OBTAINING CONTRADICTION MEASURES ON INTUITIONISTIC FUZZY SETS FROM FUZZY CONNECTIVES

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In a previous paper,[1] we proposed an axiomatic model for measuring self-contradiction in the framework of Atanassov fuzzy sets. This way, contradiction measures that are semicontinuous and completely semicontinuous, from both below and above, were defined. Although some examples were given, the problem of finding families of functions satisfying the different axioms remained open.

The purpose of this paper is to construct some families of contradiction measures firstly using continuous t-norms and t-conorms, and secondly by means of strong negations. In both cases, we study the properties that they satisfy. These families are then classified according the different kinds of measures presented in the above paper.

Keywords: Fuzzy negations; t-norms and t-conorms; intuitionistic fuzzy negations; weakly self-contradictory sets; contradictory intuitionistic fuzzy sets; contradiction measures; semicontinuous measures; continuous measures.

1. Introduction

The significance, in both theoretical and applied fields, of the failure of many fuzzy logics to comply with the Principle of Non-Contradiction greatly justifies research into contradiction. For instance, we cannot obviate the problem, in inference processes, of obtaining consequences that are contradictory, both with each other or with some hypotheses.

Consequently, Trillas *et al.* addressed the study of contradiction in the framework of fuzzy logic introducing, with respect to some fuzzy negation, the concepts of self-contradictory set, and weakly self-contradictory set.[2,3] Along the same lines, the study of contradiction in the framework of intuitionistic fuzzy sets was initiated in Ref. 4. The need to speak not only of contradiction but also of degrees of contradiction was later raised in Ref. 5, where some functions were considered for the purpose of determining those degrees. Finally, an axiomatic model for measuring how contradictory an intuitionistic fuzzy set is was proposed in Ref. 1, establishing the definition of contradiction measure, as well as several concepts for modeling its continuity.

The aim of this paper is to construct and study some families of functions that are compliant with the stated model, using continuous t-norms and t-conorms, and strong negations. The paper is organized as follows. After setting out the necessary background in this introduction, we present two families of functions using continuous t-norms and t-conorms in Sec. 2; and we prove that these families satisfy some properties of continuity. In Sec. 3, we revisit the functions defined in Ref. 5 with the aim of studying what kind of continuity they display.

1.1. *Requirements on intuitionistic fuzzy sets*

As is well known, given a fuzzy predicate $\mathbf{A}$ in a universe of discourse $X \neq \emptyset$, an intuitionistic fuzzy set (IFS) associated with $\mathbf{A}$ (Ref. 6) is a set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ are called the membership and non-membership functions, respectively, and such that, for all $x \in X$, $\mu_A(x)+\nu_A(x) \leq 1$.

According to Goguen,[7] an IFS could be considered as an L-fuzzy set, where the lattice L is the set $\mathbb{L} = \{(\alpha_1, \alpha_2) \in [0,1]^2 : \alpha_1 + \alpha_2 \leq 1\}$ with the partial order $\leq_{\mathbb{L}}$ defined as follows: given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{L}$, $\boldsymbol{\alpha} \leq_{\mathbb{L}} \boldsymbol{\beta}$ if and only if $\alpha_1 \leq \beta_1$ and $\alpha_2 \geq \beta_2$. $(\mathbb{L}, \leq_{\mathbb{L}})$ is a bounded and complete lattice in which the least element is $0_{\mathbb{L}} = (0,1)$, and the greatest element is $1_{\mathbb{L}} = (1,0)$. So, an IFS A is an $\mathbb{L}$-fuzzy set whose $\mathbb{L}$-membership function $\chi^A \in \mathbb{L}^X = \{\chi : X \to \mathbb{L}\}$ is defined for each $x \in X$ as $\chi^A(x) = (\mu_A(x), \nu_A(x))$.

The order $\leq_{\mathbb{L}}$ of $\mathbb{L}$ naturally induces a partial order on $\mathbb{L}^X$ as follows. Given $\chi^A, \chi^B \in \mathbb{L}^X$, we say that $\chi^A \leq_{\mathbb{L}} \chi^B$ if and only if $\chi^A(x) \leq_{\mathbb{L}} \chi^B(x)$ for all $x \in X$. Thus $(\mathbb{L}^X, \leq_{\mathbb{L}})$ is a bounded and complete lattice in which the least and greatest elements are, respectively, $\chi^{0_{\mathbb{L}}}$ and $\chi^{1_{\mathbb{L}}}$ defined by $\chi^{0_{\mathbb{L}}}(x) = 0_{\mathbb{L}}$ and $\chi^{1_{\mathbb{L}}}(x) = 1_{\mathbb{L}}$ for all $x \in X$.

Let us recall that a non-increasing function $\mathcal{N} : \mathbb{L} \to \mathbb{L}$ is an intuitionistic fuzzy negation (IFN) if $\mathcal{N}(0_{\mathbb{L}}) = 1_{\mathbb{L}}$ and $\mathcal{N}(1_{\mathbb{L}}) = 0_{\mathbb{L}}$ hold. Moreover, $\mathcal{N}$ is a strong IFN if the equality $\mathcal{N}(\mathcal{N}(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$ holds for all $\boldsymbol{\alpha} \in \mathbb{L}$. Bustince *et al.* introduced in Ref. 8 intuitionistic fuzzy generators, which can be used to build IFNs, and Deschrijver *et al.* focused on this problem.[9,10] They proved that any strong IFN $\mathcal{N}$ is characterized by a strong negation N (that is, a non-increasing function such that $N(0) = 1$, $N(1) = 0$ and $N^2 = id$) by means of the formula $\mathcal{N}(\alpha_1, \alpha_2) = (N(1 - \alpha_2), 1 - N(\alpha_1))$, for all $(\alpha_1, \alpha_2) \in \mathbb{L}$.

1.2. *Requirements on contradiction measures*

As in the fuzzy case, an IFS A, or alternatively $\chi^A \in \mathbb{L}^X$, is said to be self-contradictory with respect to some strong IFN $\mathcal{N}$, or $\mathcal{N}$-contradictory for short, if $\chi^A(x) \leq_{\mathbb{L}} (\mathcal{N} \circ \chi^A)(x)$ for all $x \in X$, where χ^A is the $\mathbb{L}$-membership function of A. Also, A, or χ^A, is said to be self-contradictory, or just contradictory, if there exists a strong negation $\mathcal{N}$ such that A is $\mathcal{N}$-contradictory. Note that this definition of contradictory set matches, in the fuzzy case, the one given in Ref. 2 for weakly self-contradictory set; and this is the property that we consider to be measured in this paper.

Now, let us present the concept of contradiction measure that was introduced in Ref. 1. Out of consistency with classical set theory, where the only contradictory set is the empty set, the value of a function suitable for measuring contradiction should be the highest for $\chi^{0_{\mathbb{L}}}$ and zero for the other IFSs representing classical sets (that is, IFS taking only $\mathbb{L}$-values $0_{\mathbb{L}}$ and $1_{\mathbb{L}}$).

A second requisite for a measure of contradiction pertains to sets $\chi^A = (\mu_A, \nu_A)$ such that $\operatorname*{Inf}_{x \in X} \nu_A(x) = 0$. Note that if there is $x_0 \in X$ such that $\chi^A(x_0) \in (0,1] \times \{0\}$, then if $\mathcal{N}$ is any strong IFN and N its associated fuzzy negation, $N(\mu_A(x_0)) + \nu_A(x_0) < 1$ holds. So A is non-$\mathcal{N}$-self-contradictory,[5] and hence A is non-self-contradictory. Thus, if the range of χ^A touches segment $(0,1] \times \{0\}$, the contradiction measure of A must be 0. Moreover, we consider that the measure of A should be 0 if its range infinitely approaches the above segment, that is, if $\operatorname*{Inf}_{x \in X} \nu_A(x) = 0$. This requirement is also supported by the following: if A is a non-self-contradictory IFS, then its non-membership function ν_A satisfies $\operatorname*{Inf}_{x \in X} \nu_A(x) = 0$.[5] If $\chi^A = (\mu_A, \nu_A) \in \mathbb{L}^X$ satisfies $\operatorname*{Inf}_{x \in X} \nu_A(x) = 0$, it will be designated an $\mathbb{L}$-*normal set.*

The third requirement aims to determine when one IFS is more contradictory than another. Let us suppose that χ^A and χ^B are $\mathcal{N}$-contradictory sets and that $\chi^A \leq_{\mathbb{L}} \chi^B$, then $\chi^A \leq_{\mathbb{L}} \chi^B \leq_{\mathbb{L}} \mathcal{N} \circ \chi^B \leq_{\mathbb{L}} \mathcal{N} \circ \chi^A$, as $\mathcal{N}$ is non-increasing. Thus a greater degree of contradiction should apparently be assigned to χ^A than to χ^B, since χ^A is "farther away" than χ^B from being non-$\mathcal{N}$-contradictory. Therefore anti-monotonicity is a suitable requirement.

