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Yet another reading of three fundamental theorems on structural design theory

Discussion on the paper: Optimal design of a class of symmetric plane frameworks of least weight

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Abstract The aim of the present discussion is to look for a common basis to formulate problems of structural design with the real practise in mind in a constructive manner. The discussion is first motivated by the Reply of Sokół and Lewińsky (the Authors) to a previous discussion by us (the Writers), but also by the prevalence of some confusion in the literature about the main objectives of the structural design and of the structural optimization.

Keywords design theory · peer-review method · Michell's theorem

This discussion has to do mainly with the sections 2 and 7 of Authors' Reply (Sokół and Lewiński, 2011). As our aim is constructivelly to look for a common basis about structural design topics (including optimization), we will organise the Discussion as a historical trip for the fundamental milestones on the subject. Because the posterior published discussion by Rozvany (2011) on the same paper refers to the same subject, we add here comments on this paper too: it would be an unnecessary waste to devote another discussion to it.

In order to mimimize the misunderstanding and its derived confusion about concepts, notation, denominations, etc., we begin fixing the notation we will use. We devote the entire section 1 to Maxwell's work, putting it on the broad perspective of the concepts of cost and efficiency, and trying to remain in the perspective of own Maxwell's concepts, in an effort to not reinterpret his ideas with our modern interpretations of terms or concepts[1]. After, we will examine very carefully the fundamental paper by Michell, commenting under its assertions an example after Rozvany (1996) into the section 2. In this two sections we must analyse the *gory* details of the statements of Maxwell and Michell for two reasons: i) it seems that both their works (e.g. Sokół and Lewiński, 2011) and our ideas and notions about (e.g. Rozvany, 2011) are difficult to understand; ii) it will be necessary to show which are *exactly* our coincidences and discrepancies with the Authors. We use "gory" herein with the exact semantic defined by Wall et al (2000:45). The section 3 is dedicated to examine cited works that, albeit very interesting on mathematical techniques or optimization problems in general, have little to do with the present discussion. We add the short section 4 to point out minor mistakes in Authors' Reply. Finally, we conclude with §5 outlining the interesting points of discussion with the Authors.

Definitions: Bold italic lower-case letters, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{u}$, $\boldsymbol{x}$, etc., denote vector, regarded as one-columns matrices; normal letters, b, c, etc., denote components of the aforementioned vectors; the meaning of letters will be defined when firstly used—all quantities being real. A prime ($'$) is used to denote transposition.

$\star$ $\square$: end of $\star$
$\langle\star\rangle$: denotes the logic "true" of the statement $\star$
$|\star|$: absolute value of $\star$
$\star^+$: the part of $\star$ with has to do with the members of a frame in tension
$\star^-$: *idem*, but in compression

$\boldsymbol{a}$: actions contained in a set of external forces
$\mathcal{D}$: Gâteaux differential. $\mathcal{D}F(x,\psi)$ denote $\mathcal{D}$ of the F respect to the object x in the direction of change ψ
$\boldsymbol{d}$: position vectors of external forces

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[1] See the Introduction or the comments in note 90 in Cervera Bravo (1982), about our point of view on rereading historical papers.

$\boldsymbol{e}$: internal forces of a frame (*exertus*—Rankine)
$\mathbf{f}$: reference (allowable, yield and so on) stress
$\boldsymbol{l}$: member lengths of a frame
$\boldsymbol{p}$: given applied, external forces in equilibrium
$\boldsymbol{r}$: reactions contained in a set of external forces
$\mathcal{Q}(s)$: $|\boldsymbol{e}|'\boldsymbol{l}$ of a (sub)frame s (or $\int_s |e|\,\mathrm{d}l$, if appropriated)

1 The approach of Maxwell

Elsewhere (Cervera Bravo, 1982; Cervera and Vázquez, 2011) we have examined with some detail the origin and foundation of the Theory of Structural Design. But we must recall some remarks here as they are essential for the sequel.

In any structural analysis problem, we have at least actions (weights, wind, and so on), reactions (maybe temporally defined as displacement constraints) and internal forces (or stresses, strains and so on) in the structural material. As in any thermodynamical system, the definition of *the structure to be analysed* is arbitrary to some extent. The analyser could be interested at some moment in the performance of a substructure and she or he can re-arrange the former partition as appropriated: e.g. in Fig. 1 problem of Prager and Rozvany (1977) the analyser puts its attention into the bars that transmit a load to a vertical wall, delaying the analysis of the wall itself. In fact, this is customary and very useful in structural analysis but, *is it reasonable in structural optimization? As we will show, it is not because of the physical cost.*

After appropriated definition, analysis and checking —maybe with several trial-and-error cycles—, a new structure has to be built and the actual structure will comprise then all the materials that have to be laid on place. Hence there is no indefinition about what composes the structure from a themodynamical view: all things that have a physical cost to build it. Only the *useful* —design's purpose— actions, the place and the requirements of any kind that the structure has to fulfil are given to the designer as the *design problem* data.

Maxwell, well advised of thermodynamics account on cost and efficiency of systems, works on this realm and with his very practical mind looks at the structural systems as composed of three parts: actions, reactions of the Earth that exactly will equilibrate the actions and are of a given and stable cost, and 'internal' forces of the structure in a precise sense: all other forces that have a cost dependant of the design and that are necessary for the equilibrium condition of the complete system.

