Tunneling and Filtering of Degenerate Microwave Modes in a Polarization-Dependent Waveguide Containing Index Gradient Barriers

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(Received 21 December 2018; revised manuscript received 14 March 2019; published 17 April 2019)

The drastic difference between complex transmission coefficients for TE_{11} and TH_{11} GHz modes propagating through subwavelength gradient barriers in a metallic waveguide is examined. Amplitude-phase structures of the transmitted TE_{11} and TH_{11} modes are described in the framework of an exactly solvable model of these barriers. A sharp contrast in transmission is found between degenerate TE_{11} and TH_{11} modes distinguished by their polarization structures due to polarization-dependent tunneling and nonlocal energy transport through thin gradient dielectric barriers.

DOI: [10.1103/PhysRevApplied.11.044056](https://doi.org/10.1103/PhysRevApplied.11.044056)

I. INTRODUCTION

This paper is aimed at controlling the reflectance and transmittance of microwaves in a waveguide loaded by specially designed inclusions. Some of these effects could be viewed as the microwave analogies of phenomena well known in optics and quantum mechanics, such as tunneling of microwave modes through opaque barriers formed by undersized segments of metallic waveguides [1,2]. These results provided the basis for the concept of reflectionless tunneling of waveguide modes in a diaphragm-loaded waveguide [3] expressed by plasmon-assisted transfer of wave energy through a diaphragm fabricated from a metamaterial [4]. This analogy was manifested by simulation of optical left-handed media realized in split-ring-resonator-loaded waveguides [5]. The growing applications of metamaterials in the design of waveguides gave rise to the concept of waveguide metatronics [6], opening new directions in waveguide functionality, such as the use of epsilon-near-zero metamaterials for squeezing energy flows [7–9], frequency-selective energy tunneling in wire-loaded narrow waveguide channels [10,11], and waveguide structural dispersion [12]. Structural dispersion proved to be useful for controlling the dynamics of voltage and current waves in transmission lines with continuously distributed parameters [13]. Note that these phenomena were examined for the waveguide propagation of fundamental modes.

In contrast, this paper focuses on the controlled transmission of higher transverse-electric TE_{11} and transverse-magnetic TH_{11} microwave modes through a gradient dielectric barrier in a waveguide. These modes have equal frequencies and the propagation constants form degenerate modes distinguished only by their polarization structures [14]. The appearance of degenerate modes poses the problem of filtering the flow of radiation, thereby weakening the undesirable “parasitic” mode. A simple example of such filtration, based on the different transmission of TE_{11} and TH_{11} modes through the heterogeneous dielectric nonmagnetic layer with a specially designed continuous distribution of dielectric permittivity located inside the waveguide, is examined below. This difference is determined by the polarization-dependent transmittance of these layers, fabricated from the dielectric without free carriers (gradient wave barriers).

Consider the transmittance spectra of microwave radiation with frequency ω propagating in a rectangular metallic waveguide with sides a and b ($a > b$) containing gradient dielectric layers (Fig. 1); these spectra depend upon the distribution of dielectric permittivity $\varepsilon(z)$ inside the layers disposed along the waveguide axis (z direction). The x and y axes are parallel to the waveguide cross section with sides a and b respectively. The sides of the gradient layer, located in the x - y plane, are also equal to a and b ; these layers can be viewed as gradient wave barriers. The transmittance spectra of both single-layer and multilayer gradient structures are examined below by means of exact analytical solutions of Maxwell’s equations

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FIG. 1. Geometry of the problem. Microwaves are traveling in the z direction along the rectangular waveguide with sides a and b ; the x and y axes are oriented along these sides. Gradient dielectric barriers with thickness d are disposed across the waveguide.

for isotropic nonmagnetized lossless media; these solutions are obtained independently of any suppositions about smallness or slowness of variations of fields or the spatial distributions: $\varepsilon(z)$:

$$\text{rot} \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}; \quad \text{rot} \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t}; \quad \text{div} \vec{D} = 0; \quad \text{div} \vec{H} = 0. \quad (1)$$

The electric displacement $\vec{D}$ and electric field $\vec{E}$ are linked by the usual correlation $\vec{D} = \varepsilon(z) \vec{E}$. To describe the fields of TE_{11} and TH_{11} modes in the rectangular waveguide, it is expedient to introduce the generating functions presented by vectors $\vec{A} = (A_x, A_y, 0)$ and $\vec{P} = (P_x, P_y, 0)$. The transverse-electric $\vec{E}$ and transverse-magnetic $\vec{H}$ components of these modes, located in the x - y plane, are expressed via the vectors $\vec{A}$ and $\vec{P}$, respectively:

$$\begin{aligned} \text{TE}_{11} \text{ mode: } \vec{E} &= -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}; \quad \vec{H} = \text{rot} \vec{A}; \\ \text{TH}_{11} \text{ mode: } \vec{H} &= -\frac{1}{c} \frac{\partial \vec{P}}{\partial t}; \quad \vec{E} = \frac{\text{rot} \vec{P}}{\varepsilon(z)}. \end{aligned} \quad (2)$$

From Maxwell's equations (1), we have that vectors $\vec{A}$ and $\vec{P}$ are governed by

$$\text{rot} \text{rot} \vec{A} + \frac{\varepsilon(z)}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0, \quad \text{rot} \frac{\text{rot} \vec{P}}{\varepsilon(z)} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0. \quad (3)$$

