WHAT IS...

an Affine Sphere?

Daniel J. F. Fox

Linear maps and translations generate the group Aff of *affine motions* comprising diffeomorphisms of $\mathbb{R}^{n+1}$ that preserve straight lines. The simplest interesting class of hypersurfaces preserved by Aff is the *affine spheres*. These are defined by the affinely invariant condition that their *affine normal* lines meet in a point, possibly at infinity. While affine spheres include nondegenerate quadrics, there are many others. A simple example is the surface $\{xyz = 1 : x > 0,\ y > 0\}$. It resembles a flattened hyperboloid asymptotic to the positive orthant bounded by the coordinate planes. As will be explained, an infinitude of inequivalent affine spheres results from solving certain PDEs of *Monge-Ampère* type, meaning that they involve the determinant of the Hessian of an unknown function. The construction of affine spheres has played an important role in the development of techniques for solving Monge-Ampère equations (see [4]), while affine spheres arise in studying such apparently different topics as flat projective structures (see [1]) and convex optimization (see [3]).

Most intuitive geometrical notions relate to describing position (e.g., distance, angle) and relative position (e.g., midpoints, parallels), size (e.g., volume, length), and shape (e.g., round or flat). What *geometry* means in a particular context is determined operationally by those features of space to be distinguished or identified. Rotations and translations generate the *Euclidean motions* Euc comprising affine motions preserving the standard inner product $\langle\ ,\ \rangle$. The *Euclidean* and *affine geometry* of $\mathbb{R}^{n+1}$ refer to those constructs preserved, respectively, by Euc and Aff. For example, Euclidean geometry distinguishes between a sphere and an ellipsoid, because their surfaces bend in different ways. Though they cannot be superimposed by a Euclidean motion, they can be by composing one with appropriate shears and dilations. Since Aff contains Euc, affine equivalence and invariance are coarser notions than their Euclidean counterparts. For example, any two ellipsoids or any two triangles are affinely equivalent. As Aff preserves the standard directional derivative operator D but not $\langle\ ,\ \rangle$, distance and angle have no sense in affine geometry, but straight lines, parallels, and midpoints do. Ordinary calculus (based on D) still makes sense, though there is no natural way to identify the differential of a function with a vector field.

A *smooth hypersurface* in $\mathbb{R}^{n+1}$ means a submanifold Σ locally representable as the graph $\Sigma_f = \{x^0 = f(x^1, \dots, x^n)\}$ of a smooth function f. Assume Σ is *two sided*, meaning that there is a nonvanishing vector field N everywhere *transverse* to Σ. This excludes examples such as the Möbius band, but always holds locally, and simplifies the discussion. For vector fields X and Y tangent to Σ, the part of the directional derivative $D_X Y$ in the direction of a transversal N has the form $h(X,Y)N$ for a field h of symmetric bilinear forms. If N is changed, h is multiplied by a nonvanishing function. The *second fundamental form* of Σ is the equivalence class $[h]$ of h under the identification of a field of bilinear forms with its multiple by a nonvanishing function. Since $[h]$ depends only on D, it is preserved by Aff. Henceforth the hypersurface Σ is assumed to be *nondegenerate*,

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meaning h is nondegenerate. If h is definite, then Σ is *locally uniformly convex*, in which case it is moreover locally strictly convex. For a graph Σ_f, the Hessian $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ is the representative of $[h]$ corresponding to the transversal $\frac{\partial}{\partial x^0}$. Hence Σ_f is nondegenerate (resp. locally uniformly convex) if and only if f_{ij} is nondegenerate (resp. definite). For example, the saddle $z = xy$ is nondegenerate but not convex, while $z = x^4 + y^4$ is strictly convex but not uniformly so, for it is degenerate where $xy = 0$. That Aff preserves $[h]$ means that such local convexity properties of Σ_f are affinely invariant. The scaling of N, though not its direction, can be fixed by requiring that the corresponding h equal the *equiaffine* (or *Blaschke*) *metric* given locally by $|\det f_{ij}|^{-1/(n+2)} f_{ij}$. The prefix *equi* indicates a structure preserved by the group SAff of volume-preserving affine motions. Local uniform convexity implies that the intersection of Σ with a small ball B centered on $P \in \Sigma$ is contained in one of the half-spaces delimited by the tangent plane $T_P\Sigma$ at P and divides B into an *exterior* region intersecting $T_P\Sigma$ and an *interior* region not intersecting $T_P\Sigma$. In this case, h is positive definite if N is chosen to point to the interior.

