

DETECTING SERIES PERIODICITY WITH HORIZONTAL VISIBILITY GRAPHS

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The *horizontal visibility algorithm* was recently introduced as a mapping between time series and networks. The challenge lies in characterizing the structure of time series (and the processes that generated those series) using the powerful tools of graph theory. Recent works have shown that the visibility graphs inherit several degrees of correlations from their associated series, and therefore such graph theoretical characterization is in principle possible. However, both the mathematical grounding of this promising theory and its applications are in its infancy. Following this line, here we address the question of detecting hidden periodicity in series polluted with a certain amount of noise. We first put forward some generic properties of horizontal visibility graphs which allow us to define a (graph theoretical) noise reduction filter. Accordingly, we evaluate its performance for the task of calculating the period of noisy periodic signals, and compare our results with standard time domain (autocorrelation) methods. Finally, potentials, limitations and applications are discussed.

Keywords: Horizontal visibility graph; time series; complex networks; periodicity detection; noise filter.

1. Introduction

In the last years, some methods mapping time series to network representations have been proposed (see for instance [Xu *et al.*, 2008; Zhang & Small, 2006; Lacasa *et al.*, 2008; Luque *et al.*, 2009] and a recent review on this topic [Donner *et al.*, 2011]). The general purpose is to investigate the properties of the series through graph theoretical tools recently developed at the core of the celebrated complex network theory, opening the possibility of building bridges between time series analysis, nonlinear dynamics, and graph theory. Along this line, the family of visibility algorithms [Lacasa *et al.*, 2008; Luque *et al.*, 2009; Gutin *et al.*, 2011] was introduced recently. It has been shown that several degrees of correlations (including periodicity [Lacasa *et al.*, 2008], fractality [Lacasa *et al.*, 2009]

or chaoticity [Luque *et al.*, 2009; Lacasa & Toral, 2010]) can be captured by the algorithm and translated into the associated visibility graph. Accordingly, several works applying such algorithm in several contexts ranging from geophysics [Elsner *et al.*, 2009] or turbulence [Liu *et al.*, 2010] to physiology [Shao, 2010] or finance [Yang *et al.*, 2009] have started to appear. Here we focus on a specific algorithm within this family called the *horizontal* visibility graph [Luque *et al.*, 2009], which has been recently considered for the task of discriminating chaotic from correlated stochastic processes [Lacasa & Toral, 2010]. While the first steps for a rigorous mathematical grounding have been reported recently [Gutin *et al.*, 2011], this method is currently largely unexplored, both from a purely theoretical or from an applied point

of view. To partially solve such issues, in this work, we address the task of filtering a noisy signal with a hidden periodic component within the horizontal visibility formalism, that is, we explore the possibility of using the method for noise filtering purposes. Periodicity detection algorithms (see for instance [Parthasarathy *et al.*, 2006]) can be classified in essentially two categories, namely the time domain (autocorrelation based) and frequency domain (spectral) methods. Here we make use of the horizontal visibility algorithm to propose a third category, namely graph theoretical methods.

The rest of the paper goes as follows: in Sec. 2 we present the method, and provide some theorems regarding several topological properties of horizontal visibility graphs. In Sec. 3, we introduce the concept of a graph-theoretical noise filter, and provide some examples of noisy periodic series, comparing in each case the performance of the proposed method with an autocorrelation function analysis. A pathological case that yields misleading results from the autocorrelation function is also considered. We finally provide a discussion on the potentials and limitations of this approach.

2. Horizontal Visibility Algorithm

The horizontal visibility algorithm was recently introduced [Luque *et al.*, 2009] as a map between a time series and a graph and it is defined as follows. Let $\{x_i\}_{i=1,\dots,N}$ be a time series of N real data. The algorithm assigns each datum of the series to a node in the *horizontal visibility graph* (HVG). Accordingly, a series of N data map to an HVG with N nodes. Two nodes i and j in the graph are connected if one can draw a horizontal line in the time series joining x_i and x_j that does not intersect any intermediate data height. That is, i and j are two connected nodes if the following geometrical criterion is fulfilled within the time series:

$$x_i, x_j > x_n, \quad \forall n | i < n < j. \quad (1)$$

Basic properties of this graph can be found in [Luque *et al.*, 2009], and the first steps for a rigorous mathematical characterization can be found in [Gutin *et al.*, 2011]. Among the possible applications of the method for time series analysis purposes, discrimination between chaotic and stochastic signals has been recently addressed [Lacasa & Toral, 2010].

In this section we provide some theorems regarding some specific topological properties of the

horizontal visibility graphs, and in the following sections we will rely on these theorems to define a noise filtering technique.