The Maxwell's approach to structural design focuses *the structural forms and their difference on physical cost*, which was the interesting cost for him and his colleagues—see the concluding remarks of Clausius (1885), who died three years later, unfortunately ignored by orthodox economics, essentially the same one that has driven us to the actual ecological crisis. The Maxwell's approach is undoubtedly contemporaneous as the reader can check, e.g. "The theory [of structural design] ought to be in a position to tackle the design problem directly, that is, to begin with the given forces and to produce by calculation the best structure that will safely carry them" (Hemp, 1958:1). The interested reader can have a look too to a few exemplary works, e.g., Cross (1936)—including a outstanding discussion, three times lengthier than the original paper—, Cox (1965), Owen (1965), Prager (1970), or Arup (1984), so as to realise the several flavors and approaches which the structural theory is wrote with in respect to design and/or optimization.

The paper "On reciprocal figures, frames and diagrams of forces" was first read in 1870 (Maxwell, 1890:161–207). It seems that the Authors disappoint that "the main result of Maxwell was described in words". But there is not difficult to make a mathematic contemporary translation. In fact, the main results —for our business here— are derived after a former theorem in p. 175 that it is formally proved in p. 177 by two methods. Let us try the translation of results in pp. 175–176:

• **"Theorem":** if $\boldsymbol{a}$=$\boldsymbol{r}$=$\boldsymbol{p}$=$\boldsymbol{0}$, then $\mathcal{Q}^+ = \mathcal{Q}^-$ (p. 175). □

• **"Rule":** the aforementioned "main result" in words is an application of the theorem to a structure in the Maxwell's view: if $\boldsymbol{p} \neq \boldsymbol{0}$, then $\boldsymbol{a}'\boldsymbol{d}_a + \mathcal{Q}^+ = -\boldsymbol{r}'\boldsymbol{d}_r + \mathcal{Q}^-$, denoting the subscripts a and r, actions and reactions respectively (p. 176). □

Hence we can conclude that

$$\mathcal{Q}^+ - \mathcal{Q}^- = -\boldsymbol{a}'\boldsymbol{d}_a - \boldsymbol{r}'\boldsymbol{d}_r = f(\boldsymbol{p}, \boldsymbol{d}) \tag{1}$$

is an invariant for a given set of applied forces. This can be computed by the opposite of the work of the those forces in a virtual displacement field that homogeneously contracts the whole space to a point, p. 177. This quantity, $\mathcal{M}$ in the sequel, was named "statical constant" by Owen (1965:50), and afterwards *Maxwell's number of a structural problem* by Cervera Bravo (1989) honouring the work of James Clerk Maxwell. $\mathcal{M}$ *plays a fundamental role in the structural design theory.* In fact, $|\,\mathcal{M}\,|$ is a inferior bound for $\mathcal{Q}$. Unfortunately, it is completely meaningless when a problem has $\mathcal{M} = 0$ (e.g. simple bending) *or* both $\mathcal{Q}^+ \neq 0$ and $\mathcal{Q}^- \neq 0$ holds.

• **"The importance of this theorem"** to the designer ("the engineer" in the Maxwell's view): the total cost of the tension and compression members of the frame will be 'proportional' to $\mathcal{Q}^+$ and $\mathcal{Q}^-$ respectively, for a given set of applied forces (p. 176). □

Maybe it is difficult to understand completely the value of these results—more difficult if the reader has no experience on accounting cost of real, built structures. Maxwell himself contributed to, as his main interest was then on reciprocal figures, and he passed quickly over these fundamental —for us, practical designers— annotations.

It is also tempting, and this has been a main driver for prior misunderstandings, to try to extend rapidly the class of problems addressed by Maxwell, that can be reputed as a very reduced class of self-equilibrated (or *isostatic*) problems, but doing so neglecting the reaction's influence on the costs or in the design, is nor respectful with Maxwell's purpose, nor a clever way to improve the engineering practice or to derive useful layouts. Thus we welcome the new attention given to such costs as a result of the discussion recently engaged.

Anyway, it is clear for the Writers that Maxwell had in mind two key concepts: the quantity $\mathcal{M}$, as an invariant of *well-formulated problems in structural design*; and the quantity $\mathcal{Q}$ as a *main property of the feasible solutions for those problems, relative to their cost.*

Let us outline two important points. First: *when designing* the engineer or the architect is mainly interested in related structural problems whose Maxwell's number $\mathcal{M}$ is independent of the form of the structure adopted as final solution. These problems was named *Maxwell's problems* (Cervera Bravo, 1989), honouring again his work. Undoubtedly, there are interesting problems out of this class, but they are of very especial nature and they do not belong to the kernel of real problems with high physical cost solutions (bridges, buildings, towers, aircrafts and so on). Second: the engineer or the architect is mainly interested to have some estimate —like $\mathcal{Q}$— about the cost of alternative solutions while she or he is designing and looking for a reasonable solution for the design problem on hand. Furthermore, as the design problem is many-sided in nature, optimal solutions has a role as valuable reference, but frequently they are unattainable in real project (Hemp, 1958:1; Owen, 1965:60–62). *The key concepts of Maxwell are included into this realm completely.*

We can not agree with the Authors in any way when they say:

• "Hence the latter condition [$\mathcal{M}$ is constant] is redundant if the equilibrium conditions are involved." $\mathcal{M}$ is invariant only in Maxwell's problems and this condition has nothing to do with equilibrium condition, e.g., the problem of Fig. 1 of Authors' Reply is not a Maxwell's problem, hence $\mathcal{M}$ depends on the solutions and, of course, the equilibrium equations holds. See Prager, 1970:10 who speakes of "purely static boundary conditions", or Rozvany, 1996:246, who explains that apropos of (in)dependant reactions.

