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# Ion-Ion Correlations Effect on Nonlinear High-Frequency Plasma Conductivity

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## ABSTRACT

On the basis of the BBGKY hierarchy of equations an expression is derived for the response of a fully ionized plasma to a strong, high-frequency electric field in the limit of infinite ion mass. It is found that even in this limit the ion-ion correlation function is substantially affected by the field. The corrections to earlier nonlinear results for the current density appear to be quite essential. The validity of the model introduced by Dawson and Oberman to study the response to a vanishingly small field is confirmed for larger values of the field when the correct expression for the ion-ion correlations is introduced; the model by itself does not yield such an expression. The results have interest for the heating of the plasma and for the propagation of a strong electromagnetic wave through the plasma. The theory seems to be valid for any field intensity for which the plasma is stable.

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## I. INTRODUCTION

The high-frequency conductivity of a fully ionized plasma was first properly computed by Dawson and Oberman,[1,2] who used a simple model for a plasma of infinitely heavy ions. In this limit, they regarded the ions as a set of fixed scatterers; first the ions were assumed to be randomly distributed[1] and later ion-ion thermal equilibrium correlations were taken into account.[2] Although only the linear limit (when the driving field is taken to be vanishingly small) was considered, it was realized that the model in principle could be used to study nonlinear effects. It was essential to bring into the model, as an external datum, an expression for the ion-ion correlation function.

In a parallel development it was shown[3] that a linearized treatment of the BBGKY hierarchy gives the same result obtained in Ref. 2; the thermal ion-ion correlations were introduced from the outset. It may be concluded that, in the <u>linear</u> limit, the Dawson-Oberman model gives correct results. There is actually no problem about choosing a proper expression for the ion-ion correlations because the plasma is near thermal equilibrium; moreover, no substantial difference was found even between the results of Refs. 1 and 2.

In the present paper we use the BBGKY hierarchy of equations to obtain the plasma response to a stronger, high-frequency electric field. While the concept of conductivity, strictly, is defined only for vanishingly small fields, an expression for the current is needed nonetheless to study the nonlinear absorption of the field energy by the plasma and the propagation of a strong electromagnetic wave.

Nonlinear effects have been considered by Silin,[4] who used the BBGKY equations but failed to include collective phenomena, and Albini and Rand,[5] who assumed a driving frequency, $\omega$, large compared with the electron plasma frequency, $\omega_p$. In both these papers the typical resonant plasma effects[1] were missed. Kaw and Salat[6] extended the Dawson-Oberman model to moderate fields (in the sense that the resulting directed component of electron velocity is smaller than the electron thermal velocity) and found new resonances in the current. In both Refs. 5 and 6 the assumption of uncorrelated ions was made. While the model has been shown to be valid only in the linear limit, one would feel its validity could be extended to larger fields (a first result of our study is, indeed, the confirmation of this validity). The real difficulty in using the model for nonlinear considerations lies in the selection of a proper ion-ion correlation function.

The limit of an infinite ion mass, $m_i$, was taken in Ref. 6, as will be the case here. It could be thought that in this limit the field would not modify the ion-ion correlations, because there would be no response to the field from the infinitely heavy ions, thereby allowing the use of the thermal correlations, i.e., the correlations existing before the field had been switched on. This consideration (together with the facts that no substantial difference was found between the results of Refs. 1 and 2 and that computations would be somewhat simpler for uncorrelated ions than for thermal correlations) would seem to justify the selection of zero ion-ion correlations made in Ref. 6. The most important results of the present investigation are

(1) the conclusion that the argument just given is wrong and (2) the derivation of an expression for the ion-ion correlations quite different from the thermal value (but to which it reduces for very weak fields). The expression contains additional resonances, thus modifying substantially in some cases the results of Ref. 6 .

The change in the correlations, in the limit $m_i \to \infty$, is originated actually by the finiteness of $m_i$. Generally, the infinite $m_i$ limit is taken within the conception that finite $m_i$ corrections would be of the order of the electron-to-ion mass ratio $m_e/m_i$, or perhaps $(m_e/m_i)^{1/2}$, negligible in a certain approximation; the correction found here for finite $m_i$ does not go to zero as $m_i \to \infty$. This apparent paradox can be explained as follows: In order to avoid transient effects one assumes generally that the field was switched on at time $\tau$, very far in the remote past ($\tau \to -\infty$) If we first let $m_i \to \infty$, the ions will not respond to the field no matter how long we wait (i.e., even when we let $\tau \to -\infty$), as discussed in the preceding paragraph. On the other hand, if $m_i$ is first taken to be finite, the ions will move under the action of the field and after a long time some sort of non-thermal, oscillating equilibrium will be reached; if <u>now</u> we let $m_i \to \infty$, some static field-dependent part of the ion-ion correlations will not go to zero , not being dependent on $m_i$ (as the thermal correlations are not).[7]

The time required for the ion-ion correlations to reach equilibrium is an average of the ion Landau damping time $\bar{\gamma}_i^{-1}$. We take the limit $|\tau|\bar{\gamma}_i \to \infty$ and <u>then</u> the limit $m_i \to \infty$ ($\bar{\gamma}_i$ depends on $m_i$ and goes to zero as $m_i \to \infty$.) For times such that $|\tau|\bar{\gamma}_i = O(1)$, complicated ion transient effects would enter the picture, while for $|\tau|\bar{\gamma}_i < O(1)$ the formulation of Ref. 6 would seem to be valid. In the last case, some questions could arise on whether other transient effects would have decayed off so early.

There is a second requirement in the Dawson-Oberman model , and that is the selection of a zero-order electron distribution function. This shortcoming will not be eliminated here. However, this problem is physically much less interesting than that connected with the selection of an ion-ion correlation function. Moreover, for moderate fields, to be considered in detail in the last section, the non-thermal modification of the ion correlations affect much more the nonlinear results than the deviation of the electron distribution function from its Maxwellian value.

Finally we point out that the equations to be considered are formally solved for an arbitrary field intensity. However, some physical assumptions are introduced; the restrictions that they may impose on the field intensity are discussed in the last section.

## II. THE NONLINEAR CURRENT DENSITY

We consider an infinite, homogeneous plasma composed of electrons and one species of ions, inside which there is a spatially uniform, oscillating electric field . (For waves of long but finite wavelength, the uniform field assumption is equivalent to the dipole approximation.) From the Liouville equation, the usual BBGKY hierarchy of equations[8] is derived. The equation for the one-particle distribution function of the $\alpha$ species, $f_\alpha$, reads

$$\frac{\partial f_\alpha^1}{\partial t} + \frac{q_\alpha}{m_\alpha} \underset{\sim}{E} \sin \omega t \cdot \frac{\partial f_\alpha^1}{\partial \underset{\sim}{v}_1} = \frac{q_\alpha}{m_\alpha} \frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \int d\underset{\sim}{r}_2 \, d\underset{\sim}{v}_2 \, \frac{\partial \phi^{12}}{\partial \underset{\sim}{r}_1} \sum_\beta q_\beta n_\beta \, g_{\alpha\beta}^{12} \tag{1}$$

where $\underset{\sim}{E} \sin \omega t$ is the field prevailing inside the plasma, $\phi^{12} = | \underset{\sim}{r}_1 - \underset{\sim}{r}_2 |^{-1}$, and $g_{\alpha\beta}$ is the two-particle, phase-space correlation function between the $\alpha$ and $\beta$ species (both $\alpha$ and $\beta$ stand for electrons, e, or ions, i). The superscripts are particle indices.