Theorem 1 [Mean degree of periodic series]. *The mean degree of an horizontal visibility graph associated to an infinite periodic series of period T (with no repeated values within a period) is*

$$\bar{k}(T) = 4 \left(1 - \frac{1}{2T} \right).$$

Proof. Without lack of generality, represent the series as $\{\dots, x_0, x_1, \dots, x_T, x_1, x_2, \dots\}$, where $x_0 = x_T$ corresponds to the largest value of the series. By construction, the associated HVG is composed as a concatenation of identical motifs, each of these motifs being itself an HVG of $T+1$ nodes associated to the subseries x_0, x_1, \dots, x_T , and the mean degree of the HVG \bar{k} corresponds to the mean degree of the motif constructed with T nodes (the nodes associated to x_0 and x_T only introduce half of their actual degree in the motif, what is equivalent to effectively reducing one node). Suppose that the motif is a graph with V edges, and let x_i be the smallest datum of the subseries (since no repetitions are allowed in the motif, x_i will always be well defined). By construction, the associated node i will have degree $k = 2$. Extract now from the motif this node and its two edges. The resulting motif will have $V-2$ edges and T nodes. Iterate this operation $T-1$ times (see Fig. 1 for a graphical illustration of this process). The resulting graph will have only two nodes, associated to x_0 and x_T , linked by a single edge, and the total number of deleted edges will be $2(T-1)$. Hence,

$$\begin{aligned} \bar{k} &\equiv 2 \frac{\# \text{ edges}}{\# \text{ nodes}} \\ &= \frac{2(2(T-1) + 1)}{T} \Rightarrow \bar{k} = 4 \left(1 - \frac{1}{2T} \right). \quad \blacksquare \end{aligned}$$

An interesting consequence of the previous result is that every time series extracted from a dynamical system has an associated HVG with a mean degree $2 \leq \bar{k} \leq 4$, where the lower bound is reached for constant series, whereas the upper bound is reached for aperiodic series (random, chaotic [Luque *et al.*, 2009]).

Theorem 2 [Degree distribution associated to uncorrelated random series]. *Let $\{x_i\}$ be a bi-infinite sequence of independent and identically distributed*

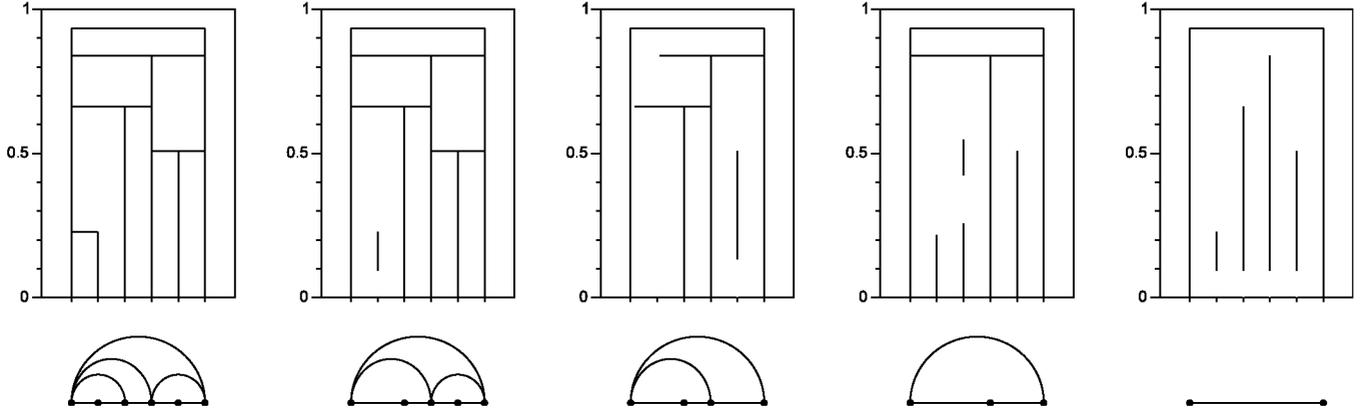


Fig. 1. Graphical illustration of the constructive proof of Theorem 1, considering a motif extracted from a periodic series of period $T = 5$.

random variables extracted from a continuous probability density $f(x)$. Then, the degree distribution of its associated horizontal visibility graph is

$$P(k) = \frac{1}{3} \left(\frac{2}{3}\right)^{k-2}, \quad k = 2, 3, 4, \dots \quad (2)$$

A lengthy constructive proof can be found in [Luque *et al.*, 2009]. Here we propose two alternative, shorter proofs for this theorem.

Proof. Let x be an arbitrary datum of the aforementioned series. The probability of its horizontal visibility being interrupted by a datum x_r on its right and another datum x_l on its left is, independent of $f(x)$,

$$\begin{aligned} \Phi_2 &= \int_{-\infty}^{\infty} \int_x^{\infty} \int_x^{\infty} f(x) f(x_r) f(x_l) dx_l dx_r dx \\ &= \int_{-\infty}^{\infty} f(x) [1 - F(x)]^2 dx = \frac{1}{3}. \end{aligned} \quad (3)$$

The probability $P(k)$ of the datum seeing exactly k data may be expressed as

$$P(k) = Q(k) \Phi_2 = \frac{1}{3} Q(k) \quad (4)$$

where $Q(k)$ is the probability of x seeing at least k data. $Q(k)$ may be recurrently calculated as

$$Q(k) = Q(k-1)(1 - \Phi_2) = \frac{2}{3} Q(k-1) \quad (5)$$

from which the following expression can be deduced:

