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Multiscale Expansion and Integrability Properties of the Lattice Potential KdV Equation

Rafael Hernandez Heredero [a], Decio Levi [b], Matteo Petrera [c,b] and Christian Scimiterna [d,b]

[a] *Departamento de Matemática Aplicada, Escuela Universitaria de Ingeniería Técnica de Telecomunicación, Universidad Politécnica de Madrid (UPM), Ctra de Valencia km. 7, 28031-Madrid, Spain*
E-mail: rafahh@euitt.upm.es

[b] *Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy*
E-mail: levi@fis.uniroma3.it

[c] *Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D-85747, Garching bei München, Germany*
E-mail: petrera@ma.tum.de

[d] *Dipartimento di Fisica, Università degli Studi Roma Tre and Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy*
E-mail: scimiterna@fis.uniroma3.it

Abstract

We apply the discrete multiscale expansion to the Lax pair and to the first few symmetries of the lattice potential Korteweg-de Vries equation. From these calculations we show that, like the lowest order secularity conditions give a nonlinear Schrödinger equation, the Lax pair gives at the same order the Zakharov and Shabat spectral problem and the symmetries the hierarchy of point and generalized symmetries of the nonlinear Schrödinger equation.

1 Introduction

Reductive perturbation techniques [19, 20] have proved to be important tools for finding approximate solutions of many physical problems, by reducing a given nonlinear partial differential equation to a simpler equation, often integrable [3], and for proving integrability [3–5,10,21]. Recently, after various attempts to carry over this approach to partial difference equations [1,11,13] we have presented a procedure for carrying out a multiscale expansion on the lattice [7,12,14] which seems to preserve the integrability properties [8]. To get a better understanding of the application of the reductive perturbation technique on difference equations, after an introduction in Section 2 on multiscale expansions on the lattice potential KdV equation (lpKdV), we discuss in Section 3 its application to the spectral operator, as was done by Zakharov and Kuznetsov in their pioneering work in 1986 [21] for the KdV equation. Later on we apply, in Section 4, the multiscale expansion to the symmetries of the lpKdV [15]. Section 5 is devoted to a few conclusive remarks.

2 Multiscale expansion on the lattice

The aim of this Section is to give a terse survey on the multiscale analysis on the lattice and its application to the reduction of the lpKdV. We refer to [7, 12, 14] for further details.

2.1 Shift operators defined on the lattice

Let $u_n : \mathbb{Z} \to \mathbb{R}$ be a function defined on a lattice of index $n \in \mathbb{Z}$. One can always extend it to a function $u(x) : \mathbb{R} \to \mathbb{R}$ by defining a real continuous variable $x = n\sigma_x$, where $\sigma_x \in \mathbb{R}$ is the constant lattice spacing.

An equation defined on the lattice is a functional relation between the function u_n and its shifted values $u_{n\pm1}$, $u_{n\pm2}$, etc, expressed in terms of a shift operator T_n such that $T_n u_n = u_{n+1}$.

For the continuous function $u(x)$ we can introduce an operator T_x, such that $T_x u(x) = u(x+\sigma_x)$. The Taylor expansion of $u(x+\sigma_x)$ centered in x reads

$$T_x u(x) = \sum_{i=0}^{\infty} \frac{\sigma_x^i}{i!} u^{(i)}(x), \tag{2.1}$$

where $u^{(i)}(x) = d^i u(x)/dx^i = d_x^i u(x)$, with d_x the total derivative. Eq. (2.1) suggests the following formal expansion for the differential operator T_x:

$$T_x = e^{\sigma_x d_x} = \sum_{i=0}^{\infty} \frac{\sigma_x^i}{i!} d_x^i.$$

Introducing a formal derivative with respect to the index n, say δ_n, we can define, by analogy with T_x, the operator T_n as

$$T_n = e^{\delta_n} = \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}. \tag{2.2}$$

The formal expansion (2.2) can be inverted, yielding

$$\delta_n = \ln T_n = \ln(1+\Delta_n) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_n^i, \tag{2.3}$$

where $\Delta_n = T_n - 1$ is the discrete right difference operator w.r.t. the variable n (i.e. $\Delta_n u_n = u_{n+1} - u_n$).

