Two applications of the subnormality of the Hessenberg matrix related to general orthogonal polynomials

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ARTICLE INFO

AMS classification:
42C05
15A09
47B20

Keywords:
Orthogonal polynomials
Hessenberg matrix
Subnormal operator

ABSTRACT

In this paper we prove two consequences of the subnormal character of the Hessenberg matrix D when the hermitian matrix M of an inner product is a moment matrix. If this inner product is defined by a measure supported on an algebraic curve in the complex plane, then D satisfies the equation of the curve in a noncommutative sense. We also prove an extension of the Krein theorem for discrete measures on the complex plane based on properties of subnormal operators.

1. Introduction

Let μ be a positive and finite Borel measure with real support. It is well known that there exists a sequence of orthonormal polynomials (NOPS), $\{p_n(x)\}_{n=0}^{\infty}$, satisfying a three term recurrence relation,

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

with coefficients $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ and initial conditions $p_0(x) = 1$ and $p_{-1}(x) = 0$. These coefficients are the non-zero entries of the tridiagonal Jacobi matrix J.

Recently, interest in extending the results of the real case to Borel measures supported in some bounded set of the complex plane has increased; see [14,15]. The role of the tridiagonal Jacobi matrix is now played by the upper Hessenberg matrix D, which corresponds to the operator of multiplication by

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z, with respect to the basis given by the NOPS. The connection between the matrix J as an operator and orthogonal polynomials has been extensively studied by Dombrowski; see [8,9]. In another different way Cantero has studied the relation between O.P. and five-diagonal operators; see [3,4].

It is well known that when the support of a measure is real and bounded, then the associated infinite Jacobi matrix J defines a bounded operator in ℓ^2. This operator is also algebraic in the sense that J is a zero of the equation $z - \bar{z} = 0$ defining the support. Here, we show that this property extends to the case of measures with bounded support on curves given by polynomials in z and $\bar{z}$. In this case the role of J is played by D. Orthogonal polynomials associated with measures supported on arbitrary curves have been extensively studied; see [17, Chapter XVI]. For closed bounded sets in the complex plane, see [20]. For some particular curves different from the unit circle, see, for instance, [2,18].

In the theory of spectral measures it is natural to ask under what conditions the support of the measure is a countable set with a finite number of limit points. An answer is provided by Krein's theorem from 1938. A matrix version of this theorem, (see [5, pp. 128–141]), establishes that if M is a real moment matrix with bounded support in the real line and J is the associated Jacobi matrix, then the measure has as the only accumulation points of its support the finite set $\sigma_1, \sigma_2, \ldots, \sigma_m \in \mathbb{R}$ if and only if $Q(J)$ is a compact operator, where $Q(x) = \prod_{k=1}^{m}(x - \sigma_k)$.

More recently, Golinskii has proved the analogue of this theorem for the unit circle, (see [11, p. 68]), and Zhedanov has constructed, using the symmetrized Al-Salam-Carlitz polynomials, examples of orthogonal polynomials for a discrete measure on the unit circle having one or two limit points; see [21, pp. 89–90].

In this paper we prove a sort of noncommutative Cayley–Hamilton theorem for the matrix D, when the support is bounded on a curve expressible as a polynomial in z and $\bar{z}$. Also, we have proved a theorem that generalizes the Krein theorem for measures not necessarily on the real line.

We work with the 2×2 matrix representation of normal extensions of subnormal operators, and we can obtain results for N through this matrix representation. We obtain also weaker results for D by restricting to ℓ^2 the results for N.

The paper is organized as follows. In Section 2, we give some preliminaries on orthogonal polynomials, the Hessenberg matrices and subnormal operators. In Section 3, we prove a result about orthogonal polynomials on algebraic curves. Finally, Section 4 contains a proof of a general case of the Krein theorem.

