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Discussion on "Optimal design of a class of symmetric plane frameworks of least weight"

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Abstract This paper is an answer to the criticism by Sokół and Lewiński (the Authors) to a previous Discussion by us (the Writers). Although with the Reply by the Authors—where several new results have been shown— some agreements have been achieved, there are still several points deserving further discussion.

Keywords design theory · Maxwell's problems · Michell's theorem

1 Introduction

The Writers cannot agree with some of the statements of the Authors' Reply (Sokół and Lewiński, 2011). The main aim of the Discussion (Vázquez Espí and Cervera Bravo, 2011) was "to fix the meaning of *Michell class* concept, applied to structural problems or solutions". The Writers show that problems of Reply's Figs. 1, 2 and 3 are very different in nature, and whereas problems 2 and 3 fulfil the definition of an *optimal design* problem by Michell (1904), problem 1 does not. The main motivation of our previous Discussion was that in Sokół and Lewiński (2010) the Authors speak about *optimal design* in spite of only this last problem being considered. The controversy regarding this point remains in the Reply. Let us stress the main idea: although problem of Fig. 1 is a legitimate *optimization problem* for which the Authors contribute a new solution, we find it is less easy to accept the application of the expression "*optimal design*" to it, both in Michell's sense and in view of engineer's design challenges.

We will list the discussion points as they appeared in the Reply and use the same section titles. Simultaneously, we will point out some mistakes in the Authors' Reply.

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2 On Maxwell's theorem and its relation to Michell's results

The Authors argued that the Writers "pay too much attention to this [Maxwell's result], since its direct application in the minimum weight design of trusses confines to a rather narrow class of trusses for which the virtual strain is identical in all bars [. . .] the Michell class encompasses all Maxwell-like optimal solutions [. . .] Hence there is no need now to consider Maxwell's solutions separately". However neither the Writers nor Maxwell considered *solutions* but rather *Maxwell's problems*, as a kind of structural problems the solutions to which are comparable from the designer's point of view, as we show in the sequel. Let us remind the reader that for this class of problems Maxwell's number $\mathcal{M}$ is constant, depending only on *given external forces* and not on internal ones. This quantity —with energy units— can be computed by the opposite of the work of those forces in a virtual displacement field that homogeneously contracts the whole space to a point (Maxwell, 1890:177) or, following Maxwell's rule (p. 176), by the difference between quantities of structure —or stress volumes— in tension and compression, $\mathcal{Q}^+ - \mathcal{Q}^-$.

As Hemp (1973:72, Eq. (4.10)) shows, Maxwell's lemma (as Cox named the rule) covers a larger class of *problems* for which we all can agree with Cox (1965:86) that the Maxwell lemma is an insufficient "guide to the best layout". But not being a "guide to the best layout" is not the same as not being a guide to tackling well formulated *design problems*. The lemma in Maxwell's view was an instrument for the definition, in a consistent manner, of the structural *design problem* according with fundamental rules of thermodynamical efficiency—Cox describes it as "the basic theorem governing the design of single purpose structures" (p. 83). This fact is very clear when Maxwell's results are put in the context of his whole scientific work (Cervera and Vázquez, 2011). However from this lemma several corollaries can be derived as in fact Cox (1965:83) did, noting that if all members have stresses of equal sign the frame is optimum, although it is to be noted that this restricted class does not cover the whole Maxwell class *of problems* to which the lemma can be applied. As designers we make regular use in such problems of another useful corollary of Maxwell's lemma that states that if a modification of a known solution increases the tension volume, $\mathcal{Q}^+$, it will increase the compression volume $\mathcal{Q}^-$ by the same amount.

But let us continue and cite Cox two pages before that the quotation by the Authors: "Only for a very restricted class of loading systems is Clerk Maxwell's lemma a sufficient guide [for the best design], but it is convenient to describe one such class [three forces with tension and compression members] as a preliminary to establishing the complete set of rules to which optimum layouts must in general all conform"(p. 84). What rules? Of course, those of Michell's work that Cox (1965:86 and ff.) introduces immediately in the sequel after the description of "such class".