Following [12, 14] we say that u_n is a *slow-varying function of order* ℓ iff $\Delta_n^{\ell+1} u_n = 0$. Hence the δ_n operators are formal series containing infinite powers of Δ_n, but, acting on slow-varying functions of order ℓ, they reduce to polynomials in Δ_n of order at most ℓ.

2.2 Dilations on the lattice

Let us introduce a second lattice, obtained from the first by a dilation. For $x \in \mathbb{R}$ we can visualize the problem as a change of variable between x and $x_1 = \varepsilon x$, $0 < \varepsilon \ll 1$. On the lattice this corresponds to a change from the index $n = x/\sigma_x$ to the new index $n_1 = x_1/\sigma_{x_1}$, where σ_{x_1} is the new lattice spacing. Assuming that $\sigma_{x_1} \gg \sigma_x$ we can set $\sigma_x = \epsilon\sigma_{x_1}$, $0 < \epsilon \ll 1$, so that $n_1 = \varepsilon\epsilon n$. As $n, n_1 \in \mathbb{Z}$, $\varepsilon\epsilon$ is a rational number and one can define in all generality $\varepsilon\epsilon = M_1/N \ll 1$ with

$M_1, N \in \mathbb{N}$. However, if we want the lattice of index n_1 to be a sublattice of the lattice of index n, we have also to require that $M_1/N = 1/M$ with $M \in \mathbb{N}$.

The relation between the discrete derivatives defined in the two lattices is given by [7,9,13,14]

$$\Delta_n^j u_n = j! \sum_{i=j}^{\infty} \frac{P_{i,j}}{i!} \Delta_{n_1}^i u_{n_1}. \tag{2.4}$$

The coefficients $P_{i,j}$ read

$$P_{i,j} = \sum_{k=j}^{i} \left(\frac{M_1}{N} \right)^k \mathcal{S}_i^k \mathfrak{S}_k^j,$$

where $\mathcal{S}_i^k$ and $\mathfrak{S}_k^j$ are the Stirling numbers of the first and second kind respectively.

If u_n is a function of infinite order of slow-varyness, i.e. $\ell = \infty$, then Eq. (2.4) implies that a finite difference in the discrete variable n depends on an infinite number of differences on the variable n_1.

2.3 Discrete multiscale expansion

Let us now consider $u_n = u_{n;n_1}$ as a function depending on a fast index n and a slow index $n_1 = n(M_1/N)$. At the continuous level, the total derivative d_x acting on functions $u(x;x_1)$ is the sum of partial derivatives, i.e. $d_x = \partial_x + \varepsilon \partial_{x_1}$. As

$$T_x = e^{\sigma_x d_x} = e^{\sigma_x \partial_x} e^{\varepsilon \sigma_x \partial_{x_1}}, \tag{2.5}$$

we can write the total shift operator T_n as

$$T_n = e^{\delta_n} e^{(M_1/N)\delta_{n_1}} = \mathcal{T}_n \mathcal{T}_{n_1}^{(M_1/N)}, \tag{2.6}$$

where the partial shift operators $\mathcal{T}_n, \mathcal{T}_{n_1}$, defined by $\mathcal{T}_n u_{n;n_1} = u_{n+1;n_1}$ and $\mathcal{T}_{n_1} u_{n;n_1} = u_{n;n_1+1}$, are given by

$$\mathcal{T}_n = \sum_{i=0}^{\infty} \frac{\delta_n^i}{i!}, \qquad \mathcal{T}_{n_1}^{(M_1/N)} = \sum_{i=0}^{\infty} \frac{(M_1/N)^i}{i!} \delta_{n_1}^i,$$

and δ_{n_1} is given by Eq. (2.3) with n substituted by n_1.

Eq. (2.5) can be extended to the case of K slow variables $x_i = \varepsilon^i x$, $1 \le i \le K$. Then the action of the shift operator T_n on a function $u_{n;\{n_i\}_{i=1}^K}$ depending on both fast and slow variables can be written in terms of the partial shifts $\mathcal{T}_n, \mathcal{T}_{n_i}$ as

$$T_n = \mathcal{T}_n \prod_{i=1}^{K} \mathcal{T}_{n_i}^{(\varepsilon_{n_i})}, \tag{2.7}$$

where the ε_{n_i}'s are suitable functions of ε and ε depending parametrically on some integer coefficients $M_i \in \mathbb{N}$, $1 \le i \le K$.