$$Q(k) = \left(\frac{2}{3}\right)^{k-2} \quad (6)$$

Proof. (2) The same result for the distribution $P(k)$ can be deduced by the expression Φ_2 along with combinatoric arguments: if the arbitrary datum x is connected to exactly k data, there exist two data $x_l > x$ and $x_r > x$ that close the left and right visibility of x . As we have proven, this happens with a probability $\Phi_2 = 1/3$. The $k-2$ remaining data will therefore be smaller than x and will be distributed in a monotonically decreasing sequence on its left and a monotonically increasing sequence on its right respectively. The number of possible distributions with i data on its left and $k-2-i$ on its right is $\binom{k-2}{i}$, where $i = 0, 1, 2, \dots, k-2$. All these configurations can all be decomposed in $k-2$ groups of three data with the central datum being closed by the two others, therefore, all of them are equiprobable with a probability ϕ_2^{k-2} , then

$$\begin{aligned} P(k) &= \frac{1}{3} \sum_{n=0}^{k-2} \left(\frac{1}{3}\right)^{k-2} \binom{k-2}{n} \\ &= \left(\frac{1}{3}\right)^{k-1} 2^{k-2} \\ &= \frac{1}{3} \left(\frac{2}{3}\right)^{k-2} \end{aligned} \quad (7)$$

Observe that the mean degree \bar{k} of the horizontal visibility graph associated to an uncorrelated random process is then:

$$\bar{k} = \sum k P(k) = \sum_{k=2}^{\infty} \frac{k}{3} \left(\frac{2}{3}\right)^{k-2} = 4, \quad (8)$$

in good agreement with the prediction of the previous theorem for aperiodic series. ■

2.1. Stochastic, chaotic and periodic processes

Deviations from $P(k) = (\frac{1}{3})(\frac{2}{3})^{k-2}$ are, according to the previous theorem, univoquely associated to series which are not generated by a purely uncorrelated process. Several possibilities arise: first, the process can still be of a stochastic nature, while some correlations can be present. As a matter of fact, in [Lacasa & Toral, 2010] it has been shown that such kind of correlated stochastic series map into HVGs with a degree distribution which is also exponentially decaying, albeit with a larger slope than Eq. (2). A second situation involves a deterministic process. Two opposite possibilities arise: the process can be either regular, that yields a periodic series of a given period T , or chaotic, that yields an aperiodic series. Periodic series have an associated HVG with a degree distribution formed by a finite number of peaks, these peaks being related to the series period, that is reminiscent of the discrete Fourier spectrum of a periodic series [Lacasa *et al.*, 2008; Luque *et al.*, 2009]. The reason is straightforward: a periodic series maps into an HVG which, by construction, is a repetition of a root motif. The second possibility has been addressed in [Luque *et al.*, 2009; Lacasa & Toral,

2010] the conclusion being that chaotic processes have an HVG whose degree distribution has an exponential tail with smaller slope than Eq. (2), and evidences a net deviation from the exponential shape for small values of the degree, this deviation being associated to short-range memory effects. Last, an interesting situation takes place when a given regular process (periodic series) is polluted with a given amount of noise.

Indeed, if we superpose a small amount of noise to a periodic series (a so-called *extrinsic* noise), whereas the degree of the nodes with associated small values will remain rather similar, the nodes associated to higher values will eventually increase their visibility and hence reach larger degrees. Accordingly, the delta-like structure of the degree distribution (associated with the periodic component of the series) will be perturbed, and an exponential tail will arise due to the presence of such noise [Luque *et al.*, 2009; Lacasa & Toral, 2010]. Can the algorithm characterize such kind of series? The answer is positive, since the degree distribution can be analytically calculated as it follows.

Consider for simplicity a period-2 time series polluted with white noise (see Fig. 2(a) for a graphical illustration). The HVG is formed by two