The electric displacement $\vec{D}$ and electric field $\vec{E}$ are linked by $\vec{D} = \varepsilon(z) \vec{E}$. Omitting the phase factor $\exp(-i\omega t)$ for simplicity, supposing the waveguide walls to be perfectly conducting, and using the boundary conditions for the tangent components of the electric fields on these walls, $E_{tg}|_{x=0,a} = E_{tg}|_{y=0,b} = 0$, one can present the components

of the generating vector functions $\vec{A}$ and $\vec{P}$ as

$$\begin{cases} A_x = A_1 \varrho_1(x, y) \Psi_s(z), & A_y = A_2 \varrho_2(x, y) \Psi_s(z), \\ P_x = P_1 \varrho_2(x, y) \Psi_p(z), & P_y = P_2 \varrho_1(x, y) \Psi_p(z). \end{cases} \quad (4)$$

The functions ϱ_1 and ϱ_2 in Eq. (4) provide the well-known expressions for the components of the TE_{11} and TH_{11} modes in a rectangular waveguide [14]:

$$\begin{aligned} \varrho_1(x, y) &= \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right), \\ \varrho_2(x, y) &= \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right), \end{aligned} \quad (5)$$

where coefficients $A_{1,2}$ and $P_{1,2}$ are the normalization constants. The coefficients $A_{1,2}$ as well as $P_{1,2}$ are linked: thus, in the case of the TE_{11} mode ($E_z = 0$), substitution of the generating function in Eq. (4) into Eq. (2) gives the values of components H_x and H_y . Further substitution of these components into the equation $\text{div} \vec{D} = 0$ gives the correlation

$$bA_1 + aA_2 = 0. \quad (6)$$

Similar operations for the TH_{11} mode ($H_z = 0$) and the equation $\text{div} \vec{H} = 0$ provide the link

$$bP_1 + aP_2 = 0. \quad (7)$$

The functions $\Psi_s(z)$ and $\Psi_p(z)$ represent the variation of the TE_{11} and TH_{11} modes upon the z coordinate. Outside the gradient structure, these functions are given by the harmonic waves [14]

$$\begin{aligned} \Psi_s(z) &= \Psi_p(z) = \exp(\pm i k_z z), \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_{\perp}^2}, \\ k_{\perp}^2 &= \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right). \end{aligned} \quad (8)$$

Both TE_{11} and TH_{11} modes have equal cutoff frequencies $\omega_1 = k_{\perp} c$.

Inside the gradient structures, these functions $\Psi_s(z)$ and $\Psi_p(z)$ depend on the spatial distribution of the dielectric permittivity $\varepsilon(z)$. To highlight the effects associated with the nonuniformity of ε , we assume that wave absorption and material dispersion are insignificant in the range of microwave frequencies ω under consideration. In this case, the spatial dependence of the isotropic dielectric permittivity in the layer, $\varepsilon(z)$, can be written as

$$\varepsilon(z) = n_0^2 U^2(z), \quad U|_{z=0} = 1. \quad (9)$$

Here, n_0 is the refractive index of the layer at the boundary $z = 0$ and the dimensionless function $U^2(z)$ describes the

normalized spatial variation of the permittivity. Gradient dielectric layers with fixed profiles $U^2(z)$ can be fabricated from the porous dielectrics with a controlled distribution of air-filled pores [15–17]. Using Eq. (9) and substituting Eqs. (2)–(5) into Maxwell's equations (1) gives the pair of equations that govern the unknown functions $\Psi_s(z)$ and $\Psi_p(z)$:

$$\frac{d^2\Psi_s}{dz^2} + \left[\frac{\omega^2 n_0^2 U^2(z)}{c^2} - k_\perp^2 \right] \Psi_s = 0, \quad (10)$$

$$\frac{d^2\Psi_p}{dz^2} + \left[\frac{\omega^2 n_0^2 U^2(z)}{c^2} - k_\perp^2 \right] \Psi_p = \frac{2}{U(z)} \frac{dU(z)}{dz} \frac{d\Psi_p}{dz}. \quad (11)$$

Note that the functions $\Psi_s(z)$ and $\Psi_p(z)$, related to the degenerate TE₁₁ and TH₁₁ modes in the same waveguide, are determined by two different equations (10) and (11). The flexible distribution of the refractive index (9), containing two free parameters, is shown below in Sec. II to reduce the set in Eqs. (10) and (11) to a pair of similar master equations. The amplitude-phase transmittance spectra based on the exact analytical solutions of the master equations for TE₁₁ and TH₁₁ modes traveling through the gradient barriers $U(z)$ are presented in Sec. III: here, we illustrate the possibility of effectively controlled quenching and filtration of these degenerate modes due to the drastic difference between their polarization-dependent transmittance. The main results of this analysis are summarized in the conclusion (Sec. IV). To obtain the solutions of the master equations and find the amplitude-phase transmittance spectra for degenerate TE₁₁ and TH₁₁ modes, some tedious algebra is needed: the corresponding calculations are removed to the appendixes.