To carry out the multiscale expansion of the fields appearing in partial difference equations with two independent discrete variables, one has to consider the action of the operator (2.7) on a function depending on two fast indices n and m, and on a set of $K_n + K_m$ slow variables $\{n_i\}_{i=1}^{K_n}$ and

$\{m_i\}_{i=1}^{K_m}$ (we shall use the notation $u_{n,m;\{n_i\}_{i=1}^{K_n},\{m_i\}_{i=1}^{K_m}}$ for such functions). Notice that in principle it is possible to consider $K_n = K_m = \infty$. We assume a common definition of the small parameter ε for both discrete variables n and m but we denote with M_i the integers for the slow variables n_i and with $\widetilde{M}_i$ the ones for m_i. We have:

$$\varepsilon_{n_i} = \frac{M_i}{N^i}, \quad 1 \le i \le K_n, \qquad \varepsilon_{m_i} = \frac{\widetilde{M}_i}{N^i}, \quad 1 \le i \le K_m.$$

Hereafter we shall assume $K_n = 1$ and $K_m = K$.

2.4 Multiscale expansion of the lattice potential KdV equation

The lpKdV is given by [17]:

$$[\mu(T_nT_m - 1) + \zeta(T_n - T_m)]u_{n,m} - (T_n - T_m)u_{n,m}(T_nT_m - 1)u_{n,m} = 0, \tag{2.8}$$

where $\mu = p - q$ and $\zeta = p + q$, and p, q, $p \neq q$, are two real parameters. The linear part of Eq. (2.8) has a travelling wave solution of the form $u_{n,m} = \exp\{\mathrm{i}[\kappa n - \omega(\kappa)m]\}$ with

$$\omega(\kappa) = -2\arctan\left(\frac{\zeta+\mu}{\zeta-\mu}\tan\frac{\kappa}{2}\right). \tag{2.9}$$

According to [7] the multiscale expansion of Eq. (2.8) is performed taking into account that

$$u_{n,m} = \sum_{\alpha\in\mathbb{Z}}\sum_{k=1}^{\infty}\frac{1}{N^k}u_k^{(\alpha)}(n_1, \{m_i\}_{i=1}^{K})e^{\mathrm{i}\alpha(\kappa n-\omega m)}, \qquad u_k^{(-\alpha)} = \bar{u}_k^{(\alpha)}. \tag{2.10}$$

The following statement, proved in [7], provides the multiscale expansion of the lpKdV (2.8) at the lowest orders of $1/N$.

Theorem 1. *The multiscale expansion of Eq. (2.8) gives the following results:*

1. $O(1/N)$*:*
   - $\alpha = 0$*: the equation is identically satisfied.*
   - $\alpha = 1$*: one gets a linear equation identically satisfied by taking into account the dispersion relation (2.9).*
   - $|\alpha| \ge 2$*: one gets a linear equation whose only solution is* $u_1^{(\alpha)} = 0$.
2. $O(1/N^2)$*:*
   - $\alpha = 1$*: one gets a linear equation whose solution is*

     $$u_1^{(1)} = u_1^{(1)}(n_2, \{m_i\}_{i=2}^{K}), \qquad n_2 = n_1 \mp m_1, \tag{2.11}$$

     provided that

     $$M_1 = \mp S\left(\mu - \zeta e^{\mathrm{i}\kappa}\right), \qquad \widetilde{M}_1 = Se^{\mathrm{i}\kappa}\frac{\zeta^2-\mu^2}{\mu e^{\mathrm{i}\kappa}-\zeta}.$$

     Here $S = r\exp(\mathrm{i}\theta)$*, with* $r > 0$ *and* $\theta = -\arctan[(\zeta\sin\kappa)/(\zeta\cos\kappa - \mu)]$*, assures that* M_1 *and* $\widetilde{M}_1$ *are positive integers.*

- $\alpha = 0$*: one gets*

$$\delta_{n_2} u_1^{(0)} = \tau_1 |u_1^{(1)}|^2, \qquad \tau_1 = \pm \frac{2\left(1+e^{i\kappa}\right)^2}{Se^{i\kappa}(\mu+\zeta)\left(\mu-\zeta e^{i\kappa}\right)},$$

where $u_1^{(0)} = u_1^{(0)}(n_2, \{m_i\}_{i=2}^K)$.

