Reliability analysis based on non-dimensional parameters

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Abstract

A reliability analysis method is proposed that starts with the identification of all variables involved. These are divided in three groups: (a) variables fixed by codes, as loads and strength project values, and their corresponding partial safety coefficients, (b) geometric variables defining the dimension of the main elements involved, (c) the cost variables, including the possible damages caused by failure, (d) the random variables as loads, strength, etc., and (e) the variables defining the statistical model, as the family of distribution and its corresponding parameters. Once the variables are known, the Π-theorem is used to obtain a minimum equivalent set of non-dimensional variables, which is used to define the limit states. This allows a reduction in the number of variables involved and a better understanding of their coupling effects. Two minimum cost criteria are used for selecting the project dimensions. One is based on a bounded probability of failure, and the other on a total cost, including the damages of the possible failure. Finally, the method is illustrated by means of an application.

1 Introduction

Since the pioneering works of Freudenthal [6] in the fifties, the establishment of the structural safety has been based on probabilistic concepts and the computation of the probability of failure in a more or less direct fashion. On the other hand, the engineering tradition has been the use of safety factors weighting in some form both the structural resistance and the loads acting on it.

The probabilistic approach is too abstruse to be used by the designer and this is why some efforts have been directed to the reinterpretation of the weighting factors from the probabilistic point of view.

In the reliability of a structure, there are many variables involved. They belong to an n-dimensional space, which can be divided in two regions: the safe and the failure regions (see Figure 1). The boundary of such a regions is defined by the system limit states.

To illustrate we represent in Figure 1 the case of two variables X_1 and X_2. Assume that X_1 and X_2 are two design variables, that become more dangerous as they increase and decrease their values, respectively. Then, a code based on partial safety coefficients states that a set of values (x_1, x_2) is safe if and only if the point $(\alpha_1 x_1, x_2/\beta_2)$ is in the safe region, where α_1 and β_2 are the partial safety coefficients for both variables. Other combinations can arise for other pairs of variables (points $(\alpha_1 x_1, \alpha_2 x_2)$, $(x_1/\beta_1, \alpha_2 x_2)$ and $(x_1/\beta_1, x_2/\beta_2)$)) (see Figure 1).

The next step consists of defining a joint probability density

$$g_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$$

for all variables involved. Then, the probability of failure is:

$$P_f = \int_{f(x_1,\ldots,x_n)\leq 0} g_{X_1,\ldots,X_n}(x_1,\ldots,x_n)dx_1\ldots dx_n \quad (1)$$

where $f(x_1,\ldots,x_n) \leq 0$ defines the failure region.

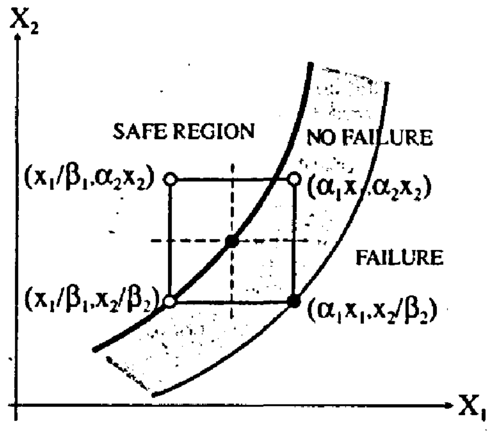

Figure 1: Illustration of the partial safety coefficients.

However, the computation of this integral is not an easy task, and some simplifications are required. This, leads to three methods used in reliability analysis:

1. LEVEL 1: Partial safety coefficients related to some design variables (loads, strengths, etc.) are used. It is the classical method.
2. LEVEL 2: Approximate methods are used to calculate the failure probability. They are normally based on an approximate representation of the probability distribution using first and second moments of the joint distribution, and on an approximation of the failure region. This method is exact when normal variables are used, because first and second order moments characterize these variables.
3. LEVEL 3: The safety coefficient is computed using the full representation of the joint distribution and the exact failure region.

