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A first approach to an axiomatic model of multi-measures

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Abstract

In this paper, we establish an axiomatic model of multi-measures, capturing some classes of measures studied in the fuzzy sets literature, where they are applied to only one or two arguments. Specifically, we look at multi-measures for determining the degrees of incompatibility and supplementarity between any number of fuzzy sets. Additionally, we introduce multi-measures for ranking their opposite properties, that is, compatibility and unsupplementarity.

Key words: Lattice, aggregation function, t-norm, t-conorm, incompatibility multi-measure, compatibility multi-measure, supplementarity multi-measure, unsupplementarity multi-measure.
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1 Introduction

Traditionally, a measure has been defined as a real-valued set function on Boolean algebras or on σ-algebras of classical sets and, more recently, on fuzzy sets (like vagueness or ambiguity measures, etc.). However there are other measurable properties that it makes sense to apply on not only one but also two or more sets. Several works concerning how to measure the contradiction, the incompatibility or the supplementarity, among other properties, between two fuzzy sets have already been published in this respect. Nevertheless, it might be worth studying these properties for more than two fuzzy sets, since we are left wondering how incompatible or supplementary a set of n fuzzy premises is. This is why multi-argument functions should be addressed.

Of the multi-argument functions, aggregation functions [3, 6, 9] deserve a special mention. Their purpose is to combine several inputs into a single output. The nature of the inputs depends on the context: they can be degrees of membership in fuzzy sets, degrees of preference, and so on. They play a significant role in many applications, which has driven their constant growth. They are useful in different areas like multi-criteria decision making, group decision making, fuzzy logic and rule-based systems, etc.

Other works have dealt with other kinds of multi-argument functions. In this respect, consider Martín and Mayor's recent papers on multi-distances [10] introducing a way to measure "how separated" the points of a collection of more than two elements are.

The main aim of this paper is to establish a general axiomatic model of multi-argument measures in order to capture some measures that we have examined previously and, in particular, incompatibility measures [7, 8] and supplementarity measures [8].

The remainder of the paper is organized as follows. Section 2 provides the axioms or conditions of multi-measures and describes some examples. Section 3 focuses on two multi-measures on the lattice of fuzzy sets with the order induced by the order of real numbers, whereas Section 4 studies two multi-measures on the lattice again composed of fuzzy sets but with the order induced by the reverse order of the real numbers. Finally, the paper ends with a summary of the results, and some future lines of research.

2 Multi-argument measures on lattices

Let $\mathcal{L} = (L, \leq_L, 0_L, 1_L)$ (or simply $(L, \leq_L)$) be a bounded lattice [4, 5] whose minimum and maximum elements are denoted by 0_L and 1_L, respectively. For each $n \in \mathbb{N}$, let us consider the set

$$L^n = \{(a_1, \ldots, a_n) \mid a_i \in L, \forall i \in \{1, \ldots, n\}\}$$

and the order relation $\leq_n$ induced by $\leq_L$, that is, given $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots b_n) \in L^n$,

$$\bar{a} \leq_n \bar{b} \iff a_i \leq_L b_i, \forall i \in \{1, \ldots, n\}.$$

We have that L^n with the order relation $\leq_n$ is also a bounded lattice, whose minimum element is $0_{L^n} = (0_L, \overset{n)}{\ldots}, 0_L)$ and whose maximum element is $1_{L^n} = (1_L, \overset{n)}{\ldots}, 1_L)$, and we say that $\mathcal{L}^n = (L^n, \leq_n, 0_{L^n}, 1_{L^n})$ is induced by $\mathcal{L}$. Moreover, if $\mathcal{L}$ is complete, then $\mathcal{L}^n$ is also complete.

Definition 2.1. Let $\mathcal{L} = (L, \leq_L, 0_L, 1_L)$ be a bounded lattice. Also, for each $n \in \mathbb{N}$, let $\mathcal{L}^n = (L^n, \leq_n, 0_{L^n}, 1_{L^n})$ be the lattice induced by $\mathcal{L}$. Consider the bounded and complete lattice of real numbers $([0, 1], \leq, 0, 1)$. A map $M : \bigcup_{n \in \mathbb{N}} L^n \to [0, 1]$ is said to be a *multi-argument $\leq_L$-measure or multi-measure on* $(L, \leq_L)$ (or, simply, on L if there is not likely to be confusion) if, for each $n \in \mathbb{N}$, the function restriction of M to L^n, $M|_{L^n}$, satisfies:

i) *(Boundary conditions)* $M(0_{L^n}) = 0$ and $M(1_{L^n}) = 1$.

ii) (*Monotony condition*) For all $\bar{a}, \bar{b} \in L^n$ such that $\bar{a} \leq_n \bar{b}$, $M(\bar{a}) \leq M(\bar{b})$; that is, for each n, M is increasing with respect to the orders of the lattices $(L^n, \leq_n)$ and $([0,1], \leq)$.