The *shape operator* is the endomorphism S of $T_P\Sigma$ associating with a tangential vector field X the tangential part of the derivative $-D_X N$. Its eigenvalues are the *principal curvatures* of Σ. These encode how Σ bends and twists in $\mathbb{R}^{n+1}$ relative to N. A Euclidean invariant transversal is determined up to sign by the requirements that it be perpendicular to Σ and have unit norm. Via the Euclidean metric the corresponding shape operator S and bilinear form h are identified. In affine geometry there is no notion of orthogonality with which to single out a transversal. Nonetheless, it is possible to define a distinguished *affine normal* direction preserved by Aff, although the corresponding S and h are *not* naturally identified.

The following description of the affine normal, due to W. Blaschke, is based on the affine equivariance of the center of mass. If $[a,b] \subset \mathbb{R}$ and $x \in (a,b)$, then $|x-a|$ and $|x-b|$ are unchanged by a translation and scale identically under a dilation so the ratio $A(x) = |x-a|/|x-b|$ is unchanged by an affine transformation applied to a, b, and x. The unique $x \in (a,b)$ such that $A(x) = 1$ is the *midpoint* of $[a,b]$. This is the one-dimensional specialization of the usual notion of the *centroid* (also *center of mass* or *barycenter*) of a region. It follows from the change-of-variables formula for integrals that the centroid is affinely equivariant in the sense that the centroid of the image of a region under $g \in \mathsf{Aff}$ is the image under g of the centroid of the region. Consider a point P on a locally uniformly convex hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ and the plane K tangent to Σ at P. Because of Σ's convexity, its intersection with the parallel translate $K(t)$ of K at distance t from P encloses an $(n-1)$-dimensional convex subset $\Omega(t) \subset K(t)$. The tangent at P to the curve $n(t)$ formed by the centroids of the $\Omega(t)$ spans the *affine normal subspace*. Since affine transformations preserve centroids and parallelism, this construction is manifestly affinely equivariant, though, since the distance t is not preserved by Aff, the vector tangent to $n(t)$ at $t = 0$ is not. However, for $s(t)$ equal to an appropriate function of the volume enclosed by $K(t)$ and Σ, the derivative at $s = 0$ of $n(t(s))$ is the manifestly SAff equivariant *equiaffine normal*. Equivalently, the scaling of the equiaffine normal is determined by the requirement that the corresponding h be the equiaffine metric. The corresponding S is the *equiaffine shape operator*.

For example, if Σ is a sphere, then its intersection with a parallel translate of its tangent plane at P is a round ball in this plane, and this ball's centroid is the plane's intersection with the radius of Σ through P. Thus a sphere's affine normals meet in its center. By affine invariance the same is true for any ellipsoid. The abscissa of the midpoint of the segment of any secant to the parabola $y = x^2$ demarcated by the parabola equals the abscissa of the point at which the tangent to the parabola is parallel to the secant. Extending this reasoning shows that the affine normals of an upward-opening elliptic paraboloid are vertical lines.

