

SOLUTION OF OPTIMAL CONTINUOUS LOW-THRUST TRANSFER USING LIE TRANSFORMS

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This paper addresses the problem of optimal constant continuous low-thrust transfer in the context of the restricted two-body problem (R2BP). Using the Pontryagin's principle, the problem is formulated as a two point boundary value problem (TPBVP) for a Hamiltonian system. Lie transforms obtained through the Deprit method allow us to obtain the canonical mapping of the phase flow as a series in terms of the order of magnitude of the thrust applied. The reachable set of states starting from a given initial condition using optimal control policy is obtained analytically. In addition, a particular optimal transfer can be computed as the solution of a non-linear algebraic equation.

INTRODUCTION

The computation of optimal continuous space trajectories has been a subject of interest since the birth of the space age. This problem fostered the development of numerical and analytical methods to tackle optimal control problems. The literature devoted to space trajectory optimization is plentiful and rich. Concerning the analytical tools, there are synthesis of analytical results on the problem of optimal transfers with continuous thrust as early as 1968.¹ In turn, a thorough review of the state-of-art numerical methods for trajectory optimization in the late 90s can be found in reference.²

In the early sixties, Lawden³ introduced the primer vector theory and laid the foundations to deal with optimal trajectories. Applying the Pontryagin's principle, the problem of finding an optimal continuous transfer is formulated as a two point boundary value problem (TPBVP) for a Hamiltonian system. The main obstacle in solving the TPVBP comes from the need of providing an initial guess in the co-states or adjoints, which have no physical meaning. In addition, the solution usually turns out to be very sensitive to the initial guess. Thus, iterative methods result in time-consuming computations with no guaranty on the convergence. This fact motivated the quest of analytical tools to ease the numerical burden. Classical mathematical methods of celestial mechanics were revisited to answer this need.⁴⁻⁶ Lately, efforts have been made to tackle the TPBVP using elements of the Hamilton-Jacobi theory.⁷ The generating functions of the canonical transformation of the phase flow are used to solve the TPBVP. This method has been successfully applied to optimal control as well as formation flying or rendezvous problems.⁸⁻¹⁰

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In this paper, we use perturbation theory in a novel way to solve the optimal control problem of a low-thrust continuous transfer. The use of perturbation theory in this problem is explained by the consideration of a small absolute value of the control, i.e., thrust is small compared to the gravitational acceleration. Although perturbation theory has been already applied in this context, the conventional approach consists of a direct manipulation of the equations of motion of the state vector. Moreover, the analysis was usually restricted to special cases whose validity was limited to certain regions of the parameter space, as in the seminal work of Edelbaum¹¹ and subsequent work. The method proposed in this paper is the use of Lie transforms to perform a convenient canonical transformation in the configuration space of state and adjoints. Lie transforms, as developed by Deprit,¹² have been widely and successfully used in celestial mechanics in initial value problems. Here, the scope of the method is broadened to solve TPBVPs. The main advantage of the method is that it allows one to build an explicit formulation of the transformation as part of the solution.

PROBLEM SETTINGS

The problem addressed in this work is the minimum-fuel constant continuous low-thrust maneuver with fixed final time. For the sake of simplicity, the mass of the spacecraft is not considered as a state variable. Accordingly, given that the thrust is continuous, the cost function considered here is integral and has the form:

$$J = \frac{1}{2} \int_{\tau_0}^{\tau_F} (\mathbf{u}^T \mathbf{u}) dt \quad (1)$$

where τ is the non-dimensional time ($\tau = t\sqrt{\mu/a_0^3}$, μ is the gravitational parameter of the primary and a_0 is the initial or characteristic value of the osculating semi-major axis). Besides, \mathbf{u} is the non-dimensional thrust acceleration (the value of the characteristic acceleration is μ/a_0^2). In turn, the spacecraft is orbiting a single large mass (R2BP). The state of the spacecraft is described in terms of the classical orbital elements. Other representations are possible¹³ and even more suitable in terms of avoiding singularities. Nevertheless, the method exposed hereafter is not tied to the state description of the spacecraft and the required modifications for using a different set of state variables are straightforward. Using classical orbital elements, the dynamics can be expressed in terms of the Gauss' planetary equations¹⁴:

$$\frac{d\xi}{d\tau} = \mathbf{B}\xi + \mathbf{C}(\xi)\mathbf{u} \quad (2)$$

where $\xi = \{\Omega, i, \omega, n, e, M\}$, $\mathbf{B} \in \mathcal{M}_{6 \times 6}$, $\mathbf{C} \in \mathcal{M}_{6 \times 3}$. Note that n is a non-dimensional variable and the characteristic value of the mean motion is $n_0 = \sqrt{\mu/a_0^3}$. The only non-null term of \mathbf{B} is $B_{65} = 1$ and matrix \mathbf{C} is given below:

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & -\frac{\cos i}{a n \sin i} \left[\cos \omega \sin E + \frac{\sin \omega (\cos E - e)}{\sqrt{1-e^2}} \right] \\ 0 & 0 & \frac{1}{a n} \left[\frac{\cos \omega (\cos E - e)}{\sqrt{1-e^2}} - \sin \omega \sin E \right] \\ -\frac{\sqrt{1-e^2} \cos E - e}{e n a (1-e \cos E)} & \frac{1}{n a (1-e \cos E)} \left[\frac{\sin E}{e} (2 - e^2) - \frac{1}{2} \sin 2E \right] & -\frac{\cos i}{a n \sin i} \left[\cos \omega \sin E + \frac{\sin \omega (\cos E - e)}{\sqrt{1-e^2}} \right] \\ -\frac{3e \sin E}{a(1-e \cos E)} & -\frac{3\sqrt{1-e^2}}{a(1-e \cos E)} & 0 \\ \frac{\sin E(1-e^2)}{n a(1-e \cos E)} & -\frac{\sqrt{1-e^2}}{2 n a(1-e \cos E)} [3e + e \cos 2E - 4 \cos E] & 0 \\ \frac{1}{n a e} \left[\frac{\cos E - e^3 \cos 2E}{1-e \cos E} - 3e \right] & -\frac{\sqrt{1-e^2} \sin E}{n a e} \left[1 + \frac{1-e^2}{1-e \cos E} \right] & 0 \end{pmatrix}$$