2. Preliminaries

Given an infinite Hermitian positive definite (HPD) matrix, $M = (c_{ij})_{i,j=0}^{\infty}$, whether it comes from a measure or not, we call M' the matrix obtained by removing from M its first column. Let M_n and M'_n be the corresponding sections of order n of M and M', respectively, i.e., the principal submatrices of order n of M and M'.

Suppose that M is an HPD matrix and let $M_n = T_n T_n^*$ be the Cholesky decomposition of M_n, which is unique if $t_{ii} > 0$. There can be built an infinite upper Hessenberg matrix $D = (d_{ij})_{i,j=1}^{\infty}$ with sections of order n satisfying

$$D_n = T_n^{-1} M'_n (T_n^*)^{-1} = T_n^* F_n (T_n^*)^{-1},$$

where F_n is the Frobenius matrix associated to $P_n(z)$, and $\{P_n(z)\}$ is the monic OPS associated to M, with

$$P_n(z) = \frac{1}{|M_n|} \begin{vmatrix} c_{00} & c_{10} & c_{20} & \cdots & c_{n0} \\ c_{01} & c_{11} & c_{21} & \cdots & c_{n1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{0,n-1} & c_{1,n-1} & c_{2,n-1} & \cdots & c_{n,n-1} \\ 1 & z & z^2 & \ldots & z^n \end{vmatrix}.$$

Throughout $P_n(z)$ will be the monic polynomial and $p_n(z)$ will be the normalized polynomial. The fact that T_n is a lower triangular matrix, implies that

$$D = T^{-1}M'(T^*)^{-1} = T^*S_R(T^*)^{-1},$$

where S_R is the infinite matrix associated to the shift-right operator in ℓ^2. We must be careful, because T^{-1}, T^* and $(T^*)^{-1}$ are infinite triangular matrices but they do not necessarily define operators in ℓ^2.

An important result of Atzmon (see [1]) established conditions on an infinite HPD matrix $M = (c_{j,k})_{j,k=0}^{\infty}$ to be the moment matrix of a measure on the unit disk, i.e., for there to exist $\Omega \subset \mathbb{C}$ and a probability measure $\mu : \Omega \to \mathbb{R}_+$, with $c_{j,k} = \int_\Omega z^j \bar{z}^k d\mu(z)$.

This result was extended in [19] to a bounded set on the complex plane using only the subnormal character of this matrix as an operator $D : \ell^2 \to \ell^2$.

An operator S on a Hilbert space $\mathcal{H}$ is *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $S = N|\mathcal{H}$. In what follows, S will always denote a subnormal operator on $\mathcal{H}$ and N will be its minimal normal extension on $\mathcal{K} \supset \mathcal{H}$. If we write $\mathcal{K} = \mathcal{H} \bigoplus \mathcal{H}^\perp$, then N has the 2×2 matrix representation (see [7, p. 41])

$$N = \begin{pmatrix} S & X \\ 0 & R \end{pmatrix}.$$

If M is a moment matrix with measure μ supported on a bounded set in the complex plane, the infinite matrix D defines a bounded subnormal operator. In this case $\mathcal{H} = \ell^2$ and $\mathcal{K} = \ell^2 \bigoplus (\ell^2)^\perp$. We use the same symbol D to denote the infinite matrix and the matrix as an operator in ℓ^2. It is well known that there is an isometric isomorphism between $L^2(\mu)$ and $\mathcal{K}$.

As usual $P(\mu)$ denotes the linear space of polynomials with complex coefficients associated to the measure μ. We denote by S_μ the operator of multiplication by z in $P^2(\mu)$, the closure in $L^2(\mu)$ of the space $P(\mu)$, and N_μ will be the operator of multiplication by z in $L^2(\mu)$. It is known that N_μ is the minimal normal extension of S_μ. In this case all the operators are bounded because the support is bounded. It is easy to prove that S_μ is unitarily equivalent to the infinite Hessenberg matrix D as an operator in ℓ^2, and N_μ is unitarily equivalent to the operator N, which is the minimal normal extension of D.