Among all methods, it is worthy mentioning the first order (FORM) and second order (SORM) methods are worth of mention. The "First Order Reliability Methods" (FORM) appeared in the field of structural reliability with Freudenthal [6] in 1956, and has been expanded by Hasofer and Lind [8], Rackwitz and Fiessler [9], Breitung [2], Wirsching and Wu [10] and Wu, Burnside and Cruse [11], Fraile, del Rey, Retana, Gómez and Alarcón [5], Castillo, Sarabia, Solares and Gómez [3], etc. The main idea of the FORM methods consists of transforming the initial set of variables in a multinormal set and using a linear approximation to eliminate the need of complex numerical integrations.

More precisely, the first order reliability methods reduce the target variable to a normal random variable, replace the integration region by a hyperplane and use the cdf of a normal random variable as an approximation to the integral to be calculated.

These methods have shown to give precise results and have demonstrated to be much more efficient than Monte Carlo simulation techniques for estimating extreme percentiles (see, for example, Wirsching [10], or Haskin, Staple and Ding [7]). Unfortunately, in some cases the first order approximation is not sufficient and we have to use a second order approximation.

The main idea of SORM methods consists of improving the above linear approximation by using quadratic expressions to approximate the integration region. Then, the well known results on quadratic forms in normal random variables are used to calculate the corresponding probabilities (integrals). In this way an important improvement can be obtained.

However, these methods work with the initial set of variables, and seem unaware of the possibility of reducing this set to a smaller set that has more information about the role of the subsets of variables. This has important advantages as:

1. *The number of variables can be reduced without changing the final results.*
2. The results are extrapolable to many other situations in which the values of the non-dimensional variables are kept constant.
3. The role of variables is better understood.

Thus, in Sections 2 we discuss the problem of non-dimensional variables and its practical implications, and we propose:

1. Working with limit state functions in non-dimensional form. This implies a reduction in the number of variables involved and avoiding the problem of dimensions.
2. Use the partial security coefficients as part or all the non-dimensional variables. This implies a clear connection of the modern and old statement of reliability problems and a better understanding of the role played by partial security coefficients.

On the other hand, the above mentioned methods are based only on the probability of failure and do not consider the cost.

To understand the implications of this assessment let us recall the connection among the FORM and the weighting factors approach of most modern Codes.

Imagine the cases of a normalized resistance variable ξ_R and a normalized load variable ξ_L. The probability of failure can be related to the integral of the probability density function on the unsafe region (see Figure 2). After linearization, that integral can be related to the distance β to the tangent plane Π at the nearest point to the origin. This point P^*, called traditionally design point, is used to connect with the Code approach using weighted values. Clearly, ξ_R^* is reduced with respect to the mean value while ξ_L^* is increased. The weighted values are then interpreted as those that should be used in a classical format to get structures which reliability be equal to that obtained in the integration over the unsafe region.

But from this approach, it is not correct to deduce that point P^* is the most "desirable" design point.

In fact, any combination of variables defining a point belonging to the C curve comply with the failure condition and then, the failure probability is the same. Which of those points should be chosen as the "design point" does not depend on any instrumental decision directed to simplify the task of computing that probability. In fact a Code should define the C curve letting the designer to choose the most "desirable" point based on the particular conditions under which the structure is to be built up. One of those conditions is the cost. In this sense, there are two decision making alternatives: either we design for the point with minimum cost irrespective of any probability of failure consideration, or we look for the point that, being contained in a curve with a probability of failure determined by the Code, leads to a minimum cost.

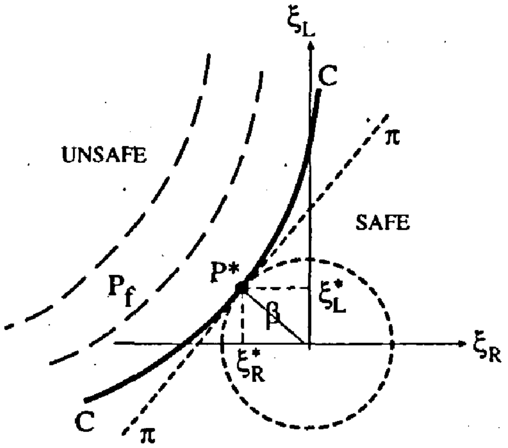

Figure 2: Illustration of the safe and unsafe regions and the probability of failure.

In Section 3 we propose a method based on two different cost functions, one of them including the possible damages caused by failure.

