Inverses of regular Hessenberg matrices

J. Abderramán Marrero[1] and Venancio Tomeo[2]

[1] *Department of Mathematics Applied to Information Technologies,
Telecommunication Engineering School, U.P.M. Tech. University of Madrid*

[2] *Department of Algebra, School of Statistics, U.C.M. University Complutense*

emails: `jc.abderraman@upm.es`, `tomeo@estad.ucm.es`

Abstract

A new proof of the general representation for the entries of the inverse of any unreduced Hessenberg matrix of finite order is found. Also this formulation is extended to the inverses of reduced Hessenberg matrices. Those entries are given with proper Hessenbergians from the original matrix. It justifies both the use of linear recurrences for such computations and some elementary properties of the inverse matrix. As an application of current interest in the theory of orthogonal polynomials on the complex plane, the resolvent matrix associated to a finite Hessenberg matrix in standard form is calculated. The results are illustrated with two examples on the unit disk.

*Key words: General orthogonal polynomials, Hessenberg matrix, Hessenbergian, inverse matrix, lower semiseparable (plus diagonal) matrix, resolvent matrix
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1 Introduction

The significance of matrix inversion in many parts of science and engineering and the methods used for its resolution are well known. The Cayley formula for the entries of the inverse matrix in terms of the adjoint matrix involves determinants. Both the computation and the expansion as a sum of products of these determinants present difficulties. These problems can be avoided by taking advantage of the special structure of certain matrices, for example tridiagonal, band, or Hessenberg matrices, to develop less costly algorithms or to identify properties invariant under matrix inversion.

In this direction, algorithms for the inversion of unreduced symmetric tridiagonal matrices were introduced in [3]. These algorithms were generalized to unreduced Hessenberg matrices in [10] and to banded unreduced matrices in [21]. In all of them the entries of the matrix inverse were represented as a product of two linear recurrences. The relation between certain elements of the inverse matrix of an unreduced Hessenberg

matrix and the product of two linear recurrences was proven in [6]. These recurrences were obtained from a closed formula for the entries on and above the diagonal of the inverse matrix of a lower Hessenberg matrix in terms of the Hessenbergians, [20], of its proper principal submatrices. In parallel, the low rank properties of submatrices of the inverse matrices of tridiagonal and banded matrices were outlined in [1], based on explicit representations of their minors. The results of [1, 6] were closely related with the nullity theorem. The subsequent development of this theorem and its implications for the invariance of low rank properties in the inversion of semiseparable matrices, [4, 7, 8, 14, 19], have dominated research up to the present time. A first closed and general representation for all entries of the inverse of any unreduced Hessenberg matrix was given by one of the authors, V. Tomeo, in [17]. Later, analogous expressions are obtained in [22]. In addition, there is an abundant literature related to general or specialized algorithms for the inverse of structured matrices. These only work for unreduced Hessenberg matrices. Recent algorithms for the inversion of Hessenberg matrices can be found in [2, 5] .

Without loss of generality we work with upper Hessenberg matrices. The compact expression for the entries below and on the diagonal is straightforward when using the Cayley formula and the Sylvester theorem on determinants,

$$(H^{-1})_{i,j} = \frac{A_{j,i}}{\det\ H} = (-1)^{i+j} \left(\prod_{k=0}^{i-j-1} h_{i-k,i-k-1} \right) \frac{\det\ H_{j-1} \det H_{n-i}^{(i)}}{\det H} \qquad (1)$$

The submatrix H_{j-1} is the left principal one of order $j-1$. The submatrix $H_{n-i}^{(i)}$ is the right principal one of order $n-i$, which begins in the $i+1$-th row and column and finishes in the n-th row and column. This formula is equivalent those given in [6] for the entries on and above the diagonal, $i \leq j$, for the inverse matrix of a lower Hessenberg matrix.

