Boundary-layer growth on a rotating and accelerating sphere

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Boundary-layer growth on a sphere is studied when it is set into motion with constant acceleration and constant angular velocity, the latter being normal to the former. Analytic expressions are derived for the velocity components of the incompressible fluid in terms of a power series of the time of motion as well as for the skin friction.

On étudie la croissance de la couche limite de fluide sur une sphère mise en mouvement avec une accélération constante et une vitesse angulaire constante, la seconde étant perpendiculaire à la première. On établit des expressions analytiques pour les composantes de la vitesse du fluide incompressible en termes de série de puissances de la durée du mouvement, ainsi que pour le frottement pelliculaire.

[Traduit par la revue]

1. Introduction

Impulsive or accelerated motion of a body of revolution is of great importance in fluid mechanics, and it has been studied extensively. Illingworth (1) considered the boundary-layer growth on a body moving along its axis of symmetry and spinning about the same axis. Squire (2) treated the case of the impulsive translational motion of a three-dimensional body by employing the method of successive approximations to produce a time-series solution, which was also used by Goldstein and Rosenhead (3) in their description of the initial stages of motion of a cylinder. Wadhwa (4) determined the boundary-layer growth on a body of revolution that simultaneously has axial and angular acceleration. Karahalios (5) described the time-dependent formation of the boundary layer on a body of revolution having translation and spin, placing emphasis on the case of a sphere.

In the present work, a sphere is moving with constant acceleration while at the same time it starts to rotate impulsively about an axis normal to the former direction. The object is to study the boundary layer, derive analytic expressions for the skin friction, and also determine the conditions for which skin friction becomes zero.

2. Statement of the problem and governing equations

At $t = 0$, a sphere of radius R starts to rotate impulsively about, say, the z axis with a constant angular velocity Ω. Simultaneously, it starts to move along the x axis with velocity Q varying with time according to the equation $Q = bt$, where b is the acceleration (see Fig. 1). The frame of reference $O(x,y,z)$ moves forward with the sphere but does not rotate. We consider a local frame of reference (ξ,θ,ζ), where ξ is measured along a meridian curve from the pole, θ is the azimuthal angle, and ζ measures distance, along the outward normal, from the surface of the sphere. Hence, a point P is defined by the coordinates ξ, θ, and ζ. The scale factors are $h_\xi = 1$, $h_\theta = r$, and $h_\zeta = 1$, where r is the radius of the cross section through P, normal to the z axis.

Let u, υ, and w be the components of the fluid velocity. Then, the equations of motion of the boundary layer for incompressible flow are

$$[1] \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + \frac{\upsilon}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial \zeta} - \frac{\upsilon^2}{r}\frac{dr}{d\xi} - \nu\frac{\partial^2 u}{\partial \zeta^2} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial \xi} + \frac{V}{r}\frac{\partial U}{\partial \theta} - \frac{V^2}{r}\frac{dr}{d\xi}$$

$$[2] \quad \frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial \xi} + \frac{\upsilon}{r}\frac{\partial \upsilon}{\partial \theta} + w\frac{\partial \upsilon}{\partial \zeta} + \frac{u\upsilon}{r}\frac{dr}{d\xi} - \nu\frac{\partial^2 \upsilon}{\partial \zeta^2} = \frac{\partial V}{\partial t} + U\frac{\partial V}{\partial \xi} + \frac{V}{r}\frac{\partial V}{\partial \theta} + \frac{UV}{r}\frac{dr}{d\xi}$$

and the continuity equation is

$$[3] \quad \frac{\partial (ru)}{\partial \xi} + \frac{\partial \upsilon}{\partial \theta} + \frac{\partial (rw)}{\partial \zeta} = 0$$

Here U and V are the velocity components in the ξ and θ directions, respectively, that the main stream would have in an incident stream with velocity Q in the negative x direction,

$$U = -\frac{3}{2}Q\cos\frac{\xi}{R}, \qquad V = \frac{3}{2}Q\sin\frac{\xi}{R}\sin\theta$$