Usually a kinetic equation for $f_\alpha^1$ is derived[8] by solving the equation for $g_{\alpha\beta}^{12}$ [see Eqs. (14) and (25) later] assuming that $f_\alpha^1$ and $f_\beta^2$ remain frozen on the time scale during which $g_{\alpha\beta}^{12}$ changes and introducing this value of $g_{\alpha\beta}^{12}$ $(f_\alpha^1, f_\beta^2)$ into (1); that assumption is certainly not satisfied when $\omega/\omega_p \geq 0(1)$.[1] On the other hand, $\omega$ is then large compared with the collision frequency, $\nu$: for most fully ionized plasmas $\omega_p/\nu = 0(N_D)$ where $N_D$, the number of particles in a Debye sphere, is a large number. The first and third terms in (1) are of order of $\omega f_\alpha^1$ and $\nu f_\alpha^1$, respectively, so that to lowest order in $\nu/\omega$ the driving field is balanced by the inertial force; we may then solve (1) by expanding $f_\alpha^1$ in powers of $\nu/\omega$ whenever $\nu/\omega \ll 1$. [We point out here that $\omega_p/\nu = 0(N_D)$ is only valid, a priori, for a weak field, E; for a strong field $\nu$ will depend on E. At this point, we simply assume $\omega$ and E such that $\nu(E)/\omega < 0(1)$, and leave the matter until the restrictions on the field intensity are discussed in the last section.]

Writing

$$f^1_\alpha = f^1_{\alpha o} + \nu/\omega \, f^1_{\alpha 1} + \ldots \tag{2}$$

we obtain from (1) the zero-order equation

$$\frac{\partial f^1_{\alpha o}}{\partial t} + \frac{q_\alpha}{m_\alpha} \underset{\sim}{E} \sin \omega t \cdot \frac{\partial f^1_{\alpha o}}{\partial \underset{\sim}{v}_1} = 0 \tag{3}$$

which describes a purely reactive response to the field; the solution of Eq. (3) is an arbitrary function of $\underset{\sim}{u}^\alpha_1 \equiv \underset{\sim}{v}_1 + \omega \underset{\sim}{\varepsilon}_\alpha \cos \omega t$, where $\underset{\sim}{\varepsilon}_\alpha = q_\alpha / m_\alpha \, \underset{\sim}{E}/\omega^2$. We normalize $f^1_{\alpha o}$ : $\int f^1_{\alpha o} \, d\underset{\sim}{v}_1 = 1$.

Consider now the equation for $f^1_{e1}$ . It is a straightforward matter to find out that the naive expansion (2) breaks down for long times because of secular behavior: there is, for instance, a slow heating of the plasma and $\int f^1_{eo} \, \underset{\sim}{v}^2_1 \, d\underset{\sim}{v}_1 \sim \nu t$. (This problem does not exist in the linear limit because for small E, the heating vanishes like $E^2$.) The relaxation frequency for the ion-ion correlations is some average of the ion Landau damping rate; when this is of the order of or smaller than $\nu$ we will run (for the long-time limit stipulated at the end of Sec. I) into times such that the accumulated heating can be neglected no more. To include such a case in out treatment, the heating has to be taken into account and the mathematical secularities avoided. This is done by using the known method of multiple scales.[9] We introduce a second time scale so that $f^1_{\alpha o}$ can be written as $f^1_{\alpha o} = f^1_{\alpha o} ( \underset{\sim}{v}_1 + \omega \underset{\sim}{\varepsilon}_\alpha \cos \omega t, t^* \omega/\nu)$, $dt^*/dt = 1$. Now $f^1_{\alpha o}$ satisfies (3) in the fast scale only and there is a slow variation of $f^1_{\alpha o}$ which appears in the next-order equation. The slow scale is introduced, of course, into the other terms of (2) too, but since we will not go beyond first-order terms, this is irrelevant here. The equation of order $\nu/\omega$ obtained from (1), for $\alpha = e$, reads now:

$$\nu/\omega \left[ \frac{\partial f^1_{eo}}{\partial(\nu/\omega t^*)} + \frac{\partial f^1_{el}}{\partial t} + \frac{q_e}{m_e} \underset{\sim}{E} \sin \omega t \cdot \frac{\partial f^1_{el}}{\partial \underset{\sim}{v}_1} \right]$$

$$= \frac{q_e^2 n_e}{m_e} \frac{\partial}{\partial \underset{\sim}{v}_1} \cdot \int d\underset{\sim}{r}_2 d\underset{\sim}{v}_2 \frac{\partial \phi^{12}}{\partial \underset{\sim}{r}_1} ( g^{12}_{ee} - g^{12}_{ei}) \qquad (4)$$

where use has been made of the neutrality condition, $q_e n_e + q_i n_i = 0$. We integrate over t in (4) and obtain

$$f^1_{el} \sim - \int^t dt \left[ \frac{\partial f^1_{eo}}{\partial(\nu/\omega t^*)} + \frac{q_e}{m_e} \underset{\sim}{E} \sin \omega t \cdot \frac{\partial f^1_{el}}{\partial \underset{\sim}{v}_1} - \frac{\omega}{\nu} C \right] \qquad (5)$$

where C is the right-hand side of (4). (We point out here that the expansion $g^{12}_{\alpha\beta} = g^{12}_{\alpha\beta(o)} + \nu/\omega \; g^{12}_{\alpha\beta(1)} + \ldots$ should have been made and only $g^{12}_{e\beta(o)}$ should appear in C; $g^{12}_{\alpha\beta(o)}$ is to be found later from the equation for $g^{12}_{\alpha\beta}$ where $f^1_{\alpha o}$ and $f^2_{\beta o}$ will substitute for $f^1_\alpha$ and $f^2_\beta$. Since we will not go beyond the zero-order term in $g^{12}_{\alpha\beta}$ we write throughout the paper $g^{12}_{\alpha\beta}$ for $g^{12}_{\alpha\beta\,(o)}$.)

The introduction of a new time scale has given us an additional freedom; this has to be used to avoid secularities in the expansion. We want $f^1_{e1}$ to be bounded in time. If the integrand in (5) were of fixed sign for large t, $\lim_{t\to\infty} f^1_{e1}$ could be finite only if that integrand would vanish in the limit $t\to\infty$. Thus, an equation for the slow variation of $f^1_{eo}$ would be obtained. However, if some terms inside the bracket in (5) are oscillating, as will be actually the case, the separation of the secular terms cannot be made until the form of C is known in detail. (The analysis would seem to be simpler for any particular moment of $f^1_{eo}$ than for $f^1_{eo}$ itself, but in any event a knowledge of C is required.)

The slow variation of $f^1_{eo}$, therefore, can be determined only after both $g^{12}_{ei}$ and $g^{12}_{ee}$ are known. The equation for $g^{12}_{ee}$ in the presence of a strong, high-frequency field is complicated and does not become simplified in the limit $m_i\to\infty$, as is the case for $g_{ei}$ and $g_{ii}$, which will be studied in Secs. III and IV. Thus, $f^1_{eo}$ will remain unknown and our results will be given for an arbitrary $f^1_{eo}$. On the other hand, something important can be established about the slow-variable dependence of $f^1_{eo}$: it can only involve even powers of $\underset{\sim}{E}$. In effect, along an axis aligned with the field there is no preferential direction for the long times covering many periods of the field because $\underset{\sim}{E}\ \sin\omega t$ changes sign every half-period; it is quite apparent that if $f^1_{eo}$ is even in $\underset{\sim}{u}^e$, at a given time, it will remain so throughout the slow-scale development. After transient effects in the fast scale have disappeared, we should take $f^1_{eo}$ to be an even function of $\underset{\sim}{u}^e_1$. The definite selection made in Ref. 6 of a Maxwellian distribution is arbitrary, although quite probably the correct results for the current would differ only by simple numerical factors. In the linear limit, of course, we can take

$f^1_{eo}$ to be Maxwellian because corrections would be of the order of $E^2$.