The validity of the representation in closed form for all entries of the inverse matrix H^{-1}, in terms of proper Hessenbergians given in [17, 22] for unreduced Hessenberg matrices is extended here to the reduced case. In addition, a new and more compact proof is introduced. This class of expressions allows us to solve for all the entries of the matrix using homogeneous linear recurrences, [11, 12], with well defined coefficients for each Hessenberg matrix. This approach has been applied in the case of tridiagonal matrices. A solution for the elements of the inverse matrix using a set of linear recurrences was introduced in [9]. A more sophisticated method was given in [13], where the solutions of second order linear difference equations were used in a boundary value problem. Thus, a compact representation for the inverse matrix of any unreduced tridiagonal matrix was obtained via combinatorial expressions, equivalent to the Leibniz formula for determinants.

In Section 2 we introduce a new proof for the representation of all entries of the inverse matrix of any unreduced Hessenberg matrix, in terms of proper Hessenbergians. The representation is extended to reduced Hessenberg matrices, although we must consider the avoidable indeterminacies that could arise. Section 3 is devoted to the linear recurrences involved in the computation of Hessenbergians and recalls some of

the elementary properties of the inverse of a Hessenberg matrix. As an interesting application of our results, in Section 4 a closed formula is given for the elements of the finite sections of the resolvent matrix associated to any sequence of monic orthogonal polynomials on a bounded region of the complex plane. It is illustrated with two examples on the unit disk.

2 Inverses of regular Hessenberg matrices

To begin with there is proved a preliminary lemma which will simplify the later proofs. For $1 \le k \le m-1 < n$, we define the upper Hessenberg submatrix H_{m-1-k}^C of order $m-1-k$ associated to a left principal submatrix H_{m-1}. Its first $m-2-k$ columns are equal to those of H_{m-1-k}, while the last column comprises the elements of the last column, $m-1$, of H_{m-1}. For example, for $m=8$, $k=3$, the resulting matrix H_4^C is

$$H_4^C = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{17} \\ h_{21} & h_{22} & h_{23} & h_{27} \\ 0 & h_{32} & h_{33} & h_{37} \\ 0 & 0 & h_{43} & h_{47} \end{bmatrix}.$$

Lemma 1 *The proper Hessenbergians,* $\det H_{i-1}^C$, $\det H_{n-i}^{(i)}$, *and first order minors* $M_{j;i}$ *with* $1 \le j < i$, *of an upper Hessenberg matrix H of order n satisfy the following equations*

$$(-1)^{m-i}h_{i,i-1} \det H_{i-1}^C \det\ H_{n-i}^{(i)} = \sum_{j=1}^{i-1} h_{j,m-1}(-1)^{m-1-j}M_{j;i} \qquad (2)$$

The submatrix H_{i-1}^C is defined relative to the left principal submatrix H_{m-1}, $i < m$.

Proof. Expanding $\det H_{i-1}^C$ along its last row and using (1), we have

$$(-1)^{m-i}h_{i,i-1} \det H_{i-1}^C \det H_{n-i}^{(i)} =$$

$$= (-1)^{m-1-(i-1)}h_{i-1,m-1}M_{i-1;i} + (-1)^{m-i-1}h_{i,i-1}h_{i-1,i-2} \det H_{i-2}^C \det H_{n-i}^{(i)}$$

Iterating the procedure, the left side of (2) is equal to

$$\sum_{j=i-2}^{i-1} h_{j,m-1}(-1)^{m-1-j}M_{j;i} + (-1)^{m-i-2}\left(\prod_{k=0}^{2} h_{i-k,i-k-1}\right)\det H_{i-3}^C \det H_{n-i}^{(i)}$$

