Soil porous system as heterogeneous complex network

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A B S T R A C T

In this paper we present a complex network model based on a heterogeneous preferential attachment scheme as a new way to quantify the porous structure of soils and to relate them with soil texture. We consider the pores as nodes, the properties of which, such as position and size, are described by fixed states in a metric space. An affinity function is introduced to bias the attachment probabilities of links according to these properties. We perform an analytical study of the degree distributions in the model and develop a numerical analysis of the degree distributions in the model for a combination of parameters corresponding to eleven empirical soil samples with different physical properties and five different textures.

1. Introduction

After soil texture, the geometry of pore space as well as pore size is probably one of the most important factors to understanding the transport of water, gas and solute in soils (Ball et al., 2007). However, a quantitative and explicit characterization, by means of a physical interpretation, is difficult because of the complexity of the pore space (VandenBygaart et al., 1999; Leij et al., 2002). This is the main reason why most theoretical approaches to soil porosity are idealizations to simplify the complex structure of the wiring and spatial location of pores (Bird et al., 2006). The complex interplay between the size and continuity of pores is considered to be the key in understanding the transport properties (Klute and Dirksen, 1986; VandenBygaart et al., 1999). Despite the empirical observation of the pore size distribution (PSD) and pore geometry distribution (PGD) (Lin et al., 1999; Vogel and Roth, 2001), the understanding of the emergence of these PSD and PGD is not entirely clear (Horgan and Ball, 1994). Considering this problem, in this work we propose a network model in order to analyze porous structures previously mentioned.

A network model is a flexible way of representing objects (nodes) and their relationships (links or connections) from a purely topological point of view showing it as a graph (Biggs et al., 1986). During the last decade, a wide range of real complex systems was studied from this perspective by means of Network Theory (Strogatz, 2001; Albert and Barabási, 2002). The results of these studies showed non-intuitive and non-trivial topologies in the studied networks, irrespective of their origin. This means that the patterns of connection between their nodes are neither purely regular nor purely random, reason why they are called complex networks.

Many real complex networks are scale-free and display properties of small-world networks (Watts and Strogatz, 1998). A network is named scale-free if its degree distribution, i.e., the probability that a node selected uniformly at random has a certain number of links (degree), follows a power law implying that the distribution of connectivities of the network has no characteristic scale (Dorogovtsev and Mendes, 2002; Newman, 2003) denoting a high inhomogeneity in the connectivity of the system elements. In scale-free topologies some nodes, called "hubs", present a degree that is orders of magnitude larger than the average. These hubs are absent in random graphs where most nodes have a connectivity close to the average (Erdős and Rényi, 1959). On the other hand, a complex network is called small-world when it shows an average short path length between nodes and a high mean clustering, denoting respectively an efficient pattern of connectivity and high levels of transitivity in the network. The presence of these properties has created a large follow-up debate among physicists because of their ubiquity and seems to be a characteristic feature of critical-point behavior and a fingerprint of self-organized systems (Bak et al., 1987).

In our proposed model we interpret porous soils as heterogeneous networks where pores are represented by nodes with distinct properties (such as area and spatial location) and the links represent fluid flows between them. The networks of pores are generated by a model known as Heterogeneous Preferential Attachment (HPA) (Santiago and Benito, 2007a,b, 2008, 2009), a generalization of the Barabási–Albert (BA) model (Barabási and Albert, 1999) to heterogeneous networks. The BA model is based on the mechanisms of growth and preferential attachment (Price, 1965, 1976) and provides a minimal account of the process leading to the emergence of scale-free networks (Barabási et al., 2002).
Within these, dynamical network models (Dorogovtsev and Mendes, 2002) are stochastic discrete-time dynamical systems that evolve networks by the iterated addition and subtraction of nodes and links. Until recently, soil pore network has been a detailed model of a porous medium, generally incorporating pore-scale descriptions of the medium and the physics of pore-scale events. It has been used to describe a wide range of properties from capillary pressure characteristics to interfacial area and mass transfer coefficients. The pores and throats are assigned some idealized geometry and rules are developed to determine the multiphase fluid configurations and transport in these elements (Berkowitz and Ewing, 1998) based on a random or lattice structure. Recently, taking into account the topology of real complex systems, two works have introduced a pore network concept from a topological point of view (Santiago et al., 2008; Mooney and Koroljuk, 2009) opening a new line in soil science context.

