Two applications of the subnormality of the Hessenberg matrix related to general orthogonal polynomials

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\textbf{ABSTRACT}

In this paper we prove two consequences of the subnormal character of the Hessenberg matrix $D$ when the hermitian matrix $M$ of an inner product is a moment matrix. If this inner product is defined by a measure supported on an algebraic curve in the complex plane, then $D$ satisfies the equation of the curve in a noncommutative sense. We also prove an extension of the Krein theorem for discrete measures on the complex plane based on properties of subnormal operators.

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\section{1. Introduction}

Let $\mu$ be a positive and finite Borel measure with real support. It is well known that there exists a sequence of orthonormal polynomials (NOPS), $\{p_n(x)\}_{n=0}^\infty$, satisfying a three term recurrence relation,

\[ xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \]

with coefficients $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=0}^\infty$ and initial conditions $p_0(x) = 1$ and $p_{-1}(x) = 0$. These coefficients are the non-zero entries of the tridiagonal Jacobi matrix $J$.

Recently, interest in extending the results of the real case to Borel measures supported in some bounded set of the complex plane has increased; see [14, 15]. The role of the tridiagonal Jacobi matrix is now played by the upper Hessenberg matrix $D$, which corresponds to the operator of multiplication by

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z, with respect to the basis given by the NOPS. The connection between the matrix $J$ as an operator and orthogonal polynomials has been extensively studied by Dombrowski; see [8,9]. In another different way Cantero has studied the relation between O.P. and five-diagonal operators; see [3,4].

It is well known that when the support of a measure is real and bounded, then the associated infinite Jacobi matrix $J$ defines a bounded operator in $\ell^2$. This operator is also algebraic in the sense that $J$ is a zero of the equation $z - \bar{z} = 0$ defining the support. Here, we show that this property extends to the case of measures with bounded support on curves given by polynomials in $z$ and $\bar{z}$. In this case the role of $J$ is played by $D$. Orthogonal polynomials associated with measures supported on arbitrary curves have been extensively studied; see [17, Chapter XVI]. For closed bounded sets in the complex plane, see [20]. For some particular curves different from the unit circle, see, for instance, [2,18].

In the theory of spectral measures it is natural to ask under what conditions the support of the measure is a countable set with a finite number of limit points. An answer is provided by Krein's theorem from 1938. A matrix version of this theorem, (see [5, pp. 128–141]), establishes that if $M$ is a real moment matrix with bounded support in the real line and $J$ is the associated Jacobi matrix, then the measure has as the only accumulation points of its support the finite set $\sigma_1, \sigma_2, \ldots, \sigma_m \in \mathbb{R}$ if and only if $Q(J)$ is a compact operator, where $Q(x) = \prod_{k=1}^{m}(x - \sigma_k)$.

More recently, Golinskii has proved the analogue of this theorem for the unit circle, (see [11, p. 68]), and Zhedanov has constructed, using the symmetrized Al-Salam-Carlitz polynomials, examples of orthogonal polynomials for a discrete measure on the unit circle having one or two limit points; see [21, pp. 89–90].

In this paper we prove a sort of noncommutative Cayley–Hamilton theorem for the matrix $D$, when the support is bounded on a curve expressible as a polynomial in $z$ and $\bar{z}$. Also, we have proved a theorem that generalizes the Krein theorem for measures not necessarily on the real line.

We work with the $2 \times 2$ matrix representation of normal extensions of subnormal operators, and we can obtain results for $N$ through this matrix representation. We obtain also weaker results for $D$ by restricting to $\ell^2$ the results for $N$.

The paper is organized as follows. In Section 2, we give some preliminaries on orthogonal polynomials, the Hessenberg matrices and subnormal operators. In Section 3, we prove a result about orthogonal polynomials on algebraic curves. Finally, Section 4 contains a proof of a general case of the Krein theorem.

### 2. Preliminaries

Given an infinite Hermitian positive definite (HPD) matrix, $M = (c_{ij})_{i,j=0}^{\infty}$, whether it comes from a measure or not, we call $M'$ the matrix obtained by removing from $M$ its first column. Let $M_n$ and $M'_n$ be the corresponding sections of order $n$ of $M$ and $M'$, respectively, i.e., the principal submatrices of order $n$ of $M$ and $M'$.