• "It seems that the Authors [i.e. we, the Writers] of the Discussion pay too much attention to this result [of Maxwell], since its direct application in the minimum weight design of trusses confines to a rather narrow class of trusses for which the virtual strain is identical in all bars. Let us repeat after Cox (1965, p.86): «the need for this limitation arises from the insufficiency of the Clerk Maxwell lemma as a guide to the best layout»". The Authors misunderstand completely both the aim of Cox and his statement—that it is very clear when comparing it with the Maxwell's paper (both cited by the Authors as at their disposal). The "Maxwell's lemma" of Cox is his "rule" —(1) here—, that obviously it is not a "minimum weight" result, but has many corollaries, as the $|\,\mathcal{M}\,| \leq \mathcal{Q}$ pointed out above. Cox clearly viewed and used several of these to show optimal solutions for some "narrow class [in fact, 'classes'] of trusses". Cox stated two pages before Authors' quotation (p. 84): "Only for a very restricted class of loading systems is Clerk Maxwell's lemma a sufficient guide, but it is convenient to describe one such class as a preliminary to establishing the complete set of rules to which optimum layouts must in general all conform". Note that he refers here to a class with different strain in bars, not "identical" as the reader can easily check. Class that Cox or Owen, not Maxwell, established in his very remarkable books, following Michell.

2 Three fundamental theorems

In spite of the minor interest of those three pages for Maxwell himself, Michell (1904), himself engineer, realised its importance and clearly stated Maxwell's results and, even more, adds new ones. Let us continue with the translation to contemporary language of "The Limits of Economy of Material. . . ":

• **Eq (1), p. 589:** $\mathcal{M}$ is constant for all frames for a given set of external, applied *over them*, forces in equilibrium, i.e., for a given Maxwell's problem. This equation is the mathematical translation that the Authors asked for Maxwell's words and is frequently and exactly named "Maxwell's *lemma*" (e.g. Cox, 1965; French, 1999), because accordingly with the Webster Dictionary, we are speaking of "A preliminary proposition accepted for immediate use in the demonstration of some other proposition", i.e. this is the passport to enter in the Maxwell's world. □

• **Eq. (2), p. 589:** Michell defines a cost (volume cost, in fact) for any frame that solves the Maxwell's problem. Let us generalise the Michell's definition to:

$$\mathcal{C} = k^+\mathcal{Q}^+ + k^-\mathcal{Q}^-, \quad \text{with } k^+ \geq 0,\ k^- \geq 0,\ k^+ + k^- > 0 \tag{2}$$

In Michell's equation, k^+ is the reciprocal of the allowable stress in tension, $1/\mathbf{f}^+$, and k^-, the same in compression, $1/\mathbf{f}^-$, and then it follows that $\mathcal{C}$ is the material volume of the frame. □

The 'structural' costs k^+ and k^-—i.e. the relative costs to $\mathcal{Q}$— may be whichever: volume, CO_2 and other green-houses-gas emissions, exergy consumption in life cicle, etc. (Vázquez Espí, 2001; see Prager, 1970:2 too). Let us outline: the only condition is *proportion to* $\mathcal{Q}$, e.g. (2) does not generally hold with painting cost.

• **Eq. (3), p. 590:** Michell proves that the problem of minimizing a $\mathcal{C}$ defined by Eq. (2) is equivalent to the problem of minimizing $\mathcal{Q}$ if his Eq. (1) holds, i.e., for a Maxwell's problem again. □

The cost $\mathcal{C}$ as a function of $\mathcal{M}$ and $\mathcal{Q}$ results (Cox, 1965:87, Eq.(121); Owen, 1965:53, Eq.(18);
Barnett, 1966:20, Eq.(5)):

$$\mathcal{C} = \frac{1}{2}\{(k^+ + k^-)\mathcal{Q} + (k^+ - k^-)\mathcal{M}\} \qquad (3)$$

Let us outline that "equivalent" means that for any frame s, $\langle \mathcal{DC}(s,\psi) \geq 0\rangle$ *if and only if* $\langle \mathcal{DQ}(s,\psi) \geq 0\rangle$, i.e. the frame (or frames) that solves a problem solves the other —*s is a graph* for mathematical reasoning (Maxwell, 1890:161).

The Michell's Eq.(3) and its equivalent (3) herein translate to mathematics the insight of Maxwell about the "engineering importance" of the work of the later, as pointed out above.

We have realised writing the present discussion that the complete understanding of this Michell's equation is for the sequel indispensable, but unfortunately his next statement, the named Michell's theorem, shadowed and left it as a kind of introductory ornament. It is our fault not to have outlined its importance in our previous discussion, making harder to the Authors and prof. Rozvany a complete understanding of our argument and we apologise to them. Let us give ourselves as penance to write the Michell's logic up to now:

$$\langle \forall s,\psi:\ \mathcal{DM}(s,\psi){=}0 \ \wedge\ \mathcal{C}(s) = k^+\mathcal{Q}^+s{+}k^-\mathcal{Q}^-s\rangle \Rightarrow \\ \Rightarrow \langle\ \mathcal{DC}(s,\psi) \geq 0 \ \Leftrightarrow\ \mathcal{DQ}(s,\psi) \geq 0\ \rangle \qquad (4)$$

From this equation, it is clear the importance of the quantity $\mathcal{Q}$, "the quantity" in Michell's words, as he did not name it in any other form. Owen (1965:53, Eq. 19) did not named at all; Barnett (1966:20) named it "quantity of material"; Prof. Ricardo Aroca named it "quantity of structure" or "structural volume" in his Madrid Lectures, depending of the epoch; de Miguel Rodríguez (1974) named it "structural work"; French (1999:738), "absolute pertinacity"; other name is "stress volume"—perhaps is this last the unambiguous one?