- $\alpha = 2$*: one gets*

$$u_2^{(2)} = \tau_2 (u_1^{(1)})^2, \qquad \tau_2 = \frac{1+e^{i\kappa}}{(1-e^{i\kappa})(\mu+\zeta)},$$

where $u_2^{(2)} = u_2^{(2)}(n_2, \{m_i\}_{i=2}^K)$.

3. $O(1/N^3)$:

- $\alpha = 1$*: one gets the following (defocusing) dNLS:*

$$i\delta_{m_2} u_1^{(1)} = \rho_1 \delta_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)} |u_1^{(1)}|^2, \tag{2.12}$$

where

$$\rho_1 = -\frac{\mu\zeta r^2(\zeta^2-\mu^2)\sin\kappa}{\widetilde{M}_2\left(\zeta^2+\mu^2-2\zeta\mu\cos\kappa\right)}, \qquad \rho_2 = \frac{8\zeta\mu(\zeta-\mu)(1+\cos\kappa)^2\sin\kappa}{\widetilde{M}_2(\mu+\zeta)\left(\zeta^2+\mu^2-2\zeta\mu\cos\kappa\right)^2}.$$

- $\alpha = 0$*: one gets*

$$\delta_{n_2} u_2^{(0)} = \tau_1 \left(u_1^{(1)} \bar{u}_2^{(1)} + \bar{u}_1^{(1)} u_2^{(1)}\right) - \tau_3 \left(\bar{u}_1^{(1)} \delta_{n_2} u_1^{(1)} - u_1^{(1)} \delta_{n_2} \bar{u}_1^{(1)}\right),$$

with

$$\tau_3 = \frac{2i\sin\kappa}{\mu+\zeta},$$

where $u_2^{(0)} = u_2^{(0)}(n_2, \{m_i\}_{i=2}^K)$ *and* $u_2^{(1)} = u_2^{(1)}(n_2, \{m_i\}_{i=2}^K)$.

- $\alpha = 2$*: one gets*

$$u_3^{(2)} = \tau_4 u_1^{(1)} (\delta_{n_2} u_1^{(1)}) + 2\tau_2 u_1^{(1)} u_2^{(1)}, \qquad \tau_4 = \pm \frac{2Se^{i\kappa}(\alpha+\beta e^{i\kappa})}{(e^{i\kappa}-1)^2(\mu+\zeta)},$$

where $u_3^{(2)} = u_3^{(2)}(n_2, \{m_i\}_{i=2}^K)$.

We have given above just those results necessary to get a discrete nonlinear Schrödinger equation (dNLS) as a secularity condition and its symmetries.

3 Multiscale expansion of the lpKdV spectral problem

As shown in [14] there are many forms for the linear problems associated with the lpKdV. The first to be introduced [17] is given by first order 2×2 matrix difference equations. Later on [14] it was shown that the matrix Lax pair could be easily reduced to a scalar non-symmetric difference equation of second order, used by Boiti et. al. [2] to integrate an alternative form of the equations of the Volterra hierarchy. In [15] it was moreover shown that by a Miura transformation it is possible to associate the lpKdV with the Toda spectral problem introduced by Manakov and Flaskha [6] when the field $b_n(t) = 0$.

One could start from any of the three linear problems delined in the previous paragraph to do the multiscale expansion. However we choose as starting spectral problem the one whose second derivative is expressed in a symmetric form, i.e. the discrete Schrödinger spectral problem used to integrate the Toda and Volterra equations.

The n-evolution equation of the (scalar) spectral problem of the lpKdV (2.8) may be written as [15]:

$$\phi_{n-1} + a_n \phi_{n+1} = \mu \phi_n, \tag{3.1}$$

with

$$a_n = \frac{4p^2}{\left[2p - (T_n^2 + 1)u_{n,m}\right]\left[2p - (T_n + T_n^{-1})u_{n,m}\right]}.$$

Here $\mu \in \mathbb{C}$ is the spectral parameter.