where $a = \sqrt[3]{1/n^2}$ is non-dimensional, and E is the eccentric anomaly. The relation between eccentric and mean anomalies is given by the Kepler equation: $M = E - e \sin E$. This fact makes more convenient to express the dynamics in terms of the eccentric anomaly instead of the true anomaly.

The boundary conditions depend on the kind of problem to solve. For the reachable set problem, the initial state of the spacecraft and the final costate is completely specified:

$$\boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0 \quad (3)$$

$$\mathbf{p}(\tau_F) = \mathbf{0} \quad (4)$$

Whereas, for the computation of the optimal trajectory between given initial and final states, the boundary conditions are:

$$\boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0 \quad (5)$$

$$\boldsymbol{\xi}(\tau_F) = \boldsymbol{\xi}_F \quad (6)$$

Control Modulated by Small Parameter

Given that the uncontrolled problem is solvable, perturbation theory tools can be applied providing that the control acceleration is small compared to the gravitational acceleration. This is usually the case for low-thrust engines operating in strong gravity fields. Thus, the control is expressed as:

$$\mathbf{u} = \varepsilon \tilde{\mathbf{u}} \quad (7)$$

where $\varepsilon = T / (\mu/a_0^2)$ (T is the magnitude of the constant continuous thrust acceleration). Therefore, dynamics are rewritten as:

$$\frac{d\boldsymbol{\xi}}{d\tau} = \mathbf{B}\boldsymbol{\xi} + \varepsilon \mathbf{C}(\boldsymbol{\xi})\tilde{\mathbf{u}} \quad (8)$$

NECESSARY CONDITIONS FOR OPTIMALITY

In a general manner, the problem we are facing is described mathematically as follows: finding the control function \mathbf{u} of the system

$$\dot{\boldsymbol{\xi}} = f(\boldsymbol{\xi}, \mathbf{u}) \quad \boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0 \quad \psi(\boldsymbol{\xi}(\tau_F), \tau_F) = 0 \quad \tau \in [\tau_0, \tau_F] \quad (9)$$

such that the cost function

$$J = \phi_T(\boldsymbol{\xi}_F) + \int_{\tau_0}^{\tau_F} \mathcal{L}(\boldsymbol{\xi}, \mathbf{u}) dt \quad (10)$$

is minimized. For convenience, a scalar ‘‘Hamiltonian’’ function is introduced as follows¹⁵:

$$\mathcal{H}(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau) = \mathcal{L}(\boldsymbol{\xi}, \mathbf{u}) + \mathbf{p}_\xi^T f(\boldsymbol{\xi}, \mathbf{u}) \quad (11)$$

With the previous description, the necessary conditions for optimality when the control is modulated by a small parameter reads:

$$\bar{\mathcal{H}}(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau) = \bar{\mathcal{H}}(\boldsymbol{\xi}, \mathbf{p}_\xi, \varepsilon \tilde{\mathbf{u}}^*(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau), \tau) \quad (12)$$

$$\dot{\boldsymbol{\xi}} = \bar{\mathcal{H}}_{\mathbf{p}_\xi}(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau) \quad (13)$$

$$\dot{\mathbf{p}}_\xi = -\bar{\mathcal{H}}_\xi(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau) \quad (14)$$

$$\tilde{\mathbf{u}}^*(\boldsymbol{\xi}, \mathbf{p}_\xi, \tau) = \arg \min_{\tilde{\mathbf{u}}} \bar{\mathcal{H}}(\boldsymbol{\xi}, \mathbf{p}_\xi, \varepsilon \tilde{\mathbf{u}}, \tau) \quad (15)$$

subject to:

$$\boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0 \quad \boldsymbol{\psi}(\boldsymbol{\xi}(\tau_F), \tau_F) = \mathbf{0} \quad (16)$$

The necessary conditions for optimality (12-16) constitute the formulation of the problem to solve.

In our case, the Hamiltonian of the system, with the previous performance index, has the form:

$$\mathcal{H} = \mathbf{p}_\xi^T \mathbf{B} \boldsymbol{\xi} + \varepsilon \mathbf{p}_\xi^T \mathbf{C}(\boldsymbol{\xi}) \tilde{\mathbf{u}} + \frac{1}{2} \varepsilon^2 \mathbf{u}^T \mathbf{u} \quad (17)$$

From equation (15), it is obtained that,

$$\frac{\partial \mathcal{H}}{\partial \tilde{\mathbf{u}}} = \mathbf{0} = \mathbf{p}_\xi^T \mathbf{C}(\boldsymbol{\xi}) + \varepsilon \tilde{\mathbf{u}}^T \quad (18)$$

The components of $\mathbf{C}(\boldsymbol{\xi})$ are of order unity. Thus the adjoints should be of order ε . Rescaling the adjoints as: $\mathbf{p}_\xi = \varepsilon \tilde{\mathbf{p}}_\xi$, and the Hamiltonian $\mathcal{H} = \varepsilon \tilde{\mathcal{H}}$, the latter turns out to be:

$$\tilde{\mathcal{H}} = \tilde{\mathbf{p}}_\xi^T \mathbf{B} \boldsymbol{\xi} - \frac{1}{2} \varepsilon \tilde{\mathbf{p}}_\xi^T \mathbf{C} \mathbf{C}^T \tilde{\mathbf{p}}_\xi$$

