LOW-DIMENSIONAL MODELLING FOR OPTIMAL STREAKS IN THE BLASIUS BOUNDARY LAYER

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Summary This paper is concerned with the low dimensional structure of optimal streaks in the Blasius boundary layer. Optimal streaks are well known to exhibit an approximate self-similarity, namely the streamwise velocity re-scaled with their maximum remains almost independent of both the spanwise wavenumber and the streamwise coordinate. However, the reason of this self-similar behavior is still unexplained as well as unexploited. After revisiting the structure of the streaks near the leading edge singularity, two additional approximately self-similar relations involving the velocity components and their wall normal derivatives are identified. Based on these properties, we derive a low dimensional model with two degrees of freedom. The comparison with the results obtained from the linearized boundary layer equations shows that this model is consistent and provide good approximations.

INTRODUCTION AND FORMULATION

When moderate levels of free-stream turbulence or surface roughness of the wall are able to induce 3D perturbations undergoing non-negligible spanwise variations, alternating low and high momentum within the boundary layer, which amplify and trigger the transition to turbulence (see, [1], and references therein). These perturbations are low frequency, streamwise elongated, and the associated streamwise velocity is much larger than (namely, $O(\sqrt{\mathrm{Re}})$ times) the cross flow velocity components. Due to their structure they are called *streaks*. The streaks show an approximately self-similar behavior was experimentally observed [2], namely the spatially evolving streamwise velocity component distribution along the wall normal direction is approximately independent of both the streamwise coordinate and the spanwise wavenumber. Such approximate self-similarity unexpected since it is not directly related to the exact boundary layer, opens up the possibility of obtaining low dimensional descriptions.

The starting point is the incompressible continuity and Navier-Stokes equations in the usual boundary layer scaling/approximation, assuming a spanwise oscillation with a period comparable to the boundary layer thickness, $\delta^* = L^*/\sqrt{\mathrm{Re}}$, where L^* is the distance to the leading edge and $\mathrm{Re} = u^* L^*/\nu \gg 1$ is the Reynolds number based on the free stream velocity u^*. Nondimensionalization is made in terms of the usual units: L^* for the stream wise coordinate x, δ^* for the wall normal and spanwise coordinates, u^* for the streamwise velocity, $u^*/\sqrt{\mathrm{Re}}$ for the wall normal and spanwise velocity components, and $\rho^*(u^*)^2$ for the pressure. Perturbations of the basic, almost parallel, 2D steady state are of the form,

$$(u, v, w, p) = (u_b, v_b, 0, p_b) + (U, V, \mathrm{i}W, P/\mathrm{Re})\mathrm{e}^{\mathrm{i}\alpha z} + \ldots, \tag{1}$$

where the basic flow may be given in terms of the self similar, normal coordinate, $\zeta = y/\sqrt{x}$, and the streamfunction F, which obeys Blasius equation $F''' + FF''/2 = 0$ with boundary conditions $F(0) = F'(0) = 0$ and $F'(\infty) = 1$, and behaves as $F \simeq \zeta - a$ as $\zeta \to \infty$, where $a = 1.721$. The velocity components U, V and W as well as the pressure P associated with the perturbations satisfy the linerized boundary layer (LBL) equations with aproppiated boundary conditions at the plate and at the free stream.

The asymptotic analysis near the free stream of the LBL-equations was carried out in [3], to conclude that the behavior of streaky perturbations near the free stream is slaved to the solution inside of the boundary layer, and such that (see [3] for details)

$$(U, \hat{V}, \hat{W}, \hat{P}) = \left[\left(0, \hat{V}_\infty(\hat{x}), \hat{W}_\infty(\hat{x}), \hat{P}_\infty(\hat{x})\right) + O(\mathrm{e}^{-(\zeta-a)^2/2}\right] \mathrm{e}^{-\sqrt{\hat{x}}(\zeta-a)}, \quad \text{with} \quad \hat{x} = \alpha^2 x. \tag{2}$$

The functions $\hat{V}_\infty$, $\hat{W}_\infty$, and $\hat{P}_\infty$ are such that $\hat{x}\hat{V}'_\infty = \hat{V}_\infty/2 + \sqrt{\hat{x}}\hat{P}_\infty$ and $\hat{V}_\infty + \hat{W}_\infty = 0$; note that $\hat{V}$, $\hat{W}$, and $\hat{P}$ converge quite slowly near the leading edge, when $\hat{x}$ is small. This result clearly suggests the convenience of replacing $\hat{W}$ by the variable $H = \hat{W} + \hat{V}$, which behaves as $\exp[-(\zeta - a)^2/2]$ at large ζ.

As $\hat{x} \to 0$, the flow variables exhibit a power law behavior, whose exponent is given by an eigenvalue problem formulated and solved by Luchini [4]. This problem was re-interpreted in [3] invoking (2), which suggests rewriting the flow variables as

$$(U, \hat{V}, H, \hat{P}) \simeq \hat{x}^{-\lambda}(\hat{x}\tilde{U}, \hat{x}\tilde{V}, \sqrt{\hat{x}}\tilde{H}, \hat{x}\tilde{P}) \exp[-\sqrt{\hat{x}}(\zeta - a)]. \tag{3}$$

The most dangerous eigenvalue is $\lambda_1 \simeq 0.787$, which means invoking (3) that it promotes algebraic growth in the streamwise and wall normal velocity components. Thus, this eigenvalue is the only one responsible for the growing part of transient growth.