These observations suggest the following definition.

Definition 1.[1] Let $X \neq \emptyset$ be a universe of discourse, a function $\mathcal{C} : \mathbb{L}^X \to [0,1]$ is a *measure of contradiction* on the set of all IFS, or on $\mathbb{L}^X$, if it satisfies the following statements:

(c.i) $\mathcal{C}(\chi^{0_{\mathbb{L}}}) = 1$.
(c.ii) If $\chi \in \mathbb{L}^X$ is $\mathbb{L}$-normal, then $\mathcal{C}(\chi) = 0$.
(c.iii) Antimonotonicity: If $\chi^A, \chi^B \in \mathbb{L}^X$ with $\chi^A \leq_{\mathbb{L}} \chi^B$, then $\mathcal{C}(\chi^A) \geq \mathcal{C}(\chi^B)$.

The set of all measures of contradiction on $\mathbb{L}^X$ was denoted by $\mathcal{CM}(\mathcal{IF}(X))$, or more concisely $\mathcal{CM}(\mathbb{L}^X)$.

Furthermore, as the above definition does not guarantee that the degrees of contradiction vary gradually, other axioms were also introduced to model the continuity of the contradiction measures, from both below and above, as follows.

Definition 2.[1] Let $X \neq \emptyset$ be a universe of discourse, a contradiction measure $\mathcal{C} : \mathbb{L}^X \to [0,1]$ is said to be

• *completely semicontinuous from below* if the following axiom is satisfied:

(c.iv) For any indexed family $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$,

$$\operatorname*{Inf}_{i\in\mathcal{I}} \mathcal{C}(\chi^i) = \mathcal{C}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)$$

holds, $\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i \in \mathbb{L}^X$ being defined for all $x \in X$ as $\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)(x) = \operatorname*{Sup}_{i\in\mathcal{I}} \chi^i(x)$.

• *completely semicontinuous from above* if the following axiom is satisfied:

(c.v) For any indexed family $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X \setminus \mathbb{L}_0^X$, where $\mathbb{L}_0^X = \{\chi \in \mathbb{L}^X\colon \chi \text{ is } \mathbb{L}\text{-normal}\}$,

$$\operatorname*{Sup}_{i\in\mathcal{I}} \mathcal{C}(\chi^i) = \mathcal{C}\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right)$$

holds, $\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i \in \mathbb{L}^X$ being defined for all $x \in X$ as $\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right)(x) = \operatorname*{Inf}_{i\in\mathcal{I}} \chi^i(x)$.

Observe that if $\chi^i = (\mu_i, \nu_i)$ for all $i \in \mathcal{I}$ then $\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i = \left(\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i, \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i\right)$ and $\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i = \left(\operatorname*{Inf}_{i\in\mathcal{I}} \mu_i, \operatorname*{Sup}_{i\in\mathcal{I}} \nu_i\right)$.

The set of all contradiction measures that are completely semicontinuous from below on $\mathbb{L}^X$ was denoted by $\mathcal{CM}_{csc}(\mathbb{L}^X)$, and the set of all contradiction measures that are completely semicontinuous from above, by $\mathcal{CM}^{csc}(\mathbb{L}^X)$.

Nevertheless, axioms (c.iv) and (c.v) of complete continuity would appear to be too restrictive because they are not satisfied by some contradiction measures with gradually changing values, such as the functions proposed in Ref. 5. For this reason, we established other weaker axioms using semilattices. Before we state these axioms, let us recall that[11,12] a set $S \subset \mathbb{L}^X$ is an upper semilattice, or semilattice from below, if $\operatorname{Sup}\{\chi^A, \chi^B\} \in S$ for all $\chi^A, \chi^B \in S$; and $S \subset \mathbb{L}^X$ is a lower semilattice, or semilattice from above, if $\operatorname{Inf}\{\chi^A, \chi^B\} \in S$ for all $\chi^A, \chi^B \in S$.

Definition 3.[1] Let $X \neq \emptyset$ be a universe of discourse, a contradiction measure $\mathcal{C} : \mathbb{L}^X \to [0,1]$ is said to be:

• *semicontinuous from below* if the following axiom is satisfied:

(c.vi) For any semilattice from below $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$, where $\mathcal{I}$ is an arbitrary set, the following holds

$$\operatorname*{Inf}_{i\in\mathcal{I}} \mathcal{C}(\chi^i) = \mathcal{C}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right);$$

• *semicontinuous from above* if the following axiom is satisfied:
(c.vii) For any semilattice from above $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X \setminus \mathbb{L}_0^X$, where $\mathcal{I}$ is an arbitrary set, the following holds

$$\operatorname*{Sup}_{i\in\mathcal{I}} \mathcal{C}(\chi^i) = \mathcal{C}\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right).$$

The set of all contradiction measures that are semicontinuous from below on $\mathbb{L}^X$ was denoted by $\mathcal{CM}_{sc}(\mathbb{L}^X)$, and the set of all contradiction measures that are semicontinuous from above, by $\mathcal{CM}^{sc}(\mathbb{L}^X)$.

Remark 1. Note that:
(a) In Ref. 1, it was shown that any axiom of continuity established in Definitions 2 and 3 implies antimonotonicity;
(b) The following relations hold among the sets of contradiction measures:

$$\emptyset \neq \mathcal{CM}_{csc}(\mathbb{L}^X) \subsetneq \mathcal{CM}_{sc}(\mathbb{L}^X) \subsetneq \mathcal{CM}(\mathbb{L}^X)$$

$$\emptyset \neq \mathcal{CM}^{csc}(\mathbb{L}^X) \subsetneq \mathcal{CM}^{sc}(\mathbb{L}^X) \subsetneq \mathcal{CM}(\mathbb{L}^X).$$

2. Obtaining Contradiction Measures from t-Norms and t-Conorms

In this section, we are going to construct contradiction measures using continuous t-norms and t-conorms. First let us recall some necessary definitions and results related to t-norms and t-conorms.[13,14] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a t-norm if it is commutative, associative, non-decreasing in each variable and satisfies $T(1,\alpha) = \alpha$ for all $\alpha \in [0,1]$; analogously, $S : [0,1] \times [0,1] \to [0,1]$ is a t-conorm if it is commutative, associative, non-decreasing in each variable and satisfies $T(0,\alpha) = \alpha$ for all $\alpha \in [0,1]$. Moreover, T is said to be Archimedean if $T(\alpha,\alpha) < \alpha$ for all $\alpha \in]0,1[$, and S is Archimedean if $S(\alpha,\alpha) > \alpha$ for all $\alpha \in]0,1[$.

If T is a continuous and Archimedean t-norm, then either it is strict (if $\beta < \gamma$ then $T(\alpha,\beta) < T(\alpha,\gamma)$ for all $\alpha \neq 0$) or it is nilpotent (for all $\alpha \neq 1$ there exists $n \in \mathbb{N}$ such that 0 is obtained by operating n times T on α, that is, $T(\alpha, T(\alpha, (...T(\alpha,\alpha)...) = 0)$. In the second case, there exists an automorphism $\varphi : [0,1] \to [0,1]$ such that $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$, where $W(\alpha,\beta) = \mathrm{Max}\{0, \alpha+\beta-1\}$ for all $(\alpha,\beta) \in [0,1]^2$ defines the Łukasiewicz t-norm. Analogously, if S is a continuous and Archimedean t-conorm, then either it is strict or it is nilpotent (for all $\alpha \neq 0$ there exists $n \in \mathbb{N}$ such that 1 is obtained by operating n times S on α). Again, in the second case, there exists an automorphism φ such that $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$, where $W^*(\alpha,\beta) = \mathrm{Min}\{1, \alpha+\beta\}$ for all $(\alpha,\beta) \in [0,1]^2$ defines the dual t-conorm of W.

Now we can introduce and study families of measures associated with t-norms and with t-conorms.

Theorem 1. *Let T be a continuous t-norm and let $f : [0,1] \to [0,1]$ be a continuous and non-increasing function such that $f(0) = 1$ and $f(1) = 0$. Then the* *function* $\mathcal{C}_{T,f} : \mathbb{L}^X \to [0,1]$ *defined for each* $\chi = (\mu, \nu) \in \mathbb{L}^X$ *by*

$$\mathcal{C}_{T,f}(\chi) = \operatorname*{Inf}_{x \in X} T(f(\mu(x)), \nu(x))$$

is a contradiction measure that is semicontinuous from below.