Illustrative example (1): Rozvany (1996:244, Table 1) made a mistake solving the Eq. (3) of Michell, very illustrative here of the relevance of that misunderstood equation: he forgets the term in $\mathcal{M}$ in our (3). Its erroneous "dual formula" was only correct to calculate the volume in two special cases: $k^+ – k^- = 0$, i.e., with equal allowable stress in tension and compression; or $\mathcal{M} = 0$. As Michell worked only with examples of this last special subset, Rozvany did not advert his error when he checked them. Unfortunately in fact, because Rozvany was trying "to provide a constructive explanation of the apparent discrepancy between Hemp's and Michell's criteria", while with this error Rozvany stated that the "discrepancy" was not "apparent", in fact it existed in his view and had to do with the volume calculated with the erroneous "dual formula". □

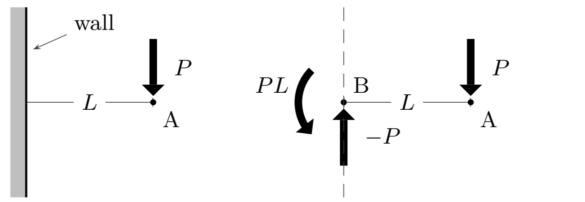
• **Michell's theorem, pp. 590–591:** We direct the reader to our full explanation in Vázquez Espí and Cervera Bravo (2011) as it is a contemporary translation — see two basic examples in Vázquez Espí (2011b); Hemp (1958) and Cox (1965:87–90) offer valuable, complementary explanations; Owen (1965:53-55) offers a detailed proof and examples (he use "Michell structures" for solutions that fulfil Michell's criterion, and "Minimum structures" for solutions with only tension or compresion members but no both, p. 55). But let us outline the important, out of doubt *facts* about it—no opinions:

i) The theorem is to be applied on the realm of the above statements, i.e., (4) holds.
ii) The logic of the theorem is as follow: ***if*** ⟨a test displacement field T exists for a frame O such that the well-known Michell's criterion is fulfilled by the couple (T,O)⟩ ***then*** ⟨the frame O has minimal $\mathcal{Q}$⟩.
iii) Corollary from Eq. (3): if a frame has minimal $\mathcal{Q}$ for a Maxwell's problem, it has minimal linear cost for any cost defined by our (2) *and* also —for fully stressed design— minimal volume.

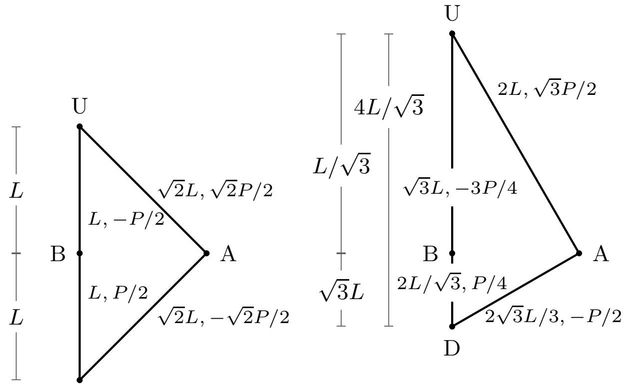
□

Unfortunately, the fact iii appears mixed with the theorem, but it depends only on Eq. (3) and Hemp (1958:4) copies this mixing as such. And this has been unfortunate as the belief that Michell worked in frames with equal stress in tension than compression, in least-weight frames (e.g. Barnett, 1966), etc., become popular because of this mistake of Michell and Hemp (recall that Michell was mainly concerned with fluids mechanics, area in which his works are more known, making several patents and pointing out important results in lubrication; and note indeed that the word "weight" does simply appear no where in his paper). Let us remark: *in any Maxwell's problem, the two facts i and iii above hold simultaneously, and only with lucky we will have also ii* (as in Michell's examples).

Illustrative example (and 2): Rozvany (1996:245) worked on a non-Maxwell problem as "illustrative example" —the before mentioned problem of Prager and Rozvany (1977:Fig. 1), see Fig. 1(a) here—, so he made two consecutive errors: to apply Michell's theory to that problem and to use his "dual formula" with unequal allowable stresses. But the story was worse by coincidence because the "Michell's solution" he got (Fig. 1(c))



(a) Rozvany's problem (b) Maxwell's version



(c) Solution "1b" (d) Solution "1c"

Fig. 1 Illustrative example after Rozvany (1996). Top, the two problems. Bottom, two solutions (Fig. 1b and Fig. 1c *ibidem*); for each member the (l,e) couple is given; note that the two figures are self-reciprocal in Maxwell's sense, i.e. if in (d) the drawing scale is such that $\mathrm{DU} = P$, the internal force of each member can be measured directly in the drawing. P is the "useful load" and L the problem's size. For Rozvany's problem the design has *only* two members: AU y AD; the simpliest Maxwell's Frame has at least four, i.e., BU ant BD must be added. The allowable stress is $\mathbf{f}$ in tension and $\mathbf{f}/3$, in compression, hence we must use $k^+ + k^- = 4/\mathbf{f}$ and $k^+ - k^- = -2/\mathbf{f}$ in (3).

has $\mathcal{M} = 0$, see Table 1 here, so our (3) and his "dual formula" did not make any difference that Rozvany would be able to discover in the form of a discrepancy between the "primal" (correct) and "dual" (erroneous) volumes: hence the selected example does not result "illustrative" at all!

Anyway Rozvany found that Hemp's criterion lead to a better solution (Fig. 1(d), with $\mathcal{M} = 2/\sqrt{3}$). Rozvany neither calculates $\mathcal{M}$ —as we do—, so he could not advert that the two solutions are not commensurable by $\mathcal{Q}$ in the scalar metric proposed by Maxwell and Michell, $\mathcal{DQ}$. Let us assume that the "Hemp" solution is better. Then a question arises again: what is the cost of the mechanical role of the wall? (The analougus to the one stated in Vázquez Espí and Cervera Bravo (2011) apropos of the solution of Fig. 1 problem of Authors's Reply. Apropos that subject, Owen (1965:64) said: "It must be remembered, nevertheless, that the reactions such as those at [fixed supports], are in any case carried by some other bodies acting as structures and the true picture of the economy achieved should include the abutments.")