Going back to Eq. (4), we multiply the equation by $n_e q_e \underset{\sim}{v}_1$ and integrate over $\underset{\sim}{v}_1$. There results

$$\frac{\nu}{\omega} \frac{\partial \underset{\sim}{j}_o}{\partial(\nu/\omega t^*)} + \frac{\partial \underset{\sim}{j}_1}{\partial t} = \frac{q_e^3 n_e^2}{m_e} \int d\underset{\sim}{r}_2 \frac{\partial \phi^{12}}{\partial \underset{\sim}{r}_{12}} \int g^{12}_{ei} \, d\underset{\sim}{v}_1 \, d\underset{\sim}{v}_2 \tag{7}$$

where $\underset{\sim}{j}_o = n_e q_e \int f^1_{eo} \underset{\sim}{v}_1 d\underset{\sim}{v}_1$ and $\underset{\sim}{j}_1 = n_e q_e \int \nu/\omega \, f^1_{e1} \underset{\sim}{v}_1 d\underset{\sim}{v}_1$. Obviously there is no contribution to (7) from $g^{12}_{ee}$; as for the last term on the left-hand side of (4) we point out that our normalization for $f^1_{eo}$ implies $\int f^1_{e1} \, d\underset{\sim}{v}_1 = 0$. We can write

$$\underset{\sim}{j}_o = n_e q_e \int f^1_{eo} \left[ \underset{\sim}{u}^e_1 - \omega \underset{\sim}{\mathcal{E}}_e \cos \omega t \right] d\underset{\sim}{u}^e_1 = \underset{\sim}{j}^*_o - n_e q_e \, \omega \underset{\sim}{\mathcal{E}} \cos \omega t.$$

Thus, $\partial \underset{\sim}{j}_o / \partial(\nu/\omega t^*) = \partial \underset{\sim}{j}^*_o / \partial(\nu/\omega t^*)$; since $f^1_{eo}$ is even in $\underset{\sim}{u}^e_1$, $\underset{\sim}{j}^*_o = 0$ and we can drop the first term in (7). The zero-order current $\underset{\sim}{j}_o$ is purely reactive, then, and satisfies the equation

$$\frac{\partial \underset{\sim}{j}_o}{\partial t} - \frac{q_e}{m_e} \underset{\sim}{E} \sin \omega t = 0$$

obtained from (3) for $\alpha = e$.

Equation (7) reduces to

$$\frac{\partial \underset{\sim}{j}_1}{\partial t} = - \frac{q_e^3 n_e^2}{m_e} (2\pi)^{-3} \int i\underset{\sim}{k} \, d\underset{\sim}{k} \, \tilde{\phi} \int \tilde{g}^{12}_{ei} \, d\underset{\sim}{v}_1 \, d\underset{\sim}{v}_2 \tag{9}$$

where for later convenience we have made some rearrangement by introducing the Fourier transforms

$$\tilde{\phi}(\underset{\sim}{k}) = \int d\underset{\sim}{r}_{12}\, \phi^{12} \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{r}_{12}\right] = \frac{4\pi}{k^2} , \tag{10}$$

$$\tilde{g}^{12}_{ei}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2) = \int d\underset{\sim}{r}_{12}\, g^{12}_{ei}(\underset{\sim}{r}_{12}, \underset{\sim}{v}_1, \underset{\sim}{v}_2) \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{r}_{12}\right] , \tag{11}$$

where we have taken advantage of the spatial homogeneity. In the next two sections we shall obtain an expression for the right-hand side of (9) in terms of $f^1_{eo}$.

Finally, let us multiply (4) by $n_e m_e \underset{\sim}{v}_1^2/2$ and integrate over $\underset{\sim}{v}_1$ and t. We obtain

$$U_1 \sim -\int^t dt\left[\frac{\nu}{\omega} \frac{\partial U_o}{\partial(\nu/\omega t^*)} - \underset{\sim}{j}_1 \cdot \underset{\sim}{E} \sin \omega t - n_e \frac{m_e}{2}\int C\underset{\sim}{v}_1^2\, d\underset{\sim}{v}_1\right] \tag{12}$$

where $U_o = 1/2\, n_e\, m_e \int f^1_{eo}\, \underset{\sim}{v}_1^2\, d\underset{\sim}{v}_1$ and $U_1 = 1/2\, n_e m_e \int \nu/\omega\, f^1_{e1} \underset{\sim}{v}_1^2\, d\underset{\sim}{v}_1$. It follows imediately that

$$\frac{\partial U_o}{\partial(\nu/\omega t^*)} = \frac{\partial}{\partial(\nu/\omega t^*)} 1/2 n_e\, m_e \int f^1_{eo} \left[\underset{\sim}{u}^e_1\right]^2 d\underset{\sim}{u}^e_1$$

so that the first term inside the bracket of (12) does not depend on t. The third term involves only $g^{12}_{ei}$ because there can be no self-heating of the electrons. In the limit $m_i \to \infty$, both $\underset{\sim}{j}_1$ and $\int C\underset{\sim}{v}_1^2\, d\underset{\sim}{v}_1$ will come out to be purely oscillating in t (there are additional dependences on $\nu/\omega t^*$). To avoid secularities, therefore,

$$\frac{\partial U_o}{\partial t^*} = \langle \underset{\sim}{j}_1 \cdot \underset{\sim}{E} \sin \omega t \rangle \tag{13}$$

in the limit $t \to \infty$, where $\langle\ \rangle$ indicates averaging over the fast variable; the average does not vanish because of those terms in $\underset{\sim}{j}_1$ that vary as $\sin \omega t$. Equation (13) describes the slow heating of the electrons. We point out that if $f^1_{eo}$ were Maxwellian, there would be only one unknown parameter, the temperature $T_e$. Hence, (13) would describe completely the evolution of $f^1_{eo}$, i.e., the slow nonlinear change in $T_e$ ($\underset{\sim}{j}_1$ being a nonlinear function of $T_e$).

## III. THE ELECTRON-ION CORRELATION FUNCTION

The equation for the electron-ion correlation function, $g_{ei}$, reads[8]

$$\left[\frac{\partial}{\partial t} + (\underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot \frac{\partial}{\partial \underset{\sim}{r}_{12}} + \underset{\sim}{E} \sin \omega t \cdot \left(\frac{q_e}{m_e} \frac{\partial}{\partial \underset{\sim}{v}_1} + \frac{q_i}{m_i} \frac{\partial}{\partial \underset{\sim}{v}_2}\right)\right] g^{12}_{ei}$$

$$= q_e q_i \frac{\partial \phi^{12}}{\partial \underset{\sim}{r}_{12}} \cdot \left(\frac{1}{m_e} \frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{1}{m_i} \frac{\partial}{\partial \underset{\sim}{v}_2}\right) f^1_e f^2_e + \frac{q_e}{m_e} \frac{\partial f^1_e}{\partial \underset{\sim}{v}_1} \cdot \frac{\partial}{\partial \underset{\sim}{r}_{12}} \int d\underset{\sim}{r}_3$$

$$\times \phi^{13} \int d\underset{\sim}{v}_3 \sum_{\alpha} q_\alpha n_\alpha \, g^{23}_{i\alpha} - \frac{q_i}{m_i} \frac{\partial f^2_i}{\partial \underset{\sim}{v}_2} \cdot \frac{\partial}{\partial \underset{\sim}{r}_{12}} \int d\underset{\sim}{r}_3 \phi^{23} \int d\underset{\sim}{v}_3 \sum_{\alpha} q_\alpha n_\alpha g^{13}_{e\alpha} \, . \tag{14}$$

Again the homogeneity of the plasma has been used, by writing $g^{mn}_{\alpha\beta}(\underset{\sim}{r}_m, \underset{\sim}{r}_n) = g^{mn}_{\alpha\beta}(\underset{\sim}{r}_m - \underset{\sim}{r}_n)$. To obtain (14) we have dropped two terms from the exact BBGKY equation, as is usually done in plasma kinetic theory. One is $\partial\phi^{12}/\partial\underset{\sim}{r}_{12} \cdot (\partial/m_e \partial\underset{\sim}{v}_1 - \partial/m_i \partial \underset{\sim}{v}_2)\, g^{12}_{ei}$; the other involves the three-particle correlation function $h^{123}_{ei\alpha}$. (Similar terms will be dropped from the BBGKY equation for $g_{ii}$ in the next section.) In general, those terms can be neglected whenever $|g_{ei}/f_e f_i| \ll 1$ and $|h_{ei\alpha}| \ll f_e|g_{i\alpha}|$, $f_i|g_{e\alpha}|$, respectively. [The *first term can be also dropped on the basis of a* comparison to the second term on the left side of (14); their ratio is of the order of $e^2/|\underset{\sim}{r}_{12}|\kappa T$. Thus, (14) can be valid only for $|\underset{\sim}{r}_{12}| \gg e^2/\kappa T$. The exclusion of the small impact region is common and amounts to a weak indeterminacy in a cutoff to be introduced in the subsequent computation of some integrals.] Not far from thermal equilibrium, a comparison with known results provides confirmation of the previously mentioned inequalities. For the arbitrary field intensity we are considering, we now make the *ansatz that the terms dropped are negligible and later, in the last section,* discuss the restrictions that this may impose on the field intensity.