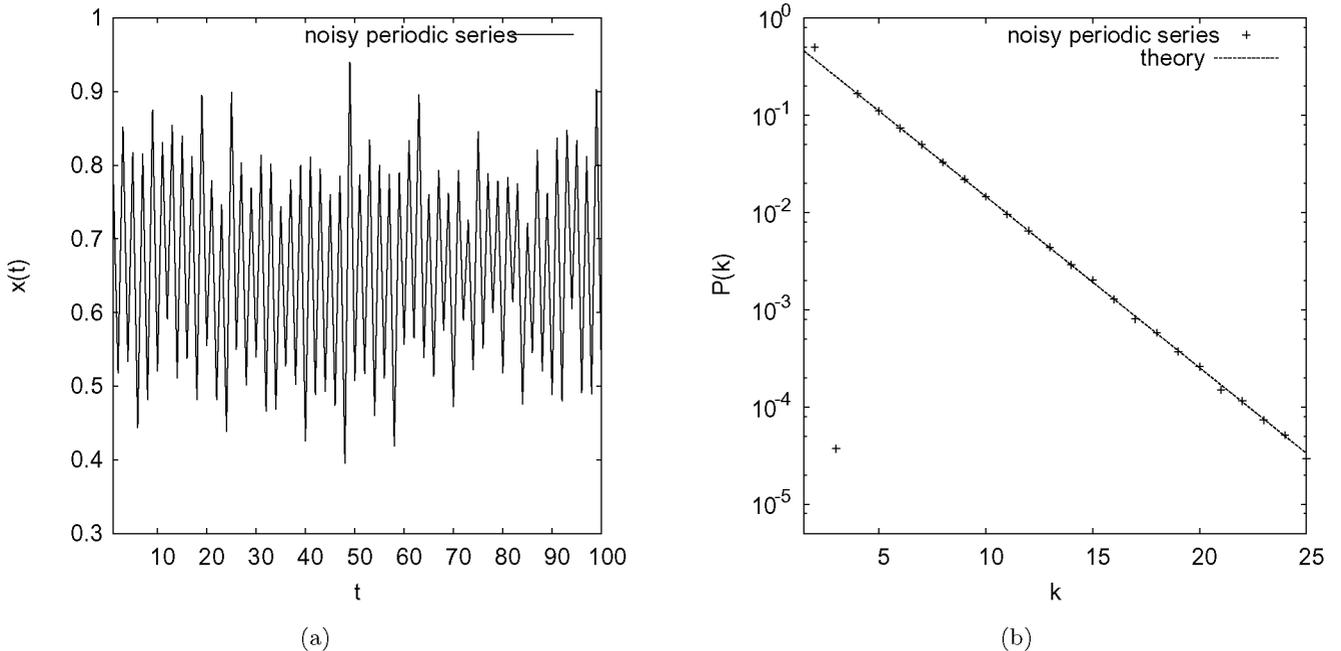


Fig. 2. (a) Periodic series of 2^{20} data generated through the logistic map $x_{n+1} = \mu x_n(1 - x_n)$ for $\mu = 3.2$ (where the map shows periodic behavior with period 2) polluted with extrinsic white Gaussian noise extracted from a Gaussian distribution $N(0, 0.05)$. (b) Dots represent the degree distribution of the associated HVG, whereas the straight line is Eq. (9) (the plot is in semi-log). Note that $P(2) = 1/2$, also as theory predicts, and that $P(3)$ is not exactly zero due to boundary effects in the time series. The algorithm efficiently detects both signals and therefore easily distinguishes extrinsic noise.

kind of nodes: those associated to high data with values (x_1, x_3, x_5, \dots) in the figure and those associated to data with small values (x_2, x_4, x_6, \dots) . These latter nodes will have, by construction, degree $k = 2$. On the other hand, the sub-graph formed by the odd nodes associated to data (x_1, x_3, x_5, \dots) will essentially reduce to the one associated to an uncorrelated series, i.e. its degree distribution will follow Eq. (2). Now, considering the whole graph, the resulting degree distribution will be such that

$$\begin{aligned} P(2) &= \frac{1}{2}, \\ P(3) &= 0, \\ P(k+2) &= \frac{1}{3} \left(\frac{2}{3}\right)^{k-2}, \quad k \geq 2, \quad \text{or} \\ P(k) &= \frac{1}{4} \left(\frac{2}{3}\right)^{k-3}, \quad k \geq 4, \end{aligned} \quad (9)$$

that is to say, introducing a small amount of extrinsic uncorrelated noise in a periodic signal introduces an exponential tail in the HVG's degree distribution with the same slope as the one associated to a purely uncorrelated process. The mean degree \bar{k} reads

$$\bar{k} = \sum_{k=2}^{\infty} kP(k) = 4,$$

which, according to Eq. (1), suggests aperiodicity, as expected. In Fig. 2(a) we plot in semi-log the degree distribution of a periodic-2 series of 2^{20} data polluted with an extrinsic white Gaussian noise extracted from a Gaussian distribution $N(0, 0.05)$. Numerical results confirm the validity of Eq. (9). Note that this methodology can be extended to every integrable deterministic system, and therefore we conclude that extrinsic noise in a mixed time series is, in principle, well captured by the algorithm. Based on the previous theorems, in the next section, we introduce a method to calculate the hidden periodicity in a noisy periodic signal.

3. A Graph-Theoretical Noise Filter

3.1. Definition and examples

Let $S = \{x_i\}_{i=1, \dots, n}$ be a periodic series of period T (where $n \gg T$) polluted by a certain amount of extrinsic noise (without loss of generality, suppose

a white noise extracted from a uniform distribution $U[-0.5, 0.5]$), and define the filter f as a real valued scalar such that $f \in [\min x_i, \max x_i]$. The so-called filtered Horizontal Visibility Graph (f-HVG) associated to S is constructed as follows:

- (i) each datum x_i in the time series is mapped to a node i in the f-HVG,
- (ii) two nodes i and j are connected in the fHVG if the associated data fulfill

$$x_i, x_j > x_n + f, \quad \forall n | i < n < j. \quad (10)$$

The procedure of filtering the noise from a noisy periodic signal goes as follows: one generates the f-HVG associated to S for increasing values of f , and in each case proceeds to calculate the mean degree \bar{k} . For the proper interval $f_{\min} < f < f_{\max}$, the f-HVG of the noisy periodic series S will be equivalent to the noise free HVG of the pure (periodic) signal, which has a well-defined mean degree as a function of the series period. In this interval, the mean degree will therefore remain constant, and from Eq. (1) the period can be inferred.

As an example, we have artificially generated a noisy periodic series of hidden period $T = 2$, that we plot in Fig. 3(a). The results of the graph filtering technique are shown in Fig. 3(b), where we plot the values of \bar{k} as a function of f . Notice that the graph filtering yields a net decrease of the mean degree, which has an initial value of 4 (as expected for the HVG ($f = 0$) of an aperiodic series such as a noisy periodic signal) and an asymptotic value of 2 (lower bound of the mean degree). The plateau is clearly found at $\bar{k} = 3$, which according to Eq. (1) yields a period

$$T = \left(2 - \frac{\bar{k}}{2}\right)^{-1} = 2,$$

as expected. For comparison, the autocorrelation function $\text{ACF}(\tau)$ of the series is also calculated, according to the following definition

$$\text{ACF}(\tau) = \langle x(t) \cdot x(t - \tau) \rangle_t,$$

such that ACF is not bounded in $[-1, 1]$ since it is not normalized. This expression has a periodic shape of period T when the series is itself periodic with period T , whereas aperiodic structures yield an autocorrelation function that lacks any structure. In the Fig. 3(c), we plot the values of the autocorrelation, showing a period-2 structure as expected.

