with all finite sequences of natural numbers. It is called the “defining system” of $A$. 
Order of growth

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$

- If $f$ is holomorphic, its order of growth can be defined as,

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}$$

where $M(r, f) = \max_{|z|=r} |f(z)|$. 
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where $T(r, f) = \int_0^r \frac{n(t)}{t} \, dt + \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$ and $n(t)$ is the number of poles of $f$ in the disk $D(0, t)$. 

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Polynomials and rational functions have $\rho = 0$ (and $\#\text{As}(f) = 1$). The exponential function has $\rho = 1$ (and $\#\text{As}(f) = 2$).
Holomorphic case

The bigger the order of growth, the richer the behavior of $f$ near infinity?

Theorem (Ahlfors)

*If $f$ is entire of order $\rho$ then,*

$$\#\text{As}(f) \leq 2\rho + 1.$$  

Notice that $\infty$ is always an asymptotic value for an entire $f$. 
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*Given an analytic set* $A$ *such that* $\infty \in A$ *there exists an entire function so that*

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Given an analytic set $A$ such that $\infty \in A$ there exists an entire function so that

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If $A$ is not finite, the order of growth of $f$ is infinite.
Meromorphic case

Theorem (Valiron)

- There exists a meromorphic function in $\mathbb{C}$ of finite order such that $\text{As}(f)$ is an infinite set (with the cardinality of the continuum).
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- Let \( f \) be a meromorphic function in \( \mathbb{C} \). If \( T(r, f) = O(\log^2 r) \) then \( f \) has at most one asymptotic value.
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Theorem (Eremenko)
For any $0 \leq \rho \leq \infty$ there exists a meromorphic function $f$ in $\mathbb{C}$ of order $\rho$ such that

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Moreover, for any increasing function $\psi(r) \nearrow \infty$ ($r \to \infty$) there exists $f$ meromorphic in $\mathbb{C}$ such that

$$T(r, f) = O(\psi(r) \log^2 r) \quad \text{and} \quad \text{As}(f) = \mathbb{C} \cup \{\infty\}.$$
Meromorphic case. The remaining case

Theorem (C., Drasin, Granados)

Given an analytic set \( \mathcal{A} \in \mathbb{C} \) and given \( 0 \leq \rho \leq \infty \) there exists a meromorphic function, \( f \), defined in \( \mathbb{C} \) of order \( \rho \) such that

\[
\text{As}(f) = \mathcal{A}.
\]

Moreover, for any increasing function \( \psi(r) \uparrow \infty \ (r \to \infty) \) there exists \( f \) meromorphic in \( \mathbb{C} \) such that

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T(r, f) = O(\psi(r) \log^2 r) \quad \text{and} \quad \text{As}(f) = \mathcal{A}.
\]
Outline of the proof

WLOG $A$ contains 0 and $\infty$.
Assume $A := A \setminus \{\infty\} \subset D(0, 1)$ and $\psi$ are given (case $\rho = 0$).
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1. Construct a suitable \( \delta \)-subharmonic function \( U \), with \( T(r, U) \) controlled in terms of \( \psi \) and with asymptotic values \( \{-\infty, \infty\} \).
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2. Approximate the Riesz charge of $U$ by point masses, that will conform the Riesz mass of $\log |g|$, for $g$ meromorphic in $\mathbb{C}$. 
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2. Approximate the Riesz charge of $U$ by point masses, that will conform the Riesz mass of $\log |g|$, for $g$ meromorphic in $\mathbb{C}$.

3. The approximation is “good” outside a (small) set $E$, and in $E \log |g|$ is small (large) whenever $U$ is small (large) so $\log |g|$ will “mimic” the behavior of $U$. 
Outline of the proof

4. $T(r, g)$ is controlled in terms of $\psi$ and the set of asymptotic values of $g$ is $\{0, \infty\}$. 
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5. Modify $g$ by a quasiconformal map $\Phi$, so asymptotic paths that approach 0 will approach $a \in A$. Now $F = \Phi \circ g$ is quasiregular of dilatation $\sigma_F$. 
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6. Solve the Beltrami equation $\overline{\overline{\partial}} \phi = \sigma_F \partial \phi$ to find another quasiconformal map so that $f = F \circ \phi$ is meromorphic, $T(r, f) = O(\psi(r) \log^2 r)$ and $\text{As}(f) = A$. 
Sketch of the proof. Construction of $U$

$U(re^{i\theta})$ is a piecewise linear function on $\theta$ (for fixed $r$) so that is symmetric: $U(z) = -U(-z)$ and $U(r) = U(-r) = 0$, therefore we will only show $U$ in the upper-half plane.