After $i-2$ iterations, with the convention that $\det H_0 = 1$, there results

$$(-1)^{m-i}h_{i,i-1} \det H_{i-1}^C \det H_{n-i}^{(i)} =$$

$$= \sum_{j=2}^{i-1} h_{j,m-1}(-1)^{m-1-j}M_{j;i} + (-1)^{m-2}\left(\prod_{k=0}^{i-2} h_{i-k,i-k-1}\right)\det H_1^C \det H_{n-i}^{(i)}$$

$$= \sum_{j=1}^{i-1} h_{j,m-1}(-1)^{m-1-j} M_{j;i}$$

∎

The centered principal submatrices $H_{j-i-1}^{(i)}$ or order $j-i-1$ appear in the next theorem in a role analogous to that played by the submatrices of H in the definitions given in (1). These matrices are formed from the matrix H, taking the elements from $(i+1)$-st to the j-th rows and columns .

Theorem 1 *Any upper Hessenberg matrix H of order n with complex coefficients satisfies the equations*

$$\det H_{j-i-1}^{(i)} \det H = \det H_{j-1} \det H_{n-i}^{(i)} - \left(\prod_{k=0}^{j-i-1} h_{j-k,j-k-1} \right) M_{j;i} \qquad (3)$$

for $1 \le i < j \le n$, where $M_{j;i}$ is the corresponding first order minor of the matrix H.

Proof. For fixed i, $1 \le i < n$, we proceed by induction on j, $i < j \le n$.

If $j = i+1$, the result follows straightforwardly from expanding $\det H_n$ along its j-th row:

$$1 \cdot \det H = \det H_{j-1} \det H_{n-j-1}^{(j-1)} - h_{j,j-1} M_{j;i} \qquad (4)$$

We suppose the statement is true for $i < j \le m-1 < n$. Then, for $j = m \le n$, expanding the Hessenbergians in (3) for the matrices depending on m along their last rows and using the induction hypothesis, we have,

$$\det H_{m-1} \det H_{n-i}^{(i)} - \det H_{m-i-1}^{(i)} \det H =$$

$$= \left(\prod_{k=0}^{m-i-2} h_{(m-1)-k,(m-1)-k-1} \right) h_{m-1,m-1} M_{m-1;i}$$

$$-h_{m-1,m-2} \left(\det H_{(m-1)-1}^{C} \det H_{n-i}^{(i)} - \det H_{(m-1)-i-1}^{(i)C} \det H \right)$$

The Hessenberg matrices $H_{(m-1)-1}^{C}$ and $H_{(m-1)-i-1}^{(i)C}$ are evident in this context and they are associated to the left principal submatrix H_{m-1} and the centered principal submatrix $H_{(m-1)-i}^{(i)}$, respectively.

If we expand the determinants indexed by m along their last rows one more time and use the induction hypothesis,

$$\det H_{m-1} \det H_{n-i}^{(i)} - \det H_{m-i-1}^{(i)} \det H =$$

$$= \left(\prod_{k=0}^{m-i-2} h_{(m-1)-k,(m-1)-k-1} \right) \left(\sum_{l=m-2}^{m-1} h_{l,m-1}(-1)^{m-1-l} M_{l;i} \right) +$$

$$+h_{m-1,m-2}h_{m-2,m-3} \left(\det H_{(m-2)-1}^{C} \det H_{n-i}^{(i)} - \det H_{(m-2)-i-1}^{(i)C} \det H \right)$$

After $m - i$ iterations and using the induction hypothesis,

$$\det H_{m-1} \det H_{n-i}^{(i)} - \det H_{m-i-1}^{(i)} \det H =$$

$$= \left(\prod_{k=0}^{m-i-2} h_{(m-1)-k,(m-1)-k-1} \right) \times$$

$$\times \left(\sum_{l=i}^{m-1} h_{l,m-1}(-1)^{m-1-l} M_{l;i} + (-1)^{m-i} h_{i,i-1} \det H_{i-1}^{C} \det H_{n-i}^{(i)} \right) \quad (5)$$

The induction hypothesis can be used up to here. We make the convention that any Hessenbergian of negative order is null. Thus $\det H_{-1}^{(i)C} = 0$.