Our aim is now to perform the multiscale expansion of Eq. (3.1) in order to get the corresponding evolution equation of the spectral problem of the dNLS (2.12). We refer to [21] for the continuous counterpart of this analysis.

To expand Eq. (3.1) we consider the development (2.10) for the field $u_{n,m}$, with the restriction (2.11), while the function ϕ_n will be expanded according to the formula:

$$\phi_n = \sum_{\alpha\,\mathrm{odd}} \sum_{k=0}^{\infty} \frac{1}{N^k} \phi_k^{(\alpha)}(n_2, \{m_i\}_{i=2}^K) e^{\mathrm{i}\alpha(\kappa n - \omega m)/2}, \qquad \phi_k^{(-\alpha)} = \bar{\phi}_k^{(\alpha)}. \tag{3.2}$$

At order $O(1)$, the multiscale analysis of Eq. (3.1) suggests the following expansion for the spectral parameter μ:

$$\mu = 2\cos\left(\frac{\kappa}{2}\right) + \sum_{k=1}^{\infty} \frac{\mu_k}{N^k}. \tag{3.3}$$

Taking into account Eq. (3.3) we proceed to the order $1/N$ of the multiscale expansion of Eq. (3.1). We have:

$$\delta_{n_2} \phi_0^{(1)} + \frac{2u_1^{(1)}}{p} \cos^2\left(\frac{\kappa}{2}\right) \bar{\phi}_0^{(1)} = -\frac{\mathrm{i}\mu_1}{2\sin\left(\frac{\kappa}{2}\right)} \phi_0^{(1)}, \tag{3.4}$$

for $\alpha = 1$. The corresponding equation for $\alpha = -1$ is given by performing the complex conjugation of Eq. (3.4). The coefficients of the higher harmonics in Eq. (3.2) can be written in terms of $\phi_0^{(1)}$. For instance, for $\alpha = 3$, we have:

$$\phi_1^{(3)} = \frac{e^{2\mathrm{i}\kappa} + e^{\mathrm{i}\kappa}}{1 - e^{\mathrm{i}\kappa}} u_1^{(1)} \phi_0^{(1)},$$

By a proper rescaling of $\phi_0^{(1)}$ and μ_1 Eq. (3.4) is equivalent to the standard Zakharov-Shabat spectral problem of the integrable NLS [18].

4 Multiscale expansion of the first two generalized symmetries

Lie symmetries of a lattice equation $\mathbb{D}(u_{n,m}, T_n^{\pm} u_{n,m}, T_m^{\pm} u_{n,m}, \dots) = 0$ are given by those continuous transformations which leave the equation invariant. Here $T_n^{\pm k} u_{n,m} = u_{n\pm k,m}$ and $T_m^{\pm k} u_{n,m} = u_{n,m\pm k}$, $k \in \mathbb{N}$. From the infinitesimal point of view they are obtained by requiring the infinitesimal invariant condition

$$\mathrm{pr}\,\widehat{X}_{n,m}\,\mathbb{D}\Big|_{\mathbb{D}=0} = 0, \tag{4.1}$$

where

$$\widehat{X}_{n,m} = F_{n,m}(u_{n,m}, T_n^{\pm} u_{n,m}, T_m^{\pm} u_{n,m}, \dots)\partial_{u_{n,m}}. \tag{4.2}$$

By $\mathrm{pr}\,\widehat{X}_{n,m}$ we mean the prolongation of the infinitesimal generator $\widehat{X}_{n,m}$ to all points appearing in $\mathbb{D} = 0$.

If $F_{n,m} = F_{n,m}(u_{n,m})$ then we get *point symmetries* and the procedure to get them from Eq. (4.1) is purely algorithmic [16]. *Generalized symmetries* are obtained when $F_{n,m} = F_{n,m}(u_{n,m}, T_n^{\pm} u_{n,m}, T_m^{\pm} u_{n,m}, \dots)$. In the case of nonlinear discrete equations, the Lie point symmetries are not very common, but, if the equation is integrable and there exists a Lax pair, it is possible to construct an infinite family of generalized symmetries.