Moreover,

iii) M is *increasing with respect to the argument* n or n*-increasing* if $M(a_1, \ldots, a_n) \leq M(a_1, \ldots, a_n, a_{n+1})$ holds for all $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n, a_{n+1} \in L$.

iv) M is *decreasing with respect to the argument* n or n*-decreasing* if $M(a_1, \ldots, a_n) \geq M(a_1, \ldots, a_n, a_{n+1})$ holds for all $n \in \mathbb{N}$ and for all $a_1, \ldots, a_n, a_{n+1} \in L$.

Remark 2.2. If M is a multi-measure on $(L, \leq_L)$, note that:

1. M is n-increasing if and only if $M(\bar{a}) \leq M(\bar{a}, \bar{b})$ holds for all $n, m \in \mathbb{N}$ and for all $\bar{a} = (a_1, \ldots, a_n) \in L^n$ and $\bar{b} = (b_1, \ldots, b_m) \in L^m$, where $(\bar{a}, \bar{b})$ denotes $(a_1, \ldots, a_n, b_1, \ldots, b_m) \in L^{n+m}$.

2. M is n-decreasing if and only if $M(\bar{a}) \geq M(\bar{a}, \bar{b})$ holds for all $n, m \in \mathbb{N}$ and for all $\bar{a} \in L^n$ and $\bar{b} \in L^m$.

Example 2.3. Let X be a non-empty and finite set, and let $\mathcal{P}(X)$ denote the set of all subsets of X, that is, the power set of X. Consider the bounded lattice $(\mathcal{P}(X), \subseteq, \emptyset, X)$, which is, in fact, a Boolean algebra, and let us define two multi-argument measures on $(\mathcal{P}(X), \subseteq)$.

a) Let $M_I : \bigcup_{n \in \mathbb{N}} \mathcal{P}(X)^n \to [0,1]$ be the map defined for each $(A_1, \ldots, A_n) \in \mathcal{P}(X)^n$ as

$$M_I(A_1, \ldots, A_n) = \frac{|A_1 \cap \cdots \cap A_n|}{|X|},$$

where $|A|$ means cardinal of the set A. Then M_I satisfies:

i) $M_I(\emptyset, \ldots, \emptyset) = 0$ for all n-tuple of empty sets; and $M_I(X, \ldots, X) = 1$ for all n-tuple of elements X.

ii) $M_I(\bar{A}) \leq M_I(\bar{B})$ holds for all $n \in \mathbb{N}$ and for all $\bar{A} = (A_1, \ldots, A_n), \bar{B} = (B_1, \ldots, B_n) \in \mathcal{P}(X)^n$ such that $A_i \subseteq B_i$ for each $i \in \{1, \ldots, n\}$.

iv) $M_I(A_1, \ldots, A_n, A_{n+1}) \leq M_I(A_1, \ldots, A_n)$ for all $n \in \mathbb{N}$ and for all $A_1, \ldots, A_n, A_{n+1} \in \mathcal{P}(X)$.

Hence, M_I is an n-decreasing multi-argument $\subseteq$-measure on $\mathcal{P}(X)$; it provides a measure of the size of the intersection of any finite family of subsets of X.

b) Let $M_U : \bigcup_{n \in \mathbb{N}} \mathcal{P}(X)^n \to [0,1]$ be the map defined for each $(A_1, \dots, A_n) \in \mathcal{P}(X)^n$ as

$$M_U(A_1, \dots, A_n) = \frac{|A_1 \cup \dots \cup A_n|}{|X|}.$$

Then, M_U also satisfies axioms i and ii of the multi-argument measure and, moreover, axiom iii, therefore M_U is an n-increasing multi-argument $\subseteq$-measure on $\mathcal{P}(X)$; it provides a measure of the size of the union of any finite family of subsets of X. $\lhd$

Example 2.4. Any aggregation function is a multi-argument measure on $([0,1], \leq)$. Indeed, recall that an aggregation function [3, 6, 9] is a map $\mathcal{A} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ such that

1. $\mathcal{A}(0, \dots, 0) = 0$ and $\mathcal{A}(1, \dots, 1) = 1$.
2. $\mathcal{A}(a) = a$ for all $a \in [0,1]$.
3. For all $n \in \mathbb{N}$ and for all $\bar{a} = (a_1, \dots, a_n), \bar{b} = (b_1, \dots, b_n) \in [0,1]^n$ such that $a_i \leq b_i$ with $i = 1, \dots, n$, $\mathcal{A}(\bar{a}) \leq \mathcal{A}(\bar{b})$ holds. $\lhd$

Thus, the occurrence of symmetric aggregation functions suggests the following definition. We denote $S_n = \{\pi : \{1, \dots, n\} \to \{1, \dots, n\} \mid \pi \text{ is a bijection}\}$, that is, S_n is the set of *permutations* of $\{1, \dots, n\}$.