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The integration of the LBL equations (see Fig.1, left plot) shows that the wall-normal profiles of U and H are almost linearly dependent and almost constant along $\hat{x}$ when rescaled with their maxima, which means that this property holds true in $0 < \hat{x} \leq 1$. Thus, the wall-normal profiles of both U and H can be approximately described in terms of just one mode. Now, a look at the continuity equation reveals that because U and H can be both approximately described using

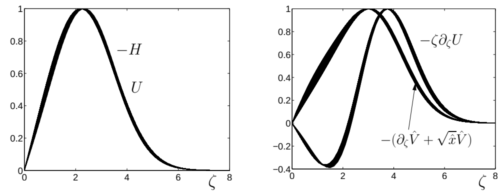

Figure 1. Left: profiles of U and $-H$ obtained from the LBL equations in the wall normal coordinate, re-scaled with their maxima, for 400 values of logarithmically equispaced values of $\hat{x}$, between 10^{-7} and 1. Rigth: as in left plot, but for the quantities $-\zeta\partial_\zeta U$ and $-(\partial_\zeta \hat{V} + \sqrt{\hat{x}}\hat{V})$.

one mode, the term $\zeta\partial_\zeta U/2$ is also described by one mode; these two modes are linearly independent, even approximately. Therefore, the remaining terms in the continuity equation, namely $\partial_\zeta \hat{V} + \sqrt{\hat{x}}\hat{V}$, must be also approximately described as a linear combination of at most two modes. In fact, Fig.1, right plot, shows that $\partial_\zeta \hat{V} + \sqrt{\hat{x}}\hat{V}$ is approximately described in terms of only one mode.

Low dimensional model

Thus, we have three quantities (depending on the velocity components) that can be described in terms of three modes:

$$U = A_1(\hat{x})\tilde{U}_1(\zeta), \quad \partial_\zeta \hat{V} + \sqrt{\hat{x}}\hat{V} = A_2(\hat{x})\tilde{V}_1', \quad H = A_3(\hat{x})\tilde{H}_1(\zeta). \tag{4}$$

The modes in (4), $\tilde{U}_1$, $\tilde{V}_1'$, and $\tilde{H}_1$, are taken from the eigenfunction associated to the only eigenvalue that promotes algebraic growth streamwise near the leading edge. In other words, the modal description of the whole solution is governed by the behavior near the leading edge. The streamwise evolution of the amplitudes A_1, A_2, and A_3 are given by a system of three linear ordinary differential equations. This system together with eqs. (4) constitute a low dimensional model that provides a quite good approximation of the streaks in the Blasius boundary layer. This is illustrated in Fig.2a where wall normal profiles of the the the cross flow velocity components obtained from the reduced model are compared with the ones obtained from the LBL-equations, for various representative values of $\hat{x}$. Also, Fig.2b shows the ability of this low dimensional model to reconstruct the optimal perturbations. This approximation is illustrated with dashed lines for two values of the initial stage, $x_{\text{in}} = 0.01$ and $x_{\text{in}} = 10^{-5}$; the exact counterparts calculated in [3] are also plotted with solid lines. Note that the approximation is good. In particular, the maxima of the gain curves are attained at $\alpha = 0.445$ and 0.472 for $x_{\text{in}} = 0.01$ and 10^{-5}, respectively, which compares quite well with their exact counterparts, which are (see [3]) $\alpha \simeq 0.45$ and 0.484, respectively.

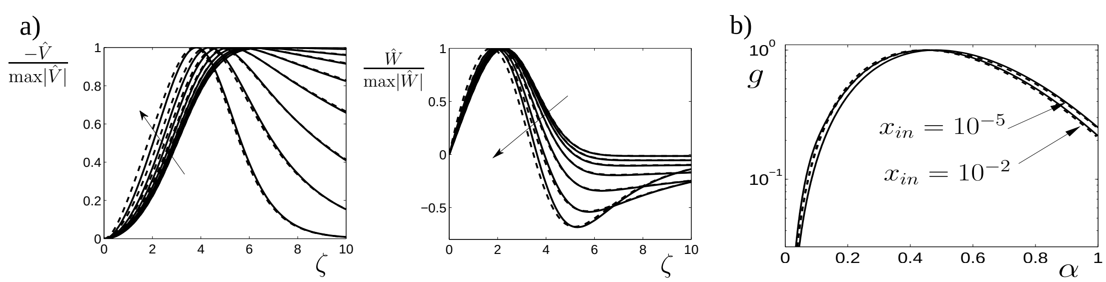

Figure 2. a) Cross flow velocity profiles, re-scaled with their maxima, as calculated from the exact LBL equations (solid lines) and from the low dimensional model (dashed lines) at $\hat{x} = 10^{-7}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ and 1; arrows indicate increasing values of $\hat{x}$. b) Maximum of 'perturbed energy gain' vs. the spanwise wavenumber α, as calculated with the exact model (solid lines) and with the low dimensional model (dashed lines), in the cases $x_{\text{in}} = 0.01$ and $x_{\text{in}} = 10^{-5}$.

References

[1] Kachanov, Y.S. Physical mechanisms of laminar-boundary-layer transition, *A*nnu. Rev. Fluid. Mech. **26**:411-482, 1994.

[2] Westin, K.J.A., Boiko, A.V., Klingmann, B.G.B, Kozlov, V.V. & Alfredsson, P.H. Experiments in a boundary layer subjected to free stream turbulence. Part 1. boundary structure and receptivity, *J. Fluid Mech.* **281** 193-228, 1994.

[3] Higuera M. & Vega J.M. Modal description of internal optimal streaks, *J.* Fluid Mech. **626**: 21-31, 2009.

[4] Luchini, P. Reynolds-number-independent instability of boundary layer over a flat surface, *J.* Fluid Mech. **327**: 101-115, 1996.