Under this new approach to the complex structure of the porous system we present this work as follows. In the next section we motivate and describe the formulation of the porous soil model based on the HPA model. In the third section we present the analytical solution for the degree distribution in the pore networks generated by the HPA model. In the fourth section we present the numerical results of the model corresponding to simulations of eleven real samples of soil with different structure and physical properties. In the last section we present our conclusions derived from this work.

2. Model formulation

The structure of a porous soil is modeled as a heterogeneous complex network where nodes \( v_i \) correspond to pores and links \( e_{ij} \) correspond to fluid flows between them. Fig. 1 briefly illustrates this idea. The links will be considered undirected \( e_{ij} = e_{ji} \) and thus the connectivity degree \( k_i \) of a node \( v_i \) will be a measure of the number of pores directly connected with the associated pore. The properties of a pore are described by the node state \( (r_i, s_i) \) which account for the position \( r_i \) of the pore center in the soil and the pore size \( s_i \).

The evolution of the porous soil structure is modeled as a stochastic growth process described by a dynamic network model. Assuming that at a given time a new pore is created, the likelihood that it will connect with any of the already existing pores will be proportional to the size of the older pores and inversely proportional to the distance between them. Likewise, the higher the number of connections accruing to an existing pore, the higher the likelihood that a new flow of fluids will intercept an existing connection and connect the new pore with the older one. Thus the attachment visibility of an existing network node \( \Pi(v_i) \) when a new node \( v_{n+1} \) is added is a monotonic function of \( k_i, s_i \) and \( d^{-1}(r_i, r_{n+1}) \), where \( d \) is the Euclidean metric.

The previous considerations prompt us to model the dynamics of the porous structure of soils as a particular case of HPA model defined by a state space \( R \), a probability distribution \( p(x) \) of the node states \( x \in R \), and an affinity function, \( \sigma(x, y) \), of the interactions between the existing and added nodes that depends on their states. This formalism prescribes the evolution of a network according to the following rules:

(i) The nodes \( v_i \) are characterized by their state \( x_i \in R \) deemed constant in the timescale of evolution of the network.
(ii) The growth process starts with a seed composed by \( N_0 \) nodes (with arbitrary states \( x_i \in R \) and \( L_o \) links).
(iii) A new node \( v_{n+1} \) (with a fixed number \( m \) of links attached) is added to the network at each iteration. The newly added node is randomly assigned a state \( x_{n+1} \) following the distribution \( p(x) \).
(iv) The \( m \) links attached to \( v_{n+1} \) are randomly connected to the network nodes following a distribution \( (\Pi(v)) \) given by an extended attachment rule,

\[
\Pi(v_i) = \frac{\pi(v_i)}{\sum \pi(v_j)} , \quad \pi(v_i) = k_i \cdot \sigma(x_i, x_i) .
\] (1)

The attachment kernel or visibility \( \pi \) of a node \( v_i \) in the rule is given by the product of its degree \( k_i \) and its affinity \( \sigma \) with the newly added node \( v_{n+1} \), which is itself a function of the states \( x_i \) and \( x_{n+1} \). It thus can be seen that for each interaction \( \sigma \) biases the degree \( k_i \) of the candidate node. Steps (iii) and (iv) are iterated until a certain number of nodes have been added to the network. This number is determinate by the number of soil pores estimated in the 2D image. To sum up, the choice of the triple \((R, p, \sigma)\) determines the form of heterogeneity in the attachment mechanism.