Definition 2.5. A multi-argument measure M on a bounded lattice $(L, \leq_L)$ is *symmetric* if, for each $n \in \mathbb{N}$, the function $M|_{L^n}$ is symmetric, that is, $M(a_1, \dots, a_n) = M(a_{\pi(1)}, \dots, a_{\pi(n)})$ holds for any $\pi \in S_n$ and for any $(a_1, \dots, a_n) \in L^n$.

Example 2.6. The maps M_I and M_U defined in Example 2.3 are both symmetric multi-measures on $\mathcal{P}(X)$. $\lhd$

Example 2.7. The aggregation functions Max, Min : $\bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$, defined as $\mathrm{Max}(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\}$ and $\mathrm{Min}(a_1, \dots, a_n) = \min\{a_1, \dots, a_n\}$ for each $(a_1, \dots, a_n) \in [0,1]^n$, are symmetric multi-measures on $([0,1], \leq)$.

Nevertheless, for each $k \in \mathbb{N} \setminus \{1\}$, the function $\mathcal{A}_k : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$, defined as

$$\mathcal{A}(a_1, \dots, a_n) = a_1 \prod_{i=2}^{n} a_i^k$$

for each $(a_1, \dots, a_n) \in [0,1]^n$, is a non-symmetric multi-measure on $([0,1], \leq)$

In what follows, we study the particular instance of multi-measures on fuzzy set lattices: if X is a non-empty set, the set of all fuzzy sets on X is identified with the set of all its membership functions, $L = [0,1]^X$, with an order relation $\preceq$ such that $\mathcal{L} = ([0,1]^X, \preceq, \mu_\wedge, \mu_\vee)$ is a bounded lattice, where $\mu_\wedge$ and $\mu_\vee$ denote the minimum and maximum elements, respectively. For each $n \in \mathbb{N}$, if $\mathbb{I}_n$ denotes $[0,1]^n$, lattice $\mathcal{L}$ induces the bounded lattice $\mathcal{L}^n = (\mathbb{I}_n^X, \preceq_n, \bar{\mu}_\wedge, \bar{\mu}_\vee)$ as $\mathbb{I}_n^X = ([0,1]^n)^X = [0,1]^X \times \overset{n)}{\cdots} \times [0,1]^X$. Thus, a multi-measure on $([0,1]^X, \preceq)$ is a multi-argument function $M : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ where i) $M(\bar{\mu}_\wedge) = 0$ and $M(\bar{\mu}_\vee) = 1$; and ii) $M(\bar{\mu}) \leq M(\bar{\sigma})$ holds for all $\bar{\mu}, \bar{\sigma} \in \mathbb{I}_n^X$ such that $\bar{\mu} \preceq_n \bar{\sigma}$.

3 Multi-measures on lattice $([0,1]^X, \leq)$

In this section, we deal with multi-measures on $([0,1]^X, \preceq)$ when $\preceq$ is the order induced by the usual order on the real line, that is, if $\mu, \sigma \in [0,1]^X$, $\mu \preceq \sigma$ if and only if $\mu(x) \leq \sigma(x)$ for all $x \in X$; and we naturally denote this by $\mu \leq \sigma$. In this case, $\mu_\wedge = \mu_\emptyset$ and $\mu_\vee = \mu_X$, and thus $\mathcal{L} = ([0,1]^X, \leq, \mu_\emptyset, \mu_X)$.

Let us look at two types of multi-measures on $([0,1]^X, \leq)$: multi-measures that evaluate how compatible a set of fuzzy sets is and multi-measures that evaluate how supplementary the set is. Remember that, in classical logic, two statements are compatible if they can both be true at the same time. As we can identify a statement on a universe X with the set of elements of X that satisfy that statement, we can translate this concept to set theory: $A, B \subset X$ are compatible if $A \cap B \neq \emptyset$. On the other hand, supplementarity can, in a sense, be understood as a symmetric property of incompatibility: A an B are supplementary if $A \cup B = X$. These concepts are extended to the fuzzy set framework and studied in the following sections.

