Measuring contradiction regarding a negation on AIFS*

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Abstract

In [5], an axiomatic model for contradiction measures on Atanassov Intuitionistic fuzzy sets was presented; there, different kinds of those measures, depending on the continuity conditions required, were established. But in previous papers (see [4]), not only the contradiction in general, but also the contradiction with respect to a given strong intuitionistic fuzzy negation were studied. This is due to the fact that in some applications, in order to fix a suitable model, not any negation is valid, but it is necessary to use a particular one. Thus, the problem of the axiomatization of the different types of contradiction measures regarding a given strong negation remained open. This is the main aim of the present work.

Keywords: Atanassov Intuitionistic fuzzy sets, $\mathcal{N}$-contradiction measures, continuity from below and from above.

1 Preliminaries

1.1 An Atanassov intuitionistic fuzzy set (AIFS) is a set $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ are called the membership and non-membership functions, respectively, and such that, for all $x \in X$, $\mu_A(x) + \nu_A(x) \leq 1$ (see [1]). Let us denote the set of all intuitionistic fuzzy sets on X as $\mathcal{IF}(X)$.

An AIFS could also be considered as an L-fuzzy set as defined by Goguen in [10], where the lattice L is the set $\mathbb{L} = \{(\alpha_1, \alpha_2) \in [0,1]^2 : \alpha_1 + \alpha_2 \leq 1\}$, with the partial order $\leq_{\mathbb{L}}$ defined as follows: given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{L}$,

$$\boldsymbol{\alpha} \leq_{\mathbb{L}} \boldsymbol{\beta} \iff \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \geq \beta_2 .$$

$(\mathbb{L}, \leq_{\mathbb{L}})$ is a complete lattice with smallest element $0_{\mathbb{L}} = (0,1)$, and greatest element $1_{\mathbb{L}} = (1,0)$.

So, an AIFS A is an $\mathbb{L}$-fuzzy set whose $\mathbb{L}$-membership function $\chi^A \in \mathbb{L}^X = \{\chi : X \to \mathbb{L}\}$ is defined for each $x \in X$ as $\chi^A(x) = (\mu_A(x), \nu_A(x))$. The order $\leq_{\mathbb{L}}$ induces, in a natural way, a partial order in $\mathbb{L}^X$, that we denote in the same way. In this way $(\mathbb{L}^X, \leq_{\mathbb{L}})$ is a bounded and complete lattice.

Furthermore, let us recall that a decreasing function $\mathcal{N} : \mathbb{L} \to \mathbb{L}$ is an intuitionistic fuzzy negation (IFN) if $\mathcal{N}(0_{\mathbb{L}}) = 1_{\mathbb{L}}$ and $\mathcal{N}(1_{\mathbb{L}}) = 0_{\mathbb{L}}$ hold. Moreover, $\mathcal{N}$ is a strong IFN if the equality $\mathcal{N}(\mathcal{N}(\boldsymbol{\alpha})) = \boldsymbol{\alpha}$ holds for all $\boldsymbol{\alpha} \in \mathbb{L}$.

Bustince *et al.* introduced in [3] the intuitionistic fuzzy generators, which can be used to construct intuitionistic fuzzy negations, and Deschrijver *et al.* focused on this problem in [8] and [9], and proved that any strong IFN $\mathcal{N}$ is characterized by a strong negation $N : [0,1] \to [0,1]$ by means of the formula $\mathcal{N}(\alpha_1, \alpha_2) = (N(1-\alpha_2), 1 - N(\alpha_1))$, for all $(\alpha_1, \alpha_2) \in \mathbb{L}$. It will be said that N is the

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negation associated to $\mathcal{N}$.

1.2 The study of contradiction in the framework of intuitionistic fuzzy sets was initiated in [6]. Similarly to the fuzzy case, an AIFS A, or alternatively χ^A, is said to be contradictory with respect to some strong IFN $\mathcal{N}$, or, to be short, $\mathcal{N}$-contradictory, if $\chi^A(x) \leq_{\mathbb{L}} (\mathcal{N} \circ \chi^A)(x)$ for all $x \in X$. Also A, or χ^A, is said to be contradictory (without depending on any specific negation) if there exists a strong negation $\mathcal{N}$, such that A is $\mathcal{N}$-contradictory.

Nevertheless, it is interesting to know not only if a set is contradictory, but also the extent to which this property holds; that is, it is necessary to measure somehow the degree of contradiction of any AIFS. In order to do this, in [4] some functions were proposed to measure both the degree of $\mathcal{N}$-contradiction with respect to a strong negation $\mathcal{N}$, and the degree of contradiction of an AIFS. And in [5], an axiomatic model to measure contradiction is given. In a similar way, this paper focuses on establishing an axiomatic model to measure $\mathcal{N}$-contradiction.

1.3. In the previous paper [4], Castiñeira *et al.* analyzed the regions of $\mathbb{L}$ in which contradictory sets with respect to a given negation are located, with the purpose of suggesting the way to measure how contradictory an AIFS is. In [6] it was proved that, given $\chi^A = (\mu_A, \nu_A) \in \mathbb{L}^X$, and $\mathcal{N}$ a strong IFN associated with the strong negation N, χ^A is $\mathcal{N}$-contradictory if and only if $N(\mu_A(x)) + \nu_A(x) \geq 1$, for all $x \in X$. Thus a region free of contradiction is determined in $\mathbb{L}$, as well as other region where contradictory sets remain. Being more specific, if $\chi^A(X) = \{\chi^A(x) : x \in X\}$ is the range of χ^A, the set A is $\mathcal{N}$-contradictory if and only if

$$\chi^A(X) \subset \{(\alpha_1, \alpha_2) \in \mathbb{L} \,|\, N(\alpha_1) + \alpha_2 \geq 1\}$$

Moreover, let $\mathbb{L}_{\mathcal{N}} = \{(\alpha_1, \alpha_2) \in \mathbb{L} : N(\alpha_1) + \alpha_2 \leq 1\}$, and the boundary curve $N(\alpha_1) + \alpha_2 = 1$ satisfies the following properties:

1) It determines an increasing function of α_1.
2) It contains the point (0,0).
3) Its intersection with the line $\alpha_1 + \alpha_2 = 1$ is the point $(\alpha_N, 1 - \alpha_N)$, being α_N the equilibrium point of the negation N.

