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CONSTANT SOLUTIONS IN SECOND ORDER LINEAR HOMOGENEOUS DIFFERENCE EQUATION, WITH VARIABLE COEFICCIENTS

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Abstract. The constant solutions of second order linear and homogeneous difference equation with variable coefficients are analyzed. Thus, a sufficient condition for the existence of the constant solution, with any initial solution, is provided. Also, fixed points and invariant subsets of the solutions are considered. Finally, a necessary and sufficient condition to maintain the existence of non-trivial fixed points in the product of transfer matrices of two dimensions, is given.

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1 Transfer Matrices for Second Order Linear Difference Equations

The homogeneous linear difference equation of second order, with variable coefficients, has solutions $X = \{x_i\}_{i=0}^{\infty}$, with x_i the component functions, that satisfy the following equation, for $n \geq 1$.

$$x_{n+1} = b_n x_n + a_n x_{n-1}. \tag{1}$$

$\{a_i\}_{i=1}^{\infty}$ y $\{b_i\}_{i=1}^{\infty}$ are two arbitrary sequences of real numbers. The necessary and sufficient condition for the conservation of non trivial constant solutions $X = x_*$ is immediate, $b_i + a_i = 1$, for $i = 1, \ldots$ In general, an explicit solution in closed form, of the Eq.(1) is not well-known, [1]. The study of qualitative properties of the solution has generated many work, [2], like oscillation theory, asymptotic solutions of particular cases, and so on.

Eq.(1) can be considered like the projection, on the real axis, of a linear application in the plane, $\vec{x}_{i+1} = \mathbf{M}^{(i)}\vec{x}_i$, [3], with $\vec{x}_i$ the vector solution at step i, that takes two consecutive solutions of difference equation. The matrix $\mathbf{M}^{(i)}$ is the auxiliary transfer matrix, that takes the state of planar system from the vector solution, $\vec{x}_i$, to the vector $\vec{x}_{i+1}$. In matrix form:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} \tag{2}$$

Transfer matrix method has many applications in Physics, Engineering, Economy and Psychology, among others. The applicability of the difference equations is amply well-known. The study is made on the solutions of the linear system given by the Eq.(2), with the hypothesis of conservation of constant solutions, $b_i + a_i = 1$, for $i = 1, ...$ Results and conclusions will be discussed under a applied approach, avoiding as far as possible the use of abstract Algebra.

By elementary theory on matrices $\mathbf{M} \in \Re^{2x2}$, [4], if the vector $\vec{x}_*$ is a fixed point, non-trivial, then $det(\mathbf{M} - \mathbf{I}_2) = 0$. This is fulfilled iff the following equation is satisfied between the trace, $tr(\mathbf{M})$, and the determinant, $det(\mathbf{M})$:

$$tr(\mathbf{M}) - det(\mathbf{M}) = 1. \tag{3}$$

By coherence, if $\mathbf{M}$ is of the form given by the Eq.(2), the condition of the Eq.(3) is equivalent to our hypothesis. In general, for matrices $\{\mathbf{M_1}, \mathbf{M_2}\} \in \Re^{2x2}$, that satisfy the condition of the Eq.(3), their product does not satisfy it.

Example 1.1 *The matrices* $\mathbf{M}_1$ *and* $\mathbf{M}_2$*:*

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \mathbf{M}_2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}; \mathbf{M}_2\mathbf{M}_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Both matrices accomplish the condition of Eq.(3)*, but their product does not accomplish it. In addition, it must be observed that the matrices* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are similar.*

A necessary and sufficient condition will be showed later, so that every transfer matrices, $\{\mathbf{M_1}, \mathbf{M_2}\} \in \Re^{2x2}$ that accomplish the condition from Eq.(3), their product also accomplishes it.