Consistently with the previous formalism, the porous soil model is defined by a state space \( R = M \times S \), where \( M \) is a Euclidean box with dimension 2 or 3 that represents the medium geometry and \( S \) is an interval of the real line that represents the spectrum of possible pore sizes; a state distribution \( p(x) = p(r, s) \) that represents the probability for a new pore having a certain position \( r \) and size \( s \); and an affinity function

\[
\sigma(x, y) = \frac{s_y^2}{(\delta + |r_x - r_y|)^\beta} ,
\] (2)

where \( \delta \) is a small nonnegative offset that ensures the continuity of \( \sigma \) on \( R^2 \), while \( \alpha \) and \( \beta \) are free parameters that measure the relative importance of the size and distance of the pores in the affinity of the attachment mechanism. Defining the porosity of the medium \( \phi = 3V_0/2 = \sum s_i \int_{\Omega} d\Omega \) as the fraction of void space in the material, then the desired number of nodes \( N \) that ends the growth process is the least number that yields the porosity of the simulated soil \( \phi > \phi_0 \) for a certain threshold \( \phi_0 \), which is the porosity of the sample medium.

Fig. 1. Cartoon of a soil pore network. Red lines correspond to links between pores.
3. Analysis

In this section we present an analytical study of the stationary degree distribution $P(k)$ of the proposed soil model. The solution is obtained by rate equations (Dorogovtsev et al., 2000; Kruglovsky and Redner, 2001) which establish a balance in the connections degree densities over a partition of the network. The exposition will be brief and the reader is directed to Santiago and Benito (2008) for a more detailed discussion of the solution, and to Santiago and Benito (2008) for the general class of preferential attachment (PA) models. Let us first define a sequence of functions $\{f(k, x, N)\}_{N=0}^\infty$ which measure the probability density of a randomly chosen node having degree $k$ and state $x$ in a network at the iteration $t = N$. The degree densities are local metrics, thus they uniformly converge when $N \to \infty$ to a stationary density distribution $P(k)$. Finally, the stationary degree distribution $P(k)$ measures the probability of a randomly chosen node having degree $k$ in the thermodynamic limit.

We will denote by $V(x) = \{v_i, x_i = x\}$ the subset of nodes in the network with state $x$. Assuming that the assignation of states $x$ is uncorrelated with the topology of the growing network, and that there are no linking events between existing nodes, the sequence $\{f\}$ can be modeled on each $V(x)$. For each $x$, the form of the equation will be $L_1 - L_2 = R_1 - R_2$, where:

- $L_1$: density of nodes with degree $k$ at $t = N + 1$;
- $L_2$: density of nodes with degree $k$ at $t = N$;
- $R_1$: increase in density due to nodes with degree $k - 1$ that have gained a link at $t = N$;
- $R_2$: decrease in density due to nodes with degree $k + 1$ that have lost a link at $t = N$.

The resulting density rate equation for $k > m$ is

$$\langle N + 1 \rangle f(k, x, N + 1) - N f(k, x, N) =$$

$$m \int \frac{\sigma(x, y)}{\psi(y, N)} \left(1 - 1 \right) f(k - 1, x, N) - k f(k, x, N).$$

(3)

where $\psi(y, N)$ is defined as the partition factor

$$\psi(y, N) = x \int \frac{\sigma(x, y)}{\psi(y, N)} \left(\frac{N}{N} f(k - 1, x, N) - k f(k, x, N)\right) dx,$$

(4)

and the brackets mean averaging over the random variable $y$.

$$\langle g \rangle_y = \int g(y) dy.$$

(5)

For $k = m$ the resulting density rate equation is

$$\langle N + 1 \rangle f(m, x, N + 1) - N f(m, x, N) =$$

$$m \int \frac{\sigma(x, y)}{\psi(y, N)} \left(\frac{N}{N} f(k - 1, x, N) - m f(m, x, N)\right).$$

(6)

There are no nodes with degree $k < m$, since when $N \to \infty$ all the links attached to newly added nodes find receptive nodes in the network, therefore the previous equations define all the possible cases in each iteration.