In correspondence with the infinitesimal generator (4.2) we can in principle construct a group transformation by integrating the initial boundary problem

$$\frac{d\widetilde{u}_{n,m}(\lambda)}{d\lambda} = F_{n,m}(\widetilde{u}_{n,m}(\lambda), T_n^{\pm}\widetilde{u}_{n,m}(\lambda), T_m^{\pm}\widetilde{u}_{n,m}(\lambda), \dots), \qquad \widetilde{u}_{n,m}(\lambda = 0) = u_{n,m}, \tag{4.3}$$

where $\lambda \in \mathbb{R}$ is the continuous Lie group parameter. This can be done effectively only in the case of point symmetries, as in the generalized case we have a differential-difference equation for which we cannot find the solution for a generic initial data, but, at most, we can find some particular solutions. Eq. (4.1) is equivalent to the request that the λ-derivative of the equation $\mathbb{D} = 0$, written for $\widetilde{u}_{n,m}(\lambda)$, is identically satisfied when the λ-evolution of $\widetilde{u}_{n,m}(\lambda)$ is given by Eq. (4.3). This is also equivalent to say that the flows (in the group parameter space) given by Eq. (4.3) are compatible or commute with $\mathbb{D} = 0$.

In [15] one can find an infinite hierarchy of integrable generalized symmetries for the lpKdV (2.8) constructed by looking at the isospectral deformations of the Lax pair. The first two symmetries of this hierarchy are given by

$$\frac{d\widetilde{u}_{n,m}}{d\lambda} = \frac{1}{2p + (T_n^{-} - T_n)\widetilde{u}_{n,m}} - \frac{1}{2p}, \tag{4.4}$$

$$\frac{d\widetilde{u}_{n,m}}{d\lambda} = \frac{1}{[2p + (T_n^{-} - T_n)\widetilde{u}_{n,m}]^2}\left[\frac{1}{2p + (1 - T_n^{2})\widetilde{u}_{n,m}} + \frac{1}{2p + (T_n^{-2} - 1)\widetilde{u}_{n,m}}\right] - \frac{1}{4p^3}. \tag{4.5}$$

The constant terms appearing in the r.h.s. of Eqs. (4.4,4.5) ensure that the above flows go asymptotically to zero as $\widetilde{u}_{n,m} \to \mathrm{cost}$.

To perform the multiscale expansion of the generalized symmetries (4.4,4.5) we consider the following development for the field $\widetilde{u}_{n,m}$, see Eq. (2.10):

$$\widetilde{u}_{n,m} = \sum_{\alpha\in\mathbb{Z}} \sum_{k=1}^{\infty} \frac{1}{N^k} \widetilde{u}_k^{(\alpha)}(n_2, \{m_i\}_{i=2}^{K}, \{\lambda_i\}_{i=0}^{K'}) e^{\mathrm{i}\alpha(\kappa n-\omega m)}, \qquad \widetilde{u}_k^{(-\alpha)} = \overline{\widetilde{u}}_k^{(\alpha)}, \tag{4.6}$$

where $\lambda_i = \lambda/N^i$ are the slow-varying group parameters, n_2 is given by Eq. (2.11) and $\widetilde{u}_{n,m}(\{\lambda_i = 0\}_{i=0}^{K'}) = u_{n,m}$.

Since Eq. (2.12) involves the harmonic $u_1^{(1)}$ we are actually interested just in those equations, arising from the multiscale expansions of the symmetries (4.4, 4.5), which are written in terms this harmonic. The following statement holds.

Theorem 2. *The multiscale expansion up to order $1/N^4$ of the symmetry (4.4) gives the following symmetries for the dNLS (2.12) (after a reparametrization of the group parameters):*

$$O(1/N): \qquad \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda} = \mathrm{i}\widetilde{u}_1^{(1)}, \tag{4.7}$$

$$O(1/N^2): \qquad \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda_1} = \delta_{n_2}\widetilde{u}_1^{(1)}, \tag{4.8}$$

$$O(1/N^3): \qquad \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda_2} = \delta_{m_2}\widetilde{u}_1^{(1)}, \tag{4.9}$$

$$O(1/N^4): \qquad \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda_3} = \rho_1\delta_{n_2}^3\widetilde{u}_1^{(1)} + 3\rho_2|\widetilde{u}_1^{(1)}|^2\delta_{n_2}\widetilde{u}_1^{(1)}, \tag{4.10}$$

with initial condition $\widetilde{u}_1^{(1)}(\lambda = 0, \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0) = u_1^{(1)}$. Eqs. (4.7,4.8,4.9) provide point symmetries of Eq. (2.12), while Eq. (4.10) is a generalized symmetry of Eq. (2.12).