Suppose that $M$ is an HPD matrix and let $M_n = T_n T_n^*$ be the Cholesky decomposition of $M_n$, which is unique if $t_{ii} > 0$. There can be built an infinite upper Hessenberg matrix $D = (d_{ij})_{i,j=1}^{\infty}$ with sections of order $n$ satisfying

$$D_n = T_n^{-1} M_n' (T_n^*)^{-1} = T_n^* F_n (T_n^*)^{-1},$$

where $F_n$ is the Frobenius matrix associated to $P_n(z)$, and $\{P_n(z)\}$ is the monic OPS associated to $M$, with

\[
\begin{pmatrix}
c_{00} & c_{10} & c_{20} & \cdots & c_{n0} \\
c_{01} & c_{11} & c_{21} & \cdots & c_{n1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{0,n-1} & c_{1,n-1} & c_{2,n-1} & \cdots & c_{n,n-1} \\
1 & z & z^2 & \cdots & z^n
\end{pmatrix}
\]
Throughout $P_n(z)$ will be the monic polynomial and $p_n(z)$ will be the normalized polynomial. The fact that $T_n$ is a lower triangular matrix, implies that

\[ D = T^{-1} M' (T^*)^{-1} = T^* S_R (T^*)^{-1}, \]

where $S_R$ is the infinite matrix associated to the shift-right operator in $\ell^2$. We must be careful, because $T^{-1}, T^*$ and $(T^*)^{-1}$ are infinite triangular matrices but they do not necessarily define operators in $\ell^2$.

An important result of Atzmon (see [1]) established conditions on an infinite HPD matrix $M = (c_{j,k})_{j,k=0}^\infty$ to be the moment matrix of a measure on the unit disk, i.e., for there to exist $\Omega \subset \mathbb{C}$ and a probability measure $\mu : \Omega \to \mathbb{R}_+$, with $c_{j,k} = \int_\Omega z^j \bar{z}^k d\mu(z)$.

This result was extended in [19] to a bounded set on the complex plane using only the subnormal character of this matrix as an operator $D : \ell^2 \to \ell^2$.

An operator $S$ on a Hilbert space $\mathcal{H}$ is subnormal if there is a Hilbert space $K$ containing $\mathcal{H}$ and a normal operator $N$ on $K$ such that $NH \subset \mathcal{H}$ and $S = N|\mathcal{H}$. In what follows, $S$ will always denote a subnormal operator on $\mathcal{H}$ and $N$ will be its minimal normal extension on $K \supset \mathcal{H}$. If we write $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, then $N$ has the $2 \times 2$ matrix representation (see [7, p. 41])

\[ N = \begin{pmatrix} S & X \\ 0 & R \end{pmatrix}. \]

If $M$ is a moment matrix with measure $\mu$ supported on a bounded set in the complex plane, the infinite matrix $D$ defines a bounded subnormal operator. In this case $\mathcal{H} = \ell^2$ and $\mathcal{K} = \ell^2 \oplus (\ell^2)^\perp$.

We use the same symbol $D$ to denote the infinite matrix and the matrix as an operator in $\ell^2$. It is well known that there is an isometric isomorphism between $L^2(\mu)$ and $\mathcal{K}$.

As usual $P(\mu)$ denotes the linear space of polynomials with complex coefficients associated to the measure $\mu$. We denote by $S_\mu$ the operator of multiplication by $z$ in $P^2(\mu)$, the closure in $L^2(\mu)$ of the space $P(\mu)$, and $N_\mu$ will be the operator of multiplication by $z$ in $L^2(\mu)$. It is known that $N_\mu$ is the minimal normal extension of $S_\mu$. In this case all the operators are bounded because the support is bounded. It is easy to prove that $S_\mu$ is unitarily equivalent to the infinite Hessenberg matrix $D$ as an operator in $\ell^2$, and $N_\mu$ is unitarily equivalent to the operator $N$, which is the minimal normal extension of $D$.