Therefore, the optimization of the integral cost function subject to a dynamic constraint is a two-point boundary value problem (16), in which the dynamics of the state and adjoints is determined by (13-14). Besides, the problem is stated as a integrable one plus a perturbation. It is worth noting that the integrable part of the problem does not fulfill the boundary conditions. The perturbation order, in this case, is,

$$\begin{aligned} \tilde{\mathcal{H}} &= \tilde{\mathcal{H}}_0 + \varepsilon \tilde{\mathcal{H}}_1 \\ \tilde{\mathcal{H}}_0 &= \tilde{\mathbf{p}}_\xi^T \mathbf{B} \boldsymbol{\xi} = \tilde{p}_M^n \\ \tilde{\mathcal{H}}_1 &= -\frac{1}{2} \tilde{\mathbf{p}}_\xi^T \mathbf{C} \mathbf{C}^T \tilde{\mathbf{p}}_\xi \end{aligned}$$

The problem formulated in this way is suitable to be dealt with perturbation methods. Next section describes the method proposed to be used in this case.

DEPRIT PERTURBATION METHOD

In this section, an outline of the Deprit's perturbation theory is sketched as well as the main characteristics and properties of the method. The reader is encouraged to find out more details about the method in the original paper¹² or in reference books¹⁷.

The objective of the technique is the construction of a canonical transformation of the original Hamiltonian system to achieve a transformed Hamiltonian function with specific requirements. The original Hamiltonian system should have a perturbation order based on a power series of a small parameter. Let $\mathcal{H}(x, X; t; \varepsilon) = \mathcal{H}_0 + \varepsilon \mathcal{H}_1 + \frac{1}{2!} \varepsilon^2 \mathcal{H}_2 + \dots$ be the original Hamiltonian and $\mathcal{K}(y, Y; t; \varepsilon) = \mathcal{K}_0 + \varepsilon \mathcal{K}_1 + \frac{1}{2!} \varepsilon^2 \mathcal{K}_2 \dots$ the transformed Hamiltonian, where \mathcal{H}_0 is solvable. The canonical transformation is carried out by means of a Lie transform. The transform is completely determined by the Lie generator \mathcal{W} equivalent to the generating function of the canonical transformation. The condition the Lie generator must satisfy to fulfill the requirements on the transformed Hamiltonian is¹²:

$$(\mathcal{H}_0; \mathcal{W}) - \frac{\partial \mathcal{W}}{\partial t} = \mathcal{K}_1 - \mathcal{H}_1 \quad (19)$$

where $(f; g) = \sum_j \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial x_j} \right)$ is the Poisson bracket of f and g over the phase space (\mathbf{x}, \mathbf{X}) . The previous condition is a linear partial differential equation for \mathcal{W} .

The method also provides with the explicit map of the canonical transformation in the form of power series.

$$x = y + \varepsilon y^{(1)}(y, Y; t) + \frac{1}{2!} \varepsilon^2 y^{(2)}(y, Y; t) + \dots \quad (20)$$

$$X = Y + \varepsilon Y^{(1)}(y, Y; t) + \frac{1}{2!} \varepsilon^2 Y^{(2)}(y, Y; t) + \dots \quad (21)$$

The process is recursive and is based on the use of Lie transforms and the associated Lie series.

In our case, the objective is to eliminate the first order terms in ε from the transformed Hamiltonian: $\mathcal{K}_1 \equiv 0$. The partial differential equation for the generating function reads:

$$\left(\frac{\partial \mathcal{H}_0}{\partial \boldsymbol{\xi}} \frac{\partial \mathcal{W}}{\partial \tilde{\mathbf{p}}_\xi} - \frac{\partial \mathcal{H}_0}{\partial \tilde{\mathbf{p}}_\xi} \frac{\partial \mathcal{W}}{\partial \boldsymbol{\xi}} \right) - \frac{\partial \mathcal{W}}{\partial t} = -\mathcal{H}_1 \Rightarrow \tilde{\mathbf{p}}_M \frac{\partial \mathcal{W}}{\partial \tilde{\mathbf{p}}_n} - n \frac{\partial \mathcal{W}}{\partial M} = -\frac{1}{2} \tilde{\mathbf{p}}_\xi^T \mathbf{C} \mathbf{C}^T \tilde{\mathbf{p}}_\xi$$

The partial derivative of \mathcal{W} with respect to time has been set to zero because the Hamiltonian is not an explicit function of time. Given that \mathcal{H}_1 is a summation, the solution of this equation can be written as the sum of three terms:

$$\mathcal{W} = \mathcal{W}_{IN} + \mathcal{W}_{LI} + \mathcal{W}_{QU} \quad (22)$$

Each of the terms corresponds to three different kinds of addends in \mathcal{H}_1 : independent of $\tilde{\mathbf{p}}_n$, linear in $\tilde{\mathbf{p}}_n$ and quadratic in $\tilde{\mathbf{p}}_n$. The explicit form of the three terms of the generating function is gathered below:

$$\mathcal{W}_{IN} = \tilde{\mathbf{p}}_\xi^T \mathcal{A}(\boldsymbol{\xi}) \tilde{\mathbf{p}}_\xi \quad (23)$$

$$\mathcal{W}_{LI} = \tilde{\mathbf{p}}_\xi^T \left(\sum_{k=0}^{\infty} \mathcal{B}_k(\boldsymbol{\xi}) \right) \tilde{\mathbf{p}}_\xi \quad (24)$$

$$\mathcal{W}_{QU} = \tilde{\mathbf{p}}_\xi^T \left(\sum_{k=0}^{\infty} \mathcal{C}_k(\boldsymbol{\xi}) \right) \tilde{\mathbf{p}}_\xi \quad (25)$$