3.1 Compatibility multi-measures on fuzzy sets

In order to define compatible fuzzy sets, we need a function that models the intersection of fuzzy sets, that is, a t-norm. Remember that a *t-norm* [1, 2, 11] is a binary aggregation function T on the unit interval $[0,1]$, which is commutative, associative, monotone increasing with respect to the usual order on the real line, and whose neutral element is 1. As in the classical case, given a t-norm T, two fuzzy sets on X, or their membership functions $\mu, \sigma \in [0,1]^X$, are T-compatible if $T(\mu, \sigma) \neq \mu_\emptyset$, where $T(\mu, \sigma) \in [0,1]^X$ is defined by $T(\mu, \sigma)(x) = T(\mu(x), \sigma(x))$ for each $x \in X$. This can be generalized similarly as follows.

Definition 3.1. Given $X \neq \emptyset$ and a t-norm T, then

1. $\{\mu\} \subset [0,1]^X$ is said to be *T-compatible* if $\mu \neq \mu_\emptyset$.
2. If $n > 1$, $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ is said to be *T-compatible* if $T(\mu_1, \ldots, \mu_n) \neq \mu_\emptyset$.

The following definition determines the conditions that a multi-argument function must satisfy to fittingly assign a degree of compatibility to every $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$.

Definition 3.2. Let T be a t-norm and $X \neq \emptyset$. A function $\mathcal{C}_T : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ is a *T-compatibility multi-measure* on $[0,1]^X$ if it is a symmetric and n-decreasing multi-measure on $([0,1]^X, \leq)$ satisfying $\mathcal{C}_T(\mu_1, \ldots, \mu_n) = 0$, provided that $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ satisfies $T(\mu_1, \ldots, \mu_n) = \mu_\emptyset$.

Remark 3.3. Note that $\mathcal{C}_T$ is a T-compatibility multi-measure on $[0,1]^X$ if and only if it satisfies the following axioms:

- c.i) $\mathcal{C}_T(\mu_X, \overset{n)}{\ldots}, \mu_X) = 1$ for each $n \in \mathbb{N}$.
- c.ii) If $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ is not T-compatible, then $\mathcal{C}_T(\mu_1, \ldots, \mu_n) = 0$.
- c.iii) $\mathcal{C}_T(\mu_1, \ldots, \mu_n) = \mathcal{C}_T(\mu_{\pi(1)}, \ldots, \mu_{\pi(n)})$ holds for all $\pi \in S_n$ and $\mu_1, \ldots, \mu_n \in [0,1]^X$.
- c.iv) If $\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n \in [0,1]^X$ satisfy $\mu_i \leq \sigma_i$ for all $i \in \{1, \ldots, n\}$, then $\mathcal{C}_T(\mu_1, \ldots, \mu_n) \leq \mathcal{C}_T(\sigma_1, \ldots, \sigma_n)$.
- c.v) $\mathcal{C}_T(\mu_1, \ldots, \mu_{n+1}) \leq \mathcal{C}_T(\mu_1, \ldots, \mu_n)$ holds for all $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_{n+1} \in [0,1]^X$.

As is well known, if $\varphi \in \mathcal{A}([0,1]) = \{\psi : [0,1] \to [0,1] \,|\, \psi \text{ is an increasing bijection}\}$ and T is a t-norm, then $T^\varphi : [0,1]^2 \to [0,1]$ defined for each $(a,b) \in [0,1]^2$ by $T^\varphi(a,b) = \varphi^{-1}(T(\varphi(a), \varphi(b)))$ is also a t-norm, and we say that it is the t-norm φ-conjugated with T. One of the main t-norms is the so-called Lukasiewicz t-norm that is defined for each $(a,b) \in [0,1]^2$ by $T_L(a,b) = \max\{0, a+b-1\}$. Moreover, if T_L^φ is the t-norm φ-conjugated with T_L, then $T_L^\varphi(a_1, \ldots, a_n) = \varphi^{-1}(\max\{0, \varphi(a_1) + \cdots + \varphi(a_n) - (n-1)\})$ for all $a_1, \ldots, a_n \in [0,1]$. For more details about t-norms see [1, 2].

Given $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$, since $T_L^\varphi(\mu_1, \ldots, \mu_n) \neq \mu_\emptyset$ if and only if there exists $x \in X$ such that $\sum_{i=1}^n \varphi(\mu_i(x)) > n-1$, then a natural way to measure the T_L^φ-compatibility of $\{\mu_1, \ldots, \mu_n\}$ could be to conveniently take into account the difference between $\sum_{i=1}^n \varphi(\mu_i(x))$ and $n-1$, as follows.