**Lemma 1** ([1,19]). Let $M = (c_{j,k})_{j,k=0}^\infty$ be an infinite HPD matrix and $\|D\| < +\infty$, then $M$ is a moment matrix if and only if $D$ is subnormal.

It is not difficult to prove that this moment problem is always determined when the support of the measure is bounded as a consequence of the Stone–Weierstrass theorem in the bidimensional case.

### 3. Polynomials in $D$ and $D^*$

The following theorem extends the results of [18] about orthogonal polynomials on harmonic algebraic curves.

**Theorem 2.** Let $\mu$ be a probability measure with bounded support and $\text{supp}(\mu) \subset \gamma \subset \mathbb{C}$, where $\gamma$ is an algebraic curve which can be expressed as a polynomial in $z$ and $\bar{z}$, that is $\sum_{j,k=0}^m a_j,k z^j \bar{z}^k = 0$, with $a_j,k \in \mathbb{C}$. Then the infinite matrix $D$ associated to $M = (c_{j,k})_{j,k=0}^\infty$, such that $c_{j,k} = \int_\gamma z^j \bar{z}^k d\mu(z)$, satisfies

\[ \sum_{j,k=0}^m a_{j,k} (D^*)^k D^j = 0, \]

where we have replaced $z$ by $D$ and $\bar{z}$ by $D^*$ in the equation of $\gamma$, and the two products $\bar{z}z$ and $z \bar{z}$ by $D^* D$. 
**Proof.** Since \( \mu \) is a probability measure, \( D \) is subnormal. We denote by \( N \) the minimal normal extension of \( D, N = \text{mne}(D) \). As before, we denote by \( N_{\mu} \) multiplication by \( z \) in \( L^2_{\mu} \). We know that \( \sigma (N) = \sigma (N_{\mu}) = \text{supp}(\mu) \), and there is a spectral measure \( E(\lambda) \) on the Borel subsets of \( \text{supp}(\mu) \) such that \( N = \int_{\sigma (N)} z dE(z) \). Therefore, by means of the spectral theorem for normal operators we have

\[
\sum_{j,k=0}^{m} a_{jk}N^j(N^*)^k = \int_{\sigma (N)} \left( \sum_{j,k=0}^{m} a_{jk} z^j \bar{z}^k \right) dE(z).
\]

By hypothesis, the points of \( \sigma (N) \) satisfy the equation of the curve and \( \sigma (N) = \text{supp}(\mu) \). Thus

\[
\sum_{j,k=0}^{m} a_{jk}N^j(N^*)^k = 0.
\]

From the \( 2 \times 2 \) matrix representation of a subnormal operator we have

\[
N = \begin{pmatrix} D & X \\ 0 & Y \end{pmatrix} \quad \text{and} \quad N^* = \begin{pmatrix} D^* & 0 \\ X^* & Y^* \end{pmatrix}.
\]

Hence

\[
N^j = \begin{pmatrix} D^j & 0 \\ 0 & Y^j \end{pmatrix}, \quad (N^*)^k = \begin{pmatrix} (D^*)^k & 0 \\ 0 & (Y^*)^k \end{pmatrix}.
\]

This yields

\[
N^*N = \begin{pmatrix} D^*D & D^*X \\ X^*D & X^*X + Y^*Y \end{pmatrix}, \quad NN^* = \begin{pmatrix} DD^* + XX^* & XY^* \\ YX^* & YY^* \end{pmatrix}.
\]

We already know that \( N^*N = NN^* \). At this point, we consider the product \( N^*N \) to obtain an equation in \( D \) and \( D^* \) in the \([1, 1]\) entry of the \( 2 \times 2 \) matrix \( \sum_{j,k=0}^{m} a_{jk}(N^*)^kN^j \). It is easy to check that

\[
\sum_{j,k=0}^{m} a_{jk}(N^*)^kN^j = \begin{pmatrix} \sum_{j,k=0}^{m} a_{jk}(D^*)^kD^j & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

and finally we obtain

\[
\sum_{j,k=0}^{m} a_{jk}(D^*)^kD^j = 0.
\]

From the proof it can be seen how to replace \( z \) by \( D \) and \( \bar{z} \) by \( D^* \). Consequently, \( (\bar{z})^k \bar{z}^j = z^j(\bar{z})^k \) takes the form \( (D^*)^kD^j \), but not \( D^j(D^*)^k \).