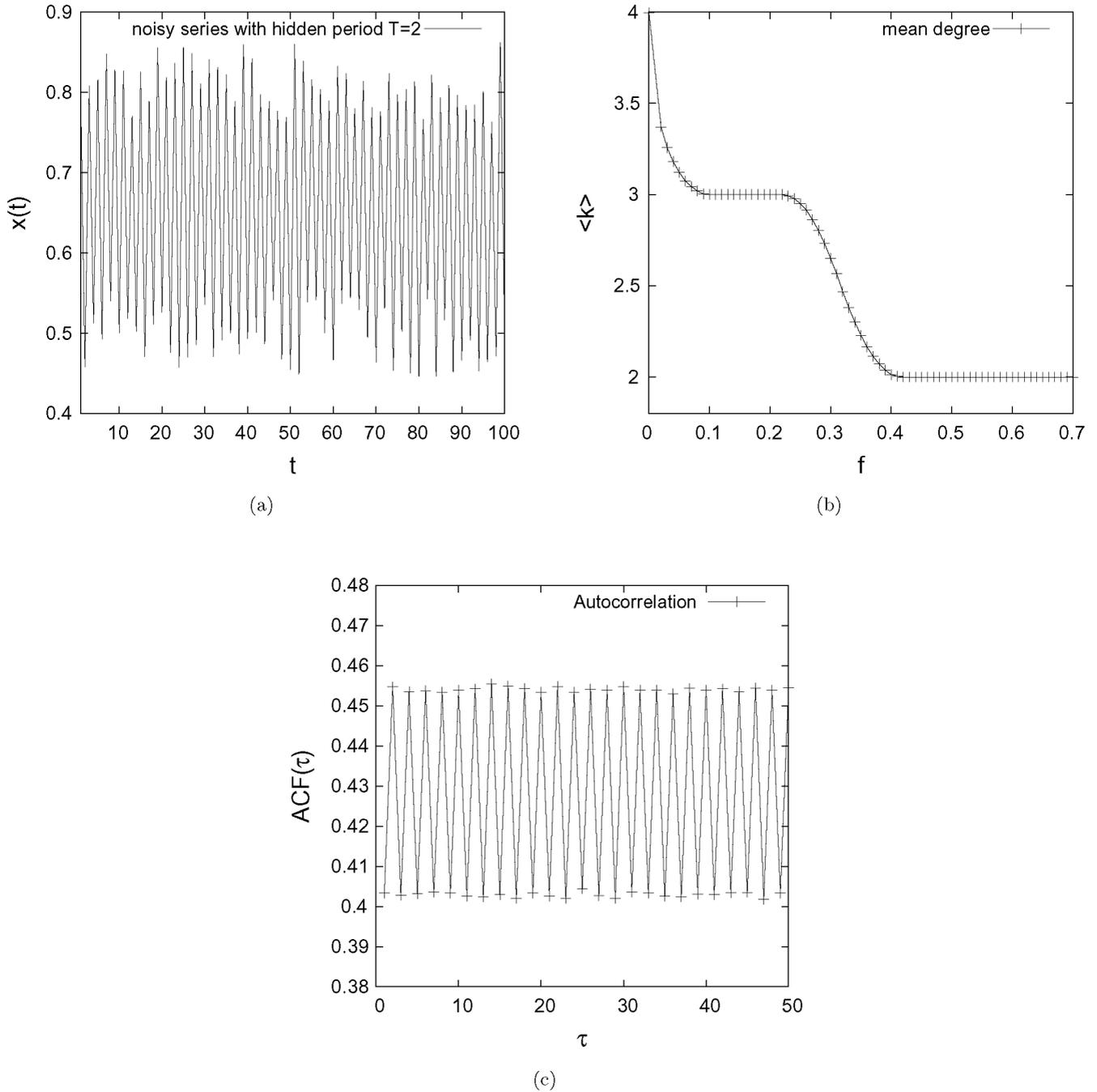
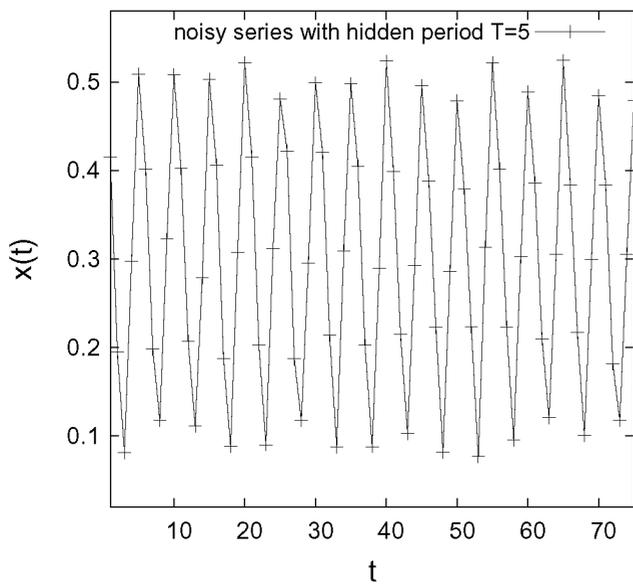
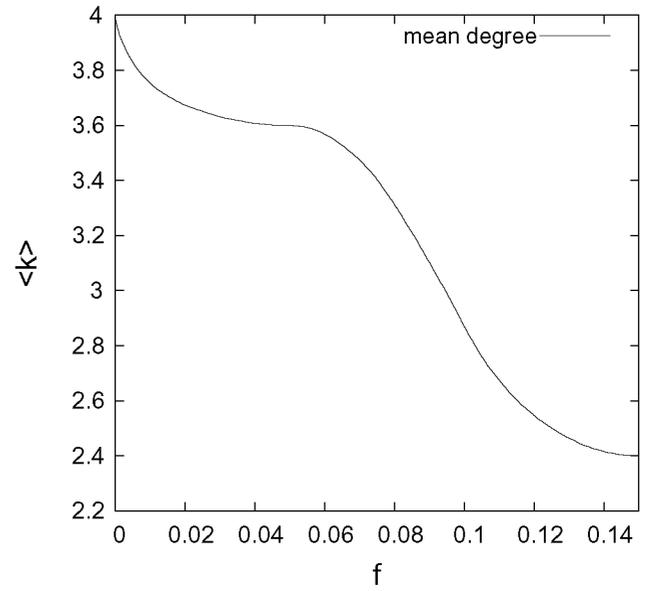