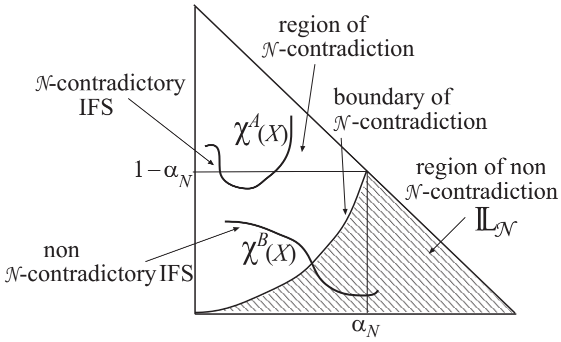

Figure 1: Regions of $\mathcal{N}$-contradiction and non-$\mathcal{N}$-contradiction

2 Measures of $\mathcal{N}$-Contradiction

In [4], in order to measure the $\mathcal{N}$-contradiction of AIFS, the following functions $\mathcal{C}_i^{\mathcal{N}} : \mathbb{L}^X \to [0,1]$, $i = 1,2,3$, were proposed. If $\chi = (\mu, \nu) \in \mathbb{L}^X$, then:

$\mathcal{C}_1^{\mathcal{N}}(\chi) = \mathrm{Max}(0, \operatorname*{Inf}_{x \in X}(N(\mu(x)) + \nu(x) - 1))$

$\mathcal{C}_2^{\mathcal{N}}(\chi) = \mathrm{Max}(0, 1 - \operatorname*{Sup}_{x \in X}(g(\mu(x)) + g(1 - \nu(x))))$,

where $g : [0,1] \to [0,1]$ is an order automorphism satisfying $N(x) = g^{-1}(1 - g(x))$ for all $x \in [0,1]$.

$\mathcal{C}_3^{\mathcal{N}}(\chi) = \frac{d(\chi(X), \mathbb{L}_{\mathcal{N}})}{d(0_L, \mathbb{L}_{\mathcal{N}})}$, where d is the Euclidean distance.

But it is necessary to determine what is understood as a measure of $\mathcal{N}$-contradiction. That is, which are the properties demanded to a function to accept it measures adequately the $\mathcal{N}$-contradiction.

Before introducing the $\mathcal{N}$-contradiction measures, we need a previous definition.

Definition 2.1. Let $\chi \in \mathbb{L}^X$; we say that χ is $\mathbb{L}_{\mathcal{N}}$-*normal* if $\overline{\chi(X)} \cap \mathbb{L}_{\mathcal{N}} \neq \emptyset$, where $\overline{\chi(X)}$ is the closure of $\chi(X)$ in the usual topology in $\mathbb{R}^2$.

Furthermore, χ is said to be $\mathbb{L}$-*normal* if $\overline{\chi(X)} \cap \{(\alpha_1, \alpha_2) \in \mathbb{L}\ ;\ \alpha_2 = 0\} \neq \emptyset$.

The set of all $\mathbb{L}_{\mathcal{N}}$-normal AIFS will be denoted by $\mathbb{L}_{\mathcal{N}}^X$. And the set of all $\mathbb{L}$-normal AIFS, $\mathbb{L}_0^X$.

Let us observe that $\chi \in \mathbb{L}^X$ is $\mathbb{L}$-normal if and only if it is $\mathbb{L}_{\mathcal{N}}$-normal for all strong IFN $\mathcal{N}$. That is, $\bigcap_{\mathcal{N}} \mathbb{L}^X_{\mathcal{N}} = \mathbb{L}^X_0$.

Now a first proposal is given.

Definition 2.2. Let $X \neq \emptyset$ be a universe of discourse and $\mathcal{N}$ a strong IFN; a function $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ is a *measure of $\mathcal{N}$-contradiction* on $\mathcal{IF}(X)$, or equivalently on $\mathbb{L}^X$, if the following is satisfied:

(c.i) $\mathcal{C}_{\mathcal{N}}(\chi^{0_{\mathbb{L}}}) = 1$, where $\chi^{0_{\mathbb{L}}}(x) = 0_{\mathbb{L}}$ for all $x \in X$.

(c.ii) If $\chi \in \mathbb{L}^X_{\mathcal{N}}$, then $\mathcal{C}_{\mathcal{N}}(\chi) = 0$.

(c.iii) Anti-monotonicity: If $\chi^A, \chi^B \in \mathbb{L}^X$ verify $\chi^A(x) \leq_{\mathbb{L}} \chi^B(x)$ for all $x \in X$, then $\mathcal{C}_{\mathcal{N}}(\chi^A) \geq \mathcal{C}_{\mathcal{N}}(\chi^B)$.

Remark. If in the axiom (c.ii) we replace $\mathbb{L}^X_{\mathcal{N}}$ with $\mathbb{L}^X_0$, the definition is just that of contradiction measure given in [5].

The set of all measures of $\mathcal{N}$-contradiction on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}(\mathbb{L}^X)$. Recall that the set of all contradiction measures is denoted by $\mathcal{CM}(\mathbb{L}^X)$.