Lemma 1 ([1,19]). *Let $M = (c_{ij})_{ij=0}^{\infty}$ be an infinite HDP matrix and $\|D\| < +\infty$, then M is a moment matrix if and only if D is subnormal.*

It is not difficult to prove that this moment problem is always determined when the support of the measure is bounded as a consequence of the Stone–Weierstrass theorem in the bidimensional case.

3. Polynomials in D and D^*

The following theorem extends the results of [18] about orthogonal polynomials on harmonic algebraic curves.

Theorem 2. *Let μ be a probability measure with bounded support and* $\text{supp}(\mu) \subset \gamma \subset \mathbb{C}$*, where γ is an algebraic curve which can be expressed as a polynomial in z and $\bar{z}$, that is $\sum_{j,k=0}^{m} a_{j,k} z^j \bar{z}^k = 0$, with $a_{j,k} \in \mathbb{C}$. Then the infinite matrix D associated to $M = (c_{jk})_{jk=0}^{\infty}$, such that $c_{j,k} = \int_\gamma z^j \bar{z}^k d\mu(z)$, satisfies*

$$\sum_{j,k=0}^{m} a_{jk} (D^*)^k D^j = 0,$$

where we have replaced z by D and $\bar{z}$ by D^* in the equation of γ, and the two products $\bar{z}z$ and $z\bar{z}$ by D^*D.

Proof. Since μ is a probability measure, D is subnormal. We denote by N the minimal normal extension of D, $N = \text{mne}(D)$. As before, we denote by N_μ multiplication by z in L^2_μ. We know that $\sigma(N) = \sigma(N_\mu) = \text{supp}(\mu)$, and there is a spectral measure $E(\lambda)$ on the Borel subsets of $\text{supp}(\mu)$ such that $N = \int_{\sigma(N)} z dE(z)$. Therefore, by means of the spectral theorem for normal operators we have

$$\sum_{j,k=0}^{m} a_{jk} N^j (N^*)^k = \int_{\sigma(N)} \left(\sum_{j,k=0}^{m} a_{jk} z^j \bar{z}^k \right) dE(z).$$

By hypothesis, the points of $\sigma(N)$ satisfy the equation of the curve and $\sigma(N) = \text{supp}(\mu)$. Thus

$$\sum_{j,k=0}^{m} a_{jk} N^j (N^*)^k = 0.$$

From the 2×2 matrix representation of a subnormal operator we have

$$N = \begin{pmatrix} D & X \\ 0 & Y \end{pmatrix} \quad \text{and} \quad N^* = \begin{pmatrix} D^* & 0 \\ X^* & Y^* \end{pmatrix}.$$

Hence

$$N^j = \begin{pmatrix} D^j & \Box \\ 0 & Y^j \end{pmatrix}, \quad (N^*)^k = \begin{pmatrix} (D^*)^k & 0 \\ \Box & (Y^*)^k \end{pmatrix}.$$

This yields

$$N^*N = \begin{pmatrix} D^*D & D^*X \\ X^*D & X^*X + Y^*Y \end{pmatrix}, \quad NN^* = \begin{pmatrix} DD^* + XX^* & XY^* \\ YX^* & YY^* \end{pmatrix}.$$

We already know that $N^*N = NN^*$. At this point, we consider the product N^*N to obtain an equation in D and D^* in the $[1, 1]$ entry of the 2×2 matrix $\sum_{j,k=0}^{m} a_{jk} (N^*)^k N^j$. It is easy to check that

$$\sum_{j,k=0}^{m} a_{jk} (N^*)^k N^j = \begin{pmatrix} \sum_{j,k=0}^{m} a_{jk} (D^*)^k D^j & \Box \\ \Box & \Box \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and finally we obtain

$$\sum_{j,k=0}^{m} a_{jk} (D^*)^k D^j = 0.$$

From the proof it can be seen how to replace z by D and $\bar{z}$ by D^*. Consequently, $(\bar{z})^k z^j = z^j (\bar{z})^k$ takes the form $(D^*)^k D^j$, but not $D^j (D^*)^k$.