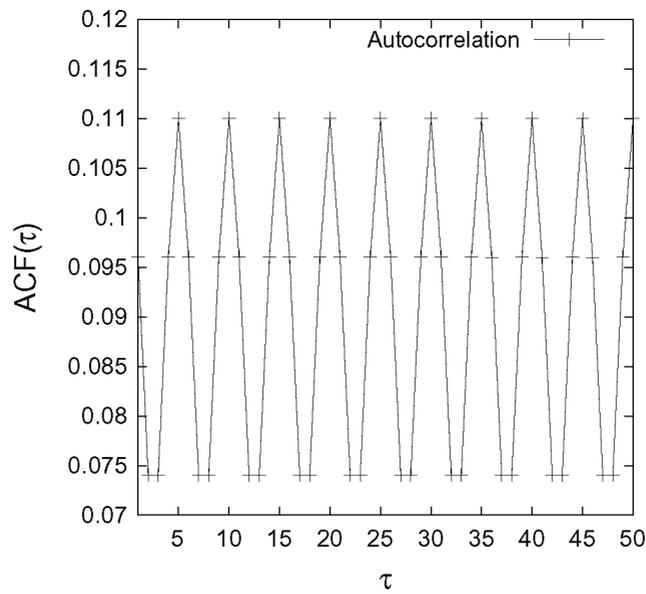
Fig. 3. (a) Periodic series of period $T = 2$ polluted with extrinsic noise extracted from a uniform distribution $U[-0.5, 0.5]$ of amplitude 0.1. (b) Values of the HVG's mean degree \bar{k} as a function of the amplitude of the graph theoretical filter. The first plateau is found for $\bar{k} = 3$, which renders a hidden period $T = (2 - \bar{k}/2)^{-1} = 2$. The second plateau corresponding to $\bar{k} = 2$ is found when the filter is large enough to screen each datum with its first neighbors, such that the mean degree reaches its lowest bound. (c) Autocorrelation function of the noisy periodic series, which is itself an almost periodic series with period $T = 2$, as it should.



(a)



(b)



(c)

Fig. 4. (a) Periodic series of period $T = 5$ polluted with extrinsic noise extracted from a uniform distribution $U[-0.5, 0.5]$ of amplitude 0.05. (b) Values of the HVG's mean degree \bar{k} as a function of the amplitude of the graph theoretical filter. The first plateau is found for $\bar{k} = 3.6$, which renders a hidden period $T = (2 - \bar{k}/2)^{-1} = 5$. (c) Autocorrelation function of the noisy periodic series, which is itself an almost periodic series with period $T = 5$, as it should.

At this point we conclude that the noise filtering is yet another feature of standard time series analysis that can be recovered in the visibility theory. An example of a noisy periodic series of period $T = 5$ is plotted in Fig. 4.

