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Section 1

THEORETICAL AND EXPERIMENTAL STUDY OF SPACE STRUCTURES FOR STATIC AND SEISMIC EFFECTS

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DYNAMIC ANALYSIS OF TRANSLATIONAL SHELLS

In this paper a dynamic analysis of translational shells is presented. The general linear shell theory is used in conjunction with additional shallow and curved plate approximations. In order to apply some type of extended Levy solution, the shell is assumed to be limited by a rectangular plan form, with two opposite edges simply supported (gable boundary conditions).

First, the shells free vibrations are studied in the usual way, obtaining for each Fourier term the natural frequencies as solutions of a transcendental equation. However, solving these equations arises enormous computational difficulties. This paper deals specifically with this problem, trying to reduce its dimension by a discretization procedure, in the shell dynamic characteristics, namely the mass. The shell mass is lumped along a family of coordinate lines. Therefore, the natural frequencies for each harmonic term can be found from the solution of a typical matrix eigenvalues problem and standard numerical techniques can be applied. The shell response to forced vibrations, particularly to earthquake excitation, can be determined by using conventional procedure either in the time or in the frequency domain.

Finally, extending the above procedure, any system of translational shells under dynamic loading can be studied. Then, by using matrix methods, a general computer program is written and applied to some illustrative examples. Numerical results has been obtained in two cases: circular cylindrical shell and box girder bridge.

Symbols

a_1, a_2	= coordinate lines of the middle surface of the shell
u_1, u_2, w	= displacements at a point of the middle surface of the shell
K_1, K_2	= curvature of the coordinate lines of the middle surface of the shell
X_1, X_2, Z	= forces per unit area on the middle surface of shell in the directions a_1, a_2 and normal to the surface
$X_{i,j}$	= derivative of X_i with respect to the coordinate a_j
N_{11}, N_{12}, N_{22}	= stress-resultants (extensional group)
M_{11}, M_{12}, M_{22}	= stress-couples (flexural group)
Q_1, Q_2	= shear stress-resultants
R_1, R_2	= Kirchhoff shear stress-resultants
e_{11}, e_{12}, e_{22}	= extensional strain-resultants
k_{11}, k_{12}, k_{22}	= flexural strain-resultants

h = thickness of the shell

E = Young's modulus

ν = Poisson's ratio

Φ = stress function

Introduction

Using some kind of mass lumping procedure, it is possible to study the dynamic behaviour of a large family of translational shells. This paper represents a natural application of ideas, already published[1,2,3]. For this reason only main equations and results will be presented.

General Governing Equations

The elastic shallow shell theory assuming curved plate approximations is described by means of the following differential equations besides the existing boundary conditions:

General governing equations

$$\frac{1}{E.h}\nabla^4\Phi + \nabla_k^2 w = \frac{1}{E.h}\left[\int X_{2,11}\,da_2 + \int X_{1,22}\,da_1 + \nu(X_{2,2} + X_{1,1})\right]$$

$$D\,\nabla^4 w - \nabla_k^2\Phi = -K_1\int X_1 da_1 - K_2\int X_2 da_2 + Z \qquad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial a_1^2} + \frac{\partial^2}{\partial a_2^2}$ is the Laplacian operator and

$$\nabla_k^2 = K_2\frac{\partial^2}{\partial a_1^2} + K_1\frac{\partial^2}{\partial a_2^2}$$

If only small normal vibrations are studied, the above w – Φ formulation is convenient. Otherwise, the formulation in three differential equations in u_1, u_2 and w would be more suitable.

Natural Frequencies

In this case $X_1 = 0$ $X_2 = 0$ $Z = \rho\cdot h\,\frac{\partial^2 w}{\partial t^2}$

where ρ is the mass per unit area.

Introducing the Ambartsumyan function $\overline{W}$ defined as

$$w = \nabla^4 \overline{W}$$

$$\Phi = -Eh \nabla_k^2 \overline{W}$$

The governing equations (1) reduce to one single equation: to find the natural frequencies ω, let

$$\overline{W} = W(a_1, a_2) e^{i\omega t}$$

and the above equations become

$$\nabla^8 W + \frac{12(1 - \nu^2)}{h^2} \nabla_k^2 W - p^2 \nabla^4 W = 0 \qquad (2)$$

where $p^2 = \frac{\rho \cdot h}{D} \omega^2$

Assuming gable boundary conditions along the edges to be $a_1 = 0$ and $a_1 = \ell_1$ the solution can be expressed in the following form:

$$W(a_1, a_2) = \sum_{m=1}^{\infty} F_m(a_2) \sin \lambda_m a_1 \qquad (3)$$

where $\lambda_m = \frac{m\pi}{\ell_1}$

After introducing expresion (3) into equation (2) this equation becomes:

$$\left[\frac{d^2}{da_2^2} - \lambda_m^2\right]^2 F_m(a_2) + \frac{12(1 - \nu^2)}{h^2} \left[K_1 \frac{d^2}{da_2^2} - K_2 \lambda_m^2\right]^2 F_m(a_2) - p^2 \left[\frac{d^2}{da_2^2} - \lambda_m^2\right] F_m(a_2) = 0$$

an ordinary differential equation with constant coefficients and where characteristic equation is as follows:

$$\left[y^2 - \lambda^2\right]^4 + \frac{12\,(1 - \nu^2)}{h^2}\left[K_1 y^2 - K_2\,\lambda^2\right]^2 - p^2\left[y^2 - \lambda^2\right]^2 = 0$$

or with the new variable $\theta = y^2 - \lambda^2$

$$\theta^4 + \frac{12(1 - \nu^2)}{h^2}\left[K_1\theta - (K_1 - K_2)\,\lambda^2\right]^2 - p^2\,\theta^2 = 0$$

This equation can be solved algebraically, by a standard procedure.