3. Approximate power-series solution

Equations [1] and [2] can be simplified if we assume a solution with successive approximations of u, υ, and w, each velocity component expressed in a power series of the time t. Initially, the effect of the impulsively started rotational motion is much more significant than that of the translational motion. Hence, if we put $\upsilon = \upsilon_0$ in [2], we get

$$[4] \quad \frac{\partial \upsilon_0}{\partial t} - \nu\frac{\partial^2 \upsilon}{\partial \zeta^2} = 0$$

By setting $\upsilon_0 = \Omega R(1 - g_0)\sin(\xi/R)$, where g_0 is a function of the similarity variable

$$\eta = \frac{\zeta}{2(\nu t)^{1/2}}$$

we obtain from [4],

$$g_0'' + 2\eta g_0' = 0$$

where primes denote differentiation with respect to η, and the boundary conditions are

$$g_0(0) = 0, \qquad g_0(\infty) = 1$$

The solution of [4] is

$$g_0 = \operatorname{erf}\eta = 2a\int_0^\eta e^{-s^2}\,ds$$

where $a = \pi^{-1/2}$. By setting $u = u_1$ in [1], we find that [1] takes the form

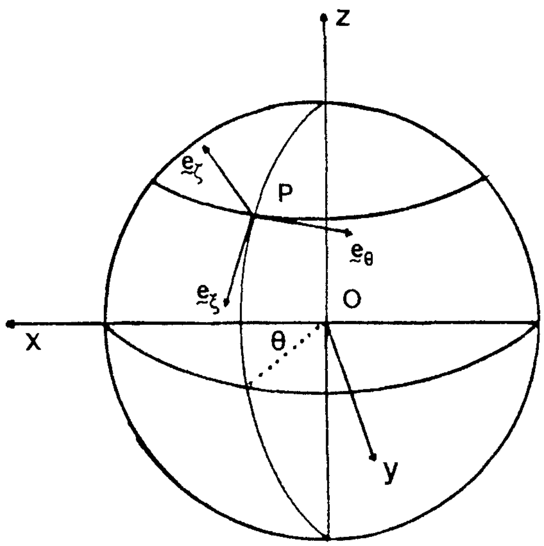

FIG. 1. Notation.

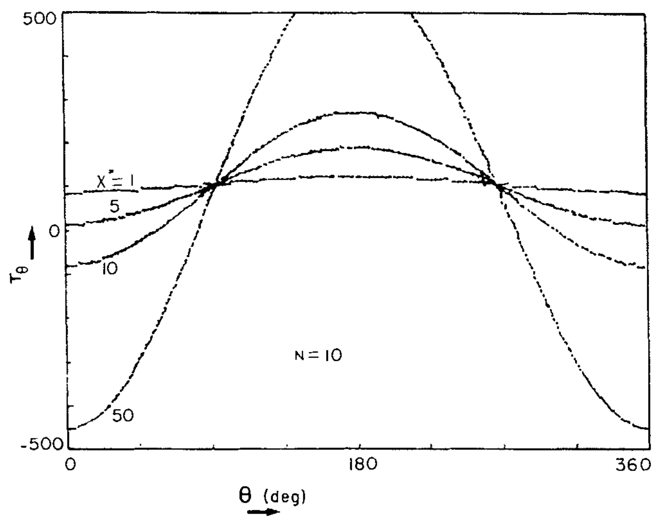

FIG. 2. Variation of skin friction with the azimuthal angle θ at the equatorial plane. N = 10.