Proof. The proof is done by a direct computation by taking into account the results contained in Theorem 1.

Inserting Eq. (4.6) in the first symmetry (4.4) we get the following determing equations:

$$O(1/N): \qquad \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda} = \frac{\mathrm{i}}{2p^2}\sin\kappa\,\widetilde{u}_1^{(1)}, \tag{4.11}$$

$$O(1/N^2): \qquad \frac{\partial\widetilde{u}_2^{(1)}}{\partial\lambda} + \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda_1} = \frac{\mathrm{i}}{2p^2}\left(\sin\kappa\,\widetilde{u}_2^{(1)} - \mathrm{i}M_1\cos\kappa\,\delta_{n_2}\widetilde{u}_1^{(1)}\right), \tag{4.12}$$

$$\begin{aligned} O(1/N^3): \qquad & \frac{\partial\widetilde{u}_3^{(1)}}{\partial\lambda} + \frac{\partial\widetilde{u}_2^{(1)}}{\partial\lambda_1} + \frac{\partial\widetilde{u}_1^{(1)}}{\partial\lambda_2} = \\ & = \frac{\mathrm{i}}{2p^2}\left(\sin\kappa\,\widetilde{u}_3^{(1)} - \mathrm{i}M_1\cos\kappa\,\delta_{n_2}\widetilde{u}_2^{(1)} + \frac{M_1^2}{2}\sin\kappa\,\delta_{n_2}^2\widetilde{u}_1^{(1)}\right) + \\ & + \frac{\mathrm{i}}{p^3}\left(-\mathrm{i}\sin\kappa\,\sin(2\kappa)\overline{\widetilde{u}}_1^{(1)}u_2^{(2)} + M_1\sin\kappa\,\widetilde{u}_1^{(1)}\delta_{n_2}\widetilde{u}_1^{(0)}\right) + \\ & + \frac{3\mathrm{i}}{2p^4}\sin^3\kappa\,|\widetilde{u}_1^{(1)}|^2\widetilde{u}_1^{(1)}, \end{aligned} \tag{4.13}$$

$$O(1/N^4): \quad \frac{\partial \widetilde{u}_4^{(1)}}{\partial \lambda} + \frac{\partial \widetilde{u}_3^{(1)}}{\partial \lambda_1} + \frac{\partial \widetilde{u}_2^{(1)}}{\partial \lambda_2} + \frac{\partial \widetilde{u}_1^{(1)}}{\partial \lambda_3} = \tag{4.14}$$
$$= \frac{\mathrm{i}}{2p^2}\left(\sin\kappa\,\widetilde{u}_4^{(1)} - \mathrm{i}M_1\cos\kappa\,\delta_{n_2}\widetilde{u}_3^{(1)} + \frac{M_1^2}{2}\sin\kappa\,\delta_{n_2}^2\widetilde{u}_1^{(1)} - \right.$$
$$\left. - \frac{\mathrm{i}M_1^3}{3}\cos\kappa\,\delta_{n_2}^3\widetilde{u}_1^{(1)}\right) +$$
$$+ \frac{\mathrm{i}}{p^3}\left[-\mathrm{i}\sin\kappa\sin(2\kappa)\left(\overline{\widetilde{u}}_1^{(1)}\widetilde{u}_3^{(2)} + \overline{\widetilde{u}}_2^{(1)}\widetilde{u}_2^{(2)}\right) + \right.$$
$$+ M_1\sin\kappa\left(\overline{\widetilde{u}}_1^{(1)}\delta_{n_2}\widetilde{u}_2^{(2)} + \widetilde{u}_1^{(1)}\delta_{n_2}\widetilde{u}_2^{(0)} + \widetilde{u}_2^{(1)}\delta_{n_2}\widetilde{u}_1^{(0)}\right) -$$
$$\left. - \mathrm{i}M_1^2\cos\kappa\,\widetilde{u}_1^{(1)}\delta_{n_2}^2\widetilde{u}_1^{(0)}\right] +$$
$$+ \frac{3\mathrm{i}}{2p^4}\left[-\mathrm{i}M_1\cos\kappa\sin^2\kappa\,(\widetilde{u}_1^{(1)})^2\delta_{n_2}\overline{\widetilde{u}}_1^{(1)} + \right.$$
$$\left. + \sin^3\kappa\left(\overline{\widetilde{u}}_2^{(1)}(\widetilde{u}_1^{(1)})^2 + 2\widetilde{u}_2^{(1)}|\widetilde{u}_1^{(1)}|^2\right)\right].$$