To answer the question, a simple Maxwell's problem could be this one: transport horizontally the load from the point A to some point of the wall, say to B (Fig. 1(b)), thinking in loads and stresses as a kind of fluid as probably Michell thought— the paper of Prager (1965) suggests that too. We must add now the cost of the mechanical action of transporting a part of the load from the upper joint —in the wall— to B (BU member), and conversely from bottom (DB). Making the appropriated corrections it results that $\mathcal{M} = 0$ in both solutions (i.e., they are now commensurable in Maxwell's world), the volume of the solution "1b" (Fig. 1(c)) is proportional to 6, and the volume of "1c" (Fig. 1(d)) is greater, to $13/\sqrt{3}$ (the optimal value, to $2 + \pi/2$!), being both proportional to the respective values of $\mathcal{Q}$, since $\mathcal{M} = 0$.

What is the discrepancy? In our view, there is not! Hemp's criterion operates on non-Maxwell's problems and Michell's on Maxwell's ones. Or in the real world: Hemp's criterion is useful for the problem in a restricted environment as could be domestic housing (the wall is specified for sound conditions or other non-structural requirements and it is supposed to have enough strength for any "domestic" load), and Michell's criterion for engineering problems as usual (the wall is a trick, denoting a symmetric axis or so, or generally speaking it will be defined, analysed and checked for structural requirements, and indeed from this process a wall-cost results, small or great —the cost of the BD and BU members in our version—, and we, the designers, are interested in the cost of the whole design not only in the cost of the bars out of the wall). Cox (1965:95-96) noticing this "apparent discrepancy" when confronting a similar case said: "When the supports [...] are actually fixed the *nature of the design problem is vitally altered*. The directions of the reactions at the supports are then determined in part by the structure itself, so that [$\mathcal{M}$] is variable and Clerk Maxwell's lemma, while still true, is of no use." (the emphasis is our). Recall that no support condition —nor displacement constraints— appears in Michell's paper.

Note that Michell's criterion although constrained to a non-Maxwell problem leads to a good solution, in fact a better one —but not optimal— when the cost of the mechanical "wall" is considered through any appropriated Maxwell's problem with $\mathcal{M} = 0$, i.e., with equal $\mathcal{M}$ for both solution and problem.[2]

□

We must add now that it is clear that Michell proved only a *sufficient condition for a frame be of optimal $\mathcal{Q}$* (see Hemp 1958:2,31; Prager 1965:326; Rozvany 1996: 245–246 and Pichugin et al 2011). Of course, that is very much! (Although all we —Authors, Writers and readers— would have prefered a necessary condition for obvious reasons—and the search continue nowadays—, but the theorem is as it does!) It is customary to name "Michell's solution" to an optimal solution for a problem if it fulfils Michell's criterion—"Michell truss", "Michell layout" and so on, see a worthy discussion by

[2] In our opinion, the Rozvany's rereading of Michell's paper (Rozvany, 1996) had the fault that we essay to avoid with the position pointed out in note 1: when he gives "A critical examination of Michell's proof" he speaks about "$\mathcal{C}$" [our $\mathcal{M}$] as "*the same constant* for any statically admissible truss layout *considering a given set of external loads*". But neither Maxwell nor Michell spoke about *loads* (and never employed the term *reactions*) but about *forces*, about an equilibrated set of forces, so it is not surprising at all the *finding* (Rozvany, 1996:246) that Maxwell's theorem requires for problems posed as *load problems* that they must be "*statically determinate*".

Design:		Fig. 1(c)	Fig. 1(d)
Members's data:	$\mathcal{Q}^+$(AU)	PL	$\sqrt{3}PL$
	$\mathcal{Q}^+$(BD)	$\frac{1}{2}PL$	$\frac{1}{12}\sqrt{3}PL$
	$\mathcal{Q}^-$(AD)	PL	$\frac{1}{3}\sqrt{3}PL$
	$\mathcal{Q}^-$(BU)	$\frac{1}{2}PL$	$\frac{3}{4}\sqrt{3}PL$
Rozvany's problem:	$\mathcal{Q}$	$2PL$	$\frac{4}{3}\sqrt{3}PL$
	$\mathcal{M}$	0	$\frac{2}{3}\sqrt{3}PL$
	volume	$4\frac{PL}{\mathsf{f}}$	$2\sqrt{3}\frac{PL}{\mathsf{f}}$
Maxwell's version:	$\mathcal{Q}$	$3PL$	$\frac{13}{6}\sqrt{3}PL$
	$\mathcal{M}$	0	0
	volume	$6\frac{PL}{\mathsf{f}}$	$\frac{13}{3}\sqrt{3}\frac{PL}{\mathsf{f}}$
Michell's number:	$\mu = \mathcal{Q} \div PL$	3.00	3.75

Table 1 Data for the problems and designs of Fig. 1.

Pichugin et al (2011): unfortunately sometimes "Michell's layout" is used as synonimum of orthogonal nets (*only one* of Michell's classes), but this use is misleading (see Prager, 1970:Fig. 4 for a "non-orthogonal net" Michell's layout). There is another point to note: this terminology joins an optimal solution *only* with Michell's criterion—not with optimum condition. But in pure logic, a Maxwell's problem could have a optimal solution that would not fullfil Michell's criterion. Conversely, a structural, non-Maxwell problem can have an optimal solution that be a Michell's one; this last fact can lead to some confusion because although optimal solution fulfils Michell's criterion —that must be appropriated for the objective function defined by that problem, as it would be the case for many optimization problems accordingly with the literature cited by the Authors— if the problem is not a Maxwell's one, i.e., if Eq. (1) does not hold, and the objective function is not $\mathcal{Q}$, that solution could be not *simultaneously* minimal in quantity of structure and all other costs defined by our (2), that is to say, the last corollary iii above could not hold—and in general it does not.