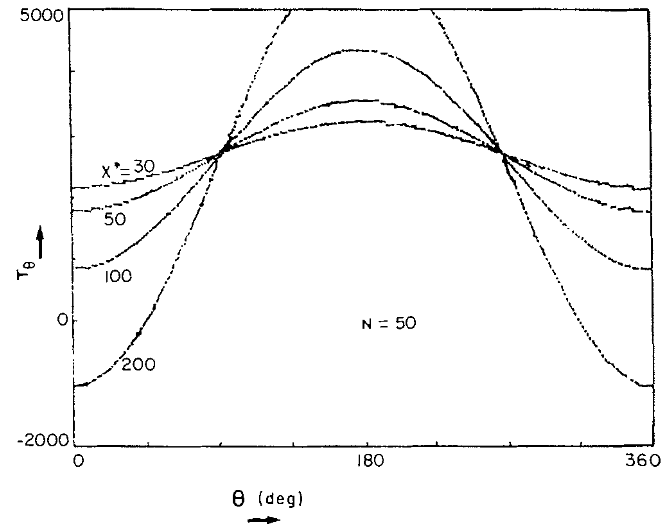

FIG. 3. Variation of skin friction with the azimuthal angle θ at the equatorial plane. N = 50.

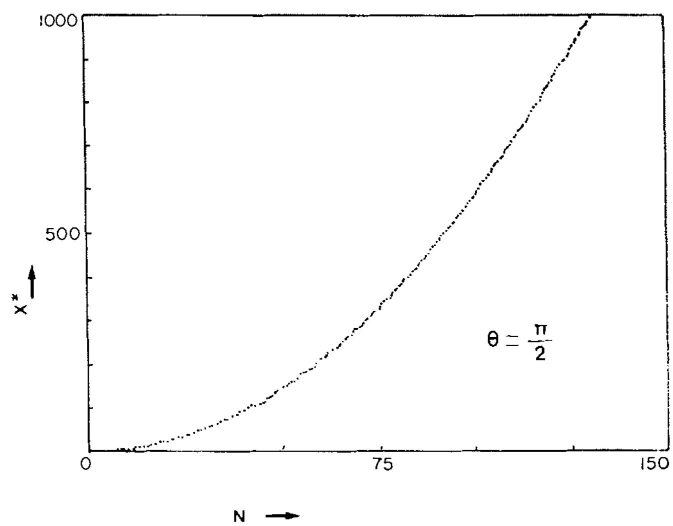

FIG. 4. Relation between distance x^* and number N of revolutions before the skin friction becomes zero.

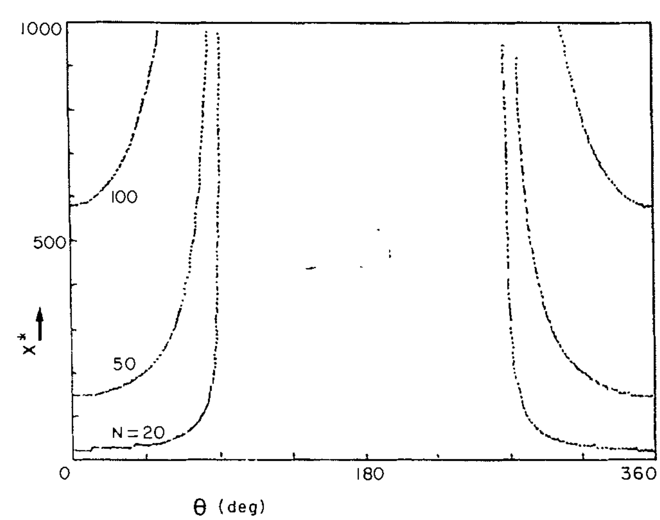

FIG. 5. Variation of distance x^* with angle θ for zero skin friction.

$$[5] \qquad \nu\frac{\partial^2 u_1}{\partial \zeta^2} - \frac{\partial u_1}{\partial t} = a_1 + a_2(1 - E)^2$$

where

$$E = \text{erf}\ \eta, \qquad a_1 = \frac{3}{2}b \cos\frac{\xi}{R}$$

and

$$a_2 = -\Omega^2 R \cos\frac{\xi}{R} \sin\frac{\xi}{R}$$

We seek a solution of [5] of the form

$$u_1 = t(a_1 f_1 + a_2 f_2)$$

Hence, the following equations are derived:

$$[6] \qquad f_1'' + 2\eta f_1' - 4f_1 = 4$$

$$[7] \qquad f_2'' + 2\eta f_2' - 4f_2 = 4(1 - E)^2$$

with boundary conditions $f_1(0) = 0$, $f_1(\infty) = 1$, $f_2(0) = 0$, and $f_2(\infty) = 0$. The solutions of [6] and [7] are