In the thermodynamic limit $N \to \infty$, $f(k, x, N + 1) = f(k, x, N) = f(k, x)$ and the rate equations become

$$f(k, x) =$$

$$m \int \frac{\sigma(x, y)}{\psi(y)} \left(\frac{N}{N} f(k - 1, x) - k f(k, x)\right)$$

(7)

for $k > m$.

$$p(x) = m \int \frac{\sigma(x, y)}{\psi(y)} \left(\frac{N}{N} f(k, x)\right)$$

(8)

for $k = m$.

where $\psi(y)$ is the stationary partition factor, defined as

$$\psi(y) = \lim_{N \to \infty} \psi(y, N) =$$

$$\int \frac{k f(k, x)}{\psi(y)} dx.$$

(9)

Through a mean-field approach it is possible to decouple the previous equations under the approximation criteria

$$\langle \psi(y) - \psi \rangle_y = 0, \quad \left(\frac{\psi(y) - \psi(y)}{\psi(y)}\right)_y = 0,$$

(10)

where $\psi = \langle \psi(y) \rangle_y$ is the mean-field partition factor of the network, $\psi(x) = \langle \psi(x,y) \rangle_y$, $\psi$ is the mean-field fitness of network nodes with state $x$, and $\psi_1$ and $\psi_2$ are accuracy bounds. This renders the decoupled system

$$f(k, x) =$$

$$\psi(x) [k f(k, x) / k f(k, x) - k f(k, x) / k f(k, x)] / 2 \quad \text{for} \quad k > m,$$

$$p(x) = \psi(x) \int \frac{\psi(y) - \psi(y)}{\psi(y)} \left(\frac{N}{N} f(k, x)\right)$$

(11)

and integrating the density in Eq. (11) over the state space $R$ the stationary degree distribution is

$$P(k) =$$

$$\int \frac{N}{N} f(k, x) dR.$$

(12)

The solution given by Eq. (12) is valid for any variant of the soil model, irrespective of the geometry of the medium $M$, or the pore sizes $S$. The solution is restricted by the extent to which the approximation criteria in Eq. (9) hold. The mean-field approximation can be considered accurate in the measure that the averages of $\langle \sigma(x, y) \rangle$ along its two arguments do not exhibit large variations over $R$.

Furthermore, given that any pore in the soil has a non-zero size, it follows that by definition the mean-field fitness $o(x)$ is strictly

Table 1 Classification, horizon, depth and physical composition of soil samples.

<table>
<thead>
<tr>
<th>Soil Classification</th>
<th>Horizon Depth</th>
<th>Org. mat.</th>
<th>Sand</th>
<th>Silt</th>
<th>Clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latier</td>
<td>Aequalta Argisol</td>
<td>54-40</td>
<td>0.4</td>
<td>3.1</td>
<td>65.2</td>
</tr>
<tr>
<td>Kampang</td>
<td>Tropic Haplotrophic</td>
<td>44-35</td>
<td>1.2</td>
<td>20.0</td>
<td>22.0</td>
</tr>
<tr>
<td>Muncheng</td>
<td>Tropic Haplotrophic</td>
<td>70-78</td>
<td>0.7</td>
<td>22.0</td>
<td>60.0</td>
</tr>
<tr>
<td>Site</td>
<td>Incipient</td>
<td>20-25</td>
<td>0.3</td>
<td>84.8</td>
<td>6.9</td>
</tr>
<tr>
<td>Bae</td>
<td>Spodosol</td>
<td>75-60</td>
<td>0.1</td>
<td>50.0</td>
<td>15.0</td>
</tr>
<tr>
<td>Ech</td>
<td>Entisol</td>
<td>1-10</td>
<td>0.8</td>
<td>27.0</td>
<td>30.0</td>
</tr>
<tr>
<td>Abak</td>
<td>Orthic Tropudult</td>
<td>32-40</td>
<td>0.5</td>
<td>48.0</td>
<td>21.0</td>
</tr>
<tr>
<td>Denny</td>
<td>Mollic Albaqualf</td>
<td>47-52</td>
<td>0.2</td>
<td>24.7</td>
<td>67.1</td>
</tr>
<tr>
<td>Durian</td>
<td>Pteric Plinthudult</td>
<td>1-8</td>
<td>1.7</td>
<td>12.0</td>
<td>51.0</td>
</tr>
<tr>
<td>Mike</td>
<td>Entisol</td>
<td>20-32</td>
<td>1.3</td>
<td>2.7</td>
<td>63.5</td>
</tr>
<tr>
<td>Ady</td>
<td>Entisol</td>
<td>6-14</td>
<td>1.1</td>
<td>1.4</td>
<td>65.2</td>
</tr>
</tbody>
</table>
positive over \( R \) and the stationary degree density in Eq. (11) may be expressed for \( k \geq m \) as

\[
f(k, x) = \frac{2\rho / \omega \cdot B(k, 1 + 2/\omega)}{m + 2/\omega B(m, 1 + 2/\omega)}.
\]