Remark. Obviously, $\mathcal{NCM}(\mathbb{L}^X) \subset \mathcal{CM}(\mathbb{L}^X)$.

In [4] it was proved that the functions $\mathcal{C}^{\mathcal{N}}_1$, $\mathcal{C}^{\mathcal{N}}_2$, $\mathcal{C}^{\mathcal{N}}_3$ defined above satisfy the axioms (c.i) and (c.iii), moreover it is not difficult to show that they also satisfy axiom (c.ii); hence $\mathcal{C}^{\mathcal{N}}_1$, $\mathcal{C}^{\mathcal{N}}_2$, $\mathcal{C}^{\mathcal{N}}_3$ are measures of $\mathcal{N}$-contradiction.

Furthermore, those $\mathcal{N}$-contradiction measures seem to vary their values in a gradual way; nevertheless the previous definition does not guarantee any kind of continuity in the measures, as the following example shows: The function $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$, given by

$$\mathcal{C}_{\mathcal{N}}(\chi) = \begin{cases} 1, & \text{if } \chi = \chi^{0_{\mathbb{L}}} \\ 0, & \text{otherwise} \end{cases}$$

is a measure of $\mathcal{N}$-contradiction, that changes sharply in $\chi^{0_{\mathbb{L}}}$.

So, if we want to modelize the continuity in the $\mathcal{N}$-contradiction measures, we need to impose some additional conditions. The following two sections are devoted to this subject.

3 Completely Semi-continuous $\mathcal{N}$-Contradiction measures

In order to demand a measure changes smoothly, we propose a new definition.

Definition 3.1. Let $X \neq \emptyset$ and $\mathcal{N}$ a strong IFN; an $\mathcal{N}$-contradiction measure $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ is to be said *completely semi-continuous from below* on $\mathbb{L}^X$ if the following axiom is satisfied:

(c.iv) For all $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$, where $\mathcal{I}$ is an arbitrary set of indexes,

$$\operatorname*{Inf}_{i\in\mathcal{I}} \mathcal{C}_{\mathcal{N}}(\chi^i) = \mathcal{C}_{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)$$

holds, where $\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i \in \mathbb{L}^X$ is defined as $\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)(x) = \operatorname*{Sup}_{i\in\mathcal{I}} \chi^i(x)$, for all $x \in X$.

It is easy to prove that (c.iv) implies (c.iii).

The set of all completely semi-continuous from below $\mathcal{N}$-contradiction measures on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Remark. $\mathcal{NCM}_{csc}(\mathbb{L}^X) \subset \mathcal{CM}_{csc}(\mathbb{L}^X)$, where $\mathcal{CM}_{csc}(\mathbb{L}^X)$ is the set of contradiction measures satisfying axiom (c.iv).

Proposition 3.2. Let $\mathcal{N}$ be a strong IFN, N the strong fuzzy negation associated with $\mathcal{N}$ and α_N the equilibrium point of N. For each $p \in (0, \alpha_N]$, let $\mathcal{C}_{\mathcal{N},p} : \mathbb{L}^X \to [0,1]$ be the function defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by:

$$\mathcal{C}_{\mathcal{N},p}(\chi) = \begin{cases} 0, & \text{if } \operatorname*{Sup}_{x\in X} \mu(x) > p \\ \operatorname{Max}\left(0, \frac{\operatorname*{Inf}_{x\in X} \nu(x) - 1 + N(p)}{N(p)}\right), & \text{else} \end{cases}$$

Then $\mathcal{C}_{\mathcal{N},p} \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Proof. Before confirming the axioms, let us notice that the function has a simple geometrical interpretation (see figure 2) since it can be written as

$$\mathcal{C}_{\mathcal{N},p}(\chi) = \begin{cases} 0, & \text{if } \begin{cases} \operatorname*{Sup}_{x\in X} \mu(x) > p & \text{or} \\ \operatorname*{Inf}_{x\in X} \nu(x) \leq 1 - N(p) \end{cases} \\ \frac{\operatorname*{Inf}_{x\in X} \nu(x) - 1 + N(p)}{N(p)}, & \text{otherwise} \end{cases}$$

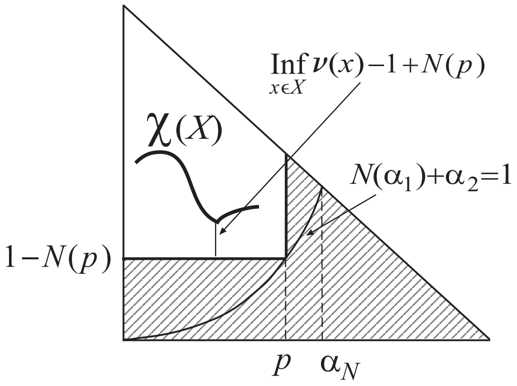

Figure 2: Measure $\mathcal{C}_{\mathcal{N},\,p} \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Now, let us prove the conditions.

(c.i) $\mathcal{C}_{\mathcal{N},p}(\chi^{0_L}) = \frac{\operatorname{Inf}_{x\in X} \nu(x)-1+N(p)}{N(p)} = 1$

(c.ii) Let $\chi = (\mu,\nu) \in \mathbb{L}_{\mathcal{N}}^X$, then if there exists $x \in X$ such that $\mu(x) > p$ or $\nu(x) < 1 - N(p)$ then $\mathcal{C}_{\mathcal{N},p}(\chi) = 0$ by the definition; if on the contrary, there is not such an x, then there exists $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} \chi(x_n) = (p, 1 - N(p))$, thus $\mathcal{C}_{\mathcal{N},p}(\chi) = \frac{\lim_{n\to\infty} \nu(x_n)-1+N(p)}{N(p)} = 0$.