If we invoke Lemma 1, then (5) yields,

$$\det H_{m-1} \det H_{n-i}^{(i)} - \det H_{m-i-1}^{(i)} \det H =$$

$$= \left(\prod_{k=0}^{m-i-2} h_{(m-1)-k,(m-1)-k-1} \right) \left(\sum_{l=1}^{m-1} h_{l,m-1}(-1)^{m-1-l} M_{l;i} \right) \quad (6)$$

In order conclude the proof, it is sufficient to show that any upper Hessenberg matrix H of order n satisfies:

$$\sum_{l=1}^{m-1} h_{l,m-1}(-1)^{m-1-l} M_{l;i} = h_{m,m-1} M_{m;i}$$

For this purpose we give the sum with cofactors of the matrix H,

$$\sum_{l=1}^{m-1} h_{l,m-1}(-1)^{m-1-l} M_{l;i} = (-1)^{m-1-i} \sum_{l=1}^{m-1} h_{l,m-1}(-1)^{l+i} M_{l;i}$$

Because $m - 1 \neq i$, the sum of alien cofactors, [20], of the matrix H is null. Taking into consideration that we are working with an upper Hessenberg matrix of order n, we have,

$$(-1)^{m-1-i} \sum_{l=1}^{n} h_{l,m-1}(-1)^{l+i} M_{l;i} = (-1)^{m-1-i} \sum_{l=1}^{m} h_{l,m-1}(-1)^{l+i} M_{l;i} = 0$$

The induction step is verified, after an appropriate change in the index k of the product from equation (6),

$$\det H_{m-1} \det H_{n-i}^{(i)} - \det H_{m-i-1}^{(i)} \det H = \left(\prod_{k=0}^{m-i-1} h_{m-k,m-k-1} \right) M_{m;i}.$$

This concludes the proof. ∎

The general representation for entries of the inverse of an upper Hessenberg matrix H as products of proper Hessenbergians, [17, 22], is also a consequence of Theorem 1.

Corollary 1 *A general expression for the elements* $(H^{-1})_{i,j}$ *of the inverse matrix of an upper Hessenberg regular matrix* H, *is*

$$(H^{-1})_{i,j} = \frac{(-1)^{i+j}(\prod_{k=2}^{i} h_{k,k-1})\left(\det H_{j-1}\det H_{n-i}^{(i)} - \det H_{j-i-1}^{(i)}\det H\right)}{(\prod_{k=2}^{j} h_{k,k-1})\det H} \qquad (7)$$

That is, the element $(H^{-1})_{i,j}$ *of the inverse matrix can be represented as,* [17, 22]

$$(H^{-1})_{i,j} = \begin{cases} (-1)^{i+j}\frac{\left(\prod_{k=0}^{i-j-1} h_{i-k,i-k-1}\right)\det H_{j-1}\det H_{n-i}^{(i)}}{\det H} & if \quad i \geq j, \\[3mm] (-1)^{i+j}\frac{\det H_{j-1}\det H_{n-i}^{(i)} - \det H_{j-i-1}^{(i)}\det H}{\left(\prod_{k=0}^{j-i-1} h_{j-k,j-k-1}\right)\det H} & if \quad i < j. \end{cases} \qquad (8)$$

The cases with $i \geq j$ are equation (1). For $i < j$, when the matrix H is not unreduced, the result is also valid as a consequence of Theorem 1. Indeed, for $i < j$,

$$\det H_{j-1}\det H_{n-i}^{(i)} - \det H_{j-i-1}^{(i)}\det H = \left(\prod_{k=0}^{j-i-1} h_{j-k,j-k-1}\right) M_{j;i}$$

Then, (7) results in,

$$(H^{-1})_{i,j} = (-1)^{i+j}\frac{\left(\prod_{k=0}^{j-i-1} h_{j-k,j-k-1}\right) M_{j;i}}{\left(\prod_{k=0}^{j-i-1} h_{j-k,j-k-1}\right)\det H} = \frac{A_{j;i}}{\det H},$$

the Cayley formula for $(H^{-1})_{i,j}$.