[8] $$f_1 = 2\eta^2 - E(1 + 2\eta^2) - 2\eta a e^{-\eta^2}$$

[9] $$f_2 = 2\eta^2E^2 + E[4(a^2 - 1)\eta^2 + 2a^2 + 4\eta a e^{-\eta^2}] + 2\eta^2(1 - 2a^2) - 2a^2 + 4ae^{-\eta^2}(a^2 - 1)\eta + 2a^2e^{-2\eta^2}$$

We now put $\upsilon = \upsilon_0 + \upsilon_1$ and $u = u_1$ in [2]. Therefore,

$$\nu\frac{\partial^2\upsilon_1}{\partial\zeta^2} - \frac{\partial\upsilon_1}{\partial t} = -\frac{\partial V}{\partial t}$$

or, if we use the expression $\upsilon_1 = t\frac{\partial V}{\partial t}g_1$,

$$g_1'' + 2\eta g_1' - 4g_1 = 4$$

with the boundary conditions $g_1(0) = 0$ and $g_1(\infty) = 1$ and the solution given by [8].

We can calculate w_1, the first approximation to w, from [3] as

$$w_1 = -2(\nu t)^{1/2}\frac{1}{r}t\left[\frac{\partial(a_1r)}{\partial\xi} + \frac{\partial^2V}{\partial\theta\partial t}h_1 + \frac{\partial(a_2r)}{\partial\xi}h_2\right]$$

where

$$h_1 = \int_0^\eta f_1(s)\,ds, \qquad h_2 = \int_0^\eta f_2(s)\,ds$$

For the second approximation we put $u = u_1 + u_2$, $\upsilon = \upsilon_0 + \upsilon_1 + \upsilon_2$, and $w = w_1 + w_2$ in [1] and [2] to obtain

[10] $$\nu\frac{\partial^2u_2}{\partial\zeta^2} - \frac{\partial u_2}{\partial t} = t2\Omega\frac{\partial V}{\partial t}\frac{dr}{d\xi}(1 - E)g_1$$

[11] $$\nu\frac{\partial^2\upsilon_2}{\partial\zeta^2} - \frac{\partial\upsilon_2}{\partial t} = t[b_1f_1(1 - E) + b_2f_2(1 - E) + b_3h_1E' + b_4h_2E']$$

where

$$b_1 = 2a_1\Omega\cos\frac{\xi}{R} + \frac{3}{2}\Omega b\sin\frac{\xi}{R}\cos\theta$$

$$b_2 = 2a_2\Omega\cos\frac{\xi}{R}$$

$$b_3 = \frac{3}{2}\Omega b\left(\cos^2\frac{\xi}{R} - \sin^2\frac{\xi}{R}\right) + \frac{3}{2}\Omega b\sin\frac{\xi}{R}\cos\theta$$

$$b_4 = -\Omega^3R\left(2\cos^2\frac{\xi}{R}\sin\frac{\xi}{R} - \sin^3\frac{\xi}{R}\right)$$

Putting

$$u_2 = t^2 2\Omega\frac{\partial V}{\partial t}\frac{dr}{d\xi}f_{21}$$

in [9], one finds

[12] $$f_{21}'' + 2\eta f_{21}' - 8f_{21} = 4(1 - E)g_1$$

with boundary conditions $f_{21}(0) = f_{21}(\infty) = 0$.