(c.iv) Let $\{\chi^i\}_{i\in I}$ be a family of AIFS.

a) If $\operatorname{Sup}_{i\in I} \chi^i = (\operatorname{Sup}_{i\in I} \mu_i, \operatorname{Inf}_{i\in I} \nu_i)$ is such that $\operatorname{Sup}_{x\in X} \operatorname{Sup}_{i\in I} \mu_i(x) > p$, by definition $\mathcal{C}_{\mathcal{N},p}(\operatorname{Sup}_{i\in I} \chi^i) = 0$ is satisfied, and furthermore, there exist $x \in X$ and $j \in I$ satisfying $\mu_j(x) > p$. Then $\mathcal{C}_{\mathcal{N},p}(\chi^j) = 0$ and $\operatorname{Inf}_{i\in I} \mathcal{C}_{\mathcal{N},p}(\chi^i) = 0 = \mathcal{C}_{\mathcal{N},p}(\operatorname{Sup}_{i\in I} \chi^i)$.

b) If $\operatorname{Sup}_{x\in X} \operatorname{Sup}_{i\in I} \mu_i(x) \le p$, then $\mathcal{C}_{\mathcal{N},p}(\operatorname{Sup}_{i\in I} \chi^i) =$

$$\operatorname{Max}\left(0, \frac{\operatorname{Inf}_{x\in X} \operatorname{Inf}_{i\in I} \nu_i(x)-1+N(p)}{N(p)}\right)$$

Furthermore, for all $x \in X$ and $i \in I$, $\mu_i(x) \le p$, and so,

$$\operatorname{Inf}_{i\in I} \mathcal{C}_{\mathcal{N},p}(\chi^i) = \operatorname{Inf}_{i\in I} \operatorname{Max}\left(0, \frac{\operatorname{Inf}_{x\in X} \nu_i(x)-1+N(p)}{N(p)}\right)$$

$$= \operatorname{Max}\left(0, \frac{\operatorname{Inf}_{i\in I} \operatorname{Inf}_{x\in X} \nu_i(x) - 1 + N(p)}{N(p)}\right)$$

$$= \operatorname{Max}\left(0, \frac{\operatorname{Inf}_{x\in X} \operatorname{Inf}_{i\in I} \nu_i(x) - 1 + N(p)}{N(p)}\right) \qquad \square$$

From now on, many proofs will be omitted due to limits of space.

Remark. Would we change in the definition of $\mathcal{C}_{\mathcal{N},p}$ the condition $\operatorname{Sup}_{x\in X} \mu(x) > p$ by $\operatorname{Sup}_{x\in X} \mu(x) \ge p$?

If we want to preserve the continuity of the measure, the answer is not. In fact, if we would have

$$\mathcal{C}(\chi) = \begin{cases} 0, & \text{if } \operatorname{Sup}_{x\in X} \mu(x) \ge p \\ \operatorname{Max}\left(0, \frac{\operatorname{Inf}_{x\in X} \nu(x)-1+N(p)}{N(p)}\right), & \text{else} \end{cases}$$

taking m, with $1 - N(p) < m < 1$, and the family of constant AIFS $\{\chi^n\}_{n\in\mathbb{N}}$, defined by (see figure 3)

$$\chi^n(x) = \left(p - \frac{p}{n}, m\right) \text{ for all } x \in X,$$

it holds $\operatorname{Sup}_{n\in N} \chi^n(x) = (p, m)$ and $\mathcal{C}(\operatorname{Sup}_{n\in\mathbb{N}} \chi^n) = 0$.

Nevertheless, for all $n \in \mathbb{N}$, $\mathcal{C}(\chi^n) = \frac{m-1+N(p)}{N(p)} > 0$, and thus

$$\operatorname{Inf}_{n\in\mathbb{N}} \mathcal{C}(\chi^n) = \frac{m - 1 + N(p)}{N(p)} \ne \mathcal{C}\left(\operatorname{Sup}_{n\in\mathbb{N}} \chi^n\right)$$

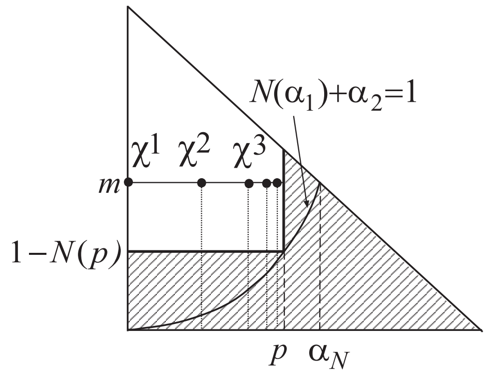

Figure 3: Counterexample.

Remark. In the extremal case $p = \alpha_N$, the measure will be given as (see figure 4)

$$C_{\mathcal{N}}^{\wedge}(\chi) = \operatorname{Max}\left(0, \frac{\operatorname{Inf}_{x\in X} \nu(x) - 1 + \alpha_N}{\alpha_N}\right).$$