Basic tasks are to identify and characterize Euclidean or affinely invariant classes of hypersurfaces. Many such classes are described by algebraic conditions on the shape operator, e.g., constant trace or determinant. The most studied Euclidean invariant hypersurfaces are the *minimal* surfaces—the critical points of the surface area functional with respect to small variations in the normal direction. These arise physically as films spanning a wire frame dipped in soapy water. They are called *minimal* because when they are extremal, they are in fact minimizing. For a graph Σ_f, the Euclidean volume element is $(1 + |df|^2)^{1/2} dx$, and minimality of Σ_f is the nonlinear second-order PDE for f given by the vanishing of the mean curvature, by definition the average of the Euclidean principal curvatures. The *equiaffine volume* of Σ is the integral of $|\det h|^{1/2}$ over Σ for the equiaffine metric h. Its critical points with respect to small normally directed variations are given by the vanishing of the *affine mean curvature*, the average of the equiaffine principal curvatures. Affine mean curvature zero hypersurfaces had been for many years called *affine minimal*, until E. Calabi calculated that, when extremal, such hypersurfaces are locally volume *maximizing*; now they are called *affine maximal*. For a graph Σ_f, $|\det h|^{1/2} = |\det D^2 f|^{1/(n+2)} |dx|$, and that Σ_f be affine maximal is a nonlinear *fourth*-order PDE for f.

Among the simplest hypersurfaces are the *umbilical* ones, those for which the principal curvatures are all equal at every point. Equivalently, the shape operator is a multiple of the identity. This means, imprecisely, that at every one of its points such a hypersurface bends in the same way in every direction. In the Euclidean case these are simply either hyperplanes or spheres. It turns out that the affine mean curvature H of an affinely umbilical hypersurface Σ must be constant. This is equivalent to the geometric condition that the affine normals of Σ either all meet in a point, its *center*, or all are parallel, in which case Σ is said to have *center at infinity*. This fact motivates calling an affinely umbilic hypersurface an *affine sphere*. An affine sphere is *improper* or *proper* as its center is or is not at infinity. For example, for any $\phi \in C^{\infty}(\mathbb{R})$ the graph of $z = xy + \phi(x)$ is an improper affine sphere with equiaffine normal $\frac{\partial}{\partial z}$.

In the convex case strong results follow from a representation of an affine sphere as a solution of an elliptic Monge-Ampère equation. The graph Σ_f of a locally strictly convex function f is a mean curvature H affine sphere centered at the origin or infinity if and only if the Legendre transform u of f solves

$$(1) \qquad \det \frac{\partial^2 u}{\partial y_i \partial y_j} = \begin{cases} (Hu)^{-n-2}, & \text{if } H \neq 0, \\ 1, & \text{if } H = 0. \end{cases}$$

Recall that $u(y)$ is the strictly convex function on the image $\Omega \subset \mathbb{R}^n$ of the differential df defined implicitly by $u = \sum_{i=1}^n x^i \frac{\partial f}{\partial x^i} - f$ with respect to the coordinates $y_i = \frac{\partial f}{\partial x^i}$ on Ω. In the proper case the *radial graph* $\{u^{-1}(-1, y) \in \mathbb{R}^{n+1} : y \in \Omega \subset \mathbb{R}^n\}$ is again an affine sphere, with mean curvature H^{-1}. In the improper case, interchanging the roles of u and f in (1) shows that the ordinary graph of u is also an improper affine sphere. That affine spheres come in pairs in this way is a deep manifestation of the Legendre transform duality.

If a convex affine sphere is improper, it is also called *parabolic*, while if proper, it is called *elliptic* or *hyperbolic* as H is positive or negative (this means its center is contained in its interior or its exterior, respectively). Using basic results about the growth of solutions to Monge-Ampère equations like (1) due to Jörgens, Calabi, and Pogorelov, it can be proved that, under some technical hypotheses, an elliptic affine sphere is an ellipsoid and a parabolic affine sphere is an elliptic paraboloid. In the hyperbolic case there are more possibilities. For $\rho(x) = -z_0^2 + z_1^2 + \cdots + z_n^2$, the hyperboloid

$$\mathbb{H}_r = \{z \in \mathbb{R}^{n+1} : \rho(z) = -r^{2(n+2)/(n+1)}, z_0 > 0\}$$