To obtain a zero-order result for $g_{ei}^{12}$ we can substitute $f_{\alpha o}^m$ for $f_\alpha^m$ in (14). Next we introduce the transformation

$$t = t, \qquad \underset{\sim}{u}_m^\alpha = \underset{\sim}{v}_m + \omega\underset{\sim}{\varepsilon}_\alpha \cos\omega t,$$

$$\underset{\sim}{\rho}_m^\alpha = \underset{\sim}{r}_m + \underset{\sim}{\varepsilon}_\alpha \sin\omega t$$

so that $\underset{\sim}{\rho}_{12} \equiv \underset{\sim}{\rho}_1^e - \underset{\sim}{\rho}_2^i = \underset{\sim}{r}_{12} + \underset{\sim}{\varepsilon}_{ei}\sin\omega t$, $(\underset{\sim}{\varepsilon}_{\alpha\beta} = \underset{\sim}{\varepsilon}_\alpha - \underset{\sim}{\varepsilon}_\beta)$. The equation for $g_{ei}^{12}$ becomes

$$\left[\frac{\partial}{\partial t} + (\underset{\sim}{u}_1^e - \underset{\sim}{u}_2^i)\cdot\frac{\partial}{\partial\underset{\sim}{\rho}_{12}}\right] g_{ei}^{12} = q_e q_i \frac{\partial\phi^{12}}{\partial\underset{\sim}{\rho}_{12}}\cdot\left(\frac{1}{m_e}\frac{\partial}{\partial\underset{\sim}{u}_1^e} - \frac{1}{m_i}\frac{\partial}{\partial\underset{\sim}{u}_2^i}\right)$$

$$\times f_{eo}^1 f_{io}^2 + \frac{q_e^2 n_e}{m_e}\frac{\partial f_{eo}^1}{\partial\underset{\sim}{u}_1^e}\cdot\frac{\partial}{\partial\underset{\sim}{\rho}_{12}}\int d\underset{\sim}{r}_3\phi^{13}\left(\int g_{ie}^{23}\,d\underset{\sim}{u}_3^e - \int g_{ii}^{23}\,d\underset{\sim}{u}_3^i\right)$$

$$- \frac{q_e q_i n_e}{m_i}\frac{\partial f_{io}^2}{\partial\underset{\sim}{u}_2^i}\cdot\frac{\partial}{\partial\underset{\sim}{\rho}_{12}}\int d\underset{\sim}{r}_3\phi^{23}\left(\int g_{ee}^{13}\,d\underset{\sim}{u}_3^e - \int g_{ei}^{13}\,d\underset{\sim}{u}_3^i\right). \qquad (15)$$

We integrate (15) over $\underset{\sim}{u}_2^i$ to obtain

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{u}_1^e \cdot \frac{\partial}{\partial \underset{\sim}{\rho}_{12}}\right] G_{ei}^{12} = \frac{q_e q_i}{m_e} \frac{\partial \phi^{12}}{\partial \underset{\sim}{\rho}_{12}} \cdot \frac{\partial f_{eo}^1}{\partial \underset{\sim}{u}_1^e} + \frac{q_e^2 n_e}{m_e} \frac{\partial f_{eo}^1}{\partial \underset{\sim}{u}_1^e} \frac{\partial}{\partial \underset{\sim}{\rho}_{12}}$$

$$\times \int d\underset{\sim}{r}_3 \phi^{13} \left(\int G_{ei}^{23}\, d\underset{\sim}{u}_3^e - \int g_{ii}^{32}\, d\underset{\sim}{u}_i^2\, d\underset{\sim}{u}_i^3\right) - \frac{\partial}{\partial \underset{\sim}{\rho}_{12}} \cdot \int \underset{\sim}{u}_2^i\, g_{ei}^{12}\, d\underset{\sim}{u}_2^i \qquad (16)$$

where $G_{ei}^{mn} = \int g_{ei}^{mn}\, d\underset{\sim}{u}_i^n$ and the equality $g_{\alpha\beta}^{mn} = g_{\beta\alpha}^{nm}$ has been used. Those terms in (15) proportional to $\partial f_{io}^2 / \partial \underset{\sim}{u}_2^i$ do not contribute to (16) since $f_{io}^2$ $(\underset{\sim}{u}_2^i \to \pm\infty) \to 0$, as required from any distribution function.

While we would have to solve an integral equation to obtain $g_{ei}^{12}$ from (15), it is very simple to obtain $G_{ei}^{12}$ from (16); of course, the last term of (16) is quite awkward but we expect that it will drop out in the infinite ion mass limit, which ultimately we shall take. We also observe that $\breve{g}_{ee}$ has disappeared from our equation; it will be considered no more in this paper.

Given any function of $\underset{\sim}{r}_{12}$, $A(\underset{\sim}{r}_{12})$, we can introduce the following Fourier transforms: $\widetilde{A}(\underset{\sim}{k}) \quad \int \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{r}_{12}\right] d\underset{\sim}{r}_{12}\, A(\underset{\sim}{r}_{12})$ [as in Eqs. (10) and (11)] and

$$\widetilde{\widetilde{A}}(k) \equiv \int \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\rho}_{12}\right] d\underset{\sim}{\rho}_{12} A(\underset{\sim}{\rho}_{12}) = \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei} \sin \omega t\right]$$

$$\times \int \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{r}_{12}\right] d\underset{\sim}{r}_{12}\, A(\underset{\sim}{r}_{12}) = \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei} \sin \omega t\right] \widetilde{A}(\underset{\sim}{k}). \quad (17)$$

Then, making a Fourier analysis of Eq. (16) with respect to the variable $\underset{\sim}{\rho}_{12}$ yields

$$(\frac{\partial}{\partial t}+ i\underset{\sim}{k} \cdot \underset{\sim 1}{u}^{e}) \widetilde{\widetilde{G}}^{12}_{ei} = \frac{q_e q_i}{m_e} i\underset{\sim}{k} \cdot \frac{\partial f^1_{eo}}{\partial \underset{\sim 1}{u}^e} \widetilde{\phi} \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim ei}{\varepsilon} \sin \omega t\right]$$

$$+\frac{q_e^2 n_e}{m_e} i\underset{\sim}{k} \cdot \frac{\partial f^1_{eo}}{\partial \underset{\sim 1}{u}^e} \frac{4\pi}{k^2}(\int \widetilde{\widetilde{G}}^{32}_{ei} \, d\underset{\sim 3}{u}^e - \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim ei}{\varepsilon} \sin \omega t\right]$$

$$\times \int \widetilde{g}^{32}_{ii} \, d\underset{\sim 3}{u}^i d\underset{\sim 2}{u}^i) - i\underset{\sim}{k} \cdot \int \underset{\sim 2}{u}^i \, \widetilde{\widetilde{g}}^{12}_{ei} \, d\underset{\sim 2}{u}^i \qquad (18)$$

where use has been made of (10) and (17).