Just after the quotation from Cox, the Authors say: "it is Michell who noted this insufficiency and introduced a new class of layouts much more useful for finding a wide class of structures of least weight". As the Authors are undoubtedly referring to Anthony George Maldon Michell (1904), this statement is very shocking. Firstly, the word "weight" (and of course, the expression "least weight") simply does not appear anywhere in the nine pages of this work (Michell refers to "quantity of material", as Maxwell did, or to "volume"). Secondly, the truth is that in this remarkable work: (i) he begins by writing in conventional mathematics the Maxwell's lemma (Eq. (1), p.589, being C our $\mathcal{M}$); (ii) he introduces a generalisable definition of cost (e.g., volume cost) as a function of $\mathcal{Q}^+$ and $\mathcal{Q}^-$ (Eq. (2), p. 589), following an annotation of Maxwell (1890:176) on "the importance [of the Maxwell lemma] for the engineer"; and (iii) he proves that for problems of given forces in equilibrium —fulfilling (i)— a solution will be of minimal cost —defined with (ii)— *if and only if* it is of minimal stress volume $\mathcal{Q}$ (Eq. (3), p. 590; a fact that Maxwell does not prove, only suggests). A structural problem that fulfils (i), and a cost that fulfils (ii) are the two ingredients of the *design problem* annotated by Maxwell (which we in the Discussion named "Maxwell's problem"). To look for the solution of minimum stress volume for a Maxwell problem is the new problem stated by Michell, an *optimization design problem* in fact.

As the reader can easily realise, it was Michell who introduced two classes of layouts that could be optimum for Maxwell's problems, not Maxwell, and Michell did not name them in any special form, and definitely not as "Maxwell's solutions". Let us stress that Michell envisaged this two class from his own theorem, not from Maxwell's lemma as Cox did several decades after.

We conclude that the statement "it is Michell who noted this insufficiency and introduced a new class of layouts" is simply false and logically impossible because it was Michell, not Maxwell, who stated the optimization problem for the Maxwell ones and discovered optimum solutions for several cases, which he classified into two sets, including the one that the Authors named "Maxwell's solutions".

So, when the Authors end this short paragraph (of only 19 lines) by saying "Hence there is no need now to consider Maxwell's solutions separately" they are undoubtedly referring to "trusses for which the virtual strain is identical in all bars", a class that only Michell mentions, not Maxwell nor the Writers in the Discussion. Whereas we agree with this statement, it should be noted that only the Authors considered this separation.

In the class of problems that Maxwell and Michell worked on there are only external force constraints, not displacement constraints. In our view this definition (since Maxwell) is consistent with the real practise of designers. Why do we outline the difference with problems showing displacement constraints? Let us cite Cox (1965) again: "When the supports [. . .] are actually fixed the nature of the *design problem* is vitally altered." (p. 95). "A rigid wall or any system of fixed points of support presupposes the existence of structure in the region adjacent to the space in which the loads are to be transmitted, and in part the *design problem* is transmuted into enquiry as to how best to utilise the *existing structure*" (p. 97). In fact, Cox devoted the next chapter, "Layout in practical design", of his remarkable book to the real world: "Nor is the scope of the basic theory limited to strictly optimal structures; it can be applied equally well to structures arbitrarily restricted. . . " (p. 115). "So long as the loading system relates only to [given forces in magnitude and position], Clerk Maxwell's lemma is always applicable and [Maxwell's number, $\mathcal{M}$] is constant" (p. 116). Etc. This class of problems of structural design was named "Free loading" by Cox (p. 117), and the others, with displacement constraint, "Fixed boundary". The problem of Reply's Fig. 1 belongs to this last class, whereas those of the other two Figs. to the former. (In Table 16, Cox (1965:117) analyses the differences on cost: the "Fixed boundary" solutions are ever cheaper as the cost of the "existing structure" is not accounted in any way, as it is the case when we compare the LP-IPM solutions for the Fig. 1a and Fig. 2a problems in Table 1 of the Reply.)

3 On problem (3) in the Discussion

The Authors say that the condition of constancy over $\mathcal{M}$ "is redundant if the equilibrium conditions are involved". This is by no means true: the solutions for the problem of Fig. 1 must fulfil the equilibrium conditions (as it must be the case with any feasible solution!) but $\mathcal{M}$ varies with solutions ("fixed boundary" case in Cox's terminology; non-Maxwell's problem in ours). So replacing the condition over $\mathcal{M}$ with equilibrium conditions, as proposed by the Authors, make no sense. Moreover, the Authors say "we rearrange problem (3) [of the Discussion] to the simple formulation of minimizing the weight but without additional assumptions concerning the stress level. Hence this formulation is incomplete." It seems to us that the Authors do not understand the profound meaning of Michell's Eq. (3): if the stress volume $\mathcal{Q}$ is minimal and Maxwell's number $\mathcal{M}$ is invariant, any cost of the structure defined as Michell's Eq. (2) will be minimal too for any conceivable (but finite) stress levels in tension and compression (see a lengthy explanation of this in Cox (1965:116): "only when there is *existing structure* to provide fixed points of support ["Fixed boundary"] does the best layout depend upon [the stress levels]"; see also a comprehensive example of "Fixed boundary" problem in Pichugin et al 2011:Fig. 4). The meaning of Eq. (3) becomes very clear if we generalise the cost definition of Michell and we write it as a function of $\mathcal{M}$ and $\mathcal{Q}$ (Cox, 1965:87, Eq.(121); Owen, 1965:53, Eq.(18); Barnett, 1966:20, Eq.(5)):