The eight roots of the characteristic equation are:

$$y_1 = r_1 + is_1 \qquad y_5 = r_2 + is_2$$
$$y_2 = r_1 - is_1 \qquad y_6 = r_2 - is_2$$
$$y_3 = -r_1 + is_1 \qquad y_7 = -r_2 + is_2$$
$$y_4 = -r_1 - is_1 \qquad y_8 = -r_2 - is_2$$

then the general solution is

If $K_1 \neq K_2$

$$F_m(a_2) = e^{-r_1 a_2}\left[A_1 \cos s_1 a_2 + A_2 \sin s_1 a_2\right] +$$

$$+ e^{-r_2 a_2}\left[A_3 \cos s_2 a_2 + A_4 \sin s_2 a_2\right] +$$

$$+ e^{-r_1 b_2}\left[A_5 \cos s_1 b_2 + A_6 \sin s_1 b_2\right] +$$

$$+ e^{-r_2 b_2}\left[A_7 \cos s_2 a_2 + A_8 \sin s_2 b_2\right]$$

If $K_1 = K_2$

$$F_m(a_2) = e^{-r_1 a_2}\left[A_1 \cos s_1 a_2 + A_2 \sin s_1 a_2\right] +$$

$$+ e^{-\lambda_m a_2}\left[A_3 a_2 + A_4\right] + e^{-r_1 b_2}\left[A_5 \cos s_1 b_2 + \right.$$

$$\left. + A_6 \sin s_1 b_2\right] + e^{-\lambda_m b_2}\left[A_7 b_2 + A_8\right]$$

where $b_2 = \ell_2 - a_2$

$A_1, A_2, A_3, \ldots\ldots\ldots A_8$ are arbitrary constants to be determined by consideration of the boundary conditions along the edges $a_2 = 0$ and $a_2 = \ell_2$.

In order to formulate these boundary conditions it is convenient to introduce the following two vectors:

Displacement vector: $\underline{d} = \left[u_1 \; u_2 \; w \;\; w_{,2}\right]^T$

Force vector: $\underline{P} = \left[N_{21} \; N_{22} \; R_2 \; M_{22}\right]^T$

After expanding these vectors into a Fourier series it is obtained:

$$\underline{d} = \begin{bmatrix} u_1 \\ u_2 \\ w \\ w_{,2} \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} u_{1m} \cos \lambda_m a_1 \\ u_{2m} \sin \lambda_m a_1 \\ w_m \sin \lambda_m a_1 \\ w_{,2m} \sin \lambda_m a_1 \end{bmatrix}$$

where u_{1m}, u_{2m}, w_m and $w_{,2m}$ are functions of the coordinate a_2 only.

In Ref.[3] it is shown:

$$\underline{d}_m = \begin{bmatrix} u_{1m} \\ u_{2m} \\ w_m \\ w_{,2m} \end{bmatrix} = \underline{G}_d \left[\underline{B}\;\underline{P}\,(a_2) \quad \underline{C}\;\underline{P}\,(b_2)\right] A \qquad (4)$$

where

$\underline{G}_d$ is the matrix (dimension 4 x 8) given in Table I

$\underline{B} = \left[\underline{B}^{(k)}\right]$ (dimension 8 x 4) and $k = 1, 2, \ldots\ldots\ldots 8$

$\underline{B}^{(k)}$ is a row-matrix (dimension 1 x 4) and its expression is $\underline{B}^{(k)} = \left[B_1^{(k)}\ B_2^{(k)}\ B_3^{(k)}\ B_4^{(k)}\right]$ if $K_1 \neq K_2$

$$\underline{B}^{(k)} = \left[B_1^{(k)}\ B_2^{(k)}\ (-1)^k\ \lambda_m^k \quad (-1)^k\ k\,\lambda_m^k\right] \qquad \text{if } K_1 = K_2$$

with

$$B_1^{(k)} = (-1)^k\ (r_1^2 + s_1^2)^{\frac{k}{2}}\ \cos\ (k.\text{arc tan}\ \frac{s_1}{r_1})$$

$$B_2^{(k)} = (-1)^k\ (r_1^2 + s_1^2)^{\frac{k}{2}}\ \sin\ (k.\text{arc tan}\ \frac{s_1}{r_1})$$

$$B_3^{(k)} = (-1)^k\ (r_2^2 + s_2^2)^{\frac{k}{2}}\ \cos\ (k.\text{arc tan}\ \frac{s_2}{r_2})$$

$$B_4^{(k)} = (-1)^k\ (r_2^2 + s_2^2)^{\frac{k}{2}}\ \sin\ (k.\text{arc tan}\ \frac{s_2}{r_2})$$

$\underline{C} = \left[(-1)^{k+1} \cdot \underline{B}^{(k)}\right]$ (dimension 8x4) and k = 1, 2, ... 8

$\underline{P}(a_2) = \begin{bmatrix} \underline{P}_1\ (a_2) & \underline{0} \\ \underline{0} & \underline{P}_2(a_2) \end{bmatrix}$ (dimension 4x4) if $K_1 \neq K_2$

$\underline{P}(a_2) = \begin{bmatrix} \underline{P}_1(a_2) & \underline{0} \\ \underline{0} & \underline{P}_3(a_2) \end{bmatrix}$ (dimension 4x4) if $K_1 = K_2$

$\underline{P}_1(a_1) = \exp(-r_1a_2)\begin{bmatrix} \cos\ s_1a_1 & \sin\ s_1a_1 \\ -\sin\ s_1a_1 & \cos\ s_1a_1 \end{bmatrix}$ (dimension 2x2)

$\underline{P}_2(a_2) = \exp(-r_2a_2)\begin{bmatrix} \cos\ s_2a_2 & \sin\ s_2a_2 \\ -\sin\ s_2a_2 & \cos\ s_2a_2 \end{bmatrix}$ (dimension 2x2)

$$\underline{P}_3(a_2) = \exp(-\lambda_m a_2) \begin{bmatrix} 1 & a_2 \\ 0 & 1 \end{bmatrix} \qquad \text{(dimension 2x2)}$$

$\underline{A} = \left[A_1, A_2, A_3, \ldots\ldots\ldots, A_8\right]^T$ column-matrix of eight arbitrary constants.

