

Far-Wake Structure in Rarefield Plasma Flows past Charged Bodies

JUAN R. SANMARTIN* AND S. H. LAM

Gas Dynamics Laboratory, Princeton University, Princeton, New Jersey 08540

(Received 11 June 1970)

The asymptotic structure of the far-wake behind a charged body in a rarefied plasma flow is investigated under the assumption of small ion-to-electron temperature ratio and of flow speed hypersonic with respect to the ions but not with respect to the electrons. It is found that waves are excited even if the flow is subacoustic (flow velocity less than the ion-acoustic speed). For both superacoustic and subacoustic velocities a steep wave front develops separating the weakly perturbed, quasineutral plasma ahead, from the region behind where ion waves appear. Near the axis a trailing front develops; the region between this and the axis is quasineutral for superacoustic speeds. The decay laws in all of these regions, the self-similar structure of the fronts and the general character of the waves are determined. The damping of the waves and special flow detail for bodies large and small compared with the Debye length are discussed. A nonlinear analysis of the leading wave front in superacoustic flow is carried out. A hyperacoustic equivalence principle is presented.

I. INTRODUCTION

The present study deals with the structure of the far-wake behind a charged body moving through a rarefied, uniform plasma. In particular, the excitation of electrostatic ion waves and the evolution of any possible wavefront are investigated; it is assumed that the ion-to-electron temperature ratio is small, and that the ions are in hypersonic flow while the electrons are not. The analysis is mostly linear but some nonlinear results are obtained in relation to present discussions on the existence of ion-acoustic shocks.

The plasma flow past a small charged body was studied by Kraus and Watson,¹ and Al'pert *et al.*²; in both papers quasineutrality was assumed in the far wake. The one-dimensional, initial value problem was considered by Mason,³ who numerically solved the linearized Vlasov equations, and by Washimi and Taniuti,⁴ who briefly discussed the behavior of a nonlinear wave front. Ion-acoustic shocks have been discussed by Kennel and Sagdeev,⁵ Lam,⁶ and Montgomery⁷; experimental evidence of such shocks exists.^{8,9}

In Sec. II the general problem is discussed and the dispersion relation is studied for zero ion temperature; it is shown that waves are excited for both $M_s > 1$ (superacoustic) and $M_s < 1$ (subacoustic) flows, M_s being the Mach number of the flow based on the ion-acoustic speed. In Sec. III a far-field formulation is presented and in Secs. IV and V asymptotic results are derived for $M_s > 1$ and $M_s < 1$, respectively. Section VI deals with warm-ion effects. In Sec. VII a hyperacoustic ($M_s \gg 1$) equivalence principle is presented and in Sec. VIII a nonlinear analysis of the $M_s > 1$ wave

front is performed. Finally, a discussion of the results is given in Sec. IX.

II. COLD IONS: DISPERSION RELATION

The steady-state flow of a rarefied plasma past a charged body may be formulated as follows: The electric potential ϕ is given by Poisson's equation,

$$\nabla^2 \phi = -4\pi e(Z_i N_i - N_e), \quad (1)$$

while the ion and electron densities, N_i and N_e , may be obtained, in terms of ϕ , from the respective time-independent Vlasov equations; in the region of interest, far from the body, these equations may be linearized. At infinity the plasma should be undisturbed, while on the surface of the body $\phi = \phi_p$ (measured with respect to the undisturbed value at infinity) and the distribution functions satisfy conditions that depend on the surface properties. Finally, we note that there are four basic similarity parameters governing the flow,²

$$\beta = \frac{T_i}{Z_i T_e}, \quad M_s^2 = \frac{U^2}{V_s^2}, \quad (2)$$

$$\chi_p = \frac{e\phi_p}{\kappa T_e}, \quad \delta = \frac{R}{\lambda_D},$$

where $V_s \equiv [Z_i \kappa T_e (1 + 3\beta)/m_i]^{1/2}$ is the ion-acoustic speed and $\lambda_D \equiv (4\pi e^2 N_\infty / \kappa T_e)^{1/2}$ is the electron Debye length; T_i , T_e , and N_∞ are the undisturbed ion and electron temperatures and electron density, m_i and Z_i are the ion mass and charge number, R is the characteristic length of the body, and U is its speed.

Let us now introduce the assumptions $\beta \ll M_s^2 \ll m_i/m_e$, $\beta \ll 1$, where m_e is the electron mass; notice that $M_s(m_e/m_i)^{1/2}$, $M_s\beta^{-1/2}$, and M_s are

roughly the Mach numbers based on the electron thermal, ion thermal, and ion-acoustic speeds, respectively, and thus the electrons are in low subsonic flow and the ions in hypersonic flow while a large range of values is allowed for M_s . The preceding assumptions imply that electron plasma oscillations will not be excited, while weakly damped ion waves will; furthermore, the assumptions allow a simplified approximate formulation for the far field: First, we can avoid using the electron Vlasov equation because ion waves are so slow that, far from absorbing boundaries, the electrons will be in equilibrium with the field; hence, for small perturbations we have

$$N_e \approx N_\infty(1 + e\phi/\kappa T_e). \quad (3)$$

Second, by letting $\beta \rightarrow 0$ we can substitute the first two moment equations

$$\hat{\nabla} \cdot N_i \mathbf{v}_i = 0, \quad (4)$$

$$N_i m_i \mathbf{v}_i \cdot \hat{\nabla} \mathbf{v}_i = -Z_i e N_i \hat{\nabla} \phi - \hat{\nabla} \cdot \mathbf{P}_i, \quad (5)$$

where the pressure term is to be considered only where crossflow is important (near the body), for the ion Vlasov equation. After introducing

$$\chi = \frac{e\phi}{\kappa T_e}, \quad \mathbf{r} = \frac{\hat{\mathbf{r}}}{\lambda_D} (\nabla = \lambda_D \hat{\nabla}) \quad (6)$$

and linearizing Eqs. (4) and (5), the system (1), (3)–(5) yields an equation for χ ,

$$\left(\frac{\partial^2}{\partial z^2} \nabla^2 - \frac{\partial^2}{\partial z^2} + M_s^{-2} \nabla^2 \right) \chi = 0 \quad (7)$$

(where the z axis is taken along the direction of the undisturbed plasma motion). In this section and in Secs. III–V, this simplified formulation will be used, and the effects of both $\beta \neq 0$ and a deviation from (3) will be considered in Sec. VI.

Equation (7) is satisfied by elementary solutions of the form $\chi \sim \exp(i\mathbf{k} \cdot \mathbf{r})$, if the dispersion relation

$$P(\mathbf{k}) \equiv k_z^2(1 + k^2) - M_s^{-2}k^2 = 0 \quad (8)$$

is satisfied. To better understand (8) we note that the well-known low-frequency oscillations of an equilibrium plasma of hot electrons and cold ions are given by¹⁰

$$\omega^2 = \frac{\omega_{pi}^2}{1 + (\lambda_D \hat{k})^{-2}}, \quad (9)$$

where ω is the frequency, $\hat{\mathbf{k}}$ is the wavenumber, and $\omega_{pi} \equiv (4\pi e^2 Z_i N_\infty / m_i)^{1/2}$ the ion-plasma frequency; in (9) $T_i \approx 0$ and electron Landau damping is neglected. If we now measure lengths in units of λ_D ($\hat{k} \lambda_D \rightarrow k$) and make the transformation $\omega \rightarrow \hat{k}_z U$,

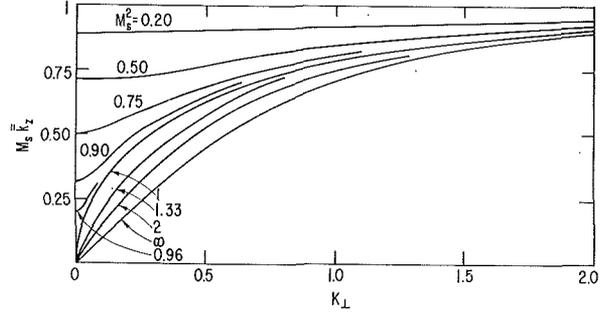


FIG. 1. \bar{k}_z vs k_\perp and M_s (cold ions).

i.e., $\partial/\partial t \rightarrow U \partial/\partial z$ (equivalent to going from a reference frame at rest with the plasma to a frame moving with our body, and asking for a steady-state solution in this frame), Eq. (8) is recovered. For $k \ll 1$, (9) yields ion-acoustic waves and for $k \gg 1$ ion-plasma waves.