1.1 Transition matrices

The transition matrices $\mathbf{T}^{(n)}$ allow us to bring, from initial vector $\vec{x}_0$, to the vector at site n, $\vec{x}_n$. The definition for transitions between any vector solutions is similar. The construction of the matrix $\mathbf{T}^{(n)}$, is made with the product of transfer matrices.

$$\mathbf{T}^{(n)} = \prod_{i=1}^{n} \mathbf{M}^{(i)} = \prod_{i=1}^{n} \begin{pmatrix} b_i & a_i \\ 1 & 0 \end{pmatrix} \tag{4}$$

With $\mathbf{T}^{(0)} = \mathbf{I}_2$. Then, the transition is expressed as $\vec{x}_n = \mathbf{T}^{(n)}\vec{x}_0$, in matrix form:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \prod_{i=1}^{n} \begin{pmatrix} b_i & a_i \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \tag{5}$$

By using traces of matrix products, of transition matrices by the projection matrix on the real axis $\mathbf{P}_x$, an expression for $\mathbf{T}^{(n)}$ can be obtained. Also, the shifted transition matrix is introduced, $\mathbf{T}^{+(k)}$, equivalent to $\mathbf{T}^{(k)}$, but with the unique difference that the coefficients b_i and a_i, are replaced by b_{i+1} y a_{i+1}, for $i = 1, ..., k$.

Lemma 1.2 *The transition matrix $\mathbf{T}^{(n)}$ from Eq.(4) is of the form:*

$$\mathbf{T}^{(n)} = \begin{pmatrix} b_n tr(\mathbf{P}_x \mathbf{T}^{(n-1)}) + a_n tr(\mathbf{P}_x \mathbf{T}^{(n-2)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-1)}) \\ tr(\mathbf{P}_x \mathbf{T}^{(n-1)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-2)}) \end{pmatrix} \tag{6}$$

With initial matrices:

$$\mathbf{T}^{(-1)} = \mathbf{T}^{+(-1)} = \mathbf{0}_2; \mathbf{T}^{(0)} = \mathbf{T}^{+(0)} = \mathbf{I}_2.$$

Proof. By induction on n. For $n = 1$, it is satisfied.

$$\mathbf{T}^{(1)} = \begin{pmatrix} b_1 & a_1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_1 tr(\mathbf{P}_x \mathbf{T}^{(0)}) + a_1 tr(\mathbf{P}_x \mathbf{T}^{(-1)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(0)}) \\ tr(\mathbf{P}_x \mathbf{T}^{(0)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(-1)}) \end{pmatrix}$$

Suppose that it is fulfill for $n - 1$.

$$\mathbf{T}^{(n-1)} = \begin{pmatrix} b_{n-1} tr(\mathbf{P}_x \mathbf{T}^{(n-2)}) + a_{n-1} tr(\mathbf{P}_x \mathbf{T}^{(n-3)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-2)}) \\ tr(\mathbf{P}_x \mathbf{T}^{(n-2)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-3)}) \end{pmatrix}$$

Then,

$$\mathbf{T}^{(n)} = \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{n-1} tr(\mathbf{P}_x \mathbf{T}^{(n-2)}) + a_{n-1} tr(\mathbf{P}_x \mathbf{T}^{(n-3)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-2)}) \\ tr(\mathbf{P}_x \mathbf{T}^{(n-2)}) & a_1 tr(\mathbf{P}_x \mathbf{T}^{+(n-3)}) \end{pmatrix}$$

The proof follows by the hypothesis of induction on $\mathbf{T}^{(n-1)}$ as well as on $\mathbf{T}^{+(n-1)}$.

By coherence, if the transition matrix $\mathbf{T}^{(n)}$ from Eq.(4), is created by the product of transfer matrices that accomplish the hypothesis to conserve the constant solutions, then the matrix $\mathbf{T}^{(n)}$ must accomplish the condition given by Eq.(3).