**Corollary 3.** Let \( \mu \) be a probability measure with bounded support. The following five assertions are satisfied for all \( z, \bar{z} \in \text{supp}(\mu) \).

1. If \( z - \bar{z} = 0 \), then \( D = D^* \).
2. If \( |z| = 1 \), then \( D^4D = I \).
3. If \( z - \beta = |z - \beta| e^{\beta i} \), then \( \alpha(D - 1\beta) = \alpha(D^* - 1\bar{\beta}) \), with \( \alpha = e^{\beta i} \).
4. If \( |z - \beta| = R \), then \( D^*D = \bar{\beta}D + \beta D^* + (R^2 - |\beta|^2)I \).
5. If \( |z - c| + |z + c| = 2a \), with \( a^2 = b^2 + c^2 \), then

\[
[D^2 + (D^*)^2](a^2 - b^2) + 4a^2b^2I = 2D^*D(a^2 + b^2).
\]
Note that the condition $DD^* = I$ in (2) is not true if $\mu$ satisfies Szegö's condition.

Another less obvious application is related to measures whose support is a cross-like set formed by the intervals $[-1, 1]$ and $[-i, i]$. The support is given by $xy = 0$ with $|x + yi| \leq 1$. The expression $xy = 0$ is equivalent to $z^2 = \overline{z}^2$. Therefore $D^2 = (D^*)^2$. Using that $D$ and $D^*$ are upper and lower Hessenberg matrices it is easy to check that $D^2$ and $(D^*)^2$ are pentadiagonal.

4. Extension of Krein's theorem

In the next theorem, we prove a generalization of the Krein theorem for the hermitian complex case.

We need first to prove two results about pure atomic distributions. Let $Z = \{z_1, z_2, \ldots\}$ be a bounded set of complex points, with weights $\{w_1, w_2, \ldots\}$, where $\sum_{n=1}^{\infty} w_n < +\infty$. For such a distribution we have the moment matrix $M = (c_{ij})_{i,j=0}^{\infty}$, where $c_{jk} = \sum_{n=1}^{\infty} z_n^j \overline{z}_n^k w_n$. Let $D$ be the associated Hessenberg matrix. Obviously the support of this measure is $\text{supp}(\mu) = Z$.

**Proposition 4.** If $C \setminus Z$ is a connected set and the interior of $Z$ is empty, then the infinite Hessenberg matrix $D$ corresponds to a normal operator in $l^2$.

**Proof.** The set $K = \overline{Z}$ is compact. As usual we denote by $C(K)$ the space of all continuous functions with support $K$. The set $K$ satisfies the hypothesis of Mergelyan's theorem (see [10, p. 97]), and consequently given $f \in C(K)$ and $\epsilon > 0$, $\exists Q(z)$ such that $|f(z) - Q(z)| < \epsilon$. This implies that $\int_{\text{supp}(\mu)} |f(z) - Q(z)|^2 d\mu(z) < \epsilon^2 c_{00}$. Clearly $C(K) = P^2(\mu)$. Since $C(K)$ is dense in $L^2_\mu(K)$ (see, for example, [13, p. 61]), we conclude that $P^2(\mu) = L^2_\mu(K)$. Therefore we are in a complete case. It follows that $S_\mu = N_\mu$, and also $D = N$. Consequently $D$ is a normal operator.