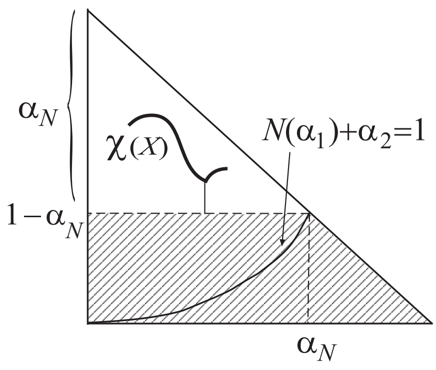

Figure 4: Measure $\mathcal{C}_{\mathcal{N}}^{\wedge} \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Proposition 3.3. Let $f : [0,1] \to [0,1]$ be a continuous and strictly decreasing function such that $f(1) = 0$ and $\alpha + f(\alpha) < 1$ for all $\alpha \in (0,1)$. Let $(p, f(p)) \in \mathbb{L}$ satisfying $f(p) + N(p) = 1$. For all $\beta \in [f(p), f(0))$ let us consider the region

$$\begin{aligned} L_\beta &= \{(\alpha_1, \beta) \mid \alpha_1 \in [0, f^{-1}(\beta)]\} \\ &\bigcup \{(f^{-1}(\beta), \alpha_2) \mid \alpha_2 \in [\beta, 1 - f^{-1}(\beta)]\} \end{aligned}$$

and $L_{f(0)} = \{(0, \alpha_2) \mid \alpha_2 \in [f(0), 1]\}$. Then the function $\mathcal{C}_{\mathcal{N}}^l : \mathbb{L}^X \to [0,1]$ defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ as (see figure 5)

$$\mathcal{C}_{\mathcal{N}}^l(\chi) = \begin{cases} 1, & \text{if } \operatorname*{Sup}_{x\in X} \chi(x) \in L_{f(0)} \\ \frac{\beta - f(p)}{1 - f(p)}, & \text{if } \operatorname*{Sup}_{x\in X} \chi(x) \in L_\beta \text{ for some } \beta \\ 0, & \text{otherwise} \end{cases}$$

satisfies that $\mathcal{C}_{\mathcal{N}}^l \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

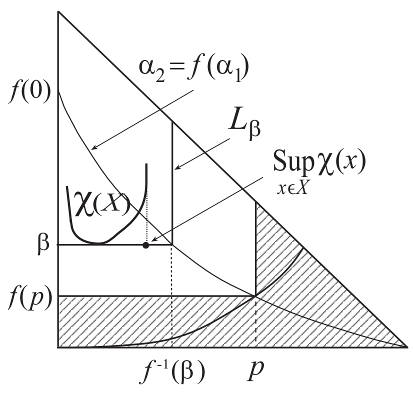

Figure 5: Measure $\mathcal{C}_{\mathcal{N}}^l \in \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

In a similar way, it is possible to define measures demanding the continuity from above.

Definition 3.4. Let $X \neq \emptyset$ and $\mathcal{N}$ a strong IFN; an $\mathcal{N}$-contradiction measure $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ is to be said *completely semi-continuous from above* on $\mathbb{L}^X$ if the following axiom is satisfied:

(c.v) For all $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X \setminus \mathbb{L}_{\mathcal{N}}^X$,

$\operatorname*{Sup}_{i\in\mathcal{I}} \mathcal{C}_{\mathcal{N}}(\chi^i) = \mathcal{C}_{\mathcal{N}}\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right)$ holds, where $\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i \in \mathbb{L}^X$ is defined as $\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right)(x) = \operatorname*{Inf}_{i\in\mathcal{I}} \chi^i(x)$ for all $x \in X$.

Remark. Notice that it is necessary to consider the AIFS are not $\mathbb{L}_{\mathcal{N}}$-normal in the previous axiom. Indeed, let $X = \{x_1, x_2\}$ and the AIFS defined as follows:

$$\chi^1(x_i) = \begin{cases} 0_{\mathbb{L}}, & \text{if } i = 1 \\ (\alpha_N, 1 - \alpha_N), & \text{if } i = 2 \end{cases}$$

$$\chi^2(x_i) = \begin{cases} (\alpha_N, 1 - \alpha_N), & \text{if } i = 1 \\ 0_{\mathbb{L}}, & \text{if } i = 2 \end{cases}$$

Then $\operatorname{Inf}\{\chi^1, \chi^2\}(x_i) = 0_{\mathbb{L}}$, for $i = 1, 2$, and thus $\mathcal{C}_{\mathcal{N}}(\operatorname{Inf}\{\chi^1, \chi^2\}) = 1$, nevertheless $\mathcal{C}_{\mathcal{N}}(\chi^1) = \mathcal{C}_{\mathcal{N}}(\chi^2) = 0$ as $\chi^1, \chi^2 \in \mathbb{L}_{\mathcal{N}}^X$.

Once again, axiom (c.v) implies axiom (c.iii). The set of all completely semi-continuous $\mathcal{N}$-contradiction measures from above on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}^{csc}(\mathbb{L}^X)$.

Remark. $\mathcal{NCM}^{csc}(\mathbb{L}^X) \subset \mathcal{CM}^{csc}(\mathbb{L}^X)$, where $\mathcal{CM}^{csc}(\mathbb{L}^X)$ is the set of contradiction measures satisfying axiom (c.iv).

Example 3.5. Let $\mathcal{C}_{\mathcal{N}}^{\vee} : \mathbb{L}^X \to [0,1]$ be a function defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by (see figure 6):

$$\mathcal{C}_{\mathcal{N}}^{\vee}(\chi) = \begin{cases} 0, & \text{if } \chi \in \mathbb{L}_{\mathcal{N}}^X \\ \operatorname*{Sup}_{x\in X} \nu(x), & \text{otherwise} \end{cases}$$

Then $\mathcal{C}_{\mathcal{N}}^{\vee} \in \mathcal{NCM}^{csc}(\mathbb{L}^X)$. Furthermore, $\mathcal{C}_{\mathcal{N}}^{\vee} \notin \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Remark. The measure $\mathcal{C}_{\mathcal{N}}^l$ is not a completely semi-continuous $\mathcal{N}$-contradiction measure from above.