Corollary 3. *Let μ be a probability measure with bounded support. The following five assertions are satisfied for all $z, \bar{z} \in \text{supp}(\mu)$.*

(1) If $z - \bar{z} = 0$, then $D = D^*$.
(2) If $|z| = 1$, then $D^*D = I$.
(3) If $z - \beta = |z - \beta| e^{\theta i}$, then $\bar{\alpha}(D - I\beta) = \alpha(D^* - I\bar{\beta})$, with $\alpha = e^{\theta i}$.
(4) If $|z - \beta| = R$, then $D^*D = \bar{\beta}D + \beta D^* + (R^2 - |\beta|^2)I$.
(5) If $|z - c| + |z + c| = 2a$, with $a^2 = b^2 + c^2$, then

$$[D^2 + (D^*)^2](a^2 - b^2) + 4a^2b^2 I = 2D^*D(a^2 + b^2).$$

Note that the condition $DD^* = I$ in (2) is not true if μ satisfies Szego's condition.

Another less obvious application is related to measures whose support is a cross-like set formed by the intervals $[-1, 1]$ and $[-i, i]$. The support is given by $xy = 0$ with $|x + yi| \leqslant 1$. The expression $xy = 0$ is equivalent to $z^2 = \bar{z}^2$. Therefore $D^2 = (D^*)^2$. Using that D and D^* are upper and lower Hessenberg matrices it is easy to check that D^2 and $(D^*)^2$ are pentadiagonal.

4. Extension of Krein's theorem

In the next theorem, we prove a generalization of the Krein theorem for the hermitian complex case.

We need first to prove two results about pure atomic distributions. Let $Z = \{z_1, z_2, \ldots\}$ be a bounded set of complex points, with weights $\{w_1, w_2, \ldots\}$, where $\sum_{n=1}^{\infty} w_n < +\infty$. For such a distribution we have the moment matrix $M = (c_{ij})_{ij=0}^{\infty}$, where $c_{jk} = \sum_{n=1}^{\infty} z_n^j \bar{z}_n^k w_n$. Let D be the associated Hessenberg matrix. Obviously the support of this measure is $\operatorname{supp}(\mu) = \overline{Z}$.

Proposition 4. *If $\mathbb{C} \setminus \overline{Z}$ is a connected set and the interior of $\overline{Z}$ is empty, then the infinite Hessenberg matrix D corresponds to a normal operator in ℓ^2.*

Proof. The set $K = \overline{Z}$ is compact. As usual we denote by $C(K)$ the space of all continuous functions with support K. The set K satisfies the hypothesis of Mergelyan's theorem (see [10, p. 97]), and consequently given $f \in C(K)$ and $\epsilon > 0$, $\exists Q(z)$ such that $|f(z) - Q(z)| < \epsilon$. This implies that $\int_{\operatorname{supp}(\mu)} |f(z) - Q(z)|^2 d\mu(z) < \epsilon^2 c_{00}$. Clearly $C(K) = P^2(\mu)$. Since $C(K)$ is dense in $L^2_{\mu}(K)$ (see, for example, [13, p. 61]), we conclude that $P^2(\mu) = L^2_{\mu}(K)$. Therefore we are in a *complete case*. It follows that $S_{\mu} = N_{\mu}$, and also $D = N$. Consequently D is a normal operator.