$P(\mathbf{k})$ in (8) may be rewritten as

$$P(\mathbf{k}) \equiv (k_z^2 + \bar{k}_z^2)(k_z^2 - \bar{k}_z^2), \quad (10)$$

where

$$\bar{k}_z(k_\perp, M_s) = 2^{-1/2} \{ [(k_\perp^2 + 1 + M_s^{-2})^2 - 4M_s^{-2}]^{1/2} + k_\perp^2 + 1 - M_s^{-2} \}^{1/2}, \quad (11)$$

$$\bar{k}_z(k_\perp, M_s) = 2^{-1/2} \{ [(k_\perp^2 + 1 + M_s^{-2})^2 - 4M_s^{-2}]^{1/2} - k_\perp^2 - 1 + M_s^{-2} \}^{1/2}. \quad (12)$$

The elliptic roots, $k_z = \pm i\bar{k}_z$, do not yield waves and their appearance is due to the fact that z acts both as a timelike variable and as a space coordinate; the hyperbolic roots, $k_z = \pm \bar{k}_z$, do yield waves of the type embodied in (9). A detailed knowledge of \bar{k}_z as a function of k_\perp and M_s is essential for the asymptotic analysis performed in later sections; we give \bar{k}_z in Fig. 1. In Fig. 2 we present $\bar{k}_z' \equiv \partial\bar{k}_z/\partial k_\perp$ schematically.

The following points are worth mentioning now:

(A) The limiting curve $M_s \rightarrow \infty$ in Fig. 1 is the

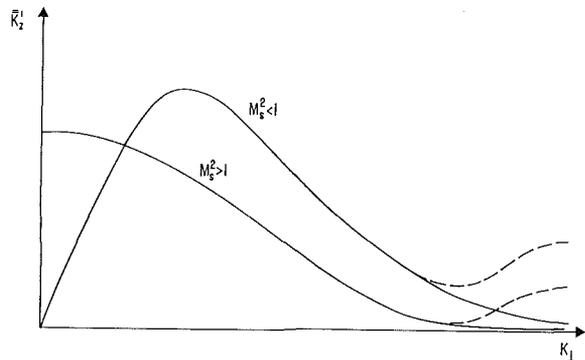


FIG. 2. Schematic representation of \bar{k}_z' vs k_\perp , for both $M_s > 1$ and $M_s < 1$ (—cold ions, --- warm ions).

same $\omega_p^{-1}\omega(k\lambda_D)$ function given in (9). This will result in a hyperacoustic equivalence principle (Sec. VII).

(B) The $M_s > 1$ curves in Fig. 1 behave like the $M_s \rightarrow \infty$ curve; this clearly shows that the roots $\pm\bar{k}_z$ correspond to ion waves. Note the maximum in \bar{k}'_z (the "group velocity") at $k_\perp = 0$ (Fig. 2); the development of an ion-acoustic wavefront is to be expected from this (Sec. IV).

(C) Predictably, the $M_s < 1$ curves show a different behavior; in fact, the existence itself of the real roots $\pm\bar{k}_z$ (i.e., the excitation of waves) for $M_s < 1$ seems to contradict general ideas on "subsonic" flows and is in contrast to earlier conclusions.¹ To clarify this we note that the phase velocity, ω/\bar{k} , of the ion waves in (9) depends on \bar{k} (contrary to the classical case of sound waves, which are not dispersive) and we have

$$0 \leq \omega/\bar{k} \leq V_s.$$

Thus, even if $M_s < 1$, the motion is "supersonic" for a part of the wave spectrum, which is given by $k \geq (M_s^{-2} - 1)^{1/2}$, as follows from the condition $\omega/\bar{k} \leq U$. In Fig. 1, as expected, $\bar{k}_z(k_\perp = 0) = (M_s^{-2} - 1)^{1/2}$ for $M_s < 1$. These curves only partially cover the wave spectrum of the $M_s > 1$ curves; this clarifies the difference in behavior.

Finally, notice that \bar{k}'_z also has a maximum for $M_s < 1$ (Fig. 2); this should result in the appearance of a subacoustic wave front. The maximum of \bar{k}'_z is

$$\bar{k}'_{zm} = 2^{-1/2}(1 - M_s^2)^{-1/2}[(1 - 3M_s^2/4)^{3/2} - 1 + 9M_s^2/8]^{1/2}, \quad (13a)$$

$$\bar{k}'_{zm} = (M_s^2 - 1)^{-1/2} \quad (13b)$$

for M_s smaller and greater than unity, respectively.

III. FAR-FIELD FORMULATION

Equation (7) is invalid near the body, where some of the boundary conditions are given. To overcome this, let us go back to Eqs. (1), (4), and (5); introducing χ , \mathbf{r} (6) and

$$n_e = \frac{N_e}{N_\infty} - 1, \quad n_i = \frac{Z_i N_i}{N_\infty} - 1,$$

$$\mathbf{u} = \frac{\mathbf{v}_i}{U} - \mathbf{1}_z, \quad \mathbf{p}_i = Z_i \mathbf{P}_i (m_i N_\infty U^2)^{-1}$$

($\mathbf{1}_z$ being an unit vector along the z axis) we obtain

$$\left(\nabla^2 \frac{\partial^2}{\partial z^2} + M_s^{-2} \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \chi = D, \quad (14)$$

$$D \equiv -\nabla \nabla : \mathbf{u} \mathbf{u} (1 + n_i) - M_s^{-2} \nabla \cdot (n_i \nabla \chi) - \nabla \nabla : \mathbf{p}_i + \frac{\partial^2}{\partial z^2} (n_e - \chi).$$

This equation is valid everywhere. For $r \gg \delta$, Eq. (7) is recovered by neglecting the nonlinear terms, the cross-flow, and the deviation from (3), that is, by writing $D = 0$ in (14).