Indeed, let X be a universe of discourse with $Card(X) \geq 2$, and $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Let us take for $i = 1, 2$ the AIFS

$$\chi^i(x) = \begin{cases} (0, f(p)), & \text{if } x = x_i \\ 0_{\mathbb{L}}, & \text{otherwise} \end{cases}$$

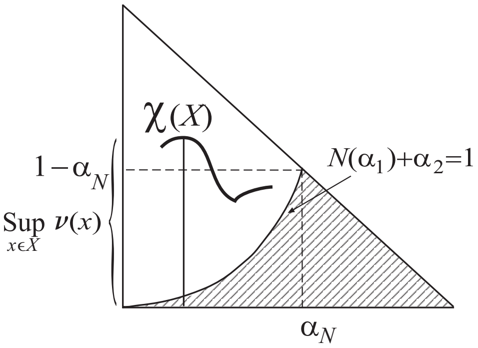

Figure 6: Measure $\mathcal{C}^{\vee}_{\mathcal{N}} \in \mathcal{NCM}^{csc}(\mathbb{L}^X)$.

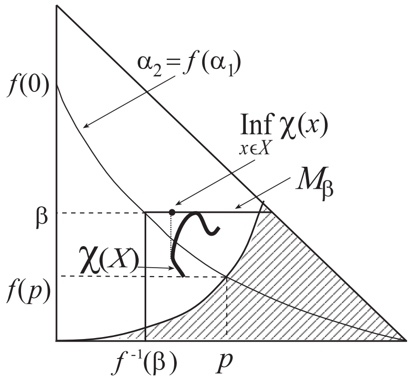

Figure 7: Measure $\mathcal{C}^{u}_{\mathcal{N}} \in \mathcal{NCM}^{csc}(\mathbb{L}^X)$.

Then $\left(\operatorname*{Inf}_{i=1,2} \chi^i\right)(x) = 0_{\mathbb{L}}$ for all $x \in X$. So, $\mathcal{C}^l_{\mathcal{N}}(\operatorname*{Inf}_{i=1,2} \chi^i) = \mathcal{C}^l_{\mathcal{N}}(\chi^{0_{\mathbb{L}}}) = 1$.

But, $\mathcal{C}^l_{\mathcal{N}}(\chi^1) = \mathcal{C}^l_{\mathcal{N}}(\chi^2) = \operatorname*{Sup}_{i=1,2} \mathcal{C}^l_{\mathcal{N}}(\chi^i) = 0$.

Proposition 3.6. Let $f : [0,1] \to [0,1]$ be a continuous and strictly decreasing function such that $f(1) = 0$ and $\alpha + f(\alpha) < 1$ for all $\alpha \in (0,1)$. Let $(p, f(p)) \in \mathbb{L}$ satisfying $f(p) + N(p) = 1$. For all $\beta \in [f(p), f(0)]$ let us consider the region

$$\begin{aligned} M_\beta &= \{(f^{-1}(\beta), \alpha_2) \mid \alpha_2 \in [0, \beta]\} \\ &\bigcup \ \{(\alpha_1, \beta) \mid \alpha_1 \in [f^{-1}(\beta), 1-\beta]\} \end{aligned}$$

and $M_\beta = \{(\alpha_1, \beta) \mid \alpha_1 \in [0, 1-\beta]\}$ if $\beta \in (f(0), 1]$. The function $\mathcal{C}^u_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ as (see fig. 7):

$$\mathcal{C}^u_{\mathcal{N}}(\chi) = \begin{cases} 0, & \text{if } \chi \in \mathbb{L}^X_{\mathcal{N}} \\ \frac{\beta - f(p)}{1 - f(p)}, & \text{if } \chi \notin \mathbb{L}^X_{\mathcal{N}} \ \& \ \operatorname*{Inf}_{x \in X} \chi(x) \in M_\beta \end{cases}$$

satisfies $\mathcal{C}^u_{\mathcal{N}} \in \mathcal{NCM}^{csc}(\mathbb{L}^X)$. Furthermore, $\mathcal{C}^u_{\mathcal{N}} \notin \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

On the other hand, measures $\mathcal{C}^{\mathcal{N}}_1, \mathcal{C}^{\mathcal{N}}_2$ and $\mathcal{C}^{\mathcal{N}}_3$ defined in [4] do not satisfy the conditions demanded in this section, as we are going to show.

Proposition 3.7. If $X \neq \emptyset$ and $\mathcal{N}$ is a strong negation, $\mathcal{N}$-contradiction measures on $\mathbb{L}^X$ $\mathcal{C}^{\mathcal{N}}_1, \mathcal{C}^{\mathcal{N}}_2$ and $\mathcal{C}^{\mathcal{N}}_3$, defined at the beginning of section 2, are neither completely semicontinuous from below nor from above.

Proof. First, let us see that, for $i = 1,2,3$, $\mathcal{C}^{\mathcal{N}}_i \notin \mathcal{NCM}_{csc}(\mathbb{L}^X)$. Let us fix β such that $0 < \beta < 1 - \alpha_N$, and let α such that $N^{-1}(1-\beta) < \alpha < \alpha_N$. We consider the AIFS

$$\left.\begin{array}{l} \chi^1(x) = (0, \beta) \\ \chi^2(x) = (\alpha, 1-\alpha) \end{array}\right\} \ \forall x \in X$$

Then $\left(\operatorname*{Sup}_{j=1,2} \chi^j\right)(x) = (\alpha, \beta)$ for all $x \in X$, and it is easy to prove that for $i = 1,2,3$,

$0 < \operatorname*{Inf}_{j=1,2} \mathcal{C}_i(\chi^j) \neq \mathcal{C}_i(\operatorname*{Sup}_{j=1,2} \chi^j) = 0$.