Anyway, only a few Michell's layouts —i.e., that fulfil Michell's criterion— are known up to date (Rozvany, 1984; Pichugin et al, 2011) for both Maxwell and non-Maxwell's problems. Meanwhile, there are many more very good solutions —for which $\langle$ it is a Michell's layout $\rangle$ or the no strictly equivalent $\langle$ it is an optimal solution $\rangle$ are not proved— than the former ones, and in fact the latter ones are the "best known" solutions in our knowledge, hence they can be considered *optimal up to date* or *practically optimal* or also, as it is nowadays customary in other disciplines, *better know-how* solutions, e.g., the solution of Hemp (1974) for the problem of a uniform load between fixed supports—see too Rozvany (2011:§4) and Sokół and Lewiński (2011:§7) about the actual status of Authors' solutions in their Reply.

Reading again Michell's paper we have been equal impressed by his profound understanding and insight of the mathematical space where Maxwell's structures live, as we had after our first reading many decades ago (see Rozvany, 1996:244, for similar feeling). As a consequence, we have proposed at home (Cervera and Vázquez, 2011) honouring Anthony George Maldon Michell naming *number of Michell*, denoted μ, to the adimensional ratio between the quantity of structure $\mathcal{Q}$ of a frame and the product of the useful load (the given load at the very begin of the design problem) and the size of the problem (span length of a bridge, height of a tower, and so on), and we will be very happy if this modest proposition is accepted by research community on these matters. The advantage of this quantity over others customarily used (e.g., the ratio between half of volume and the product of half of the load and half of the span, or the "normalized volume" in Sokół and Lewiński 2011) is that it serves to the designer to make comparison among *forms* but also across Maxwell's problem, independently of any material property and the considered cost if at least Eq. (2) will hold, i.e., linear cost. Note that as a matter of fact, when a designer confronts a given design problem she or he can have on hand several *Maxwell's problems* that fulfil the structural requirements of the *design problem* (see the examples of Sved, 1954:5–8), but she or he can not know in advance which of those problems includes the optimal solutions for the later (see our "three bridges" example in Cervera and Vázquez, 2011), hence the usefulness for the designer of a quantity that permits to compare feasible solutions for different Maxwell's problems but for the same design problem. We are here again on the thermodynamic realm of Maxwell's insights. We propose also to name "Michell's lemma" the equation (4), as it fixes mathematically those insights, hence it must be considered also a main contribution of Michell, in our view—perhaps the very best!

For the convenience of the reader we have tried to draw a summary in Figure 2 and we will happy if this terminology was of general aceptation in the future to avoid more confusions. We are not sure how much "narrow" (Sokół and Lewiński, 2011) is the set D comparing with $A - D$, but we are mainly concerned with it as practical designers. However we are sure that previously to make any comparison a "metric" has to be selected because there are a lot of them at our disposal: number of problems in each set is one, resource wasting with built designs for solving those problems is other, etc.

For the rest, Michell worked on his theorem, describing some classes of layouts that *could* fulfil Michell's criterion—for whatever structural problem where $\boldsymbol{e}$, $\boldsymbol{l}$,

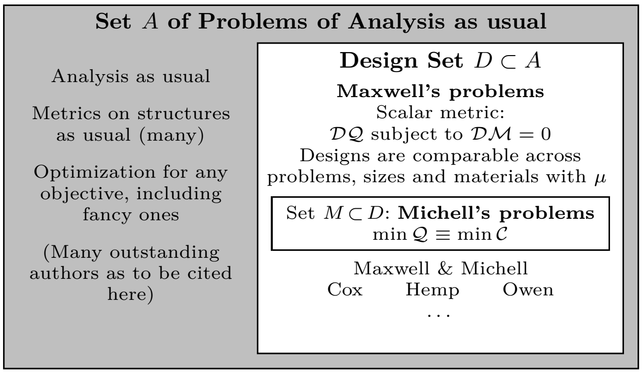


Fig. 2 Structural problems. While all kinds of calculus in $A - D$ are applicable on D, the special calculi that have sense in D have not in $A - D$.

etc., have sense—, and showed optimal solution belonging to this classes for a selection of *Maxwell's problems*.

After all the above analysis of Michell's paper, we can not agree with the Authors in any way when they make any of the following assertions:

• "it is Michell who noted this insufficiency [of Maxwell's lemma] and introduced a new class of layouts much more useful for finding a wide class of structures of least weight". (Perhaps would want the Authors to refer to another author and not to "Michell"?)

• "Let us note: Michell class encompasses all Maxwell-like optimal solutions". In our view "a Maxwell structure is any set of internal forces in self-equilibrium that added to the external forces of a Maxwell problem satisfies that every subset of forces, internal or external, acting at the same point is in equilibrium." (Vázquez Espí and Cervera Bravo, 2011) —a precise representation of its form without any material, a "Frame" in the lengthy definition of Maxwell (1890:161–163). So the set of Maxwell's structures encompass *all feasible solutions* for a Maxwell's problem, including optimal solutions that fulfil Michell's criterion —i.e., Michell's layouts— if they exist. Unfortunately, it seems another definition for "Maxwell's solutions" exists —"Maxwell trusses constitute a very simple subclass of Michell trusses that consist of a single S-region", "with members having forces of the same sign in any direction" (Rozvany, 2011)— but it has not direct relationship with Maxwell's paper or his definitions, problems, or results. Anyway, we clearly speak (Vázquez Espí and Cervera Bravo, 2011) about our own definition—our paper was self-contained, being apart the papers of Maxwell and Michell, and of course the discussed paper of Sokół and Lewiński (2010).