In the same manner, by setting

$$v_2 = t^2\sum_{i=1}^{4} b_ig_i$$

in [11], we derive

[13] $$g_{2i}'' + 2\eta g_{2i}' - 8g_{2i} = 4s_{2i}, \qquad i = 1,2,3,4$$

with the boundary conditions $g_{2i}(0) = g_{2i}(\infty) = 0$, where $s_{21} = (1 - E)f_1$, $s_{22} = f_2(1 - E)$, $s_{23} = h_1ae^{-\eta^2}$, and $s_{24} = 2h_2ae^{-\eta^2}$. Equation [13] has a solution of the form

[14] $$g_{2i} = E^3X_{2i} + E^2Y_{2i} + EZ_{2i} + T_{2i} + m\gamma_{2i} + q\delta_{2i}$$

where $m = 4\eta^4 + 12\eta^2 + 3$ and $q = mE + a(4\eta^3 + 10\eta)e^{-\eta^2}$ are the complementary functions, while the solution of [12] is $f_{21} = g_{21}$. Expressions for h_1, h_2, X_{2i}, etc. are presented in the Appendix.

The skin friction components are given by

$$\tau_\xi = \frac{1}{2}\rho\left(\frac{\nu}{t}\right)^{1/2}\left[\left.\frac{\partial u_1}{\partial\eta}\right|_0 + \left.\frac{\partial u_2}{\partial\eta}\right|_0\right]$$

$$\tau_\theta = \frac{1}{2}\rho\left(\frac{\nu}{t}\right)^{1/2}\left[\left.\frac{\partial\upsilon_0}{\partial\eta}\right|_0 + \left.\frac{\partial\upsilon_1}{\partial\eta}\right|_0 + \left.\frac{\partial\upsilon_2}{\partial\eta}\right|_0\right]$$

Substitution of the values of u_1, u_2, etc. into these expressions gives

[15] $$\tau_\xi = \frac{1}{2}\rho\left(\frac{\nu}{t}\right)^{1/2}\cos X\,t\left[\frac{3}{2}bf_1'(0) - \Omega^2R\sin X\,f_2'(0) + 3\Omega b\sin X\sin\theta\,tf_{21}'(0)\right]$$

[16] $$\tau_\theta = \frac{1}{2}\rho\left(\frac{\nu}{t}\right)^{1/2}\left\{-\Omega R\sin X\,g_0'(0) + t\frac{3}{2}b\sin X\sin\theta\,g_1'(0) + t^2\left[\left(\frac{3}{2}\Omega b\cos^2X + \frac{3}{2}\Omega b\sin X\cos\theta\right)g_{21}'(0) - 2\Omega^3R\cos^2X\sin X\,g_{22}'(0) + \left(3\Omega b\cos 2X + \frac{3}{2}\Omega b\sin X\cos\theta\right)g_{23}'(0) - \Omega^2R(2\cos^2X - \sin^2X)\sin X\,g_{24}'(0)\right]\right\}$$

where $X = \xi/R$ and the values of $g'(0)$, $f'(0)$, etc. are tabulated in the Appendix.

4. Results and discussion

For $X = \pi/2$, [15] gives $\tau_\xi = 0$, a result also occurring in the case of impulsive translational motion with spin (5), while [16] gives, for the same value of X,

[17] $$\tau_\theta = \frac{1}{2}\rho\left(\frac{\nu}{t}\right)^{1/2}\Omega R\left(-1.12838 - 1.07752\frac{x^*}{N}\sin\theta - 17.57335x^*\cos\theta - 1.12459x^* + 1.08447N^2\right)$$

where N is the number of revolutions and $x^* = \frac{1}{2}bt^2/R$. It is, therefore, deduced that τ_θ becomes zero for

[18] $$x^* = \frac{1.08447N^2 - 1.12838}{(1.12459 + 17.57335\cos\theta)N + 1.07752\sin\theta}N$$

Hence, for $\theta = \pi/2$, and for large values of N, the number x^* of radii travelled by the sphere before τ_θ becomes zero is large.