Therefore, the Lie generator function is written as:

$$\mathcal{W} = \tilde{\mathbf{p}}_\xi^T \left(\mathcal{A}(\boldsymbol{\xi}) + \sum_{k=0}^{\infty} \mathcal{B}_k(\boldsymbol{\xi}) + \sum_{k=0}^{\infty} \mathcal{C}_k(\boldsymbol{\xi}) \right) \tilde{\mathbf{p}}_\xi = \tilde{\mathbf{p}}_\xi^T \mathcal{D}(\boldsymbol{\xi}) \tilde{\mathbf{p}}_\xi \quad (26)$$

In the previous expressions, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathcal{M}_{6 \times 6}$ and their coefficients are functions of $\boldsymbol{\xi}$. The coefficients that enter into \mathcal{W}_{LI} and \mathcal{W}_{QU} are Fourier series in the mean anomaly whose coefficients are also series of Bessel functions of the first kind in the eccentricity¹⁶. The explicit expression of the coefficients can be found in appendix A.

The generating function determines completely the canonical transformation. Thus, the transformed Hamiltonian is just: $\mathcal{K}_0 = \mathcal{H}_0$, with solutions:

$$\begin{aligned}\Omega' &= \Omega'_0 & \tilde{p}'_{\Omega} &= \tilde{p}'_{\Omega,0} \\ i' &= i'_0 & \tilde{p}'_i &= \tilde{p}'_{i,0} \\ \omega' &= \omega'_0 & \tilde{p}'_{\omega} &= \tilde{p}'_{\omega,0} \\ n' &= n'_0 & \tilde{p}'_n &= -\tilde{p}'_{M,0}\tau + \tilde{p}'_{n,0} \\ e' &= e'_0 & \tilde{p}'_e &= \tilde{p}'_{e,0} \\ M' &= n'_0\tau + M'_0 & \tilde{p}'_M &= \tilde{p}'_{M,0}\end{aligned}$$

where the prime in the coordinates denotes that they are the transformed ones. The previous relations can be expressed in a compact form as: $\xi' = (\mathbf{V}\tau + \mathcal{I})\xi'_0$ and $\tilde{p}'_{\xi} = (\mathbf{S}\tau + \mathcal{I})\tilde{p}'_{\xi,0}$, where \mathcal{I} is the identity matrix, the single non-null term in \mathbf{V} is $\mathcal{V}_{64} = 1$ and the single non-null term in \mathbf{S} is $\mathcal{S}_{46} = -1$. The Lie series can be used to connect these solutions with the solutions of the original problem. The perturbation theory provides us with explicit transformation of coordinates¹² :

$$\xi = \xi' + \varepsilon \frac{\partial \mathcal{W}}{\partial \tilde{p}'_{\xi}} = \xi' + 2\varepsilon \mathcal{D}(\xi')\tilde{p}'_{\xi} \quad (27)$$

$$\tilde{p}_{\xi} = \tilde{p}'_{\xi} + \varepsilon \frac{\partial \mathcal{W}}{\partial \xi'} \quad (28)$$

In this manner, an explicit analytic solution is obtained as a function of twelve parameters, the initial conditions of the transformed state and transformed adjoints. Depending on the available information on the boundary conditions, two different kind of problems can be treated: the reachable set of states with optimal control policy and the optimal trajectory between two given initial and final states.

REACHABLE SET WITH OPTIMAL CONTROL POLICY

The configuration space of the Hamiltonian is constituted by the state and adjoint spaces. Nevertheless, in this paper, the reachable set $R(x_0, \mathcal{U})$ is defined in the state space: $R(x_0, \mathcal{U}) \in \mathcal{X}$. The reachable set is the set of all states that are visited by any trajectories that start at x_0 and are obtained for some $\tilde{u} \in \mathcal{U}$ (\mathcal{U} is the set of allowable controls). This can be expressed formally as:

$$R(x_0, \mathcal{U}) = \{x_R \in \mathcal{X} \mid \exists \tilde{u} \in \mathcal{U} \text{ and } \exists t \in [0, \infty) \text{ such that } x(t) = x_R\}$$

Within this reachable set, we are interested in the reachable subset \tilde{R} of states that are accessible using the optimal continuous control. If the set of optimal controls is denoted as \mathcal{U}^* the optimal continuous reachable set can be formally defined as:

$$\tilde{R}(x_0, \mathcal{U}^*) = \{x_{OR} \in \mathcal{X} \mid \exists \tilde{u} \in \mathcal{U}^* \text{ and } \exists t \in [0, \infty) \text{ such that } x(t) = x_{OR}\}$$

The concept of the optimal reachable subset is similar to the range of a vehicle because the amount of fuel at disposal is directly related to the burning time when the thrust is continuous. Range is an important parameter and a useful mission planning tool¹⁸. In the same manner, the optimal control reachable set is a valuable piece of information for the mission planning. In addition, the optimal reachable set constitutes a limit of possible states that can be accessed with a given amount of propellant and a given maximum thrust.