II. EXACTLY SOLVABLE MODEL OF GRADIENT BARRIER IN A WAVEGUIDE

It is convenient to reduce Eqs. (10) and (11) to a single equation with different parameters. By introducing the new functions f_s and f_p and a new variable η ,

$$f_s = \Psi_s \sqrt{U}, \quad f_p = \frac{\Psi_p}{\sqrt{U}}, \quad \eta = \int_0^z U(z_1) dz_1, \quad (12)$$

one can represent Eqs. (10) and (11) for the TE₁₁ and TH₁₁ modes in a similar form:

$$\frac{d^2 f_s}{d\eta^2} + f_s \left(K - \frac{U_{\eta\eta}}{2U} + \frac{U_\eta^2}{4U^2} \right) = 0, \quad (13)$$

$$\frac{d^2 f_p}{d\eta^2} + f_p \left(K + \frac{U_{\eta\eta}}{2U} - \frac{3U_\eta^2}{4U^2} \right) = 0, \quad (14)$$

$$K = \left(\frac{\omega n_0}{c} \right)^2 - \frac{k_\perp^2}{U^2(z)}, \quad U_\eta = \frac{dU}{d\eta}, \quad U_{\eta\eta} = \frac{d^2 U}{d\eta^2}. \quad (15)$$

Equations (10) and (11) and Eqs. (13) and (14) are valid for the arbitrary profiles $U(z)$ and $U(\eta)$, respectively. However, in our problem, one has to consider the profiles $U(z)$, ensuring the exact analytical solutions for both Eqs. (10) and (11). One of these profiles is given by the so-called, Rayleigh distribution $U(z) = (1 + z/L)^{-1}$ [18], containing only one free parameter: the length scale L . In terms of the variable η defined in Eq. (12), this distribution can be expressed as $U(\eta) = \exp(-\eta/L)$. Substituting this into Eqs. (13) and (14) results in the same equations for both the TE₁₁ and TH₁₁ modes and, therefore, this distribution is invalid for filtration of these degenerate modes. In contrast, the flexible distribution

$$U(z) = \left[\cos\left(\frac{z}{L}\right) + M \sin\left(\frac{z}{L}\right) \right]^{-1}, \quad (16)$$

containing two free parameters, L and M , is shown below to provide an essential difference in transmission for the degenerate modes suitable for their effective filtration. The values of L and M depend on the minimum of the refractive index, n_m , and on the layer thickness d . Considering the symmetrical profile $U(0) = U(d) = 1$ (Fig. 2), one can link these quantities to the physical and geometrical parameters of the layer:

$$n_m = \frac{n_0}{\sqrt{1 + M^2}}, \quad \frac{d}{L} = 2 \arctan M. \quad (17)$$

Substituting the distribution (16) into the definition of the new variable η in Eq. (12) yields the link between

FIG. 2. Geometry of the problem: distribution of the refractive index $n(z) = n_0 U(z)$ with U given by Eq. (16) across the adjoining barriers. The values n_0 and n_m correspond, respectively, to the boundaries of each barrier and to the minimum value of $n(z)$.

variables η and z :

$$\frac{\eta}{L} = \frac{1}{\sqrt{1+M^2}} \ln \left(\frac{1+m_+ \tan(\frac{z}{2L})}{1-m_- \tan(\frac{z}{2L})} \right),$$

$$m_{\pm} = \sqrt{1+M^2} \pm M. \quad (18)$$

Introducing a new variable ς , determined in the interval $-\varsigma_0 \leq \varsigma \leq \varsigma_0$, corresponding to the values $0 \leq z \leq d$,

$$\varsigma = \frac{\eta\sqrt{1+M^2}}{L} - \varsigma_0, \quad \varsigma_0 = \ln(m_+),$$

$$\varsigma|_{z=0} = -\varsigma_0, \quad \varsigma|_{z=d} = \varsigma_0, \quad (19)$$

one can write the distribution function $U(z)$ (16) via the new variable ς :

$$U(\varsigma) = \frac{\cosh(\varsigma)}{\sqrt{1+M^2}}, \quad U(\varsigma_0) = U(-\varsigma_0) = 1. \quad (20)$$

Consider the propagating modes incident on the interface ($z = 0$) of the gradient layer (16). To obtain an exact analytical solution to the wave equations (13) and (14) for this barrier, one needs to rewrite these equations by introducing the new variable ς defined in Eq. (19). It is noteworthy that, once these transformations are made, the equations governing the functions f_s and f_p can be presented in the same form:

$$\frac{d^2 f_{s,p}}{d\varsigma^2} + f_{s,p} \left(h^2 - \frac{\aleph_{s,p}}{\cosh^2(\varsigma)} \right) = 0. \quad (21)$$

Note that Eq. (21) is formally identical to the one-dimensional, time-dependent, Schrödinger equation with the potential $\cosh^{-2}(\varsigma)$ [19]. The values of the parameter h^2 in Eq. (21) are equal for Eqs. (13) and (14), governing the f_s and f_p functions, while the parameters $\aleph_s$ and $\aleph_p$ are related to functions f_s and f_p , respectively:

$$h^2 = \frac{1}{1+M^2} \left(\frac{\omega n_0 L}{c} \right)^2 - \frac{1}{4}, \quad \aleph_s = (k_{\perp} L)^2 + \frac{1}{4},$$

$$\aleph_p = (k_{\perp} L)^2 - \frac{3}{4}. \quad (22)$$

Because of the expressions (21) and (22), the forthcoming analysis of polarization-dependent transmittance of different modes can be unified.