Similarly the force vector can be expressed as:

$$\underline{P} = \begin{bmatrix} N_{21} \\ N_{22} \\ R_2 \\ M_{22} \end{bmatrix} = \sum_{m=1}^{\infty} \begin{bmatrix} N_{21m} \cos \lambda_m a_1 \\ N_{22m} \sin \lambda_m a_1 \\ R_{2m} \sin \lambda_m a_1 \\ M_{22m} \sin \lambda_m a_1 \end{bmatrix}$$

where N_{21m}, N_{22m}, R_{2m} and M_{22m} are functions of the coordinate a_2 only.

The expression of the m-th term coefficients is:

$$\underline{P}_m = \begin{bmatrix} N_{21m} \\ N_{22m} \\ R_{2m} \\ M_{22m} \end{bmatrix} = \underline{G}_p \left[\underline{B}\ \underline{P}(a_2) \quad \underline{C}\ \underline{P}(b_2)\right] \underline{A} \qquad (5)$$

where $\underline{G}_p$ is the matrix (dimension 4 x 8) given in Table II.

The matrices $\underline{B}$, $\underline{C}$, $\underline{P}(a_2)$, $P(b_2)$ and $\underline{A}$ have been already defined.

General time independent and homogeneous boundary conditions can be included in the following expression

$$\underline{K}_d \begin{bmatrix} \underline{d}_1 \\ \underline{d}_2 \end{bmatrix} + \underline{K}_p \begin{bmatrix} \underline{P}_1 \\ \underline{P}_2 \end{bmatrix} = \underline{0} \qquad (6)$$

where $\underline{d}_1$, $\underline{d}_2$ are the displacement vectors along the coordinate lines $a_2 = 0$ and $a_2 = l_2$, respectively. Similarly $\underline{P}_1$ and $\underline{P}_2$ are the force vectors along the coordinate lines a_1 and a_2.$\underline{K}_d$ and $\underline{K}_p$ are constant given matrices (dimension 8 x 8).

The above condition (6) implies that for every m-th Fourier term the following equation is fulfilled:

$$\underline{K}_d \begin{bmatrix} \underline{d}_{1m} \\ \\ \underline{d}_{2m} \end{bmatrix} + \underline{K}_p \begin{bmatrix} \underline{p}_{1m} \\ \\ \underline{p}_{2m} \end{bmatrix} = \underline{0} \qquad (7)$$

where $\underline{d}_{1m} = \underline{d}_m$ and $\underline{p}_{1m} = \underline{p}_m$ when $a_2 = 0$

$\underline{d}_{2m} = \underline{d}_m$ and $\underline{p}_{2m} = \underline{p}_m$ when $a_2 = l_2$

Using equations (4) and (5) it is obtained:

$$\begin{aligned} \underline{d}_{1m} &= \left[\underline{G}_d \, \underline{B} \qquad \underline{G}_d \, \underline{C} \, \underline{P}(l_2)\right]\underline{A} \\ \underline{d}_{2m} &= \left[\underline{G}_d \, \underline{B} \, \underline{P}(l_2) \qquad \underline{G}_d \, \underline{C}\right]\underline{A} \\ \underline{p}_{1m} &= \left[\underline{G}_p \, \underline{B} \qquad \underline{G}_p \, \underline{C} \, \underline{P}(l_2)\right]\underline{A} \\ \underline{p}_{2m} &= \left[\underline{G}_p \, \underline{B} \, \underline{P}(l_2) \qquad \underline{G}_p \, \underline{C}\right]\underline{A} \end{aligned}$$

Now equation (7) can be changed as follows:

$$\left\{\underline{K}_d \begin{bmatrix} \underline{G}_d \, \underline{B} & \underline{G}_d \, \underline{C} \, \underline{P}(l_2) \\ \underline{G}_d \, \underline{B} \, \underline{P}(l_2) & \underline{G}_d \, \underline{C} \end{bmatrix} + \underline{K}_p \begin{bmatrix} \underline{G}_p \, \underline{B} & \underline{G}_p \, \underline{C} \, \underline{P}(l_2) \\ \underline{G}_p \, \underline{B} \, \underline{P}(l_2) & \underline{G}_p \, \underline{C} \end{bmatrix}\right\}\underline{A} = \underline{0}$$

or

$$\begin{bmatrix} \underline{H}_{11} \, \underline{B} & \underline{H}_{12} \, \underline{C} \, \underline{P}(l_2) \\ \underline{H}_{21} \, \underline{B} \, \underline{P}(l_2) & \underline{H}_{22} \, \underline{C} \end{bmatrix}\underline{A} = \underline{0}$$

where

$$\underline{H}_{ij} = \underline{K}_{dij} \, \underline{G}_d + \underline{K}_{pij} \, \underline{G}_p \qquad (i, j = 1, 2)$$

assuming the partitioned matrices

$$\underline{K}_d = \begin{bmatrix} \underline{K}_{d11} & \underline{K}_{d12} \\ \underline{K}_{d21} & \underline{K}_{d22} \end{bmatrix} \quad \text{and } \underline{K}_p = \begin{bmatrix} \underline{K}_{p11} & \underline{K}_{p12} \\ \underline{K}_{p21} & \underline{K}_{p22} \end{bmatrix}$$

the matrix equation (8) corresponds to a system of eight linear simultaneous equations, where nontrivial solutions exist when

$$\det \begin{bmatrix} \underline{H}_{11}\ \underline{B} & \underline{H}_{12}\ \underline{C}\ \underline{P}(l_2) \\ \underline{H}_{21}\ \underline{B}\ \underline{P}(l_2) & \underline{H}_{12}\ \underline{C} \end{bmatrix} = 0 \qquad (9)$$