To solve Eq. (14), where we assume that D is known, we define the transform

$$J_k = \int_V d\mathbf{r} \exp(-i\mathbf{k} \cdot \mathbf{r}) J(\mathbf{r}) \quad (15)$$

of any function $J(\mathbf{r})$; the integral covers the whole space except the volume of the body. Multiplying (14) by $\exp(-i\mathbf{k} \cdot \mathbf{r})$, integrating over a volume bounded by the body and an enclosing surface Σ , using Gauss' theorem and then removing Σ to infinity, we get

$$P(\mathbf{k})\chi_k = D_k + \int_S ds \exp(-i\mathbf{k} \cdot \mathbf{r}) \cdot \left[(\nabla + i\mathbf{k}) \cdot \left(\frac{\partial^2}{\partial z^2} + M_s^{-2} \right) - \mathbf{1}_z (k^2 + 1) \left(\frac{\partial}{\partial z} + ik_z \right) \right] \chi; \quad (16)$$

in the above integral over the surface of the body the normal is taken positive outward. The right-hand side of (16) can be rewritten in a variety of ways; the most convenient form can be arrived at by introducing the transform (15) directly in (1), (4), and (5). In this way we obtain

$$P(\mathbf{k})\chi_k = \mathbf{k} \mathbf{k} : [(1 + n_i) \mathbf{u} \mathbf{u} + \mathbf{p}_i]_k - M_s^{-2} i\mathbf{k} \cdot (n_i \nabla \chi)_k - k_z^2 (n_e - \chi)_k + \int_S ds \exp(-i\mathbf{k} \cdot \mathbf{r}) \cdot [i\mathbf{k} \cdot \mathbf{b} - 2ik_z \mathbf{a} - k_z^2 (\nabla \chi + i\mathbf{k} \chi)], \quad (17)$$

where

$$\mathbf{a} = (N_\infty U)^{-1} Z_i N_i \mathbf{v}_i - \mathbf{1}_z,$$

$$\mathbf{b} = (N_\infty U^2)^{-1} Z_i N_i \mathbf{v}_i \mathbf{v}_i - \mathbf{1}_z \mathbf{1}_z + \mathbf{p}_i + M_s^{-2} \chi \mathbf{1};$$

\mathbf{a} and \mathbf{b} are nondimensional perturbations of the current and momentum flux on the surface of the body.

Using (17) and the relation

$$(2\pi)^{-3} \int d\mathbf{k} \chi_k \exp(i\mathbf{k} \cdot \mathbf{r}) = \chi(\mathbf{r}), \quad (18)$$

which holds for all \mathbf{r} outside the body, we obtain

$$\chi(\mathbf{r}) = \int_V d\mathbf{r}' \left((n_e - \chi) \frac{\partial^2}{\partial z'^2} - [\mathbf{p}_i + (1 + n_i) \mathbf{u} \mathbf{u}] : \nabla' \nabla' + M_s^{-2} (n_i \nabla' \chi) \cdot \nabla' \right) \cdot F(\mathbf{r} - \mathbf{r}') + \int_S ds' \cdot \left(2\mathbf{a} \frac{\partial}{\partial z'} - \mathbf{b} \cdot \nabla' + (\nabla' \chi) \frac{\partial^2}{\partial z'^2} - \chi \frac{\partial^2}{\partial z'^2} \nabla' \right) F(\mathbf{r} - \mathbf{r}'), \quad (19)$$

where the functions inside the brackets are evaluated at \mathbf{r}' , and

$$F(\mathbf{r}) = (2\pi)^{-3} \int d\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) [P(\mathbf{k})]^{-1}, \quad (20)$$

which is the Green's function of (7). To simplify (20) we use (10) to write

$$F(\mathbf{r}) = \frac{1}{4\pi^2} \int_0^\infty k_\perp dk_\perp J_0(k_\perp \rho) \int_{-\infty}^\infty \frac{dk_z \exp(ik_z z)}{(k_z^2 + \bar{k}_z^2)(k_z^2 - \bar{k}_z^2)},$$

where J_0 is the Bessel function and ρ is a polar coordinate in a plane perpendicular to the z axis. The k_z integration can be evaluated by the method of residues (in order to satisfy the condition of no waves at infinity upstream we must go below the \bar{k}_z poles in the complex \bar{k}_z plane); we finally arrive at

$$\begin{aligned} F &= F_1, & z < 0, \\ F &= F_1 + F_2, & z > 0, \end{aligned}$$

where

$$F_1 = - \int_0^\infty \frac{k_\perp J_0(k_\perp \rho) dk_\perp}{4\pi \bar{k}_z (\bar{k}_z^2 + \bar{k}_z'^2)} \exp(-|z| \bar{k}_z), \quad (21a)$$

$$F_2 = - \int_0^\infty \frac{k_\perp J_0(k_\perp \rho) dk_\perp}{2\pi \bar{k}_z (\bar{k}_z^2 + \bar{k}_z'^2)} \sin \bar{k}_z z. \quad (21b)$$

Note that the first discontinuous derivative at $z = 0$ is $\partial^4 F / \partial z^4$.

For $|z| \gg 1$, asymptotic results for F may be derived from (21). Furthermore if $|z| \gg \delta$ (that is, if $|z|$ is large compared with both R and λ_D) asymptotic results for χ may be obtained from (19) in terms of the conditions on the surface of the body and in a region of order R around it. This is discussed in the next two sections.

IV. ASYMPTOTIC RESULTS FOR SUPERACOUSTIC FLOW

A. Evaluation of $F(r)$

When $M_s > 1$, $\bar{k}_z(k_\perp)$ never vanishes [$\bar{k}_z \geq (1 - M_s^{-2})^{1/2}$] so that as $|z| \rightarrow \infty$ F_1 becomes exponentially small,

$$F_1 \leq 0 \{ \exp[-|z| (1 - M_s^{-2})^{1/2}] \};$$

thus, we only have to study F_2 (z being positive).

Let us think of F_2 as a function of z and $\sigma \equiv \rho/z$. Assume $z \rightarrow \infty$ with σ fixed. First consider $0 < \sigma < \sigma_1$ where $\sigma_1 = \bar{k}_z'(k_\perp^*)$ (13b); Fig. 2 shows that for any such σ there is a value of k_\perp , k_\perp^* , such that

$$\bar{k}_z'(k_\perp^*) = \sigma. \quad (22)$$

We may then show that

$$\begin{aligned} F_2 &= \frac{-1}{2\pi z} \left(\frac{M_s^2}{(1 - \sigma^2/\sigma_1^2)^{1/2}} \right. \\ &\quad \left. + \frac{(k_\perp^*)^{1/2} \sin z[\bar{k}_z(k_\perp^*) - \sigma k_\perp^*]}{[\sigma |\bar{k}_z''(k_\perp^*)|]^{1/2} \bar{k}_z(k_\perp^*) [\bar{k}_z^2(k_\perp^*) + \bar{k}_z'^2(k_\perp^*)]} \right) \\ &\quad + O(z^{-3/2}), \end{aligned} \quad (23)$$

where $k_\perp^*(\sigma)$ is given implicitly by (22). The first and second terms inside the bracket are, respectively, the contributions to (21b) from the regions $k_\perp \approx 0$ and $k_\perp \approx k_\perp^*$; the second term is $O(z^{-1})$ (although k_\perp^* is a stationary phase point¹¹) because $J_0(k_\perp^* \rho)$ itself is $O(z^{-1/2})$.

When the limit $\sigma \rightarrow 0$ is taken in (23), we find $F_2 \sim z^{-1} \sigma^{-2/3} \sin z M_s^{-1}$. This result fails to be valid for $\rho = O(z^{-1/2})$; for $\rho \ll z^{-1/2}$ one may easily show that

$$F_2 \sim - \frac{\sin z M_s^{-1}}{2\pi M_s^{-1}} \ln \rho^{-1} \quad (24)$$

so that as $\rho \rightarrow 0$, the singularity is only logarithmic. To describe the transition region, $\rho = O(z^{-1/2})$ or $\sigma = O(z^{-3/2})$, a detailed analysis would be required. There is hardly need for it, however, because (1) even though F_2 is singular at the axis, χ is not, as we shall show later in this section, and (2) for nonzero β (no matter how small) the singularity of F_2 itself disappears (see Sec. VI). The special behavior of the present case ($\beta = 0$) may be clarified by pointing out that the focusing effect of an axisymmetric field on completely cold ions produces an infinite ion density on the axis downstream of the body (for two-dimensional flow there is *no* singularity on the axis).