The solution obtained in the previous section allows one to build an implicit description of the reachable set of optimal control policy. At $\tau = \tau_0$, the initial state $\boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0$ is known. In addition, $\boldsymbol{\xi}' = \boldsymbol{\xi}'_0$ and $\tilde{\boldsymbol{p}}'_{\boldsymbol{\xi}} = \tilde{\boldsymbol{p}}'_{\boldsymbol{\xi}_0}$. Therefore, from (27):

$$\tilde{\boldsymbol{p}}'_{\boldsymbol{\xi}_0} = \frac{1}{2\varepsilon} \mathcal{D}^{-1}(\boldsymbol{\xi}'_0) (\boldsymbol{\xi}_0 - \boldsymbol{\xi}'_0) \quad (29)$$

This equation indicates that the difference between the initial state and the initial transformed state is of order ε . On the other hand, at $\tau = \tau_F$, the transversality conditions imposes that $\tilde{\boldsymbol{p}}_{\boldsymbol{\xi}_F} = \mathbf{0}$. The implicit equation that must be solved for $\boldsymbol{\xi}'_0$ is:

$$\mathbf{f}(\boldsymbol{\xi}'_0; \boldsymbol{\xi}_0, \tau_F) = \frac{1}{2\varepsilon} (\mathcal{S}_{\tau_F} + \mathcal{I}) \mathcal{D}^{-1}(\boldsymbol{\xi}'_0) (\boldsymbol{\xi}_0 - \boldsymbol{\xi}'_0) + \varepsilon \frac{\partial \mathcal{W}}{\partial \boldsymbol{\xi}'} = \mathbf{0} \quad (30)$$

Note that in the previous function the last term $\frac{\partial \mathcal{W}}{\partial \boldsymbol{\xi}'}$ involves again a double infinite series that should be truncated for numerical computation. The explicit form of the reachable set turns out to be:

$$\boldsymbol{\xi} = [(\mathcal{V}_{\tau} + \mathcal{I}) + \mathcal{D}(\boldsymbol{\xi}')(\mathcal{S}_{\tau} + \mathcal{I})\mathcal{D}^{-1}(\boldsymbol{\xi}'_0)] \boldsymbol{\xi}'_0 \quad (31)$$

where $\boldsymbol{\xi}'_0$ is the solution of (30).

OPTIMAL TRAJECTORY BETWEEN GIVEN INITIAL AND FINAL STATES

In this section, the problem of obtaining the optimal trajectory between given initial and final states with fixed final time is addressed. This is the rendezvous problem and can be seen as a generalization of the Lambert's problem. In the Lambert's problem the initial and final positions can be connected by a Keplerian trajectory and the match in velocity is not required. In such a case, therefore, among the possible connecting trajectories between the initial and final position, the solution without thrust is optimal from the point of view of the mass consumption.

The problem is posed in the following terms: knowing that at $\tau = \tau_0$, the initial state $\boldsymbol{\xi}(\tau_0) = \boldsymbol{\xi}_0$ is known and at the known final time τ_F , the final state is also known $\boldsymbol{\xi}(\tau_F) = \boldsymbol{\xi}_F$, compute the trajectory and the control that minimizes the propellant consumption. Following the development of the previous section, this problem can be stated as: determine $\boldsymbol{\xi}'_0$ such that $\mathbf{f}(\boldsymbol{\xi}'_0; \boldsymbol{\xi}_0, \boldsymbol{\xi}_F, \tau_F) \equiv \mathbf{0}$, where $\mathbf{f}(\boldsymbol{\xi}'_0; \boldsymbol{\xi}_0, \boldsymbol{\xi}_F, \tau_F)$ is:

$$\mathbf{f}(\boldsymbol{\xi}'_0; \boldsymbol{\xi}_0, \boldsymbol{\xi}_F, \tau_F) = \boldsymbol{\xi}_F - [(\mathcal{V}_{\tau_F} + \mathcal{I}) + \mathcal{D}(\boldsymbol{\xi}')(\mathcal{S}_{\tau_F} + \mathcal{I})\mathcal{D}^{-1}(\boldsymbol{\xi}'_0)] \boldsymbol{\xi}'_0 \quad (32)$$

In this manner, the solution of the TPBVP is obtained as the zero of a six-dimension non-linear algebraic vector function. In principle, this fact represents a reduction in the complexity of the problem. Nevertheless, it is important to notice that there exist a couple of concerns related to the evaluation of \mathbf{f} . The first one is the definition of \mathcal{D} as a double series. Thus, it should be truncated at some order in both the Taylor series in mean anomaly and the series of Bessel functions in eccentricity. The choice of the order should be a trade-off between accuracy and computational cost. Moreover, although less relevant, in the evaluation of \mathcal{D} the Kepler equation must be solved for the given final time.

CONCLUSION AND FUTURE WORK

In this work, the perturbation theory is applied to the canonical mapping of the Hamiltonian system that results from the optimal control problem. In particular, we exploit the advantage of Lie transforms to build explicit canonical transformations. Lie transforms allow us to build the perturbed solution starting from the unperturbed integrable Hamiltonian system. A solution of the state as a series in terms of the small parameter is provided. In this manner, the reachable set of states and adjoints starting from a given initial condition following an optimal control policy can be built as a function of the small parameter, i.e., the order of magnitude of the thrust. Analytical expressions for the reachable set has been shown. Moreover, the particular solution of an optimal transfer for given final time and state can be found by solving a non-linear algebraic equation. In this way, the TPBVP is transformed into a problem of finding a zero of a non-linear vector function.

Future work involves the development of a fine-tuned algorithm to compute optimal reachable sets as well as optimal solutions of the TPBVP. The simulation and validation of the algorithm is also required. In addition, a formulation of the dynamics in terms of non-singular elements will be mandatory to deal with circular or polar orbits. The change in the state coordinates varies the derivation of the auxiliary functions but does not alter the main conclusions of the present development.