Let us consider Eq. (4.11); by the reparametrization $\lambda \mapsto 2p^2\lambda/\sin\kappa$, Eq. (4.11) is equivalent to Eq. (4.7). This is the first point symmetry of the dNLS (2.12) and it corresponds to a phase symmetry.

Eq. (4.12) has to be split into the following equations to avoid secularities:

$$\frac{\partial \widetilde{u}_1^{(1)}}{\partial \lambda_1} = \frac{M_1}{2p^2}\cos\kappa\,\delta_{n_2}\widetilde{u}_1^{(1)}, \tag{4.15}$$
$$\frac{\partial \widetilde{u}_2^{(1)}}{\partial \lambda} = \frac{\mathrm{i}}{2p^2}\sin\kappa\,\widetilde{u}_2^{(1)}. \tag{4.16}$$

From Eq. (4.16) we see that $\widetilde{u}_2^{(1)}$ depends on λ as $\widetilde{u}_1^{(1)}$. Eq. (4.15) provides the second point symmetry (4.8) of the dNLS (2.12), corresponding to translations w.r.t. the index n_2, after the reparametrization $\lambda_1 \mapsto 2p^2\lambda_1/(M_1\cos\kappa)$.

From Eq. (4.13), taking into account Eqs. (4.11,4.16) and the secularity conditions, a straightforward algebra and the reparametrization $\lambda_2 \mapsto 4p^2\rho_1\lambda_2/(M_1^2\sin\kappa)$ leads to

$$\mathrm{i}\frac{\partial \widetilde{u}_1^{(1)}}{\partial \lambda_2} = \rho_1\delta_{n_2}^2\widetilde{u}_1^{(1)} + \rho_2\widetilde{u}_1^{(1)}|\widetilde{u}_1^{(1)}|^2,$$

which leads to Eq. (4.9) thanks to Eq. (2.12). Eq. (4.9) means that the dNLS (2.12) is invariant under translations w.r.t. the index m_2.

Finally, Eq. (4.14) gives Eq. (4.10) after a long computation by taking into account Eqs. (4.11,4.12,4.13). In this last case the reparametrization of the group parameter reads $\lambda_3 \mapsto 12p^2\rho_1\lambda_3/(M_1^3\cos\kappa)$.

■

A computation up to order $1/N^4$, similar to the one just done for the symmetry (4.4), shows that the multiscale expansion of the second generalized symmetry (4.5) of the lpKdV (2.8) gives the same symmetries (4.7,4.8,4.9,4.10), after suitable reparametrizations of the group parameters.

5 Concluding remarks

In this paper we have considered the multiscale expansion of the spectral problem and of the symmetries of the partial difference integrable lattice potential KdV equation. By a proper choice of the spectral problem of the lpKdV we have been able to derive from it the spectral problem of the reduced equation, a nonlinear Schrödinger equation. We then did the multiscale expansion of two generalized symmetries. A generalized symmetry provides us with the point and generalize symmetries of the nonlinear Schrödinger equation. At each order of the multiscale approximation, we get by reduction from the request that no secular condition exists, a higher order symmetry. The same calculation for other generalized symmetries do not provide anything new. All the information concerning the whole hierarchy of generalized symmetries for the NLS is contained in the first generalized symmetry for the lpKdV.

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