This transcendental equation (9) in p^2, gives all the natural frequencies corresponding to m or number of half-waves along the a_1 direction:

$$P_{mn} = \sqrt{\frac{\rho h}{D}} \cdot \omega_{mn}$$

For every solution or root of (9) there exists a set of nontrivial values of the constants A_i (i = 1, 2, ... 8) satisfying the homogeneous simultaneous equations (8) and representing the corresponding modes of vibration of the shell. They can be designated by $\overline{W}_{mn}$, where m is the number of the harmonics along the a_1 direction and n is the root number, ordered according to increasing values.

<u>Forced Vibrations</u>

The load is restricted to the case $P(a_1, a_2) \cdot \varphi(t)$. The problem is to find the solution of

$$\nabla^8 \overline{W} + \frac{12(1-\nu^2)}{h^2} \nabla_k^4 \overline{W} + \frac{\rho h}{D} \nabla^4 \left(\frac{\partial^2 \overline{W}}{\partial t^2} \right) = \frac{P(a_1, a_2)}{D} \varphi(t) \quad (1$$

with the boundary conditions given by equation (6).

Let ω_{mn} and $\overline{W}_{mn}$ be the natural frequency and corresponding mode of vibration (m, n = 1, 2, ...) obtained from the free vibration problem, i.e. satisfying the homogeneous equation

$$\nabla^8 \overline{W}_{mn} + \frac{12(1-\nu^2)}{h^2} \nabla_k^4 \overline{W}_{mn} - \frac{\rho h}{D} \omega_{mn} \nabla^4 \overline{W}_{mn} = 0 \qquad (11)$$

and the boundary conditions (6).

It is interesting to point out that the actual mode of deflection is:

$$W_{mn} = \nabla^4 \overline{W}_{mn} = \left[F_m^{IV}(a_2) - 2\lambda_n^2 F_m''(a_2) + \lambda_n^4 F_m(a_2) \right] \sin \lambda_n a_2 = f_m^{(n)}(a_2) \sin \lambda_n a_2$$

The set of functions $f_m^{(n)}(a_2)$ is a complete set of linear independent functions, but, in general, is not an orthogonal set.

It is possible to expand the load in the following form:

$$p(a_1, a_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{nm} \, w_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{nm} \, \nabla^4 \overline{W}_{mn}$$

where g_{nm} are constant coefficients.

Assuming the solution of equation (10) is

$$\overline{W}(a_1, a_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overline{W}_{mn} \, T_{mn}(t)$$

where $T_{mn}(t)$ is a function of time t to be determined.
Then the mn-th term in (10) becomes

$$T_{mn}(t) \left[\nabla^8 \overline{W}_{mn} + \frac{12(1-\nu^2)}{h^2} \nabla_k^4 \overline{W}_{mn} \right] +$$

$$+ \frac{\rho h}{D} T''_{mn}(t) \nabla^4 \overline{W}_{mn} = \frac{g_{nm}}{D} \nabla^4 \overline{W}_{mn} \varphi(t)$$

i.e. considering equation (11), it is obtained:

$$T''_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = \frac{g_{nm}}{\rho h} \cdot \varphi(t) \tag{12}$$

If some orthogonal damping exists, then the above equation (12) becomes

$$T''_{mn}(t) + 2 \eta_{mn} \omega_{mn} T'_{mn}(t) + \omega_{mn}^2 T_{mn}(t) = \frac{g_{nm}}{\rho h} \varphi(t)$$

The solution of this equation can be obtained from the Duhamel integral:

$$T_{mn}(t) = \frac{1}{\rho h \cdot \omega^*_{mn}} \int_0^t \varphi(r) \; h^* \cdot (t - r) \, dr +$$

$$+ e^{-\eta_{mn} \omega_{mn} t} \left[T_{mn}(0) \cos \omega^*_{mn} t + \right.$$

$$\left. + \frac{T'_{mn}(0) + \eta_{mn} \omega_{mn} T_{mn}(0)}{\omega^*_{mn}} \sin \omega^*_{mn} t \right] \tag{13}$$

where

$$h^*(t) = \left\{ e^{-\eta_{mn}\,\omega_{mn} t} \quad \sin \omega^*_{mn} t \right\} \begin{matrix} \text{if } t < 0 \\ \\ \text{if } t > 0 \end{matrix} \qquad \text{and} \quad \omega^*_{mn} = \omega_{mn}\sqrt{1 - \eta^2_{mn}}$$

$T_{mn}(0)$ and $T'_{mn}(0)$ are known values corresponding to the initial time $t = 0$ and they can be calculated from the equations

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \nabla^4 \overline{W}_{mn} \; T_{mn}(0) = \nabla^4 \overline{W}(a_1, a_2, 0) = w(a_1, a_2, 0)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \nabla^4 \overline{W}_{mn} \; T'_{mn}(0) = \nabla^4 \frac{\partial}{\partial t} \overline{W}(a_1, a_2, t)\Big|_{t=0} =$$

$$= \frac{\partial}{\partial t} w(a_1, a_2, 0)$$

where $w(a_1, a_2, 0)$ and $\frac{\partial}{\partial t} w(a_1, a_2, 0)$ are the initial shape of the shell and its initial velocity field.

Approximated Solution

Solution of the transcendental equation (9) presents an enormous computational task with many numerical difficulties involved.

In order to overcome this troublesome solution the following approximated procedure is shown.