$$\mathcal{C} = \frac{1}{2}\left\{(k^+ + k^-)\mathcal{Q} + (k^+ - k^-)\mathcal{M}\right\} \quad (1)$$

k^+ and k^- being costs for the unity of stress volume. For the geometrical volume these costs are the reciprocals of stress levels —P and Q in Michell's Eq. (2). So the variation of the cost with the solution is:

$$\delta\mathcal{C} = \frac{1}{2}(k^+ + k^-) \cdot \delta\mathcal{Q} \quad (2)$$

since $\delta\mathcal{M} = 0$ in Maxwell's problems.

In writing the present paper we have realised that the complete understanding of Michell's Eq. (3) is indispensable for this subject, but unfortunately his next statement, that of Michell's theorem, occluded it and left it as a kind of introductory ornament. It is our fault not to have outlined the importance of this in the Discussion, making it harder to the Authors to reach a complete understanding of our argument for which we now apologise.

4 On remarks in Section 3 of the Discussion concerning the half-plane problem

The Authors say that the Writers "claim that the aim of optimization should not be the weight of the structure; this weight functional should be augmented by a term measuring the cost of some forces…". This is not the case, as the reader can check in the Discussion. The Writers were only proposing following Maxwell and Michell to transform the original problem of the Authors (Fig.1 in the Reply) into a Maxwell one (e.g., Fig. 2), and to minimize the stress volume $\mathcal{Q}$ according with the Michell's Eq. (3) —our main interest is the formulation of the problem. As a consequence it is not the case —as the Writers are suggesting— "that the new functional is naturally inferred from Maxwell's equality": we are simply using Michell's functional and that is by no means new!

5 On numerical results of Table 1 in the Discussion

There are two mistakes in table 1 of the Reply. In the column "SA", the figures for the solution of the "Fig. 1a" row must be 3.815256 and 1.18 instead of 4.66312 and 23.66. These figures result from the data of the Discussion (Vázquez Espí and Cervera Bravo, 2011:note 1). It is clear that this note was ignored by the Authors, as they said that "probably" the Writers "studied only the problems of Figs. 2a and 3a".

6 On construction of the virtual displacement field

The last discussion point is about whether Michell's criterion is a sufficient and necessary condition. It is clear in our view that although the "Free loading" and "Fixed boundary" problems are intimately related, they are clearly different problems. In fact, the argument of the theorem from Michell leads to similar but different criteria (e.g. Michell's and Hemp's, see e.g. Rozvany 1996). The Authors cite a few works but none of them deal with the problem stated by Michell, as the reader can easily check; e.g., Rozvany (1976:48) says "we shall consider a slightly modified version of Michell's problem". This is a subtle but precise difference in fact: the same that there is between a vector set and a analytic vector field, the last one being unnecessary because while Michell's theorem requires that a virtual displacement field exist, the theorem does not require an internal force field, only a set —perhaps an infinite one depending on the given external forces (see e.g. Hemp, 1973:70-71).

Anyway, without a general agreement about the previous discussion points an useful discussion about this intriguing point will not be possible for the time being.

7 Conclusion

Two different approaches to structural optimization —with different aims, even if they can share a restricted class of common solutions or if they can converge based on extensions of their interpretations— can be considered.

One: Michell's approach, following the ideas of Maxwell, focussed its attention on the overall cost of the structure, or at least on to comparing the cost of alternative solutions that require exactly the same external forces to attain its equilibrium state, the external forces being useful loads and reactions, so that whatever the cost of the latter it will make no difference in favour of any feasible solution.

Two: the "fixed boundary" approach, that is a direct application of optimization techniques to any problem of structural analysis, focussed its attention only on the cost of the analysed structure, considering that the supports (with variable reactions in Cox's "existing structure") are free of cost.

Unfortunately, as Michell-like criteria play fundamental roles in both approaches, the use of "Michell" as a dummy adjective has lead to confusion between this two basic approaches. The Writers are now working on a proposition about terminology for structural design matters that (i) can be accepted by all the researchers in the area, and (ii) can become a common basis for these two and other approaches.

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