3 Recurrences for the computation and some elementary properties of the inverse matrix

Although fast numerical algorithms can be used for the computation of Hessenbergians from (8), we concentrate on the homogeneous linear recurrences found for them.

Determinants of left principal submatrices, $\det H_i$, $i = 1, \cdots, n$ and $\det H_n = \det H$, of any upper Hessenberg matrix of order n satisfy the following large recurrence relations, with $\det H_0 = 1$,

$$\det H_i = \sum_{m=1}^{i} (-1)^{m-1}\left(\prod_{k=1}^{m-1} h_{i-k+1,i-k}\right) h_{i-m+1,i}\det H_{i-m} \qquad (9)$$

For Hessenbergians of right principal submatrices, $\det H_{n-i}^{(i)}$, for $i < j \leq n$, the recurrences are similar, now with the initial conditions $\det H_0^{(i)} = 1$ for $j = i$.

$$\det H_{j-i}^{(i)} = \sum_{m=1}^{j-i} (-1)^{m-1}\left(\prod_{k=1}^{m-1} h_{j-k+1,j-k}\right) h_{j-m+1,j}\det H_{j-i-m}^{(i)} \qquad (10)$$

The recurrences for Hessenbergians of the centered principal submatrices, $\det H^{(i)}_{j-i-1}$, can be obtained as particular cases of (10). Therefore, when the matrix H is unreduced, the computation of the elements of its inverse presents no difficulty.

When the matrix H is reduced, the numerical computation of the elements of its inverse presents no difficulty if $i \geq j$, or if $i < j$ and the null elements from the subdiagonal do not appear in the product of the denominator of (8). The computational difficulty appears when $i < j$ and one or more null elements of the subdiagonal of the matrix H appear more than once in the product of the Hessenbergians of the numerator of (8). This can happen when the minor associated to this element of the inverse matrix is not a Hessenbergian and it has in its second subdiagonal null and non-null elements from the subdiagonal of the matrix H.

We overcome indeterminacies by introducing auxiliary parameters in place of the zeros that can appear in the product of the denominator of (8). We use as an illustration a reduced Hessenberg matrix of order 6, obtained in a random way

$$
H = \begin{bmatrix}
-4 & 2 & -4 & 1 & -4 & -1 \\
-1 & 3 & 2 & 0 & -1 & 4 \\
0 & 0 & 1 & 2 & 4 & 1 \\
0 & 0 & 4 & -4 & 2 & -2 \\
0 & 0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\tag{11}
$$

Elements h_{32} and h_{54} of the subdiagonal are null. The element $(H^{-1})_{2,5} = \frac{331}{60}$ has associated the minor $M_{5;2}$, which is not a Hessenbergian. In this case, by (8), using parameters instead of the zeros in h_{32} and h_{54} in the Hessenbergians of the numerator and in the product of the denominator, we have

$$
(H^{-1})_{2,5} = (-1) \frac{\det H_4 \det H_4^{(2)} - \det H_2^{(2)} \det H}{\left(\prod_{k=0}^{2} h_{4-k,3-k} \right) \det H} =
$$

$$
= \frac{2648\alpha\beta}{480\alpha\beta} = \frac{331}{60}.
$$

We can obtain the right numerical result using the previous recurrences, replacing the parameters α and β by the value 1.