(13)

where Legendre's Beta function

\[
B(y, z) = \int_0^1 t^{y-1} (1-t)^{z-1} dt
\]

for \( y, z > 0 \) satisfies the functional relation

\[
f(a)/f(a + b) = B(a, b)/f(b)
\]

for Euler's Gamma function \( f(x) \). Likewise, integrating the density in Eq. (13) over \( R \) we obtain an expression for the stationary degree distribution \( P(k) \) for \( k \geq m \) in terms of Legendre's Beta function which is equivalent to Eq. (12).

Notice that the stationary densities for a given pore state, as given by Eq. (13), follow a Beta function with arguments \( (k, 1 + 2/\omega) \). This implies that the degree densities exhibit a multiscaling according to power laws \( k^{-\gamma(\omega)} \) along the continuous spectrum of normalized fitness \( \omega \), with scaling exponents \( \gamma(\omega) = 1 + 2/\omega \) spanning themselves a continuum. The multiscaling phenomenon is a general property of heterogeneous PA networks (Santiago and Benito, 2007a, 2008) that is exhibited by any variant of the proposed soil model, irrespective of the particular details. As \( \omega \) increases (resp. decreases), the exponent \( \gamma(\omega) \) decreases (resp. increases). Nodes with states more fit than the average (\( \omega > 1 \)) adopt densities with \( \gamma < 3 \), which exhibit a slower asymptotical decay and tend to produce more hubs. The increase of either \( \alpha \) or \( \beta \) results in an increase in the variability of \( \omega \) over \( R \), and in the case of \( \beta \) such an increase is more acute as the inhomogeneity of \( \rho_M \) increases. The higher variability of \( \omega \) translates into a larger spread of the distribution of exponent \( \gamma \) of the density components. This evidences a signature of heterogeneous PA in the structure of porous soils, by which the density components \( f(k, x) \) of pores will exhibit different scaling exponents according to their intrinsic properties (Santiago et al., 2008).

Fig. 2. Binary images of soil samples used in the study. Image size: 1500 x 1000 pixels. Credit: Richard Heck.
Eq. (12) shows that the degree distribution $P(k)$ is obtained by the integration of power laws with varying exponents $\gamma(x)$. As we have seen, little fluctuations of $\omega$ over $R$ will yield density components with similar scaling exponents and thus distributions similar to the homogeneous PA case. On the other hand, large fluctuations of $\omega$ over $R$ (as in the case of large $\alpha$ or large $\beta$ parameters) will yield a wider spectrum of $\gamma$ and thus a larger deviation of $P(k)$ from the BA model. The asymptotic behavior of $P(k)$ will be dominated by the slowest decaying components, associated to properties with highest $\omega$. Furthermore, given that $\omega$ is distributed by definition around 1, the

![Fig. 3. Pore size distribution, $P(size)$, for soil samples. The value of $\phi$ corresponds to the scaling exponent of the fit to a power law denoted by the straight lines. Pore size is expressed as number of pixels.](image-url)
highest value of the spectrum will verify \( \beta_{\text{max}} \approx 1 \) and therefore the degree distributions \( P(k) \) of all the soil variants will exhibit exponents satisfying \( 1 < \gamma < \beta \leq 3 \). Finally, it is worth noting that as the sample size becomes larger, the finite-size effects will become more important, yielding a growing discrepancy with the analytical results, such as a decrease in the scaling exponent \( \gamma \) of the degree distribution and the presence of an exponential cutoff regime. This problem could nevertheless be circumvented by modeling a larger soil sample. In our case, size sample and resolution of the images were enough for the purpose of our study.