When we take $\sigma \rightarrow \sigma_1$ in Eq. (23), we find

$$\begin{aligned} F_2 &= \frac{-M_s^2 \sigma_1^{1/2}}{2^{1/2} \pi z (\sigma_1 - \sigma)^{1/2}} \\ &\quad \cdot \sin^2 \left(z (\sigma_1 - \sigma)^{3/2} 2^{1/2} 3^{-3/2} \sigma_1^{-5/2} M_s^{-2} + \frac{\pi}{4} \right). \end{aligned} \quad (25)$$

We see in (25) that the region $\sigma \approx \sigma_1$ has to be studied in detail too. Although such an analysis could be performed on (21b) itself, it is more illuminating to go back to Eq. (7); for $\sigma \approx \sigma_1$ this equation admits a self-similar solution which is completely determined by the requirement that it match (25).

We introduce the new variables

$$\eta = \epsilon \sigma_1 M_s \rho, \quad \xi = \epsilon^{1/3} (\sigma_1 M_s)^{-1} z (\sigma - \sigma_1) \quad (26)$$

into (7); ϵ must be a small, arbitrary quantity of $O(z^{-1})$ because $\rho = O(z)$ and we wish η to be $O(1)$.

Retaining the lowest order terms, Eq. (7) becomes

$$\chi_{\xi\xi\xi\xi} + 2\chi_{\xi\eta} + \eta^{-2}\chi_{\xi} = 0 \quad (27)$$

and introducing $\psi = \eta^{1/2}\chi$ and integrating (27) over ξ we get

$$\psi_{\xi\xi\xi} + 2\psi_{\eta} = C(\eta).$$

With the ansatz $C(\eta) \equiv 0$, the above equation admits a self-similar solution of the type $\psi = \eta^{-\nu}G(y)$ where ν is arbitrary and $y = 6^{-1/3}\xi\eta^{-1/3}$; G obeys the equation

$$G'''' - 4(yG' + 3\nu G) = 0. \quad (28)$$

To match to (25) we must take $y \rightarrow -\infty$ ($\sigma_1 - \sigma \neq 0$ and positive, $z \rightarrow \infty$); in terms of η and y , Eq. (25) yields

$$F_2 \sim \eta^{-2/3}(-y)^{-1/2} \sin^2 \left(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4} \right). \quad (29)$$

Choosing $\nu = \frac{1}{6}$ we find $G = KA_i^2(y)$, where K is a constant and

$$A_i(y) \equiv \pi^{-1} \int_0^{\infty} dt \cos \left(\frac{t^3}{3} + yt \right)$$

is the Airy function¹²; then for $y \rightarrow -\infty$, $\eta^{-1/2-\nu}G(y)$ has the behavior indicated in (29). We finally obtain that for $\sigma \approx \sigma_1$

$$F_2 \approx \frac{M_s^{4/3} \sigma_1^{-1/3}}{2^{2/3} 3^{1/6}} z^{-2/3} A_i^2(y). \quad (30)$$

For $y \rightarrow -\infty$ ($\sigma_1 - \sigma \gg z^{-2/3}$), (25) is recovered from (30). For $y \rightarrow \infty$, $A_i(y)$ decays exponentially. Thus, F is exponentially small for $\sigma > \sigma_1$; the region $\sigma \approx \sigma_1$ is called the wave front (ξ and η are wave-front coordinates). Note that the width of the wave front is $O(z^{1/3})$.

B. Asymptotic Behavior of χ

Let us write $\chi = I_v + I_s$, where I_v and I_s are the volume and surface integrals in (19), respectively. To evaluate either term we need $F(\bar{\mathbf{r}})$ ($\bar{\mathbf{r}} = \mathbf{r} - \mathbf{r}'$) instead of $F(\mathbf{r})$. We now assume both $|z| \gg 1$ and $|z'| \gg \delta$. First consider I_s ; for $\sigma \neq 0$, we may then use the approximations $\bar{z} \approx z$ and

$$\bar{\sigma} \equiv (\rho^2 + \rho'^2 - 2\rho\rho' \cos \alpha')^{1/2} (z - z')^{-1} \approx \rho/z \equiv \sigma$$

except in rapidly varying factors where $\bar{\sigma} \approx (\rho - \rho' \cos \alpha')(z - z')^{-1}$ and $\bar{z} = z - z'$ must be used. [The azimuthal angle α' is measured from the axial plane containing the point (ρ, z) .] Since $\bar{\sigma} \approx \sigma$, we conclude that I_s , as F , is exponentially small ahead of the wave front (Mach cone), while for

$0 < \sigma < \sigma_1$ we may use (23) to obtain

$$I_s = \frac{-(k_{\perp}^*)^{1/2} [\bar{k}_z(k_{\perp}^*) + \bar{k}_z^2(k_{\perp}^*)]^{-1}}{2\pi\sigma^{1/2} \bar{k}_z(k_{\perp}^*) |\bar{k}_z'(k_{\perp}^*)|^{1/2} z} \cdot \int_s^{\infty} ds' \cdot \left(2\mathbf{a} \frac{\partial}{\partial z'} - \mathbf{b} \cdot \nabla' + (\nabla' \chi) \frac{\partial^2}{\partial z'^2} - \chi \frac{\partial^2}{\partial z'^2} \nabla' \right) \cdot \sin [(z - z') \bar{k}_z(k_{\perp}^*) - (\rho - \rho' \cos \alpha') k_{\perp}^*] + O(z^{-3/2}), \quad (31)$$

where k_{\perp}^* is given by (22); the first term in (23) made no contribution to (31), because the \mathbf{r}' derivatives acting on z' produce a z^{-2} factor. Otherwise, I_s behaves like F_2 . {The variables \mathbf{r} and \mathbf{r}' may be separated in (31) and the factors $\sin, \cos [z \bar{k}_z(k_{\perp}^*) - \rho k_{\perp}^*]$ taken outside the integral.}

The $\sigma \approx \sigma_1$ region can be studied in two different ways: (1) Since I_s obeys Eq. (7), we can perform an analysis similar to that carried out for F_2 . The matching to (31) now requires choosing $\nu = \frac{1}{2}$ in Eq. (28) with the result

$$I_s \sim z^{-1} \frac{d}{dy} [A_i(y)]^2. \quad (32)$$

(2) We can insert (30) directly into (19); the \mathbf{r}' derivatives operating on $A_i^2(y)$ produce a $z^{-1/3}$ factor and (32) is recovered. (The \mathbf{r}' derivatives in the integrand of I_s correspond to \mathbf{k} factors in its Fourier spectrum which is therefore weaker for $k \rightarrow 0$ than the F_2 spectrum; thus, the value of I_s at the wave front, which is dominated by small- k waves, decays faster than F_2 .)

We notice here that in both methods we have to assume a body small compared with the wave-front width; since this is $O(z^{1/3})$, we require

$$\hat{z} \gg \delta^2 R \equiv \delta^3 \lambda_D. \quad (33)$$

In the case of large bodies, $\delta \gg 1$, (33) is more restrictive than the already assumed conditions $\hat{z} \gg R, \lambda_D$; for fixed R , the smaller λ_D , the longer the distance to reach the self-similar behavior of the front (32).

For $\sigma \approx 0$, the exact value of $\bar{\sigma}$ has to be retained in I_s . Since $\rho = O(\delta)$ (on the average) at the body the effective minimum value of $\bar{\sigma}$ is $O(\delta/z)$; according to our results for F_2 at $\sigma \approx 0$, Eq. (31) is valid for points near and on the axis if $z\delta^2 \gg 1$. This is always true for $\delta \geq O(1)$ when $z \gg 1$; for $\delta \ll 1$, $I_s \sim \ln \delta^{-1}$ if $z\delta^2 \ll 1$ so that I_s is never singular. A transition region exists for $z\delta^2 = O(1)$. The condition $z\delta^2 \gg 1$ reads

$$\hat{z} \gg \lambda_D / \delta^2 \equiv R / \delta^3. \quad (34)$$

This may be compared to (33).