3.2. Noisy periodic versus chaotic

The autocorrelation function is an extremely useful tool to unveil periodic structures in noisy data, in this sense, the aforementioned filter is not meant to be used *instead* of an autocorrelation analysis, but rather as a complementary study. As a matter of fact, in specific situations it is probable that an autocorrelation analysis may provide misleading results. This is so, for instance, in the case of chaotic maps with disconnected attractors. Consider the well-known Logistic map

$$x_{t+1} = \mu x_t(1 - x_t),$$

with $\mu \in [0, 4]$ and $x \in [0, 1]$. This map generates periodic series for $\mu < \mu_\infty = 3.569\dots$, while for $\mu > \mu_\infty$ the map generates chaotic (deterministic and aperiodic) series (besides regions where the orbit turns regular again, called islands of stability [Peitgen *et al.*, 1992]). In the chaotic region, the chaotic attractor is the whole interval $[0, 1]$ only for $\mu = 4$. Concretely, for $\mu \in [3.6, 3.67]$ the attractor is partitioned in two disconnected chaotic bands, and the chaotic orbit makes an alternating

journey between both bands. In Fig. 5 we have plotted a time series of 2^{18} data generated through the Logistic map at $\mu = 3.65$. Note that the map is ergodic, but the attractor is not the whole interval, as there is a gap between both chaotic bands. In this situation, the chaotic series is by definition not periodic, however, an autocorrelation function analysis indeed suggests the presence of periodicity (see Fig. 6(a)), that is reminiscent of the disconnected two-band structure of the attractor. Interestingly enough, applying the aforementioned noise filter technique, at odds with the autocorrelation function, the result suggests that the method does not find any periodic structure, as it should (Fig. 6(b)). Furthermore, information of both the phase space structure and the chaotic nature of the map becomes accessible from an analysis of the HVGs degree distribution (plotted in semi-log scale in Fig. 6(c)): first, we find $P(2) = 1/2$, that indicates that half of the data are located in the bottom chaotic band, in agreement with the alternating nature of the chaotic orbit. This is reminiscent of the misleading result obtained from the autocorrelation function. Second, the tail of the degree distribution is exponential, with an asymptotic slope smaller than the one obtained rigorously (Theorem 2 and [Luque *et al.*, 2009]) for a purely uncorrelated process. This is, according to a recent study on HVGs [Lacasa & Toral, 2010], characteristic of an underlying chaotic process.

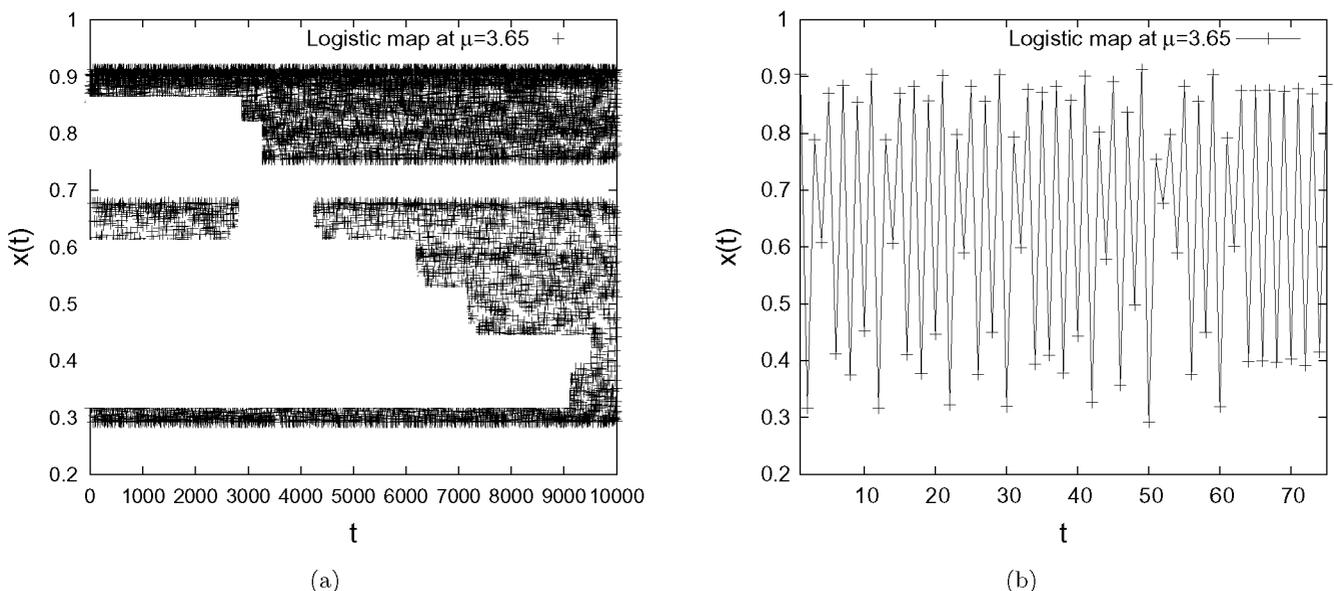


Fig. 5. (a) Series extracted from the Logistic map at $\mu = 3.65$, where the map is chaotic and the attractor is partitioned in two disconnected chaotic bands. (b) Same plot as the left panel, for the first 75 values of the series.