We make the ansatz $\widetilde{g}^{12}_{ii} = \sum_p \widetilde{g}^p_{ii}(1, 2)\, e^{-ip\omega t}$ and define $\widetilde{Q}^p_{ii} = \int \widetilde{g}^p_{ii}(m, n)\, d\underset{\sim m}{v}\, d\underset{\sim n}{v}$. Next, we solve (18) under the assumption of vanishing perturbations in the remote past; a formal integration of (18) yields

$$\widetilde{\widetilde{G}}^{12}_{ei} = \frac{i\underset{\sim}{k}}{m_e} \cdot \frac{\partial f^1_{eo}}{\partial \underset{\sim 1}{u}^e} \int_{-\infty}^{t} d\tau \, \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim 1}{u}^e (t-\tau)\right] \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim ei}{\varepsilon} \sin \omega\tau\right]$$

$$\times \left(q_e q_i \widetilde{\phi} - q_e^2 \frac{4\pi}{k^2} n_e \sum_p \widetilde{Q}^p_{ii} \, e^{-ip\omega\tau}\right) + \frac{i\underset{\sim}{k}}{m_e} \cdot \frac{\partial f^1_{eo}}{\partial \underset{\sim 1}{u}^e} q_e^2 n_e$$

$$\times \frac{4\pi}{k^2} \int_{-\infty}^{t} d\tau \, \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim 1}{u}^e (t-\tau)\right] \int \widetilde{\widetilde{G}}^{32}_{ei}(\tau) \, d\underset{\sim 3}{u}^e$$

$$- i\underset{\sim}{k} \cdot \int \underset{\sim 2}{u}^i \, d\underset{\sim 2}{u}^i \int_{-\infty}^{t} d\tau \, \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim 1}{u}^e (t-\tau)\right] \widetilde{\widetilde{g}}^{12}_{ei}(\tau).$$

If this equation is integrated over $\underset{\sim 1}{u}^e$ we obtain the following integral equation for $\widetilde{\widetilde{Q}}_{ei} \equiv \int \widetilde{\widetilde{G}}_{ei} \, d\underset{\sim}{u}^e$ :

$$\tilde{\tilde{Q}}_{ei}(t) - \frac{\omega_p^2}{k^2}\int d\underset{\sim}{u}_1^e \, i\underset{\sim}{k} \cdot \frac{\partial f_{eo}^1}{\partial \underset{\sim}{u}_1^e}\int_{-\infty}^t d\tau \, \tilde{\tilde{Q}}_{ei}(\tau) \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{u}_1^e (t-\tau)\right]$$

$$= -\frac{\omega_p^2}{k^2}\int d\underset{\sim}{u}_1^e \, i\underset{\sim}{k} \cdot \frac{\partial f_{eo}^1}{\partial \underset{\sim}{u}_1^e}\int_{-\infty}^t d\tau \, \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{u}_1^e (t-\tau)\right]\exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei} \sin \omega t\right]$$

$$\times \left(\frac{1}{n_i} + \sum_p \tilde{Q}_{ii}^p \, e^{-ip\omega\tau}\right) - \int_{-\infty}^t d\tau \int\int \tilde{\tilde{g}}_{ei}^{12}(\tau) \exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{u}_1^e (t-\tau)\right] \, i\underset{\sim}{k} \cdot \underset{\sim}{u}_2^i d\underset{\sim}{u}_2^i d\underset{\sim}{u}_1^e \; . \tag{19}$$

The solution to (19) can be written as

$$\tilde{\tilde{Q}}_{ei} = Q_1 + Q_2 \equiv \sum_p e^{-ip\omega t}\left[\frac{J_p(\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei})\chi_{-p}}{n_i D_{-p}}(1 + n_i \tilde{Q}_{ii}^o) + B_p\right] + Q_2 \tag{20}$$

where $Q_1$ and $Q_2$ are due to the first and second terms on the right-hand side of (19), respectively. To find $Q_1$ we have used the identity $\exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei} \sin \omega t\right]$ $\sum_p J_p(\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_{ei}) \, e^{-ip\omega t}$ . In (20),

$$D_{-p} \equiv 1 + \chi_{-p} = 1 + \frac{\omega_p^2}{k^2}\int \frac{\underset{\sim}{k} \cdot \frac{\partial f_{eo}^1}{\partial \underset{\sim}{u}_1^e} d\underset{\sim}{u}_1^e}{p\omega - \underset{\sim}{k} \cdot \underset{\sim}{u}_1^e + i\varepsilon} \; ;$$

$$B_p = \frac{\chi_{-p}}{D_{-p}}\sum_{s \neq 0} J_{p-s} \tilde{Q}_{ii}^s \; . \tag{21}$$

The plasma frequency is $\omega_p = (4\pi q_e^2 n_e/m_e)^{1/2}$, and $\varepsilon > 0$ and goes to zero.

If the limit $m_i \to \infty$ we expect $Q_{ii}^s$ $(s \neq 0)$ and $\int g_{ei}^{12} u_2^i du_2^i$ to vanish; if this is the case, $B_p$ and $Q_2$ will vanish and

$$\tilde{\tilde{Q}}_{ei} = \sum_p e^{-ip\omega t} \frac{J_p \chi_{-p}}{n_i D_{-p}} (1 + n_i \tilde{Q}_{ii}^o) \tag{22}$$

so that using (17), Eq. (9) reduces to

$$\frac{\partial j_1}{\partial t} = \frac{-q_e^3 n_e^2}{2\pi^2 m_e} \int \frac{ik\, dk}{k^2} \exp\left[ik \cdot \varepsilon_{ei} \sin \omega t\right] \sum_p e^{-ip\omega t} \frac{J_p \chi_{-p}}{n_i D_{-p}} (1 + n_i \tilde{Q}_{ii}^o)$$

$$= -\frac{ie^3 n_e Z}{2\pi^2 m_e} \int \frac{k\, dk}{k^2} \sum_\ell e^{i\ell\omega t} \sum_p \frac{J_p J_{p+\ell}}{D_{-p}} (1 + n_i \tilde{Q}_{ii}^o) \quad . \tag{23}$$

If we write above $\tilde{Q}_{ii}^o = 0$, we recover the result found in Ref. 6 by using the Dawson-Oberman model. The proper value to be used for $\tilde{Q}_{ii}^o$ will be obtained in the following section.

Also, if the limit $E \to 0$ is taken in (23) and terms linear in E are retained, the results of Ref. 1 and Refs. 2 and 3 are recovered by choosing $\tilde{Q}_{ii}^o = 0$ and

$$\tilde{Q}_{ii}^o = -\frac{1}{n_i} \frac{k_i^2}{k_i^2 + k_e^2 + k^2} , \tag{24}$$

respectively. Equation (24) is the known expression for the thermal ion-ion correlations, where $k_\alpha^2 = 4\pi q_\alpha^2 n_\alpha (\kappa T_\alpha)^{-1}$, $\kappa$ being Boltzman's constant.

## IV. THE ION-ION CORRELATION FUNCTION

We now proceed to determine $\tilde{Q}^{o}_{ii}$ and to show that $\tilde{Q}^{s}_{ii}$ $(s \neq 0)$ and $\int \approx g^{12}_{ei}\, \underset{\sim}{u}^{i}_{2}\, d\underset{\sim}{u}^{i}_{2}$ vanish in the limit $m_i \to \infty$ . The equation for $g^{12}_{ii}$ reads

$$\left[\frac{\partial}{\partial t} + (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)\cdot\frac{\partial}{\partial \underset{\sim}{r}_{12}} + \frac{q_i}{m_i}\underset{\sim}{E}\sin\omega t \cdot \left(\frac{\partial}{\partial \underset{\sim}{v}_1} + \frac{\partial}{\partial \underset{\sim}{v}_2}\right)\right] g^{12}_{ii}$$

$$= \frac{q_i^2}{m_i}\frac{\partial \phi^{12}}{\partial \underset{\sim}{r}_{12}}\cdot\left(\frac{\partial}{\partial \underset{\sim}{v}_1} - \frac{\partial}{\partial \underset{\sim}{v}_2}\right) f^1_{io} f^2_{io} + \frac{q_i q_e n_e}{m_i}\frac{\partial f^1_{io}}{\partial \underset{\sim}{v}_1}\cdot\frac{\partial}{\partial \underset{\sim}{r}_{12}}\int d\underset{\sim}{r}_3\, \phi^{13}\left(\int d\underset{\sim}{v}_3\, g^{23}_{ie}\right.$$

$$\left. - \int d\underset{\sim}{v}_3\, g^{23}_{ii}\right) - \frac{q_i q_e n_e}{m_i}\frac{\partial f^2_{io}}{\partial \underset{\sim}{v}_2}\cdot\frac{\partial}{\partial \underset{\sim}{r}_{12}}\int d\underset{\sim}{r}_3\, \phi^{23}\left(\int d\underset{\sim}{v}_3\, g^{13}_{ie} - \int d\underset{\sim}{v}_3\, g^{13}_{ii}\right) \cdot \qquad (25)$$