**Proposition 5.** Let $Z$ be as in Proposition 4 and $Z' \cap Z = \emptyset$, where $Z'$ is the set of accumulation points of $Z$. Then

$$D = U^8(\delta_{ij}z_i)_{i,j=1}^{\infty} U \quad \text{and} \quad U^* U = UU^* = I,$$

where $U = V(T^+)^{-1}$ and $T$ is the Cholesky factor in the decomposition $M = TT^*$, and $V$ is the Vandermonde matrix of the atoms

$$V = \begin{pmatrix}
\sqrt{w_1} & \sqrt{w_1 z_1} & \sqrt{w_1 z_1^2} & \cdots \\
\sqrt{w_2} & \sqrt{w_2 z_2} & \sqrt{w_2 z_2^2} & \cdots \\
\sqrt{w_3} & \sqrt{w_3 z_3} & \sqrt{w_3 z_3^2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

**Proof.** Let $L = (\delta_{ij}z_i)_{i,j=1}^{\infty}$. It is clear that $M' = V^*LV$. Using $D = T^{-1}M'(T^+)^{-1}$, it follows that $D = T^{-1}V^*LV(T^+)^{-1}$. The elements of the $i$th column of the infinite matrix $(T^+)^{-1}$ are the coefficients of $p_{i-1}(z)$ with respect to the basis $(z^k)_{k=0}^{\infty}$. Therefore $U = V(T^+)^{-1} = (\sqrt{w_i} p_{j-1}(z_i))_{i,j=1}^{\infty}$. Now we calculate $U^* U$,

$$(U^* U)_{i,j} = \sum_{k=1}^{\infty} p_i(z_k) p_j(z_k) w_k = \delta_{ij},$$
due to the orthogonality of the NOPS on the set $Z = \{z_1, z_2, \ldots\}$. On the other hand, the product $UU^*$ is

$$UU^* = \left( \sqrt{w_i} p_{k-1}(z_i) \right)_{i,k=1}^{\infty} \left( \sqrt{w_j} p_{k-1}(z_j) \right)_{k,j=1}^{\infty} = \left( \sum_{k=0}^{\infty} p_k(z_i) \overline{p_k(z_j)} \right)_{i,j=1}^{\infty}.$$ 

To prove the statement we need also that $(UU^*)_{ij} = \delta_{ij}$. For that we introduce the bounded functionals $L_i : P^2(\mu) \to P^2(\mu)$ defined by $L_i(f) = f(z_i)$. Recall that the inner product in $P^2(\mu)$ is $(Q(z), R(z)) = \sum_{k=0}^{\infty} Q(z) R(z) w_k$. It is extended to $P^2(\mu)$ as usual. Obviously $\|L_i\| < 1/\sqrt{w_i}$. It is clear that the n-kernel $K_n(z, z_j) = \sum_{k=0}^{n} p_k(z) \overline{p_k(z_j)}$, with $n > i$, has the reproducing property, that is $(Q(z), K_n(z, z_j)) = Q(z_j)$. The function $K(z, z_j) = \lim_{n \to \infty} K_n(z, z_j)$ defined on $Z = \{z_1, z_2, \ldots\}$ has the same property.

Now we consider the characteristic function $\chi_{z_i}(z)$, defined by $\chi_{z_i}(z) = 1$ if $z = z_i$ and 0 otherwise. Then $\chi_{z_i}(z)/w_i$ is a continuous function because we have $Z' \cap Z = \emptyset$ and it is only defined in isolated points. Hence $\chi_{z_i}(z)/w_i$ is defined for every $f \in C(K)$, agrees with $K(z, z_i)$, and for all $f \in P^2(\mu) = L^2(\mu)$ we have

$$\langle f(z), K(z, z_i) \rangle = f(z_i) = \left( f(z), \frac{\chi_{z_i}(z)}{w_i} \right) = \sum_{k=1}^{\infty} f(z_k) \frac{\chi_{z_i}(z)}{w_i} w_k.$$ 

Then $\chi_{z_i}(z)/w_i = K(z, z_i)$, a.e. in $L^2(\mu)$. In particular $\chi_{z_i}(z)/w_i = K(z, z_i)$ at the points with positive measure, i.e., $K(z_j, z_i) = \chi_{z_i}(z_j)/w_i = \delta_{ij}/w_i$ on $Z$ and therefore $UU^* = I$.