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APPENDIX A. AUXILIARY FUNCTIONS

This appendix gathers the auxiliary functions which have been used in the analytical description of the generating function. The generating function \mathcal{W} is written in compact form as:

$$\mathcal{W} = \tilde{\mathbf{p}}_{\xi}^T \mathcal{D}(\xi) \tilde{\mathbf{p}}_{\xi} \quad (33)$$

where

$$\mathcal{D}(\xi) = \mathcal{A}(\xi) + \sum_{k=0}^{\infty} \mathcal{B}_k(\xi) + \sum_{k=0}^{\infty} \mathcal{C}_k(\xi) \quad (34)$$

In turn, matrix $\mathcal{A}(\xi)$ and its components are functions of the state ξ :

$$\mathcal{A}(\xi) = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & 0 & 0 & 0 \\ \mathcal{A}_{12} & \mathcal{A}_{22} & \mathcal{A}_{23} & 0 & 0 & 0 \\ \mathcal{A}_{13} & \mathcal{A}_{23} & \mathcal{A}_{33} & 0 & \mathcal{A}_{35} & \mathcal{A}_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_{35} & 0 & \mathcal{A}_{55} & \mathcal{A}_{56} \\ 0 & 0 & \mathcal{A}_{36} & 0 & \mathcal{A}_{56} & \mathcal{A}_{66} \end{bmatrix} \quad (35)$$

where:

$$\begin{aligned} \mathcal{A}_{11} &= \frac{1}{a^2 n^3 \sin^2 i (1 - e^2)} (\mathcal{G}_{11a}(e, E) + \mathcal{G}_{11b}(e, E) \cos^2 \omega + \mathcal{G}_{11c}(e, E) \cos \omega \sin \omega) \\ \mathcal{A}_{12} &= \frac{1}{a^2 n^3 \sin i (1 - e^2)} (\mathcal{G}_{12a}(e, E) + \mathcal{G}_{12b}(e, E) \cos^2 \omega + \mathcal{G}_{12c}(e, E) \cos \omega \sin \omega) \\ \mathcal{A}_{13} &= \frac{\cos i}{a^2 n^3 \sin^2 i (1 - e^2)} (\mathcal{G}_{13a}(e, E) + \mathcal{G}_{13b}(e, E) \cos^2 \omega + \mathcal{G}_{13c}(e, E) \cos \omega \sin \omega) \\ \mathcal{A}_{22} &= \frac{1}{a^2 n^3 (1 - e^2)} (\mathcal{G}_{22a}(e, E) + \mathcal{G}_{22b}(e, E) \cos^2 \omega + \mathcal{G}_{22c}(e, E) \cos \omega \sin \omega) \\ \mathcal{A}_{23} &= \frac{\cos i}{a^2 n^3 \sin i (1 - e^2)} (\mathcal{G}_{23a}(e, E) + \mathcal{G}_{23b}(e, E) \cos^2 \omega + \mathcal{G}_{23c}(e, E) \cos \omega \sin \omega) \\ \mathcal{A}_{33} &= \frac{1}{a^2 n^3 \sin^2 i (1 - e^2) e^2} (\mathcal{G}_{33a}(e, E) + \cos^2 i (\mathcal{G}_{33b}(e, E) + \mathcal{G}_{33c} \cos^2 \omega + \mathcal{G}_{33d}(e, E) \cos \omega \sin \omega)) \\ \mathcal{A}_{35} &= \frac{2\sqrt{1 - e^2}}{a^2 n^3 e} \mathcal{G}_{35}(e, E) \\ \mathcal{A}_{36} &= \frac{2\sqrt{1 - e^2}}{a^2 n^3 e^2} \mathcal{G}_{36}(e, E) \\ \mathcal{A}_{55} &= \frac{1 - e^2}{a^2 n^3} \mathcal{G}_{55}(e, E) \end{aligned}$$