4. Model application

In this section we present results concerning the numerical simulation of the porous soil model using as input data from eleven intact soil samples with different physical properties. We simulate pore networks with the same porosity, size distribution and spatial location of the empirical 2D soil samples. The soil samples were primarily selected to include contrasting soil structures and textures (Table 1). Each of the samples was prepared for image analysis following the procedure described by Protz and Vandenberg (1998). These sample soils were kindly provided by University of Guelph, Department of Land Resource Science. For further information about image acquisition and soil properties we refer to the Canadian Soil Thin Section Collection website at URL: http://gis.lrs.uoguelph.ca/cstsc/ focused on sections prepared for the 1978 KSS in Edmonton. The data were obtained by imaging thin sections with a Kodak 460 RGB camera using transmitted and circularly polarized illumination. The data were cropped from 3000 × 2000 pixels to 3000 × 2000 pixels. Then EASY/SPACE software classified the data and the void bitmap separated (individual pixel size was 18.6 × 18.6 µm).

The samples showed a wide range for the percentages of sand (1.4–94.6%), silt (4.9–67.1%), and clay (0.4–72.0%). Organic matter content was relatively low (0.1–1.7%) because mainly subsurface horizons were sampled. Binary images of the soil samples are shown in Fig. 2.

Due to the fact that our model considers the pore size (surface = \( s \)) we analyze the distribution of pore sizes in the soil samples. The PSD for the samples, given by \( F(\text{size}) \), is presented in Fig. 3.

As can be seen, all the soils display a distribution that follows a power law denoting a scale-free character in the distribution of pore sizes, \( F(\text{size}) \approx \text{const} \cdot \text{size}^{-\phi} \), except in the extremes values. We could divide size axis in two or three ranges to better fit in each one the scaling exponent, nevertheless we prefer to follow previous works (Mooney and Mooney, 2009). Therefore, the parameter \( \phi \) is presented in Fig. 3.

Table 2

<table>
<thead>
<tr>
<th>Soil</th>
<th>Number of pores</th>
<th>Mean size</th>
<th>Max size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inter</td>
<td>5435</td>
<td>8.78</td>
<td>100</td>
</tr>
<tr>
<td>Kamppong</td>
<td>9834</td>
<td>12.12</td>
<td>110</td>
</tr>
<tr>
<td>Munchong</td>
<td>7102</td>
<td>14.80</td>
<td>155</td>
</tr>
<tr>
<td>Site</td>
<td>6275</td>
<td>16.09</td>
<td>112</td>
</tr>
<tr>
<td>Bogo</td>
<td>6101</td>
<td>10.85</td>
<td>116</td>
</tr>
<tr>
<td>Evl</td>
<td>2272</td>
<td>7.46</td>
<td>62</td>
</tr>
<tr>
<td>Atok</td>
<td>4541</td>
<td>7.95</td>
<td>73</td>
</tr>
<tr>
<td>Denzy</td>
<td>4120</td>
<td>8.65</td>
<td>96</td>
</tr>
<tr>
<td>Durian</td>
<td>2847</td>
<td>12.92</td>
<td>110</td>
</tr>
<tr>
<td>Mts</td>
<td>3347</td>
<td>10.28</td>
<td>76</td>
</tr>
<tr>
<td>Ads</td>
<td>4115</td>
<td>11.68</td>
<td>99</td>
</tr>
</tbody>
</table>

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Table 3

<table>
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<tr>
<th>Soil</th>
<th>Texture</th>
<th>( \alpha )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inter</td>
<td>Silt loam</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Kamppong</td>
<td>Clay</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Munchong</td>
<td>Clay</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Site</td>
<td>Sand</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Bogo</td>
<td>Sandy clay loam</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Bogo</td>
<td>Sandy clay loam</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Atok</td>
<td>Sandy clay loam</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>Denzy</td>
<td>Silt clay loam</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Durian</td>
<td>Silt clay loam</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Mts</td>
<td>Silt clay loam</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Ads</td>
<td>Silt clay loam</td>
<td>0.5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

In the model we analyze the distribution of pore sizes in the soil samples. The PSD for the samples, given by \( F(\text{size}) \), is presented in Fig. 3.