• "The Authors of the Discussion claim that the aim of optimization should not be the weight of the structure: this weight functional should be augmented by a term measuring the cost of some forces". Let us remember our own words: "the material cost can be estimated in this problem [Fig. 1 of Authors' Reply], following Maxwell, by the sum of the internal cost [internal forces of the structure], calculated by the authors as $2V$, and the cost of the horizontal reactions, V_N [as they depend of the structure]." (Vázquez Espí and Cervera Bravo, 2011) So, as the reader can check, neither the Authors nor the Writers speak about weight but "volume" or "material cost" respectively in theirs original papers. (Self-weight is the *next* problem! It *plays in other league*, so to speak, and we can not cover it here, see Cervera Bravo 1990 and Vázquez Espí 2011b.) And we did not talk of "some forces", but explicitly of the external forces that are not constant for all the feasible structures for the non-Maxwell problem stated by the Authors so, accordingly with the approach of Maxwell, they are 'internal' forces (they are neither actions nor given-cost reactions), that is to say, we must account their cost, whichever will be (see again Rozvany, 1996:246). Let us quote our own words again: "So we have to be sure that for each compared solution, we account the same cost. It is accomplished in Maxwell problems as the cost of external, known forces (whichever it may be) is the same for every solution, so this cost makes no difference (it does not depend of the considered solution), and we can focus our attention on the cost of internal forces only, see (Cervera Bravo, 1989; Vázquez Espí and Vázquez Espí, 1997)." (Vázquez Espí and Cervera Bravo, 2011). Also recall the second part of our illustrative example, specially the cited comments of Cox or Owen.

It is clear that a very big communication problem exists hence the second motivation of this discussion should be clear for the reader. We hope that our part of blame was completely corrected just now.

3 Cited and useless papers for this discussion

Some papers that are claimed to by Authors or prof. Rozvany in support of their opinions have little to do in this discussion. Let us briefly explain why.

Rozvany (1976:48) dedicated his very known book to non-Maxwell problems. In the Chapter dedicated to layout optimization, he says "we shall consider a slightly modified version of Michell's problem". This is a subtle but precise difference in fact: the same that there is between a vector set and a analytic vector field, the last one being unnecessary because while Michell's theorem requires that a virtual displacement field exist, the theorem does not require an internal force field, only a set —perhaps an infinite one depending on the given external forces (see also on this subject Hemp, 1973:70-71).

Prager and Rozvany (1977) devoted this paper mainly to grillage optimization —in fact a non-Maxwell's problem, see Aroca and Cervera (1987) for a Maxwell's approach, and Jaenicke Cendoya (1984) for another including structure's self price and weight. And "trusses are primarily discussed on account of the simplicity of their structural analysis, which will enable us to present the theory of optimal layout of grillages by convenient analogy" (p. 265). Moreover, they showed only some examples of optimal trusses for non-Maxwell's problems (pp. 266–276, see too our illustrative example).

Strang and Kohn (1983) said in their abstract: "We study minimum weight [...] truss-like continua [...] subject to technological constraints which limit the member forces". Hence they work on a problem that is not equivalent to the problem stated by Michell—the P problem cited by the Authors (the study of this class of problems seems to be started by Prager and Taylor, 1968). Latter, they said "(b) We may remove the limitation imposed by $\mathbf{f}$ [...]. The value of $\mathbf{f}$ does not affect Michell's design in a fundamental way (it appears just as a multiplying factor) [...] in this paper we use the language of (b), with unlimited stresses in Michell's problem and cost proportional to stress." The reader can check that in this way the Eq. (2) of Michell has no sense ($k^+ = k^- = 0$)—it seems that it is simply ignored.

4 A short list of minor mistakes of the Authors

It seems incorrect the citation of Maxwell's paper because the Authors refers to the original one (read in 1870) but with the page numbering of the reprint by Cambridge University Press (Maxwell, 1890).

It seems confusing to name "Michell's problem" to the instance of Fig. 1(c) of Authors' Reply (see the caption there): Michell (1904) did not solve this problem—in fact neither stated it.

In several places the Authors confuse Maxwell's problem with "Michell's problem". We hope that the Fig. 2 makes the difference clear now: the "Michell's problem" can be understood following Michell himself as the problem of finding a feasible solution of minimal cost —defined with (2)— for a *Maxwell's problem*. The Fig. 1 problem of the Authors is not a Michell's problem in this very sense, whereas the Fig. 2 and 3 problems are. This mistake is also in Rozvany (2011:§2).

Passim in Authors' paper, when they said "exact solution" it seems better to say "analitical solution", except when they refer to the solutions of Michell himself, when "exact optimal solution" is correct, as he found the test field that requires his own theorem.

The Authors said that the Writers pretend that their derived solutions from that of the Authors are solutions for minimizing problems, but that is a mistake, as we said in the Introduction section of our first discussion "The solved problem [by the Authors] does not belong to the Maxwell class and, as a consequence, the solution does not belong to the corresponding Michell class. There are two transformations of the solution which can be used as feasible solutions [i.e., probably good ones, but no optimal in any sense] for the related Maxwell problem." (Vázquez Espí and Cervera Bravo, 2011), i.e., we clearly spoke of *feasible solutions*—not optimal ones.

Finally, there are two mistakes in table 1. In the column "SA", the figures for the solution of the "Fig. 1a" row must be 3.815256 and 1.18 instead of 4.66312 and 23.66. These figures result from the data of the Writers (Vázquez Espí and Cervera Bravo, 2011:note 1).