A similar analysis may be carried out for I_v , and the same behavior is found. The essential point is that although the integration in I_v extends to infinity, only the region near the body, where perturbations are $O(1)$, makes a dominant contribution to the integral.

V. SUBACOUSTIC FLOW

Contrary to the $M_s > 1$ case, we now find $\bar{k}_z = 0$ at $k_\perp = 0$ so that F_1 will not be exponentially small; indeed, for large $|z|$

$$F_1 = \frac{-M_s^2}{4\pi[(1 - M_s^2)\rho^2 + z^2]^{1/2}} + O(z^{-2}) \quad (M_s < 1).$$

Now consider F_2 . For $0 < \sigma < \sigma_1 \equiv \bar{k}'_{zm}$ (13a), there are two values of k_\perp , k_\perp^* , and k_\perp^{**} , such that

$$\bar{k}'_z(k_\perp^*) = \bar{k}'_z(k_\perp^{**}) = \sigma, \quad k_\perp^* < k_\perp^{**};$$

notice that $\bar{k}''_z(k_\perp^*) > 0$, $\bar{k}''_z(k_\perp^{**}) < 0$ (see Fig. 2). On the other hand the small k_\perp range in the integral in (21b) gives a contribution of the order of z^{-2} for $M_s < 1$. Therefore, only the neighborhoods of the stationary points, k_\perp^* and k_\perp^{**} , yield dominant contributions:

$$F_2 = \frac{-(k_\perp^*)^{1/2} \cos z[\bar{k}_z(k_\perp^*) - \sigma k_\perp^*]}{2\pi z \bar{k}_z(k_\perp^*) \sigma^{1/2} [\bar{k}'_z(k_\perp^*)]^{1/2} [\bar{k}_z^2(k_\perp^*) + \bar{k}_z^2(k_\perp^*)]} - \frac{(k_\perp^{**})^{1/2} \sin z[\bar{k}_z(k_\perp^{**}) - \sigma k_\perp^{**}]}{2\pi z \bar{k}_z(k_\perp^{**}) \sigma^{1/2} |\bar{k}'_z(k_\perp^{**})|^{1/2} [\bar{k}_z^2(k_\perp^{**}) + \bar{k}_z^2(k_\perp^{**})]} + O(z^{-3/2}). \quad (35)$$

This may be compared with Eq. (23).

As $\sigma \rightarrow 0$, we have $k_\perp^{**} \rightarrow \infty$ and the second term in (35) has the same singular behavior found in Sec. IV for $M_s > 1$; on the other hand, $k_\perp^* \rightarrow 0$ and the first term in (35) approaches the finite limit

$$F_2 = -\frac{(k_\perp^m)^{1/2} \sin \{z[\bar{k}_z(k_\perp^m) - \sigma k_\perp^m] + \pi/4\} A_i(y)}{2^{1/6} \pi^{1/2} z^{5/6} \sigma^{1/2} [\bar{k}'_z(k_\perp^m)]^{1/6} \bar{k}_z(k_\perp^m) [\bar{k}_z^2(k_\perp^m) + \bar{k}_z^2(k_\perp^m)]} + O(z^{-7/6}), \quad (36)$$

where $y = 2^{1/3} [\bar{k}'_z(k_\perp^m)]^{-1/3} z^{2/3} (\sigma_1 - \sigma)$. One may easily verify that as $y \rightarrow -\infty$ ($\sigma < \sigma_1$), Eq. (35) is recovered from (36). As $y \rightarrow \infty$ ($\sigma > \sigma_1$), F_2 decays exponentially.

The behavior of χ may be analyzed in the manner of Sec. IV. We notice briefly the following points: (1) Ahead of the wave front ($\sigma \approx \sigma_1$), we find $\chi = O(z^{-2})$ because the r' derivatives inside the integrals in (19) operate upon $F_1 \sim z^{-1}$ to yield a z^{-2} decay law. On the other hand, the appearance of the sine and cosine factors in (35) and (36) result in $z^{-5/6}$ and z^{-1} decay laws at the front and behind

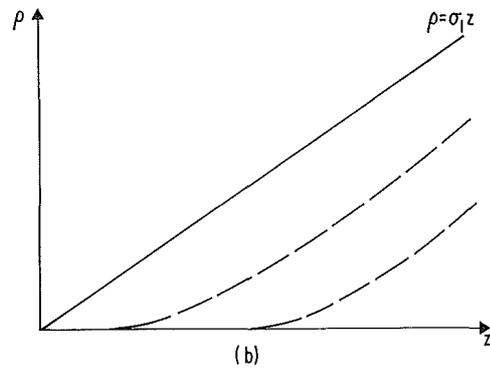
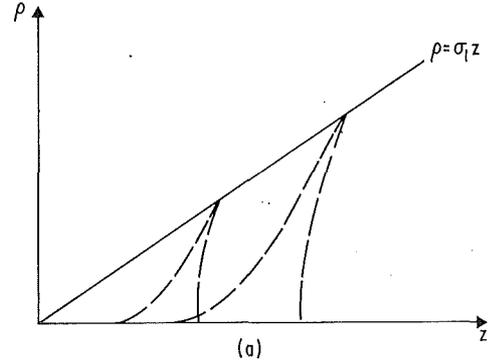


FIG. 3. Steady-state wave patterns (cold ions); (a) $M_s < 1$, (b) $M_s > 1$.

$$-(2\pi z)^{-1} \cos z(M_s^{-2} - 1)^{1/2}.$$

As $\sigma \rightarrow \sigma_1$, k_\perp^* and k_\perp^{**} coalesce. At the common limit k_\perp^m we have $\bar{k}''_z(k_\perp^m) = 0$; thus, k_\perp^m is a stationary point of higher order and (35) is invalid for $\sigma \approx \sigma_1$. A local analysis of Eq. (7) may be carried out in this region, as done for the superacoustic wave front. It is simpler, however, to analyze the integral in (21b) in more detail. We find

it, respectively (same decay of F_2). Therefore, perturbations are weaker ahead of the wave front, as for $M_s > 1$. (2) Conditions (33) and (34) and the accompanying discussions hold for $M_s < 1$. [Condition (34) now only refers to the part of χ arising from the second term in (35).]

The general characteristics of subacoustic and superacoustic flows are shown schematically in Fig. 3. Ahead of a wave front at $\sigma = \sigma_1$, the plasma is quasineutral and only weakly perturbed (very weakly for $M_s > 1$); the angle α that the front forms with the axis is given in Fig. 4 [$\tan \alpha =$

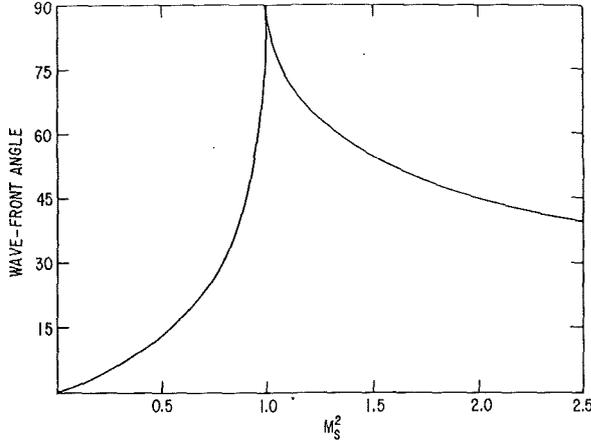


Fig. 4. Wave-front angle versus Mach number (cold ions).