Finally, in Section 4 we give some conclusions.

2 Dimensional analysis

In this section first we see the important result:

> "The only physical formulas relating ratio scale variables are those of the form:
>
> $$u(x_1,\ldots,x_n) = a\prod_{i=1}^{n} x_i^{c_i},$$
>
> where a and $c_i; i = 1,\ldots,n$ are constants."

This will lead to the Π-theorem.

2.1 Valid physical formulas

In a physical system there are some fundamental variables, such as length, time and space; from them, secondary or derived variables are obtained by some, more or less complicate, formulas.

When a variable is allowed for location and scale changes, as time, temperature and location, we say that it has an interval scale. If a variable is allowed for scale changes only, as length, area, volume, speed and acceleration, we say that it has a ratio scale.

In practical situations, formulas relate different variables not necessarily fundamental. However, not every formulae generates a valid variable, but only those satisfying some extra conditions (see Aczél[1], pp.35-70 or Castillo and Ruiz-Cobo [4]). For a formula to be valid, it is necessary that a change of location or/and scale of the independent variables keep the same formulae structure up to a change of location or/and scale of the derived or dependent variable. In other words, the formulae should remain invariant under location and scale changes. This condition can be written as the following functional equation:

$$\begin{aligned} &u(r_1x_1+p_1, r_2x_2+p_2, \ldots, r_nx_n+p_n) = \\ &= R(r_1, r_2, \ldots, r_n; p_1, p_2, \ldots, p_n)u(x_1, x_2, \ldots, x_n) \\ &+P(r_1, r_2, \ldots, r_n; p_1, p_2, \ldots, p_n); \\ &\qquad r_i, x_i > 0, (i = 1, 2, \ldots, n) \end{aligned} \quad (2)$$

where $u(x_1, x_2, \ldots, x_n)$ is the formulae that gives the derived variable as a function of the initial variables and P and R are functions associated with the location and scale changes of the derived variable, respectively.

Equation (2) is very important and states that when changing the units of the X-variables, a change in units of the U variable is obtained, but the formula (function $u(\cdot)$) remains the same.

Equation (2) shows the more general situation, in that it includes a maximum of restrictions. However, some simpler cases can occur, depending on whether or not the location or the scale changes exist for the fundamental or the derived variables or whether they are homogeneous or heterogeneous for all the variables. In this manner, we can deal with very many different situations. In this paper we only need the following case.

Theorem 1 *The general form of dependent real-valued variables with ratio scale non-constant and continuous at a point when all fundamental or independent variables have ratio scale, i.e. the general solution of the functional equation*

$$u(r_1x_1, \ldots, r_nx_n) = R(r_1, \ldots, r_n)u(x_1, \ldots, x_n), \quad r_i, x_i > 0; \forall i = 1, \ldots, n. \quad (3)$$

is

$$u(x_1, \ldots, x_n) = a\prod_{i=1}^{n} x_i^{c_i}; \quad R(r_1, \ldots, r_n) = \prod_{i=1}^{n} r_i^{c_i} \quad (4)$$

with $a \neq 0$ *and* $\sum_{i=1}^{n} c_i^2 \neq 0$.

Next, we give the π-theorem, which plays an essential role in dimensional analysis.

2.2 The Π–Theorem

The Π–Theorem is the fundamental theorem used in dimensional analysis and allows knowing the minimum set of variables involved in a given problém.

Theorem 2 (The Π–Theorem) *If a physical phenomena can be expressed in a given measure-system in which they exist m fundamental magnitudes by means of a function of r parameters, which represent other magnitudes, then, if n is the rank of the matrix A which elements are the dimensions of the parameters with respect to the fundamental magnitudes, there exist $r - n$ non-dimensional monomials by means of which the physical phenomena can be represented. They are formed by products of powers of such magnitudes.*

Example 1 Consider the simply supported beam in Figure 3, where p is the uniform load per unit length, L is the beam length, and a and b are the dimensions of its rectangular cross section.