$$\mathcal{A}_{56} = \frac{2}{a^2 n^3 e} \mathcal{G}_{56}(e, E)$$

$$\mathcal{A}_{66} = \frac{2}{a^2 n^3 e} \mathcal{G}_{66}(e, E)$$

$$\begin{aligned} \mathcal{G}_{11a} &= \left(\frac{1}{2}e^2 + \frac{1}{4}\right) \sin(2E) + \left(-\frac{11}{4}e - e^3\right) \sin(E) + \frac{1}{2}E - \frac{1}{12}e \sin(3E) + 2e^2E \\ \mathcal{G}_{11b} &= \left(-\frac{1}{12}e^3 + \frac{1}{6}e\right) \sin(3E) + \left(-\frac{1}{4}e^2 - \frac{1}{2}\right) \sin(2E) + \left(\frac{5}{4}e^3 + \frac{5}{2}e\right) \sin(E) - \frac{5}{2}e^2E \\ \mathcal{G}_{11c} &= \sqrt{1-e^2} \left[\frac{1}{2}(1-e^2) \cos(2E) + \frac{1}{6}e \cos(3E) + \frac{5}{2}e \cos(E) - \frac{1}{2}e^2 - \frac{1}{2}\right] \\ \mathcal{G}_{12a} &= \sqrt{1-e^2} \left[\frac{1}{2}(e^2+1) \cos(2E) - \frac{1}{6}e \cos(3E) + \frac{1}{2}(e^2+1) - \frac{5}{2}e \cos(E)\right] \\ \mathcal{G}_{12b} &= \sqrt{1-e^2} \left[-(e^2+1) \cos(2E) + \frac{1}{3}e \cos(3E) - (1+e^2) + 5e \cos(E)\right] \\ \mathcal{G}_{12c} &= \left(-\frac{1}{3}e + \frac{1}{6}e^3\right) \sin(3E) + -5\left(e - \frac{1}{2}e^3\right) \sin(E) + \left(\frac{1}{2}e^2 + 1\right) \sin(2E) + 5e^2E \\ \mathcal{G}_{13a} &= -\left(e^2 + \frac{1}{2}\right) \sin(2E) + \left(\frac{11}{2}e + 2e^3\right) \sin(E) - 4e^2E - E + \frac{1}{6}e \sin(3E) \\ \mathcal{G}_{13b} &= -\left(\frac{1}{2} - e^2\right) \sin(2E) + \left(2e^3 + \frac{11}{2}e\right) \sin(E) - E + \frac{1}{6}e \sin(3E) - 4e^2E \\ \mathcal{G}_{13c} &= -5e \left(1 - \frac{1}{2}e^2\right) \sin(E) + \left(\frac{1}{2}e^2 + 1\right) \sin(2E) - \frac{1}{3}e \sin(3E) + \frac{1}{6}e^3 \sin(3E) + 5e^2E \\ \mathcal{G}_{22a} &= \frac{1}{12}e(1-e^2) \sin(3E) - \frac{1}{4}(1-e^2) \sin(2E) - \frac{1}{4}e(1-e^2) \sin(E) + \frac{1}{2}E(1-e^2) \\ \mathcal{G}_{22b} &= -\frac{1}{6}e \left(1 - \frac{1}{2}e^2\right) \sin(3E) + \frac{1}{2} \left(1 + \frac{1}{2}e^2\right) \sin(2E) - \frac{5}{2}e \left(1 + \frac{1}{2}e^2\right) \sin(E) + \frac{5}{2}e^2E \\ \mathcal{G}_{22c} &= \sqrt{1-e^2} \left[\frac{1}{2}(e^2+1) \cos(2E) - \frac{1}{6}e \cos(3E) + \frac{1}{2} + \frac{1}{2}e^2 - \frac{5}{2}e \cos(E)\right] \\ \mathcal{G}_{23a} &= \sqrt{1-e^2} \left[-\frac{1}{2}(1-e^2) \cos(2E) - \frac{1}{2} + \frac{1}{6}e \cos(3E) + \frac{5}{2}e \cos(E) - \frac{1}{2}e^2\right] \\ \mathcal{G}_{23b} &= \sqrt{1-e^2} \left[(e^2+1) \cos(2E) + e^2 - \frac{1}{3}e \cos(3E) - 5e \cos(E) + 1\right] \\ \mathcal{G}_{23c} &= \frac{1}{3} \left(1 - \frac{1}{2}e^2\right) \sin(3E) + 5e \left(1 + \frac{1}{2}e^2\right) \sin(E) - \left(1 + \frac{1}{2}e^2\right) \sin(2E) - 5e^2E \\ \mathcal{G}_{33a} &= \frac{1}{12}e(1-e^2) \sin(3E) + \frac{1}{2} \left(-\frac{3}{2} + \frac{5}{2}e^2 - e^4\right) \sin(2E) + \left(\frac{9}{4}e^3 - \frac{5}{4}e - e^5\right) \sin(E) + \\ &\quad + \arctan\left(\frac{\cos(E)-1}{\sin(E)}\right) (-4e^4 + 9e^2 - 5) \\ \mathcal{G}_{33b} &= 5e \left(\frac{1}{4} - e^2\right) \sin(E) + \left(\frac{3}{4} + e^4 - e^2\right) \sin(2E) - \frac{1}{12}e \sin(3E) + \arctan\left(\frac{\cos(E)-1}{\sin(E)}\right) 5(1-2e^2) \\ \mathcal{G}_{33c} &= \frac{5}{2}e \left(1 + \frac{1}{2}e^2\right) \sin(E) - \frac{1}{2} \left(1 + \frac{1}{2}e^2\right) \sin(2E) + \frac{1}{6}e \left(1 - \frac{1}{2}e^2\right) \sin(3E) + 5e^2 \arctan\left(\frac{\cos(E)-1}{\sin(E)}\right) \\ \mathcal{G}_{33d} &= e^2 \sqrt{1-e^2} \left(-\frac{1}{2}(1+e^2) \cos(2E) + \frac{5}{2}e \cos(E) + \frac{1}{6}e \cos(3E) - \frac{1}{2}(1+e^2)\right) \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{35} &= \frac{1}{4}(e^2 - 3) \cos(2E) + \frac{1}{12}e \cos(3E) + \frac{5}{4}e \cos(E) + \frac{1}{4}e^2 - \frac{3}{4} \\
\mathcal{G}_{36} &= e \left(\frac{13}{4} + e^2 \right) \sin(E) + \left(\frac{3}{4} - e^2 \right) \sin(2E) - \frac{1}{12}e \sin(3E) - e^2E - \frac{5}{2}E \\
\mathcal{G}_{55} &= -\frac{1}{12}e \sin(3E) - \frac{15}{4} \sin(E)e + \frac{3}{4} \sin(2E) + \frac{5}{2}E \\
\mathcal{G}_{56} &= -\frac{1}{12}e(1 - e^2) \cos(3E) + \frac{3}{4}(1 - e^2)^2 \cos(2E) + \frac{3}{4}e(1 - e^2) \cos(E) + \frac{3}{4}(1 - e^2)^2 \\
\mathcal{G}_{66} &= \left(2e^4 + \frac{5}{2} + \frac{11}{2}e^2 \right) E + \left(-\frac{39}{4}e^3 - \frac{21}{4}e \right) \sin(E) + \frac{3}{2} \left(e^4 - \frac{1}{2} + \frac{3}{2}e^2 \right) \sin(2E) + \frac{1}{3}e \left(\frac{1}{4} - e^4 - \frac{1}{4}e^2 \right)
\end{aligned}$$