III. CONTROLLED QUENCHING AND FILTRATION OF DEGENERATE MODES PASSING THROUGH GRADIENT DIELECTRIC STRUCTURES IN A WAVEGUIDE

To visualize the amplitude-phase spectra of the transmitted TE₁₁ (B11) and TH₁₁ (C7) modes, it is convenient to introduce the characteristic frequency Ω and the

dimensionless parameter u :

$$\Omega = \frac{c\sqrt{1+M^2}}{2n_0 L}, \quad u = \frac{\Omega}{\omega}. \quad (23)$$

Now it is worthwhile to rewrite Eq. (21) by expressing the term h^2 (22) through the frequency-dependent parameter u :

$$\frac{d^2 f_{s,p}}{d\varsigma^2} + f_{s,p} \left[\frac{1}{4} \left(\frac{1}{u^2} - 1 \right) - \frac{\aleph_{s,p}}{\cosh^2(\varsigma)} \right] = 0. \quad (24)$$

The condition of absolute convergence of solutions of Eq. (21), $h^2 \leq 0$ (see Appendix A), implies the restriction of values u in Eq. (24): $u \leq 1$ ($\omega < \Omega$). Thus, the expression in the square brackets in Eq. (24) for the function f_s (the TE₁₁ mode) is always negative. Meanwhile, for the function f_p (the TH₁₁ mode), this expression can become positive. This result means that the TE₁₁ mode is always passing through the gradient layer (16) in the tunneling regime, while the TH₁₁ mode can travel through the same layer in the propagating regime. This case, which provides a strong contrast between the transmitted TE₁₁ and TH₁₁ modes, is examined below.

Consider, for instance, a waveguide with sides $a = 2$ cm and $b = 1$ cm. The cutoff frequency for these modes is [14] $\omega_1 = 0.5\pi c^{-1}\sqrt{a^2 + b^2} = 10.53 \times 10^{10}$ rad/s (16.77 GHz). On the other hand, considering the parameters of the gradient layer to be $n_0 = 2$, $M = 1$, and $L = 0.075$ cm, we obtain the characteristic frequency $\Omega = 14.142 \times 10^{10}$ rad/s (22.52 GHz). Recalling the condition for TE₁₁ and TH₁₁ modes to propagate $\omega > \omega_1 = 10.53 \times 10^{10}$ rad/s, one can determine the spectral range of the modes discussed: $1 < u < 1.343$; all the spectra considered here are computed inside this range. Some salient features of the interaction of TE₁₁ and TH₁₁ modes with the gradient dielectric barriers are listed below.

(1) The amplitude-phase transmittance spectra for degenerated TE₁₁ and TH₁₁ modes passing through a single gradient layer and computed by means of the formulas (B11) and (C7), respectively, are presented in Figs. 3 and 4.

Figure 3 shows that the transmittance of the TE₁₁ mode exceeds the transmittance of the TH₁₁ mode over the entire spectral range $1 < u < 1.343$. Thus, in the case $u = 1.15$, the value of the transmission coefficient $|T|^2$ for mode TE₁₁ is twice the value of $|T|^2$ for mode TH₁₁.

The transmittance of both modes vanishes ($|T|^2 \rightarrow 0$) near the cutoff frequency $u \rightarrow 1.343$. The phase of the traveling TH₁₁ mode (Fig. 4, curve 2) is characterized by an effective frequency dispersion, while the dependence of the phase of the tunneling mode TE₁₁ on the frequency is weak (Fig. 4, curve 1).

(2) The difference in the transmittance of the degenerate modes passing through the single gradient layer is

FIG. 3. Transmittance spectra of the degenerate TE_{11} and TH_{11} modes propagating in a waveguide with sides $a = 2$ cm, $b = 1$ cm (as represented in Fig. 1) through a single subwavelength gradient barrier (16) with thickness $d = 0.115$ cm and parameters $n_0 = 2$, $M = 1$. The transmission coefficients with respect to the energy, $|T|^2$, of the transmitted waves are plotted vs the normalized frequency u (23). Curves 1 and 2 correspond to the TE_{11} and TH_{11} modes, respectively.

increased drastically due to the use of two similar adjoining barriers (Figs. 5 and 6).

For the frequency $u = 1.15$, one can see in Fig. 3 that the value of the transmission coefficient $|T|^2$ for the TE_{11} mode (Fig. 5) exceeds the value of $|T|^2$ for the TH_{11} mode by a factor of 32. Unlike the monotonic variation of the phase for the traveling TH_{11} mode (Fig. 6), the phase of the tunneling TE_{11} mode tends to saturation. This feature is characteristic of tunneling [2].

To visualize the effect of the gradient profile (16), the transmission coefficients for the TE_{11} and TH_{11} modes passing through the homogeneous dielectric layer with

FIG. 5. Transmittance spectra of the degenerate TE_{11} and TH_{11} modes propagating through a pair of adjoining gradient barriers (16) with the same parameters as in Fig. 3. The transmission coefficients, $|T|^2$, of the transmitted waves are plotted vs the normalized frequency u (23). Curves 1 and 2 correspond to the TE_{11} and TH_{11} modes, respectively. Curves 3 and 4 represent the transmission coefficients $|T|^2$ for the propagation of the same modes through the homogeneous layer with the same refractive index $n_0 = 2$ and thickness $d = 0.23$ cm.

the same refractive index n_0 and the same thickness $2d$ are presented in Fig. 5 by spectra 3 and 4 respectively. Comparison of these curves with spectra 1 and 2 shows that, unlike in the gradient barriers, transmittance of the TH_{11} mode through the homogeneous barriers exceeds the transmittance of the TE_{11} mode. However, the physically meaningful difference in the transmittance of these modes is manifested in the spectral range $1 < u < 1.2$ only; the values of $|T|^2$ for both modes become less than 1% for the smaller frequencies $u \geq 1.2$.

FIG. 4. Same conditions as in Fig. 3. The phases of the transmission coefficients, φ , for the transmitted waves are plotted vs the normalized frequency u (23). Curves 1 and 2 correspond to the TE_{11} and TH_{11} modes, respectively.