Graph of $U(re^{i\theta})$ for fixed $r$. 
Sketch of the proof. Graphs of $U(re^{i\theta})$

- the slope of the linear pieces of $U(re^{i\theta})$ is $L(r)$ for some increasing function $L(r) \nearrow \infty$ as $(r \to \infty)$. The function $L$ depends on $\psi$, 
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- the slope of the linear pieces of $U(re^{i\theta})$ is $L(r)$ for some increasing function $L(r) \uparrow \infty$ as $(r \to \infty)$. The function $L$ depends on $\psi$,
- the number of the linear components of $U$ increase with $r$, by splitting local minima into two local minima.
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Sketch of the proof. Paths of local minima and maxima

In the upper-half plane

- As the local minima split their paths follow the structure of a dyadic tree.
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Paths of local minima, \( \gamma \).
Sketch of the proof. Paths of local minima and maxima

In the upper-half plane

- As the local minima split their paths follow the structure of a dyadic tree.
- The paths of the local maxima separate the branches of the tree and remain undivided.

Paths of local minima, $\gamma$. Paths of local minima and maxima.
Sketch of the proof. Riesz mass of $U$

The Riesz (signed) mass of $U$ is

$$\Delta U = \mu + \mu_e$$

where

- $\mu$ is supported on the paths of local maxima and minima, $\Gamma$.
- $\mu_e$ is supported in $\mathbb{C} \setminus \Gamma$. 
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- $\mu_e$ is supported in $\mathbb{C} \setminus \Gamma$.

The measure $\mu_e$ is “small” and only $\mu$ will be approximated by point masses located on the branches of $\Gamma$ following standard approximation techniques (Yulmukhametov, Liubarskii-Malinnikova...).
Sketch of the proof. The role of the analytic set $A$

The function $U$ is constructed in such a way that

- $U \to +\infty$ on paths of local maxima,
- $U \to -\infty$ on paths of local minima,

(so $g$ approaches $\infty$ on paths of local maxima and 0 on paths of local minima).
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(so $g$ approaches $\infty$ on paths of local maxima and 0 on paths of local minima).

The rate at which $U$ approaches $\pm \infty$ depends on the analytic set $A$ by associating to each point of $A$ a branch of $\Gamma$. 
Sketch of the proof. The role of the analytic set $A$

Example with $A$ a Cantor set

Let $A$ be a Cantor set given by $A = \bigcap_{j=1}^{\infty} E_j$ with $E_j = \bigcup_n Q^n_j$ where,

- $Q^n_j$ are closed cubes in $\mathbb{C}$.
- For every $Q^n_j \subset E_j$ there exists a $Q^{j-1}_m \subset E_{j-1}$ such that $Q^n_j \subset Q^{j-1}_m$. 
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- For every $Q_n^j \subset E_j$ there exists a $Q_{m}^{j-1} \subset E_{j-1}$ such that $Q_n^j \subset Q_{m}^{j-1}$.

Assume further that
- $\text{diam}(Q_n^j) = \delta_j$ for some given sequence $\delta_j \searrow 0$. 
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Assume further that

- $\text{diam}(Q^j_n) = \delta_j$ for some given sequence $\delta_j \downarrow 0$.

Pick a fixed $a^j_n \in Q^j_n$ for all $n$ and $j$. Each $a \in A$ is given by $a = \bigcap_j Q^j_n$ so there is a sequence $a^j_n \rightarrow a$ ($j \rightarrow \infty$) in such a way that $|a^j_n - a| \leq \delta_j$. 
Sketch of the proof. The role of the analytic set $A$

Example with $A$ a Cantor set

The association of points in $A$ and branches of $\gamma$ is done in the obvious way (recall $0 \in A$).
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The association of points in $A$ and branches of $\gamma$ is done in the obvious way (recall $0 \in A$).

Associate $0$ to the first branches, until in generation $k$,
\[ \# \{ Q_n^1 \} \leq 2^k \text{ for some } k = k(1). \]
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[Diagram of the Cantor set with branches labeled $a_{1}^{1}, a_{2}^{1}, a_{N-1}^{1}, a_{N}^{1}$]
Sketch of the proof. The role of the analytic set $A$

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The association of points in $A$ and branches of $\gamma$ is done in the obvious way (recall $0 \in A$).

Notice $|a_n^1| \leq \text{diam}(A) \leq 2$
Sketch of the proof. The role of the analytic set \( A \)

Example with \( A \) a Cantor set

The association of points in \( A \) and branches of \( \gamma \) is done in the obvious way (recall \( 0 \in A \)).

Assume \( a^n_i \) is associated to a branch.
Sketch of the proof. The role of the analytic set $A$

Example with $A$ a Cantor set

The association of points in $A$ and branches of $\gamma$ is done in the obvious way (recall $0 \in A$).

Assume $a^j_n$ is associated to a branch.