Proposition 1.3 *If $\mathbf{M}^{(i)}$, for $i = 1, .., n$, are transfer matrices as in Eq.(2), with coefficients b_i and a_i, that satisfy $b_i + a_i = 1$. Then, transition matrix $\mathbf{T}^{(n)}$ from Eq.(4), accomplishes Eq.(3) and, in each one of their rows the matrix elements add 1.*

Proof. First, it is verified that if the sum of all matrix elements of each row from $\mathbf{T}^{(n)}$ is equal to 1, then Eq.(3) is fulfilled. If

$$\mathbf{T}^{(n)} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

With $\alpha_n + \beta_n = 1$, and $\gamma_n + \delta_n = 1$. Then, $tr(\mathbf{T}^{(n)}) - det(\mathbf{T}^{(n)}) = \alpha_n + \delta_n - \alpha_n\delta_n + \gamma_n\beta_n = \alpha_n + \beta_n(\delta_n + \gamma_n) = \alpha_n + \beta_n = 1$. Thus, it is sufficient to demonstrate by induction that all matrix elements of each row from $\mathbf{T}^{(n)}$ add 1. For $n = 1$ is right, by hypothesis. Suppose that it is fulfilled for $n - 1$. For $\{\alpha_{n-1}, \gamma_{n-1}\} \in \Re$

$$\mathbf{T}^{(n-1)} = \begin{pmatrix} \alpha_{n-1} & 1-\alpha_{n-1} \\ \gamma_{n-1} & 1-\gamma_{n-1} \end{pmatrix}.$$

$$\mathbf{T}^{(n)} = \begin{pmatrix} b_n & a_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{n-1} & 1-\alpha_{n-1} \\ \gamma_{n-1} & 1-\gamma_{n-1} \end{pmatrix} = \begin{pmatrix} b_n\alpha_{n-1} + a_n\gamma_{n-1} & b_n(1-\alpha_{n-1}) + a_n(1-\gamma_{n-1}) \\ \alpha_{n-1} & 1-\alpha_{n-1} \end{pmatrix}$$

By hypothesis of induction, the elements of the first row add 1. In the second row, the result is evident.

1.2 Constant Solutions in Two-dimensional Transfer Matrices

Next, a useful result is given on the product of two transfer matrices that accomplish the condition of Eq.(3). This result claims that the product matrix saves a eigenvalue equal to 1. It shows the family of matrices, companionable with one given, that permits the existence of, non trivial, fixed point for the matrix product.

Theorem 1.4 *Let* $\mathbf{M_1}$ *,* $\mathbf{M_2}$ *be two-dimensional transfer matrices that accomplish Eq.*(3)*. The matriz product also accomplishes Eq.*(3) *iff,*

$$det(\mathbf{M_2} - \mathbf{M_1}) = 0. \tag{7}$$

Proof.

$$\mathbf{M_2M_1} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} = \begin{pmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 + \beta_2\delta_1 \\ \gamma_2\alpha_1 + \delta_2\gamma_1 & \gamma_2\beta_1 + \delta_2\delta_1 \end{pmatrix}.$$

$tr(\mathbf{M_2M_1}) - det(\mathbf{M_2M_1}) = \alpha_2\alpha_1 + \beta_2\gamma_1 + \gamma_2\beta_1 + \delta_2\delta_1 - (\alpha_2\delta_2 - \gamma_2\beta_2)(\alpha_1\delta_1 - \gamma_1\beta_1)$. Using the hypothesis in the product of the determinants, yields: $\alpha_2\alpha_1 + \beta_2\gamma_1 + \gamma_2\beta_1 + \delta_2\delta_1 - (\alpha_2 + \delta_2 - 1)(\alpha_1 + \delta_1 - 1) = (\alpha_2 + \delta_2) + (\alpha_1 + \delta_1) + \beta_2\gamma_1 + \gamma_2\beta_1 - \alpha_2\delta_1 - \delta_2\alpha_1 - 1$. Now, the hypothesis is used in the sum of traces, yields: $tr(\mathbf{M_2M_1}) - det(\mathbf{M_2M_1}) = \alpha_2\delta_2 - \gamma_2\beta_2 + \alpha_1\delta_1 - \gamma_1\beta_1 + \beta_2\gamma_1 + \gamma_2\beta_1 - \alpha_2\delta_1 - \delta_2\alpha_1 + 1 = (\alpha_2 - \alpha_1)(\delta_2 - \delta_1) - (\gamma_2 - \gamma_1)(\beta_2 - \beta_1) + 1$. Eq.(3) is fulfilled iff, $det(\mathbf{M_2} - \mathbf{M_1}) = 0$.