Proposition 3.4. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$. The function $\mathcal{C}_L^\varphi : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ defined for each $\bar{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{I}_n^X$, by*

$$\mathcal{C}_L^\varphi(\bar{\mu}) = \max\left\{0, \sup_{x\in X} \sum_{i=1}^n \varphi(\mu_i(x)) - (n-1)\right\}$$

is a T_L^φ-compatibility multi-measure on $[0,1]^X$

Proof. First, note that $\mathcal{C}_L^\varphi$ is well defined; indeed, since $\sum_{i=1}^n \varphi(\mu_i(x)) \leq n$ then $\sum_{i=1}^n \varphi(\mu_i(x)) - (n-1) \leq 1$.

Axioms c.i and c.iii follow straightforwardly from the definition of $\mathcal{C}_L^\varphi$. Regarding axiom c.ii, if $\mu = \mu_\emptyset$, then $\mathcal{C}_L^\varphi(\mu) = 0$; if $\bar{\mu} \in \mathbb{I}_n^X$ satisfies $T_L^\varphi(\mu_1, \ldots, \mu_n) = \mu_\emptyset$, then $\sum_{i=1}^n \varphi(\mu_i(x)) \leq n-1$ for all $x \in X$ and thus $\mathcal{C}_L^\varphi(\bar{\mu}) = 0$. $\mathcal{C}_L^\varphi$ is monotonic increasing: given $\bar{\mu} = (\mu_1, \ldots, \mu_n), \bar{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{I}_n^X$ such that $\mu_i \leq \sigma_i$ for every $i \in \{1, \ldots, n\}$, as φ is increasing, then $\sum\limits_{i=1}^n \varphi(\mu_i(x)) \leq \sum\limits_{i=1}^n \varphi(\sigma_i(x))$ and thus $\mathcal{C}_L^\varphi(\bar{\mu}) \leq \mathcal{C}_L^\varphi(\bar{\sigma})$.

Finally, $\mathcal{C}_L^\varphi$ is n-decreasing: given $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n, \mu_{n+1} \in [0,1]^X$, it follows that $\mathcal{C}_L^\varphi(\mu_1, \ldots, \mu_{n+1}) \leq \mathcal{C}_L^\varphi(\mu_1, \ldots, \mu_n)$ as $\sum\limits_{i=1}^{n+1} \varphi(\mu_i(x)) \leq \sum\limits_{i=1}^{n} \varphi(\mu_i(x)) + 1$. □

3.2 Supplementary multi-measures on fuzzy sets

As applies in the case of compatible fuzzy sets, we need a tool to model the union of fuzzy sets in order to define supplementary fuzzy sets, and t-conorms are suitable functions for this purpose. Remember that a *t-conorm* [1, 2] is a binary aggregation function S on the unit interval $[0,1]$, which is commutative, associative, monotone increasing with respect to the usual order on the real line, whose neutral element is 0. Given a t-conorm S, two fuzzy sets on X or their membership functions $\mu, \sigma \in [0,1]^X$ are S-supplementary [8] if $S(\mu, \sigma) = \mu_X$, where $S(\mu, \sigma) \in [0,1]^X$ is defined by $S(\mu, \sigma)(x) = S(\mu(x), \sigma(x))$ for each $x \in X$. This can be generalized similarly as follows.

Definition 3.5. Given $X \neq \emptyset$ and a t-conorm S, then

1. $\{\mu\} \subset [0,1]^X$ is said to be S-*supplementary* if $\mu = \mu_X$.
2. If $n > 1$, $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ is said to be S-*supplementary* if $S(\mu_1, \ldots, \mu_n) = \mu_X$.

Definition 3.6. Let S be a t-conorm and $X \neq \emptyset$. A function $\mathcal{S}_S : \bigcup_{n \in \mathbb{N}} \mathbb{I}_n^X \to [0,1]$ is an S-*supplementarity multi-measure* on $[0,1]^X$ if it is a symmetric and n-increasing multi-measure on $([0,1]^X, \leq)$ satisfying $\mathcal{S}_S(\mu_1, \ldots, \mu_n) = 0$ provided that $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ is not S-supplementary.