Proof. Taking into account Remark 1, we only need to confirm the following axioms:
(c.i) $\mathcal{C}_{T,f}(\chi^{0_{\mathbb{L}}}) = T(f(0), 1) = 1$.
(c.ii) If $\chi \in \mathbb{L}^X$ is $\mathbb{L}$-normal, then $\mathcal{C}_{T,f}(\chi) = \operatorname*{Inf}_{x \in X} T(f(\mu(x)), \nu(x)) \leq \operatorname*{Inf}_{x \in X} \nu(x) = 0$ as $T \leq \text{Min}$.
(c.vi) Let $\{\chi^i\}_{i \in \mathcal{I}} \subset \mathbb{L}^X$ be a semilattice from below. We must prove that $\mathcal{C}_{T,f}\left(\operatorname*{Sup}_{i \in \mathcal{I}} \chi^i\right) = \operatorname*{Inf}_{i \in \mathcal{I}} \mathcal{C}_{T,f}(\chi^i)$, which is equivalent to proving that

$$\operatorname*{Inf}_{x \in X} T\left(f\left(\operatorname*{Sup}_{i \in \mathcal{I}} \mu_i(x)\right), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right) = \operatorname*{Inf}_{x \in X} \operatorname*{Inf}_{i \in \mathcal{I}} T(f(\mu_i(x)), \nu_i(x)) \,. \tag{1}$$

We find that

$$T\left(\operatorname*{Inf}_{i \in \mathcal{I}} f\left(\mu_i(x)\right), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right) = \operatorname*{Inf}_{i \in \mathcal{I}} T(f(\mu_i(x)), \nu_i(x)) \,, \tag{2}$$

thus (1) will be demonstrated, since (2) is equivalent to

$$T\left(f\left(\operatorname*{Sup}_{i \in \mathcal{I}} \mu_i(x)\right), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right) = \operatorname*{Inf}_{i \in \mathcal{I}} T(f(\mu_i(x)), \nu_i(x)) \quad \forall x \in X,$$

because f is a continuous and non-increasing function.

As T is non-decreasing, then

$$T\left(\operatorname*{Inf}_{i \in \mathcal{I}} f\left(\mu_i(x)\right), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right) \leq T(f(\mu_j(x)), \nu_j(x))$$

hold for each $x \in X$ and for all $j \in \mathcal{I}$.

For each $x \in X$, we find that the lower bound $T\left(\operatorname*{Inf}_{i \in \mathcal{I}} f\left(\mu_i(x)\right), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right)$ of the set $\{T(f(\mu_i(x)), \nu_i(x))\}_{i \in \mathcal{I}}$ is actually its greatest lower bound.

Given $x \in X$ and $\varepsilon > 0$, as T is a continuous function, then there exists $\delta = \delta(x, \varepsilon) > 0$ such that for each $(\alpha_1, \alpha_2) \in [0,1]^2$ satisfying

$$\left|(\alpha_1, \alpha_2) - \left(\operatorname*{Inf}_{i \in \mathcal{I}} f(\mu_i(x)), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right)\right| < \delta \,,$$

then

$$\left|T(\alpha_1, \alpha_2) - T\left(\operatorname*{Inf}_{i \in \mathcal{I}} f(\mu_i(x)), \operatorname*{Inf}_{i \in \mathcal{I}} \nu_i(x)\right)\right| < \varepsilon \,. \tag{3}$$

Now, we consider $\delta/\sqrt{2} > 0$. Thus, by definition of infimum, there exist $i_1, i_2 \in \mathcal{I}$ such that

$$\begin{aligned} f\left(\mu_{i_1}(x)\right) &< \operatorname*{Inf}_{i\in\mathcal{I}} f\left(\mu_i(x)\right) + \delta/\sqrt{2} \\ \nu_{i_2}(x) &< \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x) + \delta/\sqrt{2}\,. \end{aligned} \tag{4}$$

Since $\{\chi^i\}_{i\in\mathcal{I}}$ is a semilattice from below, then there exists $i_\varepsilon \in \mathcal{I}$ such that $\chi^{i_\varepsilon} = \operatorname{Sup}\{\chi^{i_1}, \chi^{i_2}\}$. Thus from (4) we obtain:

$$\begin{aligned} f\left(\mu_{i_\varepsilon}(x)\right) &< \operatorname*{Inf}_{i\in\mathcal{I}} f\left(\mu_i(x)\right) + \delta/\sqrt{2} \\ \nu_{i_\varepsilon}(x) &< \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x) + \delta/\sqrt{2}\,, \end{aligned}$$

hence

$$\left|\left(f(\mu_{i_\varepsilon}(x)), \nu_{i_\varepsilon}(x)\right) - \left(\operatorname*{Inf}_{i\in\mathcal{I}} f(\mu_i(x)), \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x)\right)\right| < \delta$$

and applying (3), we arrive at

$$T\left(f(\mu_{i_\varepsilon}(x)), \nu_{i_\varepsilon}(x)\right) < T\left(\operatorname*{Inf}_{i\in\mathcal{I}} f(\mu_i(x)), \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x)\right) + \varepsilon\,.$$

Therefore (2) is satisfied. □

In the same way, we can obtain a "dual" family of contradiction measures that are semicontinuous from above as follows.

Theorem 2. *Let S be a continuous t-conorm and let $f : [0,1] \to [0,1]$ be a continuous and non-increasing function such that $f(0) = 1$ and $f(1) = 0$. Then the function $\mathcal{C}_{S,f} : \mathbb{L}^X \to [0,1]$ defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by*

$$\mathcal{C}_{S,f}(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is } \mathbb{L}\text{-normal} \\ \operatorname*{Sup}_{x\in X} S\left(f(\mu(x)), \nu(x)\right) & \text{otherwise} \end{cases}$$

is a contradiction measure that is semicontinuous from above.

Note that, unlike Theorem 1, we have to define the measure $\mathcal{C}_{S,f}$ on the $\mathbb{L}$-normal sets as 0 to guarantee that it satisfies the second axiom of contradiction measure.

Proposition 1. *Let T be a continuous t-norm, let S be a continuous t-conorm and let $f : [0,1] \to [0,1]$ be a continuous and non-increasing function such that $f(0) = 1$ and $f(1) = 0$, then the following holds:*

(a) *If X is an infinite universe, then the contradiction measure $\mathcal{C}_{T,f}$ is not semicontinuous from above, that is, $\mathcal{C}_{T,f} \notin \mathcal{CM}^{sc}(\mathbb{L}^X)$.*

(b) *The contradiction measure* $\mathcal{C}_{S,f}$ *is not semicontinuous from below, that is,* $\mathcal{C}_{S,f} \notin \mathcal{CM}_{sc}(\mathbb{L}^X)$.

Proof. (a) Let $\mathcal{P}_F(X)$ be the set of all finite parts of X, consider the family $\{\chi^A\}_{A\in\mathcal{P}_F(X)} \subset \mathbb{L}^X \setminus \mathbb{L}_0^X$ defined as follows. For each $A \in \mathcal{P}_F(X)$

$$\chi^A(x) = \begin{cases} 0_{\mathbb{L}}, & \text{if } x \in A; \\ (0, 1/2), & \text{if } x \notin A. \end{cases}$$

For all $A_1, A_2 \in \mathcal{P}_F(X)$ we have that $\mathrm{Inf}\{\chi^{A_1}, \chi^{A_2}\} = \chi^{A_1\cup A_2} \in \{\chi^A\}_{A\in\mathcal{P}_F(X)}$ as $A_1 \cup A_2 \in \mathcal{P}_F(X)$. Therefore, $\{\chi^A\}_{A\in\mathcal{P}_F(X)}$ is a semilattice from above. Moreover, $\mathcal{C}_{T,f}(\chi^A) = 1/2$, for all $A \in \mathcal{P}_F(X)$, and $\operatorname*{Inf}_{A\in\mathcal{P}_F(X)} \chi^A = \chi^{0_{\mathbb{L}}}$. Thus

$$1 = \mathcal{C}_{T,f}\left(\operatorname*{Inf}_{A\in\mathcal{P}_F(X)} \chi^A\right) \neq \operatorname*{Sup}_{A\in\mathcal{P}_F(X)} \mathcal{C}_{T,f}(\chi^A) = 1/2\,.$$

(b) Let $\{\chi^n\}_{n\in\mathbb{N}} \subset \mathbb{L}^X$ be such that $\chi^n(x) = (0, \frac{1}{n})$ for all $x \in X$ and for each $n \in \mathbb{N}$, then $\mathcal{C}_{S,f}(\chi^n) = S(f(0), 1/n) = 1$ for all $n \in \mathbb{N}$. Moreover $\operatorname*{Sup}_{n\in\mathbb{N}} \chi^n(x) = (0, 0)$ for all $x \in X$, thus

$$1 = \operatorname*{Inf}_{n\in\mathbb{N}} \mathcal{C}_{S,f}(\chi^n) \neq \mathcal{C}_{S,f}\left(\operatorname*{Sup}_{n\in\mathbb{N}} \chi^n\right) = 0\,,$$

as $\operatorname*{Sup}_{n\in\mathbb{N}}\{\chi^n\}$ is $\mathbb{L}$-normal.