Example 1.5 *If the necessary and sufficient condition of the theorem 1.4 is applied to the Example 1.1, the result is clear:*

$$det(\mathbf{M_2} - \mathbf{M_1}) = det(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}) = -1.$$

The condition is not fulfilled and product matrix does not satisfy Eq.(3)*. If both matrices are similar then, the matrix of the change of basis,* $\mathbf{M_3}$*, must accomplish Eq.*(7)*, with both matrices,* $det(\mathbf{M_3} - \mathbf{M_1}) = 0$*, and* $det(\mathbf{M_3} - \mathbf{M_2}) = 0$*. In Example 1.1 are not fulfilled.*

Also, the theorem 1.4 is a necessary condition for to maintain the same fixed point. Eq.(7) isn't a equivalence relation between matrices in $\Re^{2x2}$. The transitive property is not fulfilled.

Example 1.6 *The matrices* $\mathbf{M}_1$, $\mathbf{M}_2$ *and* $\mathbf{M}_3$*:*

$$\mathbf{M}_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}; \mathbf{M}_2 = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}; \mathbf{M}_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

And, $det(\mathbf{M}_2 - \mathbf{M}_1) = 0$*,* $det(\mathbf{M}_3 - \mathbf{M}_2) = 0$*, but* $det(\mathbf{M}_3 - \mathbf{M}_1) = -2 \neq 0$*. Thus, the products* $\mathbf{M}_2\mathbf{M}_3$ *and* $\mathbf{M}_3\mathbf{M}_2$ *maintain constant vectors of the type* $\vec{x} = (x, -x)^T$*. The product* $\mathbf{M}_1\mathbf{M}_2$ *maintains constant the vector* $\vec{x} = (-\frac{x}{2}, x)^T$*, the product* $\mathbf{M}_2\mathbf{M}_1$ *maintains constant the vector* $\vec{x} = (\frac{2x}{5}, x)^T$*. Nevertheless, anyone of the two products of the matrix* $\mathbf{M}_1$ *with the matrix* $\mathbf{M}_3$*, does not maintain any solution constant, except for the trivial solution.*

The result is restricted to matrices that appear in Eq.(2) and they fulfill the hypothesis $b_i + a_i = 1$. Then, transitive property is right and these matrices always maintain vector solution of the type $\vec{x} = (x, x)^T$. This is the type of vector suitable for constant solutions of difference equation from Eq.(1). Due 1 is an eigenvalue, from Eq.(3) the remaining eigenvalue is $\lambda_2 = tr(\mathbf{M}^{(i)}) - 1$. Thus $\rho(\mathbf{M}^{(i)}) = 1$ iff $0 \leq tr(\mathbf{M}^{(i)}) \leq 2$.

2 Fixed points and Stability

It is easy to observe that transfer matrices from Eq.(2), associated to difference equation from Eq.(1), can maintain constant solution only in the form $\vec{x_*} = (x_*, x_*)^T$. The line $(x, y) : x = y$, is the unique subespace in $\Re^2$ that has all its points invariants. If spectral radius is $\rho(\mathbf{M}^{(i)}) = 1$, and the absolute value of the second eigenvalue is $|\lambda^{(i)}_{min}| < 1$, the invariant line $x = y$ is stable. In this situation, the solution tends to an constant vector $\vec{x_*} = (x_*, x_*)^T$, from any initial vector $\vec{x_0}$. This sufficient condition is accomplished by particular stochastic matrices, in the form from Eq.(2), a special type of 2-dimensional Markov Chain. With $\theta \in (0, \frac{\pi}{2})$:

$$\mathbf{M}^{(i)} = \begin{pmatrix} cos^2(\theta_i) & sin^2(\theta_i) \\ 1 & 0 \end{pmatrix}.$$

Steady state matrices on Markov Chains are well-known, [5]. Thus, transition matrix $T^{(n)}$ tends towards an invariant matrix T^*.