Remark 3.7. Note that $\mathcal{S}_S$ is an S-supplementarity multi-measure on $[0,1]^X$ if and only if:

- s.i $\mathcal{S}_S(\mu_X, \overset{n)}{\ldots}, \mu_X) = 1$ for each $n \in \mathbb{N}$.
- s.ii If $\{\mu_1, \ldots, \mu_n\} \subset [0,1]^X$ is not S-supplementary, then $\mathcal{S}_S(\mu_1, \ldots, \mu_n) = 0$.
- s.iii $\mathcal{S}_S(\mu_1, \ldots, \mu_n) = \mathcal{S}_S(\mu_{\pi(1)}, \ldots, a_{\pi(n)})$ holds for all $\pi \in S_n$ and $\mu_1, \ldots, \mu_n \in [0,1]^X$.
- s.iv If $\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_n \in [0,1]^X$ satisfy $\mu_i \leq \sigma_i$ for all $i \in \{1, \ldots, n\}$, then $\mathcal{S}_S(\mu_1, \ldots, \mu_n) \leq \mathcal{S}_S(\sigma_1, \ldots, \sigma_n)$.

s.v $\mathcal{S}_S(\mu_1,\ldots,\mu_n) \leq \mathcal{S}_S(\mu_1,\ldots,\mu_{n+1})$ holds for all $n \in \mathbb{N}$ and $\mu_1,\ldots,\mu_{n+1} \in [0,1]^X$.

As in the t-norm case, if $\varphi \in \mathcal{A}([0,1])$ and S is a t-conorm, the function defined for each $(a,b) \in [0,1]^2$ by $S^\varphi(a,b) = \varphi^{-1}(S(\varphi(a),\varphi(b))$ is also a t-conorm, the t-conorm φ-conjugated with S. The Lukasiewicz t-norm has a dual t-conorm defined for each $(a,b) \in [0,1]$ by $S_L(a,b) = \min\{1, a+b\}$. Moreover, if S_L^φ is the t-conorm φ-conjugated with S_L, then $S_L^\varphi(a_1,\ldots,a_n) = \varphi^{-1}(\min\{1, \varphi(a_1)+\cdots+\varphi(a_n)\})$ for all $a_1,\ldots,a_n \in [0,1]$.

To find a way to measure the S_L^φ-supplementarity of $\{\mu_1,\ldots,\mu_n\} \subset [0,1]^X$, note that $S_L^\varphi(\mu_1,\ldots,\mu_n) = \mu_X$ if and only if $\sum_{i=1}^n \varphi(\mu_i(x)) \geq 1$ for all $x \in X$; hence we can fittingly use the difference between 1 and $\sum_{i=1}^n \varphi(\mu_i(x))$. So, we can prove the following result.

Proposition 3.8. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$. Let $\mathcal{S}_L^\varphi : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ be the function defined:*

1. *For each $\mu \in [0,1]^X$, by* $\mathcal{S}_L^\varphi(\mu) = \begin{cases} 1 & \text{if } \mu = \mu_X \\ 0 & \text{if } \mu \neq \mu_X, \end{cases}$
2. *For each $\bar{\mu} = (\mu_1,\ldots,\mu_n) \in \mathbb{I}_n^X$ with $n > 1$, by*

$$\mathcal{S}_L^\varphi(\bar{\mu}) = \min\left\{1, \max\left\{0, \inf_{x\in X} \sum_{i=1}^n \varphi(\mu_i(x)) - 1\right\}\right\}.$$

Then $\mathcal{S}_L^\varphi$ is an S_L^φ-supplementarity multi-measure on $[0,1]^X$.

Remark 3.9. We have that $\mathcal{S}_L^\varphi(\mu_1,\mu_2) = \max\{0, \inf_{x\in X}(\varphi(\mu_1(x)) + \varphi(\mu_2(x))) - 1\}$ for each $(\mu_1,\mu_2) \in \mathbb{I}_2^X$, thus the restriction of $\mathcal{S}_L^\varphi$ to $\mathbb{I}_2^X = [0,1]^X \times [0,1]^X$ is an S_L^φ-supplementarity measure regarding the definition reported in [8].

4 Multi-measures on lattice $([0,1]^X, \geq)$

In this section, we deal with multi-measures on $([0,1]^X, \preceq)$ when $\preceq$ is the order induced by the usual reverse order of the real line, that is, if $\mu, \sigma \in [0,1]^X$, $\mu \preceq \sigma$ if and only if $\mu(x) \geq \sigma(x)$ for all $x \in X$; and we naturally denote this by $\mu \geq \sigma$. In this case, $\mu_\wedge = \mu_X$ and $\mu_\vee = \mu_\emptyset$, where, for all $x \in X$, $\mu_X(x) = 1$ and $\mu_\emptyset(x) = 0$, then the lattice $\mathcal{L}$ is $([0,1]^X, \geq, \mu_X, \mu_\emptyset)$.