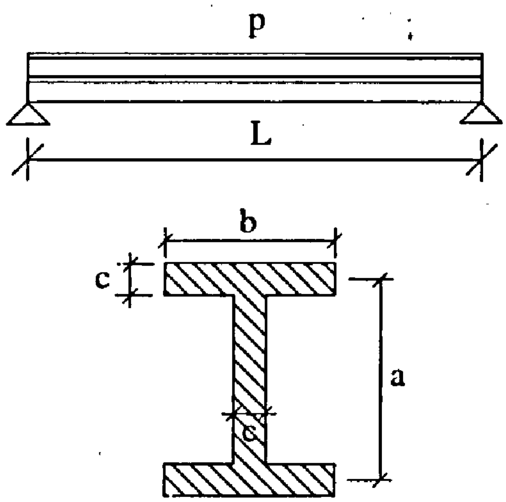

Figure 3: A simply supported beam and its cross section.

To analyze the beam problem we consider the following initial set of variables:

Random variables

1. p: the maximum load per unit length occurred during the service life of the beam.
2. f: the real strength of the beam.

Variables fixed by code rules

3. p_0: the weighted load per unit length, fixed by Code rules.
4. f_0: the maximum admissible strength fixed by Code rules.

Geometric variables

5. L the beam length.
6. a the depth of the beam.
7. b the flange width of the beam.
8. c the flange and web thickness of the beam.

Cost variables

9. S_0: the steel cost per unit volume.
10. K: the beam structural assembling cost.
11. C: the beam total cost.

12. D: the cost associated with the damage produced by the failure of the beam.

13. C_0: A reference fixed amount of money.

Probability and Statistical variables

14. P_F: the probability of failure.

15. $F(\cdot;\theta)$: assumed family of cumulative distribution functions.

16. θ: non-dimensional parameters of the family $F(x;\theta)$.

Using the Π-theorem and considering the dimensional analysis in Table 1, which matrix is of rank 3, we conclude that any existing relation among the 16 variables in the list above, can be written in terms of only 13 variables.

We can select any three variables leading to a full rank, and express the remaining variables in terms of them. For example, in Table 1 we have selected $\{p_0, f_0, C_0\}$, and the non-dimensional resulting variables are:

$$
\begin{array}{llll}
p^* = \frac{p}{p_0}, & f^* = \frac{f}{f_0}, & L_0^* = \frac{f_0 L}{p_0}, & a^* = \frac{f_0 a}{p_0}, \\
b^* = \frac{f_0 b}{p_0}, & c^* = \frac{f_0 c}{p_0}, & S_0^* = \frac{p_0^3 S_0}{f_0^3 C_0}, & K^* = \frac{K}{C_0}, \\
C^* = \frac{C}{C_0}, & D^* = \frac{D}{C_0}, & p_F^* = p_f, & F^* = F, \\
\theta^* = \theta. & & &
\end{array} \qquad (5)
$$

An alternative option consists in selecting $\{p_0, L, C_0\}$, and then, the non-dimensional resulting variables are (see Table 1):

$$\frac{p}{p_0}, \frac{fL}{p_0}, \frac{f_0 L}{p_0}, \frac{a}{L}, \frac{b}{L}, \frac{c}{L}, \frac{S_0 L^3}{C_0}, \frac{KL^2}{C_0}, \frac{C}{C_0}, \frac{D}{C_0}, p_F, F, \theta.$$

However, in this paper we use the previous set of non-dimensional variables, which are referred to using an asterisk.

3 Minimizing cost

In this section we consider the cost function as the design criteria.

We have two options:

1. Minimize the building cost, $C(p_F)$, of the structure under consideration, where we have shown its dependence on the probability of failure p_F, subject to the Code constraints.

   Figure 4 illustrates the unconstrained and constrained optima, where ξ_L and ξ_R are the normalized random variables used in the classical analysis. Note the cost contours.

2. Minimize the total cost of the beam, including its assembling and the possible damages caused by its failure, that is,

$$C(p_F)(1+p_F) + p_F D, \qquad (6)$$

   where D is the damage produced in case of failure. In this case the probability of failure is not predetermined. Note that the first term $C(p_F)(1+p_F)$, for convenient $C(p_F)$ decreases with increasing p_F, while the second $p_F D$, increases with increasing p_F.