$$\mathbf{T}^{(*)} = \begin{pmatrix} cos^2(\theta_*) & sin^2(\theta_*) \\ cos^2(\theta_*) & sin^2(\theta_*) \end{pmatrix}.$$

Indeed, the stochastic matrix $\mathbf{M}^{(i)}$ is nonnegative and the product of this type of matrices results in a positive matrix $\mathbf{T}$, $t_{i,j} > 0$, for $i, j = 1, 2$, that it accomplishes Eq.(3). From Perron's theorem on positive matrices, the spectral radius $\rho(\mathbf{M}^{(i)}) = 1$ and 1 is a simple eigenvalue. $|\lambda^{(i)}_{min}| < 1$ and the transition matrix tends to $\mathbf{T}^{(*)}$, with $det(\mathbf{T}^{(*)})$=0. Like the invariant matrix $\mathbf{T}^{(*)}$ also accomplishes Eq.(3), both conditions result in equal rows for $\mathbf{T}^{(*)}$. In addition, can be demonstrated that,

$$tr(\mathbf{T}^{(n)}) = 1 + (-1)^n \prod_{i=1}^{n} sen^2(\theta_i).$$

Example 2.1 *In the special case where the angle is constant* $\theta_i = \theta$, *for i=1,2,..., in addition to the line of fixed points* $x = y$, *a second invariant line exists. It depends on the second eigenvalue. This line always has as unique fixed point to* $\vec{0} = (0,0)$, *and its direction is provided by the equation* $tan(\varphi) = \frac{1}{\lambda_{min}} = \frac{-1}{sin^2(\theta)}$. *If a solution falls in this line, it moves, on this line, towards the null fixed point. Solutions go to the null fixed point by jumps above it. The next jump will be shorter and closer to the fixed point. The Jordan form for* $\mathbf{T}^{(n)}$ *is:*

$$\mathbf{T}^{(n)} = \begin{pmatrix} 1 & -sin^2(\theta) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n sin^{2n}(\theta) \end{pmatrix} \begin{pmatrix} \frac{1}{1+sin^2(\theta)} & \frac{sin^2(\theta)}{1+sin^2(\theta)} \\ \frac{-1}{1+sin^2(\theta)} & \frac{1}{1+sin^2(\theta)} \end{pmatrix}. \tag{8}$$

And,

$$\mathbf{T}^{(*)} = lim_{n\to\infty}\mathbf{T}^{(n)} \to \frac{1}{1+sin^2(\theta)} \begin{pmatrix} 1 & sin^2(\theta) \\ 1 & sin^2(\theta) \end{pmatrix}.$$

Stochastic matrices are not the only matrices that accomplish the sufficient condition. Thus, the following type of matrices satisfy the condition and they must be considered.

$$\mathbf{M}^{(i)} = \begin{pmatrix} 1 + cos^2(\theta_i) & -cos^2(\theta_i) \\ 1 & 0 \end{pmatrix}.$$

Also, the product of this type of matrices tends to an invariant transition matrix $\mathbf{T}^{(*)}$. Now, Perron's theorem isn't applicable, but it's possible demonstrate by induction that the transition matrix for this type of transfer matrices accomplish,

$$tr(\mathbf{T}^{(n)}) = 1 + \prod_{i=1}^{n} cos^2(\theta_i)$$

Then $tr(\mathbf{T}^{(*)}) = 1$ and for Eq.(3), $det(\mathbf{T}^{(*)}) = 0$. $\mathbf{T}^{(*)}$ is a invariant matrix, in each row the matrix elements add 1, and both rows are equal.