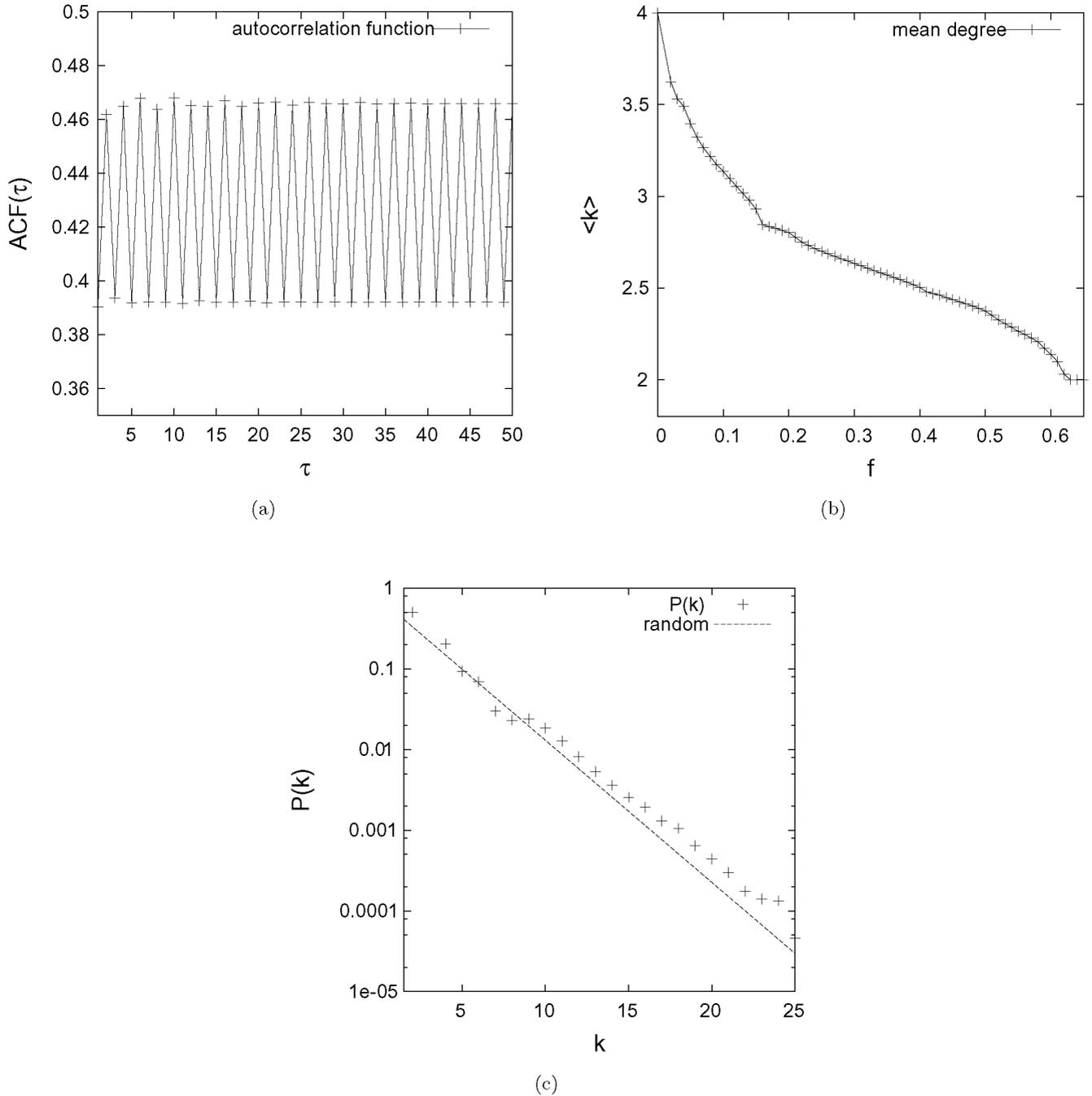


Fig. 6. (a) Autocorrelation function of the chaotic series plotted in Fig. 5, which suggests that the series has a periodic component. This misleading result is the consequence of the orbit within the attractor, more specifically to the alternated visit to the disconnected bands in the chaotic attractor. (b) Values of the HVG's mean degree \bar{k} associated to the same chaotic series, as a function of the amplitude of the graph theoretical filter. No plateau is found, which suggests that the series lacks any periodic structure, as it should. (c) Degree distribution $P(k)$ of the HVG associated to the chaotic series, in semi-log scale. $P(2) = 1/2$, that is reminiscent of the attractor structure and the order of visits to chaotic bands (half of the nodes correspond to data located in the bottom chaotic band, that by construction has degree $k = 2$). The tail of the distribution is exponential with a slope that deviates from the distribution associated to a purely uncorrelated process, which is an indication of a chaotic process, according to a previous study on HVGs [Lacasa & Toral, 2010].

4. Discussion

In this work we have outlined some properties of the HVGs associated to time series extracted from dynamical systems, and have accordingly proposed a method to detect periodicity in signals polluted with noise. The results suggest that the HVG correctly inherits the hidden periodicity of noisy signals, and can be retrieved by making use of the aforementioned filter in situations where the noise level is not very large (indeed, the maximum power of the noise is of order $O(\Delta x^2)$, where $\Delta x = \min\{|x_i - x_j|\}_{i,j}$).

Also, we have found that specific pathological cases where a classical time series analysis yields misleading results, such as in chaotic (aperiodic) series generated from chaotic maps with a disconnected attractor, can be efficiently analyzed within this network-based tool. This approach is radically different from traditional methods for time series analysis since it is based on graph theoretical properties, and therefore can be used as a complementary tool in practical situations. We have deliberately not tackled the task of systematically comparing the goodness of such graph theoretical method with other standard tools of time series analysis, and have restricted this comparison to checking that an autocorrelation function analysis provides equivalent results for the cases addressed here. In this sense, we must emphasize that the goal of this work is to push forward the state of the art in the visibility theory providing mathematically sound properties, rather than putting the practical usefulness of both methods in direct competence.

Acknowledgments

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