Example 2 Consider again the beam problem, and assume that our analysis is based on the following considerations:

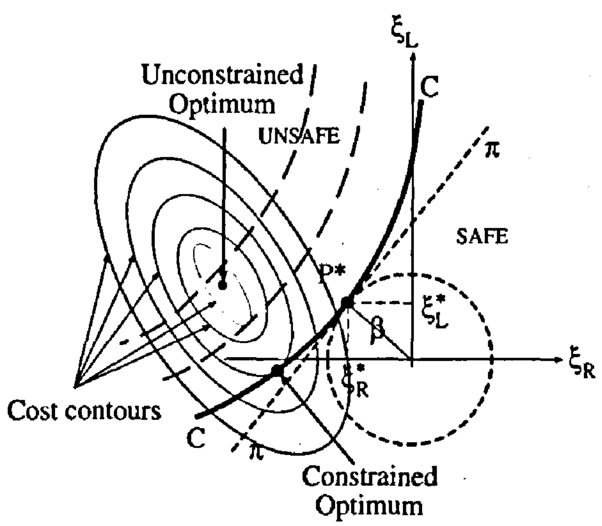

Figure 4: Illustration of constrained and unconstrained optima. In the constrained case, the probability of failure p_F is fixed by Codes (curve C).

Code rules: The Code rules define a design strength for the material f_0 which is not allowed to be trespassed. As a consequence, an optimal design fulfilling the Code has to accomplish $f = f_0$, where f is the resulting maximal stress in the material which, according to the strength of materials, is (see Figure 3)

$$
\begin{aligned}
f &= \frac{M_0}{W} = \frac{p_0 L^2}{8} \frac{(a+c)/2}{[(a+c)^3 b - (a-c)^3(c-b)]/12} \\
&= \frac{3 p_0 L^2}{4} \frac{(a+c)}{[(a+c)^3 b - (a-c)^3(c-b)]} = f_0,
\end{aligned} \qquad (7)
$$

where M_0 is the design moment for the loading p_0 and W is the modulus of the beam cross section. Thus, the last equality in (7) represents a requirement concerning the beam cross section dimensions a, b and c to be fulfilled, which allows defining a relation as a function of f_0 and p_0.

Failure law: On the other hand the real strength f, exhibited by the material and the real applied load p, are related through the same expression

$$f = \frac{3pL^2}{4} \frac{(a+c)}{[(a+c)^3 b - (a-c)^3(c-b)]}, \qquad (8)$$

where f and p are random variables, with joint distribution functions to be defined. Since L is a fixed quantity and a, b and c are deterministic quantities, though they may be freely chosen as long as (7) is satisfied, it follows that failure occurs when

$$f/p < f_0/p_0. \qquad (9)$$

Equations (7) and (9) can be rewritten in terms of the non-dimensional variables (5) as

$$1 = \frac{3L^{*2}}{4} \frac{a^* + c^*}{(a^*+c^*)^3 b^* - (a^*-c^*)^3(c^*-b^*)} \qquad (10)$$

and

$$\frac{f^*}{p^*} \le 1, \qquad (11)$$

respectively.

Table 1: Matrix of rank 3 relating variables, involved in the beam problem in Section 2, with fundamental magnitudes, and matrices showing the variables written in terms of the base (p_0, f_0, C_0) and (p_0, L, C_0).

	p	f	p_0	f_0	L	a	b	c	S_0	K	C	D	C_0	p_F	F	θ
Mass	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
Length	0	-1	0	-1	1	1	1	1	-3	0	0	0	0	0	0	0
Time	-2	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0
Money	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0

	p	f	p_0	f_0	L	a	b	c	S_0	K	C	D	C_0	p_F	F	θ
p_0	1	0	1	0	1	1	1	1	-3	0	0	0	0	0	0	0
f_0	0	1	0	1	-1	-1	-1	-1	3	0	0	0	0	0	0	0
C_0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0

	p	f	p_0	f_0	L	a	b	c	S_0	K	C	D	C_0	p_F	F	θ
p_0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
L	0	-1	0	-1	1	1	1	1	-3	0	0	0	0	0	0	0
C_0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0