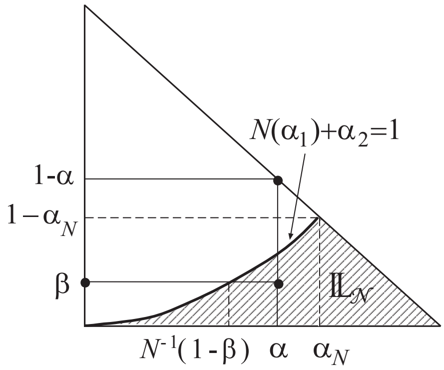

Figure 8: $\mathcal{C}^{\mathcal{N}}_1$, $\mathcal{C}^{\mathcal{N}}_2$, $\mathcal{C}^{\mathcal{N}}_3$ are not in $\mathcal{NCM}_{csc}(\mathbb{L}^X)$.

Second, let us see that, for $i = 1,2,3$, $\mathcal{C}^{\mathcal{N}}_i \notin \mathcal{NCM}^{csc}(\mathbb{L}^X)$. Let us fix α such that $1-\alpha_N < \alpha < 1$, and β with $\alpha < 1-\beta$. Now, we consider the AIFS

$$\left.\begin{array}{l} \chi^1(x) = (0, \alpha) \\ \chi^2(x) = (\beta, 1-\beta) \end{array}\right\} \ \forall x \in X$$

Then $\left(\operatorname*{Inf}_{j=1,2} \chi^j\right)(x) = (0, 1-\beta)$ for all x, and it can be proved that for $i = 1,2,3$,

$$\operatorname*{Sup}_{j=1,2} \mathcal{C}_i(\chi^j) \neq \mathcal{C}_i(\operatorname*{Inf}_{j=1,2} \chi^j). \qquad \square$$

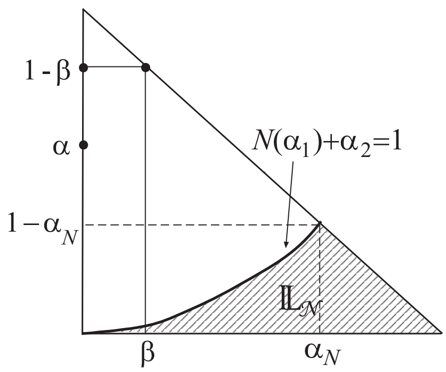

Figure 9: $\mathcal{C}_1^{\mathcal{N}}$, $\mathcal{C}_2^{\mathcal{N}}$, $\mathcal{C}_3^{\mathcal{N}}$ are not in $\mathcal{N}CM^{csc}$

So, we need to weaken the conditions, in order to accept $\mathcal{C}_1^{\mathcal{N}}$, $\mathcal{C}_2^{\mathcal{N}}$ and $\mathcal{C}_3^{\mathcal{N}}$ as $\mathcal{N}$-contradiction measures with some kind of continuity, in such a way that the mathematical model be consistent with the intuition.

4 Semi-continuous $\mathcal{N}$-Contradiction measures

Let us remember that a set $S \subset \mathbb{L}^X$ is a semilattice from below if for all χ^A, $\chi^B \in S$, $\text{Sup}\{\chi^A, \chi^B\} \in S$ holds; and similarly, a set $S \subset \mathbb{L}^X$ is a semilattice from above if for all χ^A, $\chi^B \in S$, $\text{Inf}\{\chi^A, \chi^B\} \in S$ holds (see, for example, [2]).

Definition 4.1. Let $X \neq \emptyset$ and $\mathcal{N}$ a strong IFN, an $\mathcal{N}$-contradiction measure $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ is to be said *semicontinuous from below* if the following axiom is satisfied:

(c.vi) For all semi-lattice from below $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X$, where $\mathcal{I}$ is an arbitrary set, the following is satisfied

$$\operatorname*{Inf}_{i\in\mathcal{I}} \mathcal{C}_{\mathcal{N}}(\chi^i) = \mathcal{C}_{\mathcal{N}}\left(\operatorname*{Sup}_{i\in\mathcal{I}} \chi^i\right)$$

Notice that axiom (c.vi) implies axiom (c.iii).

The set of all semi-continuous from below $\mathcal{N}$-contradiction measures on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}_{sc}(\mathbb{L}^X)$.

Remark. Obviously, $\mathcal{NCM}_{csc}(\mathbb{L}^X) \subset \mathcal{NCM}_{sc}(\mathbb{L}^X)$.

Proposition 4.2. Let $X \neq \emptyset$ and $\mathcal{N}$ and strong IFN. Given a fixed $p \in (0, +\infty)$, for all $\beta \in [0,1]$ let us consider the following region

$$L_\beta = \left\{ (\alpha_1, \alpha_2) \in \mathbb{L} \mid \begin{array}{c} \alpha_1 \in [0, \beta], \\ \alpha_2 = \frac{(\alpha_1+p)(1-\beta)}{\beta+p} \end{array} \right\},$$

that is, L_β is a segment on the line joining the points $(-p, 0)$ and $(\beta, 1-\beta)$.

Given the function $\mathcal{C}_{\mathcal{N}}^L : \mathbb{L}^X \to [0,1]$ defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by (see figure 10):

$$\mathcal{C}_{\mathcal{N}}^L(\chi) = \begin{cases} 0, \text{ if } \chi \in \mathbb{L}_{\mathcal{N}}^X \\ 1-\beta, \text{ if } \chi \notin \mathbb{L}_{\mathcal{N}}^X \ \& \ \operatorname*{Sup}_{x\in X} \chi(x) \in L_\beta \end{cases}$$

we have $\mathcal{C}_{\mathcal{N}}^L \in \mathcal{NCM}_{sc}(\mathbb{L}^X) \setminus \mathcal{NCM}_{csc}(\mathbb{L}^X)$.