Proposition 5. *Let Z be as in Proposition 4 and $Z' \cap Z = \emptyset$, where Z' is the set of accumulation points of Z. Then*

$$D = U^*(\delta_{ij} z_i)_{i,j=1}^{\infty} U \quad \text{and} \quad U^*U = UU^* = I,$$

where $U = V(T^*)^{-1}$ and T is the Cholesky factor in the decomposition $M = TT^*$, and V is the Vandermonde matrix of the atoms

$$V = \begin{pmatrix} \sqrt{w_1} & \sqrt{w_1} z_1 & \sqrt{w_1} z_1^2 & \cdots \\ \sqrt{w_2} & \sqrt{w_2} z_2 & \sqrt{w_2} z_2^2 & \cdots \\ \sqrt{w_3} & \sqrt{w_3} z_3 & \sqrt{w_3} z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Let $L = (\delta_{ij} z_i)_{i,j=1}^{\infty}$. It is clear that $M' = V^* L V$. Using $D = T^{-1} M' (T^*)^{-1}$, it follows that $D = T^{-1} V^* L V (T^*)^{-1}$. The elements of the ith column of the infinite matrix $(T^*)^{-1}$ are the coefficients of $p_{i-1}(z)$ with respect to the basis $\{z^k\}_{k=0}^{\infty}$. Therefore $U = V(T^*)^{-1} = (\sqrt{w_i}\, p_{j-1}(z_i))_{i,j=1}^{\infty}$. Now we calculate U^*U,

$$(U^*U)_{i,j} = \sum_{k=1}^{\infty} \overline{p_i(z_k)} p_j(z_k) w_k = \delta_{ij},$$

due to the orthogonality of the NOPS on the set $Z = \{z_1, z_2, \ldots\}$. On the other hand, the product UU^* is

$$UU^* = (\sqrt{w_i}p_{k-1}(z_i))_{i,k=1}^{\infty} \left(\sqrt{w_j}\overline{p_{k-1}(z_j)}\right)_{k,j=1}^{\infty} = \left(\sqrt{w_i}\sqrt{w_j}\sum_{k=0}^{\infty} p_k(z_i)\overline{p_k(z_j)}\right)_{i,j=1}^{\infty}.$$

To prove the statement we need also that $(UU^*)_{ij} = \delta_{ij}$. For that we introduce the bounded functionals $L_i : P^2(\mu) \to P^2(\mu)$ defined by $L_i(f) = f(z_i)$. Recall that the inner product in $P(\mu)$ is $\langle Q(z), R(z)\rangle = \sum_{k=1}^{\infty} Q(z_k)\overline{R(z_k)}w_k$. It is extended to $P^2(\mu)$ as usual. Obviously $\|L_i\| \leqslant 1/\sqrt{w_i}$. It is clear that the n-kernel $K_n(z, z_i) = \sum_{k=0}^{n} \overline{p_k(z)}p_k(z_i)$, with $n > i$, has the reproducing property, that is $\langle Q(z), K_n(z, z_i)\rangle = Q(z_i)$. The function $K(z, z_i) = \lim_n K_n(z, z_i)$ defined on $Z = \{z_1, z_2, \ldots\}$ has the same property.

Now we consider the characteristic function $\chi_{z_i}(z)$, defined by $\chi_{z_i}(z) = 1$ if $z = z_i$ and 0 otherwise. Then $\chi_{z_i}(z)/w_i$ is a continuous function because we have $Z' \cap Z = \emptyset$ and it is only defined in isolated points. Hence $\chi_{z_i}(z)/w_i$ is defined for every $f \in C(K)$, agrees with $K(z, z_i)$, and for all $f \in P^2(\mu) = L^2_{\mu}(K)$ we have

$$\langle f(z), K(z, z_i)\rangle = f(z_i) = \left\langle f(z), \frac{\chi_{z_i}(z)}{w_i}\right\rangle = \sum_{k=1}^{\infty} f(z_k)\frac{\overline{\chi_{z_i}(z)}}{w_i}w_k.$$

Then $\chi_{z_i}(z)/w_i = K(z, z_i)$, a.e. in L^2_{μ}. In particular $\chi_{z_i}(z)/w_i = K(z, z_i)$ at the points with positive measure, i.e., $K(z_j, z_i) = \chi_{z_i}(z_j)/w_i = \delta_{ij}/w_i$ on Z and therefore $UU^* = I$.