The mass of the shell is assumed to be lumped along a discrete set of coordinate lines (a_1 = constant), but the shell flexibility remains unchanged.

The idea is similar to the approximated analysis of beams by concentrating mass in a set of intermediate extra nodes.

In order to mathematically formulate this procedure it is convenient to deduct the shell stiffness matrix for the m-th Fourier term.

This shell stiffness matrix $\underline{K}_{sm}$ can be defined by the equation:

$$\begin{bmatrix} \underline{P}_{1m} \\ \underline{P}_{2m} \end{bmatrix} = \underline{K}_{sm} \begin{bmatrix} \underline{d}_{1m} \\ \underline{d}_{2m} \end{bmatrix}$$

where $\underline{P}_{1m}$, $\underline{P}_{2m}$, $\underline{d}_{1m}$ and $\underline{d}_{2m}$ have been already described.

Then using the notation already introduced it is obtained

$$\underline{K}_{sm} = \begin{bmatrix} -\underline{G}_p \, \underline{B} & -\underline{G}_p \, \underline{C} \, \underline{P}(l_2) \\ \underline{G}_p \, \underline{B} \, \underline{P}(l_2) & \underline{G}_p \, \underline{C} \end{bmatrix} \begin{bmatrix} \underline{G}_d \, \underline{B} & \underline{G}_d \, \underline{C} \, \underline{P}(l_2) \\ \underline{G}_d \, \underline{B} \, \underline{P}(l_2) & \underline{G}_d \, \underline{C} \end{bmatrix}^{-1}$$

If the shell mass is considered concentrated along the edges $a_1 = 0$ and $a_1 = l_2$ there exists a static situation. Then the expressions of r_1, s_1, r_2 and s_2 are independent on the unknown p, and they can be calculated previously by using standard formulae for solution of algebraic equations.

Let the shell, presented in the Fig. 1 be divided in a set of shells by means of different coordinate lines a_1 = constant. For each of these shells the expressions of its stiffness matrix can be obtained, when the m-th harmonic term is considered.

The overall shell stiffness matrix can be calculated from these elementary stiffness matrices, according to the standard rules of the matrix structural analysis. The boundary conditions can be introduced also by using conventional technique.

Let the total shell stiffness matrix be $\underline{K}^*_{sm}$ and the following matrix equation holds:

$$\underline{P}^*_{sm} = \underline{K}^*_{sm} \, \underline{D}^*_{sm}$$

where $\underline{P}^*_{sm}$ and $\underline{D}^*_{sm}$ are the vectors collecting all the applied nodal forces (i.e. acting along each coordinate line a_1) and all the corresponding unknown nodal displacements, respectively.

The vector $\underline{P}^*_{sm}$ is composed in general by the external nodal forces, the inertial forces and the damping (it can be assumed viscosous) forces. Then the equation (14) can be expressed in the following way:

$$\underline{K}^*_{sm} \, \underline{D}^*_{sm} = \underline{P}_{sm} - \underline{M} \, \ddot{\underline{D}}^*_{sm} - \underline{C}_m \, \dot{\underline{D}}^*_{sm} \qquad (15)$$

where

$\underline{P}_{sm}$ = known vector of external forces

$\underline{M}$ = mass matrix

$\underline{C}_m$ = damping matrix

$\dot{\underline{D}}_{sm}$ and $\ddot{\underline{D}}_{sm}$ = the velocity and acceleration vectors.

The equation (15) is a typical vibration equation of multidegree-of-freedom systems.

The standard computational steps, needed to solve the equation (15), are summarised as follows:

1) Find the natural frequencies ω_{im} and the modes of vibration $\underline{\Phi}_{im}$ corresponding to the order i of the harmonic term m., i.e. the solution of the algebraic eigenvalue problem

$$\underline{K}^*_{sm} \underline{\Phi}_{im} = \omega^2_{im} \underline{M} \underline{\phi}_{im}$$

2) Assuming an orthogonal damping matrix C, i.e. such that

$$\underline{\Phi}^T_{im} \underline{C}_m \underline{\phi}_{jm} = 0 \qquad \text{if } i \neq j$$

Then the following equations hold for every mode i

$$\underline{\Phi}^T_{im} \underline{M} \underline{\phi}_{im} = \overline{M}_i$$

$$\underline{\Phi}^T_{im} \underline{C}_m \underline{\phi}_{im} = C_i = 2 \eta_{im} \omega_{im} \overline{M}_i$$

$$\underline{\Phi}^T_{im} \underline{K}_{sm} \phi_{im} = \omega^2_{im} \overline{M}_i$$

$$\underline{\Phi}^T_{im} \underline{P}_{sm}(t) = \overline{\Pi}_{sm}(t)$$

3) The solution of equation (15) is given by the expression

$$\underline{D}^*_{sm} = \sum_{i=1}^{\mu} \underline{\Phi}_{im} Y_{im}(t) \qquad (16)$$

where μ is the number of modes of vibration that is equal to the number of degrees of freedom

$$Y_{im}(t) = \frac{1}{\overline{M}_i \omega^*_{im}} \int_0^t \overline{\Pi}_{sm}(\tau) \; h^*(t - \tau) \, d\tau +$$

$$+ e^{-\eta_{im} \omega_{im} t} \Big[Y_{im}(0) \cos \omega^*_{im} t +$$

$$+ \frac{\dot{Y}_{im}(0) + \eta_{im}\,\omega_{im}\,Y_{im}(0)}{\omega^*_{im}} \sin \omega^*_{im}\, t \Bigg]$$

The expressions for $h^*(t)$ and ω^*_{im} have been given before (13). If D^*_{sm} and $\dot{D}^*_{sm}$ are given, at initial time $t = 0$ the values of $Y_{im}(0)$ and $\dot{Y}_{im}(0)$ can be obtained from the equation (16) and its derivative with respect to the time t.