Regarding complete continuity, the results are poorer than for continuity, as stated below.

Theorem 3. *Let* $f : [0,1] \to [0,1]$ *be a continuous and non-increasing function such that* $f(0) = 1$ *and* $f(1) = 0$. *Then the following is satisfied:*

(a) *The function* $\mathcal{C}_{\mathrm{Min},f} : \mathbb{L}^X \to [0,1]$ *defined for each* $\chi \in \mathbb{L}^X$ *by*

$$\mathcal{C}_{\mathrm{Min},f}(\chi) = \operatorname*{Inf}_{x\in X} \mathrm{Min}(f(\mu(x)), \nu(x))$$

is a contradiction measure that is completely semicontinuous from below.

(b) *The function* $\mathcal{C}_{\mathrm{Max},f} : \mathbb{L}^X \to [0,1]$ *defined for each* $\chi \in \mathbb{L}^X$ *by*

$$\mathcal{C}_{\mathrm{Max},f}(\chi) = \begin{cases} 0 & \textit{if } \chi \textit{ is } \mathbb{L}\textit{-normal} \\ \operatorname*{Sup}_{x\in X} \mathrm{Max}\,(f(\mu(x)), \nu(x)) & \textit{otherwise} \end{cases}$$

is a contradiction measure that is completely semicontinuous from above.

We omit the proof of the above theorem because it is a particular case of Theorems 3.7 and 3.8 given in Ref. 15.

Although $\mathcal{C}_{T,f}$ and $\mathcal{C}_{S,f}$ behave well regarding complete continuity when T is the t-norm Min and S is the t-conorm Max, both $\mathcal{C}_{T,f} \in \mathcal{CM}_{csc}(\mathbb{L}^X)$ and $\mathcal{C}_{S,f} \in \mathcal{CM}^{csc}(\mathbb{L}^X)$ are generally not true. To be precise, we can state the following results.

Proposition 2. *Let T be a continuous and Archimedean t-norm, let S be a continuous and Archimedean t-conorm, and let $f : [0,1] \to [0,1]$ be a continuous and non-increasing function such that $f(0) = 1$ and $f(1) = 0$. The following statements are satisfied:*

(a) *The contradiction measure $\mathcal{C}_{T,f}$ is not completely semicontinuous from below, that is, $\mathcal{C}_{T,f} \notin \mathcal{CM}_{csc}(\mathbb{L}^X)$.*
(b) *The contradiction measure $\mathcal{C}_{S,f}$ is not completely semicontinuous from above, that is, $\mathcal{C}_{S,f} \notin \mathcal{CM}^{csc}(\mathbb{L}^X)$.*

Proof. (a) Let us distinguish two cases: T is strict or T is nilpotent

As f is continuous, then there exists $\alpha \in [0,1]$ such that $0 < f(\alpha) < 1$, and let $\beta < 1 - \alpha$. Now we consider $\chi^A, \chi^B \in \mathbb{L}^X$, defined as follows:

$$\left.\begin{array}{l} \chi^A(x) = (\alpha, 1-\alpha) \\ \chi^B(x) = (0, \beta) \end{array}\right\} \quad \forall x \in X.$$

Then $\operatorname{Sup}\{\chi^A, \chi^B\}(x) = (\alpha, \beta)$ for all $x \in X$, thus $\mathcal{C}_{T,f}(\chi^A) = T(f(\alpha), 1-\alpha)$ and $\mathcal{C}_{T,f}(\chi^B) = T(f(0), \beta) = T(1, \beta) = \beta$. However,

$$\mathcal{C}_{T,f}(\operatorname{Sup}\{\chi^A, \chi^B\}) = T(f(\alpha), \beta) < \operatorname{Min}\{\mathcal{C}_{T,f}(\chi^A), \mathcal{C}_{T,f}(\chi^B)\}$$

as T is strict and because of the choice of α and β.

Now suppose that T is nilpotent, then $T = \varphi^{-1} \circ W \circ (\varphi \times \varphi)$, φ being an order automorphism. We consider $N_\varphi \circ N_s$, where N_φ is the strong negation defined by $N_\varphi(a) = \varphi^{-1}(1 - \varphi(a))$ for all $a \in [0,1]$, and N_s is the standard negation, so $N_\varphi \circ N_s$ is an order automorphism of the unit interval. Thus we can find $\alpha_0 \in]0,1[$ such that $f(\alpha_0) = (N_\varphi \circ N_s)(\alpha_0)$, and let α be such that $\operatorname{Sup}\{a \in [0,1] : f(a) = 1\} < \alpha < \alpha_0$ (see Fig. 1(a)). Hence, on the one hand,

$$\varphi(f(\alpha)) < 1 \tag{5}$$

and, the other hand, as $\varphi^{-1}(1 - \varphi(1-\alpha)) = (N_\varphi \circ N_s)(\alpha) < (N_\varphi \circ N_s)(\alpha_0) = f(\alpha_0) \leq f(\alpha)$, then

$$\varphi(f(\alpha)) + \varphi(1-\alpha) > 1\,. \tag{6}$$

Moreover, let β be such that $0 < \beta < 1 - \alpha$, then

$$\varphi(\beta) < \varphi(1-\alpha)\,. \tag{7}$$

Now we consider $\chi^A, \chi^B \in \mathbb{L}^X$ such that $\chi^A(x) = (\alpha, 1-\alpha)$ and $\chi^B(x) = (0, \beta)$ for all $x \in X$, thus $\operatorname{Sup}\{\chi^A, \chi^B\}(x) = (\alpha, \beta)$ for all $x \in X$, and the following holds:

$$
\begin{aligned}
\mathcal{C}_{T,f}(\chi^A) &= \varphi^{-1}\left(\mathrm{Max}\left\{0, \varphi(f(\alpha)) + \varphi(1-\alpha) - 1\right\}\right) \\
&\overset{(6)}{=} \varphi^{-1}\left(\varphi(f(\alpha)) + \varphi(1-\alpha) - 1\right) \\
&\overset{(6)\text{ and }(7)}{>} \varphi^{-1}\left(\mathrm{Max}\left\{0, \varphi(f(\alpha)) + \varphi(\beta) - 1\right\}\right) = \mathcal{C}_{T,f}\left(\mathrm{Sup}\left\{\chi^A, \chi^B\right\}\right)
\end{aligned}
$$

$$
\mathcal{C}_{T,f}(\chi^B) = \varphi^{-1}\left(\mathrm{Max}\left\{0, \varphi(f(0)) + \varphi(\beta) - 1\right\}\right) \overset{(5)}{>} \mathcal{C}_{T,f}\left(\mathrm{Sup}\left\{\chi^A, \chi^B\right\}\right).
$$

(b) Again, we make a distinction between S is strict or S is nilpotent.

Suppose that S is strict. We consider $\alpha, \beta, \gamma \in]0,1[$ such that $f(\alpha) < f(\beta)$, so $\alpha > \beta$ as f is non-increasing, and $\gamma < 1-\alpha$. Now, let $\chi^A, \chi^B \in \mathbb{L}^X \setminus \mathbb{L}_0^X$ such that

$$
\left.\begin{array}{l}
\chi^A(x) = (\alpha, 1-\alpha) \\
\chi^B(x) = (\beta, \gamma)
\end{array}\right\} \quad \forall x \in X,
$$

then $\mathrm{Inf}\{\chi^A, \chi^B\}(x) = (\beta, 1-\alpha)$ for all $x \in X$. Moreover, as S is strict, then

$$
\begin{aligned}
\mathcal{C}_{S,f}(\chi^A) &= S(f(\alpha), 1-\alpha) < S(f(\beta), 1-\alpha) = \mathcal{C}_{S,f}\left(\mathrm{Inf}\left\{\chi^A, \chi^B\right\}\right) \\
\mathcal{C}_{S,f}(\chi^B) &= S(f(\beta), \gamma) < S(f(\beta), 1-\alpha) = \mathcal{C}_{S,f}\left(\mathrm{Inf}\left\{\chi^A, \chi^B\right\}\right).
\end{aligned}
$$