Theorem 6 (Extension of Krein's theorem to the complex case). *Let M be a moment matrix with bounded support and let D be the associated Hessenberg matrix. Then the measure associated to M has $\sigma_1, \sigma_2, \ldots, \sigma_m \in \mathbb{C}$, as the only accumulation points of its support, if and only if $Q(D)$ is a compact operator, where $Q(z) = \prod_{k=1}^{m}(z - \sigma_k)$.*

Proof.

Necessary condition. As the support is a bounded set and it has a finite number of limit points, necessarily the measure is atomic. Assume that $L = \operatorname{diag}(z_1, z_2, \ldots)$ is the matrix of the atoms reordered such that $d(z_i, \cup_k^m \sigma_k) \geqslant d(z_{i+1}, \cup_k^m \sigma_k)$. We have shown before that in this case D is an infinite, bounded, and normal Hessenberg matrix, satisfying $D = U^*LU$, with $U = V(T^*)^{-1}$, where $V = (\sqrt{w_j}z_j^{k-1})_{j,k=1}^{\infty}$. We have proved that U^* and U are unitary operators, and we have $D^n = U^*L^nU$, so $Q(D) = U^*Q(L)U$. L is a diagonal matrix, hence $Q(L) = (Q(z_i)\delta_{ij})_{i,j=1}^{\infty}$. The zeros of $Q(z)$ are exactly the accumulation points of the diagonal elements of L. Therefore $\lim_n Q(z_n) = 0$ and the diagonal matrix $Q(L)$ defines a compact operator. As U^* and U are bounded operators, we have finally that $Q(D)$ is a compact operator.

Sufficient condition. By Lemma 1, if M is a moment matrix then D defines a subnormal operator, and it is bounded by hypothesis. If $N = \operatorname{mne}(D)$, it is well known that $Q(N)$ is the normal extension of $Q(D)$, see [6, p. 204]. Hence $Q(D)$ is subnormal and bounded. Therefore, $Q(D)$ is hyponormal. By hypothesis $Q(D)$ is a compact operator. We find on [12, p. 206] that an operator compact and hyponormal is necessarily normal, and consequently $Q(D)$ is a compact and normal operator. The eigenvectors of $Q(D)$ are a basis of ℓ^2, $Q(D)$ is a diagonalizable operator, and the sequence of eigenvalues of $Q(D)$ converges to zero if it is an infinite set. This is the case, because the matrix $Q(D)$ has the same rank as the matrix D, which is not finite. Were this not so, the moment matrix M would have finite rank, which is not possible. We have that

$$\sigma(Q(D)) = \{\mu_1, \mu_2, \ldots, \mu_n, \ldots\}, \quad \text{with} \quad \mu_n \to 0.$$

Thus $\sigma(Q(D)) = Q(\sigma(D))$. Consequently,

(i) $\sigma(D)$ is a discrete set, because it is the inverse image via $Q(z)$ of a denumerable set.

(ii) The limit points of $\sigma(D)$ are the solutions of $Q(z) = 0$. Suppose that $S = \{z|Q(z) = \mu_n, n \in \mathbb{N}\} = Q^{-1}(\{\mu_k\}_{k\in\mathbb{N}})$, is the set of all solutions of $Q(z) = \mu_n$, for all $n \in \mathbb{N}$. Then $S = Q^{-1}(Q(\sigma(D))) \supset \sigma(D)$. Hence the limit points of $\sigma(D)$ are necessarily the zeros of the polynomial $Q(z)$.

(iii) D is a normal matrix. We know that D is hyponormal because it is subnormal and by ii), $\sigma(D)$ has a finite number of limit points. By Corollary 2, [16, p. 1455], if the spectrum of a hyponormal operator has a finite number of limit points then the operator is normal. Hence D is normal, and $\sigma(D) = \text{supp}(\mu)$.

Consequently $\text{supp}(\mu)$ is a discrete set in $\mathbb{C}$ and all the limit points of $\text{supp}(\mu)$ are zeros of $Q(z)$.

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