Wait until $\#\{Q^j_n\} \leq 2^k$ (for all $n$) for some $k = k(j)$.

Associate to those branches, $a^j_{m+1} \in Q^j_{m+1} \subset Q^j_n$. 

\[ a^j_n \]

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Sketch of the proof. The role of the analytic set $A$.

Example with $A$ a Cantor set.

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Notice

$$|a^j_n - a^{j+1}_m| \leq \text{diam}(Q^j_n) = \delta_j$$
Sketch of the proof

Example with $A$ a Cantor set

$U$ is constructed so that $|g|$ is small on the branches of $\gamma$. 
Sketch of the proof

Example with \( A \) a Cantor set

\( U \) is constructed so that \(|g|\) is small on the branches of \( \Upsilon \). Concretely, given a sequence \( \delta_j \searrow 0 \),

- In the ‘first’ generation of branches of \( \Upsilon \),
  \( \delta_1 < |g| < 2 \) (area in blue),
- in the ‘second’ generation
  \( \delta_2 < |g| < \delta_1 \) (area in yellow),
- and so on...

The set of asymptotic values is \( \text{As}(g) = \{0, \infty\} \).
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Example with $A$ a Cantor set
$\mathcal{U}$ is constructed so that $|g|$ is small on the branches of $\mathcal{U}$. Concretely, given a sequence $\delta_j \to 0$,

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The set of asymptotic values is $\text{As}(g) = \{0, \infty\}$. That is easily proved since $|g| \to 0$ uniformly on branches of $\mathcal{U}$, and $|g| \to \infty$ on the other paths of $\Gamma$ in the upper half-plane.
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Sketch of the proof. Quasiconformal translations

To “map” the set \( \{0, \infty\} \) into \( A \cup \{\infty\} = \mathcal{A} \) we compose recursively with quasiconformal translations
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Lemma

Given \( K > 1 \) and \( R > 0 \) there exists \( 0 < \delta < R \) so that if \( |a| < \delta \), then there is a \( K \)-quasiconformal map \( \varphi \) so that

\[
\varphi(z) = \begin{cases} 
z, & |z| > R, 
z + a, & |z| \leq \delta. \end{cases}
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Given a sequence \( \{K_j\} \) such that \( \prod_j K_j = K < \infty \) use lemma (starting with \( R = 2 \)) to find a sequence \( \{\delta_j\} \).
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Given a sequence \( \{K_j\} \) such that \( \prod_j K_j = K < \infty \) use lemma (starting with \( R = 2 \)) to find a sequence \( \{\delta_j\} \). This will be our “given” sequence \( \{\delta_j\} \).
Sketch of the proof. Composition with qc transformations

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Given $a \in A$, consider $a^j \to a$ and the associated branch.
Sketch of the proof. Composition with qc transformations

Example with $A$ a Cantor set

Given $a \in A$, consider $a^j \rightarrow a$ and the associated branch. Compose $g$ (on the branch) with a qc map, $\varphi_1$ such that $\varphi_1(z) = z + a^1$ if $|z| < \delta_1$. 

\[ \delta_1 < |g| < \delta_2 \]

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\[ \delta_3 < |g| < \delta_2 \]

\[ \delta_1 < |g| < 2 \]
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g(z) \xrightarrow{\varphi_1} g(z) + a^1
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Compose $\varphi_1 \circ g$ with $\varphi_2$ such that $\varphi_2(z) = z + a^2 - a^1$ if $|z - a^1| < \delta_2$. 

$$g(z) \xrightarrow{\varphi_1} g(z) + a^1$$
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Compose $\varphi_1 \circ g$ with $\varphi_2$ such that $\varphi_2(z) = z + a^2 - a^1$ if $|z - a^1| < \delta_2$.

Along this branch, after composing with qc transformations $F = \Phi \circ g$
Sketch of the proof. Final tuning

In this way,

- $F = \Phi \circ g$ a quasiregular map whose set of asymptotic values is $A \cup \{\infty\} = \mathcal{A}$. 
Sketch of the proof. Final tuning

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- $F = \Phi \circ g$ a quasiregular map whose set of asymptotic values is $A \cup \{\infty\} = \mathcal{A}$.
- The qc distortion is killed by precomposing $F$ with a qc map with the same dilatation to get a meromorphic function $f$ with $\text{As}(f) = \mathcal{A}$.
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- $T(r, f)$ is controlled by $T(r, F)$ and $K$, so by choosing an adequate $L(r)$ we get the desired growth: $T(r, f) = O(\psi(r) \log^2 r)$.
- If $A = \mathcal{A} \setminus \{\infty\}$ is unbounded, then decompose $A$ in a countable number of bounded sets of diameter at most 2, and construct countable many dyadic trees in a similar way.
Sketch of the proof. Analytic sets

When $A$ is not a Cantor set, but a general analytic set,
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- $S_{n_1...n_k}$ are (nonempty) closed sets in $\mathbb{C}$,
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This is called "regular defining system".
Sketch of the proof. Analytic sets

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This is called “regular defining system”.