FIG. 6. Same conditions as in Fig. 5. The phases of the transmission coefficients, φ , for the transmitted waves are plotted vs the normalized frequency u (23). Curves 1 and 2 correspond to the TE_{11} and TH_{11} modes, respectively.

(3) The attenuation of the transmitted TE_{11} mode due to its reflectance and tunneling is much less than the attenuation of the transmitted TH_{11} mode caused by reflectance alone. This effect can be understood as the first stage of reflectionless tunneling of electromagnetic waves through the set of gradient barriers [16], when the high transmittance is determined mainly by the nonlocal energy transport by the tunneling waves ($|T|^2 \approx 0.9\text{--}0.95$).

(4) Note that the thickness $d = 1.15$ mm (17) of the barrier in this entire spectral range $\omega_1 < \omega < \Omega$ is much smaller than the wavelengths of the traveling modes: $\lambda \geq 2$ cm. The absorption of microwaves in these thin subwavelength nonconducting layers is negligible. The quenching of the transmitted TH_{11} mode is provided not by the absorption in the barrier, but by its reflection from the barrier.

(5) The same analysis can be carried out for any other pair of degenerate TE_{mj} and TH_{mj} modes. To perform this generalization, one has to make the replacements $x \rightarrow mx$ and $y \rightarrow jy$ in functions ϱ_1 and ϱ_2 (4) that describe the spatial structure of these modes and to consider the solutions in the spectral range between the characteristic frequency Ω (23) and the cutoff frequency of the TE_{mj} and TH_{mj} modes.

Thus, even a pair of subwavelength gradient barriers ensures strong filtering of the degenerate modes, providing high contrast between the power flows in the transmitted TE_{11} and TH_{11} modes.

The analysis of the filtering of degenerate modes is presented here for the cases of one and two gradient layers. However, even these simple barriers are shown to provide a difference in the transmittance of the degenerate TE_{11} and TH_{11} modes by more than a factor of 30. If necessary for any concrete problem, the complex generalized formulas for the reflection and/or transmission coefficients for three or more layers, based on massive algebra, can be derived by means of the computational procedure presented for two layers in Appendix D.

IV. CONCLUSION

In conclusion, we consider the possibility of controlled formation of amplitude-phase spectra of transverse-electric (TE) and transverse-magnetic (TH) modes transmitted through gradient dielectric lossless layers in a waveguide. The nonlocal dispersion of these modes inside the gradient layers, determined by the spatial distribution of the dielectric permittivity inside each layer and its thickness, determines the contrast in the transmittance of the modes discussed. This contrast is shown to ensure the effective quenching and filtering of the degenerate TE_{11} and TH_{11} , distinguished only by their polarization structures. Some salient features of this polarization-dependent effect are listed below.

(i) Owing to the smooth, continuously shaped distribution of dielectric permittivity across the gradient layer, these thin subwavelength layers are shown to provide flexible control over the spectra of the transmitted modes.

(ii) The propagation of the TE_{11} mode through the gradient layer is an example of tunneling of waves with vector polarization structure characterized by one longitudinal (H_z) and four transverse (E_x, E_y, H_x, H_y) components. Unlike the well-known effect of frustrated total internal reflection of both TE- and TH-polarized waves in a homogeneous opaque medium, this polarization-dependent tunneling determined by the nonlocal dispersion of the gradient layer arises only for the TE-polarized mode in the exactly solvable model of a gradient layer (16).

(iii) Exact analytical solutions of Eqs. (10) and (11) for the model (16) presented by the expressions (B1) and (C1), respectively, are obtained without any suppositions about smallness or slowness of the spatial variations of the dielectric permittivity $U(z)$ inside the layer. These solutions may become useful for several problems of cross-disciplinary physics dealing with the incidence of waves of different physical nature on the interface of layered media at an arbitrary angle. Unlike widely used models of these media built from broken lines (boxlike [20] or “trapezoidal” [21] potentials), the smooth flexible distribution of potential (16) may pave the way to revisit the series of reflectance and/or transmittance problems in optics [22], acoustics [23], electromagnetics of heterogeneous media [24], and matter waves physics [25].

ACKNOWLEDGMENTS

The authors thank Professor O.F. Petrov for the support of this research. S.J. and L.V. thank the partial support of the Spanish Ministerio de Economía y Empresa under Grant No. ESP2016-79135-R. A.S. conceived the idea and supervised the project, N.E. conducted the initial study of microwaves in the heterogeneous media, and S.J. and L.V. carried out the computations. All authors contributed to interpreting the results and writing the manuscript.

APPENDIX A: SOLUTION OF THE MASTER EQUATIONS

To find the solutions of Eq. (21), it is worthwhile to introduce a new variable ν and a new function $F(\nu)$, given by

$$\nu = \frac{1 - \tanh(\zeta)}{2}, \quad f(\zeta) = \frac{1}{\cosh^2(\zeta)} F(\nu). \quad (\text{A1})$$

After this transformation, the equation governing the function becomes

$$\nu(1-\nu)\frac{dF^2}{d\nu^2} + [\gamma - (1+\alpha+\beta)\nu]\frac{dF}{d\nu} - \alpha\beta F = 0. \quad (\text{A2})$$