Now, suppose that S is nilpotent, then $S = \varphi^{-1} \circ W^* \circ (\varphi \times \varphi)$, where φ is an order automorphism. Let $\beta \in]0,1[$ such that $f(\beta) = (N_\varphi \circ N_s)(\beta)$. Moreover, because of the conditions satisfied by f, we can choose α such that $\mathrm{Sup}\{a \in [0,1] : f(a) = f(\beta)\} < \alpha < 1$ (see Fig. 1(b)), so

$$
f(\alpha) < f(\beta) \tag{8}
$$

and hence $\beta < \alpha$. Also we consider

$$
0 < \gamma < 1 - \alpha. \tag{9}
$$

As $f(\beta) = (N_\varphi \circ N_s)(\beta) < (N_\varphi \circ N_s)(\alpha) = \varphi^{-1}(1 - \varphi(1-\alpha))$, then

$$
(\varphi \circ f)(\beta) + \varphi(1-\alpha) < 1. \tag{10}
$$

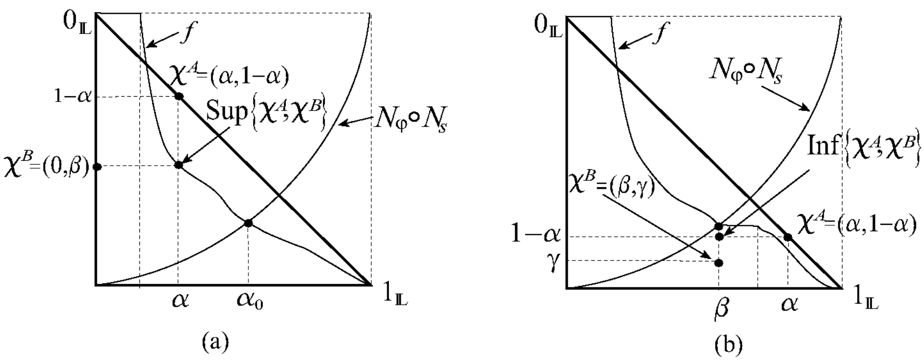

Fig. 1. Geometrical interpretation of proposition 2 for T and S nilpotent.

Now we consider $\chi^A, \chi^B \in \mathbb{L}^X \setminus \mathbb{L}_0^X$ such that $\chi^A(x) = (\alpha, 1-\alpha)$ and $\chi^B(x) = (\beta, \gamma)$ for all $x \in X$, thus $\mathrm{Inf}\{\chi^A, \chi^B\}(x) = (\beta, 1-\alpha)$ for all $x \in X$, and the following holds:

$$\begin{aligned}
\mathcal{C}_{S,f}\left(\mathrm{Inf}\left\{\chi^A, \chi^B\right\}\right) &= \varphi^{-1}\left(\mathrm{Min}\left\{1, \varphi(f(\beta)) + \varphi(1-\alpha)\right\}\right) \\
&\overset{(10)}{=} \varphi^{-1}\left(\varphi(f(\beta)) + \varphi(1-\alpha)\right) \\
&\overset{(8) \text{ and } (10)}{>} \varphi^{-1}\left(\mathrm{Min}\left\{1, \varphi(f(\alpha)) + \varphi(1-\alpha)\right\}\right) = \mathcal{C}_{S,f}(\chi^A)
\end{aligned}$$

$$\mathcal{C}_{S,f}\left(\mathrm{Inf}\left\{\chi^A, \chi^B\right\}\right) \overset{(9)}{>} \varphi^{-1}\left(\mathrm{Min}\left\{1, \varphi(f(\beta)) + \varphi(\gamma)\right\}\right) = \mathcal{C}_{S,f}(\chi^B).$$

3. Obtaining Contradiction Measures from Negations

In Ref. 4 it was shown that, given an IFN $\mathcal{N}$, $\chi = (\mu, \nu) \in \mathbb{L}^X$ is $\mathcal{N}$-contradictory if and only if $N(\mu(x)) + \nu(x) \geq 1$, for all $x \in X$, where N is the strong negation associated with $\mathcal{N}$. Thus, if there exists $x \in X$ such that $\chi(x) \in \mathbb{L}_{\mathcal{N}} = \{(\alpha_1, \alpha_2) \in \mathbb{L} : N(\alpha_1) + \alpha_2 < 1\}$, then χ is not $\mathcal{N}$-contradictory. For this reason we called $\mathbb{L}_{\mathcal{N}}$ the *region of non-$\mathcal{N}$-contradiction*. Moreover, we noted that the boundary curve $\alpha_2 = 1 - N(\alpha_1)$ delimiting $\mathbb{L}_{\mathcal{N}}$ and $\mathbb{L} \setminus \mathbb{L}_{\mathcal{N}}$ is a (strictly) increasing function of α_1, and it intersects with $\alpha_1 + \alpha_2 = 1$ on $(\alpha_N, 1-\alpha_N)$, where α_N is the fixed point of N.

On the one hand, the above considerations suggested defining the degree of $\mathcal{N}$-contradiction of χ as how far it is from $\mathbb{L}_{\mathcal{N}}$ with the following meaning:

$$\mathcal{C}_1^{\mathcal{N}}(\chi) = \mathrm{Max}\left(0, \inf_{x \in X}\{N(\mu(x)) + \nu(x) - 1\}\right). \tag{11}$$

On the other hand, as a function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists an order automorphism $\varphi : [0,1] \to [0,1]$ such that $N(\alpha) = \varphi^{-1}(1 - \varphi(\alpha))$,[16] then χ is $\mathcal{N}$-contradictory if and only if $\varphi(\mu(x)) + \varphi(1-\nu(x)) \leq 1$ for all $x \in X$, thus we can also measure the degree of $\mathcal{N}$-contradiction of χ according to the formula

$$\mathcal{C}_2^{\mathcal{N}}(\chi) = \mathrm{Max}\left(0, 1 - \sup_{x \in X}\{\varphi(\mu(x)) + \varphi(1-\nu(x))\}\right). \tag{12}$$

Furthermore, we can measure the degree of $\mathcal{N}$-contradiction of χ by calculating how far $\chi(X) = \{(\mu(x), \nu(x)) : x \in X\}$ is from $\mathbb{L}_{\mathcal{N}}$ using the Euclidean distance, that is,

$$\mathcal{C}_3^{\mathcal{N}}(\chi) = \frac{d(\chi(X), \mathbb{L}_{\mathcal{N}})}{d(0_{\mathbb{L}}, \mathbb{L}_{\mathcal{N}})}, \tag{13}$$

where d is the Euclidean distance on $\mathbb{R}^2$.

The functions given in Eqs. (11), (12) and (13) behave well in the framework of contradiction theory. In fact, it was proved in Ref. 5 that they are measures of contradiction. Also $\mathcal{C}_3^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$ was proved in Ref. 1.

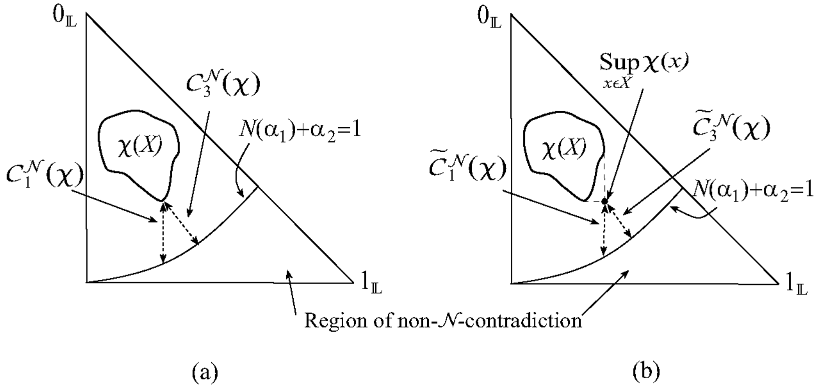

Fig. 2. Geometrical interpretation of measures $\mathcal{C}_1^{\mathcal{N}}, \mathcal{C}_3^{\mathcal{N}}, \widetilde{\mathcal{C}}_1^{\mathcal{N}}$ and $\widetilde{\mathcal{C}}_3^{\mathcal{N}}$.

Now, we are going to introduce other families of measures whose construction is suggested by (11), (12) and (13). Then, we will analyze what kind of measures the functions (11), (12) and the new families are.

For each $k = 1, 2, 3$ the function $\mathcal{C}_k^{\mathcal{N}}$ somehow quantifies how far $\chi(X)$ is from $\mathbb{L}_{\mathcal{N}}$ (see Fig. 2(a)). In the same way, we can consider how far $\operatorname{Sup}_{x\in X}\chi(x)$ is from $\mathbb{L}_{\mathcal{N}}$ (see Fig. 2(b)). Thus we arrive at the following definition.