Results

A computer program has been written, using the theory already described. However, in order to reduce the computational task, only the case $K_1 = K_2 = 0$ (folded plate structures) have been considered.

This program has been applied to the following two examples:

a) Simply supported cylindrical shell.
The first few natural frequencies of a steel cylindrical shell of 6 ft. mean diameter, $\frac{1}{2}$ in. thickness and effective length between the end supports of 8 ft. will be determined. It is assumed: $E = 30 \times 10^6$ lb/in^2, $\nu = 0.3$ and $\rho g = 0.283$ lb/in^3 for steel. This example was solved by the Rayleigh-Ritz method[4], for m = number of axial half-waves = 1.

Results from Ref. 4	91	104	117	139	cycles/sec.
Author's results	86	98	114	130	cycles/sec.

b) Simply supported concrete box bridge
The bridge cross-section is presented in Fig. 2. The bridge span is 30.00 m. The concrete properties are:

Young's modulus $E = 300.000$ kgcm^{-2}

Poisson's coefficient $\nu = 0.2$

Relative mass unit 2.5

Relative damping coefficient $\eta = 0.02$

A knife load is assumed to act along the edges 1 and 1' of the bridge. This loading varies with time according to the Fig. 3.

The structural model of the bridge is given in Fig. 4. Two cases have been considered: a) Symmetrical vibrations with respect to the axis A-A. b) Antisymmetrical vibrations with respect to the axis A-A.

In Figs. 5 and 6 the first frequencies and corresponding modes of vibrations for the two cases are shown.

An example of a time history dynamic analysis is presented in Figs. 7 and 8, where the vertical deflection at node 1 and the transversal bending moment at node 10 are given, for the center of the bridge span.

Conclusions

The numerical method of dynamic analysis of translational shells presented here can be easily implemented into an electronic digital computer. The results obtained by this method are dependent on the fine subdivision of the shell, but even for coarse subdivision, the numerical accuracy can be adequate. The computational effort is dramatically reduced by using this method in comparison with another more powerful and general shell dynamic analysis, such as, the Finite Element Method.

References

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2. Jenkins R.S., "Theory and design of cylindrical shell structures", London, 1947.
3. Samartin, A., and Munro, J., "Dynamic Analysis of Translational shells", CSTR 67/2, Imperial College of Science and Technology Report, London, 1967.
4. Warburton, G.B., "The Dynamical Behaviour of Structures", Pergamon Press, 1964.