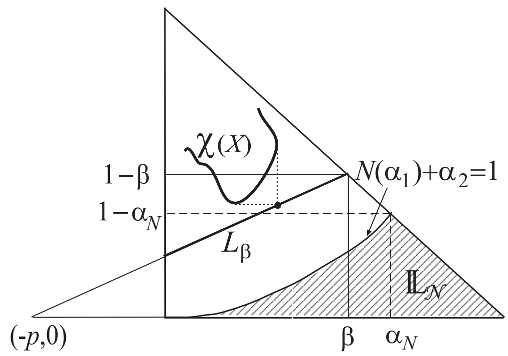

Figure 10: Measure $\mathcal{C}_{\mathcal{N}}^L \in \mathcal{NCM}_{sc}(\mathbb{L}^X)$.

Similarly, we have

Definition 4.3. Let $X \neq \emptyset$, an $\mathcal{N}$-contradiction measure $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \to [0,1]$ is to be said *semicontinuous from above* if the following axiom is satisfied:

(c.vii) For all semilattice from above $\{\chi^i\}_{i\in\mathcal{I}} \subset \mathbb{L}^X \setminus \mathbb{L}_{\mathcal{N}}^X$, where $\mathcal{I}$ is an arbitrary set, the following holds

$$\operatorname*{Sup}_{i\in\mathcal{I}} \mathcal{C}_{\mathcal{N}}(\chi^i) = \mathcal{C}_{\mathcal{N}}\left(\operatorname*{Inf}_{i\in\mathcal{I}} \chi^i\right)$$

Again, (c.vii) implies (c.iii).
The set of all semi-continuous from above $\mathcal{N}$-contradiction measures on $\mathbb{L}^X$ will be denoted by $\mathcal{NCM}^{sc}(\mathbb{L}^X)$.

Remark. $\mathcal{NCM}^{csc}(\mathbb{L}^X) \subset \mathcal{NCM}^{sc}(\mathbb{L}^X)$.

Proposition 4.4. Consider for any $\beta \in [0,1]$, the segment L_β defined in Proposition 4.2.

Let $\mathcal{C}_{\mathcal{N}}^{U} : \mathbb{L}^X \to [0,1]$ be the function defined for each $\chi = (\mu, \nu) \in \mathbb{L}^X$ by (see figure 11):

$$\mathcal{C}_{\mathcal{N}}^{U}(\chi) = \begin{cases} 0, & \text{if } \chi \in \mathbb{L}_{\mathcal{N}}^{X} \\ 1-\beta, & \text{if } \chi \notin \mathbb{L}_{\mathcal{N}}^{X} \ \& \ \operatorname*{Inf}_{x \in X} \chi(x) \in L_\beta \end{cases}$$

Then $\mathcal{C}_{\mathcal{N}}^{U} \in \mathcal{NCM}_{\mathcal{N}}^{sc}(\mathbb{L}^X) \setminus \mathcal{NCM}_{\mathcal{N}}^{csc}(\mathbb{L}^X)$.

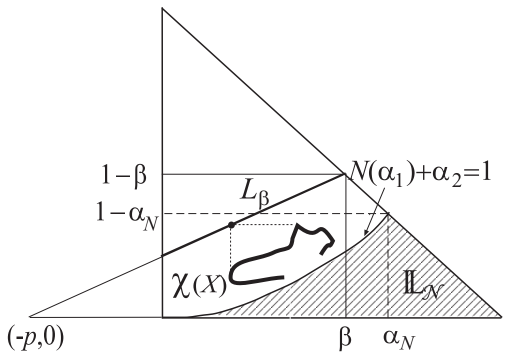

Figure 11: Measure $\mathcal{C}_{\mathcal{N}}^{U} \in \mathcal{NCM}_{\mathcal{N}}^{sc}(\mathbb{L}^X)$.

Now, we have the following result.

Proposition 4.5. For $i = 1, 2, 3$, each measure $\mathcal{C}_i^{\mathcal{N}}$ defined at the beginning of section 2 satisfies that $\mathcal{C}_i^{\mathcal{N}} \in \mathcal{NCM}_{sc}(\mathbb{L}^X)$, but, in general, $\mathcal{C}_i^{\mathcal{N}} \in \mathcal{NCM}^{sc}(\mathbb{L}^X)$ do not hold.

Finally, the functions presented through this paper show the following result.

Proposition 4.6. For any strong IFN $\mathcal{N}$, the following inequalities hold:

$$\mathcal{NCM}_{csc}(\mathbb{L}^X) \subsetneq \mathcal{NCM}_{sc}(\mathbb{L}^X) \subsetneq \mathcal{NCM}(\mathbb{L}^X)$$
$$\mathcal{NCM}^{csc}(\mathbb{L}^X) \subsetneq \mathcal{NCM}^{sc}(\mathbb{L}^X) \subsetneq \mathcal{NCM}(\mathbb{L}^X)$$

Conclusions

Contradictory sets can result inconvenient in certain applications, for instance, in the processes of fuzzy inference. Until now, a mathematic model had been defined to measure in which degree an AIFS is contradictory. However, demanding that an object have a small contradictory degree can be very restrictive and it may result more interesting to measure that degree regarding a given negation, if that negation is the one used in a specific application. That is why, in this work, we have presented a mathematic model to measure the $\mathcal{N}$-contradiction of an AIFS. Moreover, we have obtained families of measures that satisfy different kinds of continuity.

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