The previous results are illustrated in Figs. 2–5. In Figs. 2 and 3, the skin-friction component τ_θ, given by [17], has been plotted against the azimuthal angle θ. The comparison of the two diagrams shows that when the number of revolutions is increased, then the nondimensional distance x^* that has to be travelled for τ_θ to become zero or negative increases. In Fig. 4, the nondimensional distance x^* versus the number N of revolutions ([18]) has been sketched at a given azimuthal angle, $\theta = \pi/2$. It can be deduced that the variation of x^* with N is almost proportional. Evidently, when the number of revolutions is large while the time of motion is small, it is not possible for τ_θ to become zero. Finally, in Fig. 5, x^* versus θ ([18]) has been plotted for various values of N. This drawing shows again that an increase in the value of N also drastically increases the value of x^* for which τ_θ could be zero.

5. Conclusions

When a sphere is simultaneously moving with constant acceleration and rotating with constant angular velocity, the rotational motion affects the length of time it takes for the skin friction to become equal to zero. In particular, an increase in the angular velocity increases the distance that has to be travelled by the sphere for the skin friction to reach a value of zero. If this is combined with the fact that the forward velocity approaches an upper limit, this suggests that practically speaking, one could avoid zero friction in accelerated translational motion by giving the sphere a large angular velocity normal to its direction of motion.

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3. S. Goldstein and L. Rosenhead. Proc. Cambridge Philos. Soc. **32**, 392 (1936).
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Appendix

The coefficients of the error function in [14] are as follows:

$$h_1 = \frac{2a}{3} + \frac{2}{3}\eta^3 - \frac{2}{3}ae^{-\eta^2}(\eta^2 + 1) - E\left(\frac{2}{3}\eta^3 + \eta\right)$$

$$h_2 = (A + 1)\eta + \frac{2A}{3}\eta^3 + \left[\frac{2B}{3}(1 + \eta^2) - 2\right]ae^{-\eta^2} - \frac{2Ba}{3} + 2a - \frac{2}{3}\eta a^2 e^{-2\eta^2} + E\left[(B - 2)\eta + \frac{2B}{3}\eta^3 + \frac{2}{3}(1 - 2\eta^2)ae^{-\eta^2}\right] - \frac{2}{3}E^2\eta^3 - \frac{2\sqrt{2}}{3}a \operatorname{erf} \sqrt{2}\eta$$

where $A = 2a^2 - 1$ and $B = 2(1 - a^2)$.

g_{21}:

$$X_{21} = 0, \qquad Y_{21} = -2\eta^2 - 1 + \frac{1}{3}(4\eta^4 + 12\eta^2 + 3)$$

$$Z_{21} = 4\eta^2 + \frac{3}{2} + \frac{2}{3}ae^{-\eta^2}(4\eta^3 + 5\eta)$$

$$T_{21} = -2\eta^2 - \frac{1}{2} + \frac{10}{3}\eta ae^{-\eta^2} + \frac{4}{3}a^2e^{-2\eta^2}(1 + \eta^2)$$

$$\gamma_{21} = \frac{1}{6} - \frac{4}{9}a^2, \qquad \delta_{21} = \frac{4}{9}a^2 - \frac{1}{2}$$

$$f_{21} = g_{21}$$

g_{22}:

$$X_{22} = -\frac{1}{36}(20\eta^4 - 12\eta^2 - 3)$$

$$Y_{22} = \eta^4\left(\frac{5}{4} - \frac{8a^2}{3}\right) - \eta^2(1 + 4a^2) - \frac{1}{4} + \frac{1}{6}ae^{-\eta^2}(7\eta - 10\eta^3)$$

$$Z_{22} = 2\eta^2(3 - 4a^2) + \frac{3}{2} - 4a^2 + \frac{1}{3}ae^{-\eta^2}[2(5 - 8a^2)\eta^3 - (7 + 20a^2)\eta] + \frac{1}{3}a^2e^{-2\eta^2}(4 - 5\eta^2)$$