Equation (A2) is the hypergeometric equation in its standard form, where the parameters α , β , and γ are defined via the quantities h^2 and $\aleph_{s,p}$ (22):

$$\gamma = 1 + 2p, \quad 2p = \pm\sqrt{-h^2}, \quad (\text{A3})$$

$$\begin{aligned} \alpha_s, \beta_s &= 2p + \frac{1}{2} \pm ik_{\perp}L, \\ \alpha_p, \beta_p &= 2p + \frac{1}{2} \pm \sqrt{1 - (k_{\perp}L)^2}. \end{aligned} \quad (\text{A4})$$

Thus, the values of the parameters α and β are different for the TE₁₁ and TH₁₁ modes, while the parameter γ has the same value for both modes. The hypergeometric equation (A2) has two linearly independent solutions F_1 and F_2 . Since the parameters α , β , and γ are linked by the relation $\text{Re}(\alpha + \beta + 1) = 2\gamma$, these solutions are given by the hypergeometric series G [26]:

$$F_1 = G(\alpha, \beta, \gamma, \nu), \quad F_2 = G(\alpha, \beta, \gamma, 1-\nu). \quad (\text{A5})$$

The hypergeometric series G are known to converge absolutely inside the circle $|\nu| = 1$ under the condition $\text{Re}(\alpha + \beta - \gamma) < 0$ [26]. According to Eqs. (A3) and (A4), this process corresponds to $\alpha + \beta - \gamma = 2p$. This result means that the series (A5) converge absolutely if the value p is negative, which means, in turn, that the condition $h^2 < 0$ has to be satisfied. This case is considered below.

Using a standard procedure to find the reflection and/or transmission coefficients for the barrier (16) based on the continuity conditions of the components of the modes on the boundaries $z = 0$ and $z = d$ of the gradient layer, the values of the variables ς and ν at these boundaries can be shown to satisfy

$$\begin{aligned} z=0: \quad \varsigma &= -\ln(m_+), \quad \nu_{\pm} = \frac{1}{2} \left(1 \pm \frac{M}{\sqrt{1+M^2}} \right), \\ \cosh(\varsigma) &= \sqrt{1+M^2}, \\ \tanh(\varsigma) &= -\frac{M}{\sqrt{1+M^2}}, \\ z=d: \quad \varsigma &= \ln(m_+), \quad \nu = \nu_-, \quad h = \nu_+, \\ \cosh(\varsigma) &= \sqrt{1+M^2}, \quad \tanh(\varsigma) = \frac{M}{\sqrt{1+M^2}}. \end{aligned} \quad (\text{A6}) \quad (\text{A7})$$

Since the transformations (12) ensure the presentation of the equations for both the S - and the P -polarized waves

under the same form (21), the following analysis of polarization effects proves to be standardized. The continuity conditions on the boundaries of the gradient layers provide the essential difference in the reflectance and/or transmittance spectra for the TE₁₁ and TH₁₁ modes. Although we achieve a single formulation for both cases, it is convenient, in order to simplify the computations, to examine the spectra separately.

APPENDIX B: PROPAGATION OF THE TE₁₁ MODE IN A WAVEGUIDE LOADED BY GRADIENT LAYERS

The spatial structure of the TE₁₁ mode is described by Eqs. (2), (4), and (5); the function $\Psi_s(z)$ in these equations is governed by Eqs. (10) and (21). Introducing in Eq. (A3) the definition $2p = -l$ ($l > 0$), $l = \sqrt{-h^2}$, and using the functions in Eq. (A5), one can write the explicit solution of the wave equation (21) for the gradient layer (16) as a sum of forward and backward waves inside the gradient layer:

$$\Psi_s = C_s [\cosh(\varsigma)]^{l-1/2} (F_1 + Q_s F_2), \quad (\text{B1})$$

where C_s is some normalization constant; the parameter Q_s determines the contribution of the backward wave to the entire field; and the values of the parameters α and β in the functions F_1 and F_2 (A5) are given in Eq. (A4): $\alpha = \alpha_s$, $\beta = \beta_s$. Substitution of this function $\Psi(z)$ into Eqs. (2), (4), and (5) gives the transverse components of the fields $\vec{E}$ and $\vec{H}$; using these components and taking into account the correlation (6), one can calculate the longitudinal component H_z from the first of Maxwell's equations (1):

$$H_z = -\frac{A_1 \pi (a^2 + b^2) \Psi_s}{a^2 b} \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right). \quad (\text{B2})$$

Consider the radiation flow of the TE₁₁ mode incident on the boundary $z = 0$ of the gradient layer. To find the complex reflection coefficient R for the TE₁₁ mode, we use the continuity conditions for the components E_x , H_y , and H_z on the plane $z = 0$. Calculations with the components E_y , H_x , and H_z give the same result. The components H_y and H_x are expressed via the derivatives of function Ψ_s (B1) with respect to the variable z (2), which can be calculated by means of the formulas

$$\frac{d\varsigma}{dz} = \frac{\cosh(\varsigma)}{L}, \quad \frac{d\nu}{dz} = -\frac{1}{2L \cosh(\varsigma)}. \quad (\text{B3})$$

Noting that the value of the argument ν of the hypergeometric function $G(\alpha_s, \beta_s, \gamma, \nu)$ at the plane $z = 0$ is equal

to ν_+ (A6), we introduce the notations

$$\begin{aligned} G_1 &= G(\alpha_s, \beta_s, \gamma, \nu_+), \\ G_2 &= G(\alpha_s, \beta_s, \gamma, \nu_-), \quad \nu_- = 1 - \nu_+. \end{aligned} \quad (\text{B4})$$