We introduce the transformation $t = t$, $\underset{\sim}{u}^{\alpha}_{n} = \underset{\sim}{v}_n + \omega\, \underset{\sim}{\varepsilon}_{\alpha} \cos\omega t$, and Fourier analyze with respect to $\underset{\sim}{r}_{12}$ to obtain

$$\left[\frac{\partial}{\partial t} + i\underset{\sim}{k}\cdot(\underset{\sim}{u}^{i}_{1} - \underset{\sim}{u}^{i}_{2})\right]\tilde{g}^{12}_{ii} = \frac{q_i^2}{m_i} i\underset{\sim}{k}\,\tilde{\phi}\cdot\left(\frac{\partial}{\partial \underset{\sim}{u}^{i}_{1}} - \frac{\partial}{\partial \underset{\sim}{u}^{i}_{2}}\right) f^1_{io}\, f^2_{io}$$

$$+ \frac{q_i q_e n_e}{m_i} i\underset{\sim}{k}\cdot\frac{\partial f^1_{io}}{\partial \underset{\sim}{u}^{i}_{1}}\frac{4\pi}{k^2}\left\{\exp\left[i\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}_{ei}\sin\omega t\right]\int \approx g^{32}_{ei}\, d\underset{\sim}{u}^{e}_{3} - \int \tilde{g}^{32}_{ii}\, d\underset{\sim}{u}^{3}_{i}\right\}$$

$$- \frac{q_i q_e n_e}{m_i} i\underset{\sim}{k}\cdot\frac{\partial f^2_{io}}{\partial \underset{\sim}{u}^{i}_{2}}\frac{4\pi}{k^2}\left\{\exp\left[-i\underset{\sim}{k}\cdot\underset{\sim}{\varepsilon}_{ei}\sin\omega t\right]\int \approx g^{13}_{ie}\, d\underset{\sim}{u}^{e}_{3} - \int \tilde{g}^{13}_{ii}\, d\underset{\sim}{u}^{3}_{i}\right\}$$

(26)

where Eqs. (10) and (17) have been used.

Now we complete the ansatz made in Sec. III, $\tilde{g}_{ii}^{mn} = \sum_s e^{-is\omega t}\, \tilde{g}_{ii}^{s}\,(m,n)$, by assuming that for large $m_i$,

$$f_{io}(\underset{\sim}{u}^{i}) = f_{im}(\underset{\sim}{u}^{i}) + o(1), \qquad g_{ei}^{n2} = f_{io}^{2}\, G_{ei}^{n2} + o(1),$$

$$g_{ii}^{o} = f_{io}\, f_{io}\, \tilde{Q}_{ii}^{o} + o(1) \tag{27}$$

where $f_{im}$ is a Maxwellian distribution; the symbol o(1) represents a quantity that goes to zero as $m_i \to \infty$. As obtained from (26), the equation for $\tilde{g}_{ii}^{s}$ $(s \neq 0)$ reads for very large $m_i$,

$$\left[-s\omega + \underset{\sim}{k}\cdot(\underset{\sim}{u}_1^i - \underset{\sim}{u}_2^i)\right]\tilde{g}_{ii}^{s}\,(1,2) \approx \frac{q_e q_i n_e}{\kappa T_i}\,\frac{4\pi}{k^2}\,\underset{\sim}{k}\cdot\left[\underset{\sim}{u}_1^i f_{im}^1 \int \tilde{g}_{ii}^{s}(3,2)\, d\underset{\sim}{u}_3^i\right.$$

$$\left. - \underset{\sim}{u}_2^i f_{im}^2 \int \tilde{g}_{ii}^{s}(1,3)\, d\underset{\sim}{u}_3^i\right] - \frac{q_e q_i n_e}{m_i}\,\frac{4\pi}{k^2}\,\underset{\sim}{k}$$

$$\times\left[\underset{\sim}{u}_1^i f_{im}^1 f_{im}^2 \sum_\ell J_\ell\, \tilde{\tilde{Q}}_{ei}^{s+\ell} - \underset{\sim}{u}_2^i f_{im}^2 f_{im}^1 \sum_\ell J_\ell\, \tilde{\tilde{Q}}_{ei}^{s-\ell}(-\underset{\sim}{k})\right] \tag{28}$$

where use has been made of the identity $\tilde{\tilde{Q}}_{ei}(-\underset{\sim}{k}) = \tilde{\tilde{Q}}_{ie}(\underset{\sim}{k})$. We have called $\tilde{\tilde{Q}}_{ei}^{s}$ the bracket multiplying $e^{-is\omega t}$ in Eq. (20) and dropped $Q_2$ because with the ansatz (27), $\int \tilde{\tilde{g}}_{ei}^{12}\, \underset{\sim}{u}_2^i\, d\underset{\sim}{u}_2^i \equiv 0$.

If the limit $m_i \to \infty$ is taken in (28), $f_{im} \to \delta(\underset{\sim}{u}^i)$ and since $\underset{\sim}{u}^i\,\delta(\underset{\sim}{u}^i) \equiv 0$, there results

$$\left[-s\omega + \underset{\sim}{k}\cdot(\underset{\sim}{u}_1^i - \underset{\sim}{u}_2^i)\right]\tilde{g}_{ii}^{s} = 0,$$

so that $g_{ii}^{s} = 0$ for $s \neq 0$; in this limit, therefore, $B_p \to 0$, [see Eq. (21)] .

As for $\tilde{g}^o_{ii}$, we have from Eqs. (26) and (27):

$$\underset{\sim}{k} \cdot (\underset{\sim}{u}^i_1 - \underset{\sim}{u}^i_2) f^1_{im} f^2_{im} \tilde{Q}^o_{ii} \approx -\frac{q_i^2}{\kappa T_i} \tilde{\phi} \underset{\sim}{k} \cdot (\underset{\sim}{u}^i_1 - \underset{\sim}{u}^i_2) f^1_{im} f^2_{im}$$

$$+ \frac{q_e q_i n_e}{\kappa T_i} \frac{4\pi}{k^2} \underset{\sim}{k} \cdot (\underset{\sim}{u}^i_1 - \underset{\sim}{u}^i_2) f^1_{im} f^2_{im} \tilde{Q}^o_{ii} - \frac{q_e q_i n_e}{\kappa T_i} \frac{4\pi}{k^2}$$

$$\times f^1_{im} f^2_{im} \underset{\sim}{k} \cdot \left[ \underset{\sim}{u}^i_1 \sum_\ell J_\ell \tilde{\tilde{Q}}^\ell_{ei} - \underset{\sim}{u}^i_2 \sum_\ell J_\ell \tilde{\tilde{Q}}^{-\ell}_{ei} (-\underset{\sim}{k}) \right] . \tag{29}$$

We divide throughout by $(f^1_{im} f^2_{im})$ and take the limit $m_i \to \infty$; since $B_\ell \to 0$, $\tilde{\tilde{Q}}^\ell_{ei} \to J_\ell \chi_{-\ell} (n_i D_{-\ell})^{-1} (1 + n_i \tilde{Q}^o_{ii})$ and we obtain

$$\underset{\sim}{k} \cdot (\underset{\sim}{u}^i_1 - \underset{\sim}{u}^i_2) \left[ \tilde{Q}^o_{ii} + \frac{q_i^2}{\kappa T_i} \tilde{\phi} - \frac{q_e q_i n_e}{\kappa T_i} \frac{4\pi}{k^2} \tilde{Q}^o_{ii} \right.$$

$$\left. + \frac{q_e q_i n_e}{\kappa T_i} \frac{4\pi}{k^2} \sum_\ell \frac{J^2_\ell \chi_{-\ell}}{n_i D_{-\ell}} (1 + n_i \tilde{Q}^o_{ii}) \right] = 0 . \tag{30}$$