Matrix $\mathcal{B}_k(\xi)$ and its components are functions of the state ξ :

$$\mathcal{B}_0(\xi) = \frac{3M}{2a^2n^3}(1 - e^2)[e(1 + \mathcal{D}_0) - \mathcal{C}_0] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & 2n & 0 & M \\ 0 & 0 & 0 & 0 & M & 0 \end{bmatrix} \quad (36)$$

$$\mathcal{B}_k(\xi) = \frac{3}{ka^2n^3}e \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{F}_{Ak}n \cos(kM) & 0 & \frac{1}{k}\mathcal{F}_{Ak} \sin(kM) \\ 0 & 0 & \mathcal{F}_{Ak}n \cos(kM) & 0 & -\mathcal{F}_{Ck}n \sin(kM) & \mathcal{F}_{Ck}n \cos(kM) \\ 0 & 0 & 0 & -\mathcal{F}_{Ck}n \sin(kM) & 0 & \frac{1}{2}\mathcal{F}_{Bk}n \cos(kM) \\ 0 & 0 & 0 & \mathcal{F}_{Ck}n \cos(kM) & \frac{1}{2}\mathcal{F}_{Bk}n \cos(kM) & \frac{1}{k}\mathcal{F}_{Bk} \sin(kM) \end{bmatrix} \quad (37)$$

Matrix $\mathcal{C}_k(\xi)$ and its components are also functions of the state ξ :

$$\mathcal{C}_0(\xi) = -\frac{3M}{a^2n^3} [\mathcal{C}_0(e - 2) + e^2\mathcal{D}_0] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & \frac{3M}{2n} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3M}{2n} & 0 & \frac{M^2}{n^2} \end{bmatrix} \quad (38)$$

$$\mathcal{C}_k(\xi) = \frac{9}{a^2n} [\mathcal{C}_k(e - 2) + e^2\mathcal{D}_k] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\sin(kM)}{2k} & 0 & \frac{1}{2} \frac{\cos(kM)}{k^2n} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \frac{\cos(kM)}{k^2n} & 0 & \frac{\sin(kM)}{k^3n^2} \end{bmatrix} \quad (39)$$

The functions $\mathcal{F}_{Ak}, \mathcal{F}_{Ck}, \mathcal{F}_{Bk}$ has the following form:

$$\begin{aligned}
\mathcal{F}_{Ak} &= A_k(e^2 + 4) - 2e\mathcal{B}_k(e^2 + 1) + \mathcal{E}_k e^4 \\
\mathcal{F}_{Bk} &= \sqrt{1 - e^2}(2A_k - e\mathcal{B}_k) \\
\mathcal{F}_{Ck} &= e(1 - e^2)(2\mathcal{C}_k - e\mathcal{D}_k)
\end{aligned}$$

The coefficients that appear in the description of the matrices \mathcal{B} and \mathcal{C} correspond to the coefficients of the Taylor series in mean anomaly of the functions:

$$\begin{aligned}\frac{1}{(1 - e \cos E)^2} &= \sum_{k=1}^{\infty} \mathcal{O}_k \cos(kM) \\ \frac{\sin E}{(1 - e \cos E)^2} &= \sum_{k=1}^{\infty} \mathcal{A}_k \sin(kM) \\ \frac{\sin 2E}{(1 - e \cos E)^2} &= \sum_{k=1}^{\infty} \mathcal{B}_k \sin(kM) \\ \frac{\cos E}{(1 - e \cos E)^2} &= \sum_{k=0}^{\infty} \mathcal{C}_k \cos(kM) \\ \frac{\cos 2E}{(1 - e \cos E)^2} &= \sum_{k=0}^{\infty} \mathcal{D}_k \cos(kM) \\ \frac{\sin 3E}{(1 - e \cos E)^2} &= \sum_{k=1}^{\infty} \mathcal{E}_k \sin(kM)\end{aligned}$$

Explicit expressions of these coefficients are:

$$\begin{aligned}\mathcal{O}_k &= 2\mathcal{J}_k(ke) \\ \mathcal{A}_k &= \frac{2}{\sqrt{1-e^2}} \sum_{n=-\infty}^{n=\infty} \mathcal{J}_n(-ke) \left(\beta^{|n+k-1|} - \beta^{n+k+1} \right) \\ \mathcal{B}_k &= \frac{2}{\sqrt{1-e^2}} \sum_{n=-\infty}^{n=\infty} \mathcal{J}_n(-ke) \left(\beta^{|n+k-2|} - \beta^{n+k+2} \right) \\ \mathcal{C}_0 &= \frac{1}{e\sqrt{1-e^2}} - \frac{1}{e} \\ \mathcal{C}_k &= \frac{2}{\sqrt{1-e^2}} \sum_{n=-\infty}^{n=\infty} \mathcal{J}_n(-ke) \left(\beta^{|n+k-1|} + \beta^{n+k+1} \right) \quad k \geq 1 \\ \mathcal{D}_0 &= \frac{2}{e^2\sqrt{1-e^2}} - \frac{2}{e^2} - \frac{1}{\sqrt{1-e^2}} \\ \mathcal{D}_k &= \frac{2}{\sqrt{1-e^2}} \sum_{n=-\infty}^{n=\infty} \mathcal{J}_n(-ke) \left(\beta^{|n+k-2|} + \beta^{n+k+2} \right) \quad k \geq 1 \\ \mathcal{E}_k &= \frac{2}{\sqrt{1-e^2}} \sum_{n=-\infty}^{n=\infty} \mathcal{J}_n(-ke) \left(\beta^{|n+k-3|} - \beta^{n+k+3} \right)\end{aligned}$$

where,

$$\beta = \frac{1 - \sqrt{1 - e^2}}{e}$$

and \mathcal{J}_n is a Bessel function of the first kind.