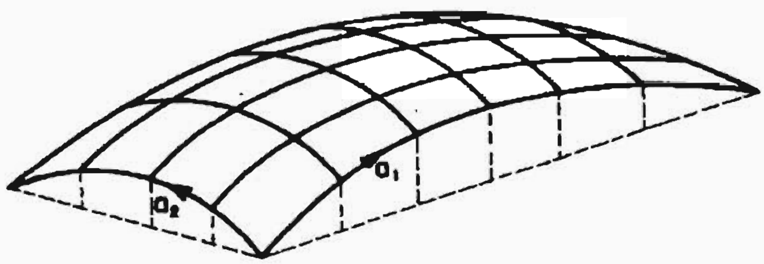

Fig. 1. Coordinate lines

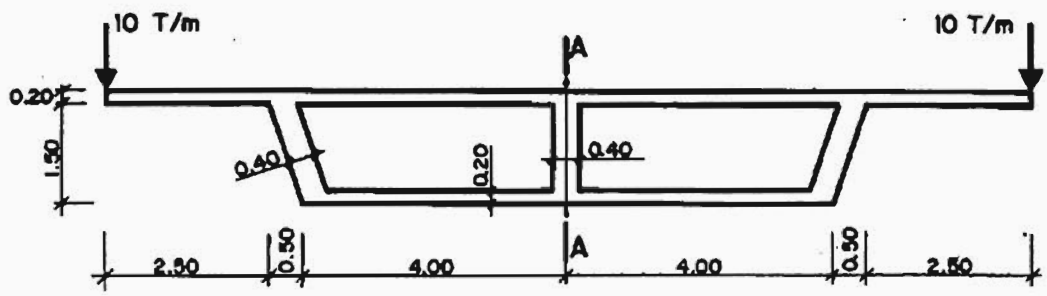

Fig. 2. Transversal cross-section

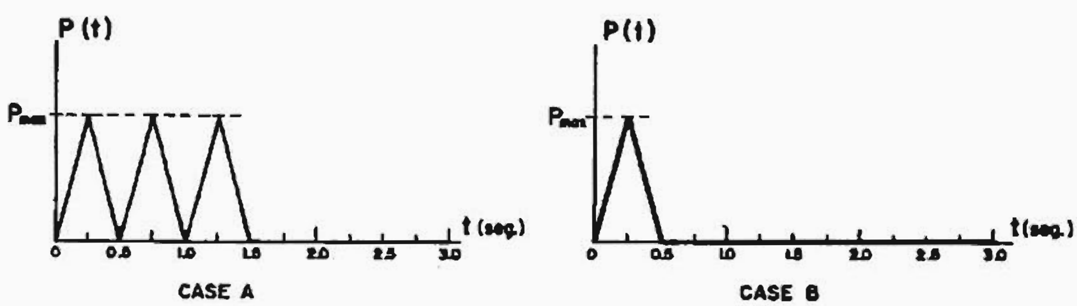

Fig. 3. Loading temporal functions

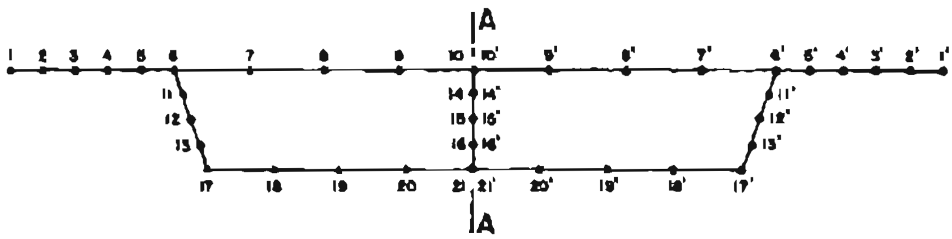

Fig. 4. Structural model