To calculate the element $(H^{-1})_{1,6} = -\frac{129}{20}$, we work with the minor $M_{6;1}$ of the matrix H, which is also not a Hessenbergian. We proceed in a similar way,

$$
(H^{-1})_{1,6} = (-1) \frac{\det H_5(\alpha,\beta) \det H_5^{(1)}(\alpha,\beta) - \det H_4^{(1)}(\alpha,\beta) \det H(\alpha,\beta)}{(-4\alpha\beta)(120)}
$$

$$
(H^{-1})_{1,6} = \frac{-256\alpha^2\beta^2 + 768\alpha^2\beta - 1016\alpha\beta^2 + 3096\alpha\beta}{-480\alpha\beta} = \frac{-32\alpha\beta + 96\alpha - 127\beta + 387}{-60}
$$

If we give now to the parameters their null values,

$$
(H^{-1})_{1,6} = -\frac{129}{20}
$$

we obtain the right result using the recurrences associated to the Hessenbergians and elementary symbolic computations. Were we to solve numerically as for the element previously obtained, replacing the parameters α and β in the recurrences involved in (8) by the value 1, it is obvious that the result would be inaccurate.

3.1 Some elementary properties of the inverse matrix

It is well known that the inverse of an upper Hessenberg matrix is a lower semiseparable (plus diagonal) matrix, as can be derived easily from (8) for $i \geq j + k$,

$$\det \begin{bmatrix} (H^{-1})_{i,j} & (H^{-1})_{i,j+k} \\ (H^{-1})_{i+l,j} & (H^{-1})_{i+l,j+k} \end{bmatrix} = 0$$

It is also known that the inverse matrix is semiseparable if and only if the matrix H is tridiagonal. In the unreduced case, more important in applications, if there are zeros in the diagonal of the inverse matrix, some principal, left or right, submatrices have non-maximal rank, with null associated Hessenbergians.

Moreover, the low rank property of some of the principal submatrices involved is a sufficient condition for the nullity of some element above the diagonal for the inverse matrix of a unreduced Hessenberg matrix H. The next illustrative matrix has an inverse with null diagonal elements and some elements above the diagonal are also null. This can be checked using (8).

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad ; \quad H^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

The entry $(H^{-1})_{1,3}$ is 0, because $\det H_2$ and $\det H_1^{(1)}$ are null. Also the entry $(H^{-1})_{2,4} = 0$ because $\det H_3$, $\det H_2^{(2)}$, and $\det H_1^{(2)}$ are null Hessenbergians.

4 General Orthogonal Polynomials: Hessenberg matrices in standard form

The computation of the resolvent matrices associated to orthogonal polynomials on the real line and on the unit circle and their associated tridiagonal and pentadiagonal Green matrices, are of current interest. Our results have application to the more general case of finite sections of the resolvent matrix associated with any sequence of orthogonal polynomials on an arbitrary and bounded domain in the complex plane. We will give two particular examples on the unit disk.

Given an infinite HPD (Hermitian positive definite) matrix, $M = (c_{ij})_{i,j=0}^{\infty}$, whether it comes from a measure or not, we denote by M' the matrix obtained by eliminating from M its first column. Let M_n and M_n' be the corresponding sections of order n of M and M', respectively, i.e., the corresponding left principal submatrices. As M is an HPD matrix, an infinite upper Hessenberg matrix $D = (d_{ij})_{i,j}^{\infty}$ can be built. Matrix

D is in standard form. That is, it has a positive subdiagonal. Its sections of order n satisfy, [18],

$$D_n = T_n^{-1} M_n' T_n^{-H}, \tag{12}$$

where $M_n = T_n T_n^H$ is the Cholesky decomposition of M_n, and $\{\widetilde{P}_n(z)\}_{n=0}^{\infty}$ is its associated orthogonal sequence of monic polynomials, [16]. From the properties of the matrix D_n, we have a determinantal expression for $\widetilde{P}_n(z)$. That is, the zeros of the orthogonal polynomials are the eigenvalues of the Hessenberg matrix:

$$\widetilde{P}_n(z) = det\,(zI_n - D_n) \tag{13}$$

If this Hessenbergian is expanded along the last row and the procedure is iterated, we obtain, as a particular case of (9), the *large recurrence relation* for monic orthogonal polynomials:

$$\widetilde{P}_n(z) = (z - d_{n,n})\widetilde{P}_{n-1}(z) - \sum_{k=1}^{n-1} d_{k,n} \left[\prod_{m=k}^{n-1} d_{m+1,m} \right] \widetilde{P}_{k-1}(z), \tag{14}$$

with initial condition $\widetilde{P}_0(z) = 1$.