Probability of failure: The probability of failure is:

$$\begin{aligned} p_F^* &= Prob\left[\frac{f}{p} \le \frac{f_0}{p_0}\right] = F_{\frac{f}{p}}\left(\frac{f_0}{p_0};\theta^*\right) \\ &= F^*_{\frac{f^*}{p^*}}(1;\theta^*) \\ &= F_{\frac{f}{p}}\left(\frac{3L^{*2}(a^*+c^*)}{4[(a^*+c^*)^3 b^* - (a^*-c^*)^3(c^*-b^*)]};\theta^*\right), \end{aligned} \tag{12}$$

where $F(x;\theta)$ and $F^*(x;\theta)$ are the cumulative distribution functions (cdf) of the random variables $\frac{f}{p}$ and $\frac{f^*}{p^*}$, respectively. Note that the Code, fixing the values of f_0 and p_0, controls the value of p_F, through formula (12).

Cost minimization: We assume that the design values a^* and b^* are those minimizing the cost. As mentioned before, we have two alternatives:

1. Minimize the cost of the beam including its assembling, that is,

$$C^* = c^*(a^* + 2b^* - c^*)L^*S_0^* + K^*,$$

with respect to a^*, b^* and c^*, subject to (10) and (12), that is, minimizing the cost subject to the Code rule (10). This implies a given (α) probability of failure.

From (10) we obtain:

$$b^* = \frac{3L^{*2}(a^*+c^*) + 4c^*(a^*-c^*)^3}{4[(a^*+c^*)^3 + (a^*-c^*)^3]}. \tag{13}$$

Thus, the project value can be obtained by minimizing

$$c^*(a^* + \frac{3L^{*2}(a^*+c^*) + 4c^*(a^*-c^*)^3}{2[(a^*+c^*)^3 + (a^*-c^*)^3]} - c^*)L^*S_0^* + K^* \tag{14}$$

with respect to a^* and c^*, which is an unconstrained minimization. Once a^* and c^* are known, b^* can be computed using (13), and the probability of failure, using (12).

2. Minimize the cost of the beam including its assembling and the possible damages caused by failure of the beam, that is,

$$[c^*(a^* + 2b^* - c^*)L^*S_0^* + K^*](1 + p_F^*) + p_F^*D^*. \tag{15}$$

with respect to a^*, b^* and c^*, subject to (10). In this case the probability of failure is not predetermined.

Replacing (10) into (15) we get

$$[c^*(a^* + \frac{3L^{*2}(a^*+c^*) + 4c^*(a^*-c^*)^3}{2[(a^*+c^*)^3 + (a^*-c^*)^3]} - c^*)L^*S_0^* + K^*] \times (1 + p_F^*) + p_F^*D^*. \tag{16}$$

To illustrate more precisely the proposed method, we present below a simple example. We suppose that c is fixed ($c = 0.02m$). Assume that the Code forces $p_F = 4 \times 10^{-5}$, and that $L = 8m$, $S_0 = 3855$ euros, $K = 120$ euros, $D_1 = 60,240$ euros, and $C_0 = 1$ euro.

Assume also that $\log p$ and $\log f$ are lognormal distributions with percentiles, $p_{0.99}$ and $f_{0.01}$, and variation coefficients, $v_{\log p}$ and $v_{\log f}$:

$$\begin{array}{lllll} p_{0.99} &=& 45000N/m, & v_{\log p} &=& 0.10. \\ f_{0.01} &=& 240MPa; & v_{\log f} &=& 0.03. \end{array} \tag{17}$$

Then, we have

$$\begin{array}{rrcl} \mu_{\log f} & -2.326\sigma_{\log f} &=& \log(240 \times 10^6) \\ 0.03\mu_{\log f} & -\sigma_{\log f} &=& 0 \end{array}$$

which leads to

$$\begin{array}{rcl} \mu_{\log f} &=& 20.74\log(MPa) \\ \sigma_{\log f} &=& 0.622\log(MPa). \end{array}$$

and

$$\begin{array}{rrcl} \mu_{\log p} & +2.326\sigma_{\log p} &=& \log(45000) \\ 0.10\mu_{\log p} & -\sigma_{\log p} &=& 0 \end{array}$$

which leads to

$$\begin{array}{rcl} \mu_{\log p} &=& 8.693\log(N/m) \\ \sigma_{\log p} &=& 0.869\log(N/m). \end{array}$$

Consequently, $\log(f/p) = \log f - \log p$ is a normal distribution $N(12.05, 1.069)$.