Example 2.2 *For constant angle* $\theta_i = \theta$ *the other invariant line is similar to the stochastic case, but now* $tan(\varphi) = \frac{1}{\lambda_{min}} = \frac{1}{cos^2(\theta)}$, *the solutions move towards the fixed point, on this line, without jumps over the fixed point. The Jordan form for* $\mathbf{T}^{(n)}$ *is:*

$$\mathbf{T}^{(n)} = \begin{pmatrix} 1 & cos^2(\theta) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & cos^{2n}(\theta) \end{pmatrix} \begin{pmatrix} \frac{1}{1-cos^2(\theta)} & \frac{-cos^2(\theta)}{1-cos^2(\theta)} \\ \frac{-1}{1-cos^2(\theta)} & \frac{1}{1-cos^2(\theta)} \end{pmatrix}. \tag{9}$$

And,

$$\mathbf{T}^{(*)} = lim_{n\to\infty}\mathbf{T}^{(n)} \to \frac{1}{1-cos^2(\theta)} \begin{pmatrix} 1 & -cos^2(\theta) \\ 1 & -cos^2(\theta) \end{pmatrix}.$$

Any mixed product of both types of matrices, accomplish the sufficient condition and tends to an invariant transition matrix, $\mathbf{T}^{(*)}$ too.

Example 2.3 *If* $\mathbf{M}^{(i)} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$, *if i is even, and* $\mathbf{M}^{(i)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$, *if i is odd. Then transition matrices* $\mathbf{T}^{(n)}$ *tend to* $\mathbf{T}^{(*)} = \frac{1}{5}\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$.

It can be demonstrated that the product of the matrices $\mathbf{T} \in \Re^{2x2}$, where in each one of their rows the matrix elements add 1, with $\alpha, \beta \in \Re$,

$$\mathbf{T} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

forms an algebra $\mathcal{T}$. A bilateral ideal $\mathcal{C} \subset \mathcal{T}$ is the set of the matrices $\mathbf{C}$ that have the two equal rows, with $\gamma \in \Re$.

$$\mathbf{C} = \begin{pmatrix} \gamma & 1-\gamma \\ \gamma & 1-\gamma \end{pmatrix}$$

In addition, each matrix $\mathbf{C}$ is an left ideal of $\mathcal{T}$, $\forall \mathbf{T} \in \mathcal{T} : \mathbf{TC} = \mathbf{C}$. These matrices $\mathbf{C}$ are the only matrices of algebra $\mathcal{T}$ whose determinants are equal to 0. With the exception of the unit matrix, they are the unique ones that can be invariant in the study of the constant, non trivial, solutions of Eq.(2) for any initial condition.

When spectral radius is above 1, the line $x = y$ continues invariant, but it is unstable. In general, the solutions are unbounded. Asymptotic behavior of solutions is complex. In some particular cases, Poincaré-Perron theory is applicable. Birkhoff-Adams theory in other special situations, see [2]. Anyway, with the main hypothesis of this work, initial solutions belong to the invariant line $x = y$, always are fixed points.

3 Conclusions

The previous analysis is applied to the solutions of the difference equation, Eq.(1), with the hypothesis $b_i + a_i = 1$. If the coefficients of this equation are combinations of both particular cases, $b_i = cos^2(\theta_i), a_i = sin^2(\theta_i)$, or $b_i = 1 + cos^2(\theta_i), a_i = -cos^2(\theta_i)$, con $\theta_i \in (0, \frac{\pi}{2})$, these solutions tend to a constant value x_*, independently of the initial solutions x_0 and x_1.

If the sufficient condition $\rho(\mathbf{M}^{(i)}) = 1$, for $i \geq 1$, is not fulfilled and the initial solutions are $x_0 = x_1 = x_*$ the solution remains constant. For other initials conditions, the solution becomes unbounded, except for the possibility of existence of periodic solutions. Because the condition is not necessary, other forms can exist to approach the solutions towards a constant solution for the Eq.(1).

Due sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ are of real numbers and transfer matrices are of discrete steps on time, Jacobians aren't no considered in this work. If the sequences will depend on a real parameter λ, $\{a_i(\lambda)\}_{i=1}^{\infty}$ and $\{b_i(\lambda)\}_{i=1}^{\infty}$, Jacobians must be considered. Also, Eq.(2) can be seen as the linear approximation of a non linear system of two particular difference equations, [2]. In this context, matrix $\mathbf{M}^{(i)}$ is the Jacobian at step i.

The extension of the method, to obtain sufficient conditions on the constant solutions of linear and homogenous difference equations of greater order with variable coefficients, seems reasonable.

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