For each $n_1 \ldots n_k$ pick $a_{n_1\ldots n_k} \in S_{n_1\ldots n_k}$.

Each $a \in A$ is given by $a = \bigcap S_{n_1\ldots n_k}$ so there is a sequence $\{a_{n_1\ldots n_k}\}$ so that

$$a_{n_1\ldots n_k} \to a \ (k \to \infty) \text{ and } |a_{n_1\ldots n_k} - a| \leq \delta_k.$$
Sketch of the proof. Analytic sets and binary expansions

The association of points in $A$ and branches of $\Gamma$ is done in a similar way.
Sketch of the proof. Analytic sets and binary expansions

The association of points in $A$ and branches of $T$ is done in a similar way. To a finite sequence $n_1, \ldots, n_k$ associate a finite sequence $\xi_1, \ldots, \xi_N$ of 0's and 1's (and this latter one, with a (finite) branch of a dyadic tree) in the following way:

$$\sum_{j=1}^{N} \frac{\xi_j}{2^j} = \sum_{j=1}^{k} \frac{1}{2^{m_1+\cdots+m_j}}$$

Notice that $N = n_1 + \cdots + n_k$. 
Sketch of the proof. Analytic sets and binary expansions

The association of points in \( A \) and branches of \( \mathcal{T} \) is done in a similar way. To a finite sequence \( n_1, \ldots, n_k \) associate a finite sequence \( \xi_1, \ldots, \xi_N \) of 0's and 1's (and this latter one, with a (finite) branch of a dyadic tree) in the following way:

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Notice that \( N = n_1 + \cdots + n_k \).

Example

\[1, 3, 1 \leftrightarrow \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} \leftrightarrow 1, 0, 0, 1, 1\]
Sketch of the proof. Analytic sets and binary expansions

The association of points in $A$ and branches of $\mathcal{T}$ is done in a similar way. To a finite sequence $n_1, \ldots, n_k$ associate a finite sequence $\xi_1, \ldots, \xi_N$ of 0's and 1's (and this latter one, with a (finite) branch of a dyadic tree) in the following way:

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$$1, 3, 1 \iff \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^5} \iff 1, 0, 0, 1, 1, 0$$
Sketch of the proof. Analytic sets and trees

To $0 \in A$ associate any finite sequence of 0’s.
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With this association the distance between two adjacent points of consecutive generations does not decrease with the generation!

In the second generation:

- $|a_1 - a_{1,1}| \leq \delta_1$ since $a_1, a_{1,1} \in S_1$
- $|a_2| \leq 2$ since $0, a_1 \in A$
Sketch of the proof. Analytic sets and trees

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In the third generation:
- $|a_{1,1} - a_{1,1,1}| \leq \delta_2$ since $a_{1,1}, a_{1,1,1} \in S_{1,1}$
- $|a_2 - a_{2,1}| \leq \delta_1$ since $a_2, a_{2,1} \in S_2$ and $|a_1 - a_{1,2}| \leq \delta_1$ since $a_1, a_{1,2} \in S_1$
- $|a_3| \leq 2$ since $0, a_3 \in A$
Sketch of the proof. Analytic sets and trees

Again, $U$ is constructed so that $|g|$ is small on the branches of $\Upsilon$ depending on the distances $\{\delta_k\}$ of two adjacent points of different generations.
Sketch of the proof. Analytic sets and trees

Again, $U$ is constructed so that $|g|$ is small on the branches of $T$ depending on the distances $\{\delta_k\}$ of two adjacent points of different generations. Graphically:

$\delta_1 < |g| < 2$

$\delta_2 < |g| < \delta_1$

$\delta_3 < |g| < \delta_2$
Sketch of the proof. Analytic sets and trees

Again, $U$ is constructed so that $|g|$ is small on the branches of $\Gamma$ depending on the distances $\{\delta_k\}$ of two adjacent points of different generations. Graphically:

Now there are branches of the tree where $|g|$ is bounded and some work has to be done to show that $g$ does not take any asymptotic value on those branches to get $\text{As}(g) = \{0, \infty\}$. 
Sketch of the proof. Analytic sets and trees

Again, $U$ is constructed so that $|g|$ is small on the branches of $\Gamma$ depending on the distances $\{\delta_k\}$ of two adjacent points of different generations. Graphically:

Now there are branches of the tree where $|g|$ is bounded and some work has to be done to show that $g$ does not take any asymptotic value on those branches to get $\text{As}(g) = \{0, \infty\}$. The rest follows as explained before.
The end

Thank you!