$\sigma_1 \equiv \bar{k}_{zm}$ as given in (13a, b)]. Behind the wave front, ion waves appear. For $M_s < 1$ there is a set of waves, which we call transverse, corresponding to the first (k_1^*) term in (35); a second type of wave, called divergent, corresponds to the second term in (35). The curves of constant phase have the form given in Fig. 3(a). These curves are geometrically similar with respect to the origin and have a relative phase difference of $\pi/2$. For $M_s > 1$, the transverse waves disappear; the curves of constant phase now have the form given in Fig. 3(b). There is a strong resemblance to the pattern of water waves left by a moving ship¹² (although here the divergent waves do not emanate from the body but from different points on the axis); the surface elevation near the critical lines in subcritical motion in water (say, motion in water of infinite depth), first calculated in 1960,¹³ is given by a formula similar to (36).

VI. WARM-ION EFFECTS

In this section we shall study the influence that modifications of our basic dispersion relation, Eq. (8), may have on the far-wake structure. First, we shall briefly consider any effects due to deviations from (3); this may be done by using the linearized Vlasov equation but retaining $T_i = 0$. The resulting dispersion relation is¹⁰

$$1 + \frac{1}{\lambda_D k^2} \left(1 - 2x \exp(-x^2) \int_0^x \exp(y^2) dy - i\pi^{1/2} x \exp(-x^2) \right) - \frac{\omega_{pi}^2}{\omega^2} = 0, \quad (37)$$

$$x = \omega(2^{1/2} \hat{k} V_\alpha)^{-1}, \quad (37)$$

V_α being the thermal velocity of the α species, $\alpha = e, i$. Assuming $|\omega/\hat{k}V_\alpha| \ll 1$, neglecting terms $O(\omega^2/\hat{k}^2 V_\alpha^2)$ and writing $\omega = \hat{k}_z U$, $\hat{k} = \lambda_D^{-1} k$, Eq. (37)

becomes

$$1 + k^{-2} \left[1 - i \left(\frac{\pi}{2} \right)^{1/2} \left(\frac{Z_i m_e}{m_i} \right)^{1/2} \frac{M_s k_z}{k} \right] - (M_s k_z)^{-2} = 0. \quad (38)$$

To solve (38) we write $k_z = \bar{k}_z + i\gamma$ and assume $|\gamma/\bar{k}_z| \ll 1$; we obtain (12) and

$$\gamma \approx \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{Z_i m_e}{m_i} \right)^{1/2} \frac{M_s \bar{k}_z^2}{k(\bar{k}_z^2 + k^2 + 1 - M_s^{-2})}. \quad (39)$$

Our ansatz $|\omega/\hat{k}V_\alpha|$, $|\gamma/\bar{k}_z| \ll 1$ are thus verified except at $k_\perp \approx 0$ for $M_s \approx 1$. Excluding this regime from the present study, we conclude that the only new result is the appearance of a small Landau damping.

For warm ions the ion Vlasov equation must be used in place of Eqs. (4) and (5); this implies changing $-\omega_{pi}^2/\omega^2$ in (37) by an expression like the second term in that equation [that expression has $x = \omega(2^{1/2} \hat{k} V_i)^{-1}$, and $(\lambda_D \hat{k})^{-2}/\beta$ in place of $(\lambda_D \hat{k})^{-2}$]. Assuming $|\omega/\hat{k}V_\alpha| \ll 1$, $|\omega/\hat{k}V_i| \gg 1$ and neglecting terms $O(\omega^2/\hat{k}^2 V_\alpha^2)$ and $O(\hat{k}^4 V_i^4/\omega^4)$, we obtain (for $\omega = \hat{k}_z U$)

$$k_z^2(1 + k^2) - M_s^{-2} k^2 - 3\beta M_s^{-2}(1 + 3\beta)^{-1} k^4 - i \left(\frac{\pi}{2} \right)^{1/2} M_s (1 + 3\beta)^{1/2} \frac{k_z}{k} \cdot [k_z^2 - 3\beta M_s^{-2}(1 + 3\beta)^{-1} k^2] \cdot \left[\left(\frac{Z_i m_e}{m_i} \right)^{1/2} + \beta^{-3/2} \right] \cdot \exp[-k_z^2(1 + 3\beta)(2k^2\beta)^{-1} M_s^2] = 0. \quad (40)$$

The term with the i factor represents electron and ion Landau damping; the other three terms could be obtained by Fourier transforming the differential equation

$$\left(\frac{-3\beta M_s^{-2}}{1 + 3\beta} \nabla^2 \nabla^2 + \frac{\partial^2}{\partial z^2} \nabla^2 + M_s^{-2} \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \chi = 0, \quad (41)$$

which may be derived from (4) and (5) by using an one-dimensional adiabatic law for the small perturbations of P_i . Warm-ion effects, therefore, include ion Landau damping and a small, adiabatic pressure term.

To solve (40) we write $k_z = \bar{k}_z + i\gamma$ and assume $|\gamma/\bar{k}_z| \ll 1$, to find

$$\bar{k}_z \approx \left\{ \left[\left(k_\perp^2 + \frac{\epsilon + \tau}{1 + \tau} \right)^2 - \frac{4\tau}{\epsilon(1 + \tau)^2} \right]^{1/2} + k_\perp^2 \frac{\tau - 1}{\tau + 1} + \frac{\tau - \epsilon}{\epsilon(1 + \tau)} \right\}^{1/2} \left(\frac{\tau + 1}{2} \right)^{1/2}, \quad (42)$$

$$\gamma \approx \left(\frac{\pi}{8}\right)^{1/2} \left(\frac{1+\tau}{\epsilon}\right)^{1/2} \frac{k}{(k^2+1)^2 - \tau/\epsilon} \left[\left(\frac{Z_i m_e}{m_i}\right)^{1/2} + \frac{3^{3/2}}{\epsilon^{3/2}} \exp\left(\frac{-3\bar{k}_z^2}{2k^2} \frac{1+\tau}{\tau}\right) \right], \quad (43)$$

where

$$\epsilon = 3\beta, \quad \tau = \left(\frac{1+3\beta}{3\beta} M_s^2 - 1\right)^{-1}.$$

As $\beta \rightarrow 0$, (42) goes back to (12); $\partial \bar{k}_z / \partial k_\perp \equiv \bar{k}'_z$, as obtained from (42), is represented schematically in Fig. 2.¹⁴

One may easily verify that the ansatz $|\omega/\bar{k}V_e| \gg 1$ is satisfied if

$$\beta(1+k_\perp^2+M_s^{-2}) \ll 1, \quad (44)$$

while $|\gamma/\bar{k}_z| \ll 1$ is satisfied if we have (44) and

$$\beta^{-3/2} M_s^2 \exp(-M_s^2/2\beta) \ll 1, \quad (45a)$$

$$\beta^{-3/2} (k_\perp^2+1)^{-1} \exp[-2\beta(k_\perp^2+1)]^{-1} \ll 1, \quad (45b)$$

which are slightly more restrictive than (44). We also notice that $|\omega/\bar{k}V_e| \ll 1$ follows from (42), whatever the value of U/V_e ; on the other hand if $U \gg V_e$, electron plasma waves are also excited and the structure of the flow will be modified. [For $U/V_e = O(1)$, such waves are heavily damped.] Thus, we conclude that our analysis will be valid if

$$\beta \ll 1, \quad \beta \ll M_s^2 \leq O(m_i/m_e), \quad \beta k_\perp^2 \ll 1 \quad (46)$$

and if (45a, b) are satisfied. (We are again excluding the near-acoustic regime, $M_s \approx 1$.)