**Theorem 6** (Extension of Krein's theorem to the complex case). Let $M$ be a moment matrix with bounded support and let $D$ be the associated Hessenberg matrix. Then the measure associated to $M$ has $\sigma_1, \sigma_2, \ldots, \sigma_m \in \mathbb{C}$, as the only accumulation points of its support, if and only if $Q(D)$ is a compact operator, where $Q(z) = \prod_{k=1}^{m} \frac{1}{z - \sigma_k}$.

**Proof.**

Necessary condition. As the support is a bounded set and it has a finite number of limit points, necessarily the measure is atomic. Assume that $L = \text{diag}(z_1, z_2, \ldots)$ is the matrix of the atoms reordered such that $d(z_i, \cup_k^{m} \sigma_k) > d(z_{i+1}, \cup_k^{m} \sigma_k)$. We have shown before that in this case $D$ is an infinite, bounded, and normal Hessenberg matrix, satisfying $D = U^*LU$, with $U = V(T^*)^{-1}$, where $V = \left( \sqrt{w_j} f_j^{k-1} \right)_{j,k=1}^{\infty}$. We have proved that $U^*$ and $U$ are unitary operators, and we have $D^n = U^*L^nU$, so $Q(D) = U^*Q(L)U$. $L$ is a diagonal matrix, hence $Q(L) = (Q(z))_{i,j=1}^{\infty}$. The zeros of $Q(z)$ are exactly the accumulation points of the diagonal elements of $L$. Therefore $\lim_{n \to \infty} Q(z_n) = 0$ and the diagonal matrix $Q(L)$ defines a compact operator. As $U^*$ and $U$ are bounded operators, we have finally that $Q(D)$ is a compact operator.

Sufficient condition. By Lemma 1, if $M$ is a moment matrix then $D$ defines a subnormal operator, and it is bounded by hypothesis. If $N = \text{max}(D)$, it is well known that $Q(N)$ is the normal extension of $Q(D)$, see [6, p. 204]. Hence $Q(D)$ is subnormal and bounded. Therefore, $Q(D)$ is hyponormal. By hypothesis $Q(D)$ is a compact operator. We find on [12, p. 206] that an operator compact and hyponormal is necessarily normal, and consequently $Q(D)$ is a compact and normal operator. The eigenvectors of $Q(D)$ are a basis of $l^2$, $Q(D)$ is a diagonalizable operator, and the sequence of eigenvalues of $Q(D)$ converges to zero if it is an infinite set. This is the case, because the matrix $Q(D)$ has the same rank as the matrix $D$, which is not finite. Were this not so, the moment matrix $M$ would have finite rank, which is not possible. We have that

$$\sigma(Q(D)) = \{\mu_1, \mu_2, \ldots, \mu_n, \ldots\}, \text{ with } \mu_n \to 0.$$ 

Thus $\sigma(Q(D)) = Q(\sigma(D))$. Consequently,

(i) $\sigma(D)$ is a discrete set, because it is the inverse image via $Q(z)$ of a denumerable set.
(ii) The limit points of $\sigma(D)$ are the solutions of $Q(z) = 0$. Suppose that $S = \{z|Q(z) = \mu_n, n \in \mathbb{N}\} = Q^{-1}(\{\mu_k\}_{k \in \mathbb{N}})$ is the set of all solutions of $Q(z) = \mu_n$ for all $n \in \mathbb{N}$. Then $S = Q^{-1}(Q(\sigma(D))) \supset \sigma(D)$. Hence the limit points of $\sigma(D)$ are necessarily the zeros of the polynomial $Q(z)$.

(iii) $D$ is a normal matrix. We know that $D$ is hyponormal because it is subnormal and by (ii), $\sigma(D)$ has a finite number of limit points. By Corollary 2, [16, p. 1455], if the spectrum of a hyponormal operator has a finite number of limit points then the operator is normal. Hence $D$ is normal, and $\sigma(D) = \text{supp}(\mu)$.

Consequently $\text{supp}(\mu)$ is a discrete set in $\mathbb{C}$ and all the limit points of $\text{supp}(\mu)$ are zeros of $Q(z)$.

References