is the radial graph over the unit ball of the solution $u = -r^{-(n+2)/(n+1)}(1 - |x|^2)^{1/2}$ of (1) with $H = -r^2$. The $\mathbb{H}_r$ are asymptotic to the null cone $\{\rho(z) = 0\}$ and fill out its interior $\{\rho(z) < 0\}$. From a theorem of S. Y. Cheng and S. T. Yau showing that on a bounded convex domain Ω there is for $H < 0$ a unique negative convex solution of (1) extending continuously to be 0 on the boundary of Ω, it follows that a similar geometric picture holds far more generally (see [2] and [4]). Namely, suppose the convex cone K is *sharp*, meaning it contains no full straight line (this rules out, e.g., a half-space). Then the interior of K is, in a unique way, a disjoint union $\bigcup_{r>0} L_r$ of mean curvature $-r^2$ hyperbolic affine spheres L_r asymptotic to K and with centers at the vertex of K. Alternatively, by other theorems of Cheng and Yau and of Mok and Yau there is on K a unique solution of the Monge-Ampère equation $\det \frac{\partial^2 F}{\partial z_I \partial z_J} = e^{2F}$ tending to $+\infty$ on the boundary of K and such that $\frac{\partial^2 F}{\partial z_I \partial z_J}$ is a complete Riemannian metric, and the affine spheres foliating K are the level sets of F. In fact, for the solution u of (1), F equals $-(n+1)\log|z_0 u(z_i/z_0)|$ plus a constant. For example, the radial graph of

$$(2) \qquad u = -\sqrt{n+1}\,(y_1 y_2 \cdots y_{n-1} y_n)^{1/(n+1)}$$

over the orthant $\Omega = \{y \in \mathbb{R}^n : y_i > 0\}$ is a mean curvature -1 affine sphere contained in a level set of $-\log|\prod_{I=1}^{n+1} z_I|$ asymptotic to $\{\prod_{I=1}^{n+1} z_I = 0\}$. Though this theorem yields an affine sphere for every sharp convex cone, such a sphere is difficult to describe explicitly except when K is homogeneous, meaning its automorphism group G acts on it transitively. In this case orbits of $G \cap \mathsf{SAff}$ are affine spheres asymptotic to K; e.g., (2) comes from the group of diagonal linear maps. A component of a nonzero level set of the discriminant

$$x_2^2 x_3^2 + 18 x_1 x_2 x_3 x_4 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2$$

of the binary cubic form $f(u, v) = x_1 u^3 + x_2 u^2 v + x_3 u v^2 + x_4 v^3$ is a striking example arising in this way. Although such constructions yield many examples, affine spheres with indefinite signature equiaffine metric are less well understood, in part because the theory for nonelliptic Monge-Ampère equations is inadequately developed. Also, explicit representations of the hyperbolic affine spheres asymptotic to inhomogeneous cones are not known, even for seemingly simple cases like polyhedral cones.

Affine spheres occur as models in some applied contexts. The flow evolving a hypersurface along its equiaffine normal is studied in mathematical imaging. Its self-similar (soliton) solutions are affine spheres. In another direction, the function $-\log|\prod_I z_I|$ is the prototype for the *self-concordant barrier functions* that play a key role in polynomial time interior point methods for solving convex programming problems; see [3].

The Cheng-Yau theorem associates an affine sphere with the cone over the universal cover of a manifold M carrying a *flat projective structure* (see

[1]), which is *convex* in the sense that it is the quotient of a convex domain by a group of projective transformations. The equiaffine metric descends to give a canonical metric on M analogous to the Kähler-Einstein metric on a compact Kähler manifold with negative first Chern class. This metric should be fundamental in better understanding convex projective structures. For example, J. Loftin showed how to use it to identify the deformation space of convex projective structures on a genus $g > 1$ compact orientable surface M with a vector bundle of total dimension $16g - 16$ over the Teichmüller space of M.

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