From the analysis of this section the following new features of the far wake appear (for $M_s^2 > 4^{4/3} 3^{-2/3} \beta^{1/3}$):

(a) The wave-free region. First, in Fig. 2 notice the appearance of a relative minimum of \bar{k}'_z ; its value is

$$\bar{k}'_{zmi} = 4M_s^{-1} \beta^{3/4} [1 - 2^{-1}(3 + M_s^{-2})\beta^{1/2} + O(\beta)]. \quad (47)$$

For $M_s > 1$, this minimum is absolute so that if $\sigma < \sigma_3 \equiv \bar{k}'_{zmi}$, Eq. (22) admits no solution; hence, a region free of waves exists around the axis, with a z^{-2} decay law. For $M_s < 1$, however, the minimum is only relative so that there is no such quasineutral region near the axis (transverse waves do exist for $\sigma < \sigma_3$).

(b) The second wave front. The existence of a minimum of \bar{k}'_z also implies the appearance at $\sigma \approx \sigma_3$ of a second wave front, a region behind the leading front ($\sigma \approx \sigma_1$) where the perturbations steepen and the decay law is slower than in neigh-

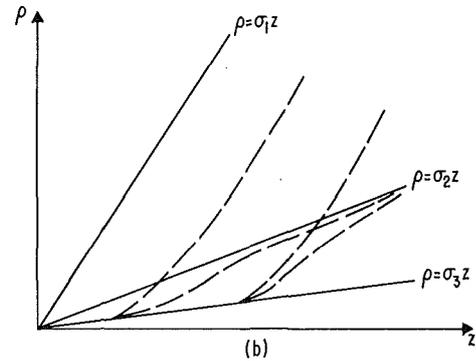
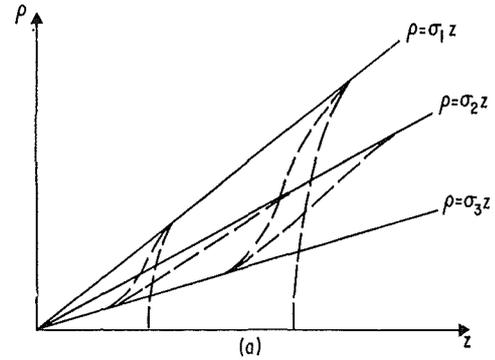


FIG. 5. Steady-state wave patterns (warm ions); (a) $M_s < 1$, (b) $M_s > 1$.

boring regions. An analysis similar to that leading to (36) may be performed for this second front; the decay law therefore is $z^{-5/6}$ and the perturbations behave as the Airy function times a modulating sinusoidal factor. This is valid for $M_s \geq 1$.

(c) Wave damping. All the asymptotic results found up to now are valid only if $z\gamma(\sigma)$ is small; $\gamma(\sigma)$ is the function obtained by solving for $k_\perp^*(\sigma)$ in (22) and using this value in (43). Thus, the damping is markedly directional.

The damping decreases as M_s increases. The ion Landau damping increases with the wavenumber; thus, for $M_s > 1$, it reaches a minimum at the Mach cone, but for $M_s < 1$ it is a minimum on the axis (the transverse waves are the less damped). The electron damping decreases as k_\perp increases. The ion damping is dominant for all k_\perp if

$$\beta^{-3/2} \exp\left(-\frac{M_s^2(1+3\beta)}{2\beta}\right) > \left(\frac{Z_i m_e}{m_i}\right)^{1/2}, \quad M_s < 1,$$

$$\beta^{-3/2} \exp\left(-\frac{1+3\beta}{2\beta}\right) > \left(\frac{Z_i m_e}{m_i}\right)^{1/2}, \quad M_s > 1;$$

it is always dominant for large k_\perp .

The general features of both superacoustic and subacoustic flows with $\beta \neq 0$ are presented in Fig. 5.

We notice that the curves of constant phase of the divergent waves do not reach the axis now but turn back to asymptotically approach the angle $\tan^{-1} \sigma_2$, where

$$\sigma_2 = \left(\frac{1 + 3\beta}{3\beta} M_s^2 - 1 \right)^{-1/2}. \quad (48)$$

Thus for $\sigma_3 < \sigma < \sigma_2$, there are two types of divergent waves for both $M_s > 1$ and $M_s < 1$. It should be remembered that the part of the constant-phase curves approaching σ_2 corresponds to $k_\perp \rightarrow \infty$ and, as indicated earlier, (42) and (43) are invalid for $k_\perp^2 \geq O(\beta^{-1})$; the waves are heavily damped in this region. We stress, however, that near σ_3 the damping is small for sufficiently small β .

VII. HYPERACOUSTIC EQUIVALENCE PRINCIPLE

The $M_s > 1$ results obtained in the preceding sections hold in the hyperacoustic limit, $M_s \gg 1$. Such a case is of special interest, nevertheless, because of the following: Let us write $\nabla^2 = \nabla_\perp^2 + \partial^2/\partial z^2$ in (41) and define $\zeta = zM_s^{-1}$; taking the limit $M_s \rightarrow \infty$ we arrive at

$$\left(\frac{-3\beta}{1+3\beta} \nabla_\perp^4 + \frac{\partial^2}{\partial \zeta^2} \nabla_\perp^2 + \nabla_\perp^2 - \frac{\partial^2}{\partial \zeta^2} \right) \chi = 0. \quad (49)$$

The highest order ζ derivative is now of second order and the structure of (49) is the same as that of the corresponding unsteady two-dimensional problem, in agreement with our previous comment on the $M_s \rightarrow \infty$ curve of Fig. 1 (Sec. II). In classical gas dynamics this results in the known hypersonic equivalence principle. Equation (49) seems to indicate that such a principle also holds for the present case of a collisionless gas with collective interaction.

However, to arrive at (49) we neglected the operator $M_s^{-2} \partial^2/\partial \zeta^2$ (as compared to ∇_\perp^2) in the original equation. To justify this we have to consider the behavior of $\partial^2/\partial \zeta^2$ as $M_s \rightarrow \infty$; from our former analysis it is evident that the behavior of $\sin(\zeta \lim_{M_s \rightarrow \infty} M_s \bar{k}_z)$, $\exp(-|\zeta| \lim_{M_s \rightarrow \infty} M_s \bar{k}_z)$ must be studied. As $M_s \rightarrow \infty$ we find

$$\begin{aligned} M_s \bar{k}_z &\rightarrow \frac{k_\perp}{1+3\beta} \left(\frac{1+3\beta(k_\perp^2+1)}{k_\perp^2+1} \right)^{1/2}, \\ M_s \bar{k}_z &\rightarrow (k_\perp^2+1)^{1/2} M_s. \end{aligned}$$

The hyperbolic root of the original differential equation, \bar{k}_z , approaches a finite limit which is the root of the transformed equation (49). On the other hand, the elliptic root \bar{k}_z , which has been lost in (49), is not well behaved as $M_s \rightarrow \infty$; terms due to this root are such that

$$\frac{\partial^2}{\partial \zeta^2} \sim M_s^2.$$

However, such terms decay at least as $\exp(-|\zeta| M_s)$ and are therefore confined to a thin sheath. Hence, the equivalence principle holds essentially; for instance, outside a sheath of width λ_D the solution of the original differential equation with boundary conditions up to $\partial^3/\partial \zeta^3$ imposed in a plane $\zeta = \zeta_0$, and the solution of the transformed equation with the same conditions up to $\partial/\partial \zeta$, will be the same within an exponentially small error.