Let us look at two types of multi-measures on $([0,1]^X, \geq)$: multi-measures that evaluate how incompatible a set of fuzzy sets is and multi-measures that evaluate how unsupplementary the set is, where the concepts of incompatibility and unsupplementarity are opposite to compatibility and supplementarity, respectively. That is, given a t-norm T and a t-conorm S, $\{\mu_1,\ldots,\mu_n\} \subset \mathbb{I}_n^X$ is T-*incompatible* if it is not T-compatible, and it is S-*unsupplementary* if it is not S-supplementary.

4.1 Incompatibility multi-measures on fuzzy sets

Although the concepts of compatibility and incompatibility are opposites, the negation of a compatibility measure cannot be used to assign degrees of incompatibility. Indeed, let $\mathcal{C}_T$ be a non-trivial T-compatibility multi-measure, that is, at least it takes a value $a \in (0,1)$, and let N be any strong negation [12] (i.e., $N : [0,1] \to [0,1]$ is an involutive and decreasing bijection); if a is achieved on $\bar{\mu} \in \mathbb{I}_n^X$, then $0 < \mathcal{C}_T(\bar{\mu}) = a < 1$, and it follows from axiom ii of Remark 3.3 that $\bar{\mu}$ is T-compatible, and also $0 = N(1) < N(\mathcal{C}_T(\bar{\mu})) < N(0) = 1$ holds. Thus $N(\mathcal{C}_T(\bar{\mu}))$ cannot be considered as a degree of the T-incompatibility of $\bar{\mu}$ since the incompatibility measure of compatible sets should be 0. Therefore, it makes sense to propose a mathematical model for the study of the incompatibility.

Definition 4.1. Let T be a t-norm and $X \neq \emptyset$. A function $\mathcal{I}_T : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ is a *T-incompatibility multi-measure* on $[0,1]^X$ if it is a symmetric and n-increasing multi-measure on $([0,1]^X, \geq)$ satisfying $\mathcal{I}_T(\mu_1,\dots,\mu_n) = 0$, provided that $\{\mu_1,\dots,\mu_n\} \subset [0,1]^X$ is T-compatible.

Remark 4.2. Note that $\mathcal{I}_T$ is a T-incompatibility multi-measure on fuzzy sets on X if and only if:

- ic.i $\mathcal{I}_T(\mu_\emptyset, \overset{n)}{\dots}, \mu_\emptyset) = 1$ for each $n \in \mathbb{N}$.
- ic.ii If $\{\mu_1,\dots,\mu_n\} \subset [0,1]^X$ is T-compatible, then $\mathcal{I}_T(\mu_1,\dots,\mu_n) = 0$.
- ic.iii $\mathcal{I}_T(\mu_1,\dots,\mu_n) = \mathcal{I}_T(\mu_{\pi(1)},\dots,\mu_{\pi(n)})$ holds for all $\pi \in S_n$ and $\mu_1,\dots,\mu_n \in [0,1]^X$.
- ic.iv If $\mu_1,\dots,\mu_n,\sigma_1,\dots,\sigma_n \in [0,1]^X$ satisfy $\mu_i \leq \sigma_i$ for all $i \in \{1,\dots,n\}$, then $\mathcal{I}_T(\sigma_1,\dots,\sigma_n) \leq \mathcal{I}_T(\mu_1,\dots,\mu_n)$.
- ic.v $\mathcal{I}_T(\mu_1,\dots,\mu_n) \leq \mathcal{I}_T(\mu_1,\dots,\mu_{n+1})$ holds for all $n \in \mathbb{N}$ and $\mu_1,\dots,\mu_{n+1} \in [0,1]^X$.

As in the case of compatibility, if T_L^φ is the t-norm φ-conjugated with the Lukasiewicz t-norm, taking into account that $T_L^\varphi(\mu_1,\dots,\mu_n) = \mu_\emptyset$ if and only if $\sum_{i=1}^n \varphi(\mu_i(x)) \leq n-1$, we can find a T_L^φ-incompatibility multi-measure by fittingly considering the difference between $n-1$ and $\sum_{i=1}^n \varphi(\mu_i(x))$, and thus we can prove the following result.