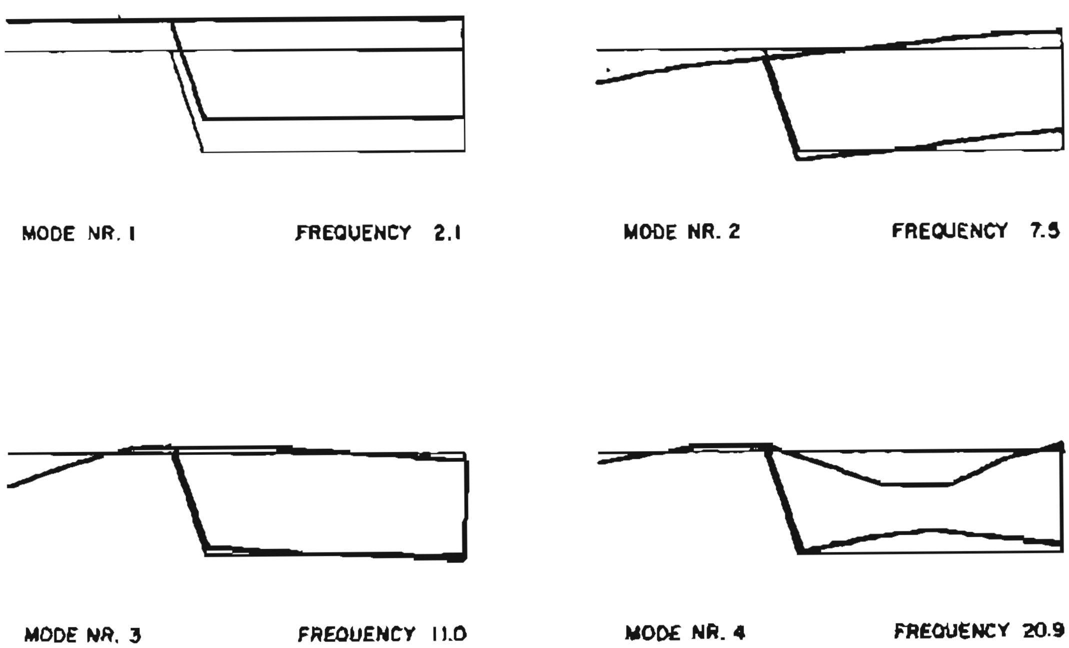

Fig. 5. Symmetrical vibrations

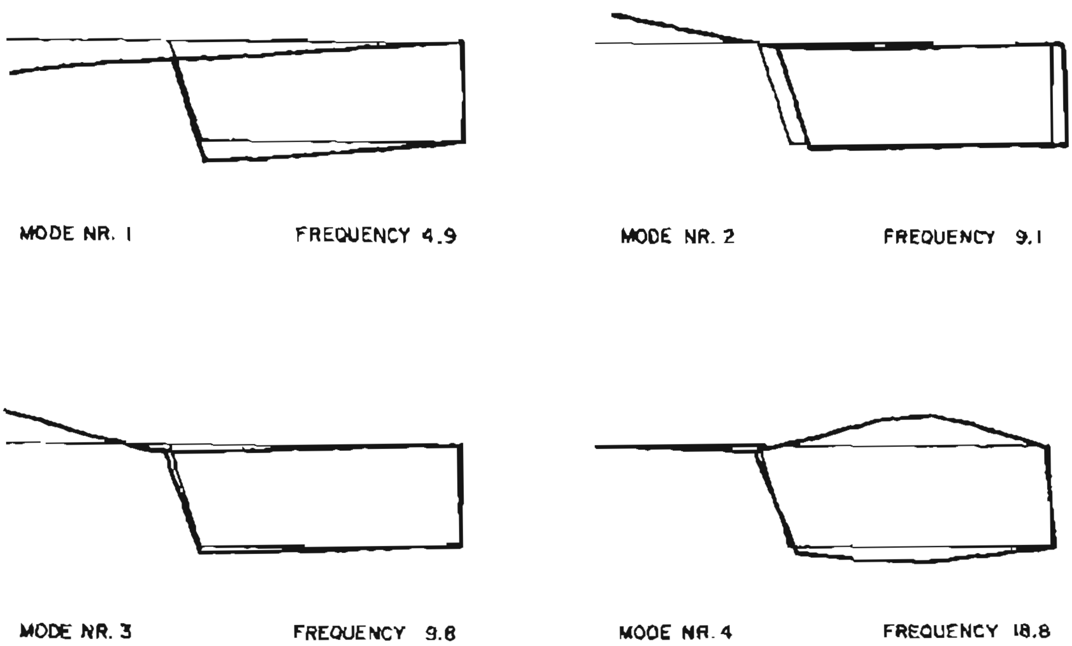

Fig. 6. Antisymmetrical vibrations

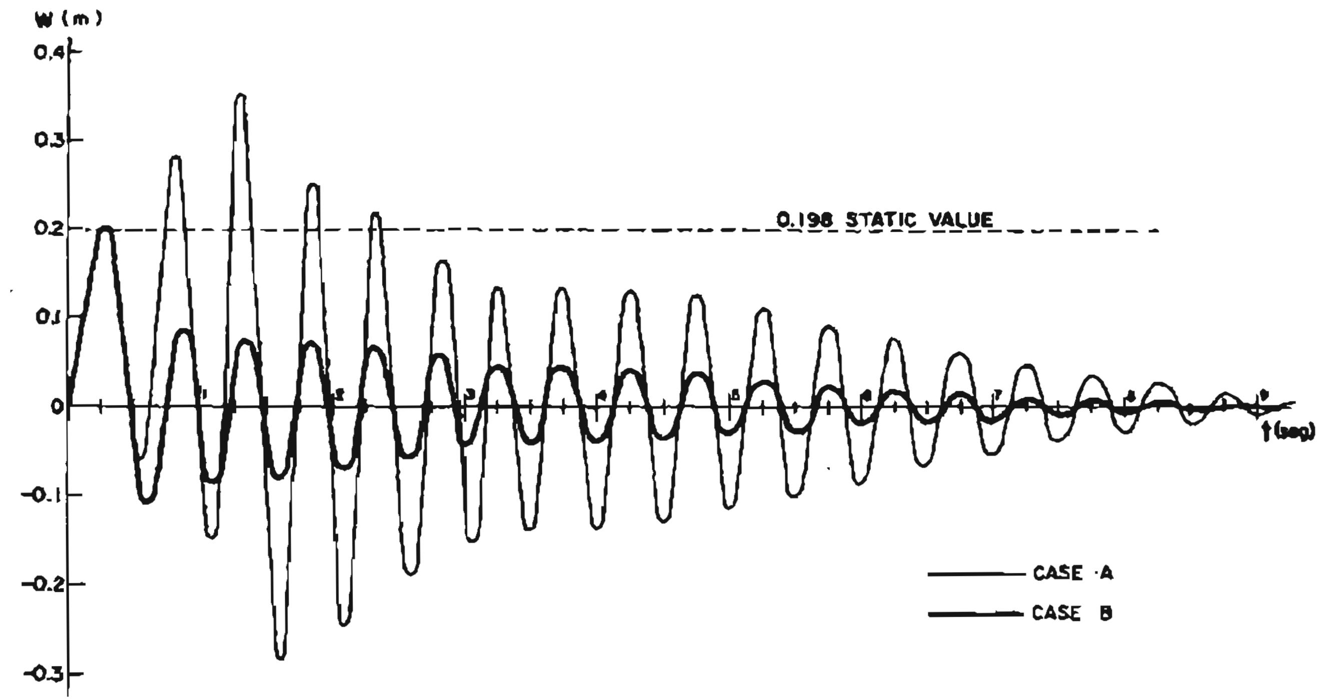

Fig. 7. Vertical deflection at node 1

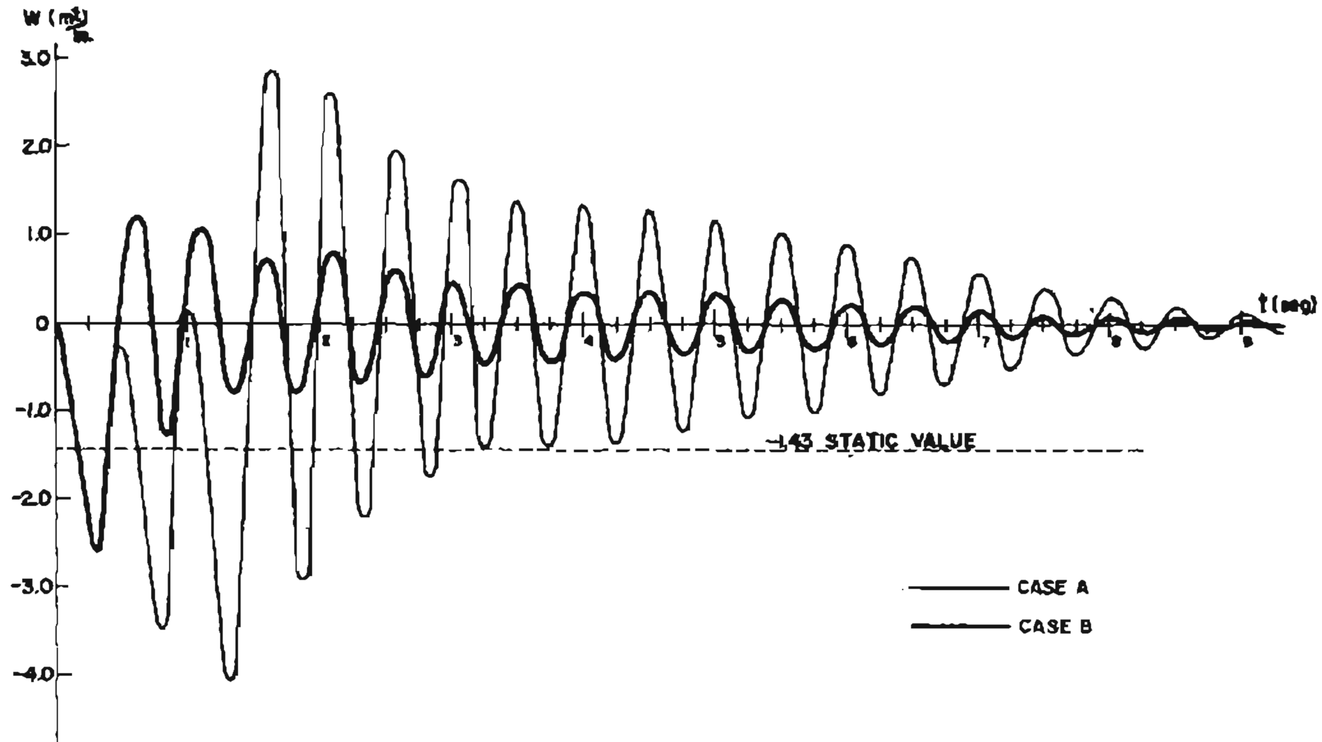

Fig. 8. Bending moment at node 10