VIII. NONLINEAR WAVE-FRONT ANALYSIS

According to the analysis of Sec. V, the subacoustic wave front decays as $z^{-5/6}$; since the waves behind it decay as z^{-1} , the perturbations build up at the front. Surprisingly, no such effect was found for the superacoustic case; the intensity of the waves in both the Mach cone and the region behind it decay as z^{-1} . Actually, the linear analysis of the past sections shows that for either type of flow, the wave fronts are unimportant in terms of energy storage: For $M_s > 1$, the energy per unit length along the Mach cone is $O(z^{-2/3})$ [energy density of order $(z^{-1})^2$ times area of order $z z^{1/3}$]; on the other hand, in the region behind the Mach cone we find an energy per unit length which is $O(1)$. Similarly, for $M_s < 1$, we find energies $O(z^{-1/3})$ and $O(1)$, respectively.

We shall now show, however, that, in superacoustic flow, nonlinear effects may in some cases substantially modify both the structure of the wave front and its decay law. The local failure of the linear theory at the front is due to the cumulative, steepening effect of the nonlinear convective derivative in the ion momentum equation; a similar phenomenon occurs in many other wave motions.

The most convenient variable to analyze is u_z . For simplicity consider $\beta = 0$. Proceeding as in Sec. II but retaining the aforementioned nonlinear term, we arrive at the following equation:

$$\left(\nabla^2 \frac{\partial^2}{\partial z^2} + M_s^{-2} \nabla^2 - \frac{\partial^2}{\partial z^2} \right) u_z = \frac{1}{2} \frac{\partial^2}{\partial z^2} (u^2 - \nabla^2 u^2). \quad (50)$$

After introducing the wave-front coordinates (26) and retaining the lowest order terms, we define $\psi = \eta^n u_z$ and integrate over ξ to get

$$\psi_{\xi\xi\xi} + 2\psi_\eta - \eta^{-n} \psi \psi_\xi = C(\eta); \quad (51)$$

$n = 0$ for two-dimensional flows and $n = \frac{1}{2}$ for three-dimensional flows (the case usually considered in this paper). With the ansatz $C(\eta) \equiv 0$, the above

equation admits a solution of the type $\psi = \eta^{-\nu} G(y)$, $y = \xi/(6\eta)^{1/3}$, if $\nu = \frac{2}{3} - n$. When the nonlinear term is further neglected in (51) a self-similar solution exists for any ν , as indicated in Sec. IV; the actual value of ν was then found by matching to the (linear) solution behind the front, with the result $\nu = \frac{1}{2}$. One may show that this is also true for $n = 0$. A comparison of the nonlinear term with the linear ones in (51) shows that if the linear wave-front structure is initially valid, it always remains so in three-dimensional flows; in two dimensions, however, the nonlinear term ultimately becomes of the order of the linear ones so that the linear front structure breaks down.

IX. DISCUSSION

The structure of the far wake behind a charged body in a rarefied plasma flow has been studied under the assumptions of small ion-to-electron temperature ratio β , and of flow speed hypersonic with respect to the ions but not with respect to the electrons.

In superacoustic flow (flow velocity larger than the ion-acoustic speed) a steep wave front develops at the ion-acoustic Mach angle; the plasma ahead is quasineutral and the perturbations are exponentially small. Transversally, this front behaves as the derivative of the square of the Airy function and is $O(z^{-1})$; its width grows as $z^{1/3}$. Behind it there is a broad region where ion waves appear; the decay law is z^{-1} . Near the axis a second wave front develops which is $O(z^{-5/6})$ and has the structure of the Airy function times a modulating sinusoidal factor; its width is $O(z^{1/3})$ and with the axis forms an angle that vanishes as the $\frac{3}{4}$ power of the temperature ratio. Between this trailing front and the axis the plasma is again quasineutral and the decay law is z^{-2} . In the region of the ion waves and within a thin angle adjoining the trailing front, there are *two* distinct sets of waves.

In subacoustic flow, waves are also excited. A steep wave front develops again; its angle goes to 90° as the acoustic Mach number M_s approaches unity, and for small M_s vanishes as M_s^2 . Ahead of this leading front, the plasma is quasineutral and the perturbations are $O(z^{-2})$. As in superacoustic flow a trailing front appears; now, however, both wave fronts are $O(z^{-5/6})$ and have the structure of the Airy function times a sinusoidal factor. In the region between them, the decay law is also z^{-1} but new waves (called transverse because of the form of the constant-phase curves) appear simultaneously with the waves (called divergent) already

existing in the superacoustic case. The transverse waves extend to the region between the trailing front and the axis which, therefore, is no longer quasineutral and has a z^{-1} decay law. There is some resemblance to the water-wave pattern left by a ship moving at supercritical and subcritical speeds.

All of these waves are damped. The damping decreases as M_s increases, and is markedly directional; in superacoustic flow the minimum damping occurs at the leading front but at subacoustic speeds it occurs at the axis. The transverse waves are the less damped, and the trailing front is, in general, more damped than the leading one.

As in classical gas dynamics, a hyperacoustic equivalence principle holds for $M_s \gg 1$. For $M_s > 1$, it is found that nonlinear effects may substantially modify the self-similar structure of the leading wave front. Also, for bodies large compared with the Debye length, the smaller this length the longer the distance to reach the self-similar structure of that front. Finally, for cold ions and small bodies, the flow near the axis is very sensitive to body size.

Within the range of parameters considered, the regimes $M_s^2 \approx 1$ and $\beta \ll M_s^2 < 4^{4/3} 3^{-2/3} \beta^{1/3}$ were excluded from this study.

ACKNOWLEDGMENT

The authors acknowledge support from the National Aeronautics and Space Administration, Grant NGR 31-001-103.

* Present address: Mechanical Engineering Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

¹ L. Kraus and K. M. Watson, *Phys. Fluids* **1**, 480 (1958).

² Ya. L. Al'pert, A. V. Gurevich, and L. P. Pitaevskii, *Space Physics with Artificial Satellites* (Consultants Bureau, New York, 1965), p. 90.

³ R. J. Mason, *Phys. Fluids* **13**, 1042 (1970).

⁴ H. Washimi and T. Taniuti, *Phys. Rev. Letters* **17**, 996 (1966).

⁵ C. F. Kennel and R. Z. Sagdeev, *J. Geophys. Res.* **72**, 3303 (1967).

⁶ S. H. Lam, in *Rarefied Gas Dynamics*, edited by C. L. Brundin (Academic, New York, 1967), Vol. 4, p. 1603.

⁷ D. Montgomery, *Phys. Rev. Letters* **19**, 1465 (1967).

⁸ D. W. Koopman and D. A. Tidman, *Phys. Rev. Letters* **18**, 533 (1967).

⁹ S. G. Alikhanov, V. G. Belan, and R. Z. Sagdeev, *Zh. Eksp. Teor. Fiz. Pis. Red.* **7**, 405 (1968) [*JETP Lett.* **7**, 318 (1968)].

¹⁰ A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, *Collective Oscillations in a Plasma* (MIT Press, Cambridge Mass., 1967), p. 22.

¹¹ H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, London, 1950), Chap. 17.

¹² T. H. Havelock, *Proc. Roy. Soc. (London)* **A81**, 398 (1908).

¹³ F. Ursell, *J. Fluid Mech.* **8**, 418 (1960).

¹⁴ In this figure it has been assumed that $M_s^2 > 4^{4/3} 3^{-2/3} \beta^{1/3}$; for lower M_s , k_z' is substantially modified. The above condition will be assumed in the rest of the paper.