5 Conclusion: the Writers are hopeful

From great to less importance, we think the more interesting points of a fruitfull —we hope— discussion are the following:

⋆ First: There is a big communication problem—at least. We know for self-experience that many times the researchers are very busy, with little money and very pressed in many countries for publishing a lot of papers per year, and all these facts could be sometimes a justification for the present research situation. Because it seems that nowadays lesser care is taken for the precision of concepts, words, equations, etc., than at Maxwell's times. This is a big problem in our view precisely with problems that involve a lot of mathematics and where the probability of unadvised error would be great. This kind of problems are noway exclusive of structural research: e.g. the situation in standard economics is worse (or the worst?), and the built, "flashy", fancy, or so-called landmark structures —wasting a disproportional quantity of no-renewable resources in the last decades all over the world— do not put the structural design in a better position. Returning to the rigorous manner of Maxwell is pressing in both theory and practise. And we are afraid that the time ago useful, money-free peer-review method is not useful now at this point. All we must look for solutions and we must do quickly.

⋆ Second: Press et al (1988:473–474) say apropos the measure of the variability of the median value of a distribution: "Statisticians have historically sniffed at the use of *average deviation* or *mean absolute deviation* instead of *variance*, since the absolute brackets $[|\star|]$ in the former are «non-analytic» and make theorem-proving difficult." They follow pointing out that confronting the real world, more robust measures becomes popular as almost the unique useful, and in fact they are those of lower moments: linear sums or indeed simple counting. So the concerned researchers nowadays use more frequently robust *average deviation* instead of *variance*, at the cost of less support by theorems and at the risk of leaving only with numerical methods that supports their results—i.e. simple counting at their profound being—, but with the certainity of accomplishing their original, real task.

Naredo (1987:393–394) make similar, but opposite comments on standard economics: "…it is good to enjoy of those mattresses of equations[[3]] among which to hide the action responsability. Hence stakeholders, technicians and advisers prefer frequently the most complex and imperspicuous models, without any certainty of they are coming to result on better predictions than other simpler and more tractable ones, [as in that way] «if one formula does not pass the test, we can always add another variable, deflate by another, and so on. By cleverly choosing one's chisels, one can always prove that inside any log there is a beautiful *Madonna*» (Georgescu-Roegen, 1971:340)".

Hall and Day, Jr. (2009) made a similar analysis apropos geology, ecology and environmental science in their outstanding paper, and arrive to similar conclusions: with a few exceptions to the rule, the important, general issues of resource scarcity and population growth —at any analysis level, simple or complex— disappeared from college curricula, being substituted for very special and "challenger" models and problems, owning by small and special communities of researchers.

So as many people pointed out many times since many decades ago (Vázquez Espí, 2011a), our methods come with two extreme flavors—at least. In one side, methods for solving actual problems in the real, dramatic world of today; in the other side, beautiful equations solving *ad hoc* challenger "models" problems—formulated as a mean to get the formers in a tautology circle. In the discussion about the relative relevance of optimization objectives these two flavors —and many more, of course— arise too. We, the Writers, are concerned as architects with the *design theory*, and mainly with its formal —or shape-like— and physical cost aspects. This concern can be traced undoubtedly to Vitruvius, twenty-one centuries ago, and was a common basis with the Renaissance Engineering. Nevertheless, the optimization methods are indispensable tools for us in both research and practical matters, as the reader can check in our handbooks (e.g. Cervera Bravo, 2008). But it seems the Authors assume without criticism that new mathematical problems —easily gotten changing here or there a objective function or a constraint— have to be the main guide for research. At this point, we think that so many imaginative minds would be both economically and socially very most useful working in a well-stated problem as that posed by Maxwell and Michell and that we have named here as *structural design*, which furthermore offers a lot of open, unsolved questions up to date, and in simple and well-stated extensions on that class of problems as such that could be defined or interpreted in the realm of the Prager's-Shield optimality criteria (Rozvany, 1984:), but never disregarding important commponents in costs that differ as the layout changes and that have thus a main impact in *good* layouts.

⋆ Third: although for many optimization problems it could be that it has been proved that a sufficient and necessary optimality criterion exists, it is no the case of the so called Michell's problem: Michell's theorem points out a sufficient condition for the quantity of structure to be optimal for a Maxwell's problem, and the Authors point out no proof that this condition would be also a necessary one. Furthermore, for the Fig. 2 and 3 problems of Authors' Reply, in fact one Maxwell's problem with two different permissible spaces, their solutions are the better known —the Writers thank and congratulate them—, and the probability that they are optimal is very, very great (Rozvany, 2011:§4). But the fact is that while there be no proof that they fulfil Michell's criterion, the existence of the very solutions of the Authors is not a counterproof in any way of the following statement: *it could be the case that the optimal solution for a Michell's problem does not fulfil Michell's criterion*, that is to say, "the Michell theorem is a sufficient test for a framework to be optimal [for $\mathcal{Q}$], but maybe no necessary" (Vázquez Espí and Cervera Bravo, 2011:Abstract).

Anyway, we are happy with this opportunity to revisit our own old works—as Umberto Eco (1977) says "if your PhD was good enough, you return to it all along your life many times…and with pleasure". We are hopeful that a common ground can emerge of this discussion. The Writers will be grateful to Authors and prof. Rozvany for further comments and criticism. Let us end with a exciting quotation: "…it seems necessary to accept analysis as a science and design as an art. The procedures of analysis and design are differents; that the science is defective and the art often clumsy is another matter." (Cross, 1936:1406)

[3] "J'ai fini par comprendre que […] je n'avais pas affaire à des simples idiots, mais plutôt à des pleutres auxquels la responsabilité d'affirmer la vérité d'un essai, fait en leur présence, paraissait plus lourde à porter que celle de la vérification d'un calcul. Ils s'abritaient derrière leur matelas d'équations qu'ils jugeainent d'autant plus efficaces qu'elles étaient plus compliquées." (Freyssinet, 1966:313)

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