As can be seen, all the soils display a distribution that follows a power law denoting a scale-free character in the distribution of pore sizes, \( F(\text{size}) \approx \text{const} \cdot \text{size}^{-\phi} \), except in the extremes values. We could divide size axis in two or three ranges to better fit in each one the scaling exponent, nevertheless we prefer to follow previous works (Mooney and Mooney, 2009). Therefore, the parameter \( \phi \) is presented in Fig. 3.
Fig. 4. Cartesian networks of pores generated by the soil model considering real data from soil samples. Node position reflects the spatial location of a pore center of mass in the simulated soil sample. Node color corresponds to the connectivity degree \(k\): warm colors mean high degrees, cold colors mean low degrees.

Fig. 5 depicts the degree distributions \(N \times P(k)\) obtained through numerical simulation of the generated networks with a size of \(N = 10,000\) pores. The scaling exponent \(\gamma\) of the degree distribution varies from 2.07 (munching sample) to 2.65 (mss sample) not showing a high numerical difference, however we should have in consideration that it is the exponent of a power law. In all the cases where \(a = 0.5\) and \(b = 2.0\) the degree distribution presents a \(k\) maximum range around 1000 (see Fig. 5). In the cases where \(b\) was reduce to 1 this maximum did not exist except for evh1 that presents the lower number of pores and shorter range of pore size than the rest with these parameters (see Table 1). Finally, those cases where \(a = 1\), kampong and munching, show a strong nonlinear trend in the degree distribution at the less connected pores. Even though these results do not reflect the variety of images shown in Fig. 2, they offer us information about the topology of these pore networks with more common features than expected.

The degree distributions do not fit well a power law along the studied interval, however they evidence a progressively better agreement for higher degrees. This can be explained by the fact that the power law components in the density spectrum with lower scaling exponents dominate the distribution decay, so that the asymptotic behavior of \(N \times P(k)\) over arbitrarily high degrees would fit the power law as analytically predicted. The behavior of the distribution for lower degrees can be explained by the inhomogeneity of the distribution of pore sizes, which yields a non-negligible presence of very large pores in the network. We have to remark that the connectivity pattern between pores was not observed empirically, the scale-free distributions obtained (both numerically and analytically) are the result of the implementation of a reasonable attachment rule \(\Pi(w_i)\) and the adoption of real parameters of soil samples.

Finally, in this work the general scheme established a biasing dependence on node size and distance to be parameterized by power laws, and the dependence on degree being strictly linear. Future work should consider extending the formalism so as to include nonlinear functions of the degree.

5. Conclusions

In summary, we have presented a complex network model based on a heterogeneous preferential attachment scheme in order to quantify the structure of porous soils. The proposed network model allows the specification of different medium geometries and pore dynamics through the choice of different underlying spaces, distributions of
pore properties and affinity functions. We have obtained analytical solutions for the degree densities and degree distribution of the pore networks generated by the model in the thermodynamic limit. We have shown that these networks exhibit a multiscaling of their degree densities according to power laws with exponents spanning a continuum, and that such phenomenon leaves a signature of heterogeneity in the topology of pore networks that can be empirically tested in real soil samples.

We have also shown the relationship between the variability in the scaling exponents and the parameters regulating the affinity function, as
well as the inhomogeneity of the distributions of pore properties, and the consequences of this on the asymptotic behavior of the degree distribution. In particular, it is worth emphasizing that the degree distributions of all the variants of the model exhibit power law behaviors with exponents within the limits empirically observed in real networks. We have performed a numerical analysis of the model for a combination of parameters corresponding to empirical samples with different properties. We have shown that the simulation results exhibit a good agreement with the analytical predictions, concerning the scaling behavior of the degree metrics as well as the ranges of values of the scaling exponents obtained.

Finally, it is worth remarking that the application of heterogeneous complex networks to the modeling of porous soils offers the possibility of linking microscopic traits (such as biasing parameters related to the soil texture) with non-trivial macroscopic properties (as illustrated by scale-free degree distributions and small-world properties), as well as furthering our understanding of the dynamics of soil development.

Acknowledgements

This work has been supported by the Spanish MEC under Project IV-MATH No. CSD2006-00032 and Project MTM2009-14621, GESAN and Comunidad de Madrid under Project TAGRAL1A-CM-P-AGR-Q00187-0505. We are grateful to Prof. Richard Heck, from the Dept. of Soils and Landscape Processes of Guelph University, for the images provided for this work.

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