Proposition 4.3. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$. Let $\mathcal{I}_L^\varphi : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ be the function defined:*

1. *For each $\mu \in [0,1]^X$, by* $\mathcal{I}_L^\varphi(\mu) = \begin{cases} 1 & \text{if } \mu = \mu_\emptyset \\ 0 & \text{if } \mu \neq \mu_\emptyset, \end{cases}$
2. *For each $\bar{\mu} = (\mu_1,\dots,\mu_n) \in \mathbb{I}_n^X$ with $n > 1$, by*

$$\mathcal{I}_L^\varphi(\bar{\mu}) = \min\left\{1, \max\left\{0, (n-1) - \sup_{x\in X} \sum_{i=1}^n \varphi(\mu_i(x))\right\}\right\}.$$

Then $\mathcal{I}_L^\varphi$ is a T_L^φ-incompatibility multi-measure on $[0,1]^X$.

Remark 4.4. We have that $\mathcal{I}_L^\varphi(\mu,\mu_2) = \max\{0, 1 - \sup_{x\in X}(\varphi(\mu_1(x)) + \varphi(\mu_2(x)))\}$ for each $(\mu_1,\mu_2) \in \mathbb{I}_2^X$, thus $\mathcal{I}_L^\varphi|_{[0,1]^X\times[0,1]^X}$ is a T_L^φ-incompatibility measure regarding the definition reported in [7].

4.2 Unsupplementarity multi-measures on fuzzy sets

As for incompatibility, although unsupplementary is the opposite to supplementary, it is not possible to assign degrees of unsupplementarity by means of a negation of a supplementarity multi-measure. Hence we establish a mathematical model to measure the unsupplementarity property.

Definition 4.5. Let S be a t-conorm and $X \neq \emptyset$. A function $\mathcal{U}_T : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ is an *S-unsupplementarity multi-measure* on $[0,1]^X$ if it is a symmetric and n-decreasing multi-measure on $([0,1]^X, \geq)$ satisfying $\mathcal{U}_S(\mu_1,\ldots,\mu_n) = 0$, provided that $\{\mu_1,\ldots,\mu_n\} \subset [0,1]^X$ is S-supplementary.

Remark 4.6. Note that $\mathcal{U}_S$ is an S-supplementarity multi-measure on $[0,1]^X$ if and only if:

us.i $\mathcal{U}_S(\mu_\emptyset,\overset{n)}{\ldots},\mu_\emptyset) = 1$ for each $n \in \mathbb{N}$.

us.ii If $\{\mu_1,\ldots,\mu_n\} \subset [0,1]^X$ is S-supplementary, then $\mathcal{U}_S(\mu_1,\ldots,\mu_n) = 0$.

us.iii $\mathcal{U}_S(\mu_1,\ldots,\mu_n) = \mathcal{U}_S(\mu_{\pi(1)},\ldots,\mu_{\pi(n)})$ holds for all $\pi \in S_n$ and $\mu_1,\ldots,\mu_n \in [0,1]^X$.

us.iv If $\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n \in [0,1]^X$ satisfy $\mu_i \leq \sigma_i$ for all $i \in \{1,\ldots,n\}$, then $\mathcal{U}_S(\sigma_1,\ldots,\sigma_n) \leq \mathcal{U}_S(\mu_1,\ldots,\mu_n)$.

us.v $\mathcal{U}_S(\mu_1,\ldots,\mu_{n+1}) \leq \mathcal{U}_S(\mu_1,\ldots,\mu_n)$ holds for all $n \in \mathbb{N}$ and $\mu_1,\ldots,\mu_{n+1} \in [0,1]^X$.

As in the case of supplementarity, if S_L^φ is the t-conorm φ-conjugated with the Lukasiewicz t-conorm, taking into account that $S_L^\varphi(\mu_1,\ldots,\mu_n) = \mu_X$ if and only if $\sum_{i=1}^n \varphi(\mu_i(x)) \geq 1$ for all $x \in X$, we can use the difference between $\sum_{i=1}^n \varphi(\mu_i(x))$ and 1 to assign degrees of S_T^φ-unsupplementarity. So, we can prove the following result.

Proposition 4.7. *Let $X \neq \emptyset$ and $\varphi \in \mathcal{A}([0,1])$. The function $\mathcal{U}_L^\varphi : \bigcup_{n\in\mathbb{N}} \mathbb{I}_n^X \to [0,1]$ defined for each $\bar{\mu} = (\mu_1,\ldots,\mu_n) \in \mathbb{I}_n^X$, by*

$$\mathcal{U}_L^\varphi(\bar{\mu}) = \max\left\{0, 1 - \inf_{x\in X} \sum_{i=1}^n \varphi(\mu_i(x))\right\}$$

is an S_L^φ-unsupplementarity multi-measure on $[0,1]^X$