Using the expression of the field $\Psi_s(z)$ inside the waveguide in the area $z \leq 0$ (8), one can write the continuity conditions for the components E_x , H_y , and H_z on the plane $z = 0$. These conditions give the equation governing the complex reflection coefficient R :

$$\begin{aligned} \frac{1 + R_{s1}}{ik_z(1 - R_{s1})} &= \frac{2L\sqrt{1 + M^2}}{-M\sqrt{1 + M^2}(2l - 1) - \Lambda_1}, \\ \Lambda_1 &= \frac{G'_1 - Q_s G'_2}{G_1 + Q_s G_2}, \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} G'_1 &= \left. \frac{dG(\alpha_s, \beta_s, \gamma, \nu)}{d\nu} \right|_{\nu=\nu_+}, \\ G'_2 &= \left. \frac{dG(\alpha_s, \beta_s, \gamma, \nu)}{d\nu} \right|_{\nu=\nu_-}. \end{aligned} \quad (\text{B6})$$

From Eq. (B5), the expression for R_{s1} is

$$\begin{aligned} R_{s1} &= \frac{\sqrt{1 + M^2}[2iLk_z + M(2l - 1)] + \Lambda_1}{I_s - \Lambda_1}, \\ I_s &= \sqrt{1 + M^2}[2iLk_z - M(2l - 1)]. \end{aligned} \quad (\text{B7})$$

The unknown parameter Q_s has to be determined from the continuity conditions at the far side of the gradient layer, $z = d$. Taking into account formulas (B5) and (B7), these conditions can be written as

$$ik_z = \frac{M\sqrt{1 + M^2}(2l - 1)(G_2 + Q_s G_1) + G'_2 - Q_s G'_1}{2L\sqrt{1 + M^2}(G_2 + Q_s G_1)}. \quad (\text{B8})$$

Substitution of the value of Q_s , found from Eq. (B8),

$$Q_s = \frac{G'_2 - I_s G_2}{G'_1 + I_s G_1}, \quad (\text{B9})$$

into the definition of parameter Λ_1 (B5) gives the value of Λ_1 for the mode TE_{11} :

$$\Lambda_1 = \frac{(G'_1)^2 - (G'_2)^2 - I_s (G_1 G'_1 + G_2 G'_2)}{G_1 G'_1 + G_2 G'_2 - I_s [(G_1)^2 - (G_2)^2]}. \quad (\text{B10})$$

Finally, substitution of Λ_1 (B10) into Eq. (B7) results in the explicit formula for the complex reflection coefficient R_{s1} describing the reflectance of the TE_{11} mode from the single gradient layer. Substitution of the value of R_{s1} (B7)

into the continuity conditions on the plane $z = d$ gives the complex transmission coefficient T_{s1} describing the amplitude and phase of the TE_{11} mode traveling through one gradient layer:

$$\begin{aligned} T_{s1} &= |T_{s1}| \exp(i\varphi_{s1}) \\ &= \frac{(1 + R_{s1}) [G_1 G'_1 + G_2 G'_2 - \Lambda_1 (G_1^2 - G_2^2)]}{G_1 G'_2 + G_2 G'_1}. \end{aligned} \quad (\text{B11})$$

The generalization of these results to the case of a multi-layer barrier ($q > 1$) can be carried out by means of the same algorithm. However, a more cumbersome algebra is necessary. An example of such a calculation for a pair of adjoining layers ($q = 2$) is given in Appendix D.

APPENDIX C: PROPAGATION OF THE TH_{11} MODE IN A WAVEGUIDE LOADED BY GRADIENT LAYERS

The reflection of the TH_{11} mode in a waveguide loaded by gradient layers (16) can be examined by analogy with the analysis developed above in Appendix B for the TE_{11} mode. We start, instead of Eq. (B1), from the generating function Ψ_p ,

$$\Psi_p = C_p [\cosh(\zeta)]^{l+1/2} (V_1 + Q_p V_2), \quad (\text{C1})$$

where C_p is some normalization constant and V_1 and V_2 are the solutions of Eq. (21) given by the hypergeometric functions

$$V_1 = G(\alpha_p, \beta_p, \gamma, \nu), \quad V_2 = G(\alpha_p, \beta_p, \gamma, 1 - \nu), \quad (\text{C2})$$

with the parameters α_p , β_p , and γ defined in Eqs. (A3) and (A4). We can determine by means of Eq. (2) the transverse components of the TH_{11} mode E_x , H_x , H_y , and E_y . Substituting the components H_x and H_y into the second of Maxwell's equations (1) and using the condition in Eq. (7) gives the value of the longitudinal component E_z :

$$E_z = -\frac{B_1 \pi (a^2 + b^2) \Psi_p}{a^2 b} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right). \quad (\text{C3})$$

Now, the continuity conditions on the plane $z = 0$ for the components E_x , H_y , and E_z correspond to

$$\begin{aligned} \frac{1 + R_{p1}}{ik_z(1 - R_{p1})} &= -\frac{2Ln_0^2 \sqrt{1 + M^2}}{M\sqrt{1 + M^2}(2l + 1) - \Lambda_1}, \\ \Lambda_1 &= \frac{Y'_1 - Q_p Y'_2}{Y_1 + Q_p Y_2}. \end{aligned} \quad (\text{C4})$$