$$T_{22} = -\frac{1}{36}a^3e^{-3\eta^2}(54\eta^3 + 173\eta) + \frac{1}{3}a^2e^{-2\eta^2}[2(1 - 4a^2)\eta^2 - 4(3 + a^2)] + \frac{4}{3}(2 - 5a^2)\eta ae^{-\eta^2} + \eta^2a^2e^{-2\eta^2} + 2(2a^2 - 1)\eta^2 + 2a^2 - \frac{1}{2} - \frac{3\sqrt{3}}{8}a^2(4\eta^4 + 12\eta^2 + 3) \operatorname{erf} \sqrt{3}\eta$$

$$\gamma_{22} = \frac{2}{9}a^2(4a^2 - 1) + \frac{1}{6}, \qquad \delta_{22} = -\frac{17}{18} + \frac{8}{9}a^2(1 - a^2) + \frac{3\sqrt{3}}{8}a^2$$

g_{23}:

$$X_{23} = 0, \qquad Y_{23} = -\frac{1}{24}(4\eta^4 + 12\eta^2 + 3)$$

$$Z_{23} = -\frac{1}{12}ae^{-\eta^2}(2\eta^3 + 5\eta)$$

$$T_{23} = -\frac{1}{60}a^2e^{-2\eta^2}(10\eta^3 + 5\eta + a)$$

$$\gamma_{23} = \frac{4a^2}{45}, \qquad \delta_{23} = \frac{1}{24} - \frac{4a^2}{45}$$

g_{24}:

$$X_{24} = -\frac{1}{18}(4\eta^4 + 12\eta^2 + 3)$$

$$Y_{24} = -\frac{1}{6}ae^{-\eta^2}(6\eta^3 + 11\eta) + \frac{1}{6}(a^2 + 2)(4\eta^4 + 12\eta^2 + 3)$$

$$Z_{24} = \frac{2}{3}ae^{-\eta^2}[(3 + a^2)\eta^3 + (3 + 5a^2)\eta] - \frac{4}{3}a^2e^{-2\eta^2}(\eta^2 + 1)$$

$$T_{24} = \frac{1}{180}a^3e^{-3\eta^2}(378\eta^3 + 971\eta) + \frac{4}{3}a^2e^{-2\eta^2}(\eta^2 + 1) - ae^{-\eta^2}\left[\frac{1 - 2a^2}{3}\eta^3 + \frac{1}{6}(1 - 10a^2)\eta - \frac{8a}{15}(1 + 2a^2)\right.$$
$$\left. + \frac{8\sqrt{2}}{15}a \operatorname{erf} \sqrt{2}\eta\right] + \frac{21\sqrt{3}}{40}a^2 \operatorname{erf} \sqrt{3}\eta(4\eta^4 + 12\eta^2 + 3)$$

$$\gamma_{24} = \frac{4a^2}{45}(7 - 4a^2), \qquad \delta_{24} = -\frac{1}{24} + \frac{1}{90}a^2(32a^2 - 71)$$

The first derivatives of the functions f and g at $\eta = 0$ are

$$g_0' = 2a = 1.12838, \qquad g_1' = -4a = -2.25676$$

$$g_{21}' = \frac{19}{3}a + \frac{64}{9}a^2 - 8 = -2.16325$$

$$g_{22}' = -\frac{49}{9}a - \frac{28}{3}a^3 - \frac{128}{9}a^5 + 6\sqrt{3}\,a^3 = -3.69453$$

$$g_{23}' = -\frac{271}{180}a^2 + \frac{2}{3} = 0.18743$$

$$g_{24}' = -\frac{35}{18}a + \frac{855}{45}a^3 + \frac{256}{45}a^5 - 16a^3\frac{21\sqrt{3}}{40} = 0.02747$$

$$f_1' = g_1', \qquad f_2' = -8(1 - a^2) + 4a = -0.82006$$

$$f_{21}' = g_{21}'$$