We have made use of the equality $\tilde{Q}^o_{ii}(-\underset{\sim}{k}) = \tilde{Q}^o_{ii}(\underset{\sim}{k})$, which follows from simple symmetry considerations about $\int g^{12}_{ii} \, d\underset{\sim}{v}_1 d\underset{\sim}{v}_2$; then

$$\sum_\ell J_\ell (\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e) \frac{J_\ell (\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e) \chi_{-\ell}(\underset{\sim}{k})}{D_{-\ell}(\underset{\sim}{k})} \left[ 1 + n_i \tilde{Q}^o_{ii} (\underset{\sim}{k}) \right]$$

$$= \sum_\ell J_\ell (\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e) \frac{J_{-\ell} (-\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e) \chi_\ell(-\underset{\sim}{k})}{D_\ell(-\underset{\sim}{k})} \left[ 1 + n_i \tilde{Q}^o_{ii} (-\underset{\sim}{k}) \right]$$

and the brackets in (29) can be written: $(\underset{\sim}{u}^i_1 - \underset{\sim}{u}^i_2) \sum_\ell J_\ell \tilde{\tilde{Q}}^\ell_{ei}$, as we have done in (30) . Equation (30) yields immediately

$$\tilde{Q}^{o}_{ii} = - \frac{1}{n_i} \frac{k_i^2 \sum_{\ell} J_{\ell}^2 / D_{-\ell}}{k^2 + k_i^2 \sum_{\ell} J_{\ell}^2 / D_{-\ell}} \tag{31}$$

and

$$1 + n_i \tilde{Q}^{o}_{ii} = \frac{k^2}{k^2 + k_i^2 \sum_{\ell} J_{\ell}^2 / D_{-\ell}} \quad . \tag{32}$$

$\tilde{Q}^{o}_{ii}$ is a function of $\omega$ through the dispersion functions $D_{-\ell}$.

We can go back now to Eq. (15) and observe that our ansatz $g_{ei} = f_{im} G_{ei}$ satisfies (15) in the limit $m_i \rightarrow \infty$, by simply noticing Eq. (16) and the relation $\underset{\sim}{u}^i \, \delta \, (\underset{\sim}{u}^i) \equiv 0$. We have, therefore, obtained a consistent solution to Eqs. (15) and (26) by assuming the ansatz (27). We point out here that this ansatz has put no restriction on the equation for $g_{ee}$; for $m_i \rightarrow \infty$, the equations for $g_{ei}$ and $g_{ee}$ do not involve $g_{ee}$, the equation for which, on the other hand, involves $G_{ei}$ (although not $g_{ei}$ and $g_{ii}$).

## V. DISCUSSION

We have found an expression for the ion-ion correlations, $\tilde{Q}^o_{ii}$. It is an even, real function of $\underset{\sim}{k}$ as it should. In the limit $E \to 0$ (when $f^1_{eo}$ should become Maxwellian), Eq. (31) becomes

$$\tilde{Q}^o_{ii} = -\frac{1}{n_i} \frac{k_i^2}{k_i^2 + k_e^2 + k^2} + 0(E^2) \tag{33}$$

so that the thermal correlations are recovered. The corrections for small E are $0(E^2)$ so that in the linear limit one can use the thermal value of $\tilde{Q}^o_{ii}$, as proposed in Sec. I. (That $\tilde{Q}^o_{ii}$ had to be even in E, follows from symmetry considerations.) We also observe that for $E \to 0$,

$$\tilde{Q}_{ei} = \frac{1}{n_i} \frac{k_e^2}{k_e^2 + k_i^2 + k^2} + 0(E) \quad ;$$

the first term is the expression for the electron-ion thermal correlations.

Of the two functions that the Dawson-Oberman model requires and that the model does not provide, $f_{eo}$ and $\tilde{Q}^o_{ii}$, we only determine $\tilde{Q}^o_{ii}$; to find $f_{eo}$ an analysis of the equation of $g_{ee}$ has to be made and this is left for future work. On the other hand, the physical phenomenon studied here (the finite modifications that the field produces in the ion correlations for very large ion mass) seems quite interesting by itself, while not much physics is involved in the determination of $f_{eo}$. Moreover, for moderate fields, studied in more detail later in this section, the indeterminacy in $f_{eo}$ is unimportant, while the precise form of $\tilde{Q}^o_{ii}$ is quite essential.

When Eq. (32) is introduced into Eq(23) an expression for the current density results. Carrying into Eq. (13) the component of $\underset{\sim}{j}_1$ that varies as $\sin \omega t$ provides a description of the heating, or absorption by the plasma particles of the field energy. Also, the expression for the current can be used in Maxwell's equations to study the propagation of an electromagnetic wave (of very large wavelength, $\lambda \gg |\underset{\sim}{\varepsilon}_{ei}|$, according to our uniform field assumption); we should notice that all harmonics of the fundamental frequency appear in the current and therefore will appear in the wave too, as follows from Ampère's law. However, $|\underset{\sim}{j}_1| \ll |\underset{\sim}{j}_o|$ and so an iterative procedure can be used, as in Ref. 6, whenever a detailed study of the propagation is wanted.

Our treatment is formally valid for any field intensity. However, we have made some assumptions and we want to discuss now any physical restrictions that these assumptions may impose on the intensity of the field. First of all we assumed that a (quasi-steady) equilibrium would be reached, expecting that the transients would die off. This would not be the case, certainly, if the presence of the field makes the plasma unstable. It is known[10] that for $\omega \approx \omega_p$ an instability sets in for relatively weak fields $\left[\omega |\underset{\sim}{\varepsilon}_e| \ll (\kappa T_e/m_e)^{1/2}\right]$; for $\omega < \omega_p$ there are instabilities too, but the threshold is larger: $\omega |\underset{\sim}{\varepsilon}_e| / (\kappa T_e m_e^{-1})^{1/2} = 0(1)$.[11] Therefore, when $\omega \lesssim \omega_p$ our results will be valid only if E is less than the appropiate threshold field for instability.

A second restriction may be imposed by the ansatz made at the beginning of Sec. III in connection with Eq. (14) (as well as Eq. (25)). Dropping the term $\partial\phi^{12}/\partial \underset{\sim}{r}_{12} \cdot (\partial/m_e \partial \underset{\sim}{v}_1 - \partial/m_i \partial \underset{\sim}{v}_2)\, g_{ei}^{12}$ from the BBGKY equation for $g_{ei}$, as we did by writing Eq. (14), is valid if $|g_{ei}| \ll f_e f_i$ since we retain a term, the first on the right side of Eq. (14), that would be much larger that the neglected one. A similar comment could be made for Eq. (25). For our results to be consistent we should require, then, that $|G_{ei}/f_e| \ll 1$ and $|Q_{ii}^o| \ll 1$ ( for $|\underset{\sim}{r}_{12}| \gg e^2/\kappa T$). $\tilde{\tilde{G}}_{ei}$ can be easily obtained from Eqs. (18) and (22). By Fourier inversion of $\tilde{Q}_{ii}^o$ and $\tilde{\tilde{G}}_{ei}$ the consistency could be checked. Unfortunately, the Fourier inversion results in too-complicated integrals, especially for $G_{ei}$. To clarify the validity of our dropping the aforementioned terms we have to argue in a different way, as follows:

As noticed by simply examining Eqs. (14) and (15), the electric field nearly ceases to appear explicitly in the equation for $g_{ei}$, after a proper transformation is made; this happens also in the $g_{ii}$ equation in Sec. IV (and obviously in the $g_{ee}$ equation, not studied here). In the new variables the field enters in the expression for $\phi^{12} = |\underset{\sim}{r}_{12}|^{-1} \equiv |\underset{\sim}{\rho}_{12} - \underset{\sim}{\varepsilon}_{ei} \sin \omega t|^{-1}$; this difference aside, Eq. (15) is the same as the one in the absence of the field in the original variables. If $\phi^{12}$ were not time dependent in the new frame of reference, our dropping $\partial\phi^{12}/\partial \underset{\sim}{\rho}_{12} \cdot (\partial/m_e \partial \underset{\sim}{u}_1^e - \partial/m_i \partial \underset{\sim}{u}_2^i)\, g_{ei}^{12}$ would seem to be valid because its ratio to a term retained in Eq. (15) (the second on the left-hand side) would appear to be of order of