The reflection coefficient on the single layer is determined by Eq. (C4). Here, Y_1 and Y_2 can be viewed as the result of

replacing α_s by α_p and β_s by β_p in Eqs. (B4) and (B6) that give the values of G_1 , G_2 , and their derivatives. We have, now,

$$\begin{aligned} Y_1 &= G(\alpha_p, \beta_p, \gamma, v_+), \quad Y_2 = G(\alpha_p, \beta_p, \gamma, v_-), \\ Y'_1 &= \left. \frac{dG(\alpha_p, \beta_p, \gamma, v)}{dv} \right|_{v=v_+}, \\ Y'_2 &= \left. \frac{dG(\alpha_p, \beta_p, \gamma, v)}{dv} \right|_{v=v_-}. \end{aligned} \quad (C5)$$

Proceeding in a similar way, one can calculate the parameter Q_p from the continuity condition at the plane $z = d$ and define the parameter Λ_1 (C4) for the TH₁₁ mode:

$$\begin{aligned} \Lambda_1 &= \frac{(Y'_1)^2 - (Y'_2)^2 - I_p (Y_1 Y'_1 + Y_2 Y'_2)}{Y_1 Y'_1 + Y_2 Y'_2 - I_p [(Y_1)^2 - (Y_2)^2]} \\ I_p &= \sqrt{1 + M^2 [2in_0^2 Lk_z - M(2l + 1)]}. \end{aligned} \quad (C6)$$

Finally, the complex reflection coefficient for the TH₁₁ mode passing through a single gradient layer corresponds to

$$\begin{aligned} T_{p1} &= |T_{p1}| \exp(i\varphi_{p1}) \\ &= \frac{(1 + R_{p1}) [Y_1 Y'_1 + Y_2 Y'_2 - \Lambda_1 (Y_1^2 - Y_2^2)]}{Y_1 Y'_1 + Y_2 Y'_2}. \end{aligned} \quad (C7)$$

Note that the expressions for the quantities Λ_1 , I_p (C6), and the transmission coefficient for the TH₁₁ mode (C7) can be obtained from the corresponding expressions for the TE₁₁ mode exchanging G_1 by Y_1 , G_2 by Y_2 , and replacing L by Ln_0^2 and $2l - 1$ by $2l + 1$ in Eq. (B7).

The example of computing the reflection and/or transmission coefficients for the TH₁₁ mode traveling through the pair of gradient barriers is given in Appendix D.

APPENDIX D: TRANSMITTANCE THROUGH A PAIR OF ADJOINING BARRIERS

To find the complex transmission coefficient for the waves passing through a pair of similar adjoining barriers, one can follow the computational scheme developed above for a single barrier. Assigning the value $q = 1$ to the gradient layer at the far side of this structure, we have to examine the reflection of the wave incident on the layer with value $q = 2$. The standard continuity conditions on the boundary between layers 1 and 2 have to be taken into account.

Consider, firstly, the propagation of the TE₁₁ mode. Let us introduce the parameter Λ_{2s} analogous to

parameter Λ_1 (B5):

$$\Lambda_{2s} = \frac{G'_2 - Q_{2s} G'_1}{G_2 + Q_{2s} G_1}. \quad (D1)$$

Here, the coefficient Q_{2s} describes, according to Eq. (B1), the contribution of the backward wave to the entire field in layer 2. The continuity conditions on the boundary between the adjoining layers 1 and 2 give the correlation

$$\Lambda_{2s} = 2M\sqrt{1 + M^2(2l - 1)} + \Lambda_{s1}. \quad (D2)$$

The value Λ_{s1} is defined as Λ_1 in Eq. (B10). Omitting the intermediate calculations, similar to those performed in Appendix B, we find the reflection and/or transmission coefficients for the TE₁₁ mode passing through the pair of barriers:

$$R_{s2} = \frac{\sqrt{1 + M^2 [2iLk_z + M(2l - 1)]} + \chi_{s2}}{I_s - \chi_{s2}}, \quad (D3)$$

$$\begin{aligned} \chi_{s2} &= \frac{P_s \Lambda_{s2} + K_s}{P_s + H_s \Lambda_{s2}}, \quad P_s = G_1 G'_1 + G_2 G'_2, \\ K_s &= (G'_1)^2 - (G'_2)^2, H_s = G_1^2 - G_2^2, \end{aligned} \quad (D4)$$

$$T_{s2} = |T_{s2}| \exp(i\varphi_{s2}) = \frac{(1 + R_{s2}) (P_s - H_s \Lambda_{s2})}{P_s + H_s \Lambda_{s2}}. \quad (D5)$$

The corresponding similar expressions obtained for the TH₁₁ mode are

$$R_{p2} = \frac{\sqrt{1 + M^2 [2in_0^2 Lk_z + M(2l + 1)]} + \chi_{p2}}{I_p - \chi_{p2}}, \quad (D6)$$

$$I_p = \sqrt{1 + M^2 [2in_0^2 Lk_z - M(2l + 1)]}, \quad (D7)$$

$$\begin{aligned} \chi_{p2} &= \frac{P_p v + K_p}{P_p + H_p \Lambda_{p2}}, \quad P_p = Y_1 Y'_1 + Y_2 Y'_2, \\ K_p &= (Y'_1)^2 - (Y'_2)^2, H_p = Y_1^2 - Y_2^2, \end{aligned} \quad (D8)$$

$$\Lambda_{2p} = 2M\sqrt{1 + M^2(2l + 1)} + \Lambda_{p1}, \quad (D9)$$

with the value Λ_{p1} as defined in Eq. (C6). Finally,

$$T_{p2} = |T_{p2}| \exp(i\varphi_{p2}) = \frac{(1 + R_{p2}) (P_p - H_p \Lambda_{p2})}{P_p + H_p \Lambda_{p2}}. \quad (D10)$$

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