The matrix obtained when deleting from D its first i rows and columns is denoted $D^{(i)}$. From $D^{(i)}$ we can build the infinite HPD matrix $M(i)$, for all $i \in \mathcal{Z}_+$. It defines an inner product. Then, the associated monic polynomials are defined, for $n \geq i$, as

$$\widetilde{P}_{n-i}^{(i)}(z) = det\left(zI_{n-i} - D_{n-i}^{(i)} \right) \tag{15}$$

with $\widetilde{P}_0^{(i)}(z) = 1$. They are orthogonal with respect to the inner product defined by $M(i)$. When expanding this Hessenbergian, we obtain, as a particular case of (10), the *large recurrence relation* for the associated monic polynomials,

$$\widetilde{P}_{n-i}^{(i)}(z) = (z - d_{n,n})\widetilde{P}_{n-i-1}^{(i)}(z) - \sum_{k=i+1}^{n-1} d_{k,n} \left[\prod_{m=k}^{n-1} d_{m+1,m} \right] \widetilde{P}_{k-i-1}^{(i)}(z), \tag{16}$$

Corollary 2 *The elements for finite sections* $(I_n z - D_n)^{-1}$, $n \geq 1$, *of the resolvent matrix related to the monic orthogonal polynomials coming from the matrix D are*

$$\left((I_n z - D_n)^{-1}\right)_{i,j} = \begin{cases} (-1)^{i+j} \left(\prod_{k=0}^{i-j-1} d_{i-k,i-k-1} \right) \dfrac{\widetilde{P}_{j-1}(z)\widetilde{P}_{n-i}^{(i)}(z)}{\widetilde{P}_n(z)} & if \quad j \leq i \\[3mm] \dfrac{(-1)^{i+j}}{\left(\prod_{k=0}^{j-i-1} d_{j-k,j-k-1} \right)} \left[\dfrac{\widetilde{P}_{j-1}(z)\widetilde{P}_{n-i}^{(i)}(z)}{\widetilde{P}_n(z)} - \widetilde{P}_{j-i-1}^{(i)}(z) \right] & if \quad i < j \end{cases} \tag{17}$$

If there are known expressions in closed form for the orthogonal polynomials of the sequence under analysis and those of its associated sequences, then the closed form expressions for the finite sections of the resolvent matrix are easily obtained. When expressions in closed form for the monic polynomials are not known, the entries of the resolvent matrix, for any complex number z, can be obtained numerically using the preceding recurrences.

4.1 A measure with radial symmetry on the unit disk

Let μ be a measure on the unit disk with a radially symmetric weight function constant on every circle centered on the origin. We suppose also that μ is a probability measure, i.e. $c_{00} = 1$. We have $\omega(z) = \omega(|z|)$. In this case, writing $r = |z|$, the moments are

$$c_{ij} = \int_{|z|<1} z^i \bar{z}^j \omega(r) dx dy = \int_0^1 \omega(r) r^{i+j+1} dr \int_0^{2\pi} e^{I(i-j)\theta} d\theta,$$

where the imaginary unit is denoted by I, to avoid confusion with the i index. By symmetry if $i \neq j$ then $c_{ij} = 0$. We have

$$c_{ii} = 2\pi \int_0^1 \omega(r) r^{2i+1} dr, \qquad i > 1, \quad with \quad 2\pi \int_0^1 \omega(r) dr = 1,$$