$e^2/\kappa T\ |\rho_{12} - \varepsilon_{ei} \sin \omega t|^{-1}$ and, therefore, small except in the region $|r_{12}| \equiv |\rho_{12} - \varepsilon_{ei} \sin \omega t| = 0(e^2/\kappa T)$. Although in the $\rho_{12}$ frame this region is not at the origin, it is small and our lack of knowledge of the proper equation for $g_{ei}$ inside it would result in the usual unimportant weak indeterminacy in some integrals. For the actually time-dependent $\phi^{12}$ that we have, the argument obviously holds when $|\varepsilon_{ei}|$ is small compared with the Debye length. If $|\varepsilon_{ei}|$ is comparable to or much larger than the Debye length, the region where Eq. (15) is not valid covers a large, important part in the $\rho_{12}$ space when a whole period of the field is considered. Nevertheless, we point out that at any given *time* the region of indeterminacy is still quite small, $0(e^2/\kappa T)$; although this is certainly not conclusive, we are inclined to believe on this basis that the term we are discussing can be validly dropped from the BBGKY equation, even when $|\varepsilon_{ei}|$ is comparable or large compared with the Debye length.

Whenever $|g_{\alpha\beta}| \ll f_\alpha f_\beta$, an argument can be given to justify our dropping also the three-particle correlation function, $h_{\alpha\beta\gamma}$, from the $g_{\alpha\beta}$ equation, as we did in Eqs. (14) and (25). Although the field $E \sin \omega t$ appears explicitly in the equation for $h_{\alpha\beta\gamma}$, a transformation similar to those preceding Eqs. (15) and (26) can be made and the field will disappear except for its presence in the new interparticle potential $\phi^{mn} = |\rho_{mn} - \varepsilon \sin \omega t|^{-1}$, $\varepsilon$ being $\varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha\gamma}$, or $\varepsilon_{\beta\gamma}$. For zero field, $|g_{\alpha\beta}/f_\alpha f_\beta| = 0(N_D^{-1})$; if this ratio remains small for the actual field considered it should be expected that $|h_{\alpha\beta\gamma}/f_\alpha g_{\beta\gamma}|$ will remain small too.

We conclude, tentatively, that our results will certainly be valid for $|\underset{\sim}{\varepsilon}_{ei}|$ small compared with the Debye length, and will probably be valid for $|\underset{\sim}{\varepsilon}_{ei}|$ comparable to that length if exception is made of the stability restrictions for $\omega \lesssim \omega_p$. (We point out here, finally, that when our results are valid, $\omega_p/\nu(E) \gg 1$ and therefore the expansion made in Sec. II in powers of $\nu/\omega$ is valid if $\omega/\omega_p \gtrsim 0(1)$.)

We shall discuss, as our last point, the quantitative importance of our result for the function $\tilde{Q}^{o}_{ii}$. The most interesting aspect of the result is the appearance in Eq. (32) of all the resonant factors $(D_{-p})^{-1}$. A detailed qualitative comparison with the moderate field results of Ref. 6 can easily be made. (The moderate field condition is defined by the inequality $v_E \equiv |\omega \underset{\sim}{\varepsilon}_{ei}| \ll v_e \equiv \left[\kappa T_e/m_e\right]^{1/2}$.) In particular, the component of $\underset{\sim}{j}_1$ varying as sin $\omega t$ (the only one heating the plasma) is given in Ref. 6 by

$$\underset{\sim}{j}_{1(\sin \omega t)} = \frac{\omega_p^2 \, Z \, e}{\omega 4\pi^3} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}}{k^2} \sum_p \frac{J_p \, J_{p+1} \, \mathrm{Im}\, D_1}{|D_1|^2} \sin \omega t \qquad (35)$$

while we include the factor $k^2 \left[k^2 + k_i^2 \sum J_p^2/D_{-p}\right]^{-1}$ inside the integral. Retaining up to the third power of E, our integral becomes

$$\int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}}{k^2} \; \frac{-\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k}}{2} \left[\frac{\mathrm{Im}\, D_1}{|D_1|^2} - \frac{(\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2}{6} \left\{\frac{2 \,\mathrm{Im}\, D_1}{|D_1|^2} - \frac{\mathrm{Im}\, D_2}{|D_2|^2}\right\}\right]$$

$$\times \left[\frac{k^2}{k^2 + k_i^2 \left\{\dfrac{1 - (\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2/2}{D_o} + \dfrac{(\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2}{2} \; \dfrac{\mathrm{Re}\, D_1}{|D_1|^2}\right\}}\right] \qquad (36)$$

while the second bracket does not appear in Ref. 6. Here the prefixes Re and Im stand for the real and imaginary parts, and we wrote $\underset{\sim}{\varepsilon}_e$ for $\underset{\sim}{\varepsilon}_{ei}$ since $m_i \to \infty$. We point out at this stage that the use of the small argument expansion in the Bessel functions in Eq. (35) that we have used to obtain Eq. (36), is valid under the moderate field assumption, even if $|\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k}|$ is not small. It is possible to show this by following a procedure used in Refs. 1 and 6: when $|\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e| \geq 0(1)$, we have $\omega/kv_e \ll 1$ from our condition $v_E \ll v_e$; then one solves Eq. (19) by expanding $\tilde{\tilde{Q}}_{ei}(\tau)$ and $\exp\left[-i\underset{\sim}{k} \cdot \underset{\sim}{\varepsilon}_e \sin \omega t\right]$ inside the time integral in a Taylor series in $\omega/kv_e$ and solving iteratively for $\tilde{\tilde{Q}}_{ei}(t)$. The result for the integral in $j_{\underset{\sim}{1}(\sin \omega t)}$ agrees with Eq. (36) when the small argument expansion is made in $D_p(p\omega/kv_e)$.

A subsequent expansion of the second bracket in Eq. (36) is perhaps not valid throughout the whole range of the $\underset{\sim}{k}$ integration. To make our comparison simpler we shall assume so here. Then this bracket becomes:

$$\frac{k^2+k_e^2}{k^2+k_e^2+k_i^2}\left\{1-\frac{k_i^2}{k^2+k_e^2+k_i^2}\left[-\frac{(\underset{\sim}{\varepsilon}_e\cdot\underset{\sim}{k})^2}{2}+0\left(\frac{\omega^2\varepsilon_e^2}{v_e^2}\right)+\frac{k^2+k_e^2}{k^2}\right.\right.$$

$$\left.\left.\times\frac{(\underset{\sim}{\varepsilon}\cdot\underset{\sim}{k})^2}{2}\;\frac{\mathrm{Re}D_1}{|D_1|^2}\right]\right\}$$

The term $0(\omega^2\varepsilon_e^2/v_e^2)$ is due to the fact that $f_{eo} = f_{em}\left[1+0(\omega^2\varepsilon_e^2/v_e^2)\right]$ ; a similar correction with respect to the linear result will appear in the first term inside the first bracket in Eq. (36). Since we retain only third powers of $\varepsilon_e$, $f_{em}$ can be used in the other terms in Eq. (36) involving $D_1$ and $D_2$.

We see, finally, that to the nonlinear terms given in Ref. 6 [as adding to the linear result inside the integral in Eq. (36)],

$$-\frac{(\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2}{6}\left\{\frac{2 I_m D_1}{|D_1|^2} - \frac{I_m D_2}{|D_2|^2}\right\} ,$$

we add the essential corrections

$$-\frac{I_m D_1}{|D_1|^2}\;\frac{(k_e^2 + k^2)k_i^2}{(k^2 + k_e^2 + k_i^2)^2}\left[0\left(\frac{\varepsilon_e^2\omega^2}{v_e^2}\right) - \frac{(\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2}{2} + \frac{k^2 + k_e^2}{k^2}\;\frac{(\underset{\sim}{\varepsilon}_e \cdot \underset{\sim}{k})^2}{2}\;\frac{R_e D_1}{|D_1|^2}\right] .$$

In fact, the last term inside this bracket seems to yield the most dominant nonlinear term.

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