Definition 4. Let $X \neq \emptyset$ be a universe, and let $\mathcal{N}$ be a strong IFN associated with the fuzzy negation N, given by the automorphism φ. For each $k = 1, 2, 3$ we define the function $\widetilde{\mathcal{C}}_k^{\mathcal{N}} : \mathbb{L}^X \to [0, 1]$ as follows. For each $\chi = (\mu, \nu) \in \mathbb{L}$

(a) $\widetilde{\mathcal{C}}_1^{\mathcal{N}}(\chi) = \operatorname{Max}\left\{0, N\left(\operatorname{Sup}_{x\in X}\mu(x)\right) + \operatorname{Inf}_{x\in X}\nu(x) - 1\right\}.$

(b) $\widetilde{\mathcal{C}}_2^{\mathcal{N}}(\chi) = \operatorname{Max}\left\{0, 1 - \varphi\left(\operatorname{Sup}_{x\in X}\mu(x)\right) - \varphi\left(1 - \operatorname{Inf}_{x\in X}\nu(x)\right)\right\}.$

(c) $\widetilde{\mathcal{C}}_3^{\mathcal{N}}(\chi) = \dfrac{d\left(\operatorname{Sup}_{x\in X}\chi(x), \mathbb{L}_{\mathcal{N}}\right)}{d(0_{\mathbb{L}}, \mathbb{L}_{\mathcal{N}})}.$

Theorem 4. *Let $X \neq \emptyset$ and let $\mathcal{N}$ be a strong IFN, then the contradiction measures $\mathcal{C}_1^{\mathcal{N}}, \mathcal{C}_2^{\mathcal{N}}$ defined in Eqs.* (11) *and* (12), *respectively, are semicontinuous from below, that is, $\mathcal{C}_1^{\mathcal{N}}, \mathcal{C}_2^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$.*

Proof. Let us just prove that $\mathcal{C}_1^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$, as $\mathcal{C}_2^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$ is proved similarly.

We have to prove the axiom (c.vi). Let $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$ be a semilattice from below, where $\chi^i = (\mu_i, \nu_i)$ for each $i \in \mathcal{I}$. Since $\mathcal{C}_1^{\mathcal{N}}$ is antimonotonic, then

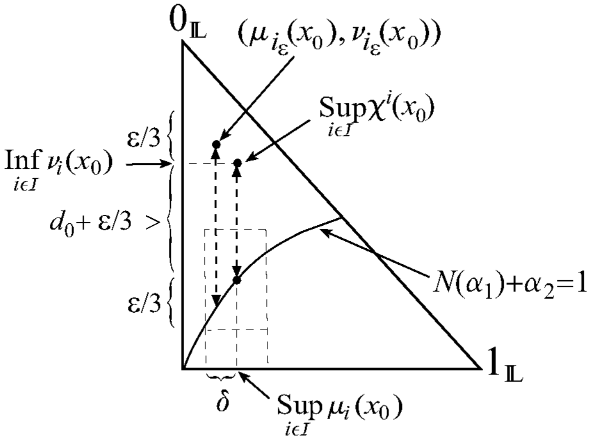

Fig. 3. Geometrical interpretation of the proof of Theorem 4.

$\mathcal{C}_1^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right) \leq \mathcal{C}_1^{\mathcal{N}}(\chi^j)$ for all $j \in \mathcal{I}$. We find that the lower bound $\mathcal{C}_1^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)$ of the set $\left\{\mathcal{C}_1^{\mathcal{N}}(\chi^i)\right\}_{i\in\mathcal{I}}$ is actually its greatest lower bound.

We write $d_0 = \operatorname*{Inf}_{x\in X}\left\{N\left(\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x)\right) + \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x) - 1\right\}$, and we distinguish two cases:

• First, let us suppose that $d_0 \geq 0$, then $\mathcal{C}_1^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right) = d_0$. Thus given $\varepsilon > 0$, by definition of infimum, there exists $x_0 \in X$ such that (see Fig. 3)

$$N\left(\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0)\right) + \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x_0) - 1 < d_0 + \frac{\varepsilon}{3}\,. \tag{14}$$

Moreover, as N is a continuous function, there exists $\delta = \delta(\varepsilon/3) > 0$ such that

$$\left|N(\alpha) - N\left(\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0)\right)\right| < \varepsilon/3 \tag{15}$$

for all $\alpha \in [0,1]$ satisfying $|\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0) - \alpha| < \delta$.

On the one hand, given that $\delta = \delta(\frac{\varepsilon}{3}) > 0$, it follows from the definition of $\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0)$ that there exists $i_1 \in \mathcal{I}$ such that $\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0) - \delta < \mu_{i_1}(x_0)$ (see Fig. 3). Thus we can apply (15) with $\alpha = \mu_{i_1}(x_0)$, obtaining

$$N(\mu_{i_1}(x_0)) < \frac{\varepsilon}{3} + N\left(\operatorname*{Sup}_{i\in\mathcal{I}} \mu_i(x_0)\right). \tag{16}$$

On the other hand, given $\varepsilon/3 > 0$, it follows from the definition of $\operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x_0)$ that there exists $i_2 \in \mathcal{I}$ such that (see Fig. 3)

$$\nu_{i_2}(x_0) < \frac{\varepsilon}{3} + \operatorname*{Inf}_{i\in\mathcal{I}} \nu_i(x_0). \tag{17}$$

Now, since $\{\chi^i\}_{i\in\mathcal{I}}$ is a semilattice from below, there exists $i_\varepsilon \in \mathcal{I}$ such that $\chi^{i_\varepsilon} = \mathrm{Sup}\{\chi^{i_1}, \chi^{i_2}\}$. Taking into account that N is a non-increasing function, it follows from (16) and (17) that

$$N(\mu_{i_\varepsilon}(x_0)) < \frac{\varepsilon}{3} + N\left(\mathop{\mathrm{Sup}}_{i\in\mathcal{I}} \mu_i(x_0)\right),$$

$$\nu_{i_\varepsilon}(x_0) < \frac{\varepsilon}{3} + \mathop{\mathrm{Inf}}_{i\in\mathcal{I}} \nu_i(x_0),$$

thus also considering (14), we have

$$\begin{aligned}\mathcal{C}_1^{\mathcal{N}}(\chi^{i_\varepsilon}) &= \mathop{\mathrm{Inf}}_{x\in X} \{N(\mu_{i_\varepsilon}(x)) + \nu_{i_\varepsilon}(x) - 1\} < N\left(\mathop{\mathrm{Sup}}_{i\in\mathcal{I}} \mu_i(x_0)\right) + \mathop{\mathrm{Inf}}_{i\in\mathcal{I}} \nu_i(x_0) - 1 + 2\frac{\varepsilon}{3} \\ &< d_0 + \varepsilon = \mathcal{C}_1^{\mathcal{N}}\left(\mathop{\mathrm{Sup}}_{i\in\mathcal{I}} \chi^i\right) + \varepsilon.\end{aligned}$$

Therefore, $\mathop{\mathrm{Inf}}_{i\in\mathcal{I}} \mathcal{C}_1^{\mathcal{N}}(\chi^i) = \mathcal{C}_1^{\mathcal{N}}\left(\mathop{\mathrm{Sup}}_{i\in\mathcal{I}} \chi^i\right)$.

• Second, let us suppose that $d_0 < 0$, then $\mathcal{C}_1^{\mathcal{N}}\left(\mathop{\mathrm{Sup}}_{i\in\mathcal{I}} \chi^i\right) = 0$. Now we consider $0 < \varepsilon < -d_0$. Thus, as in the first case, we obtain $i_\varepsilon \in \mathcal{I}$ such that

$$\mathop{\mathrm{Inf}}_{x\in X} \{N(\mu_{i_\varepsilon}(x)) + \nu_{i_\varepsilon}(x) - 1\} < d_0 + \varepsilon < 0,$$

then $\mathcal{C}_1^{\mathcal{N}}(\chi^{i_\varepsilon}) = 0$ and therefore the equality of axiom (c.vi) is also satisfied.

Before proving that each function $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$, with $k = 1, 2, 3$, is semicontinuous from below, we state a previous result. To simplify the notation, for each $\chi = (\mu, \nu) \in \mathbb{L}^X$, we introduce $\overset{\vee}{\chi} = \left(\overset{\vee}{\mu}, \overset{\wedge}{\nu}\right) \in \mathbb{L}^X$ defined for all $y \in X$ by $\overset{\vee}{\chi}(y) = \mathop{\mathrm{Sup}}_{x\in X} \chi(x) = \left(\mathop{\mathrm{Sup}}_{x\in X} \mu(x), \mathop{\mathrm{Inf}}_{x\in X} \nu(x)\right) = \left(\overset{\vee}{\mu}(y), \overset{\wedge}{\nu}(y)\right)$.