Figure 5 shows the ratio f_0/p_0 versus the probability of failure p_F.

Minimization of (14) leads to

$$a = 0.342m, \;\; b = 0.093m, \;\; c = 0.02m,$$

and minimization of (16) leads to practically the same solution, because the value of p_F is very small and D_1 not large enough to have some influence on the design value.

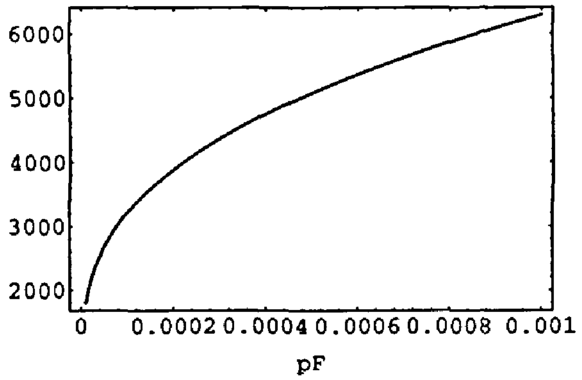

Figure 5: The ratio f_0/p_0 versus the probability of failure p_F.

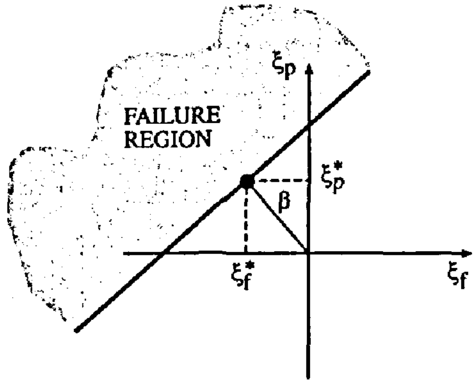

Figure 6: Failure region and design point for the beam example.

For a comparison, we give next the classical analysis to this example, using $\log p$ and $\log f$ as the two random variables instead of p and f. Then, we have

$$\log f \sim N(20.74, 0.622); \quad \log p \sim N(8.693, 0.869), \tag{18}$$

and normalizing the variables

$$\begin{array}{rcl} \xi_f &=& \dfrac{\log f - 20.74}{0.622} \sim N(0,1); \\ \xi_p &=& \dfrac{\log p - 8.693}{0.869} \sim N(0,1). \end{array} \tag{19}$$

the failure criteria is (see Figure 6)

$$\frac{f}{p} < k_0 \Rightarrow \log f - \log p < \log k_0 = k \tag{20}$$

which is equivalent to

$$0.622\xi_f - 0.869\xi_p + 12.047 - k < 0. \tag{21}$$

The critical point (ξ_f^*, ξ_p^*) can be obtained by solving the system of equations

$$\begin{array}{rcl} 0.622\xi_p^* \; +0.869\xi_f^* &=& 0 \\ -0.869\xi_p^* \; +0.622\xi_f^* &=& -12.047 + k \end{array} \tag{22}$$

leading to

$$\begin{array}{rcl} \xi_f^* &=& -0.622\dfrac{(12.047 - k)}{0.869^2 + 0.622^2}, \\ \xi_p^* &=& 0.869\dfrac{(12.047 - k)}{0.869^2 + 0.622^2}), \\ \beta &=& -\dfrac{12.047 - k}{\sqrt{0.869^2 + 0.622^2}}. \end{array} \tag{23}$$

If the Code fix $p_F = 4 \times 10^{-5}$, then we get

$$\begin{array}{rcl} \beta &=& \Phi^{-1}(4 \times 10^{-5}) = -3.9444, \\ p_0 &=& 96787.1 N/m, \\ f_0 &=& 243.8 MPa. \end{array}$$

Note that the classical analysis leads to fixed values of p and f, while the alternate proposal leads to a fixed value of f/p, since the probability of failure depends only on this ratio.

4 Conclusions

As it has been shown, the use of non-dimensional magnitudes in reliability allows not only reducing the complexity of the limit state functions, but a clear connection between the classical reliability analysis, based on partial security coefficients, and the actual reliability methods.

5 Acknowledgments

We thank Iberdrola for partial support of this work.

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