The moment matrix $M = (c_{ij})_{i,j=0}^\infty$ is diagonal and the associated Hessenberg matrix $D = (d_{ij})_{i,j=1}^\infty$ satisfies $d_{i+1,i} = \sqrt{\frac{c_{ii}}{c_{i-1,i-1}}}$ and $d_{ij} = 0$ if $i \neq j+1$. The monic polynomials are $\widetilde{P}_n(z) = z^n$ and the associated polynomials, for $n > i$, are $\widetilde{P}_{n-i}^{(i)}(z) = z^{n-i}$, with $\widetilde{P}_n^{(0)}(z) = 1$. Using Corollary 2, we obtain the resolvents of the finite sections

$$(I_n z - D_n)^{-1} = \begin{bmatrix} 1/z & 0 & 0 & \cdots & 0 \\ \sqrt{\frac{c_{11}}{c_{00}}} \frac{1}{z^2} & 1/z & 0 & \cdots & 0 \\ \sqrt{\frac{c_{22}}{c_{00}}} \frac{1}{z^3} & \sqrt{\frac{c_{22}}{c_{11}}} \frac{1}{z^2} & 1/z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{c_{n-1,n-1}}{c_{00}}} \frac{1}{z^n} & \sqrt{\frac{c_{n-1,n-1}}{c_{11}}} \frac{1}{z^{n-1}} & \sqrt{\frac{c_{n-1,n-1}}{c_{22}}} \frac{1}{z^{n-2}} & \cdots & 1/z \end{bmatrix} \quad (18)$$

4.2 A measure without radial symmetry on the unit disk

Now, we give an example partially treated in [15] with a full Hessenberg matrix in standard form. We consider the density function on the unit closed disk given by $\omega(z) = |z-1|^2$ with $|z| \leq 1$. The density function is null for $z = 1$ and positive in the rest of the disk. The moments are obtained by applying Green's formula,

$$c_{ij} = \frac{1}{2I} \int_{|z|=1} \left[-\frac{z^{i-j}}{j+1} + \left(\frac{1}{j+1} + \frac{1}{j+2} \right) z^{i-j-1} - \frac{z^{i-j-2}}{j+2} \right] dz$$

Therefore, the matrix of moments is

$$M = \begin{bmatrix} \pi\left(1 + \frac{1}{2}\right) & -\frac{\pi}{2} & 0 & 0 & \cdots \\ -\frac{\pi}{2} & \pi\left(\frac{1}{2} + \frac{1}{3}\right) & -\frac{\pi}{3} & 0 & \cdots \\ 0 & -\frac{\pi}{3} & \pi\left(\frac{1}{3} + \frac{1}{4}\right) & -\frac{\pi}{4} & \cdots \\ 0 & 0 & -\frac{\pi}{4} & \pi\left(\frac{1}{4} + \frac{1}{5}\right) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (19)$$

The elements of the matrix D are given by

$$d_{ij} = \begin{cases} \dfrac{-2\sqrt{i}}{\sqrt{(i+1)(i+2)j(j+1)(j+2)}} & \text{if} \quad i \leq j \\ \dfrac{\sqrt{j(j+3)}}{j+2} & \text{if} \quad i = j+1 \\ 0 & \text{if} \quad i > j+1 \end{cases} \tag{20}$$

The monic polynomials are obtained from (13)-(14),

$$\widetilde{P}_n(z) = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (k+2)(k+1)z^k \tag{21}$$

and the associated monic polynomials, if $n > i$, from (15)-(16),

$$\widetilde{P}_{n-i}^{(i)}(z) = z^{n-i} + \frac{2}{(i+2)(n+1)(n+2)} \sum_{k=i}^{n-1} (k+2)(k+1-i)z^{k-i} \tag{22}$$

In the particular case $n = j - 1 > i$, the monic polynomials $\widetilde{P}_{j-1-i}^{(i)}(z)$ are obtained from (22).

The resolvent matrices of D_n are readily obtained using Corollary 2, with the subdiagonal entries of D_n given in (20). The monic polynomials and their associated polynomials are obtained from (21) and (22), respectively.

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