Lemma 1. *If $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$ is a semilattice from below, then $\left\{\overset{\vee}{\chi}{}^i\right\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$ is a semilattice from below.*

Proof. Suppose that $\chi^i = (\mu_i, \nu_i)$ for all $i \in \mathcal{I}$. Let $\overset{\vee}{\chi}{}^{i_1}, \overset{\vee}{\chi}{}^{i_2} \in \left\{\overset{\vee}{\chi}{}^i\right\}_{i\in\mathcal{I}}$. Thus, as $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$ is a semilattice from below, given $i_1, i_2 \in \mathcal{I}$, there exists $j = j(i_1, i_2) \in \mathcal{I}$ such that $\mathrm{Sup}\{\chi^{i_1}, \chi^{i_2}\}(x) = \chi^j(x)$ for all $x \in X$. Then, for all $y \in X$

$$\begin{aligned}\mathrm{Sup}\left\{\overset{\vee}{\chi}{}^{i_1}, \overset{\vee}{\chi}{}^{i_2}\right\}(y) &= \mathrm{Sup}\left\{\overset{\vee}{\chi}{}^{i_1}(y), \overset{\vee}{\chi}{}^{i_2}(y)\right\} = \mathrm{Sup}\left\{\mathop{\mathrm{Sup}}_{x\in X} \chi^{i_1}(x), \mathop{\mathrm{Sup}}_{x\in X} \chi^{i_2}(x)\right\} \\ &= \mathrm{Sup}\left\{\left(\mathop{\mathrm{Sup}}_{x\in X} \mu_{i_1}(x), \mathop{\mathrm{Inf}}_{x\in X} \nu_{i_1}(x)\right), \left(\mathop{\mathrm{Sup}}_{x\in X} \mu_{i_2}(x), \mathop{\mathrm{Inf}}_{x\in X} \nu_{i_2}(x)\right)\right\} \\ &= \left(\mathrm{Max}\left\{\mathop{\mathrm{Sup}}_{x\in X} \mu_{i_1}(x), \mathop{\mathrm{Sup}}_{x\in X} \mu_{i_2}(x)\right\}, \mathrm{Min}\left\{\mathop{\mathrm{Inf}}_{x\in X} \nu_{i_1}(x), \mathop{\mathrm{Inf}}_{x\in X} \nu_{i_2}(x)\right\}\right)\end{aligned}$$

$$= \left(\operatorname*{Sup}_{x\in X} \left(\operatorname{Max}\left\{\mu_{i_1}(x), \mu_{i_2}(x)\right\}\right), \operatorname*{Inf}_{x\in X} \left(\operatorname{Min}\{\nu_{i_1}(x), \nu_{i_2}(x)\}\right) \right)$$

$$= \left(\operatorname*{Sup}_{x\in X} \mu_j(x), \operatorname*{Inf}_{x\in X} \nu_j(x) \right) = \left(\overset{\vee}{\mu}_j(y), \overset{\wedge}{\nu}_j(y) \right) = \overset{\vee}{\chi}{}^j(y)\,.$$

Theorem 5. *Let $X \neq \emptyset$ and let $\mathcal{N}$ be a strong IFN. The functions $\widetilde{\mathcal{C}}_1^{\mathcal{N}}, \widetilde{\mathcal{C}}_2^{\mathcal{N}}$ and $\widetilde{\mathcal{C}}_3^{\mathcal{N}}$ given in Definition 4 are contradiction measures that are semicontinuous from below, that is, $\widetilde{\mathcal{C}}_1^{\mathcal{N}}, \widetilde{\mathcal{C}}_2^{\mathcal{N}}, \widetilde{\mathcal{C}}_3^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$.*

Proof. First, let us show that $\widetilde{\mathcal{C}}_k^{\mathcal{N}}(\chi) = \mathcal{C}_k^{\mathcal{N}}\left(\overset{\vee}{\chi}\right)$ for all $\chi \in \mathbb{L}^X$ and $k = 1, 2, 3$.

Given $\chi = (\mu, \nu) \in \mathbb{L}$, because of the definition of $\overset{\vee}{\chi}$, the following holds:

- For $k = 1$

$$\mathcal{C}_1^{\mathcal{N}}\left(\overset{\vee}{\chi}\right) = \operatorname{Max}\left(0, \operatorname*{Inf}_{y\in X} \left\{ N\left(\overset{\vee}{\mu}(y)\right) + \overset{\wedge}{\nu}(y) - 1 \right\}\right)$$

$$= \operatorname{Max}\left(0, N\left(\operatorname*{Sup}_{x\in X} \mu(x)\right) + \operatorname*{Inf}_{x\in X} \nu(x) - 1\right) = \widetilde{\mathcal{C}}_1^{\mathcal{N}}(\chi).$$

- For $k = 2$

$$\mathcal{C}_2^{\mathcal{N}}\left(\overset{\vee}{\chi}\right) = \operatorname{Max}\left(0, 1 - \operatorname*{Sup}_{y\in X} \left\{ \varphi\left(\overset{\vee}{\mu}(y)\right) + \varphi\left(1 - \overset{\wedge}{\nu}(y)\right) \right\}\right)$$

$$= \operatorname{Max}\left(0, 1 - \left\{ \varphi\left(\operatorname*{Sup}_{x\in X} \mu(x)\right) + \left(1 - \operatorname*{Inf}_{x\in X} \nu(x)\right) \right\}\right) = \widetilde{\mathcal{C}}_2^{\mathcal{N}}(\chi).$$

- For $k = 3$

$$\mathcal{C}_3^{\mathcal{N}}\left(\overset{\vee}{\chi}\right) = \frac{d\left(\overset{\vee}{\chi}(X), \mathbb{L}^X\right)}{d(0_{\mathbb{L}}, \mathbb{L}_{\mathcal{N}})} = \frac{d\left(\left\{\operatorname*{Sup}_{x\in X} \chi(x)\right\}, \mathbb{L}^X\right)}{d(0_{\mathbb{L}}, \mathbb{L}_{\mathcal{N}})} = \widetilde{\mathcal{C}}_3^{\mathcal{N}}(\chi).$$

Second, we can trivially confirm the axioms (c.i) and (c.ii) for each $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$, with $k = 1, 2, 3$. Furthermore, from the above considerations, we can guarantee that (c.i) is satisfied because $\mathcal{C}_k^{\mathcal{N}}$ and $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$ are equal on each IFSs whose $\mathbb{L}$-membership function takes a constant value, and (c.ii) is satisfied because if $\chi \in \mathbb{L}^X$ is $\mathbb{L}$-normal, then $\overset{\vee}{\chi}$ is also $\mathbb{L}$-normal.

Finally, let us prove the axiom (c.vi): if $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$ is any semilattice from below, as, according to the above lemma, $\{\overset{\vee}{\chi}{}^i\}_{i\in\mathcal{I}}$ is also a semilattice from below and $\mathcal{C}_k^{\mathcal{N}} \in \mathcal{CM}_{sc}(\mathbb{L}^X)$ for each $k = 1, 2, 3$, then

$$\operatorname*{Inf}_{i\in\mathcal{I}} \widetilde{\mathcal{C}}_k^{\mathcal{N}}(\chi^i) = \operatorname*{Inf}_{i\in\mathcal{I}} \mathcal{C}_k^{\mathcal{N}}\left(\overset{\vee}{\chi}{}^i\right) = \mathcal{C}_k^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \overset{\vee}{\chi}{}^i\right) = \mathcal{C}_k^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}}^{\vee} \chi^i\right) = \widetilde{\mathcal{C}}_k^{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right).$$

The measures of contradiction $\mathcal{C}_k^{\mathcal{N}}$ and $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$, with $k = 1, 2, 3$, satisfy the axiom (c.vi) of continuity from below, but, in general, they do not satisfy either of the other axioms of continuity, as the following two results show.

Proposition 3. *Let $X \neq \emptyset$, it holds, for any strong IFN $\mathcal{N}$, that the contradiction* *measures $\mathcal{C}_k^{\mathcal{N}}, \widetilde{\mathcal{C}}_k^{\mathcal{N}}$, with $k = 1, 2, 3$, are not completely semicontinuous from either above or below, that is, $\mathcal{C}_k^{\mathcal{N}}, \widetilde{\mathcal{C}}_k^{\mathcal{N}} \notin \mathcal{CM}_{csc}(\mathbb{L}^X) \cup \mathcal{CM}^{csc}(\mathbb{L}^X)$.*

Proof. Let N be the strong fuzzy negation associated with $\mathcal{N}$ and φ the automorphism that determines N. We consider $0 < \alpha < \alpha_N$, where α_N is the fixed point of N, and let $\chi^A, \chi^B \in \mathbb{L}^X$ be the sets defined by

$$\left.\begin{array}{l} \chi^A(x) = (0, 1 - N(\alpha)) \\ \chi^B(x) = (\alpha, 1 - \alpha) \end{array}\right\} \quad \forall x \in X,$$

then

$$\left.\begin{array}{r} \mathrm{Sup}\{\chi^A, \chi^B\}(x) = (\alpha, 1 - N(\alpha)) \\ \mathrm{Inf}\{\chi^A, \chi^B\}(x) = (0, 1 - \alpha) \end{array}\right\} \quad \forall x \in X.$$

Thus we have:

(1) $\mathcal{C}_1^{\mathcal{N}} \notin \mathcal{CM}_{csc}(\mathbb{L}^X) \cup \mathcal{CM}_{csc}(\mathbb{L}^X)$: As $0 < \alpha < \alpha_N = N(\alpha_N) < N(\alpha) < 1$, then $\mathcal{C}_1^{\mathcal{N}}(\chi^A) = 1 - N(\alpha) > 0$, $\mathcal{C}_1^{\mathcal{N}}(\chi^B) = N(\alpha) - \alpha > 0$ and so

$$\mathcal{C}_1^{\mathcal{N}}\left(\mathrm{Sup}\{\chi^A, \chi^B\}\right) = 0 < \mathrm{Min}\{\mathcal{C}_1^{\mathcal{N}}(\chi^A), \mathcal{C}_1^{\mathcal{N}}(\chi^B)\}$$
$$\mathcal{C}_1^{\mathcal{N}}\left(\mathrm{Inf}\{\chi^A, \chi^B\}\right) = 1 - \alpha > \mathrm{Max}\{\mathcal{C}_1^{\mathcal{N}}(\chi^A), \mathcal{C}_1^{\mathcal{N}}(\chi^B)\}.$$

(2) $\mathcal{C}_2^{\mathcal{N}} \notin \mathcal{CM}_{csc}(\mathbb{L}^X) \cup \mathcal{CM}_{csc}(\mathbb{L}^X)$: As $0 < \alpha < \alpha_N = \varphi^{-1}(1/2) < 1$, then

$$\mathcal{C}_2^{\mathcal{N}}(\chi^A) = \mathrm{Max}\{0, 1 - (\varphi(N(\alpha)))\} = \mathrm{Max}\{0, \varphi(\alpha)\} = \varphi(\alpha) > 0$$
$$\mathcal{C}_2^{\mathcal{N}}(\chi^B) = \mathrm{Max}\{0, 1 - 2\varphi(\alpha)\} = 1 - 2\varphi(\alpha) > 0,$$

and thus $\mathcal{C}_2^{\mathcal{N}}(\mathrm{Sup}\{\chi^A, \chi^B\}) = 0 < \mathrm{Min}\{\mathcal{C}_2^{\mathcal{N}}(\chi^A), \mathcal{C}_2^{\mathcal{N}}(\chi^B)\}$ and

$$\mathcal{C}_2^{\mathcal{N}}(\mathrm{Inf}\{\chi^A, \chi^B\}) = 1 - \varphi(\alpha) > \mathrm{Max}\{\mathcal{C}_2^{\mathcal{N}}(\chi^A), \mathcal{C}_2^{\mathcal{N}}(\chi^B)\}.$$

(3) $\mathcal{C}_3^{\mathcal{N}} \notin \mathcal{CM}_{csc}(\mathbb{L}^X)$: On the one hand, as $N(0) + 1 - N(\alpha) > 1$, then $(0, 1 - N(\alpha)) \notin \overline{\mathbb{L}}_{\mathcal{N}}$, where $\overline{\mathbb{L}}_{\mathcal{N}}$ is the closure of $\mathbb{L}_{\mathcal{N}}$ under the usual topology on $\mathbb{R}^2$ restricted to $\mathbb{L}$, and so $d(\chi^A(X), \mathbb{L}_{\mathcal{N}}) > 0$. Therefore, $\mathcal{C}_3^{\mathcal{N}}(\chi^A) > 0$. On the other hand, as $N(\alpha) - \alpha > 0$, then $(\alpha, 1 - \alpha) \notin \overline{\mathbb{L}}_{\mathcal{N}}$ and so $d(\chi^B(X), \mathbb{L}_{\mathcal{N}}) > 0$. Therefore, $\mathcal{C}_3^{\mathcal{N}}(\chi^B) > 0$. However, $\mathcal{C}_3^{\mathcal{N}}(\mathrm{Sup}\{\chi^A, \chi^B\}) = 0$ as $(\alpha, 1 - N(\alpha)) \in \overline{\mathbb{L}}_{\mathcal{N}}$.

(4) $\mathcal{C}_3^{\mathcal{N}} \notin \mathcal{CM}^{csc}(\mathbb{L}^X)$: We find that it is easier to consider $\chi^E, \chi^F \in \mathbb{L}^X$ defined for all $x \in X$ by $\chi^E(x) = \mathrm{Sup}\{\chi^A, \chi^B\}(x) = (\alpha, 1 - N(\alpha))$ and $\chi^F(x) = (\alpha_N, 1 - \alpha_N)$, thus $\mathcal{C}_3^{\mathcal{N}}(\chi^E) = \mathcal{C}_3^{\mathcal{N}}(\chi^F) = 0$ because $(\alpha, 1 - N(\alpha)), (\alpha_N, 1 - \alpha_N) \in \overline{\mathbb{L}}_{\mathcal{N}}$. However, $\mathrm{Inf}\{\chi^E, \chi^F\}(x) = (\alpha, 1 - \alpha_N)$ for all $x \in X$, and as $N(\alpha) - \alpha_N > 0$ then $(\alpha, 1 - \alpha_N) \notin \overline{\mathbb{L}}_{\mathcal{N}}$ and thus $\mathcal{C}_3^{\mathcal{N}}(\mathrm{Inf}\{\chi^E, \chi^F\}) > 0$.

(5) $\widetilde{\mathcal{C}}_k^{\mathcal{N}} \notin \mathcal{CM}_{csc}(\mathbb{L}^X) \cup \mathcal{CM}^{csc}(\mathbb{L}^X)$ for $k = 1, 2, 3$: Since $\mathcal{C}_k^{\mathcal{N}}(\chi) = \widetilde{\mathcal{C}}_k^{\mathcal{N}}(\chi)$ holds if $\chi \in \mathbb{L}^X$ is a constant function, then the previous reasoning for replacing $\mathcal{C}_k^{\mathcal{N}}$ with $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$ is valid.

Proposition 4. *Let X be an infinite set, if $\mathcal{N}$ is a strong IFN, then $\mathcal{C}_k^{\mathcal{N}}, \widetilde{\mathcal{C}}_k^{\mathcal{N}}$ are not semicontinuous from above for $i = 1, 2, 3$, that is, $\mathcal{C}_k^{\mathcal{N}}, \widetilde{\mathcal{C}}_k^{\mathcal{N}} \notin \mathcal{CM}^{sc}(\mathbb{L}^X)$.*

Proof. The idea in this proof is the same as in Proposition 1. If N is the strong fuzzy negation associated with $\mathcal{N}$ whose fixed point is α_N, let us again consider the family of all finite parts of X, $\mathcal{P}_F(X)$, where now $\{\chi^A\}_{A\in\mathcal{P}_F(X)} \subset \mathbb{L}^X \setminus \mathbb{L}_0^X$ such that for each $A \in \mathcal{P}_F(X)$

$$\chi^A(x) = \begin{cases} 0_{\mathbb{L}}, & \text{if } x \in A; \\ (\alpha_N, 1-\alpha_N), & \text{if } x \notin A. \end{cases}$$

In the same way as in Proposition 1, $\{\chi^A\}_{A\in\mathcal{P}_F(X)}$ is a semilattice from above. Moreover, we have that $\mathcal{C}_k^{\mathcal{N}}(\chi^A) = \widetilde{\mathcal{C}}_k^{\mathcal{N}}(\chi^A) = 0$ for all $A \in \mathcal{P}_F(X)$ and for $k = 1, 2, 3$. However,

$$0 = \operatorname*{Sup}_{A\in\mathcal{P}_F(X)} \mathcal{C}_k^{\mathcal{N}}(\chi^A) \neq \mathcal{C}_k^{\mathcal{N}}\left(\operatorname*{Inf}_{A\in\mathcal{P}_F(X)} \chi^A\right) = \mathcal{C}_k^{\mathcal{N}}\left(\chi^{0_{\mathbb{L}}}\right) = 1, \ \ \forall i = 1, 2, 3,$$

and we obtain the same inequality by replacing $\mathcal{C}_k^{\mathcal{N}}$ with $\widetilde{\mathcal{C}}_k^{\mathcal{N}}$.

4. Conclusions

In an earlier paper,[1] we presented an axiomatic model to measure the degree of contradiction of an IFS, and we set some conditions to guarantee that a measure of contradiction has some sort of continuity. However, we did not address the open problem of finding families of functions satisfying the definitions given in that paper.

The main result of this paper is the construction of a family of functions using t-norms. These functions turned out to be contradiction measures that are semicontinuous from below, but not from above. In a similar way, and by means of t-conorms, we have obtained a family of contradiction measures that are semicontinuous from above, but not from below. Furthermore, both families are completely semicontinuous if the t-norm is the minimum, and the t-conorm is the maximum, but the result fails when the t-norm or the t-conorm is Archimedean.

Additionally, we have studied the continuity of some families of contradiction measures defined in Ref. 5, which are associated with strong intuitionistic negations. We have also introduced new families of measures also associated with such negations, and we have studied their continuity.

Some of the topics we intend to address in the future are the continuity of these measures in the case of finite universes; the generalization of this model to the case where negation is not necessarily strong; the construction of a general model covering self-contradiction and contradiction among two or more sets; applications of these measures to inference and decision-making problems.

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